Smoothness of the Arctangent of a Square

Proof

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Let $f(x) \coloneqq \arctan x^2$. Then, its first derivative is given by

$$\frac{df}{dx}(x) = \frac{d}{du}\left(\arctan u\right)\bigg|_{u=x^2} \frac{d}{dx}\left(x^2\right) = \frac{2x}{x^4+1} = \frac{p_1(x)}{q_1(x)} \tag{1}$$

where p_1 and q_1 are the polynomials defining the quotient that defines the first-order derivative. From Equation (1), the following can be stated:

- Since $x^4 + 1 \neq 0 \quad \forall x \in (-\infty, \infty)$, there is no finite value of x for which the first derivative is not finite.
- Since $1 = deg(p_1) < deg(q_1) = 4$, the limit of the derivative as $x \to \pm\infty$ is 0.

Thus, $\arctan x^2$ is proven to be C^1 $\forall x \in \mathbb{R}$.

Now, using the quotient rule, the second derivative can be written in terms of the polynomials p_1 and q_1 to form a new quotient; namely

$$\begin{aligned}\frac{d^2 f}{dx^2}(x) &= \frac{d}{dx}\left(\frac{df}{dx}(x)\right) = \frac{d}{dx}\left(\frac{p_1(x)}{q_1(x)}\right)\\ &= \frac{p_1'(x)q_1(x) - p_1(x)q_1'(x)}{\left(q_1(x)\right)^2} = \frac{-6x^4+2}{(x^4+1)^2} = \frac{p_2(x)}{q_2(x)}.\end{aligned} \tag{2}$$

By induction, from the expression in Equation (2), the derivative of degree $n+1$ can be written as a recursion of the form

$$\begin{aligned}\frac{d^{n+1} f}{dx^{n+1}}(x) &= \frac{d}{dx}\left(\frac{d^n f}{dx^n}(x)\right) = \frac{d}{dx}\left(\frac{p_n(x)}{q_n(x)}\right)\\ &= \frac{p_n'(x)q_n(x) - p_n(x)q_n'(x)}{\left(q_n(x)\right)^2} = \frac{p_{n+1}(x)}{q_{n+1}(x)}\end{aligned} \tag{3}$$

for all $n \geq 1$.

From Equation (3), it is easy to see that q_n is given by q_1 elevated to a non-negative power of 2; namely,

$$q_n(x) = q_1(x)^{2^{n-1}} = (x^4+1)^{2^{n-1}} \quad \forall n \in \mathbb{Z}^+. \tag{4}$$

Since $\nexists x \in (-\infty, \infty), n \in \mathbb{Z}^+$ for which Equation (4) is zero, there is no finite value of x for which the n^{th} derivative is not finite.

Furthermore, the degree dominating these polynomials can also be expressed as a function of their predecessors: let P_n and Q_n denote the maximum degrees of the polynomials p_n and q_n, respectively; then, using the polynomial definitions in Equation (3), these can be rewritten as a recursive system of equations of the form

$$\begin{cases} P_{n+1} &= P_n + Q_n - 1, \\ Q_{n+1} &= 2Q_n. \end{cases} \tag{5}$$

To get Equation (5) purely in terms of n, the first few terms of the recursion are expanded as

$$\begin{aligned}
n = 2 \implies & \begin{cases} P_2 &= 1 + 4 - 1 \\ &= (2^0 + 2^1 + 2^2 - 2^1) - 1, \\ Q_2 &= 2 \cdot 4 = 2^3, \end{cases} \\
n = 3 \implies & \begin{cases} P_3 &= ((2^0 + 2^1 + 2^2 - 2^1) - 1) + 2^3 - 1 \\ &= (2^0 + \cdots + 2^3 - 2^1) - 2, \\ Q_3 &= 2 \cdot 2^3 = 2^4, \end{cases} \\
n = 4 \implies & \begin{cases} P_4 &= ((2^0 + \cdots + 2^3 - 2^1) - 2) + 2^4 - 1 \\ &= (2^0 + \cdots + 2^4 - 2^1) - 3, \\ Q_4 &= 2 \cdot 2^4 = 2^5, \end{cases} \\
& \vdots
\end{aligned}$$

This emerging pattern can be used to rewrite Equation (5) as

$$\begin{cases} P_{n+1} &= \left(\sum_{k=0}^{n} 2^k - 2^1\right) - (n-1) \\ &= 2^{n+1} - n - 2, \\ Q_{n+1} &= 2^{n+1}. \end{cases} \tag{6}$$

Finally, using Equation (6) and L'Hôpital's rule, the limit of the nth order derivative as $x \to \pm\infty$ is calculated as

$$\begin{aligned} \lim_{x \to \pm\infty} \frac{d^n f}{dx^n}(x) = \lim_{x \to \pm\infty} \frac{p_n(x)}{q_n(x)} &\overset{H}{=} \lim_{x \to \pm\infty} \overset{|\alpha_n| < \infty}{\overset{\downarrow}{\alpha_n}} \; x^{P_n - Q_n} \\ &= \lim_{x \to \pm\infty} \alpha_n x^{-n-2} = 0 \quad \forall n \in \mathbb{Z}^+. \end{aligned} \tag{7}$$

Thus, $\arctan x^2$ is proven to be $C^\infty \, \forall x \in \mathbb{R}$.