

1 Use  $\Theta$  notation to express the statement

$$4n^6 \leq 17n^6 - 45n^3 + 2n + 8 \leq 30n^6, n \geq 3$$

Let  $A = 4$ ,  $B = 30$  and  $k = 3$  then the statement translates to

$$An^6 \leq 17n^6 - 45n^3 + 2n + 8 \leq Bn^6, n \geq k$$

hence by the definition of  $\Theta$  notation  $17n^6 - 45n^3 + 2n + 8$  is  $\Theta(n^6)$ .

2 Use  $\Omega$  notation to express the statement

1. Use  $\Omega$  notation to express the statement

$$\frac{11}{4}n^2 \leq 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2, n \geq 2$$

Let  $A = \frac{11}{4}$  and  $k = 2$  then  $An^2 \leq 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2, n \geq 2$  then the statement translates to

$$An^2 \leq 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2, n \geq k$$

which by the definition of  $\Omega$  notation,  $3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2$  is  $\Omega(n^2)$ .

2. Use  $O$  notation to express the statement

$$0 \leq 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2 \leq 6n^2, n \geq 1$$

Let  $A = 6$  and  $k = 1$  then the statement translates to

$$0 \leq 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2 \leq An^2, n \geq k$$

which by the definition of  $O$  notation,  $3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2$  is  $O(n^2)$ .

3. Justify the statement:  $3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2$  is  $\Theta(n^2)$ .

Let  $A = \frac{11}{4}$ ,  $B = 6$  and  $k = 2$  then  $A \cdot n^2 \leq 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2 \leq Bn^2, n \geq k$   
which by the definition of  $\Theta$  notation,  $3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2$  is  $\Theta(n^2)$ .

3. Given the function  $15n^3 + 11n^2 + 9$

1. Show that the function is  $\Omega(n^3)$ .

$$15n^3 \leq 15n^3 + 11n^2 + 9, n \geq 1$$

Let  $A = 15$  and  $k = 1$  then the statements translates to  $An^3 \leq 15n^3 + 11n^2 + 9, n \geq k$  which by the definition of  $\Omega$  notation,  $15n^3 + 11n^2 + 9$  is  $\Omega(n^3)$ .

2. Show that the function is  $O(n^3)$ .

$$\begin{aligned} 15n^3 + 11n^2 + 9 &\leq 15n^3 + 11n^3 + 9n^3 \\ &\leq 35n^3, n \geq 1 \end{aligned}$$

Let  $A = 35$  and  $k = 1$  then the statement translates to  $15n^3 + 11n^2 + 9 \leq An^3, n \geq k$  which by the definition of  $O$  notation,  $15n^3 + 11n^2 + 9$  is  $O(n^3)$ .

4. Given the function  $n^4 - 5n - 8$

1. Show that the function is  $\Omega(n^4)$ .

Let  $A = \frac{1}{2}$  and  $a = (|-5| + |-8|)$

$$\begin{aligned} n &\geq \frac{2}{1} \cdot (|-5| + |-8|) \\ \frac{1}{2}n^4 &\geq 5n^3 + 8n^3 \\ \frac{1}{2}n^4 &\geq 5n + 8 \\ n^4 - 5n - 8 &\geq \frac{1}{2}n^4, n \geq a \end{aligned}$$

Hence by the definition of  $\Omega$  notation,  $n^4 - 5n - 8$  is  $\Omega(n^4)$ .

2. Show that the function is  $O(n^4)$ .

$$\begin{aligned} n^4 - 5n - 8 &\leq n^4 + 5n + 8 \\ &\leq n^4 + 5n^4 + 8n^4 \\ &= 14n^4, n \geq 1 \end{aligned}$$

Let  $A = 14$  and  $k = 1$  then the statement translates to  $n^4 - 5n - 8 \leq An^4, n \geq k$  which by the definition of  $O$  notation translates,  $n^4 - 5n - 8$  is  $O(n^4)$ .

5. Show that  $15n^3 + 11n^2 + 9$  is  $\Theta(n^3)$ .

Since we have  $\Omega(n^3)$  and  $O(n^3)$  we have that there exists real positive number constants  $A$  and  $B$  such that  $Ag(n) \leq f(n) \leq Bg(n), k \geq n$  where  $k = \max(k', k'')$  obtained from the previous inequalities. By definition of  $\Theta$ ,  $15n^3 + 11n^2 + 9$  is  $\Theta(n^3)$ .

6. Show that  $n^4 - 5n - 8$  is  $\Theta(n^4)$ .

Since we have shown that the function is  $\Omega(n^4)$  and  $O(n^4)$  we have that there exists real positive number constants  $A$  and  $B$  such that  $Ag(n) \leq f(n) \leq Bg(n), k \geq n$  where  $k = \max(k', k'')$  obtained from the previous inequalities. by definition of  $\Theta$ ,  $n^4 - 5n - 8$  is  $\Theta(n^4)$ .

7. Let  $g(n) = n^4 - 5n - 8$ , show that  $g(n)$  is not  $O(n^r)$  for any positive real number  $r < 4$ .

We prove this by contradiction. Suppose that  $g(n)$  is  $O(n^r)$  for any positive real number  $r < 4$ . then

$$g(n) \leq An^r, n \geq a$$

where  $A$  and  $a$  are real positive numbers.

$$\begin{aligned} g(n) &\leq n^4 \\ &\leq An^r \\ n^{4-r} &\leq A \\ n &\leq \sqrt[4-r]{A} \end{aligned}$$

which is a contradiction. We conclude that  $g(n)$  is not  $O(n^r)$  for any positive real number  $r < 4$ .

8. Use theorem on polynomial orders to find orders for the function given by the following formulas.

1.  $f(n) = 7n^5 + 5n^3 - n + 4$  for each positive integer  $n$ .

By direct application of theorem on polynomial orders,  $7n^5 + 5n^3 - n + 4$  is  $\Theta(n^5)$ .

2.  $g(n) = \frac{(n-1)(n+1)}{4}$  for each positive integer  $n$ .

$$\begin{aligned}
\frac{(n-1) \cdot (n+1)}{4} &= \frac{n^2 + n - n + 1}{4} \\
&= \frac{n^2 + 1}{4} \\
&= \frac{n^2}{4} + \frac{1}{4}
\end{aligned}$$

Thus  $g(n)$  is  $\Theta(n^2)$ .

9. Show that for a positive integer variable  $n$ ,

$$1 + 2 + 3 \dots + n \text{ is } \Theta(n^2)$$

$$\begin{aligned}
\sum_{i=1}^n i &= \frac{n(n+1)}{2} \\
&= \frac{n^2}{2} + \frac{n}{2}
\end{aligned}$$

10. Express  $5x^8 - 9x^7 + 2x^5 + 3x - 1 \leq 6x^8, x > 3$  using  $O$  notation

Let  $A = 6$  and  $a = 3$  then  $5x^8 - 9x^7 + 2x^5 + 3x - 1 \leq Ax^8, x > a$  and by definition of  $O$  notation,  $5x^8 - 9x^7 + 2x^5 + 3x - 1$  is  $O(x^8)$ .

11. Express  $x^{\frac{7}{2}} \leq \frac{(x^2-7)^2(10x^{\frac{1}{2}}+3)}{x+1}, x > 4$  using  $\Omega$  notation

Let  $A = 1$  and  $k = 4$  then the statement translates to

$$Ax^{\frac{7}{2}} \leq \frac{(x^2-7)^2(10x^{\frac{1}{2}}+3)}{x+1}, x > k$$

which by the definition of  $\Omega$  notation,  $\frac{(x^2-7)^2(10x^{\frac{1}{2}}+3)}{x+1}$  is  $\Omega(x^{\frac{7}{2}})$ .

12. Express  $3x^6 + 5x^4 - x^3 \leq 9x^6, x > 1$  using  $O$  notation. Let  $A = 9$  and  $k = 1$  then the statement translate to

$$3x^6 + 5x^4 - x^3 \leq Ax^6, x > k$$

which by the definition of  $\Omega$  notation,  $3x^6 + 5x^4 - x^3$  is  $O(x^6)$ .

**13.** Express  $\frac{1}{2}x^4 \leq x^4 - 50x^3 + 1$  for all real numbers  $x > 101$  using  $\Omega$  notation.

Let  $A = \frac{1}{2}$  and  $k = 101$  this statement translates to  $Ax^4 \leq x^4 - 50x^3 + 1, n > k$  which by the definition of  $\Omega$  notation,  $x^4 - 50x^3 + 1$  is  $\Omega(x^4)$ .

**14.** Express  $\frac{1}{2}x^2 \leq 3x^2 - 80x + 7 \leq 3x^2, x > 25$

Let  $A = \frac{1}{2}, B = 3$  and  $k = 25$  then the statement translates to

$$Ax^2 \leq 3x^2 - 80x + 7 \leq Bx^2, x > k$$

which by the definition of  $\Theta$  notation  $3x^2 - 80x + 7$  is  $\Theta(x^2)$ .

**15.** Suppose  $g(x)$  is  $O(f(x))$  show  $f(x)$  is then  $\Omega(g(x))$

Since  $g(x)$  is  $O(f(x))$  then there exists a real positive numbers  $B$  and  $k$  such that  $g(x) \leq B \cdot f(x), n > k$ . We obtain  $\frac{g(x)}{B} \leq f(x), n > k$  which by the definition of  $\Omega$  notation  $f(x)$  is  $\Omega(g(x))$ .

**16.** Prove that if  $f(x)$  is  $O(g(x))$  and  $c$  is any nonzero real number, then  $c \cdot f(x)$  is  $O(g(x))$ .

Since  $f(x)$  is  $O(g(x))$  then there exists real positive numbers  $B$  and  $k$  such that  $g(x) \leq B \cdot f(x), n > k$ . Multiplying by the constant  $c$  we obtain  $c \cdot g(x) \leq c \cdot B \cdot f(x), n > k$  which by the definition of  $O$  notation  $c \cdot g(x)$  is  $O(g(x))$ .