

1. Prove that  $f(n) = 10 \cdot n^4 + 2 \cdot n^2 + 3$  is  $O(n^4)$ , provide the appropriate  $C$  and  $k$  constants.

$$f(n) = O(g(n)) \iff \exists c > 0, \exists k \geq 0 \text{ s.t. } f(n) \leq c \cdot g(n) \forall n \geq k$$

$$\begin{aligned} 10 \cdot n^4 + 2 \cdot n^2 + 3 &\leq 10 \cdot n^4 + 2 \cdot n^4 + 3 \\ &\leq 10 \cdot n^4 + 2 \cdot n^4 + 3 \cdot n^4 \\ &= 15 \cdot n^4, n \geq 1 \end{aligned}$$

Let  $c = 15$  and  $k = 1$  then the statement translates to

$$f(n) \leq c \cdot g(n), n \geq k$$

By the definition of  $O$  notation  $f(n) = O(g(n))$

2. Prove that  $f(n) = 2 \cdot n^2 - n \cdot \log_2(n) + 3 \cdot \log_2(n)$  is  $O(n^2)$ , provide the appropriate  $C$  and  $k$  constants.

$$f(n) = O(g(n)) \iff \exists c > 0, \exists k \geq 0 \text{ s.t. } f(n) \leq c \cdot g(n) \forall n \geq k$$

$$\begin{aligned} 2 \cdot n^2 - n \cdot \log_2(n) + 3 \cdot \log_2(n) &\leq 2 \cdot n^2 + n^2 + 3 \cdot \log_2(n) \\ &\leq 2 \cdot n^2 + n^2 + 3 \cdot n^2 \\ &= 6 \cdot n^2, n \geq 1 \end{aligned}$$

Let  $c = 6$  and  $k = 1$ , then the statement translates to

$$f(n) \leq c \cdot g(n), n \geq k$$

which by the definition of  $O$  notation  $f(n) = O(g(n))$ .

3. Prove that  $f(n) = 2 \cdot n^4 \cdot \log_2(n^4) - n^2 + 3 \cdot \log_2(n)$  is  $O(n^4 \log_2(n))$ , provide the appropriate  $C$  and  $k$  constants.

$$f(n) = O(g(n)) \iff \exists c > 0, \exists k \geq 0 \text{ s.t. } f(n) \leq c \cdot g(n) \forall n \geq k$$

$$\begin{aligned}
2 \cdot n^4 \cdot \log_2(n^4) - n^2 + 3 \cdot \log_2(n) &= 8 \cdot n^4 \cdot \log_2(n) - n^2 + 3 \cdot \log_2(n) \\
&\leq 8 \cdot n^4 \cdot \log_2(n) + n^4 \cdot \log_2(n) + 3 \cdot \log_2(n) \\
&\leq 8 \cdot n^4 \cdot \log_2(n) + n^4 \cdot \log_2(n) + 3 \cdot n \cdot \log_2(n) \\
&= 12 \cdot n^4 \cdot \log_2(n), n \geq 1
\end{aligned}$$

Let  $c = 12$  and  $k = 1$  then the statement translates to

$$2 \cdot n^4 \cdot \log_2(n^4) - n^2 + 3 \cdot \log_2(n) \leq c \cdot n^4 \cdot \log_2(n), n \geq 1$$

which by the definition  $O$  notation  $f(n) = O(g(n))$ .

4. Prove or disprove  $f(n) = 5 \cdot n^3 - n + 3$

1.  $f(n) = O(n^2)$
2.  $f(n) = \Omega(n)$
3.  $f(n) = \Theta(n^3)$
4.  $f(n) = \omega(n)$
5.  $f(n) = o(n^2)$

Provide the appropriate  $C$  and  $k$  constants if possible

5. Prove that  $(n + 5)^{100} = \Theta(n^{100})$

$$f(n) = \Theta(g(n)) \iff f(n) = O(g(n)) \text{ and } f(n) = \Omega(g(n))$$

1. Showing  $(n + 5)^{100} = O(n^{100})$

$$f(n) = O(g(n)) \iff \exists c > 0, \exists k \geq 0 \text{ s.t } f(n) \leq c \cdot g(n) \forall n \geq k$$

$$\begin{aligned}
(n + 5)^{100} &= \sum_{i=0}^{100} \binom{100}{i} \cdot n^{100-i} \cdot 5^i \\
&\leq \sum_{i=0}^{100} \binom{100}{i} \cdot n^{100}, n \geq 1
\end{aligned}$$

Let  $c = \sum_{i=0}^{100} \binom{100}{i}$  and  $k = 1$  then the statements translates to

$$f(n) \leq c \cdot g(n), n \geq k$$

which by the definition of  $O$  notation  $f(n) = O(g(n))$ .

2. Showing  $(n + 5)^{100} = \Omega(n^{100})$

$$f(n) = \Omega(g(n)) \iff \exists c > 0, \exists k \geq 0 \text{ s.t } f(n) \geq c \cdot g(n) \forall n \geq k$$

$$\begin{aligned} (n + 5)^{100} &= \sum_{i=0}^{100} \binom{100}{i} \cdot n^{100-i} \cdot 5^i \\ &\geq n^{100} \end{aligned}$$

Let  $c = 1$  and  $k = 1$  then the statement translates to

$$f(n) \geq c \cdot g(n), n \geq k$$

which by the definition of  $\Omega$  notation  $f(n) = \Omega(g(n))$ .

**6.** Prove transitivity of big- $O$ : if  $f(n) = O(g(n))$ , and  $g(n) = O(h(n))$ , then  $f(n) = O(h(n))$ .

Since  $f(n) = O(g(n))$  and  $g(n) = O(h(n))$  we have the equalities

$$\begin{aligned} f(n) &\leq c_1 \cdot g(n), n \geq k_1 \\ g(n) &\leq c_2 \cdot h(n), n \geq k_2 \end{aligned}$$

From this we obtain

$$f(n) \leq c_1 \cdot c_2 \cdot h(n), n \geq k'$$

where  $k' = \max(k_1, k_2)$ . Let  $c = c_1 \cdot c_2$  and  $k = k'$  the statement then translate to

$$f(n) \leq c \cdot h(n), n \geq k$$

which by the definition of  $O$  notation,  $f(n) = O(h(n))$ .

**7.** Prove that  $f(n) = O(g(n)) \iff g(n) = \Omega(f(n))$ .

Forward direction:  $f(n) = O(g(n)) \implies g(n) = \Omega(f(n))$ .

Since  $f(n) = O(g(n))$  there exists a number  $c$  and a number  $k$  such that  $f(n) \leq c \cdot g(n), n \geq k$  where  $c > 0$  and  $k \geq 0$ . From this we obtain  $\frac{1}{c} \cdot f(n) \leq g(n), n \geq k$ . Which by the definition of  $\Omega$  notation,  $g(n) = \Omega(f(n))$ .

Backward direction:  $g(n) = \Omega(f(n)) \implies f(n) = O(g(n))$

Since  $g(n) = \Omega(f(n))$  there exists a number  $c$  and a number  $k$  such that  $g(n) \geq c \cdot f(n), n \geq k$  where  $c > 0$  and  $k \geq 0$ . From this we obtain  $g(n) \cdot \frac{1}{c} \geq f(n), n \geq k$ . Which by the definition of  $O$  notation,  $f(n) = O(g(n))$ .

We conclude  $f(n) = O(g(n)) \iff g(n) = \Omega(f(n))$ .

**8.** Compare the growth of

**9.** Prove or disprove  $2^{n+1} = O(2^n)$

$$f(n) = O(g(n)) \iff \exists c > 0, \exists k \geq 0 \text{ s.t } f(n) = c \cdot g(n) \forall n \geq k$$

$$\begin{aligned} 2^{n+1} &= 2 \cdot 2^n \\ &\leq 3 \cdot 2^n, n \geq 1 \end{aligned}$$

Let  $c = 3$  and  $k = 1$  then the statement translates to

$$2^{n+1} \leq c \cdot 2^n, n \geq k$$

which by the definition of  $O$  notation,  $2^{n+1} = O(2^n)$ .

**10.** Prove or disprove  $2^{2 \cdot n} = O(2^n)$

$$f(n) = O(g(n)) \iff \exists c > 0, \exists k \geq 0 \text{ s.t } f(n) = c \cdot g(n) \forall n \geq k$$

$$\begin{aligned} 2^{2 \cdot n} &\leq c \cdot 2^n, n \geq k \\ 2^n &\leq c \end{aligned}$$

which is a contradiction, hence  $2^{2 \cdot n} \neq O(2^n)$ .

**11.** Prove that if  $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = C$ , for some constant  $C > 0$  then  $f(n) = \Theta(g(n))$ .

Since  $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = C$ , for every  $\epsilon > 0$ , there exists  $k \geq 0$  such that, for all  $n \geq k$ ,  $|\frac{f(n)}{g(n)} - C| < \epsilon$ . From this we obtain

$$\begin{aligned} -\epsilon &< \frac{f(n)}{g(n)} - C < \epsilon \\ C - \epsilon &< \frac{f(n)}{g(n)} < C + \epsilon \\ g(n) \cdot (C - \epsilon) &< f(n) < g(n) \cdot (C + \epsilon), n \geq k \end{aligned}$$

Let  $c_1 = (C - \epsilon)$  and  $c_2 = C + \epsilon$  then the statement translates to

$$c_1 \cdot g(n) < f(n) < c_2 \cdot g(n)$$

which by the definition of  $\Theta$  notation,  $f(n) = \Theta(g(n))$ .