1. Prove that $f(n) = 10 \cdot n^4 + 2 \cdot n^2 + 3$ is $O(n^4)$, provide the appropriate C and k constants.

$$f(n) = O(g(n)) \quad \iff \quad \exists c > 0, \exists k \geq 0 \, s.t \, f(n) \leq c \cdot g(n) \, \forall n \geq k$$

$$\begin{aligned} 10 \cdot n^4 + 2 \cdot n^2 + 3 &\leq 10 \cdot n^4 + 2 \cdot n^4 + 3 \\ &\leq 10 \cdot n^4 + 2 \cdot n^4 + 3 \cdot n^4 \\ &= 15 \cdot n^4, n \geq 1 \end{aligned}$$

Let $c = 15$ and $k = 1$ then the statement translates to

$$f(n) \quad \leq \quad c \cdot g(n), n \geq k$$

By the definition of O notation $f(n) = O(g(n))$

2. Prove that $f(n) = 2 \cdot n^2 - n \cdot log_2(n) + 3 \cdot log_2(n)$ is $O(n^2)$, provide the appropriate C and k constants.

$$f(n) = O(g(n)) \quad \iff \quad \exists c > 0, \exists k \geq 0 \, s.t \, f(n) \leq c \cdot g(n) \, \forall n \geq k$$

$$\begin{aligned} 2 \cdot n^2 - n \cdot log_2(n) + 3 \cdot log_2(n) &\leq 2 \cdot n^2 + n^2 + 3 \cdot log_2(n) \\ &\leq 2 \cdot n^2 + n^2 + 3 \cdot n^2 \\ &= 6 \cdot n^2, n \geq 1 \end{aligned}$$

Let $c = 6$ and $k = 1$, then the statement translates to

$$f(n) \quad \leq \quad c \cdot g(n), n \geq k$$

which by the definition of O notation $f(n) = O(g(n))$.

3. Prove that $f(n) = 2 \cdot n^4 \cdot log_2(n^4) - n^2 + 3 \cdot log_2(n)$ is $O(n^4 log_2(n))$, provide the appropriate C and k constants.

$$f(n) = O(g(n)) \quad \iff \quad \exists c > 0, \exists k \geq 0 \, s.t \, f(n) \leq c \cdot g(n) \, \forall n \geq k$$

$$\begin{aligned}
2 \cdot n^4 \cdot log_2(n^4) - n^2 + 3 \cdot log_2(n) &= 8 \cdot n^4 \cdot log_2(n) - n^2 + 3 \cdot log_2(n) \\
&\leq 8 \cdot n^4 \cdot log_2(n) + n^4 \cdot log_2(n) + 3 \cdot log_2(n) \\
&\leq 8 \cdot n^4 \cdot log_2(n) + n^4 \cdot log_2(n) + 3 \cdot n \cdot log_2(n) \\
&= 12 \cdot n^4 \cdot log_2(n), n \geq 1
\end{aligned}$$

Let $c = 12$ and $k = 1$ then the statement translates to

$$2 \cdot n^4 \cdot log_2(n^4) - n^2 + 3 \cdot log_2(n) \leq c \cdot n^4 \cdot log_2(n), n \geq 1$$

which by the definition O notation $f(n) = O(g(n))$.

4. Prove or disprove $f(n) = 5 \cdot n^3 - n + 3$

1. $f(n) = O(n^2)$

$$f(n) = O(n^2) \iff \exists c > 0, \exists k \geq 0 \, s.t \, f(n) \leq c \cdot n^2 \, \forall n \geq k$$

$$\begin{aligned}
5 \cdot n^3 - n + 3 &\leq c \cdot n^2 \\
5 \cdot n^2 - \frac{1}{n} + \frac{3}{n^2} &\leq c \\
lim_{n\to\infty} 5 \cdot n - lim_{n\to\infty} \frac{1}{n} + lim_{n\to\infty} \frac{3}{n^2} &\leq c \\
\infty &\leq c
\end{aligned}$$

Which is a contradiction, hence $f(n) \neq O(n^2)$.

2. $f(n) = \Omega(n)$

$$f(n) = \Omega(n) \iff \exists c > 0, \exists k \geq 0 \, s.t \, f(n) \geq c \cdot n \, \forall n \geq k$$

$$\begin{aligned}
5 \cdot n^3 - n + 3 &\geq 5 \cdot n - n + 3 \\
&\geq 5 \cdot n - n \\
&= 4 \cdot n, n \geq 1
\end{aligned}$$

Let $c = 4$ and $k = 1$ then the statement translate to

$$5 \cdot n^3 - n + 3 \geq c \cdot n, n \geq k$$

which by the definition of Ω notation, $5 \cdot n^3 - n + 3$ is $\Omega(n)$.

3. $f(n) = \Theta(n^3)$

$$f(n) = \Theta(n^3) \iff f(n) = O(n^3)\, and\, f(n) = \Omega(n^3)$$

Showing $f(n) = O(n^3)$

$$f(n) = O(n^3) \iff \exists c > 0, \exists k \geq 0\, s.t\, f(n) \leq c \cdot g(n)\, \forall n \geq k$$

$$\begin{aligned} 5 \cdot n^3 - n + 3 &\leq 5 \cdot n^3 + n^3 + 3 \cdot n^3 \\ &= 9 \cdot n^3, n \geq 1 \end{aligned}$$

Let $c = 9$ and $k = 1$ then the statement translates to

$$5 \cdot n^3 - n + 3 \leq c \cdot n^3, n \geq k$$

which by the definition of O notation, $5 \cdot n^3 - n + 3 = O(n^3)$.

Showing $f(n) = \Omega(n^3)$

$$f(n) = \Omega(n^3) \iff \exists c > 0, \exists k \geq 0\, s.t\, f(n) \geq c \cdot g(n)\, \forall n \geq k$$

$$\begin{aligned} 5 \cdot n^3 - n + 3 &\geq 5 \cdot n^3 - n^3 + 3 \\ &\geq 4 \cdot n^3 + 3 \\ &\geq 4 \cdot n^3, n \geq 1 \end{aligned}$$

Let $c = 4$ and $k = 1$ then the statement translate to

$$5 \cdot n^3 - n + 3 \geq c \cdot n^3, n \geq k$$

which by the definition of Ω notation, $5 \cdot n^3 - n + 3$ is $\Omega(n^3)$.

Since $5 \cdot n^3 - n + 3$ is $O(n^3)$ and $5 \cdot n^3 - n + 3$ is $\Omega(n^3)$ we conclude that $5 \cdot n^3 - n + 3$ is $\Theta(n^3)$.

4. $f(n) = \omega(n)$

$$f(n) = \omega(n) \quad \iff \quad lim_{n\to\infty}\frac{f(n)}{n} = \infty$$

$$lim_{n\to\infty}\frac{5\cdot n^3 - n + 3}{n} \quad = \quad \infty$$

which by the definition of ω notation, $5\cdot n^3 - n + 3$ is $\omega(n)$.

5. $f(n) = o(n^2)$

$$f(n) = o(n^2) \quad \iff \quad lim_{n\to\infty}\frac{f(n)}{n^2} = 0$$

$$lim_{n\to\infty}\frac{5\cdot n^3 - n + 3}{n^2} \quad = \quad \infty$$

hence $f(n) \neq o(n^2)$.

5. Prove that $(n+5)^{100} = \Theta(n^{100})$

$$f(n) = \Theta(g(n)) \quad \iff \quad f(n) = O(g(n))\, and\, f(n) = \Omega(g(n))$$

1. Showing $(n+5)^{100} = O(n^{100})$

$$f(n) = O(g(n)) \quad \iff \quad \exists c > 0, \exists k \geq 0\, s.t\, f(n) \leq c\cdot g(n)\, \forall n \geq k$$

$$\begin{aligned}
(n+5)^{100} \quad &= \quad \sum_{i=0}^{100}\binom{100}{i}\cdot n^{100-i}\cdot 5^i \\
&\leq \quad \sum_{i=0}^{100}\binom{100}{i}\cdot n^{100}, n \geq 1
\end{aligned}$$

Let $c = \sum_{i=0}^{100} \binom{100}{i}$ and $k = 1$ then the statements translates to

$$f(n) \quad \leq \quad c \cdot g(n), n \geq k$$

which by the definition of O notation $f(n) = O(g(n))$.

2. Showing $(n+5)^{100} = \Omega(n^{100})$

$$f(n) = \Omega(g(n)) \quad \iff \quad \exists c > 0, \exists k \geq 0 \, s.t \, f(n) \geq c \cdot g(n) \, \forall n \geq k$$

$$\begin{aligned} (n+5)^{100} &= \sum_{i=0}^{100} \binom{n}{i} \cdot n^{100-i} \cdot 5^i \\ &\geq n^{100} \end{aligned}$$

Let $c = 1$ and $k = 1$ then the statement translates to

$$f(n) \quad \geq \quad c \cdot g(n), n \geq k$$

which by the definition of Ω notation $f(n) = \Omega(g(n))$.

Since $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ by the definition of Θ notation $f(n) = \Theta(n^{100})$.

6. Prove transitivity of big-O: if $f(n) = O(g(n))$, and $g(n) = O(h(n))$, then $f(n) = O(h(n))$.

Since $f(n) = O(g(n))$ and $g(n) = O(h(n))$ we have the equalities

$$\begin{aligned} f(n) &\leq c_1 \cdot g(n), n \geq k_1 \\ g(n) &\leq c_2 \cdot h(n), n \geq k_2 \end{aligned}$$

From this we obtain

$$f(n) \quad \leq \quad c_1 \cdot c_2 \cdot h(n), n \geq k'$$

where $k' = max(k_1, k_2)$. Let $c = c_1 \cdot c_2$ and $k = k'$ the statement then translate to

$$f(n) \quad \leq \quad c \cdot h(n), n \geq k$$

which by the definition of O notation, $f(n) = O(h(n))$.

7. Prove that $f(n) = O(g(n)) \iff g(n) = \Omega(f(n))$.

Forward direction: $f(n) = O(g(n)) \implies g(n) = \Omega(f(n))$.

Since $f(n) = O(g(n))$ there exists a number c and a number k such that $f(n) \leq c \cdot g(n), n \geq k$ where $c > 0$ and $k \geq 0$. From this we obtain $\frac{1}{c} \cdot f(n) \leq g(n), n \geq k$. Which by the definition of Ω notation, $g(n) = \Omega(f(n))$.

Backward direction: $g(n) = \Omega(f(n)) \implies f(n) = O(g(n))$

Since $g(n) = \Omega(f(n))$ there exists a number c and a number k such that $g(n) \geq c \cdot f(n), n \geq k$ where $c > 0$ and $k \geq 0$. From this we obtain $g(n) \cdot \frac{1}{c} \geq f(n), n \geq k$. Which by the definition of O notation, $f(n) = O(g(n))$.

We conclude $f(n) = O(g(n)) \iff g(n) = \Omega(f(n))$.

8. Compare the growth of

1. $f(n) = n$ and $g(n) = n^{1+sin(n)}$

$$f(n) = O(g(n)) \quad \iff \quad \exists c > 0, \exists k \geq 0 \, s.t \, f(n) \leq c \cdot g(n) \, \forall n \geq k$$

   No analysis can be described for $f(n)$ and $g(n)$.

2. $f(n) = \sqrt{n}$ and $g(n) = n + sin(n)$

$$f(n) = o(g(n)) \quad \iff \quad lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

$$\begin{aligned} lim_{n \to \infty} \frac{\sqrt{n}}{n + sin(n)} &= lim_{n \to \infty} \frac{\sqrt{n}}{n} \\ &= 0 \end{aligned}$$

   Thus $\sqrt{n}$ is $o(n + sin(n))$. This implies $\sqrt{n}$ is $O(n + sin(n))$. We also have then that $n + sin(n)$ is $\Omega(\sqrt{n})$.

3. $f(n) = n$ and $g(n) = n \cdot |sin(n)|$

$$f(n) = O(g(n)) \quad \iff \quad \exists c > 0, \exists k \geq 0 \, s.t \, f(n) \leq c \cdot g(n) \, \forall n \geq k$$

$$\begin{aligned} n \cdot |sin(n)| &\leq c \cdot n \\ |sin(n)| &\leq c \end{aligned}$$

Let $c = 2$ and $k = 0$ then the following equality holds

$$n \cdot |sin(n)| \leq c \cdot n \geq k$$

by the definition of O notation $n \cdot |sin(n)|$ is $O(n)$. by part (7) we also have that n is $\Omega(n \cdot |sin(n)|)$.

9. Prove or disprove $2^{n+1} = O(2^n)$

$$f(n) = O(g(n)) \iff \exists c > 0, \exists k \geq 0 \, s.t \, f(n) \leq c \cdot g(n) \, \forall n \geq k$$

$$\begin{aligned} 2^{n+1} &= 2 \cdot 2^n \\ &\leq 3 \cdot 2^n, n \geq 1 \end{aligned}$$

Let $c = 3$ and $k = 1$ then the statement translates to

$$2^{n+1} \leq c \cdot 2^n, n \geq k$$

which by the definition of O notation, $2^{n+1} = O(2^n)$.

10. Prove or disprove $2^{2 \cdot n} = (2^n)$

$$f(n) = o(g(n)) \iff lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

$$\begin{aligned} lim_{n \to \infty} \frac{f(n)}{g(n)} &= lim_{n \to \infty} \frac{2^{2 \cdot n}}{2^n} \\ &= lim_{n \to \infty} \frac{2^n}{1} \\ &= \infty \end{aligned}$$

hence $2^{2 \cdot n} \neq o(2^n)$.

11. Prove that if $lim_{n \to \infty} \frac{f(n)}{g(n)} = C$, for some constant $C > 0$ then $f(n) \leq \Theta(g(n))$.

Since $lim_{n \to \infty} \frac{f(n)}{g(n)} = C$, for every $\epsilon > 0$, there exists $k \geq 0$ such that, for all $n \geq k$, $|\frac{f(n)}{g(n)} - C| < \epsilon$. From this we obtain

$$\begin{aligned} -\epsilon < & \quad \tfrac{f(n)}{g(n)} - C & < \epsilon \\ C - \epsilon < & \quad \tfrac{f(n)}{g(n)} < & C + \epsilon \\ g(n) \cdot (C - \epsilon) < & \quad f(n) < & g(n) \cdot (C + \epsilon), n \geq k \end{aligned}$$

Since $C > 0$ and $\epsilon > 0$, the equality implies $f(n) = \Theta(g(n))$ so long as $(C - \epsilon) > 0$. Let $\epsilon = \frac{C}{2}$ then the equality holds and by the definition of Θ notation, $f(n) = \Theta(g(n))$.