1. Where in a min heap the largest element resides? (Assume all elements are distinct) Explain.

Suppose that we have a min heap with n elements where $n > 1$. Suppose further that the largest element was an internal node, then that means it has atleast one child. This would be a contradiction since the largest element would be larger than its children which contradicts the min-heap property. we conclude that the largest element must be a leaf.

10. Use induction to prove that $1 + 2 + 4 + \ldots + 2^h = 2^{h+1} - 1$.

Proof. We prove by induction

Let $P(h) = 1 + 2 + 4 + \ldots + 2^h = 2^{h+1} - 1$

Base case: $h = 0$

$1 = 1 = 2^{0+1} - 1$.

Thus $P(0)$ holds.

Inductive step: Let $P(k)$ be true we show that $P(k+1)$ is true, that is $1 + 2 + 4 + \ldots + 2^k = 2^{(k+1)+1} - 1$

$$\begin{aligned} 1 + 2 + 4 + \ldots + 2^k &= \sum_{i=0}^{k+1} 2^i \\ &= \sum_{i=0}^{k} 2^i + 2^{k+1} \\ &= 2^{k+1} - 1 + 2^{k+1} \\ &= 2 \cdot 2^{k+1} - 1 \\ &= 2^{k+1+1} - 1 \\ &= 2^{(k+1)+1} - 1 \end{aligned}$$

Thus $P(k+1)$ holds. By the principle of mathematical induction, $P(h)$ holds for for all integers $h \geq 0$. $\square$

11. What is the minimum and maximum number of leaves in a binary heap that has height h. Explain.

When $h = 0$ then the minimum equal the maximum namely 1. Suppose that $h \geq 1$. A tree with height h must have at least one leaf at level h. The rest of the leaves are on the $h-1$ level where the leftmost node is the only parent. Thus there are a total of $2^{h-1} - 1 + 1 = 2^{h-1}$ minimum number of leafs for height h. In a perfect tree where all levels are filled we have $\sum_{i=0}^{h} 2^i = 2^{h+1} - 1$ total nodes where the leafs are the last summation term and thus contribute 2^h nodes. Thus the minimum and maximum number of leaves in a binary heap that height h is 2^{h-1} and 2^h leaves for $h \geq 1$.

12. Prove that a binary heap with n elements has height $\lfloor log_2(n) \rfloor$.

$$\begin{aligned} 2^h \leq & \quad n & \leq 2^{h+1} - 1 \\ 2^h \leq & \quad n & < 2^{h+1} \\ log_2(2^h) \leq & \quad log_2(n) & < log_2(2^{h+1}) \\ h \leq & \quad log_2(n) & < h + 1 \end{aligned}$$

which by definition of the floor $h = \lfloor log_2(n) \rfloor$. Hence we conclude a binary heap with n elements has height $\lfloor log_2(n) \rfloor$.

13. Prove that a binary heap with n nodes has exactly $\lceil \frac{n}{2} \rceil$ leaves.