1. Where in a min heap the largest element resides? (Assume all elements are distinct) Explain.

Suppose that we have a min heap with n elements where $n > 1$. Suppose further that the largest element was an internal node, then that means it has atleast one child. This would be a contradiction since the largest element would be larger than its children which contradicts the min-heap property. we conclude that the largest element must be a leaf.

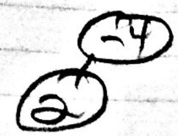
2. Make the min heap by successive insertions into an initially-empty min heap. Re-draw the heap each time an insertion causes one or more swaps.

$$[-4, 2, 10, 12, -1, -3, 15, 76]$$

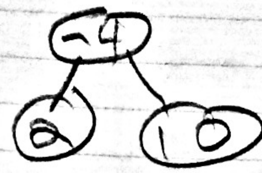
iteration	tree

1

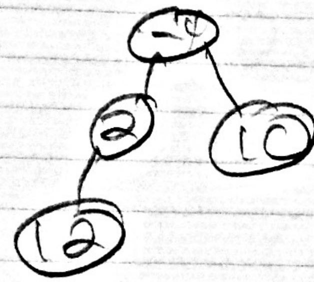
2



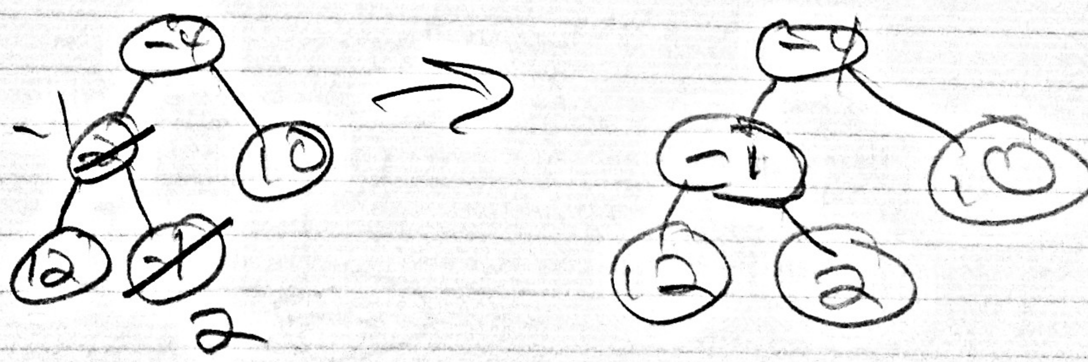
3



4

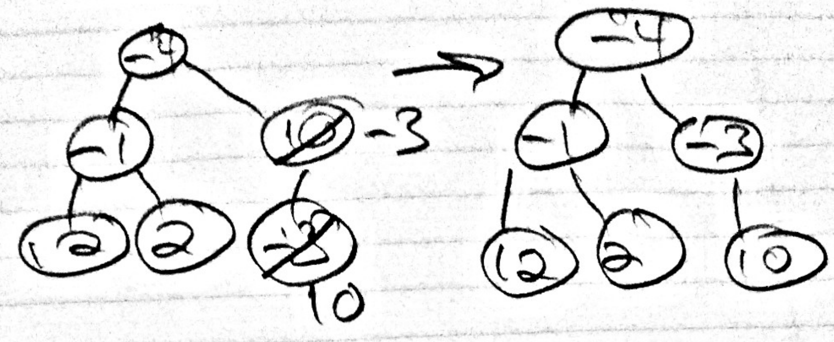


5

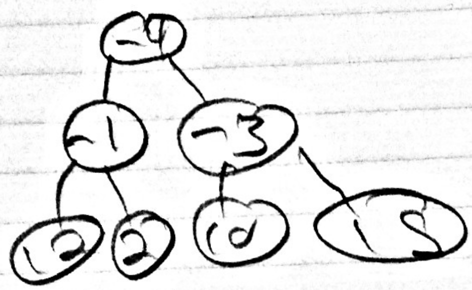


iteration tree

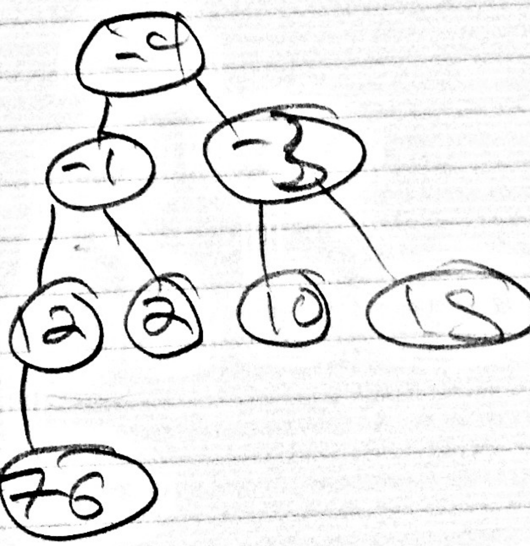
6



7



8



4. Is this a max-heap? if not, apply max-heapify function to change it to a max heap.

$$[2, -1, 10, -4, 4, 12, 15, 0, 27, 39, 3, 24]$$

6. Insert

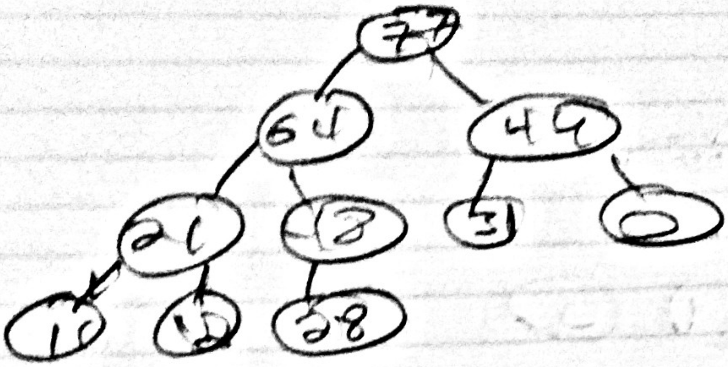
1. 95
2. 160
3. 37

into the below max-heap

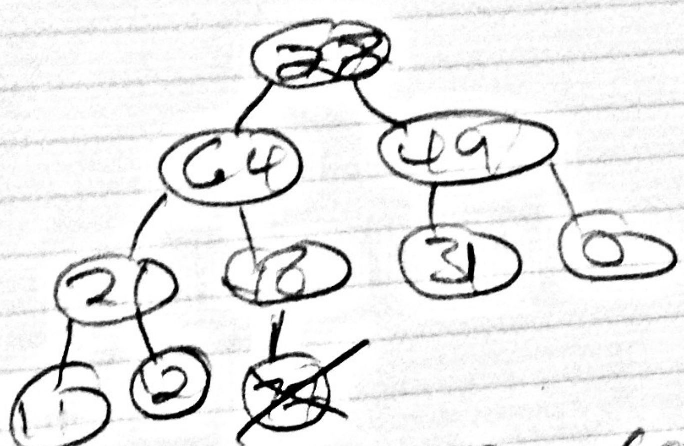
$$[100, 90, 80, 85, 6, 70, 10, 28, 1, 2, 0, 21, 27]$$

7. Delete the root of the below max-heap

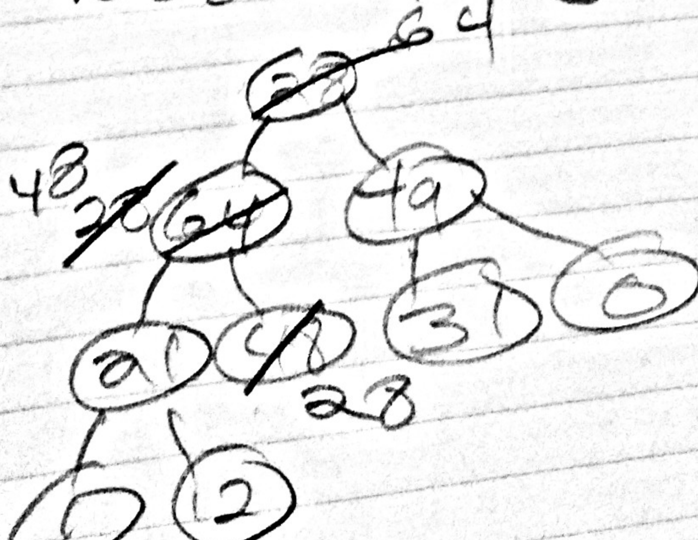
$$[77, 64, 49, 21, 48, 31, 0, 1, 12, 28]$$



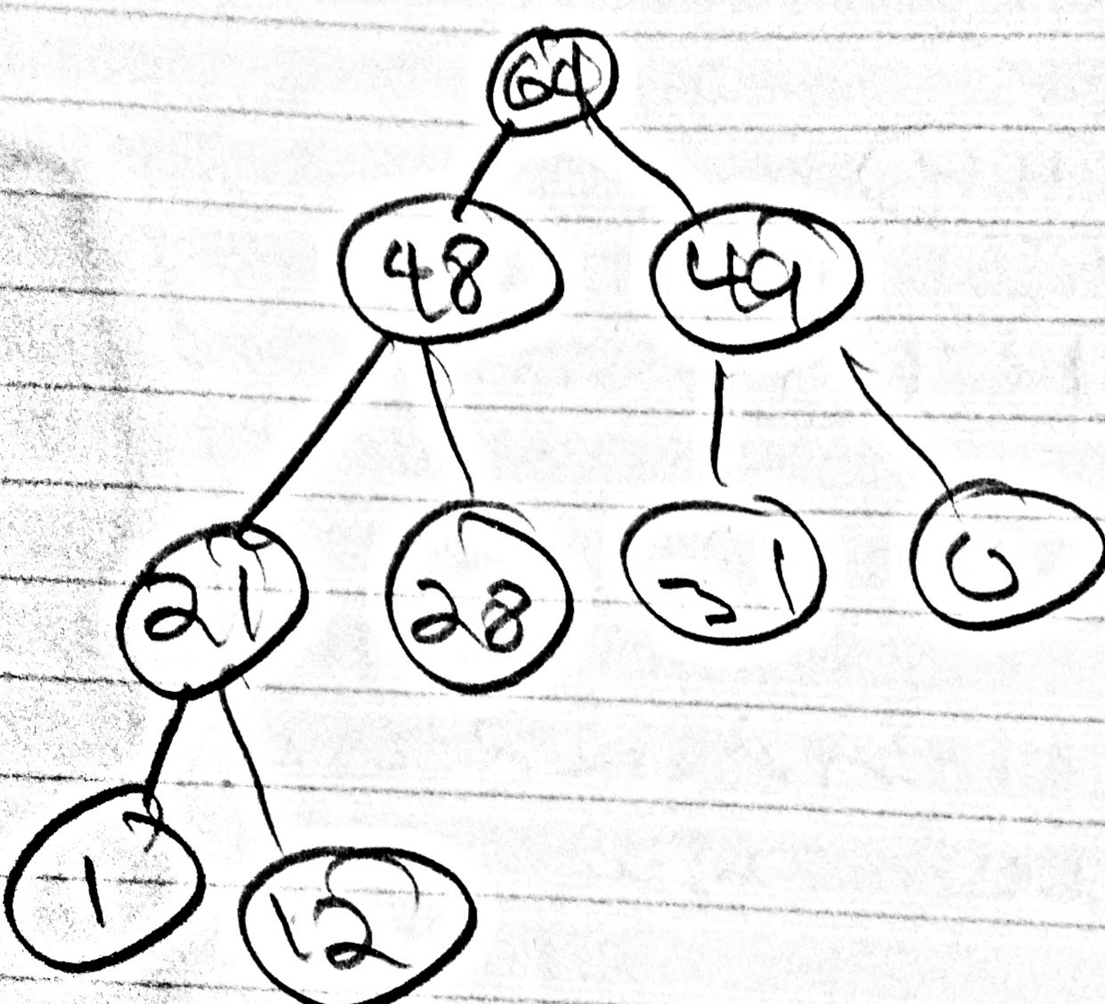
Swap 77 and 28
then remove 77



max heapify (a,0)

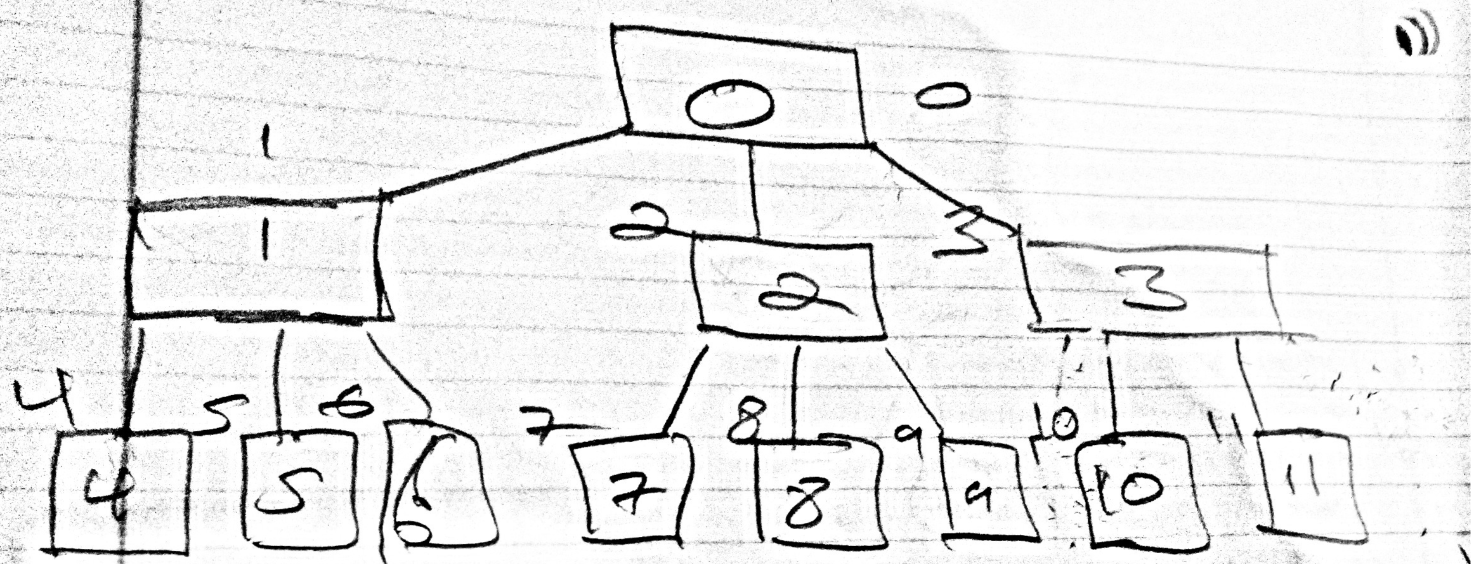


result



9. Suppose that instead of binary heaps, we wanted to work with ternary heaps. Suggest an appropriate indexing scheme so that a complete tree will yield a contigious sequence.

$$\begin{aligned} left\_child &= 3 \cdot i + 1 \\ middle\_child &= 3 \cdot i + 2 \\ right\_child &= 3 \cdot i + 3 \\ parent &= \lfloor \frac{i-1}{3} \rfloor \end{aligned}$$


0
0
1
1
2
2
3
3
4
4
5
5
6
6
7
7
8
8
9
9
10
10
11
11

10. Use induction to prove that $1 + 2 + 4 + \ldots + 2^h = 2^{h+1} - 1$.

Proof. We prove by induction

Let $P(h) = 1 + 2 + 4 + \ldots + 2^h = 2^{h+1} - 1$

Base case: $h = 0$

$1 = 1 = 2^{0+1} - 1.$

Thus $P(0)$ holds.

Inductive step: Let $P(k)$ be true we show that $P(k+1)$ is true, that is $1 + 2 + 4 + \ldots + 2^k = 2^{(k+1)+1} - 1$

$$
\begin{aligned}
1 + 2 + 4 + \ldots + 2^k &= \sum_{i=0}^{k+1} 2^i \\
&= \sum_{i=0}^{k} 2^i + 2^{k+1} \\
&= 2^{k+1} - 1 + 2^{k+1} \\
&= 2 \cdot 2^{k+1} - 1 \\
&= 2^{k+1+1} - 1 \\
&= 2^{(k+1)+1} - 1
\end{aligned}
$$

Thus $P(k+1)$ holds. By the principle of mathematical induction, $P(h)$ holds for for all integers $h \geq 0$. $\square$

11. What is the minimum and maximum number of leaves in a binary heap that has height h. Explain.

When $h = 0$ then the minimum equal the maximum namely 1. Suppose that $h \geq 1$. The minimum case is where there is only one left child in the level h. The rest of the leaves are on the $h-1$ level where the leftmost node is the only parent. Thus there are a total of $2^{h-1} - 1 + 1 = 2^{h-1}$ minimum number of leafs for height h. In a perfect tree where all levels are filled we have $\sum_{i=0}^{h} 2^i = 2^{h+1} - 1$ total nodes where the leafs are the last summation term and thus contribute 2^h nodes. Thus the minimum and maximum number of leaves in a binary heap that has height h is 2^{h-1} and 2^h leaves for $h \geq 1$.

12. Prove that a binary heap with n elements has height $\lfloor log_2(n) \rfloor$.

$$
\begin{array}{rcl}
2^h \leq & n & \leq 2^{h+1} - 1 \\
2^h \leq & n & < 2^{h+1} \\
log_2(2^h) \leq & log_2(n) & < log_2(2^{h+1}) \\
h \leq & log_2(n) & < h + 1
\end{array}
$$

which by definition of the floor $h = \lfloor log_2(n) \rfloor$. Hence we conclude a binary heap with n elements has height $\lfloor log_2(n) \rfloor$.

13. Prove that a binary heap with n nodes has exactly $\lceil \frac{n}{2} \rceil$ leaves.

Version 1:

$$\begin{aligned} n &= \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil \\ n - \lfloor \frac{n}{2} \rfloor &= \lceil \frac{n}{2} \rceil \end{aligned}$$

Hence the number of elements n subtracted by the last internal node index tells us the amount of leaves in the heap. We conclude that binary heap with n nodes has exactly $\lceil \frac{n}{2} \rceil$ leaves.

Version 2:

We prove by induction. Let $P(n)$ be the statement that a binary heap with n nodes has exactly $\lceil \frac{n}{2} \rceil$ leaves. $P(1)$ holds because $\lceil \frac{1}{2} \rceil = 1$ leaf, which is namely the root. We assume $P(k)$ holds for some integer k and show that is must hold for the $k+1$ integer. Case 1: the $\lfloor \frac{k}{2} \rfloor$ internal node has one child, then the next node will be its right child.

$$\begin{aligned} \lceil \frac{k}{2} \rceil + 1 &= \lceil \frac{k+2}{2} \rceil \\ &= \lceil \frac{k+1}{2} \rceil \, (since\, the\, last\, child\, has\, index\, k+1) \end{aligned}$$

which proves the 1^{st} case.

Case 2: The $\lfloor \frac{k}{2} \rfloor$ internal node has two children, then the next node will be the left child of the $\lfloor \frac{k+1}{2} \rfloor$.

$$\begin{aligned} \lceil \frac{k}{2} \rceil + 1 &= \lceil \frac{k+2}{2} \rceil \\ &= \lceil \frac{k+1}{2} \rceil \, (since\, the\, last\, child\, has\, index\, k+1) \end{aligned}$$

which proves the 2^{nd} case. Thus $P(k+1)$ holds. By the principle of mathematical induction, $P(n)$ holds for all integers $n \geq 1$.

14. Show that there are at most $\lceil \frac{n}{2^{h+1}} \rceil$ nodes of height h in any $n-$element heap.

We prove by induction. Let $P(h)$ be the statement that there are at most $\lceil \frac{n}{2^{h+1}} \rceil$ nodes of height h in any n-element heap. $P(0)$ holds since $\lceil \frac{n}{2^1} \rceil$ are the number of leaves in the heap which by definition have height 0. We assume $P(k)$ holds for some integer k and show that it must hold for the $k+1$ integer. We have that there at most $\lceil \frac{n}{2^{k+1}} \rceil$ nodes of height k in any $n-$element heap. We make a new heap H' which haves all the elements of the original heap H except the leaves, hence it has $\lfloor \frac{n}{2} \rfloor$ total elements. The nodes of height k in this heap will be of height $k+1$ in the original heap. $\lceil \frac{\lfloor \frac{n}{2} \rfloor}{2^{k+1}} \rceil \leq \lceil \frac{\frac{n}{2}}{2^{k+1}} \rceil = \lceil \frac{n}{2^{(k+1)+1}} \rceil$. Hence $P(k+1)$ holds. By the principle of mathematical induction, $P(h)$ holds for all integers $h \geq 0$.