1. Compute

1. $\sum_{i=0}^{n}\sum_{k=0}^{i}2^k$

$$\begin{aligned}\sum_{i=0}^{n}\sum_{k=0}^{i}2^k &= \sum_{i=0}^{n}(2^{i+1}-1)\\ &= \sum_{i=0}^{n}2^{i+1}-\sum_{i=0}^{n}1\\ &= 2\cdot\sum_{i=0}^{n}2^{i}-\sum_{i=0}^{n}1\\ &= 2\cdot(2^{n+1}-1)-n\end{aligned}$$

2. $4^{log_2(5)}+log^2(8^3)$

$$\begin{aligned}4^{log_2(5)}+log^2(8^3) &= 2^{log_2(5^2)}+3^2\cdot log_2^2(8)\\ &= 25+3^2\cdot 3^2\\ &= 25+3^4\end{aligned}$$

2. Use L'Hopital's rule to determine the limit of:

$$\begin{aligned}lim_{x\to\infty}\frac{x\cdot lnx^5+4}{(3\cdot x+1)^2} &= lim_{x\to\infty}\frac{5\cdot x\cdot ln(x)+4}{9\cdot x^2+6\cdot x+1}\\ &= 0\end{aligned}$$

3. What is the definition of $f(n)=\Omega(g(n))$.

$$f(n)=\Omega(g(n)) \quad\Longleftrightarrow\quad \exists c>0,\exists k\geq 0\, s.t\, f(n)\geq c\cdot g(n)\,\forall n\geq k$$

4. What is the growth of the below function

$$f(n) \;=\; n\cdot 2^{log_2(n)}+\sqrt{n}+5\cdot n\cdot log^3(n)+log^2(n^n)$$

1. $\Theta(n^2)$
2. $\Theta(n\cdot log^3(n))$

3. $\Theta(\sqrt{n})$
4. $\Theta(n^2 \cdot log^2(n))$
5. Neither!

$$\begin{aligned} f(n) &= n \cdot 2^{log_2(n)} + \sqrt{n} + 5 \cdot n \cdot log^3(n) + log^2(n^n) \\ &= n^2 + \sqrt{n} + 5 \cdot n \cdot log^3(n) + n^2 \cdot log^2(n) \\ &= \Theta(n^2 \cdot log^2(n)) \end{aligned}$$

(4) is the correct answer.

5. What is the growth of the below function

$$f(n) = 6 \cdot log(n) + 4 \cdot log^4 log^3(n) + 3 \cdot log(n^2) + log^2(n)$$

1. $\Theta(log(n))$
2. $\Theta(log(n^2))$
3. $\Theta(log^4 log^3(n))$
4. $\Theta(log^2(n))$
5. Neither!

$$f(n) = \Theta(log^2(n))$$

(4) is the correct answer.

8. Prove that $f(n) = 2 \cdot n \cdot log(n^2) - 6 \cdot log^2(n) + \sqrt{n}$ is $O(n \cdot log(n))$, provide the appropriate C and k constants.

$$\begin{aligned} 2 \cdot n \cdot log(n^2) - 6 \cdot log^2(n) + \sqrt{n} &= 4 \cdot n \cdot log(n) - 6 \cdot log^2(n) + \sqrt{n} \\ &\leq 4 \cdot n \cdot log(n) + 6 \cdot n \cdot log(n) + \sqrt{n} \\ &\leq 4 \cdot n \cdot log(n) + 6 \cdot n \cdot log(n) + n \cdot log(n) \\ &= 11 \cdot n \cdot log(n), n \geq 4 \end{aligned}$$

Let $c = 11$ and $k = 4$ then the statement translates to

$$2 \cdot log(n^2) - 6 \cdot log^2(n) + \sqrt{n} \leq c \cdot n \cdot log(n), n \geq k$$

which by the definition of big-O notation, $2 \cdot n \cdot log(n^2) - 6 \cdot log^2(n) + \sqrt{n}$ is $O(n \cdot log(n))$.

9. Compare the growth of $f(n) = 4^{2^{log_2(n)}}$ and $g(n) = 2^{n+log(n)}$

$$\begin{aligned} f(n) &= 4^n \\ g(n) &= n \cdot 2^n \end{aligned}$$

$$\begin{aligned} lim_{n\to\infty}\frac{2^{2\cdot n}}{n \cdot 2^n} &= lim_{n\to\infty}\frac{2^n}{n} \\ &= \infty \end{aligned}$$

$f(n) = \omega(g(n))$ this implies $f(n) = \Omega(g(n))$. We also know then that $g(n) = O(f(n))$.

10. Prove transitivity of big-O: if $f(n) = O(g(n))$, and $g(n) = O(h(n))$, then $f(n) = O(h(n))$.

Since $f(n) = O(g(n))$ and $g(n) = O(h(n))$ we have the equalities

$$\begin{aligned} f(n) &\leq c_1 \cdot g(n), n \geq k_1 \\ g(n) &\leq c_2 \cdot h(n), n \geq k_2 \end{aligned}$$

From this we obtain

$$f(n) \leq c_1 \cdot c_2 \cdot h(n), n \geq k'$$

where $k' = max(k_1, k_2)$. Let $c = c_1 \cdot c_2$ and $k = k'$ the statement then translate to

$$f(n) \leq c \cdot h(n), n \geq k$$

which by the definition of O notation, $f(n) = O(h(n))$.

12. Prove $\forall k > 0, \epsilon > 0 \implies log^k(n) = o(n^\epsilon)$.

We prove this by induction on k, let ϵ be a real number greater than 0. When $k = 1$ we have

$$\begin{aligned} lim_{n\to\infty}\frac{log(n)}{n^\epsilon} &= lim_{n\to\infty}\frac{\frac{1}{n \cdot ln(2)}}{\epsilon \cdot n^{\epsilon-1}} \\ &= lim_{n\to\infty}\frac{1}{\epsilon \cdot ln(2) \cdot n^\epsilon} \\ &= 0 \end{aligned}$$

Assume that the result holds for all n greater than or equal to some integer k, we show it must also hold for the $k+1$ integer.

$$\begin{aligned} lim_{n\to\infty}\frac{log^{k+1}(n)}{n^{\epsilon}} &= lim_{n\to\infty}\frac{(k+1)\cdot log^{k}(n)\cdot\frac{1}{n\cdot ln(2)}}{\epsilon\cdot n^{\epsilon-1}} \\ &= lim_{n\to\infty}\frac{(k+1)\cdot log^{k}(n)}{\epsilon\cdot ln(2)\cdot n^{\epsilon}} \\ &= 0 \end{aligned}$$

which is what we wanted to show. Hence by the principle of mathematical induction the result holds for all $n > 0$ and $\epsilon > 0$.

13. Use to the definition of big-O to prove that $f(n) + g(n) = O(max(f(n), g(n)))$.

$$f(n) + g(n) \quad \leq \quad 2\cdot max(f(n), g(n)), n \geq 1$$

which by the definition of big-O notation, $f(n) + g(n) = O(max(f(n), g(n)))$.

14. Prove or disprove $(n!)^{log(n)} = \omega(n^{2^{log(n)}})$

$$\begin{aligned} log(lim_{n\to\infty}\frac{n!^{log(n)}}{n^{n}}) &= lim_{n\to\infty}log(n)\cdot log(n!) - n\cdot log(n) \\ &= lim_{n\to\infty}log(n)\cdot\sum_{i=0}^{n}log(n) - n\cdot log(n) \\ &= lim_{n\to\infty}log(n)\cdot\sum_{i=0}^{n-1}log(n) \\ &= \infty \\ 2^{log(lim_{n\to\infty}\frac{n!^{log(n)}}{n^{n}})} &= 2^{\infty} \\ &= \infty \end{aligned}$$

which by the definition of little-ω notation, $n!^{log(n)} = \omega(n^{2^{log(n)}})$.