**1** Use $\Theta$ notation to express the statement

$$4n^6 \leq 17n^6 - 45n^3 + 2n + 8 \leq 30n^6, n \geq 3$$

Let $A = 4$, $B = 30$ and $k = 3$ then the statement translates to

$$An^6 \leq 17n^6 - 45n^3 + 2n + 8 \leq Bn^6, n \geq k$$

hence by the definition of $\Theta$ notation $17n^6 - 45n^3 + 2n + 8$ is $\Theta(n^6)$.

**2** Use $\Omega$ notation to express the statement

1. Use $\Omega$ notation to expres the statement

$$\frac{11}{4}n^2 \leq 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2, n \geq 2$$

   Let $A = \frac{11}{4}$ and $k = 2$ then $An^2 \leq 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2, n \geq 2$ then the statement translates to

$$An^2 \leq 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2, n \geq k$$

   which by the definition of $\Omega$ notation, $3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2$ is $\Omega(n^2)$.

2. Use $O$ notation to express the statement

$$0 \leq 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2 \leq 6n^2, n \geq 1$$

   Let $A = 6$ and $k = 1$ then the statement translates to

$$0 \leq 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2 \leq An^2, n \geq k$$

   which by the defnition of $O$ notation, $3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2$ is $O(n^2)$.

3. Justify the statement: $3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2$ is $\Theta(n^2)$.
   Let $A = \frac{11}{4}, B = 6$ and $k = 2$ then $A \cdot n^2 \leq 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2 \leq Bn^2, n \geq k$ which by the definition of $\Theta$ notation, $3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2$ is $\Theta(n^2)$.

**3.** Given the function $15n^3 + 11n^2 + 9$

1. Show that the function is $\Omega(n^3)$.

$$15n^3 \leq 15n^3 + 11n^2 + 9, n \geq 1$$

Let $A = 15$ and $k = 1$ then the statements translates to $An^3 \leq 15n^3 + 11n^2 + 9, n \geq k$ which by the definition of $\Omega$ notation, $15n^3 + 11n^2 + 9$ is $\Omega(n^3)$.

2. Show that the function is $O(n^3)$.

$$
\begin{aligned}
15n^3 + 11n^2 + 9 &\leq 15n^3 + 11n^3 + 9n^3 \\
&\leq 35n^3, n \geq 1
\end{aligned}
$$

Let $A = 35$ and $k = 1$ then the statement translates to $15n^3 + 11n^2 + 9 \leq An^3, n \geq k$ which by the definition of $O$ notation, $15n^3 + 11n^2 + 9$ is $O(n^3)$.

**4.** Given the function $n^4 - 5n - 8$

1. Show that the function is $\Omega(n^4)$.
   Let $A = \frac{1}{2}$ and $a = (|-5|+|-8|)$

$$
\begin{aligned}
n &\geq \frac{2}{1} \cdot (|-5|+|-8|) \\
\frac{1}{2}n^4 &\geq 5n^3 + 8n^3 \\
\frac{1}{2}n^4 &\geq 5n + 8 \\
n^4 - 5n - 8 &\geq \frac{1}{2}n^4, n \geq a
\end{aligned}
$$

Hence by the definition of $\Omega$ notation, $n^4 - 5n - 8$ is $\Omega(n^4)$.

2. Show that the function is $O(n^4)$.

$$
\begin{aligned}
n^4 - 5n - 8 &\leq n^4 + 5n + 8 \\
&\leq n^4 + 5n^4 + 8n4 \\
&= 14n^4, n \geq 1
\end{aligned}
$$

Let $A = 14$ and $k = 1$ then the statement translates to $n^4 - 5n - 8 \leq An^4, n \geq k$ which by teh definition of $O$ notation translates, $n^4 - 5n - 8$ is $O(n^4)$.

2

**5.** Show that $15n^3 + 11n^2 + 9$ is $\Theta(n^3)$.

Since we have $\Omega(n^3)$ and $O(n^3)$ we have that there exists real positive number constants $A$ and $B$ such that $Ag(n) \leq f(n) \leq Bg(n), k \geq n$ where $k = max(k\prime, k\prime\prime)$ obtained from the previous inequalities. By definition of $\Theta$, $15n^3 + 11n^2 + 9$ is $\Theta(n^3)$.

**6.** Show that $n^4 - 5n - 8$ is $\Theta(n^4)$.

Since we have shown tha the function is $\Omega(n^4)$ and $O(n^4)$ we have that there exists real positive number constants $A$ and $B$ such that $Ag(n) \leq f(n) \leq Bg(n), k \geq n$ where $k = max(k\prime, k\prime\prime)$ obtained from the previous inequaliies. by definition of $\Theta$, $n^4 - 5n - 8$ is $\Theta(n^4)$.

**7.** Let $g(n) = n^4 - 5n - 8$, show that $g(n)$ is not $O(n^r)$ for any positive real number $r < 4$.

We prove this by contradiction. Suppose that $g(n)$ is $O(n^r)$ for any positive real number $r < 4$. then

$$g(n) \quad \leq \quad An^r, n \geq a$$

where $A$ and $a$ are real positive numbers.

$$
\begin{aligned}
g(n) &\leq n^4 \\
&\leq An^r \\
n^{4-r} &\leq A \\
n &\leq \sqrt[4-r]{A}
\end{aligned}
$$

which is a contradiction. We conclude that $g(n)$ is not $O(n^r)$ for any positive real number $r < 4$.

**8.** Use theorem on polynomial orders to find orders for the function given by the following formulas.

1. $f(n) = 7n^5 + 5n^3 - n + 4$ for each positive integer $n$.

   By direct application of theorem on polynomial orders, $7n^5 + 5n^3 - n + 4$ is $\Theta(n^5)$.

2. $g(n) = \frac{(n-1)(n+1)}{4}$ for each positive integer $n$.

$$\frac{(n-1) \cdot (n+1)}{4} = \frac{n^2 + n - n + 1}{4}$$
$$= \frac{n^2 + 1}{4}$$
$$= \frac{n^2}{4} + \frac{1}{4}$$

Thus $g(n)$ is $\Theta(n^2)$.

**9.** Show that for a positive integer variable $n$,

$$1 + 2 + 3 \ldots + n \quad is \quad \Theta(n^2)$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$
$$= \frac{n^2}{2} + \frac{n}{2}$$

**10.** Express $5x^8 - 9x^7 + 2x^5 + 3x - 1 \le 6x^8, x > 3$ using $O$ notation

Let $A = 6$ and $a = 3$ then $5x^8 - 9x^7 + 2x^5 + 3x - 1 \le Ax^8, x > a$ and by definition of $O$ notation, $5x^8 - 9x^7 + 2x^5 + 3x - 1$ is $O(x^8)$.

**11.** Express $x^{\frac{7}{2}} \le \frac{(x^2-7)^2(10x^{\frac{1}{2}}+3)}{x+1}, x > 4$ using $\Omega$ notation

Let $A = 1$ and $k = 4$ then the statement translates to

$$Ax^{\frac{7}{2}} \le \frac{(x^2 - 7)^2(10x^{\frac{1}{2}} + 3)}{x + 1}, x > k$$

which by the definition of $\Omega$ notation, $\frac{(x^2-7)^2(10x^{\frac{1}{2}}+3)}{x+1}$ is $\Omega(x^{\frac{7}{2}})$.

**12.** Express $3x^6 + 5x^4 - x^3 \le 9x^6, x > 1$ using $O$ notation. Let $A = 9$ and $k = 1$ then the statement translate to

$$3x^6 + 5x^4 - x^3 \le Ax^6, x > k$$

which by the definition of $\Omega$ notation, $3x^6 + 5x^4 - x^3$ is $O(x^6)$.

**13.** Express $\frac{1}{2}x^4 \leq x^4 - 50x^3 + 1$ for all real numbers $x > 101$ using $\Omega$ notation.

Let $A = \frac{1}{2}$ and $k = 101$ this statement translates to $Ax^4 \leq x^4 - 50x^3 + 1, n > k$ which by the definition of $\Omega$ notation, $x^4 - 50x^3 + 1$ is $\Omega(x^4)$.

**14.** Express $\frac{1}{2}x^2 \leq 3x^2 - 80x + 7 \leq 3x^2, x > 25$

Let $A = \frac{1}{2}, B = 3$ and $k = 25$ then the statement translates to

$$Ax^2 \leq \quad 3x^2 - 80x + 7 \quad \leq Bx^2, x > k$$

which by the definition of $\Theta$ notation $3x^2 - 80x + 7$ is $\Theta(x^2)$.

**15.** Suppose $g(x)$ is $O(f(x))$ show $f(x)$ is then $\Omega(g(x))$

Since $g(x)$ is $O(f(x))$ then there exists a real positive numbers $B$ and $k$ such that $g(x) \leq B \cdot f(x), n > k$. We obtain $\frac{g(x)}{B} \leq f(x), n > k$ which by the definition of $\Omega$ notation $f(x)$ is $\Omega(g(x))$.

**16.** Prove that if $f(x)$ is $O(g(x))$ and $c$ is any nonzero real number, then $c \cdot f(x)$ is $O(g(x))$.

Since $f(x)$ is $O(g(x))$ then there exists real positive numbers $B$ and $k$ such that $g(x) \leq B \cdot f(x), n > k$. Multiplying by the constant $c$ we obtain $c \cdot g(x) \leq c \cdot B \cdot f(x), n > k$ which by the definition of $O$ notation $c \cdot g(x)$ is $O(g(x))$.

**17.** Prove that if $f(x)$ is $O(h(x))$ and $g(x)$ is $O(l(x))$, then $f(x) + g(x)$ is $O(G(x))$ where $G(x) = max(h(x), l(x))$.

$$\begin{aligned} f(x) + g(x) &\leq 2 \cdot max(h(x), l(x)) \\ &= 2 \cdot G(x), n \geq k \end{aligned}$$

where $k = max(k\prime, k\prime\prime)$ where $k\prime$ and $k\prime\prime$ are terms that satisfy the previous $O$ notations.

**18.** Prove that $f(x)$ is $\Theta(f(x))$

Let $A = 1, B = 1$ and $k = 1$ the $Af(x) \leq f(x) \leq Bf(x), n \geq k$ which by the definition of $\Theta$ notation $f(x)$ is $\Theta(f(x))$.

**19.** Prove that if $f(x)$ is $O(h(x))$ and $g(x)$ is $O(k(x))$ then $f(x)g(x)$ is $O(h(x)k(x))$.

we have that $f(x)$ is $O(h(x))$ and $g(x)$ is $O(k(x))$ hence we have the we have constants $B, B', b$ and $b'$ such that

$$
\begin{aligned}
f(x) &\leq B \cdot h(x), x > b \\
g(x) &\leq B' \cdot k(x), x > b'
\end{aligned}
$$

Let $b_1 = max(b, b')$ then

$$
\begin{aligned}
f(x) \cdot g(x) &\leq B \cdot h(x) \cdot g(x) \\
&\leq B \cdot h(x) \cdot g(x) \\
&\leq B \cdot B' \cdot h(x) \cdot k(x), x > b_1
\end{aligned}
$$

then by the definition of $O$ notation, $f(x) \cdot g(x)$ is $O(h(x) \cdot g(x))$.

**20.** Prove that if $x$ is a real number with $x > 1$, then $x^n > 1$ for all integers $n \geq 1$.

We prove this by mathematical induction, Let $P(n)$ be the statement $x^n > 1$.
   $P(1)$ is trivially true since $x^1 > 1$. Let $P(k)$ be true, thus $x^k > 1$ we now show that $P(k+1)$ is true.

$$
\begin{aligned}
x^{k+1} &= x^k x^1 \\
&> x^k \\
&> 1
\end{aligned}
$$

Thus $x^{k+1} > 1$, which implies that $P(k+1)$ is true. By the principle of mathematical induction, $P(n)$ holds for all positive integers $n$.

**21.** Prove that if $x > 1$ then $x^m < x^n$ for any integers $m$ and $n$ with $m < n$.

$$
\begin{aligned}
x^m &= \underbrace{x \cdot x \cdot \ldots \cdot x}_{m\,times} \\
&< \underbrace{x \cdot x \cdot \ldots \cdot x}_{m\,times} \cdot \underbrace{1 \cdot 1 \ldots \cdot 1}_{n-m\,times} \\
&< \underbrace{x \cdot x \cdot \ldots \cdot x}_{m\,times} \cdot \underbrace{x \cdot x \cdot \ldots \cdot x}_{n-m\,times} \\
&= x^n
\end{aligned}
$$

**22.** Given the function $g(x) = 2x^2 + 15x + 4$

   1. Show that $x^2 \leq 2x^2 + 15x + 4$

6

$$\begin{aligned} x^2 \quad &< \quad x^2 + 15 + 4 \\ &< \quad x^2 + 15x + 4 \\ &< \quad 2x^2 + 15x + 4 \\ &\leq \quad 2x^2 + 15x + 4 \end{aligned}$$

2. Show that $2x^2 + 15x + 4 \leq 21x^2$

$$\begin{aligned} 2x^2 + 15x + 4 \quad &\leq \quad x^2 + 15x^2 + 14x^2 \\ &\leq \quad 21x^2 \end{aligned}$$

3. Rewrite statement (1) using $\Omega$ notation

Let $A = 1$ and $a = 1$ then $A \cdot f(x) \leq g(x)$ and by the definition of $\Omega$ notation $g(x)$ is $\Omega(n^2)$.

4. Rewrite statement (2) using $O$ notation

Let $A = 21$ and $a = 1$ then $g(x) \leq A \cdot x^2$ and by the definition of $O$ notation $g(x)$ is $O(n^2)$.

**23.** Given the function $g(x) = x^4 \leq 23x^4 + 8x^2 + 4x$.

1. Show that if $x > 1$ then $x^4 \leq 23x^4 + 8x^2 + 4x$

Let $x$ be a real number such that $x > 1$.

$$\begin{aligned} x^4 \quad &\leq \quad 23 \cdot x^4 \\ &\leq \quad 23 \cdot x^4 + 8x^2 \\ &\leq \quad 23 \cdot x^4 + 8x^2 + 4x \end{aligned}$$

2. Show that if $x > 1$ then $23x^4 + 8x^2 + 4x \leq 23x^4$

$$\begin{aligned} 23x^4 + 8x^2 + 4x \quad &\leq \quad 23x^4 + 8x^4 + 4x^4 \\ &\leq \quad 35x^4 \end{aligned}$$

3. What can you conclude by part (1) and part (2)

the function $g(x)$ is $\Omega(x^4)$ and $O(x^4)$.

**24.** Prove $5x^3 + 65x + 30$ is $\Theta(x^3)$ by proving that $5x^3 + 65x + 30$ is $\Omega(x^3)$ and $5x^3 + 65x + 30$ is $O(x^3)$

$$
\begin{aligned}
5x^3 &\leq 5x^3 + 65x^3 + 30x^3 \\
&\leq 100x^3, x > 1
\end{aligned}
$$

$$
x^3 \leq 5x^3
$$

From these two inequalities we obtain that $x^3 \leq 5x^3 \leq 5x^3 + 65x^3 + 30x^3, x \geq 1$ hence by the definition of $\Theta$ notation $5x^3$ is $\Theta(x^3)$.

**25.** Prove that $x^2 + 100x + 88$ is $\Theta(x^2)$ by proving that $x^2 + 100x + 88$ is $\Omega(x^2)$ and $O(x^2)$.

$$
\begin{aligned}
x^2 + 100x + 88 &< x^2 + 100x \\
&< x^2 + 100x + 8, x > 1
\end{aligned}
$$

$$
\begin{aligned}
x^2 + 100x + 88 &< x^2 + 100x^2 + 88x^2 \\
&< 189x^2
\end{aligned}
$$

From these two inequalties we obtain that $x^2 \leq x^2 + 100x + 88 < 189x^2$, which by the definition of $\Theta$ notation is $\Theta(x^2)$.

**26.** Show that for any real number $x$, If $x > 1$ then $7x^4 - 95x^3 + 3 \leq 105x^4$ and use $O$ notation to express the result.

$$
\begin{aligned}
7x^4 - 95x^3 + 3 &\leq 7x^4 + 95x^4 + 3x^4 \\
&\leq 105x^4
\end{aligned}
$$

Which by the definition of $O$ notation, $7x^4 - 95x^3 + 3$ is $O(x^4)$.

**27.** Show that for any real number $x$, if $x > 1$ then $\frac{1}{5}x^2 - 42x - 8 \leq 51x^2$ and use $O$ notation to express the result.

$$
\begin{aligned}
\frac{1}{5}x^2 - 42x - 8 &\leq \frac{1}{5}x^2 + 42x^2 + 8x^2 \\
&\leq x^2 + 42x^2 + 8x^2 \\
&\leq 51x^2
\end{aligned}
$$

Which by the definition of $O$ notation, $\frac{1}{5}x^2 - 42x - 8$ is $O(x^2)$.

**28.** Show that for any real number $x$, if $x > 1$ then $\frac{1}{4}x^5 - 50x^3 + 3x + 12 \leq 66x^5$.

$$
\begin{aligned}
\frac{1}{4}x^5 - 50x^3 + 3x + 12 &\leq x^5 + 50x^5 + 3x^5 + 12x^5 \\
&\leq 66x^5
\end{aligned}
$$

which by the definition of $O$ notation, $\frac{1}{4}x^5 - 50x^3 + 3x + 12$ is $x^5$.

**29.** Show that $x^5$ is not $O(x^2)$.

Suppose that $x^5$ is in fact $O(x^2)$ then by definition there exists postive real numbers $B$ and $k$ such that

$$
\begin{aligned}
x^5 &\leq Bx^2, x > k \\
x^3 &\leq B, x > k
\end{aligned}
$$

which is absurd hence $x^5$ is not $O(x^2)$.

**30.** Suppose $a_0, a_1, a_2, \ldots, a_n$ are real numbers and $a_n \neq 0$. Use the generalization of the triangle inequality to $n$ integers to show taht $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$ is $O(x^n)$.

$$
\begin{aligned}
|a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0| &\leq |a_n x^n| + |a_{n-1} x^{n-1}| + \ldots + |a_0| \\
&\leq |a_n| x^n + |a_{n-1}| x^{n-1} + \ldots + |a_0| \\
&\leq |a_n| x^n + |a_{n-1}| x^n + \ldots + |a_0| x^n \\
&\leq x^n \sum_{i=0}^{n} |a_i|
\end{aligned}
$$

Which by the definition of $O$ notation $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$ is $O(x^n)$.

**31.** Use the theorem on polynomial orders to prove $\frac{(x+1)(x-2)}{4}$ is $\Theta(x^2)$

$$
\begin{aligned}
\frac{(x+1)(x-2)}{4} &= \frac{x^2 - 2x - x - 2}{4} \\
&= \frac{x^2}{4} - \frac{2x}{4} - \frac{x}{4} - \frac{2}{4}
\end{aligned}
$$

which by the theorem on polynomial orders $\frac{(x+1)(x-2)}{4}$ is $\Theta(x^2)$.

**32.** Use the theorem on polynomial orders to prove that $\frac{x}{3}(4x^2 - 1)$ is $\Theta(x^3)$.

$$\frac{x}{3}(4x^2 - 1) = \frac{4x^3}{3} - \frac{x}{3}$$

which by the theorem on polynomial orders is $\Theta(x^3)$.

**33.** Use the theorem on polynomial orders to prove that $\frac{x(x-1)}{2} + 3x$ is $\Theta(x^2)$.

$$\frac{x(x-1)}{2} + 3x = \frac{x^2 - x}{2} + 3x$$

which by the order of polynomial orders is $\Theta(x^3)$.

**34.** Use the theorem on polynomial orders to prove that $\frac{n(n+1)(2n+1)}{6}$ is $\Theta(n^3)$.

$$
\begin{aligned}
\frac{n(n+1)(2n+1)}{6} &= \frac{(n^2 + n)(2n+1)}{6} \\
&= \frac{2n^3 + n^2 + 2n^2 + n}{6} \\
&= \frac{2n^3}{6} + \frac{3n^2}{6} + \frac{n}{6}
\end{aligned}
$$

which by the order of polynomial orders is $\Theta(n^3)$.

**35.** Use the theorem on polynomial orders to prove that $[\frac{n(n+1)}{2}]^2$ is $\Theta(n^4)$.

$$
\begin{aligned}
[\frac{n(n+1)}{2}]^2 &= \frac{n^2(n+1)^2}{4} \\
&= \frac{n^2(n^2 + 2n + 1)}{4} \\
&= \frac{n^4 + 2n^3 + n^2}{4} \\
&= \frac{n^4}{4} + \frac{2n^3}{4} + \frac{n^2}{4}
\end{aligned}
$$

which by the theorem on polynomial orders is $\Theta(n^4)$.

**36.** Use the theorem on polynomial orders to prove that $2 \cdot (n - 1) + \frac{n(n+1)}{2} + 4 \cdot \frac{n(n-1)}{2}$ is $\Theta(n^2)$

$$2 \cdot (n - 1) + \frac{n^2}{2} + \frac{n}{2} + \frac{4n^2}{2} - \frac{4n}{2}$$

which by the theorem on polynomial orders is $\Theta(n^2)$.

**37.** Prove that $1^2 + 2^2 + 3^2 + \ldots + n^2$ is $\Theta(n^3)$.

$$\sum_{i=1}^{n} i^2 \quad = \quad \frac{(n)(n+1)(2n+1)}{6}$$

which by (34) we know is $\Theta(n^3)$.

**38.** Prove that $1^3 + 2^3 + 3^3 + \ldots + n^3$ is $\Theta(n^4)$

$$\sum_{i=1}^{n} i^3 \quad = \quad (\frac{(n)(n+1)}{2})^2$$

which by (35) we know is $\Theta(n^4)$.

**39.** Prove that $2 + 4 + 6 + \ldots + 2 \cdot n$ is $\Theta(n^2)$

$$\sum_{i=1}^{n} 2 \cdot i \quad = \quad n \cdot (n+1)$$
$$= \quad n^2 + n$$

which is by the theorem on order of polynomials is $\Theta(n^2)$.

**40.** Prove that $5 + 10 + 15 + 20 + 25 + \ldots + 5n$ is $\Theta(n^2)$

$$\sum_{i=1}^{n} 5 \cdot i \quad = \quad 5 \cdot \frac{n(n+1)}{2}$$
$$= \quad \frac{5}{2}n^2 + 5\frac{n}{2}$$

which by the theorem on order of polynomials is $\Theta(n^2)$.

**41.** Prove that $\sum_{i=1}^{n}(4i-9)$ is $\Theta(n^2)$

$$
\begin{aligned}
\sum_{i=1}^{n}(4i-9) &= \sum_{i=1}^{n}4i - \sum_{i=1}^{n}9 \\
&= \frac{4}{2}(n)(n+1) - 9n \\
&= 2n^2 + 2n - 9n
\end{aligned}
$$

whicy by the theorem on orders of polynomials is $\Theta(n^2)$.

**42.** Prove that $\sum_{k=1}^{n}(k+3)$ is $\Theta(n^2)$

$$
\begin{aligned}
\sum_{k=1}^{n}k + \sum_{k=1}^{n}3 &= \frac{n \cdot (n+1)}{2} + 3 \cdot n \\
&= \frac{n^2}{2} + \frac{n}{2} + 3 \cdot n
\end{aligned}
$$

which by theorem on orders of polynomials is $\Theta(n^2)$.

**43.** Prove that $\sum_{i=1}^{n}i(i+1)$ is $\Theta(n^3)$.

$$
\begin{aligned}
\sum_{i=1}^{n}i(i+1) &= \sum_{i=1}^{n}i^2 + \sum_{i=1}^{n}i \\
&= \frac{(n)(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}
\end{aligned}
$$

which by the theorem on orders of polynomials is $\Theta(n^3)$.

**44.** Prove that $\sum_{k=3}^{n}(k^2-2k)$ is $\Theta(n^3)$

$$
\begin{aligned}
\sum_{k=3}^{n}k^2 - 2\sum_{k=3}^{n}k &= \sum_{k=1}^{n}k^2 - \sum_{k=1}^{2}k^2 - 2\sum_{k=1}^{n}k + 2\sum_{k=1}^{2}k \\
&= \frac{n(n+1)(2n+1)}{6} - \frac{2(2+1)(2(2)+1)}{6} - \frac{n(n+1)}{2} + 2\frac{3(4)}{2}
\end{aligned}
$$

which by the theorem on orders of polynomials is $\Theta(n^3)$.

**45.** proving the theorem on order of polynomials

1. Let $a_0, a_1, \ldots, a_n$ be real numbers with $a_n \neq 0$. prove that

$$lim_{x \to \infty} \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0}{a_n x^n} = 1$$

$$
\begin{aligned}
lim_{x \to \infty} \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0}{a_n x^n} &= lim_{x \to \infty} \frac{a_n}{a_n} + lim_{x \to \infty} \frac{a_{n-1}}{a_n x} \\
&+ lim_{x \to \infty} \frac{a_{n-2}}{a_n x^2} + \ldots + lim_{x \to \infty} \frac{a_0}{a_n x^n} \\
&= 1
\end{aligned}
$$

2. Use result of part (1) and the definition of limit to prove that $\sum_{i=0}^{n} a_i x^i$ is $\Theta(x^n)$

By the definition of the limit there must be some real number $\epsilon > 0$ such that

$$1 - \epsilon < lim_{x \to \infty} \frac{\sum_{i=0}^{n} a_i x^i}{a_n x^n} < 1 + \epsilon$$

when $\epsilon = \frac{1}{2}$

$$\frac{1}{2} < lim_{x \to \infty} \frac{\sum_{i=0}^{n} a_i x^i}{a_n x^n} < \frac{3}{2}$$

$$\frac{a_n x^n}{2} < lim_{x \to \infty} \sum_{i=0}^{n} a_i x^i < \frac{3}{2} a_x x^n$$

which by the definition of $\Theta$ notation $\sum_{i=0}^{n} a_i x^i$ is $\Theta(x^n)$.