

1. Sort the below numbers using

1. Counting Sort
2. Radix Sort
3. Insertion Sort
4. Bubble Sort
5. Selection Sort

[1, 2, 0, -3, 5, -7, 10]

[0, 2, 3, 8, 9, 16]

⑨ Part 1

$$[1, 2, 0, -3, 5, -2, 10]$$

add 7 to all elements

$$[8, 9, 2, 4, 1, 0, 17]$$

new element is 17

*				1			1	1	1		1							
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	

WST count

*				1	2	3	4	5										
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	

$$[0, 4, 7, 8, 9, 10, 17]$$

subtract all elements by -7

$$[-7, -3, 0, 1, 2, 5, 10]$$

resulting array

$$[-7, -3, 0, 1, 2, 5, 10]$$

16

Par + 2

[0' 2 3 3 9 1 6]

max is 16

10110001100001
012345678910112131415

o Gorget count

~~1 2 3 3 3 3 4 5 5 5 5 5 5 5~~

$$WSD[9] = WSD[4] + LOSS[1-1]$$

$$\{0, 2, 3, 8, 9, \underline{\underline{15}}\}$$

resulting array

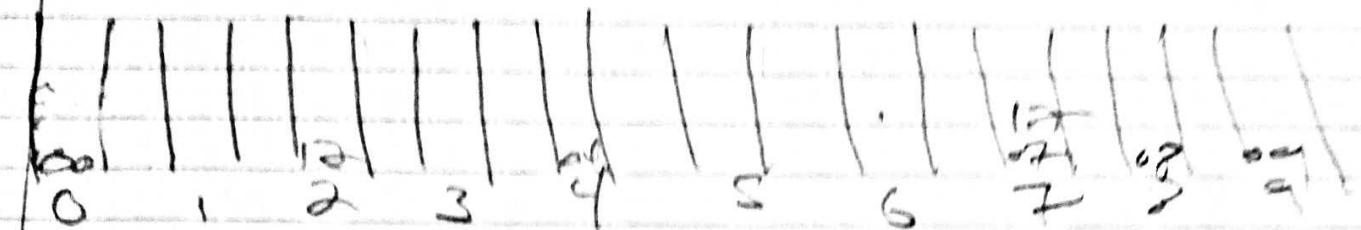
[0 2 3 8 9 (6)]

①b) Part 1

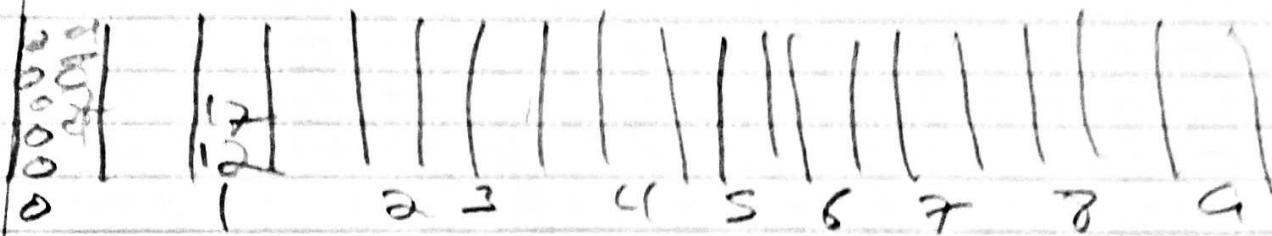
[1, 2, 0, -3, 5, -7, 10]

add 7 to all elements

[8, 9, 7, 4, 12, 0, 17]



0/12/104/107/112/108/09



0/104/107/108/09/12/17 -

Subtracting by -7

-7/-3/0/11/0/5/10

resettling array

[-7, -3, 0, 1, 2, 5, 10]

(b) Part 2

[00, 02, 03, 08, 09, 16]

00 | 1 | 1 | 62 | 63 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
0 1 2 3 4 5 6 7 8 9
00 / 02 / 03 / 16 / 08 / 09

88 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
83 | 15 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
02 | 00 | 16 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
00 | 00 | 00 | 00 | 00 | 00 | 00 | 00 | 00 | 00 | 00 | 00 |
00 | 00 | 00 | 00 | 00 | 00 | 00 | 00 | 00 | 00 | 00 | 00 |

00 / 02 / 03 / 08 / 09 / 16 ✓

resulting array

[0, 2, 3, 8, 9, 16]

(16) Part I

~~0 12~~ 0 -3 5 -7 10

~~12 5~~ -3 5 -7 10

~~0 12~~ -3 5 -7 10

~~-3 0 12~~ 5 -7 10

~~-3 5 12~~ 5 -7 10

~~12 -3 0 10~~ 5 10

~~-7 -3 0 12 5 10~~ ✓

⑫ Part 2

16 3 8 9 16

to 2 3 8 9 16
()

to 2 3 8 9 16

to 2 3 8 9 16

to 2 3 8 9 16

10 12 13 8 9 16 ✓

① Part 1

$$\left[\begin{matrix} 1 & 2 & 0 & -3 & 5 & -7 & 10 \end{matrix} \right]$$

$$\left[\begin{matrix} 1 & 2 & 0 & -3 & 5 & -7 & 10 \end{matrix} \right]$$

$$\left[\begin{matrix} 1 & 0 & 2 & -3 & 5 & -7 & 10 \end{matrix} \right]$$

$$\left[\begin{matrix} 1 & 0 & -3 & 2 & 5 & -7 & 10 \end{matrix} \right]$$

$$\left[\begin{matrix} 1 & 0 & -3 & 2 & 5 & -7 & 10 \end{matrix} \right]$$

$$\left[\begin{matrix} 1 & 0 & -3 & 2 & -7 & 5 & 10 \end{matrix} \right]$$

$$\left[\begin{matrix} 1 & 0 & -3 & 2 & -7 & 5 & 10 \end{matrix} \right]$$

$$\left[\begin{matrix} 0 & 1 & -3 & 2 & -7 & 5 & 10 \end{matrix} \right]$$

$$\left[\begin{matrix} 0 & -3 & 1 & 2 & -7 & 5 & 10 \end{matrix} \right]$$

$$\left[\begin{matrix} 0 & -3 & 1 & 2 & -7 & 5 & 10 \end{matrix} \right]$$

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$$\left[\begin{matrix} -3 & 0 & 1 & -7 & 2 & 5 & 10 \end{matrix} \right]$$

$$\left[\begin{matrix} -3 & 0 & 1 & -7 & 2 & 5 & 10 \end{matrix} \right]$$

$$[-3 \ 0 \ 1 \ -7 \ 2 \ s \ (0)]$$

$$[-3 \ 0 \ -7 \ 1 \ 2 \ s \ (0)]$$

$$[-3 \ 0 \ -7 \ 1 \ 2 \ s \ (0)]$$

$$[-3 \ 0 \ -7 \ 1 \ 2 \ s \ (0)]$$

$$[-3 \ 0 \ -7 \ 1 \ 2 \ s \ (0)]$$

$$[-3 \ 0 \ -7 \ 1 \ 2 \ s \ (0)]$$

$$[-3 \ -7 \ 0 \ 1 \ 2 \ s \ (0)]$$

$$[-3 \ -7 \ 0 \ 1 \ 2 \ s \ (0)]$$

$$[-3 \ -7 \ 0 \ 1 \ 2 \ s \ (0)]$$

$$[-3 \ -7 \ 0 \ 1 \ 2 \ s \ (0)]$$

$$[-3 \ -7 \ 0 \ 1 \ 2 \ s \ (0)]$$

$$[-7 \ -3 \ 0 \ 1 \ 2 \ s \ (0)]$$

$$[-7 \ -3 \ 0 \ 1 \ 2 \ s \ (0)]$$

$$[-7 \ -3 \ 0 \ 1 \ 2 \ s \ (0)]$$

$$[-7 \ -3 \ 0 \ 1 \ 2 \ s \ (0)]$$

$$[-7 \ -3 \ 0 \ 1 \ 2 \ s \ (0)]$$

$$[-7 \ -3 \ 0 \ 1 \ 2 \ s \ (0)]$$

$[-7 -3 \overset{0}{\cancel{3}} 1 2 S 1 0]$

No swaps so I stop

resulting array

$[-7 -3 0 1 2 S 1 0]$

(1D) Part 2.

[0 2 3 2 9 1 6]

[0 2 3 8 9 1 6]

[0 2 3 3 9 1 6]

[0 2 3 8 9 1 6]

[0 2 3 2 9 1 6]

No swaps where needed
so STOP

resulting array

[0 2 3 8 9 1 6]

① e port 1

112|0-3|5|1|0

↓
idc

new min min min min
old max max max max

Surf current idc and why
↓
↓

-2|2|0-3|5|1|0

Set idc and why
equal to 1

-2|2|0-3|5|1|0

↓
idc ↑ ↑
g min min
ydc idc idc
idc

Surf current idc and why

-2|-3|0|2|5|1|0

idx and win idx equal
to 2

-7 -3 0 2 5 | 1 | 0

↑
idx
↑
win
↑
idx

Sup idx and win idx

-7 -3 0 2 5 | 1 | 0

idx and win idx equal
to 3

-7 -3 0 2 3 | 1 | 0

↑
idx ↑
9
win
↑
idx

Sup idx and win idx
values

-7 -3 0 1 5 | 2 | 0

$i \rightarrow x$ and $m \rightarrow x$ equal
to 4

$-2|-3|0|1|2|5|2|0$

$\uparrow \quad \uparrow$
 $i \rightarrow x$ and $m \rightarrow x$
 \uparrow
~~5~~ 4

Snap $i \rightarrow x$ and $m \rightarrow x$
value

$-2|-3|0|1|2|5|2|0$

$i \rightarrow x$ and $m \rightarrow x$ equal
to 5

$-2|-3|0|1|2|5|1|0$

\uparrow
 $i \rightarrow x$
 \uparrow
new $i \rightarrow x$

Snap $i \rightarrow x$ and $m \rightarrow x$ values

$-2|-3|0|1|2|5|1|9$

i₂x and min i₂x equal
to 6

[-7 -3 0 12 5 0]

i₂x
min
i₂x

Swap i₂x and min i₂x

[-2 -3 0 10 5 0]

resulting array

[-2 -3 0 12 5 0]

• Depart 2

Set i_{1r} and i_{1w} to 0

0 | 2 | 3 | 8 | 9 | 16

↑

9_{1r}

8_{1r}

9_{1w}

Sweep i_{1r} and i_{1w} values

0 | 2 | 3 | 8 | 9 | 16

Set i_{1r} and i_{1w} to 1

0 | 2 | 3 | 8 | 9 | 16

9_{1r} ↑

8_{1r}

9_{1w}

Sweep i_{1r} and i_{1w} values

0 | 2 | 3 | 8 | 9 | 16

see idx and non idx to 2

0|1|3|8|7|1|6

idx
non

swap idx and non idx -

1|0|2|3|8|1|9|1|6

see idx and non idx to 3

0|1|2|3|8|9|1|6

idx with q3x

swap q3x and other idx values

1|0|2|3|8|9|1|6

see idx and non idx - colf

1|0|2|3|8|9|1|6

idx with q3x

swap idx and non q3x values

0|1|2|3|8|9|1|6

see 9-8 and we pick 10 as
to 12 3 8 9 15

pick 10 and write

swap 10 and write 9-8

to 12 3 8 19 15:

resulting array

10 12 3 8 19 15

- 2.** What is the running time of Insertion sort if all elements are equal?

Algorithme 1 Insertion Sort

Insertion-Sort(a) :

```
1: for j = 1 :len(A)
2:   key = a[j]
3:   i = j - 1
4:   while i > 0 and A[i] > key
5:     A[i + 1] = A[i]
6:     i = i - 1
7:   A[i + 1] = key
```

Since the array is sorted the algorithm never enters the while loop and thus does a constant amount of work per iteration, we denote this constant c . $\sum_{i=1}^n c = n \cdot c = \Theta(n)$.

- 3.** Sort the below numbers using:

1. Merge Sort
2. QuickSort

[8, 0, 2, -1, -2, 2, 3, 7, -6, -9]
[19, 7, 6, 3, 2, -1, -7, -18]

Part I

$$\text{Ex) } \begin{array}{ccccccccc} -9 & -6 & -2 & -1 & 0 & 2 & 2 & 3 & 7 & 8 \\ \hline 8 & 0 & 1 & 2 & 1 & -1 & 2 & 2 & 3 & 7 & 6 & 1 & -9 \end{array}$$

$$\begin{array}{r} -2 \sim 0 \sim 2 \cancel{8} \\ 8 \mid 0 \mid 2 \mid -1 \mid -2 \end{array} \quad \begin{array}{r} \nearrow \\ a = 6 \end{array} \quad \begin{array}{r} 2 \sim 3 \sim 3 \cancel{7} \\ 2 \mid 3 \mid 7 \mid -6 \mid -9 \end{array}$$

$$\begin{array}{cccc} \overset{\leftarrow}{\swarrow} & \overset{\rightarrow}{\swarrow} & \overset{\leftarrow}{\swarrow} & \overset{\rightarrow}{\swarrow} \\ 0 & 8 & -2 & -1 & 2 & 3 & -9 & -6 & 7 \\ \boxed{8} & \boxed{0} & \boxed{-2} & \boxed{-1} & \boxed{2} & \boxed{3} & \boxed{-9} & \boxed{-6} & \boxed{7} \end{array}$$

$$\begin{array}{ccccccccc}
 & \swarrow & & \leftarrow & \rightarrow & - & \swarrow & & \swarrow \downarrow \searrow \\
 8 & 10 & 2 & -1 & -2 & 2 & 3 & 7 & -6 & -9 \\
 & & & \downarrow & & & & & \swarrow & \downarrow \\
 & & & -1 & -2 & & & & -6 & -9
 \end{array}$$

Row 2

(3a) $\begin{array}{ccccccccc} -8 & -7 & -1 & 2 & 3 & 6 & 7 & 19 \\ \boxed{19} & \boxed{7} & \boxed{6} & \boxed{3} & \boxed{2} & \boxed{-1} & \boxed{-7} & \boxed{-18} \end{array}$

$\begin{array}{cccc} 3 & 6 & 7 & \text{pd} \\ \boxed{19} & \boxed{7} & \boxed{6} & \boxed{3} \end{array} \quad \begin{array}{cccc} -8 & -7 & -1 & 2 \\ \boxed{12} & \boxed{-11} & \boxed{-7} & \boxed{-8} \end{array}$

$\begin{array}{cccc} 2 & 4 & 3 & 6 \\ \boxed{19} & \boxed{7} & \boxed{6} & \boxed{3} \end{array} \quad \begin{array}{cccc} -1 & 2 & -8 & -7 \\ \boxed{12} & \boxed{-11} & \boxed{-7} & \boxed{-8} \end{array}$

$\begin{array}{cccc} \cancel{2} & \cancel{4} & \cancel{3} & \cancel{6} \\ \cancel{\boxed{19}} \cancel{\boxed{7}} & \cancel{\boxed{6}} \cancel{\boxed{3}} & \cancel{\boxed{12}} \cancel{\boxed{-11}} & \cancel{\boxed{-7}} \cancel{\boxed{-8}} \end{array}$

$\begin{array}{cccc} \cancel{2} & \cancel{4} & \cancel{3} & \cancel{6} \\ \cancel{\boxed{19}} \cancel{\boxed{7}} & \cancel{\boxed{6}} \cancel{\boxed{3}} & \cancel{\boxed{12}} \cancel{\boxed{-11}} & \cancel{\boxed{-7}} \cancel{\boxed{-8}} \end{array}$

3b part 1

$$\boxed{8 \ 0 \ 1 \ 2 \ -1 \ -2 \ 2 \ 3 \ 7 \ -6 \ 5 \ 1}$$

$$M(3)(-2) - 9 = -2$$

$$\boxed{8 \ 0 \ 1 \ 2 \ -1 \ -9 \ 2 \ 3 \ 7 \ -6 \ 5 \ -2}$$

X

X

swap 3 and -6

$$\boxed{-6 \ 0 \ 1 \ 2 \ -1 \ -9 \ 2 \ 3 \ 7 \ 2 \ -2}$$

↑ X ↑ D D
↓ ↓ ↓ D D

swap 3 and -9

$$\boxed{-6 \ -9 \ 1 \ 2 \ -1 \ 6 \ 2 \ 3 \ 7 \ 3 \ -2}$$

↑ ↑ ↑ ↑ ↓
↓ X ↓ ↓ ↓

swap 3 and -2

$$\boxed{-6 \ -9} \ \overbrace{\quad -2 \quad}^{\text{swap}} \ \boxed{-1 \ 0 \ 2 \ 3 \ 7 \ 8 \ 2}$$

sort
normally

$$\boxed{-9 \ -6}$$

quick sort

-1|0|2|3|7|8|2

$$MC(-1|3,2) = 2$$

L-1|0|2|3|7|8|2

↑ ↑ ↑ ↑ ↑ ↑
✓ ✓ X ✓ ✓ ✓

Swap 2 and 2

L-1|0|2|3|7|8|2

sort
numerically

quick
sort

→ |0|

|3|7|8|2|

$$MC(3,7,8,2) = 3$$

1|2|7|8|3|

↑ ↑ ↑
✓ ✓ X ✓

Swap 2 and 3

21 23

Sort
memory

18 17

Sort
various?

12

27 18

9 6 - 2 1 - 1 0 1 2 1 3 1 7 + 2

Combining all
the work

35 part 2

$$19 \boxed{7} 6 \boxed{3} 2 - 1 \boxed{-7} - 18$$

$$M(19, 3, -18) \rightarrow 3$$

$$\boxed{19} \boxed{7} \cancel{6} - 18 \boxed{2} - 1 \cancel{-7} \boxed{3}$$

↑ ↑
X X

Snap 19 and -7

$$\cancel{-7} \cancel{2} \cancel{6} - \cancel{1} \cancel{8} \cancel{2} \cancel{+} \cancel{1} \cancel{9} \cancel{3}$$

↑ ↑↑
✓ X

Snap 7 and -11

$$\cancel{-7} \cancel{-1} \cancel{6} \cancel{-1} \cancel{8} \cancel{2} \cancel{+} \cancel{-7} \cancel{1} \cancel{9} \cancel{3}$$

↑ ↑ ↑ ↑
V X V V

Snap 6 and 2

-7	-1	2	-18	6	7	19	3
↑	2	↑		↑			

end partition

-7	-1	2	-18	3	7	19	6
quicksort					sort array		

-7	-1	2	-18
----	----	---	-----

$$MC(-7, -1, 2, -18) = -7$$

1	-18	-1	2	-7
↑	↑	↑	↑	↑

Stop partition

-18 {-7} {2,-1}

sort
nearly

-18

sort nearly

-1 2

-18 -7 -1 2 3 6 7 9

Combining work

- 4.** Perform the partitioning algorithm on the below array using the median-of-three heuristic

[1, 2, 6, -3, 20, -61, 7, 8, 19, 100]
[0, 7, -6, 23, 12, 30, -71, 19]

4

60

1	2	3	4	5	6	7	8	9	10
0	1	2	3	4	5	6	7	8	9

$$M(1,2,3,100) = 20$$

1 2 6 -3 106 -61 7 2 42

Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ

Sup coo and la

11 21 G -3 19 (-6) 7 8 (100) 20

A series of handwritten marks on lined paper. It includes several upward-pointing arrows and a series of checkmarks (✓) and X's. The first two arrows have checkmarks below them. The next two arrows have X's below them. The final two arrows have checkmarks below them.

Snop 100 arkas

1	2	6	-3	14	-61	7	8	20	100
---	---	---	----	----	-----	---	---	----	-----

Final perspective
array

(4) (b)

0	7	-6	2	3	1	0	3	0	-7	1	1	9
6	1	2	3	4	5	6	7					

$$M(0, 2, 3, 1) = 1$$

0	7	-6	1	2	3	1	2	3	0	-7	1	9
✓	✓	✓	✓	✓	✗				✓			✗

Since 2 and -7

0	7	-6	1	2	3	0	2	3	1	9
✓	✓	✓	✗	✓	✓				✓	

Since 3 and 1

0	7	-6	1	2	3	0	2	3	1	9
✓	✓	✓	✓	✓	✓				✓	

final parallelized array

- 5.** What is the worst-case running time of Quicksort if the pivot is randomly chosen as the first element in the array in each recursive call?

We assume that the array is sorted. If so then the first element chosen is the smallest. This leads to the left-subproblem having no elements and the right subarray having $n - 1$ elements. This is of the form $T(n) = T(n - 1) + O(n)$, which solves to $O(n^2)$.

$$T(n) = T(n-1) + O(n)$$

level	size	tree
0	n	n
1	n-1	n-1
2	n-2	n-2
⋮	⋮	⋮
k	n-k	n-k-1

$$\begin{aligned}n-k &= 1 \\k &= n-1\end{aligned}$$

$$\sum_{i=0}^{k-1} i = \frac{(k+1)k}{2} =$$

$$\frac{(n)(n-1)}{2} = O(n^2)$$

6. Show that the average of a_{left} is $\frac{(n-1)}{2}$ when the input to Quicksort is n distinct elements and the median M is randomly chosen from one of the elements?

$$\begin{aligned}
 & \overbrace{(n-1) + (n-2) + (n-3) + \dots + (n-n)}^{\substack{a_{left} \text{ possible sizes} \\ \underbrace{n}_{\text{total cases}}}} = \frac{\sum_{i=0}^{n-1} i}{n} \\
 & = \frac{\frac{(n-1) \cdot n}{2}}{n} \\
 & = \frac{(n-1)}{2}
 \end{aligned}$$

7. Calculate the running time of a divide-and-conquer algorithm that requires three recursive calls (each with input-size $\frac{n}{2}$) and $5 \cdot n^2$ steps that include dividing the input, and using the three solutions to obtain the final solution.

$$\begin{aligned}
 T(n) &= 3 \cdot T\left(\frac{n}{2}\right) + 5 \cdot n^2 \\
 &= 3 \cdot T\left(\frac{n}{2}\right) + \Theta(n^2)
 \end{aligned}$$

$$T(n) = 3T\left(\frac{n}{2}\right) + n^2$$

level	size	+ tree
0	$\frac{n}{2}$	n^2 $(\frac{n}{2})^2$
1	$\frac{n}{4}$	
2	$\frac{n}{8}$	
3	$\frac{n}{16}$	
k	$\frac{n}{2^k}$	

$$k = \log_2(n)$$

$$\sum_{i=0}^{k-1} 3\left(\frac{n}{2^i}\right)^2 + 3^k$$

$$n^2 \sum_{i=0}^{k-1} \left(\frac{3}{4}\right)^i + 3^k$$

$$n^2 + \Theta(1) + n^{\log_3(2)}$$

$\Theta(n^2)$

$$T(n) = \Theta(n^2)$$

- 8.** Use a recursion tree for the following algorithms to find the running time.

8

$$T(n) = T\left(\frac{n}{2}\right) + 1$$

level	size	tree
0	n	1
1	$\frac{n}{2}$	1
2	$\frac{n}{4}$	1
:		
0	$\frac{n}{2^k}$	1
:		
0	1	1

$$\frac{n}{2^k} = 1$$

$$\log_2(n) = k$$

$$\sum_{i=0}^k 1 = (k+1) = (\log_2(n)) \\ = O(\log_2(n))$$

$O(\log_2(n))$

(b)

$$T(n) = T(n-1) + 1$$

level	size	cost
0	n	1
1	n-1	1
2	n-2	1
...
k	1	1

$$\text{node} = 1$$
$$k = n - 1$$

$$\sum_{i=0}^k 1 = (k+1) = (n-1+1) = n$$
$$= \Theta(n)$$

$$T(n) = \Theta(n)$$

c)

$$T(n) = 2T(n-1) + 1$$

Level size tree

0

n

\downarrow

1

$n-1$

$\{\}$

2

$n-2$

\vdots

K

1

1

$$K = n-1$$

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1 = \Theta(2^n)$$

$$T(n) = \Theta(2^n)$$

$$\textcircled{2} \quad T(n) = 3T(\frac{n}{3}) + n$$

Level	Size	Tree
0	$\frac{n}{3}$	$\frac{1}{3}$
1	$\frac{n}{9}$	$\frac{1}{3} \frac{1}{3} \frac{1}{3}$
2	$\frac{n}{27}$	$\frac{1}{9} \frac{1}{9} \frac{1}{9}$

k

$$k = \log_3(n)$$

$$\sum_{i=0}^{k-1} 3^i \frac{n}{3^i} + 3^k$$

$$n \sum_{i=0}^{k-1} (\frac{3}{4})^i + n^{\log_3(3)}$$

$$n\Theta(1) + n^{\log_3(3)}$$

$$\Theta(n)$$

$$\boxed{T(n) = \Theta(n)}$$

$$e) T(n) = 3T\left(\frac{n}{3}\right) + \sqrt{n}$$

<u>Level</u>	<u>size</u>	<u>tree</u>
0	1	\sqrt{n}
1	$\frac{1}{3}$	leaf
2	$\frac{1}{3^2}$	leaf
:	$\frac{1}{3^k}$	$\sqrt{\frac{n}{3^k}}$

$$k = \log_3(n)$$

$$\sum_{i=0}^{k-1} 3^i \sqrt{\frac{n}{3^i}} + 3^k$$

$$\sqrt{n} \sum_{i=0}^{k-1} \left(\frac{1}{3}\right)^i + n$$

$$\sqrt{n} b(1) + n$$

$$O(n)$$

$T(n) = O(n)$

(+)

$$T(n) = T\left(\frac{n}{2}\right) + n^2$$

level	size	+ rec
0	n	$\frac{n^2}{n}$
1	$\frac{n}{2}$	$\left(\frac{n}{2}\right)^2$
2	$\frac{n}{4}$	$\left(\frac{n}{4}\right)^2$
...
k	$\frac{n}{2^k}$	$\left(\frac{n}{2^k}\right)^2$

$$k = \log_2(n)$$

$$\sum_{i=0}^{k-1} \left(\frac{n}{2^i}\right)^2 + 1 =$$

$$n \geq \sum_{i=0}^{k-1} \left(\frac{n}{2^i}\right)^2 + 1$$

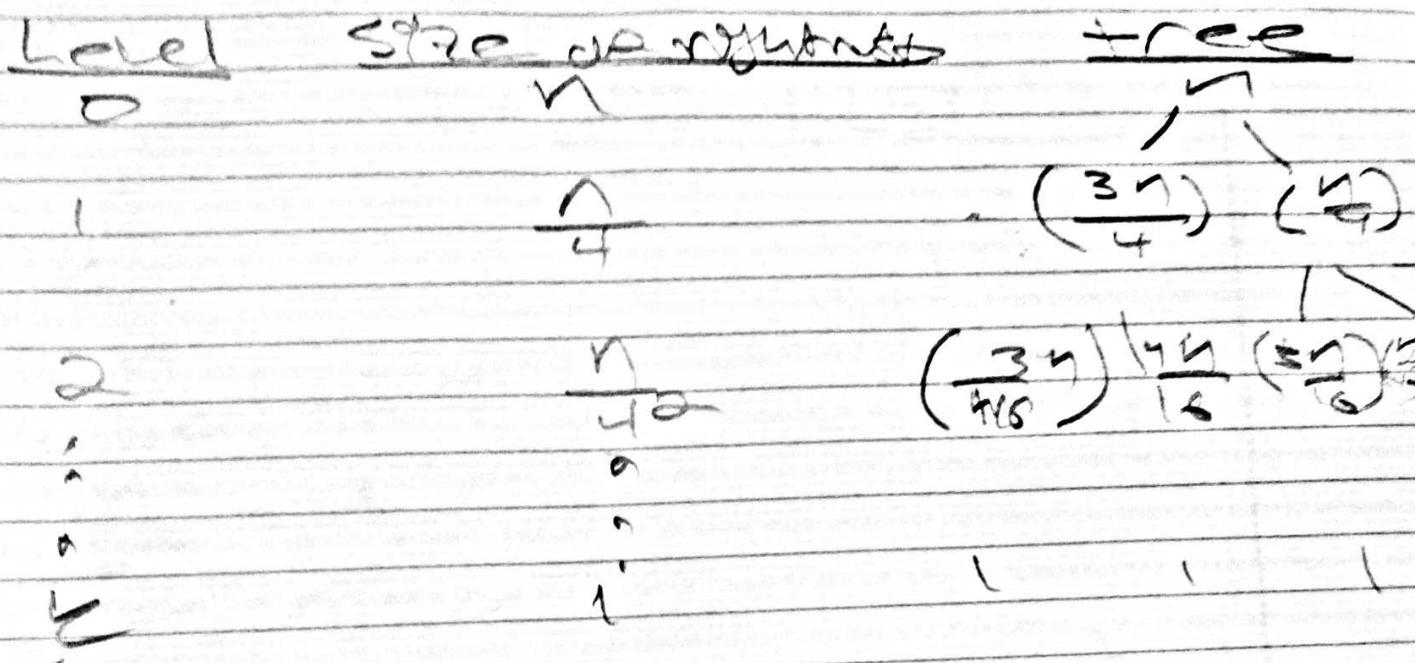
$$n \geq b^k(1) + 1$$

$$\Theta(n^2)$$

$$T(n) = \Theta(n^2)$$

$$⑨ T(n) = T\left(\frac{3n}{4}\right) + T\left(\frac{n}{4}\right) + n$$

finding lower bound



$$k = \log_4(n)$$

$$\begin{aligned} n \sum_{i=0}^k i &= n(k+1) \\ &= n(\log_4(n)+1) \\ &= \cancel{k}(n \log_4 n) \end{aligned}$$

$$T(n) = \underline{n \log n}$$

$$T(n) = T\left(\frac{3n}{4}\right) + T\left(\frac{n}{4}\right) + n$$

finding upper bound

<u>level</u>	<u>size of leftmost tree</u>
0	$\frac{n}{4}$
1	$\frac{3n}{16}$
2	$\frac{9n}{64}$
3	$\frac{27n}{256}$
4	$\frac{81n}{1024}$
5	$\frac{243n}{4096}$
6	$\frac{729n}{16384}$
7	$\frac{2187n}{65536}$
8	$\frac{6561n}{262144}$
9	$\frac{19683n}{819200}$
10	$\frac{59049n}{248832}$
11	$\frac{1770729n}{737280}$
12	$\frac{531441n}{243000}$
13	$\frac{1594323n}{72900}$
14	$\frac{4782969n}{21870}$
15	$\frac{14348523n}{65610}$
16	$\frac{43045571n}{196830}$
17	$\frac{129136713n}{590490}$
18	$\frac{387409141n}{17707290}$
19	$\frac{1162227423n}{5314410}$
20	$\frac{3486682269n}{15943230}$
21	$\frac{10459946787n}{35897930}$
22	$\frac{31379839361n}{107693080}$
23	$\frac{94139517983n}{323079240}$
24	$\frac{282418553949n}{970397760}$
25	$\frac{847255661847n}{2911193280}$
26	$\frac{2541766985541n}{8733579840}$
27	$\frac{7625299956623n}{26200739520}$
28	$\frac{22875899869869n}{78602218560}$
29	$\frac{68627699609587n}{23580665520}$
30	$\frac{205883098828761n}{65741996480}$
31	$\frac{617649296486283n}{197225989440}$
32	$\frac{1852947890458851n}{591677968320}$
33	$\frac{5558843671376533n}{1775033904960}$
34	$\frac{16676530914129609n}{5325101714880}$
35	$\frac{50029592742388827n}{15975305144640}$
36	$\frac{150088778227166481n}{47925915434560}$
37	$\frac{450266334681500043n}{14377774626880}$
38	$\frac{1350799003943500129n}{43133323880640}$
39	$\frac{4052397011826500387n}{129400006401920}$
40	$\frac{12157191035479501161n}{388200001933760}$
41	$\frac{36471573106438503483n}{1164600005761280}$
42	$\frac{109414719319315510449n}{3493800017283840}$
43	$\frac{328244157957946531347n}{10481400051451520}$
44	$\frac{984732473873839594041n}{31444200154854560}$
45	$\frac{2954197421621518782123n}{9433260046455520}$
46	$\frac{8862592264864556346369n}{2829978013936640}$
47	$\frac{26587776794593679039107n}{8489934041810080}$
48	$\frac{79763330383781037117321n}{25469802125430240}$
49	$\frac{239290001151343111351963n}{76389406376290720}$
50	$\frac{717870003453829333955891n}{2291682191288720}$
51	$\frac{2153610010361487991867673n}{6875046573865160}$
52	$\frac{6460830031084463975602919n}{20625139721595440}$
53	$\frac{19382500093253391927808757n}{61875419145385440}$
54	$\frac{58147500280760175783426271n}{18562513775065120}$
55	$\frac{174442500842280527350278813n}{55687541250217040}$
56	$\frac{523327502526841582050836439n}{16706251250705120}$
57	$\frac{1570082507576524746152509217n}{5018753751915040}$
58	$\frac{4710247522729574238507527651n}{1505625125371680}$
59	$\frac{14130742568188622715522582953n}{451875375111560}$
60	$\frac{42392227694565868146567748859n}{135562512533520}$
61	$\frac{127176683083697574439699426577n}{40672537510560}$
62	$\frac{381529949250992721318998279731n}{12191251251520}$
63	$\frac{1144589847752978163956994839193n}{36573753754560}$
64	$\frac{3433769543258934511869984517579n}{10972125125360}$
65	$\frac{10301308639776803535589953552737n}{32916375375120}$
66	$\frac{30904025919330410606769860658211n}{98750125125360}$
67	$\frac{92712077757991231819909581974633n}{296250375375120}$
68	$\frac{278136233273973695459728745923909n}{888750125125360}$
69	$\frac{834398700821910786379186237771737n}{2666250375375120}$
70	$\frac{2503196102465732359337586713315211n}{800000000000000}$

$$n \left(\frac{3}{4}\right)^k = 1$$

$$n = \left(\frac{4}{3}\right)^{-k}$$

$$\log_{\frac{4}{3}}(n) = k$$

$$\sum_{i=0}^k n = n(k+1) = n(\log_{\frac{4}{3}}(n) + 1)$$

$$= \Theta(n \log_{\frac{4}{3}}(n))$$

$$T(n) = O(n \log(n))$$

$$T(n) = O(n \log(n)) \text{ and } \Omega(n \log(n)) \rightarrow$$

$$T(n) = \Theta(n \log(n))$$

$$\textcircled{h} \quad T(n) = 4 + (\lfloor \frac{n}{2} \rfloor + 2) + n^2$$

level size tree

0

n

n^2

$\lfloor \frac{n}{2} \rfloor + 2$

$(\lfloor \frac{n}{2} \rfloor + 2)^2$

1

$\lfloor \frac{n}{4} \rfloor + 3$

$(\lfloor \frac{n}{4} \rfloor + 3)^2$

2

$\lfloor \frac{n}{8} \rfloor + 3$

$(\lfloor \frac{n}{8} \rfloor + 3)^2$

3

$\lfloor \frac{n}{32} \rfloor + 3$

$(\lfloor \frac{n}{32} \rfloor + 3)^2$

\dots

1

1

$$\lfloor \frac{n}{2^k} \rfloor + 3 = 4$$

$$\frac{n}{2^k} = 1$$

$$\log_2(n) = k$$

$$4^0 n^2 + (\lfloor \frac{n}{2} \rfloor + 2)^2 4^1 + (\lfloor \frac{n}{2^2} \rfloor + 3)^2 4^2 + \dots + 4^{k-1}$$

$$= O\left(\sum_{i=0}^{k-1} 4^i \left(\lfloor \frac{n}{2^i} \rfloor + 3\right) + 4^k\right)$$

$$= O\left(\sum_{i=0}^{k-1} 4^i \left(\frac{n}{2^i}\right) + 4^k\right)$$

$$= O\left(n \sum_{i=0}^{k-1} 2^i + 4^k\right)$$

$$= O\left(n [2^k - 1] + 4^k\right)$$

$$= O\left(n \cdot 2^{\log_2(n)} - 1 + n^{\log_2(2)}\right)$$

$$= O(n^2 + n^2)$$

$$= O(n^2)$$

- 9.** Use the formula you learned in class to determine the asymptotic growth of $T(n)$.

Ⓐ Ⓛ

$$T(n) = T\left(\frac{n}{2}\right) + 1$$

$$k = \log_2(n)$$

$$\sum_{i=0}^{k-1} 1 + n^{\log_2(1)} =$$

$$\sum_{i=0}^{k-1} 1 + 1 =$$

$$\sum_{i=0}^{k-1} 1 = (k+1) = (\log_2(n)+1)$$

$$= \Theta(\log_2(n))$$

$$\boxed{T(n) = \Theta(\log_2(n))}$$

⑥ $T(n) = 3T\left(\frac{n}{3}\right) + n$

$$\sum_{i=0}^{k-1} \cancel{\frac{3^n}{3^i}} + 3^{\log_3(n)} = k = \log_3(n)$$

$$\sum_{i=0}^{k-1} n + n =$$

$$nk + n =$$

$$n \log_3(n) + n$$

$$\boxed{\Theta(n \log_2(n))}$$

$$T(n) = 4 T\left(\frac{n}{3}\right) + n$$

$$k = \log_3(n)$$

$$n \sum_{i=0}^{k-1} 4^i \left(\frac{1}{3}\right)^i + 4^k =$$

$$n \left[\frac{\left(\frac{4}{3}\right)^k - 1}{\left(\frac{4}{3}\right) - 1} \right] + n^{\log_3(n)} =$$

$$n \Theta\left(\left(\frac{4}{3}\right)^k\right) + n^{\log_3(n)} =$$

$$n \cdot n^{\log_3\left(\frac{4}{3}\right)} + n^{\log_3(n)}$$

$$n^{\log_3(n) - \log_3(3)} + n^{\log_3(n)} =$$

$$n^{\log_3(n)} + n^{\log_3(n)}$$

$$\Theta(n^{\log_3(n)})$$

(6)

$$T(n) = 3 + \left(\frac{n}{4}\right) + \sqrt{n}$$

$$k = \log_4(n)$$

$$\sum_{i=0}^{k-1} 3^i \sqrt{\frac{n}{2^i}} + 3^k =$$

$$n \sum_{i=0}^{k-1} \left(\frac{3}{2}\right)^i + n^{\log_4(3)}$$

$$\sqrt{n} \left[\frac{(3/2)^k - 1}{(3/2) - 1} \right] + n^{\log_4(3)}$$

$$\sqrt{n} \Theta\left((\frac{3}{2})^k\right) + n^{\log_4(3)}$$

$$\sqrt{n} \cdot \sqrt{n}^{\log_4(\frac{3}{2})} + n^{\log_4(3)} =$$

$$n^{\log_4(3) - \frac{1}{2} + \frac{1}{2}} + n^{\log_4(3)} =$$

$$n^{\log_4(3)} + n^{\log_4(3)} =$$

$$\boxed{\Theta(n^{\log_4(3)})}$$

②

$$T(n) = 5T\left(\frac{n}{3}\right) + n^2$$

$$k = \log_3(5)$$

$$\sum_{i=0}^{k-1} 5^i \left(\frac{n}{3}\right)^2 + n^{\log_3(5)}$$

$$n^2 \sum_{i=0}^{k-1} \left(\frac{5}{9}\right)^i + n^{\log_3(5)}$$

$$n^2 f(1) + n^{\log_3(5)} =$$

$$\boxed{f(n^2)}$$

③ $T(n) = 6T\left(\frac{n}{3}\right) + n^3$

$$k = \log_3(6)$$

$$\sum_{i=0}^{k-1} 6^i \left(\frac{n}{3}\right)^3 + n^{\log_3(6)}$$

$$n^3 \sum_{i=0}^{k-1} \left(\frac{6}{27}\right)^i + n^{\log_3(6)}$$

$$n^3 f(1) + n^{\log_3(6)}$$

$$\boxed{f(n^3)}$$