

1 Use Θ notation to express the statement

$$4n^6 \leq 17n^6 - 45n^3 + 2n + 8 \leq 30n^6, n \geq 3$$

Let $A = 4$, $B = 30$ and $k = 3$ then the statement translates to

$$An^6 \leq 17n^6 - 45n^3 + 2n + 8 \leq Bn^6, n \geq k$$

hence by the definition of Θ notation $17n^6 - 45n^3 + 2n + 8$ is $\Theta(n^6)$.

2 Use Ω notation to express the statement

1. Use Ω notation to express the statement

$$\frac{11}{4}n^2 \leq 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2, n \geq 2$$

Let $A = \frac{11}{4}$ and $k = 2$ then $An^2 \leq 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2, n \geq 2$ then the statement translates to

$$An^2 \leq 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2, n \geq k$$

which by the definition of Ω notation, $3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2$ is $\Omega(n^2)$.

2. Use O notation to express the statement

$$0 \leq 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2 \leq 6n^2, n \geq 1$$

Let $A = 6$ and $k = 1$ then the statement translates to

$$0 \leq 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2 \leq An^2, n \geq k$$

which by the definition of O notation, $3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2$ is $O(n^2)$.

3. Justify the statement: $3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2$ is $\Theta(n^2)$.

Let $A = \frac{11}{4}$, $B = 6$ and $k = 2$ then $A \cdot n^2 \leq 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2 \leq Bn^2, n \geq k$
which by the definition of Θ notation, $3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2$ is $\Theta(n^2)$.

3. Given the function $15n^3 + 11n^2 + 9$

1. Show that the function is $\Omega(n^3)$.

$$15n^3 \leq 15n^3 + 11n^2 + 9, n \geq 1$$

Let $A = 15$ and $k = 1$ then the statements translates to $An^3 \leq 15n^3 + 11n^2 + 9, n \geq k$ which by the definition of Ω notation, $15n^3 + 11n^2 + 9$ is $\Omega(n^3)$.

2. Show that the function is $O(n^3)$.

$$\begin{aligned} 15n^3 + 11n^2 + 9 &\leq 15n^3 + 11n^3 + 9n^3 \\ &\leq 35n^3, n \geq 1 \end{aligned}$$

Let $A = 35$ and $k = 1$ then the statement translates to $15n^3 + 11n^2 + 9 \leq An^3, n \geq k$ which by the definition of O notation, $15n^3 + 11n^2 + 9$ is $O(n^3)$.

4. Given the function $n^4 - 5n - 8$

1. Show that the function is $\Omega(n^4)$.

Let $A = \frac{1}{2}$ and $a = (|-5| + |-8|)$

$$\begin{aligned} n &\geq \frac{2}{1} \cdot (|-5| + |-8|) \\ \frac{1}{2}n^4 &\geq 5n^3 + 8n^3 \\ \frac{1}{2}n^4 &\geq 5n + 8 \\ n^4 - 5n - 8 &\geq \frac{1}{2}n^4, n \geq a \end{aligned}$$

Hence by the definition of Ω notation, $n^4 - 5n - 8$ is $\Omega(n^4)$.

2. Show that the function is $O(n^4)$.

$$\begin{aligned} n^4 - 5n - 8 &\leq n^4 + 5n + 8 \\ &\leq n^4 + 5n^4 + 8n^4 \\ &= 14n^4, n \geq 1 \end{aligned}$$

Let $A = 14$ and $k = 1$ then the statement translates to $n^4 - 5n - 8 \leq An^4, n \geq k$ which by the definition of O notation translates, $n^4 - 5n - 8$ is $O(n^4)$.

5. Show that $15n^3 + 11n^2 + 9$ is $\Theta(n^3)$.

Since we have $\Omega(n^3)$ and $O(n^3)$ we have that there exists real positive number constants A and B such that $Ag(n) \leq f(n) \leq Bg(n), k \geq n$ where $k = \max(k', k'')$ obtained from the previous inequalities. By definition of Θ , $15n^3 + 11n^2 + 9$ is $\Theta(n^3)$.

6. Show that $n^4 - 5n - 8$ is $\Theta(n^4)$.

Since we have shown that the function is $\Omega(n^4)$ and $O(n^4)$ we have that there exists real positive number constants A and B such that $Ag(n) \leq f(n) \leq Bg(n), k \geq n$ where $k = \max(k', k'')$ obtained from the previous inequalities. by definition of Θ , $n^4 - 5n - 8$ is $\Theta(n^4)$.

7. Let $g(n) = n^4 - 5n - 8$, show that $g(n)$ is not $O(n^r)$ for any positive real number $r < 4$.

We prove this by contradiction. Suppose that $g(n)$ is $O(n^r)$ for any positive real number $r < 4$. then

$$g(n) \leq An^r, n \geq a$$

where A and a are real positive numbers.

$$\begin{aligned} g(n) &\leq n^4 \\ &\leq An^r \\ n^{4-r} &\leq A \\ n &\leq \sqrt[4-r]{A} \end{aligned}$$

which is a contradiction. We conclude that $g(n)$ is not $O(n^r)$ for any positive real number $r < 4$.

8. Use theorem on polynomial orders to find orders for the function given by the following formulas.

1. $f(n) = 7n^5 + 5n^3 - n + 4$ for each positive integer n .

By direct application of theorem on polynomial orders, $7n^5 + 5n^3 - n + 4$ is $\Theta(n^5)$.

2. $g(n) = \frac{(n-1)(n+1)}{4}$ for each positive integer n .

$$\begin{aligned}
\frac{(n-1) \cdot (n+1)}{4} &= \frac{n^2 + n - n + 1}{4} \\
&= \frac{n^2 + 1}{4} \\
&= \frac{n^2}{4} + \frac{1}{4}
\end{aligned}$$

Thus $g(n)$ is $\Theta(n^2)$.

9. Show that for a positive integer variable n ,

$$1 + 2 + 3 \dots + n \text{ is } \Theta(n^2)$$

$$\begin{aligned}
\sum_{i=1}^n i &= \frac{n(n+1)}{2} \\
&= \frac{n^2}{2} + \frac{n}{2}
\end{aligned}$$

10. Express $5x^8 - 9x^7 + 2x^5 + 3x - 1 \leq 6x^8, x > 3$ using O notation

Let $A = 6$ and $a = 3$ then $5x^8 - 9x^7 + 2x^5 + 3x - 1 \leq Ax^8, x > a$ and by definition of O notation, $5x^8 - 9x^7 + 2x^5 + 3x - 1$ is $O(x^8)$.

11. Express $x^{\frac{7}{2}} \leq \frac{(x^2-7)^2(10x^{\frac{1}{2}}+3)}{x+1}, x > 4$ using Ω notation

Let $A = 1$ and $k = 4$ then the statement translates to

$$Ax^{\frac{7}{2}} \leq \frac{(x^2-7)^2(10x^{\frac{1}{2}}+3)}{x+1}, x > k$$

which by the definition of Ω notation, $\frac{(x^2-7)^2(10x^{\frac{1}{2}}+3)}{x+1}$ is $\Omega(x^{\frac{7}{2}})$.

12. Express $3x^6 + 5x^4 - x^3 \leq 9x^6, x > 1$ using O notation. Let $A = 9$ and $k = 1$ then the statement translate to

$$3x^6 + 5x^4 - x^3 \leq Ax^6, x > k$$

which by the definition of Ω notation, $3x^6 + 5x^4 - x^3$ is $O(x^6)$.

13. Express $\frac{1}{2}x^4 \leq x^4 - 50x^3 + 1$ for all real numbers $x > 101$ using Ω notation.

Let $A = \frac{1}{2}$ and $k = 101$ this statement translates to $Ax^4 \leq x^4 - 50x^3 + 1, n > k$ which by the definition of Ω notation, $x^4 - 50x^3 + 1$ is $\Omega(x^4)$.