7. Prove that the maximum number of nodes in a binary tree with height h is $2^{h+1} - 1$.

When the tree is a perfect binary tree every level is filled and we have the summation $\sum_{i=0}^{h} 2^i = 2^{h+1} - 1$. We conclude the maximum number of nodes in binary tree with height h is $2^{h+1} - 1$.

8. Prove that it takes $\Omega(n \cdot log(n))$ steps in the best case to build a binary search tree having n distinct keys.

In the best case the binary search tree insertion takes $log(k)$ where k is the current numbdrer of elements in the three. we have the summation $\sum_{i=1}^{n} log(i) = \Omega(\int_1^n log(x) \cdot dx) = \Omega(n \cdot log(n))$.

9. Prove that, when a binary tree with n nodes is implemented using links to the left and right child, then there will be a total of $n + 1$ null links.

We prove by induction. When $n = 1$ there is only the root which has a null left child and null right child hence it has 2 null links, hence $P(1)$ holds. we assume it holds for all integers $i \leq k$ for some positive integer k and show it must hold for the $k + 1$ integer.

Case 1: the root only has child. WLOG we assume it is the left child. the left subtree has n elements and by the inductive hypothesis has $n+1$ null links. Thus we have $(n + 1) + 1$. Which proves the $1st$ case.

Case 2: the root has both children. Let the left subtree contain $n-k$ elements and the right subtree contain k elements where $0 \leq k \leq n$. by the inductive hypothesis the left subtree contains $n - k + 1$ null links and the right subtree contains $k + 1$ null links. we have in total $(n + 1) + 1$ null links which proves the $2nd$ case.

Since both cases hold $P(k + 1)$ is true. We conclude by the principle of mathematical induction the result holds for all integers $n \geq 1$.

10. A full node for a binary tree is one that has two children. Prove that the number of full nodes plus one equals the number of leaves on a binary tree.

The case when there are no full nodes is when there is only the root. we have one leaf which shows $P(0)$ holds. We assume that a tree with n full nodes has $n + 1$ leaves and show that $P(n + 1)$ must also be true. We find one of the full nodes that contains two leaves as children. If we cannot find a node we remove a leaf from a non-full node. since the number of leaves and number of full node have not change we continously do so until we reached the case of a full node that contains two leaves as children. We remove one of the leafs from the full node and we now have a tree with n full nodes. by the induction hypothesis we have $n + 1$ leaves. When we add back the leaf we have an extra full node and an extra leave which make $P(n+1)$ hold. By the principle of mathematical induction the result holds for all integers $n \geq 0$.

11. Prove or disprove: deleting keys x and y from a BST is commutative. In otherwords, it does not matter which order the keys are deleted. the final trees will be identical. If true, provide a proof. If false, provide a counterexample.