1 Use Θ notation to express the statement

$$4n^6 \quad \leq \quad 17n^6 - 45n^3 + 2n + 8 \leq 30n^6, n \geq 3$$

Let $A = 4$, $B = 30$ and $k = 3$ then the statement translates to

$$An^6 \quad \leq 17n^6 - 45n^3 + 2n + 8 \quad \leq Bn^6, n \geq k$$

hence by the definition of Θ notation $17n^6 - 45n^3 + 2n + 8$ is $\Theta(n^6)$.

2 Use Ω notation to express the statement

1. Use Ω notation to expres the statement

   $$\frac{11}{4}n^2 \quad \leq \quad 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2, n \geq 2$$

   Let $A = \frac{11}{4}$ and $k = 2$ then $An^2 \leq 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2, n \geq 2$ then the statement translates to

   $$An^2 \quad \leq \quad 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2, n \geq k$$

   which by the definition of Ω notation, $3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2$ is $\Omega(n^2)$.

2. Use O notation to express the statement

   $$0 \quad \leq \quad 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2 \leq 6n^2, n \geq 1$$

   Let $A = 6$ and $k = 1$ then the statement translates to

   $$0 \quad \leq 3 \cdot (\lfloor \tfrac{n}{4} \rfloor)^2 + 5n^2 \quad \leq An^2, n \geq k$$

   which by the defnition of O notation, $3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2$ is $O(n^2)$.

3. Justify the statement: $3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2$ is $\Theta(n^2)$.
   Let $A = \frac{11}{4}, B = 6$ and $k = 2$ then $A \cdot n^2 \leq 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2 \leq Bn^2, n \geq k$ which by the definition of Θ notation, $3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2$ is $\Theta(n^2)$.

3. Given the function $15n^3 + 11n^2 + 9$

1. Show that the function is $\Omega(n^3)$.

$$15n^3 \leq 15n^3 + 11n^2 + 9, n \geq 1$$

   Let $A = 15$ and $k = 1$ then the statements translates to $An^3 \leq 15n^3 + 11n^2 + 9, n \geq k$ which by the definition of Ω notation, $15n^3 + 11n^2 + 9$ is $\Omega(n^3)$.

2. Show that the function is $O(n^3)$.

$$\begin{aligned} 15n^3 + 11n^2 + 9 &\leq 15n^3 + 11n^3 + 9n^3 \\ &\leq 35n^3, n \geq 1 \end{aligned}$$

   Let $A = 35$ and $k = 1$ then the statement translates to $15n^3 + 11n^2 + 9 \leq An^3, n \geq k$ which by the definition of O notation, $15n^3+11n^2+9$ is $O(n^3)$.

4. Given the function $n^4 - 5n - 8$

1. Show that the function is $\Omega(n^4)$.
   Let $A = \frac{1}{2}$ and $a = (|-5|+|-8|)$

$$\begin{aligned} n &\geq \frac{2}{1} \cdot (|-5|+|-8|) \\ \frac{1}{2}n^4 &\geq 5n^3 + 8n^3 \\ \frac{1}{2}n^4 &\geq 5n + 8 \\ n^4 - 5n - 8 &\geq \frac{1}{2}n^4, n \geq a \end{aligned}$$

   Hence by the definition of Ω notation, $n^4 - 5n - 8$ is $\Omega(n^4)$.

2. Show that the function is $O(n^4)$.

$$\begin{aligned} n^4 - 5n - 8 &\leq n^4 + 5n + 8 \\ &\leq n^4 + 5n^4 + 8n4 \\ &= 14n^4, n \geq 1 \end{aligned}$$

   Let $A = 14$ and $k = 1$ then the statement translates to $n^4 - 5n - 8 \leq An^4, n \geq k$ which by teh definition of O notation translates, $n^4 - 5n - 8$ is $O(n^4)$.

5. Show that $15n^3 + 11n^2 + 9$ is $\Theta(n^3)$.

Since we have $\Omega(n^3)$ and $O(n^3)$ we have that there exists real positive number constants A and B such that $Ag(n) \leq f(n) \leq Bg(n), k \geq n$ where $k = max(k\prime, k\prime\prime)$ obtained from the previous inequalities. By definition of Θ, $15n^3 + 11n^2 + 9$ is $\Theta(n^3)$.

6. Show that $n^4 - 5n - 8$ is $\Theta(n^4)$.

Since we have shown tha the function is $\Omega(n^4)$ and $O(n^4)$ we have that there exists real positive number constants A and B such that $Ag(n) \leq f(n) \leq Bg(n), k \geq n$ where $k = max(k\prime, k\prime\prime)$ obtained from the previous inequaliies. by definition of Θ, $n^4 - 5n - 8$ is $\Theta(n^4)$.

7. Let $g(n) = n^4 - 5n - 8$, show that $g(n)$ is not $O(n^r)$ for any positive real number $r < 4$.

We prove this by contradiction. Suppose that $g(n)$ is $O(n^r)$ for any positive real number $r < 4$. then

$$g(n) \leq An^r, n \geq a$$

where A and a are real positive numbers.

$$\begin{aligned} g(n) &\leq n^4 \\ &\leq An^r \\ n^{4-r} &\leq A \\ n &\leq \sqrt[4-r]{A} \end{aligned}$$

which is a contradiction. We conclude that $g(n)$ is not $O(n^r)$ for any positive real number $r < 4$.

8. Use theorem on polynomial orders to find orders for the function given by the following formulas.

1. $f(n) = 7n^5 + 5n^3 - n + 4$ for each positive integer n.

   By direct application of theorem on polynomial orders, $7n^5 + 5n^3 - n + 4$ is $\Theta(n^5)$.

2. $g(n) = \frac{(n-1)(n+1)}{4}$ for each positive integer n.

$$\begin{aligned}\frac{(n-1)\cdot(n+1)}{4} &= \frac{n^2+n-n+1}{4}\\ &= \frac{n^2+1}{4}\\ &= \frac{n^2}{4}+\frac{1}{4}\end{aligned}$$

Thus $g(n)$ is $\Theta(n^2)$.

9. Show that for a positive integer variable n,

$$1+2+3\ldots+n \quad is \quad \Theta(n^2)$$

$$\begin{aligned}\sum_{i=1}^{n} i &= \frac{n(n+1)}{2}\\ &= \frac{n^2}{2}+\frac{n}{2}\end{aligned}$$

10. Express $5x^8-9x^7+2x^5+3x-1 \leq 6x^8, x > 3$ using O notation

Let $A=6$ and $a=3$ then $5x^8-9x^7+2x^5+3x-1 \leq Ax^8, x > a$ and by definition of O notation, $5x^8-9x^7+2x^5+3x-1$ is $O(x^8)$.

11. Express $x^{\frac{7}{2}} \leq \frac{(x^2-7)^2(10x^{\frac{1}{2}}+3)}{x+1}, x > 4$ using Ω notation

Let $A=1$ and $k=4$ then the statement translates to

$$Ax^{\frac{7}{2}} \leq \frac{(x^2-7)^2(10x^{\frac{1}{2}}+3)}{x+1}, x > k$$

which by the definition of Ω notation, $\frac{(x^2-7)^2(10x^{\frac{1}{2}}+3)}{x+1}$ is $\Omega(x^{\frac{7}{2}})$.

12. Express $3x^6+5x^4-x^3 \leq 9x^6, x > 1$ using O notation. Let $A=9$ and $k=1$ then the statement translate to

$$3x^6+5x^4-x^3 \leq Ax^6, x > k$$

which by the definition of Ω notation, $3x^6 + 5x^4 - x^3$ is $O(x^6)$.

13. Express $\frac{1}{2}x^4 \leq x^4 - 50x^3 + 1$ for all real numbers $x > 101$ using Ω notation.

Let $A = \frac{1}{2}$ and $k = 101$ this statement translates to $Ax^4 \leq x^4 - 50x^3 + 1, n > k$ which by the definition of Ω notation, $x^4 - 50x^3 + 1$ is $\Omega(x^4)$.

14. Express $\frac{1}{2}x^2 \leq 3x^2 - 80x + 7 \leq 3x^2, x > 25$

Let $A = \frac{1}{2}, B = 3$ and $k = 25$ then the statement translates to

$$Ax^2 \leq \quad 3x^2 - 80x + 7 \quad \leq Bx^2, x > k$$

which by the definition of Θ notation $3x^2 - 80x + 7$ is $\Theta(x^2)$.

15. Suppose $g(x)$ is $O(f(x))$ show $f(x)$ is then $\Omega(g(x))$

Since $g(x)$ is $O(f(x))$ then there exists a real positive numbers B and k such that $g(x) \leq B \cdot f(x), n > k$. We obtain $\frac{g(x)}{B} \leq f(x), n > k$ which by the definition of Ω notation $f(x)$ is $\Omega(g(x))$.

16. Prove that if $f(x)$ is $O(g(x))$ and c is any nonzero real number, then $c \cdot f(x)$ is $O(g(x))$.

Since $f(x)$ is $O(g(x))$ then there exists real positive numbers B and k such that $g(x) \leq B \cdot f(x), n > k$. Multiplying by the constant c we obtain $c \cdot g(x) \leq c \cdot B \cdot f(x), n > k$ which by the definition of O notation $c \cdot g(x)$ is $O(g(x))$.

17. Prove that if $f(x)$ is $O(h(x))$ and $g(x)$ is $O(l(x))$, then $f(x) + g(x)$ is $O(G(x))$ where $G(x) = max(h(x), l(x))$.

$$\begin{aligned} f(x) + g(x) &\leq 2 \cdot max(h(x), l(x)) \\ &= 2 \cdot G(x), n \geq k \end{aligned}$$

where $k = max(k\prime, k\prime\prime)$ where $k\prime$ and $k\prime\prime$ are terms that satisfy the previous O notations.

18. Prove that $f(x)$ is $\Theta(f(x))$

Let $A = 1, B = 1$ and $k = 1$ the $Af(x) \leq f(x) \leq Bf(x), n \geq k$ which by the definition of Θ notation $f(x)$ is $\Theta(f(x))$.

19. Prove that if $f(x)$ is $O(h(x))$ and $g(x)$ is $O(k(x))$ then $f(x)g(x)$ is $O(h(x)k(x))$.

we have that $f(x)$ is $O(h(x))$ and $g(x)$ is $O(k(x))$ hence we have the we have constants B, B', b and b' such that

$$\begin{aligned} f(x) &\leq B \cdot h(x), x > b \\ g(x) &\leq B' \cdot k(x), x > b' \end{aligned}$$

Let $b_1 = max(b, b')$ then

$$\begin{aligned} f(x) \cdot g(x) &\leq B \cdot h(x) \cdot g(x) \\ &\leq B \cdot h(x) \cdot g(x) \\ &\leq B \cdot B' \cdot h(x) \cdot k(x), x > b_1 \end{aligned}$$

then by the definition of O notation, $f(x) \cdot g(x)$ is $O(h(x) \cdot g(x))$.

20. Prove that if x is a real number with $x > 1$, then $x^n > 1$ for all integers $n \geq 1$.

We prove this by mathematical induction, Let $P(n)$ be the statement $x^n > 1$.

$P(1)$ is trivially true since $x^1 > 1$. Let $P(k)$ be true, thus $x^k > 1$ we now show that $P(k+1)$ is true.

$$\begin{aligned} x^{k+1} &= x^k x^1 \\ &> x^k \\ &> 1 \end{aligned}$$

Thus $x^{k+1} > 1$, which implies that $P(k+1)$ is true. By the principle of mathematical induction, $P(n)$ holds for all positive integers n.

21. Prove that if $x > 1$ then $x^m < x^n$ for any integers m and n with $m < n$.

$$\begin{aligned} x^m &= \underbrace{x \cdot x \cdot \ldots \cdot x}_{m\,times} \\ &< \underbrace{x \cdot x \cdot \ldots \cdot x}_{m\,times} \cdot \underbrace{1 \cdot 1 \ldots \cdot 1}_{n-m\,times} \\ &< \underbrace{x \cdot x \cdot \ldots \cdot x}_{m\,times} \cdot \underbrace{x \cdot x \cdot \ldots \cdot x}_{n-m\,times} \\ &= x^n \end{aligned}$$

22. Given the function $g(x) = 2x^2 + 15x + 4$

1. Show that $x^2 \leq 2x^2 + 15x + 4$

$$
\begin{aligned}
x^2 &< x^2 + 15 + 4 \\
&< x^2 + 15x + 4 \\
&< 2x^2 + 15x + 4 \\
&\leq 2x^2 + 15x + 4
\end{aligned}
$$

2. Show that $2x^2 + 15x + 4 \leq 21x^2$

$$
\begin{aligned}
2x^2 + 15x + 4 &\leq x^2 + 15x^2 + 14x^2 \\
&\leq 21x^2
\end{aligned}
$$

3. Rewrite statement (1) using Ω notation

   Let $A = 1$ and $a = 1$ then $A \cdot f(x) \leq g(x)$ and by the definition of Ω notation $g(x)$ is $\Omega(n^2)$.

4. Rewrite statement (2) using O notation

   Let $A = 21$ and $a = 1$ then $g(x) \leq A \cdot x^2$ and by the definition of O notation $g(x)$ is $O(n^2)$.

23. Given the function $g(x) = x^4 \leq 23x^4 + 8x^2 + 4x$.

1. Show that if $x > 1$ then $x^4 \leq 23x^4 + 8x^2 + 4x$

   Let x be a real number such that $x > 1$.

$$
\begin{aligned}
x^4 &\leq 23 \cdot x^4 \\
&\leq 23 \cdot x^4 + 8x^2 \\
&\leq 23 \cdot x^4 + 8x^2 + 4x
\end{aligned}
$$

2. Show that if $x > 1$ then $23x^4 + 8x^2 + 4x \leq 23x^4$

$$
\begin{aligned}
23x^4 + 8x^2 + 4x &\leq 23x^4 + 8x^4 + 4x^4 \\
&\leq 35x^4
\end{aligned}
$$

3. What can you conclude by part (1) and part (2)

   the function $g(x)$ is $\Omega(x^4)$ and $O(x^4)$.

24. Prove $5x^3 + 65x + 30$ is $\Theta(x^3)$ by proving that $5x^3 + 65x + 30$ is $\Omega(x^3)$ and $5x^3 + 65x + 30$ is $O(x^3)$

$$\begin{aligned} 5x^3 &\leq 5x^3 + 65x^3 + 30x^3 \\ &\leq 100x^3, x > 1 \end{aligned}$$

$$x^3 \leq 5x^3$$

From these two inequalities we obtain that $x^3 \leq 5x^3 \leq 5x^3 + 65x^3 + 30x^3, x \geq 1$ hence by the definition of Θ notation $5x^3$ is $\Theta(x^3)$.

25. Prove that $x^2+100x+88$ is $\Theta(x^2)$ by proving that $x^2+100x+88$ is $\Omega(x^2)$ and $O(x^2)$.

$$\begin{aligned} x^2 + 100x + 88 &< x^2 + 100x \\ &< x^2 + 100x + 8, x > 1 \end{aligned}$$

$$\begin{aligned} x^2 + 100x + 88 &< x^2 + 100x^2 + 88x^2 \\ &< 189x^2 \end{aligned}$$

From these two inequalties we obtain that $x^2 \leq x^2 + 100x + 88 < 189x^2$, which by the definition of Θ notation is $\Theta(x^2)$.

26. Show that for any real number x, If $x > 1$ then $7x^4 - 95x^3 + 3 \leq 105x^4$ and use O notation to express the result.

$$\begin{aligned} 7x^4 - 95x^3 + 3 &\leq 7x^4 + 95x^4 + 3x^4 \\ &\leq 105x^4 \end{aligned}$$

Which by the definition of O notation, $7x^4 - 95x^3 + 3$ is $O(x^4)$.

27. Show that for any real number x, if $x > 1$ then $\frac{1}{5}x^2 - 42x - 8 \leq 51x^2$ and use O notation to express the result.

$$\begin{aligned} \frac{1}{5}x^2 - 42x - 8 &\leq \frac{1}{5}x^2 + 42x^2 + 8x^2 \\ &\leq x^2 + 42x^2 + 8x^2 \\ &\leq 51x^2 \end{aligned}$$

Which by the definition of O notation, $\frac{1}{5}x^2 - 42x - 8$ is $O(x^2)$.

28. Show that for any real number x, if $x > 1$ then $\frac{1}{4}x^5 - 50x^3 + 3x + 12 \leq 66x^5$.

$$\begin{aligned} \frac{1}{4}x^5 - 50x^3 + 3x + 12 &\leq x^5 + 50x^5 + 3x^5 + 12x^5 \\ &\leq 66x^5 \end{aligned}$$

which by the definition of O notation, $\frac{1}{4}x^5 - 50x^3 + 3x + 12$ is x^5.

29. Show that x^5 is not $O(x^2)$.

Suppose that x^5 is in fact $O(x^2)$ then by definition there exists postive real numbers B and k such that

$$\begin{aligned} x^5 &\leq Bx^2, x > k \\ x^3 &\leq B, x > k \end{aligned}$$

which is absurd hence x^5 is not $O(x^2)$.

30. Suppose $a_0, a_1, a_2, \ldots, a_n$are real numbers and $a_n \neq 0$. Use the generalization of the triangle inequality to n integers to show taht $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ is $O(x^n)$.

$$\begin{aligned} |a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0| &\leq |a_n x^n|+|a_{n-1}x^{n-1}|+ \ldots + |a_0| \\ &\leq |a_n|x^n + |a_{n-1}|x^{n-1} + \ldots + |a_0| \\ &\leq |a_n|x^n + |a_{n-1}|x^n + \ldots + |a_0|x^n \\ &\leq x^n \sum_{i=0}^{n} |a_i| \end{aligned}$$

Which by the definition of O notation $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ is $O(x^n)$.

31. Use the theorem on polynomial orders to prove $\frac{(x+1)(x-2)}{4}$ is $\Theta(x^2)$

$$\begin{aligned} \frac{(x+1)(x-2)}{4} &= \frac{x^2 - 2x - x - 2}{4} \\ &= \frac{x^2}{4} - \frac{2x}{4} - \frac{x}{4} - \frac{2}{4} \end{aligned}$$

which by the theorem on polynomial orders $\frac{(x+1)(x-2)}{4}$ is $\Theta(x^2)$.

32. Use the theorem on polynomial orders to prove that $\frac{x}{3}(4x^2-1)$ is $\Theta(x^3)$.

$$\frac{x}{3}(4x^2-1) \quad = \quad \frac{4x^3}{3} - \frac{x}{3}$$

which by the theorem on polynomial orders is $\Theta(x^3)$.

33. Use the theorem on polynomial orders to prove that $\frac{x(x-1)}{2} + 3x$ is $\Theta(x^2)$.

$$\frac{x(x-1)}{2} + 3x \quad = \quad \frac{x^2-x}{2} + 3x$$

which by the order of polynomial orders is $\Theta(x^3)$.

34. Use the theorem on polynomial orders to prove that $\frac{n(n+1)(2n+1)}{6}$ is $\Theta(n^3)$.

$$\begin{aligned} \frac{n(n+1)(2n+1)}{6} &= \frac{(n^2+n)(2n+1)}{6} \\ &= \frac{2n^3+n^2+2n^2+n}{6} \\ &= \frac{2n^3}{6} + \frac{3n^2}{6} + \frac{n}{6} \end{aligned}$$

which by the order of polynomial orders is $\Theta(n^3)$.

35. Use the theorem on polynomial orders to prove that $[\frac{n(n+1)}{2}]^2$ is $\Theta(n^4)$.

$$\begin{aligned} [\frac{n(n+1)}{2}]^2 &= \frac{n^2(n+1)^2}{4} \\ &= \frac{n^2(n^2+2n+1)}{4} \\ &= \frac{n^4+2n^3+n^2}{4} \\ &= \frac{n^4}{4} + \frac{2n^3}{4} + \frac{n^2}{4} \end{aligned}$$

which by the theorem on polynomial orders is $\Theta(n^4)$.

36. Use the theorem on polynomial orders to prove that $2 \cdot (n - 1) + \frac{n(n+1)}{2} + 4 \cdot \frac{n(n-1)}{2}$ is $\Theta(n^2)$

$$2 \cdot (n-1) + \frac{n^2}{2} + \frac{n}{2} + \frac{4n^2}{2} - \frac{4n}{2}$$

which by the theorem on polynomial orders is $\Theta(n^2)$.