Lecture 4 (Review/ Growth & Functions)

Thursday, September 3, 2020 5:00 PM

(Reminder: HW 1 is due this Saturday)

Integrals:

Examples:

① $\int 6\, dx = \underbrace{6x}\, + \cancel{C}$

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② $\int (7x + 2)\, dx = \frac{7}{2}x^2 + 2x$

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③ $\int \left(\underbrace{\frac{5}{x}} - \underbrace{\sqrt{x}}_{x^{1/2}} + 8x^2 + \sin x\right) dx = 5\ln x - \frac{2}{3}x^{3/2} + \frac{8}{3}x^3 - \cos x$

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④ $\int \left(\underbrace{x\sqrt{x}}_{x^1 x^{\frac{1}{2}} = x^{3/2}} - \underbrace{\frac{1}{x\ln 2}} + \underbrace{e^x}\right) dx = \frac{2}{5}x^{5/2} - \underbrace{\frac{\ln x}{\ln 2}}_{= \log x} + e^x$

$= \log x = \frac{\log_a x}{\log_a 2}$

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⑤ $\underbrace{\int \underbrace{x}\underbrace{e^x}\, dx}_{\text{integration by parts}} = \int s'\, p$

$p = x \rightarrow p' = 1$

$s' = e^x \rightarrow s = e^x$

$(ps)' = p's + s'p$

$\int (ps)' = \int p's + \int s'p$

$\int s'p = ps - \int p's$

$= xe^x - \int e^x dx$

$\boxed{= xe^x - e^x}$

$$\int (PS)' = \int P'S + \int S'P \qquad = \boxed{xe^x - e^x}$$

$$PS = \int P'S + \int S'P$$

$$PS - \int P'S = \int S'P$$

$$\int xe^x\,dx = \int S'P = PS - \int P'S$$

$$= \frac{x^2}{2}e^x - \underbrace{\int \frac{x^2}{2}e^x\,dx}_{?!\ :C}$$

$P = e^x \Rightarrow P' = e^x$

$S' = x \Rightarrow S = \frac{x^2}{2}$

⑥ $\int x\ln x\,dx =$ ↙ Vivian :)

$$= \frac{1}{2}x^2\ln(x) - \int \frac{x}{2}\,dx$$

$$= \frac{1}{2}x^2\ln(x) - \frac{x^2}{4}$$

$$= \frac{1}{2}x^2\ln(x) - \frac{x^2}{4} + C$$

$P = \ln x \to P' = 1/x$

$S' = x \to S = \frac{x^2}{2}$

⑦ $\int \ln x\,dx =$ ↙ Bridget :)

⑦ $\int (\ln x)\,dx =$ $P = \ln x$ $P' = \frac{1}{x}$

$S' = 1$ $S = x$

$= x\ln x - \int \frac{1}{x}x\,dx$

$\boxed{x\ln x} - \int 1\,dx$

$= \boxed{x\ln x - x}$

Growth of functions

g

```
for i=1:n        → i = 1, 2, 3, → n      i=1
    for j=1:n
        sum++                             2
        sum += j
```

f

```
for i=1:n   ← input
    sum++                                 5
end
```

sum ++

sum += j

2

end

end

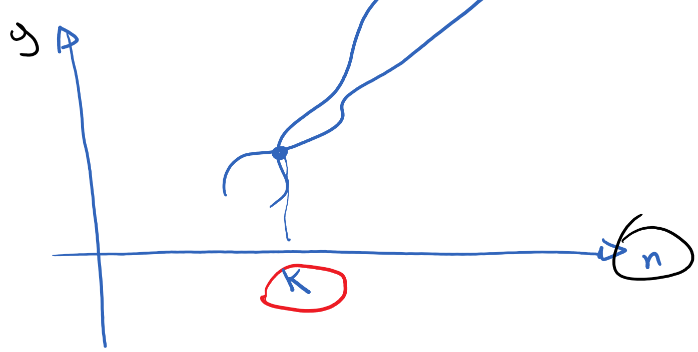
end

$f(n) = 5n$

$$g(n) = \underbrace{2n}_{i=1} + \underbrace{2n}_{i=2} + \cdots = 2n(n) = 2n^2$$

$f(n) \simeq 5n$ / $g(n) \simeq 2n^2$

$n \nearrow$



① $f(n) = O(g(n)) \underset{\text{iff}}{\Longleftrightarrow} \exists c > 0, \exists K \geq 0$ such that $f(n) \leq c\,g(n) \;\forall n \geq K$

O ↳ big-Oh; $\exists$ ↳ there exist; $\forall$ ↳ for all

② $g(n) = \Omega(f(n)) \Longleftrightarrow \exists c > 0, \exists K \geq 0$ s.t. $g(n) \geq c\,f(n) \;\forall n \geq K$

Ω ↳ big-omega

Example: Prove or disprove $f(n) = O(g(n))$

$f(n) = 2n\log n + 6n - 10$

$g(n) = n^2$

$\Rightarrow \underbrace{f(n) = O(g(n))}_{\checkmark} \Longleftrightarrow \underbrace{\exists c > 0, \exists K \geq 0 \text{ s.t. } f(n) \leq c\,g(n) \;\; \forall n \geq K}$

$$\Rightarrow \frac{2n\log n + 6n - 10}{n^2} \leq \frac{cn^2}{n^2}$$

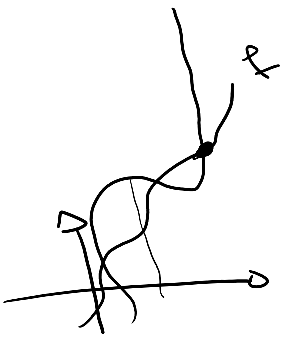
$$\Rightarrow \frac{2n\log n + \cancel{6n} - 10}{} \to 0, \; c$$

$$\Rightarrow \frac{2n\log n + 6n - 10}{n^2} \leq c \quad \xrightarrow{} 0$$

$$n \longrightarrow \infty$$

$$\Rightarrow \quad 0 \leq c \Rightarrow c = 1$$

100% sure that you are proving! :)



$$\Rightarrow 2n\log n + 6n - 10 \leq cn^2$$

$$\overset{c=1}{\Rightarrow} \quad 2n\log n + 6n - 10 \leq n^2 \qquad k = 1 \checkmark$$

$$2(1)\overset{0}{\cancel{\log 1}} + 6(1) - 10 \overset{?}{\leq} 1^2$$

$$0 + 6 - 10 \leq 1$$

$$-4 \leq 1 \checkmark \quad :)$$

Sol. 2

$$f(n) = O(g(n)) \Leftrightarrow \exists c > 0, \exists k \geq 0, \text{ s.t. } f(n) \leq c\,g(n) \ \forall n \geq k$$

$$\Rightarrow \overset{a}{2n\log n + 6n - 10} \overset{?}{\leq} \overset{x}{cn^2}$$

$a \leq b$, $b \leq x \Rightarrow a \leq x$

$$\overset{a}{2n\log n + 6n - 10} \leq \overset{b}{2n^2 + 6n^2 + 10n^2}$$

imp

$$b \leq x \Rightarrow 18\cancel{n^2} \leq c\cancel{n^2} \Rightarrow 18 \leq c \Rightarrow c = 20 \checkmark$$

$$k = 1 \Rightarrow 2n\log n + 6n - 10 \leq 20n^2$$

$$2(1)(0) + 6 - 10 \leq 20$$

$$-4 \leq 20 \checkmark$$

$$\Rightarrow \boxed{f(n) = O(g(n))}$$

Example 2: prove or disprove $f(n) = \Omega(g(n))$

$f(n) = 2n\log n + 6n - 10$

$g(n) = n^2$

$\Rightarrow f(n) = \Omega(g(n)) \Leftrightarrow \exists c > 0, \exists k \geq 0$ s.t. $f(n) \geq c\,g(n) \; \forall n \geq k$

$$\frac{2n\log n + 6n - 10}{n^2} \geq \frac{cn^2}{n^2}$$

$$\frac{2n\log n + 6n - 10}{n^2} \geq c \Rightarrow 0 \geq c \; \times$$

$n \to \infty$

disproved

Example: $f(n) = n^2\sqrt{n} + 2n + 4\log n$

$g(n) = n^3$

$f(n) \overset{?}{=} O(g(n))$

Andrew

Sol 1

Example: prove/disprove

$f(n) = n^2\sqrt{n} + 2n + 4\log n$

$g(n) = n^3$

$f(n) \overset{?}{=} O(g(n)) \iff \exists\, c>0,\ \exists k \geq 0$ s.t. $f(n) \leq c\,g(n)\ \forall n \geq k$

$n^2\sqrt{n} + 2n + 4\log n \overset{?}{\leq} cn^3$

$\Rightarrow \dfrac{n^2\sqrt{n} + 2n + 4\log n}{n^3} \leq c \Rightarrow 0 \leq c$ ✓

$c = 5$

$n \to \infty$

$c=5 \Rightarrow n^2\sqrt{n} + 2n + 4\log n \leq 5n^3$ $\quad k=1$

$1\sqrt{1} + 2(1) + 4\log(1) \leq 5(1)^3$

$1 + 2 + 0 \leq 5$

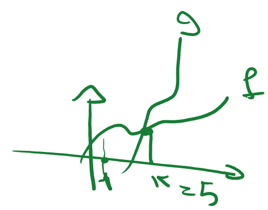
$3 \leq 5$ ✓

Sol 2

$c = 1$

$n^2\sqrt{n} + 2n + 4\log n \leq n^3$

$k=1$ $\quad 1 + 2 + 0 \leq 1 \Rightarrow 3 \leq 1$:(



✓ $k=4$ $\quad 4^2\sqrt{4} + 2(4) + 4\log_2 4 \leq 4^3$

$32 + 8 + 8 \leq 64$

$32 + 16 \leq 64 \Rightarrow 48 \leq 64$ ✓

Sol 3

$n^2\sqrt{n} + 2n + 4\log n \leq cn^3$

$a \leq b,\ b \leq x \to a \leq x$

$n^2\sqrt{n} + 2n + 4\log n \leq \underbrace{n^3 + 2n^3 + 4n^3}_{7n^3}$

$\Rightarrow 7n^3 \leq cn^3 \Rightarrow 7 \leq c \Rightarrow \boxed{c=10}$ ✓

find k $\Rightarrow n^2\sqrt{n} + 2n + 4\log n \leq 10n^3$

$k=1$ $\quad 1 + 2 + 0 \leq 10$

$(k=1)$ $\quad 1 + 2 + 0 \leq 10$

$3 \leq 10$ ✓

$$\Rightarrow \boxed{f(n) = O(g(n))}$$

Example: $f(n) = 2n^3 + 5n + \log n$

$g(n) = n^3$

I $\; f(n) \overset{?}{=} O(g(n))$ $\quad$ II Example: $f(n) \overset{?}{=} \Omega(g(n))$

$\Rightarrow$ I $\; f(n) = O(g(n)) \iff \exists c > 0, \exists k \geq 0, \text{ s.t. } f(n) \leq c\,g(n) \;\; \forall n \geq k$

Bishoy

$f(n) = 2n^3 + 5n + \log n$

$g(n) = n^3$

$f(n) \overset{?}{=} O(g(n))$ $\iff$ write the def please

$2n^3 + 5n + \log n \leq 20n^3$

$16 + 10 + 1 \leq 16$ ✗

$c = 20$ ✓

$n = 2 \quad c = 20$

$16 + 10 + 1 \leq 20 \times 8$

$27 \leq 160$ ✓ $\Rightarrow \boxed{f(n) = O(g(n))}$ ✓

$f(n) \overset{?}{=} \Omega(g(n))$ $\iff$ $\exists c > 0, \exists k \geq 0 \text{ s.t. } f(n) \leq c\,g(n) \;\; \forall n \geq k$

$c = 1 \quad n = 2$

$16 + 10 + 1 \geq 1 \times 8$

$27 \geq 8$ ✓ $\Rightarrow \boxed{f(n) = \Omega(g(n))}$ ✓

$\Rightarrow f(n) = \Theta(g(n))$ (big-theta) $\iff$ (1) $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

or

② $\begin{cases} f(n) = O(g(n)) \\ \text{and} \\ g(n) = O(f(n)) \end{cases}$

or

③ $c_1 g(n) \leq f(n) \leq c_2 g(n) \quad \exists c_1 > 0$

$\exists c_2 > 0$

$O / \Omega \quad \cancel{\forall} \quad c > c$

$$\left. \begin{array}{l} f(n) = \underline{\underline{2n^3}} + 5n + \log n \\ g(n) = \underline{\underline{n^3}} \end{array} \right) \Longleftrightarrow f(n) = \Theta(g(n))$$

1s

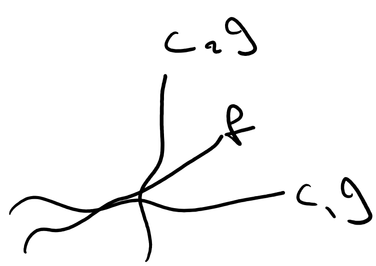
```
for i=1:n
    for j=1:n
        } 10 lines
    end
end
```

.9s

```
for i=1:n
    for j=1:n
        } 5 lines
    end
end
```

0.00001

```
for i=1:n
    }
end
```



$f(n) = 10n^2$

$g(n) = 5n^2$

$\underline{h(n) = 20n}$

proving $f(n) = \Theta(g(n))$ for previous Example

$f(n) = 2n^3 + 5n + \log n$

$g(n) = n^3$

$f(n) = O(g(n)) \Leftrightarrow \exists c > 0, \exists k \geq 0$ s.t. $f(n) \leq c\,g(n) \quad \forall n \geq k$

$$\Rightarrow \frac{2n^3 + 5n + \log n}{n^3} \leq \frac{c\,n^3}{n^3}$$

$$\Rightarrow \frac{2n^3 + 5n + \log n}{n^3} \leq c \Rightarrow 2 \leq c \Rightarrow \boxed{c = 10}$$

$n \to \infty$

finding k

$$\Rightarrow 2n^3 + 5n + \log n \leq 10 n^3$$

$k = 1$ ✓ $2 + 5 + 0 \leq 10 \Rightarrow \boxed{7 \leq 10}$ ✓

$$\Rightarrow \boxed{f(n) = O(g(n))}$$