1. Where in a min heap the largest element resides? (Assume all elements are distinct) Explain.

When $n = 1$ the largest element is trivially a leaf. Suppose that we have a min heap with n elements where $n > 1$. Suppose further that the largest element was an internal node, then that means it has atleast one child. This would be a contradiction since the largest element would be larger than its children which contradicts the min-heap property. we conclude that the largest element must be a leaf.

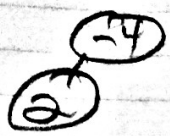
2. Make the min heap by successive insertions into an initially-empty min heap. Re-draw the heap each time an insertion causes one or more swaps.

$$[-4, 2, 10, 12, -1, -3, 15, 76]$$

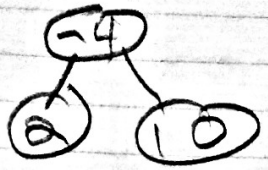
Iterative tree

1

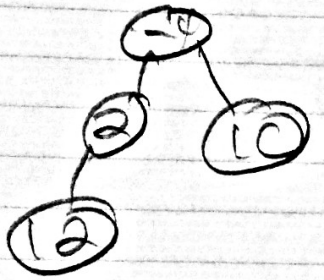
2



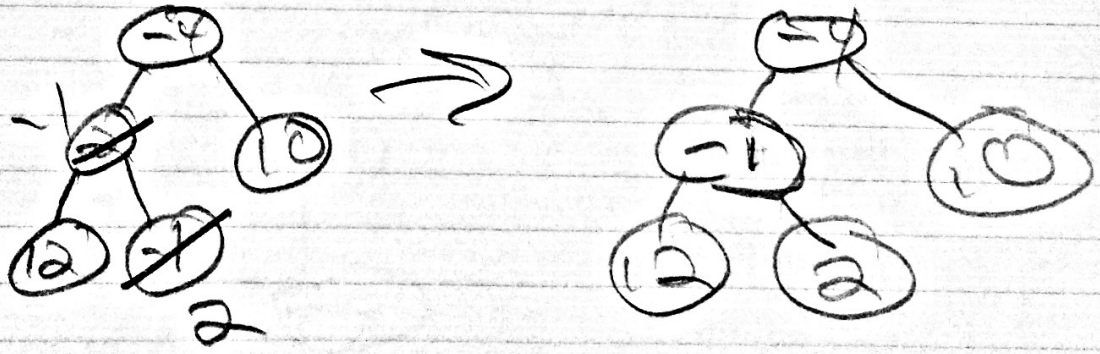
3



4

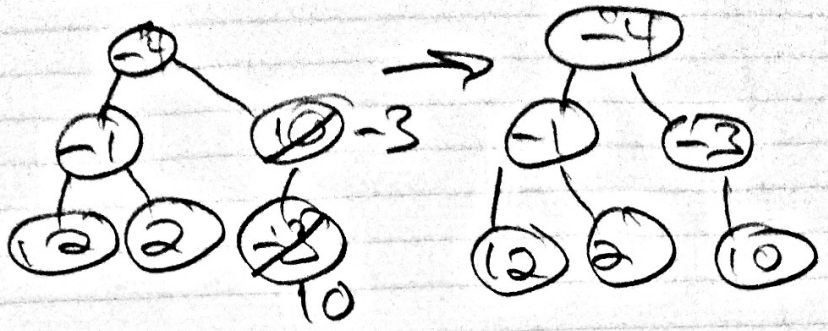


5

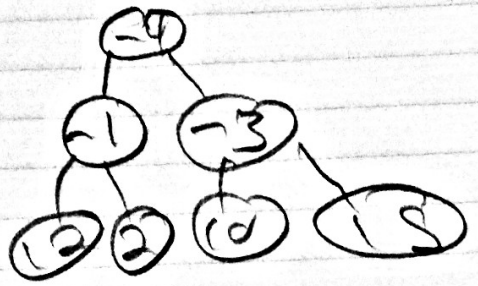


Iteration tree

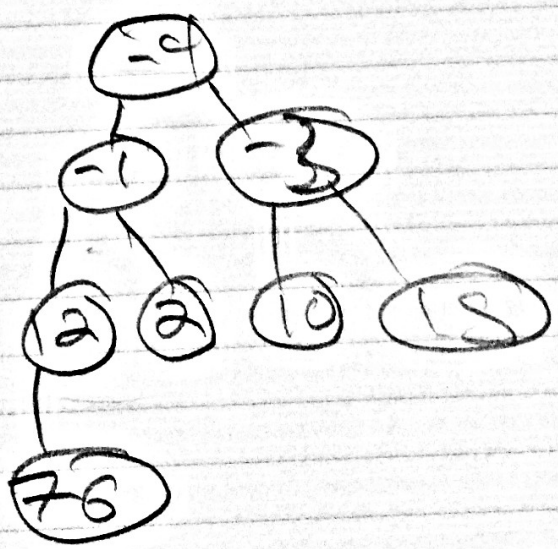
6



7



8

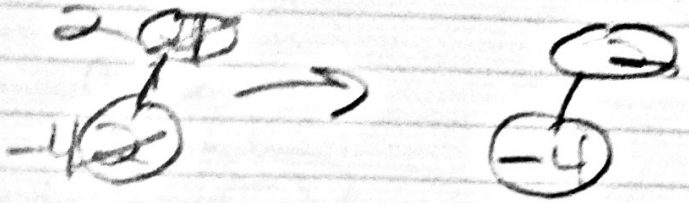


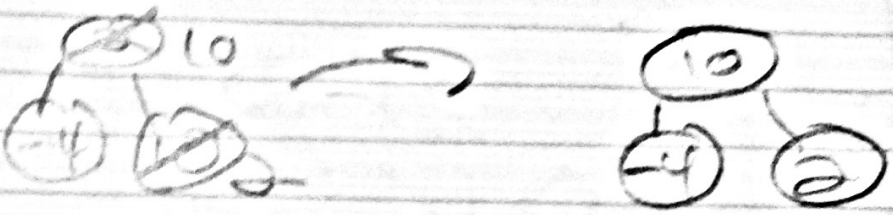
3. Make the max heap by successive insertions into an initially-empty max heap, Re-draw the heap each time an insertion causes one or more swaps

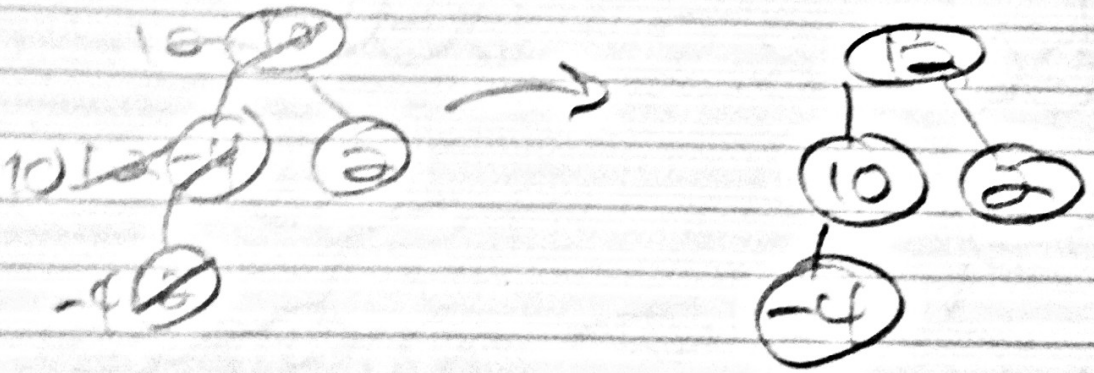
1. -4,2,10,12,-1,-3,15,76
2. 61,25,-12,16,20,97,-1,100

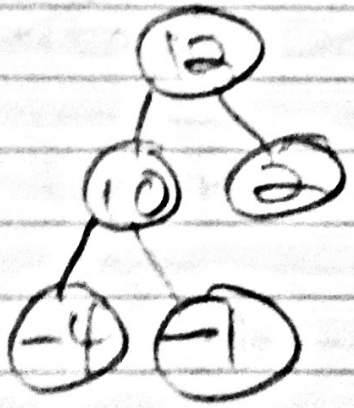
Q.1)

Iteration	Heap
1	-4
2	
3	
4	



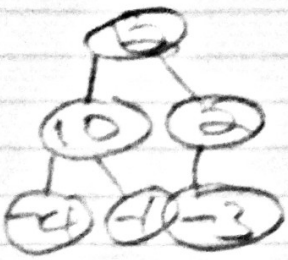




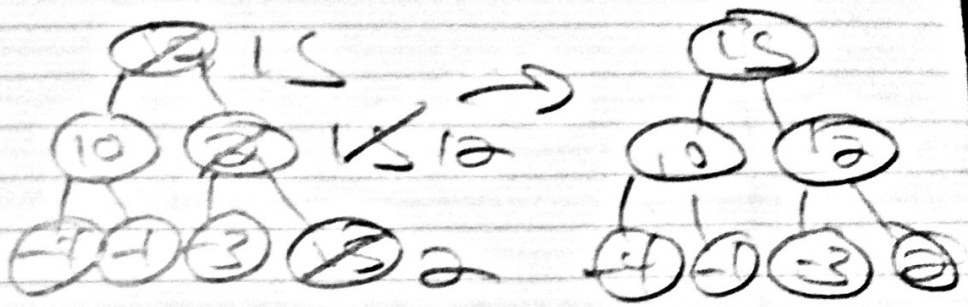


iteration heap

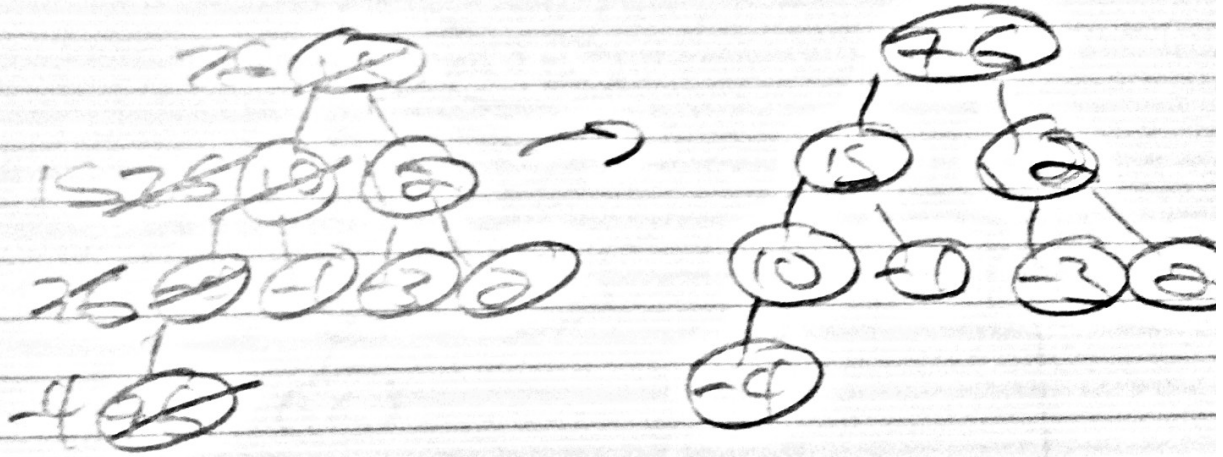
6



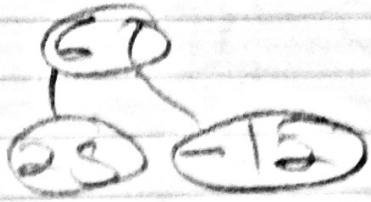
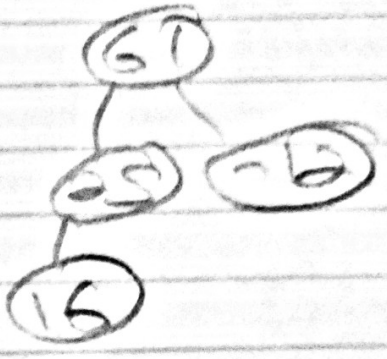
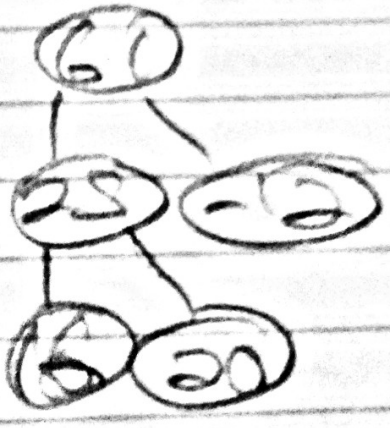
7



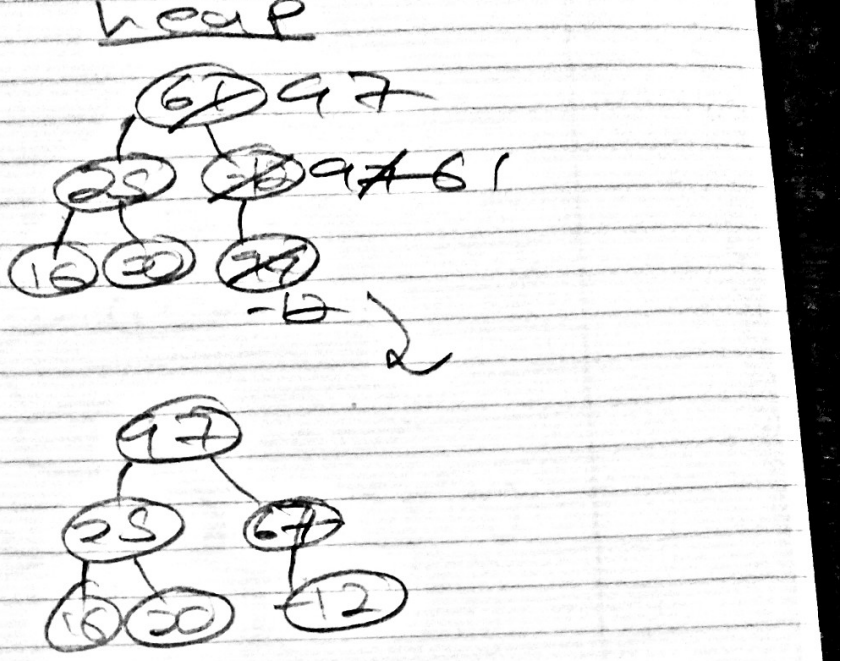
8



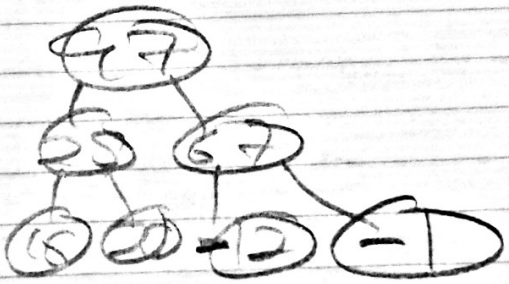
3.2

iteration	heap
1	
2	
3	
4	
5	

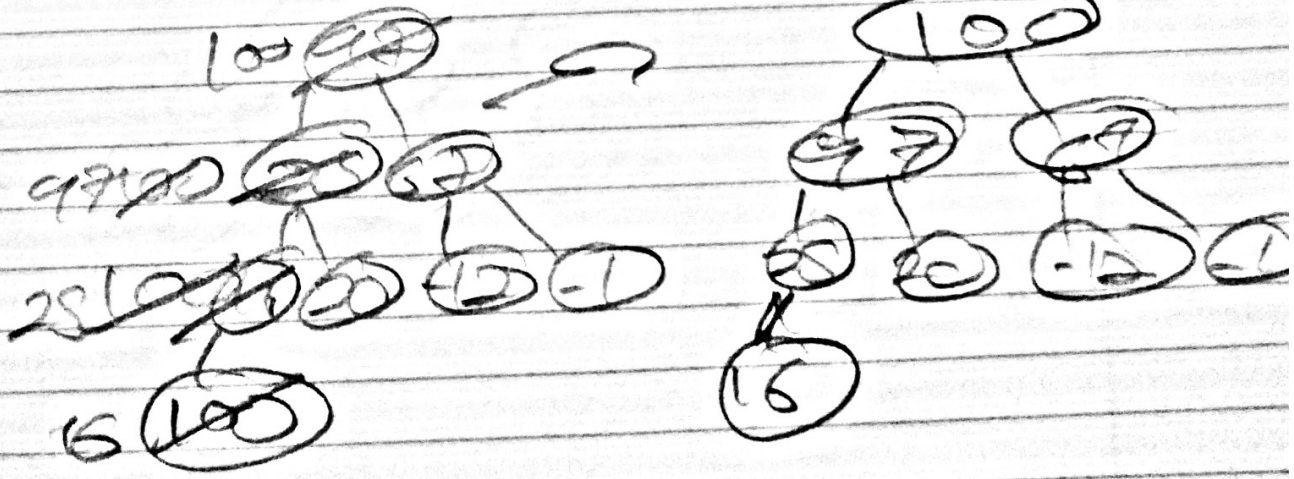
6



7

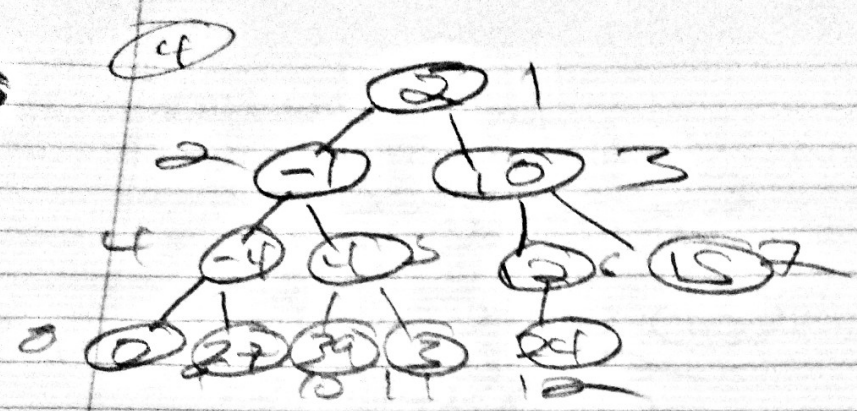


8



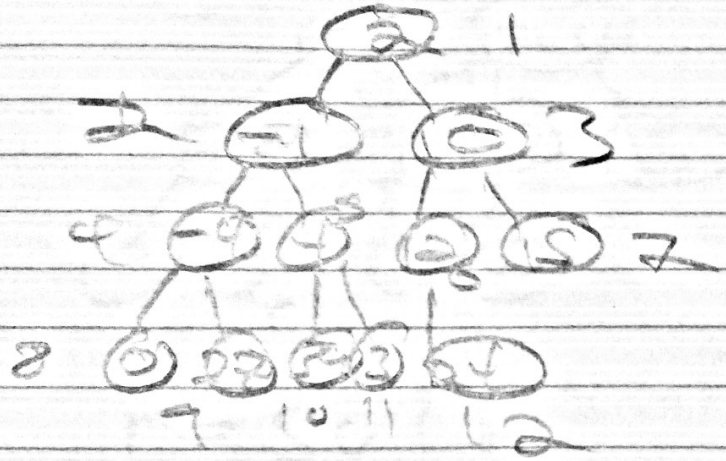
4. Is this a max-heap? if not, apply max-heapify function to change it to a max heap.

$$[2, -1, 10, -4, 4, 12, 15, 0, 27, 39, 3, 24]$$

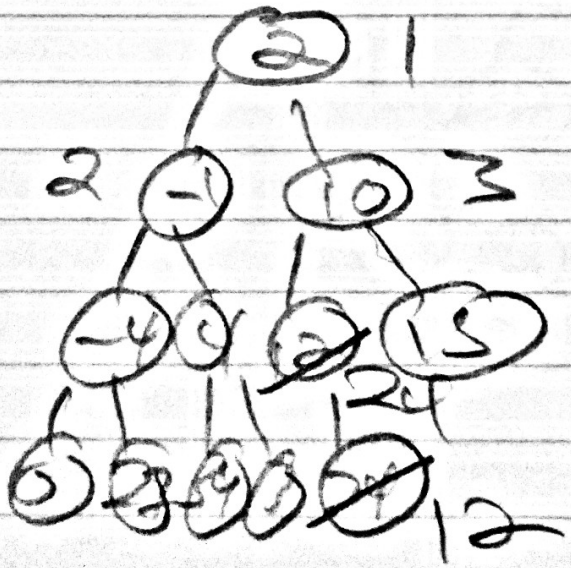


max_heapify(a, 7).

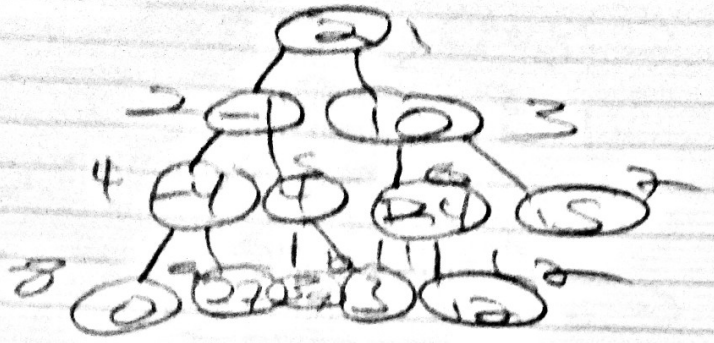
Result



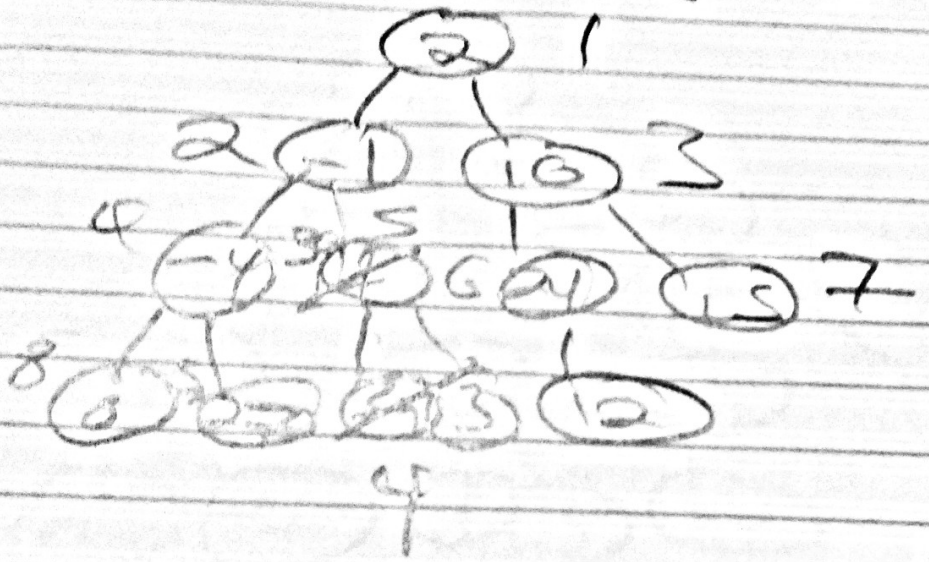
max-heapify(a, 6)



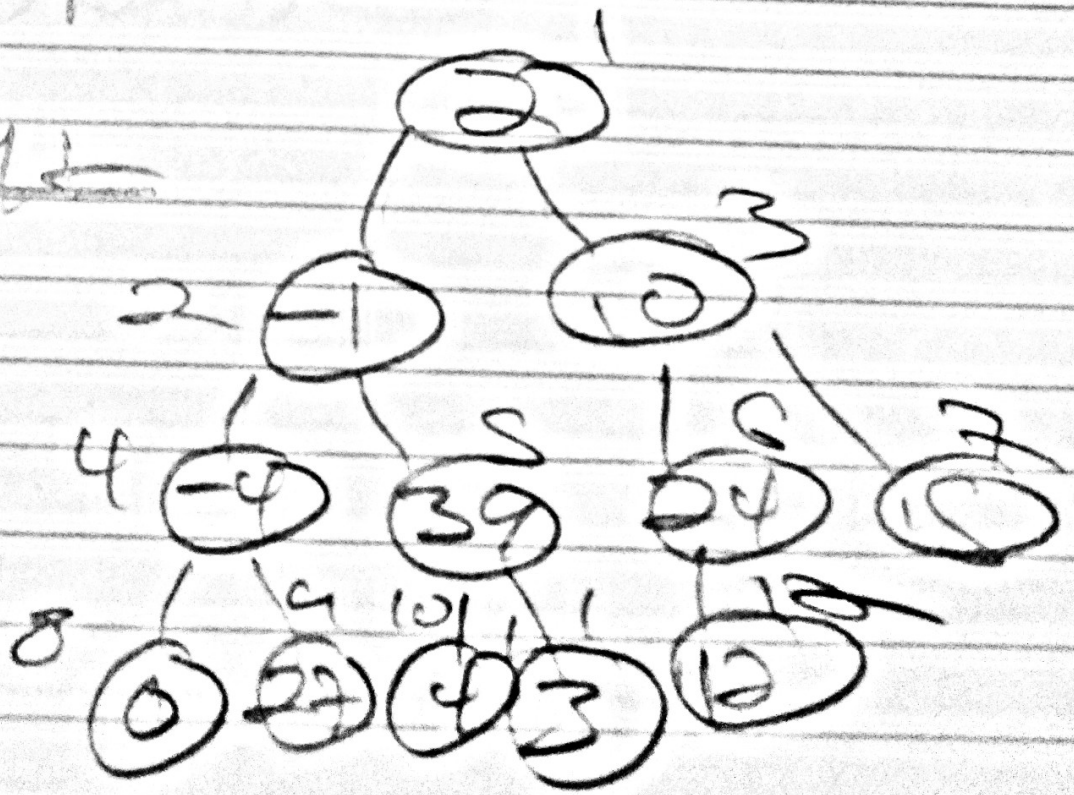
result



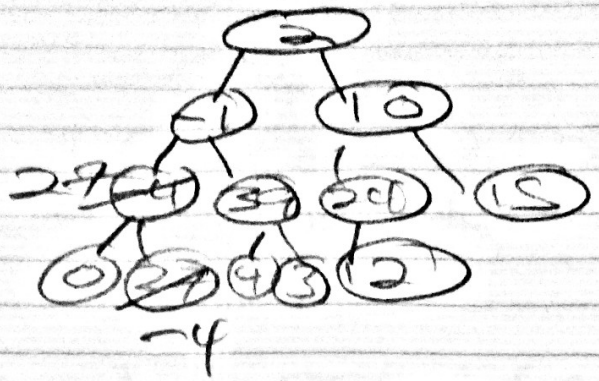
next_heapify(4,5)



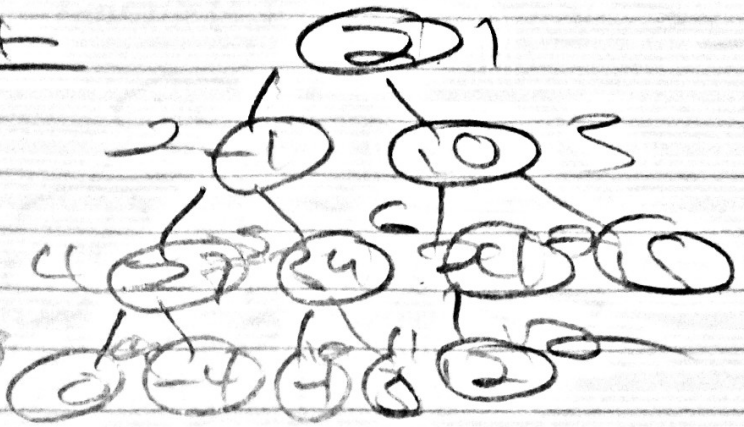
result



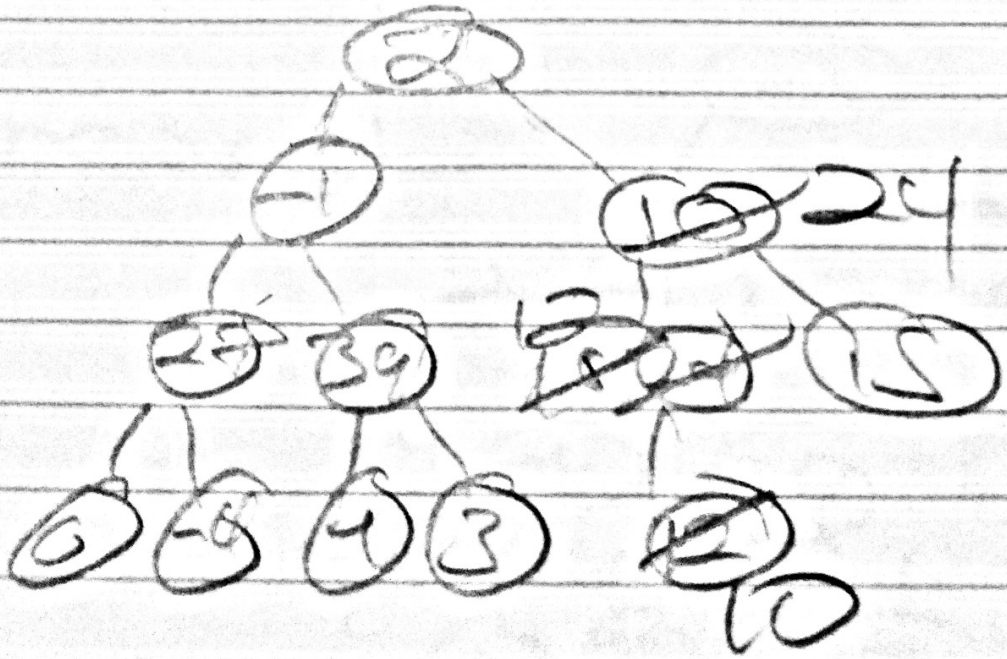
max-heapify (A, 2)-



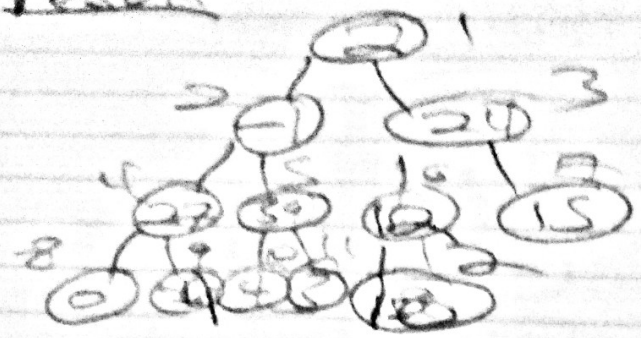
result



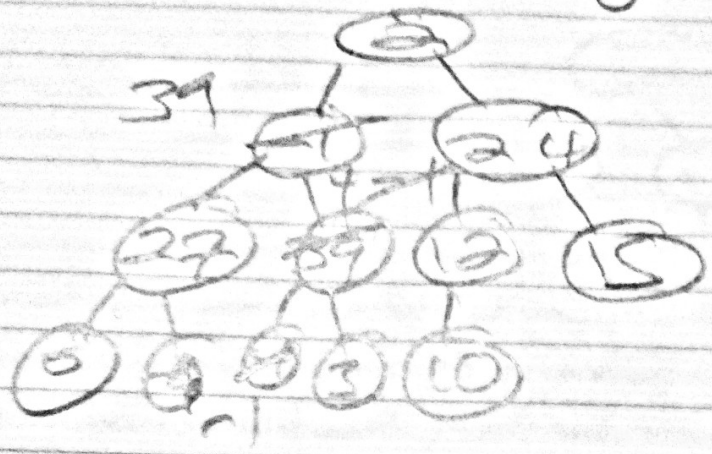
max-heapify (A, 3)



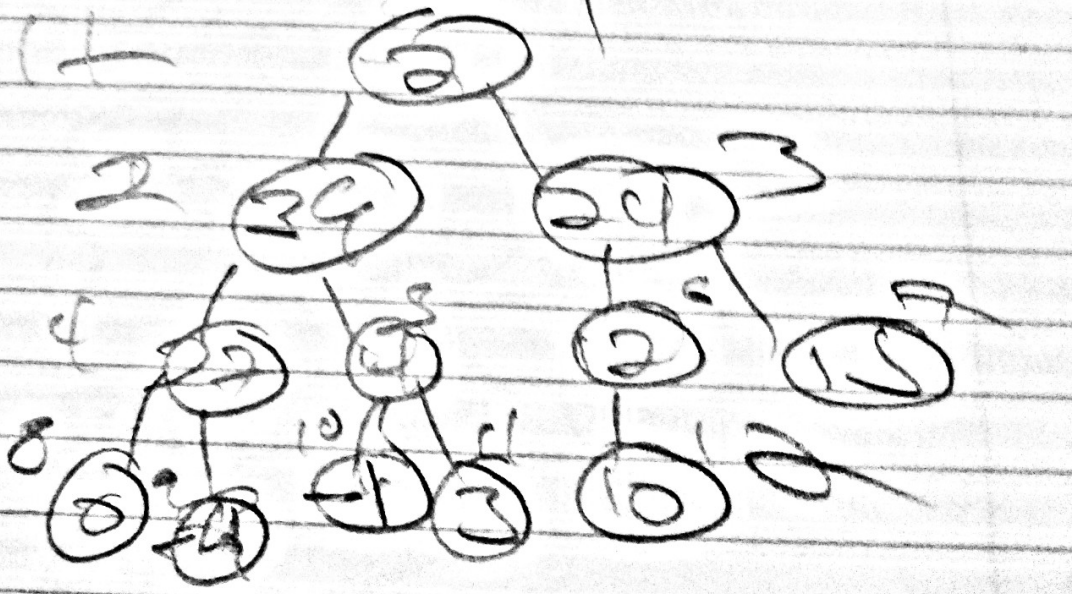
result



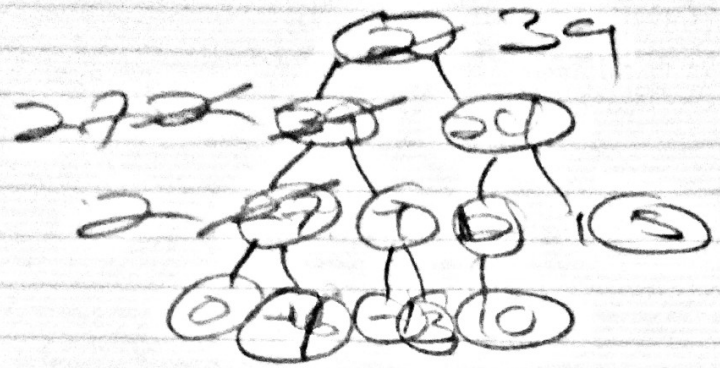
max-Heapify (A, 2)



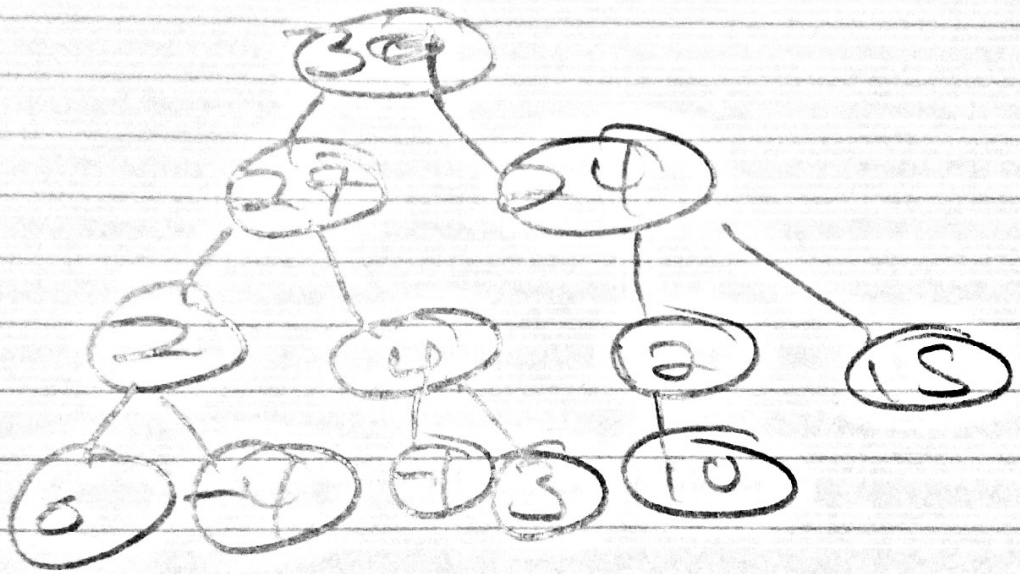
result



max-heapify(A, 1)

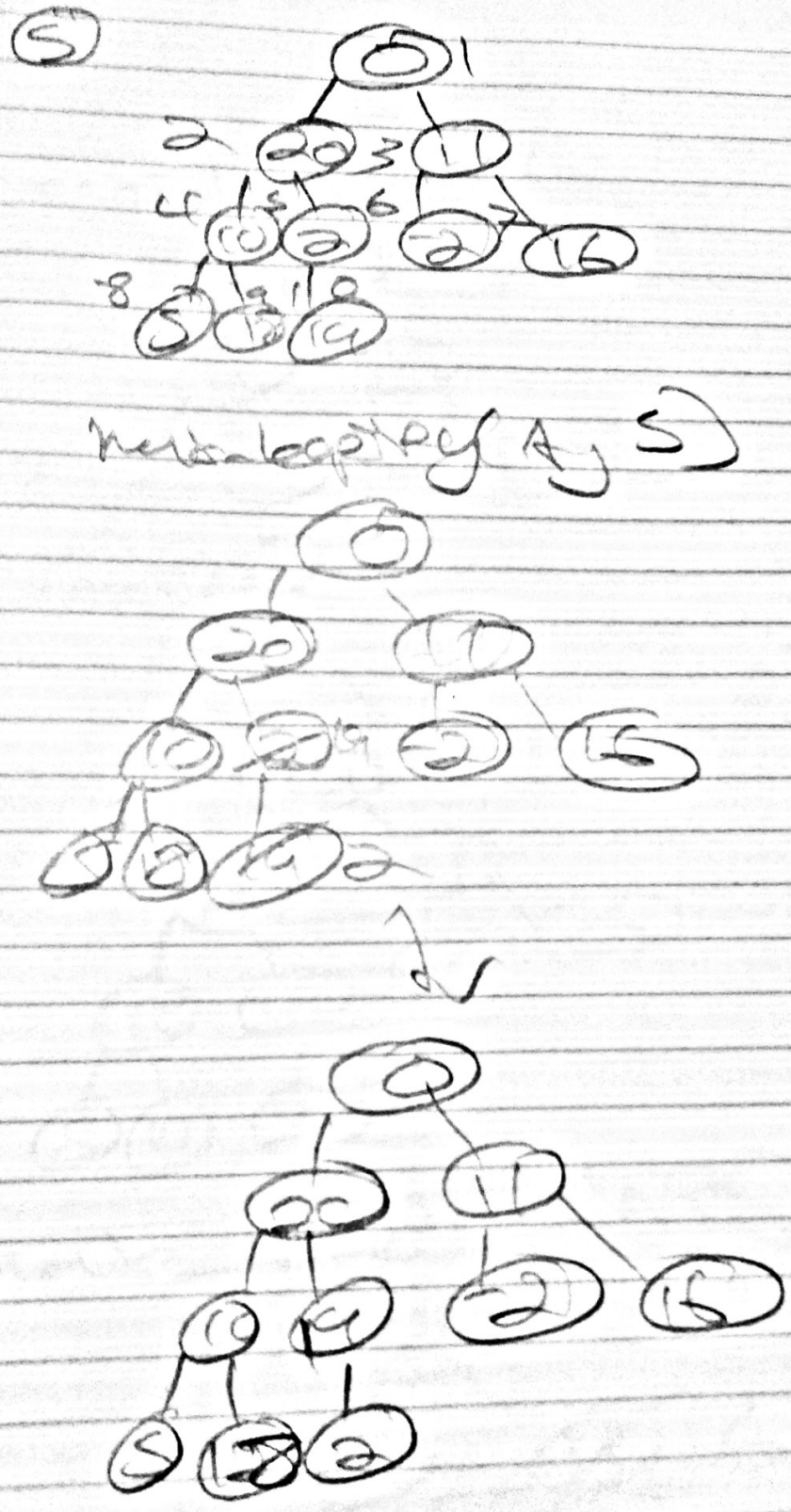


result

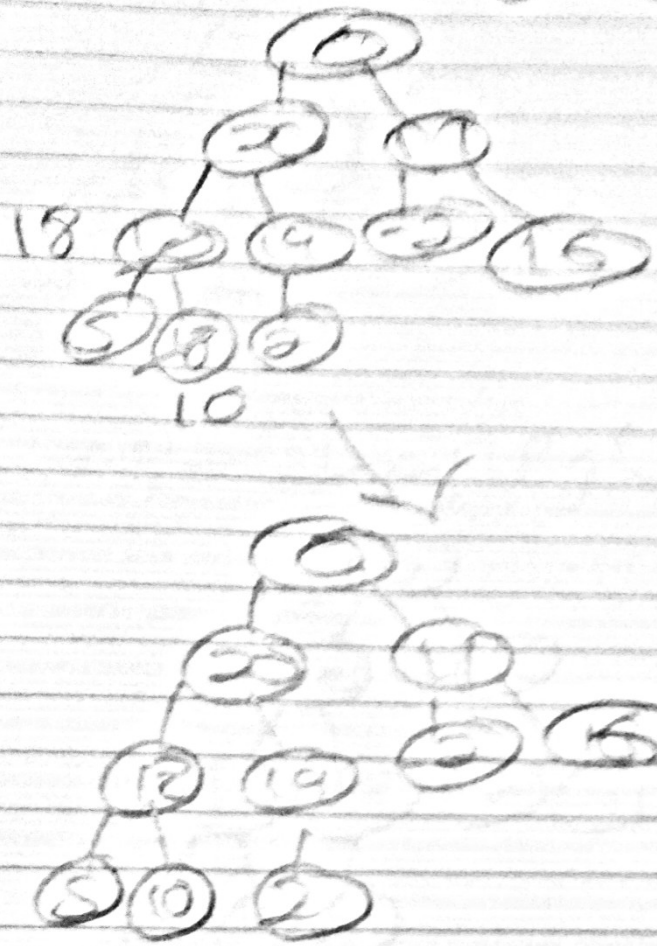


5. Is this array max-heap? If not, change it to the max heap by putting the elements on a binary tree and then applying the max-heapify function.

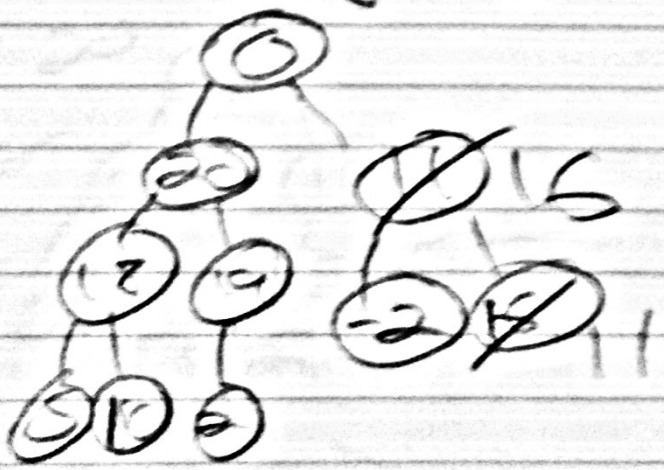
$$0, 2, 11, 2, -2, 16, 5, 18, 19$$

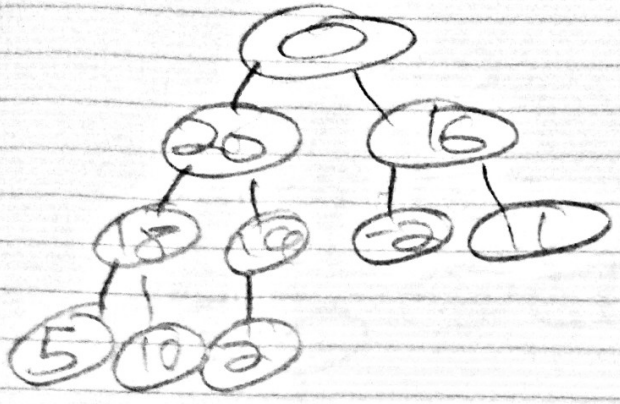


max_heapify(A, 3)

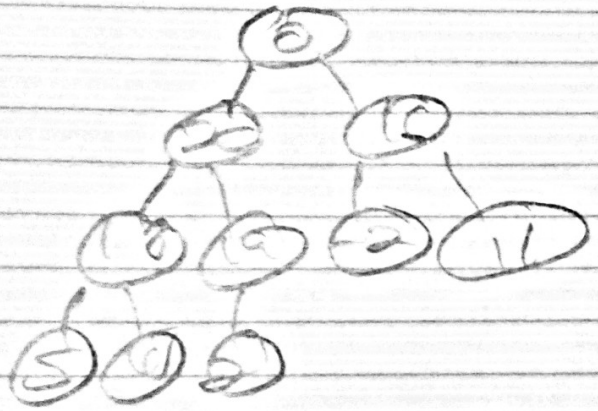


max_heapify(A, 3)

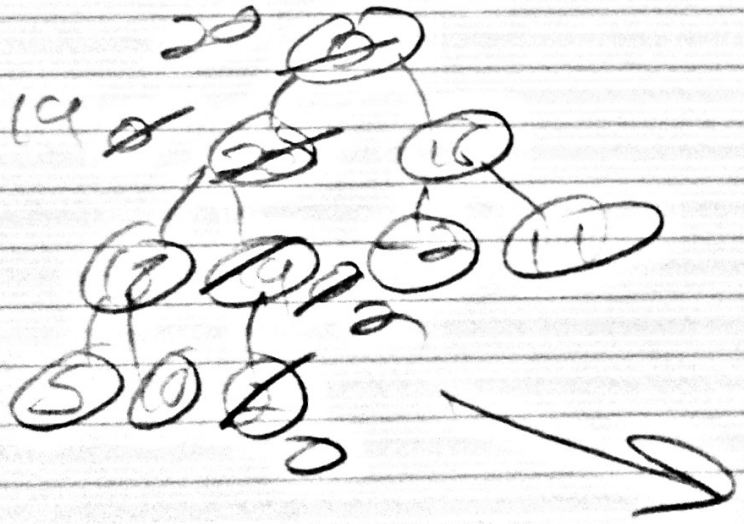


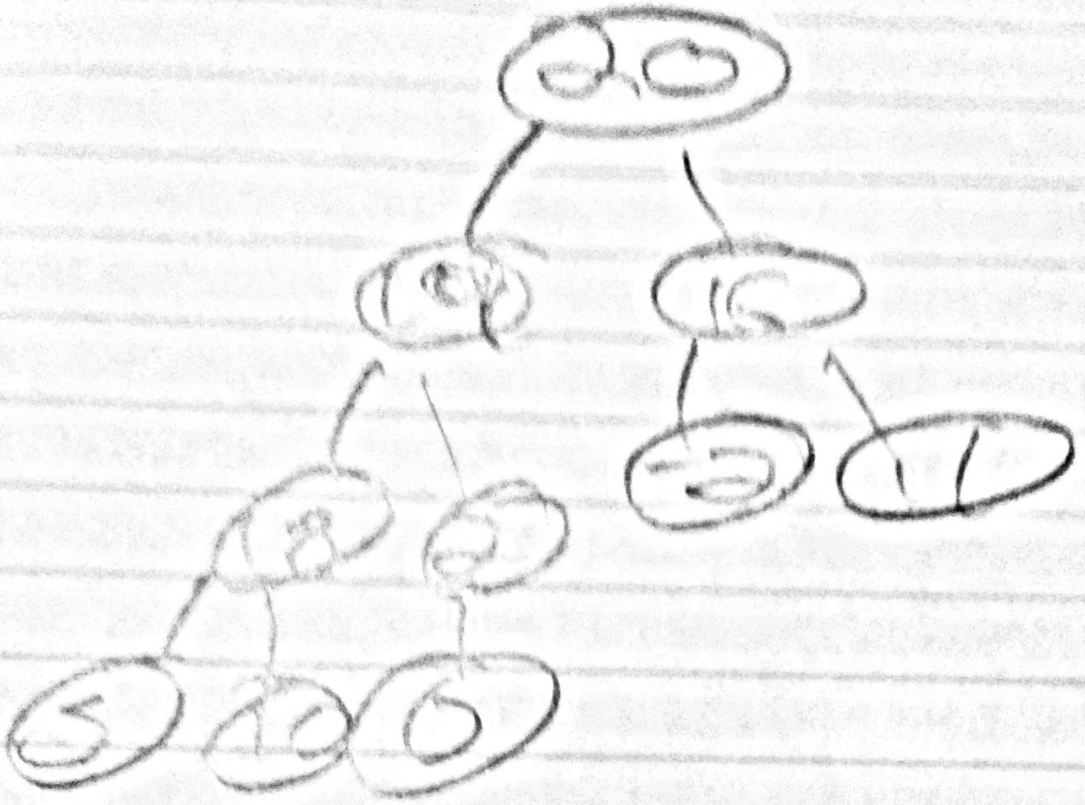


max_heapify(A, 2)



max_heapify(A, 1)





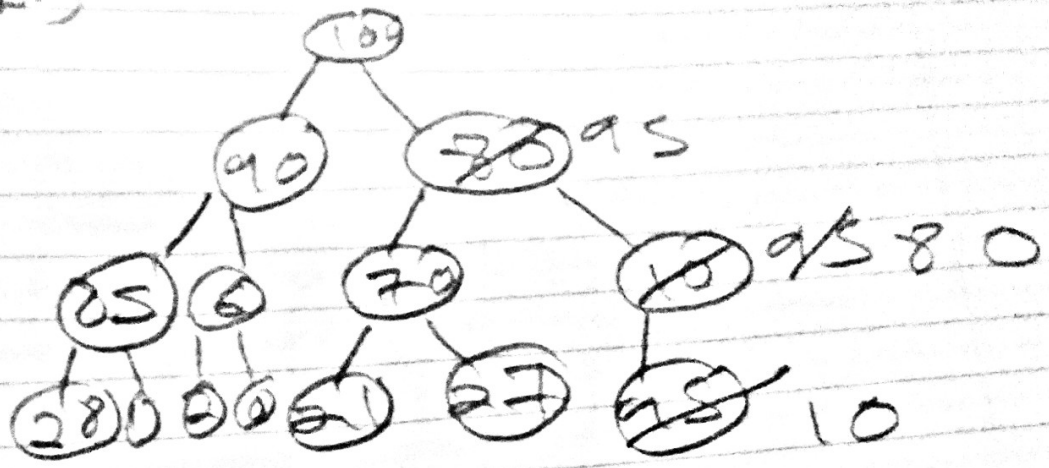
6. Insert

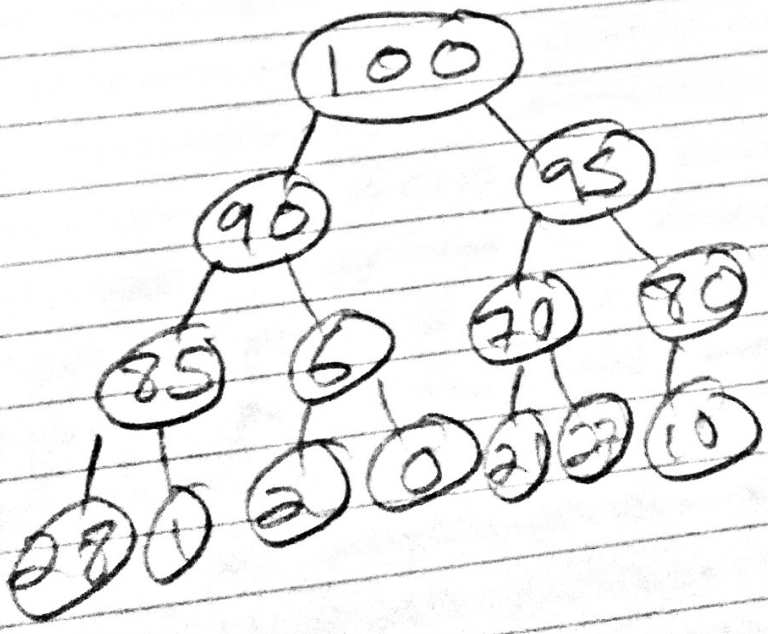
1. 95
2. 160
3. 37

into the below max-heap

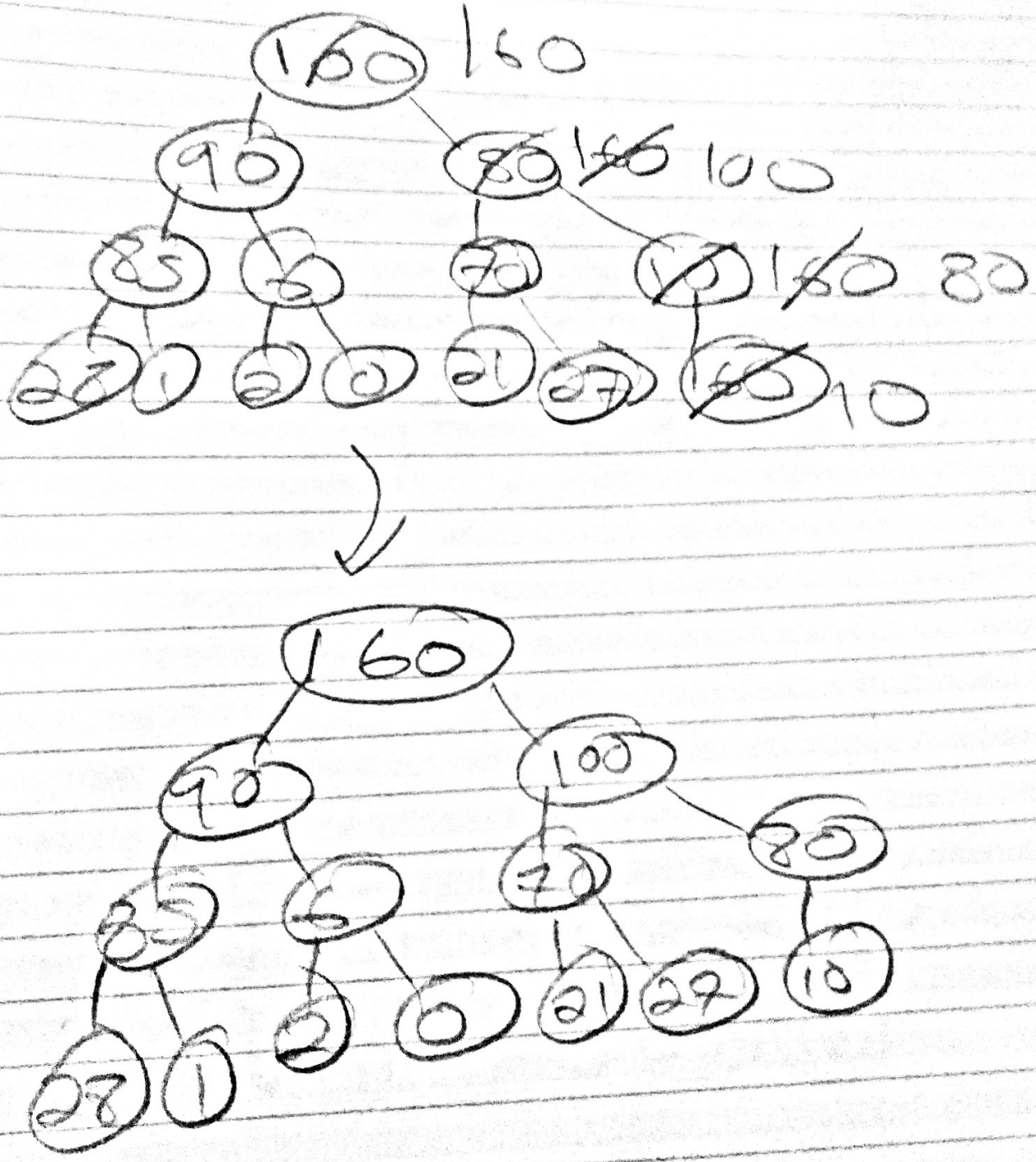
$$[100, 90, 80, 85, 6, 70, 10, 28, 1, 2, 0, 21, 27]$$

9.1 Insert 95

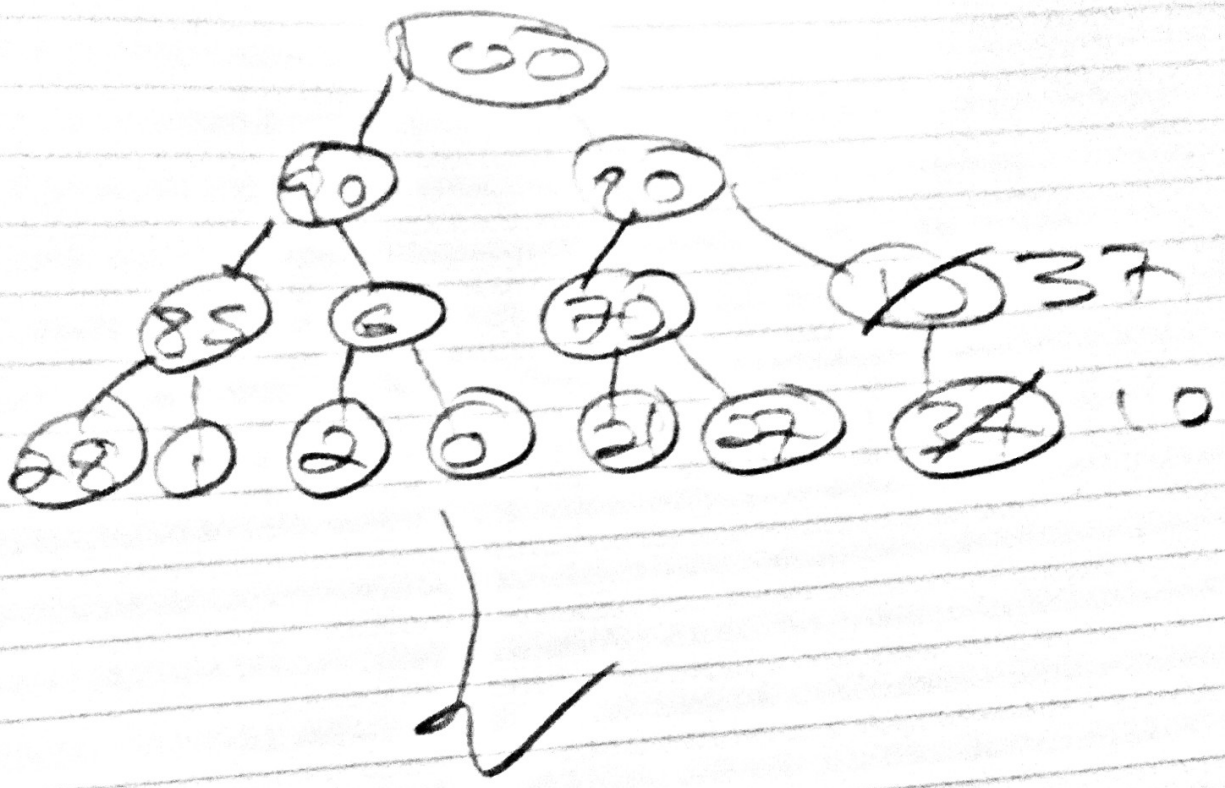


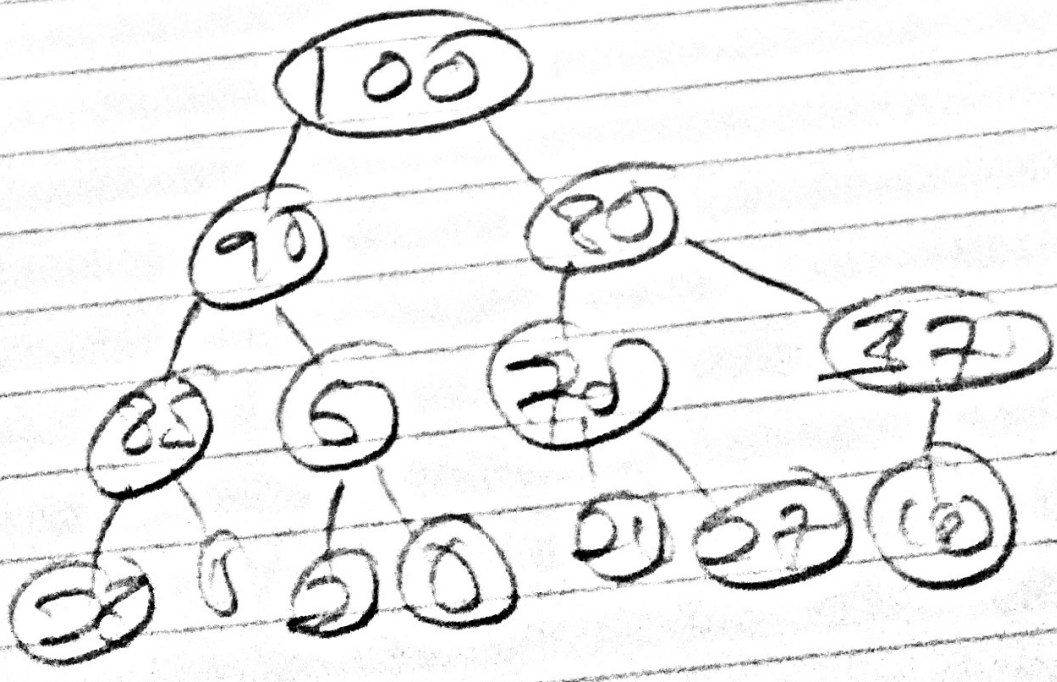


6.2 inserting 160



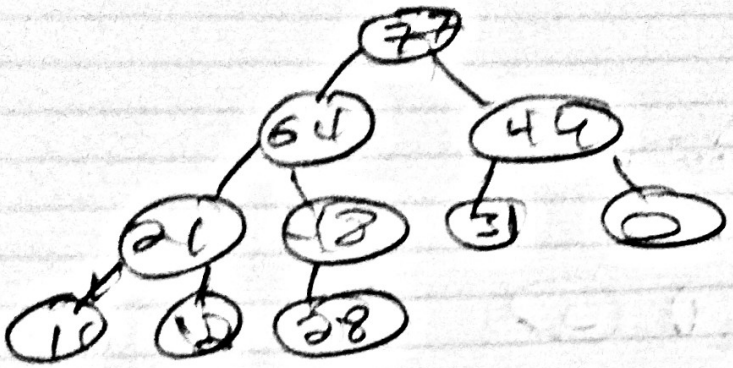
6.3 inserting 37



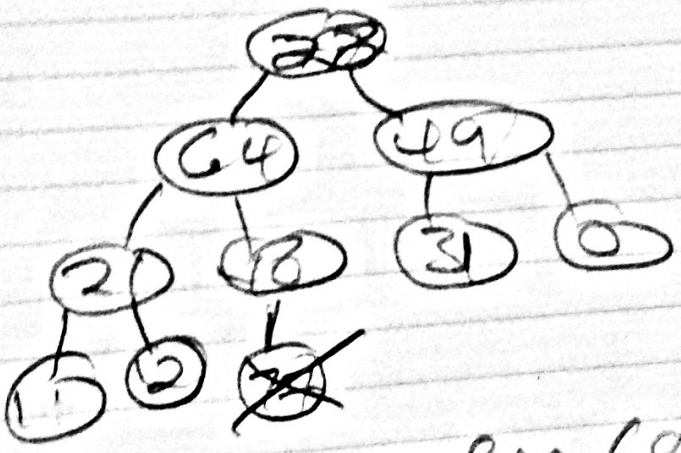


7. Delete the root of the below max-heap

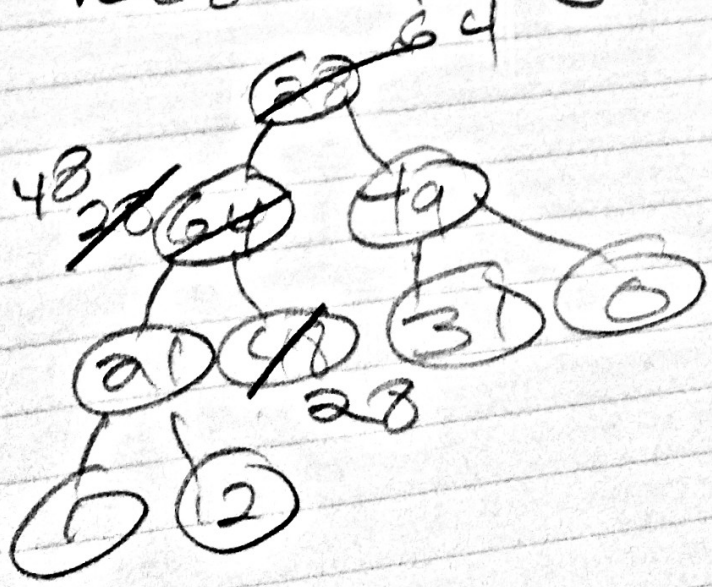
$$[77, 64, 49, 21, 48, 31, 0, 1, 12, 28]$$



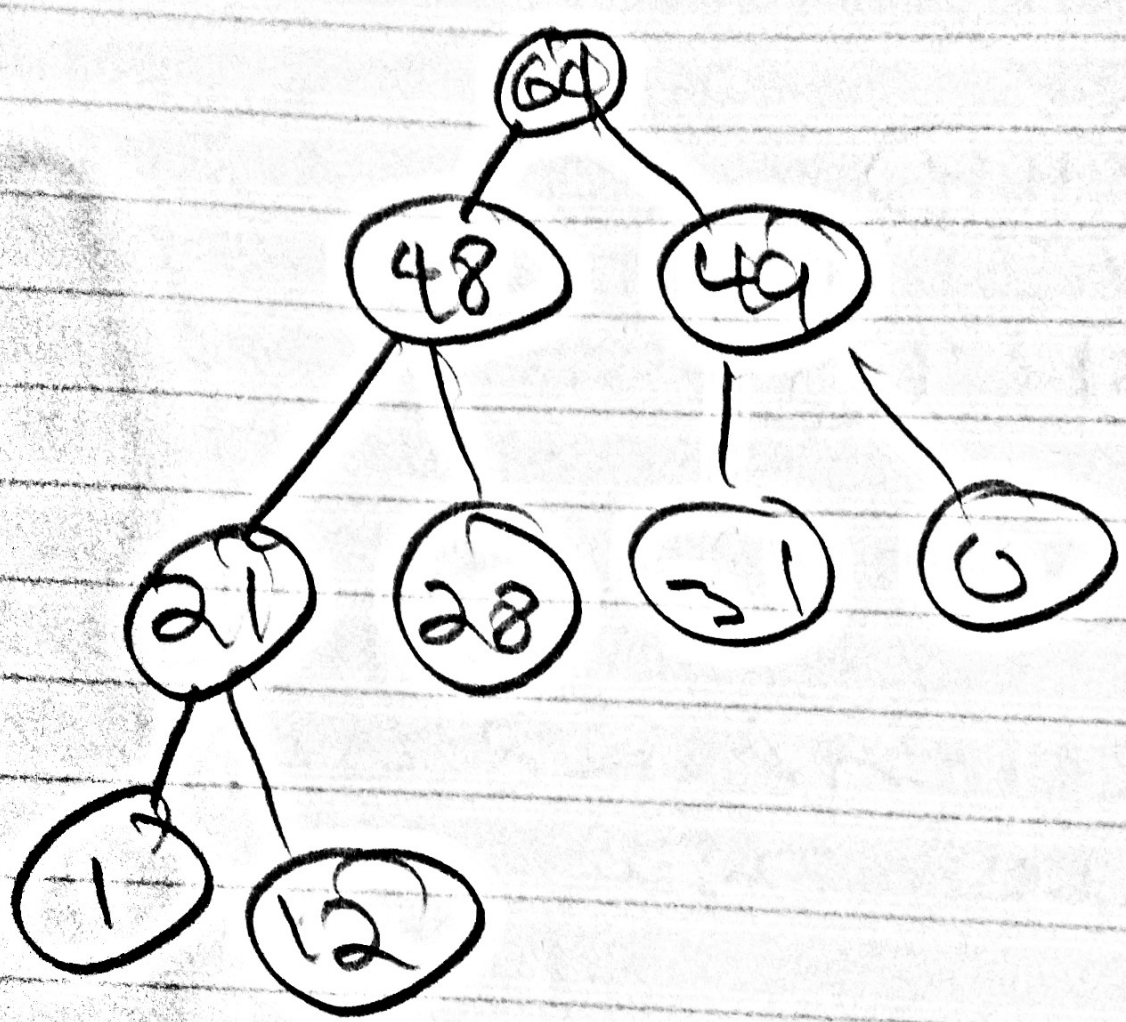
Swap 77 and 28
then remove 77



max heapify (a, 0)



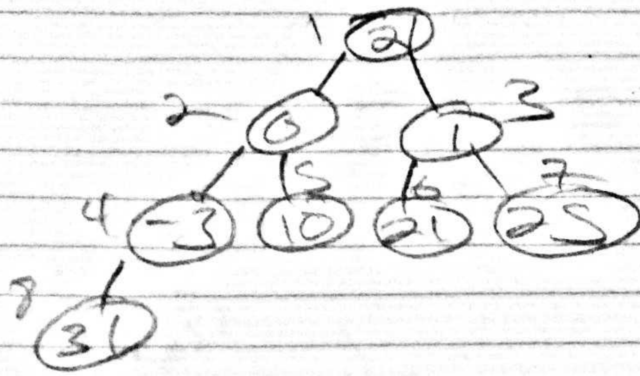
result



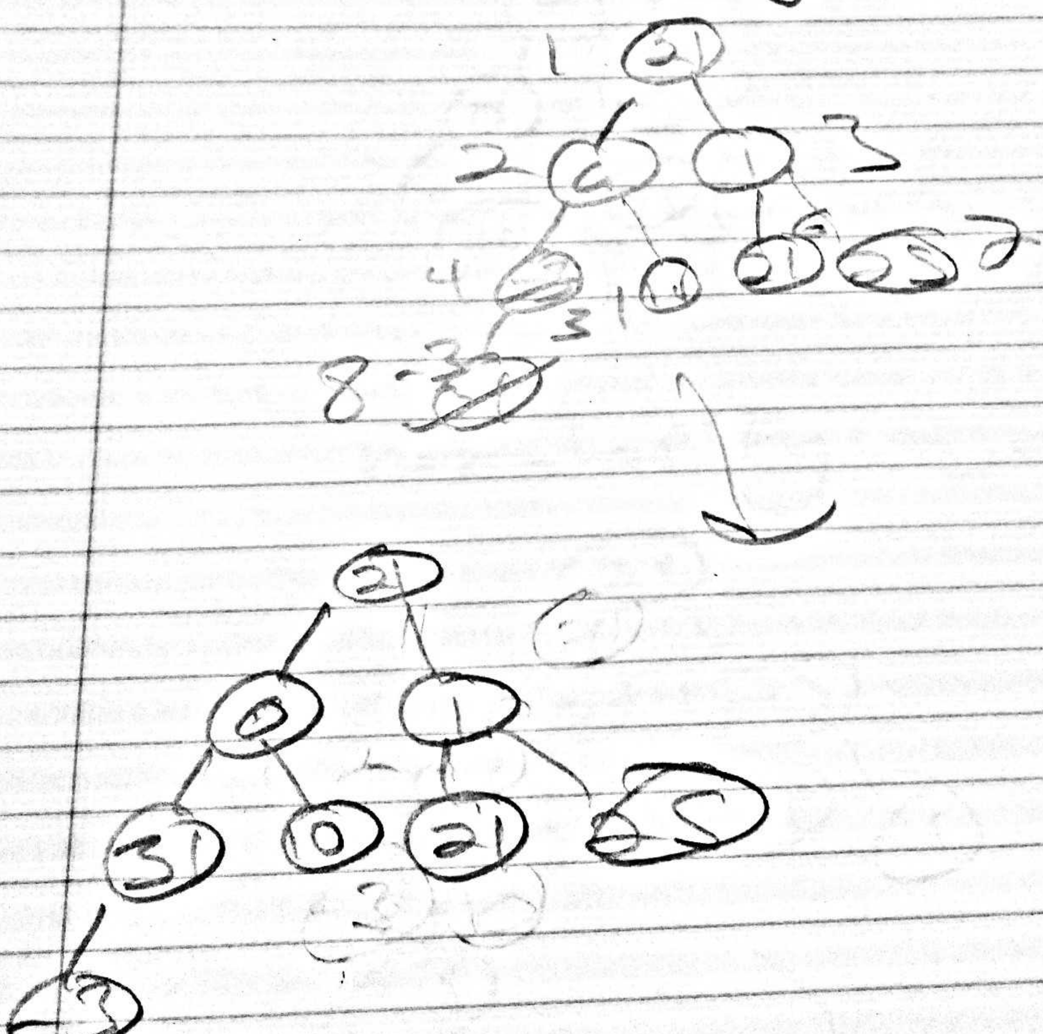
8. Sort the below array using heapsort

1. 21,0,1,-1,-3,10,21,25,31
2. 87,21,0,-1,-22,10,17,11,-3,10

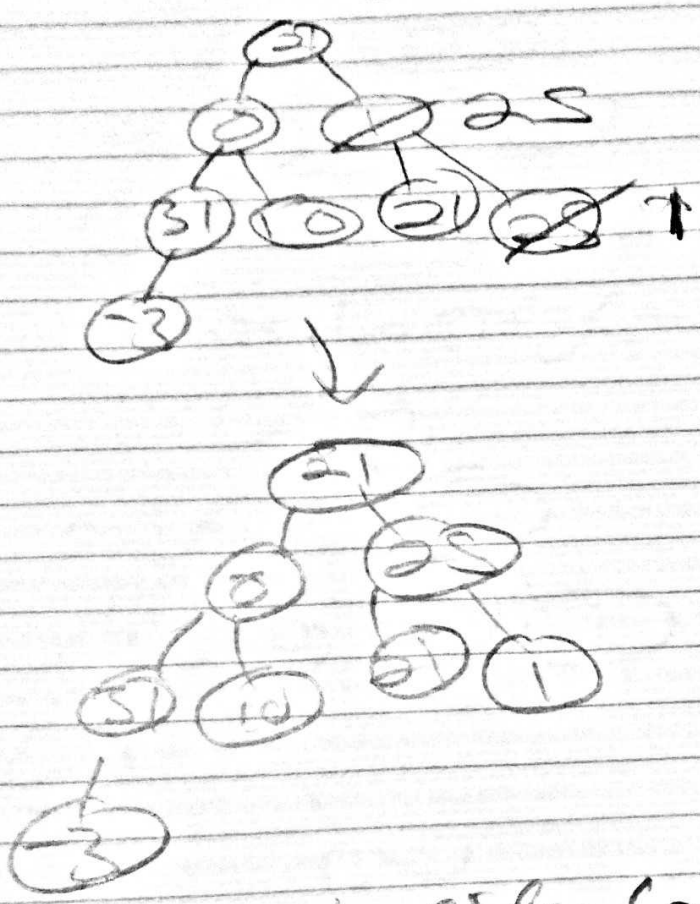
8.1 21, 0, 1, -3, 10, 21, 25, 31



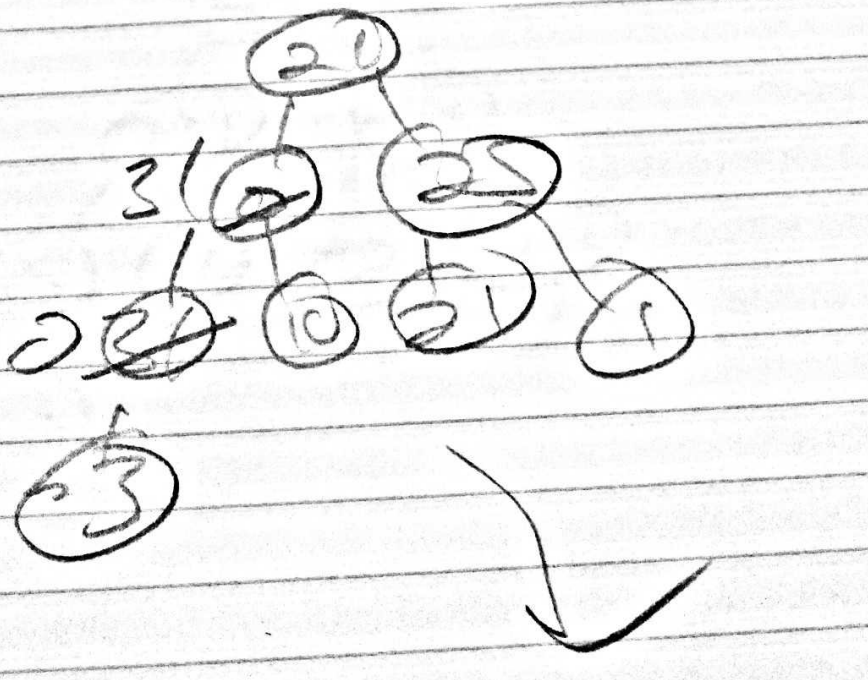
max-heapify(a, 4)

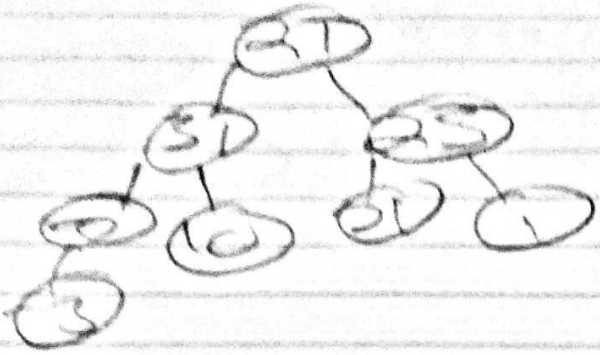


max-heapify(A, 3)

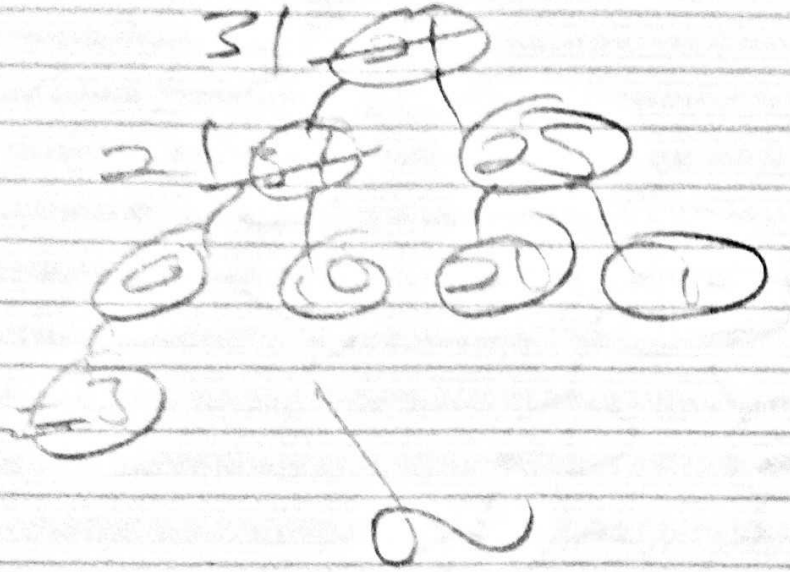


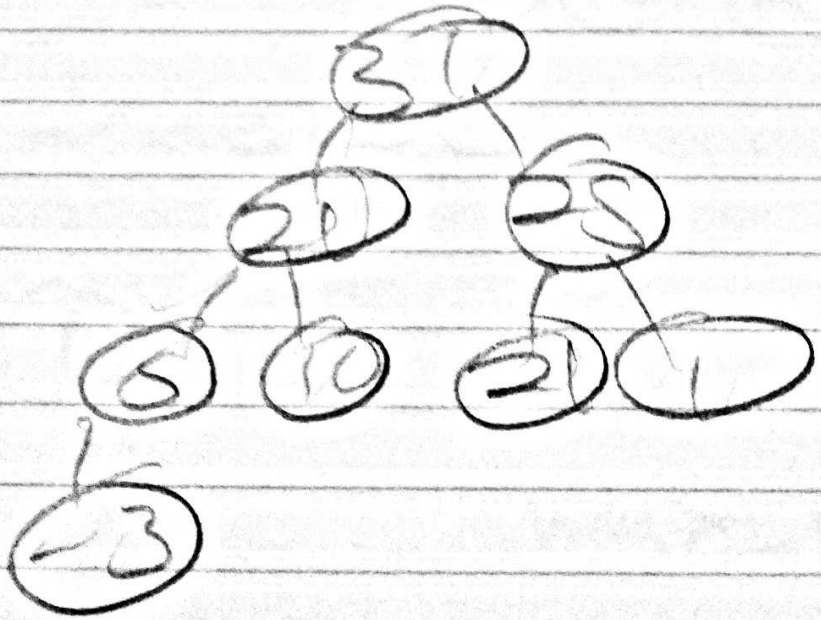
heap-heapify(A, 2)

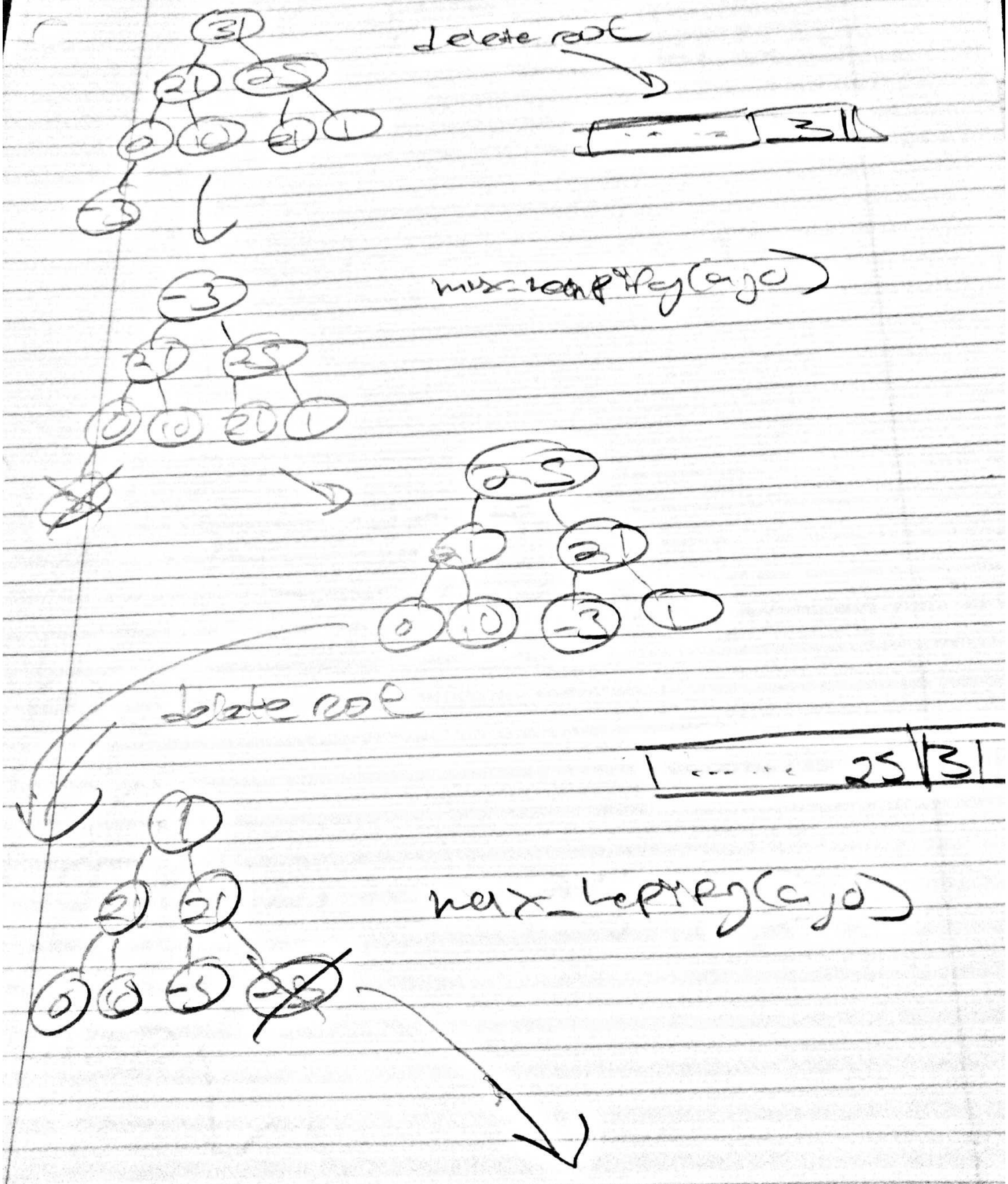


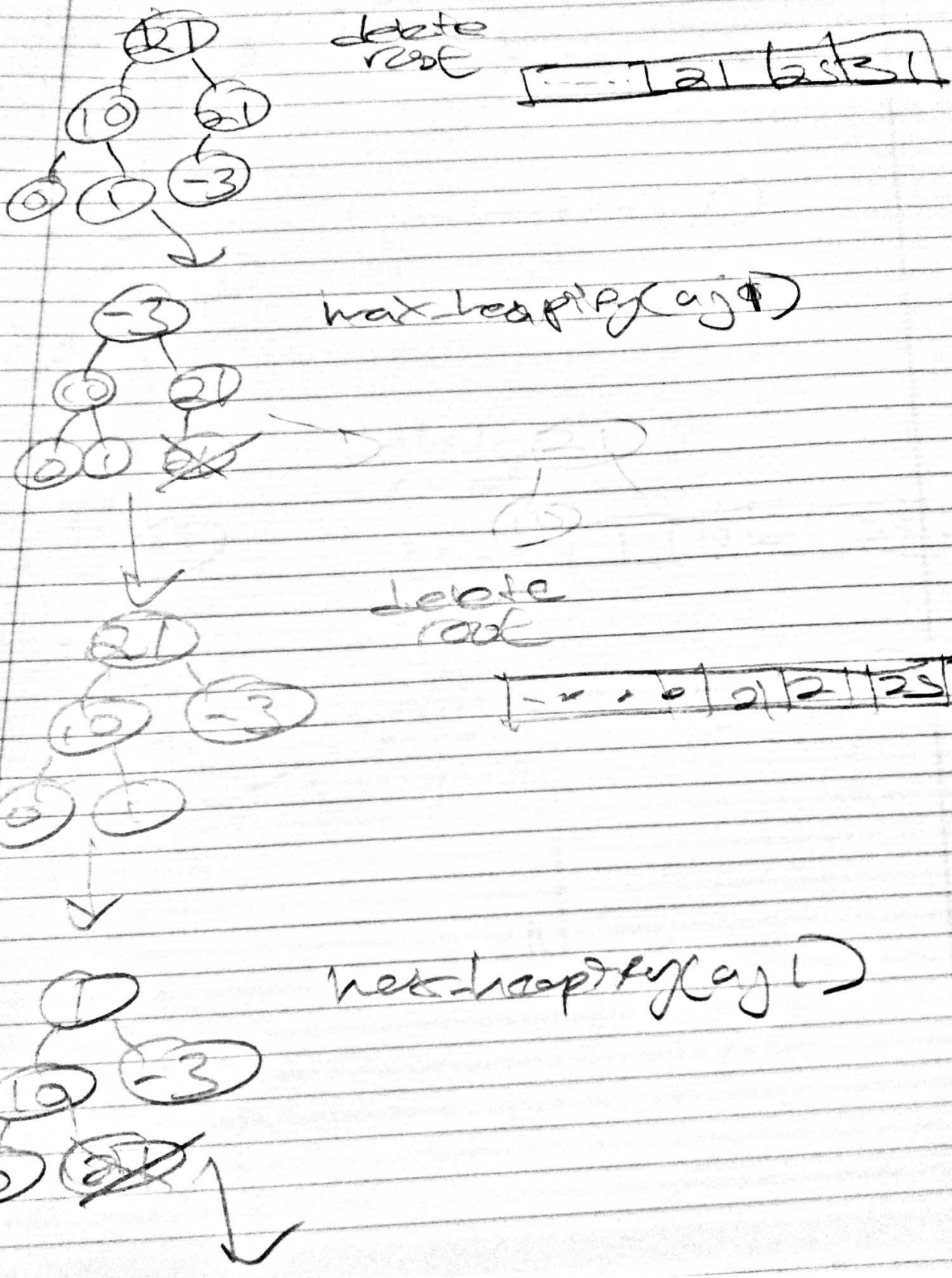


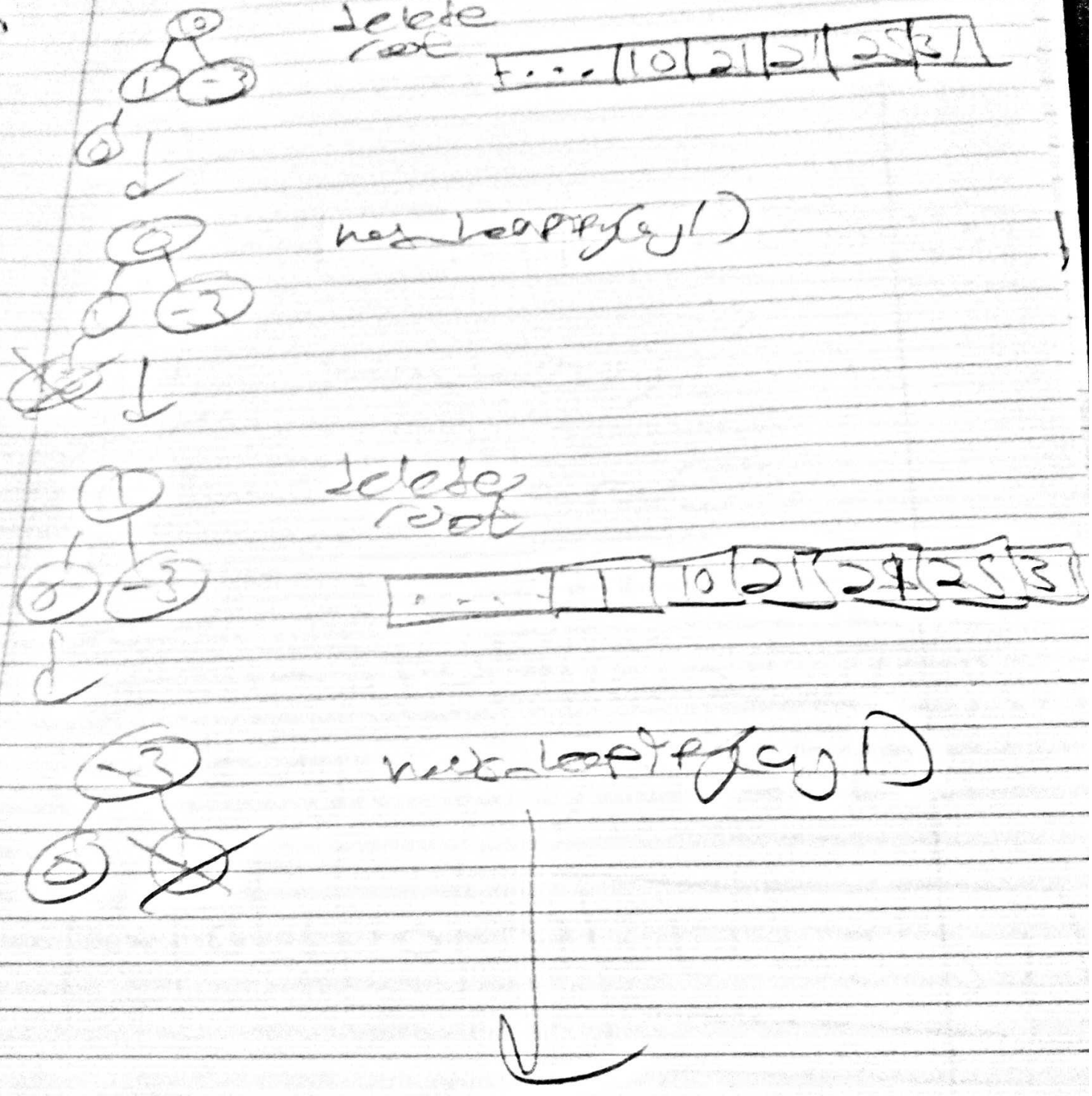
max-heapify(arg 1)

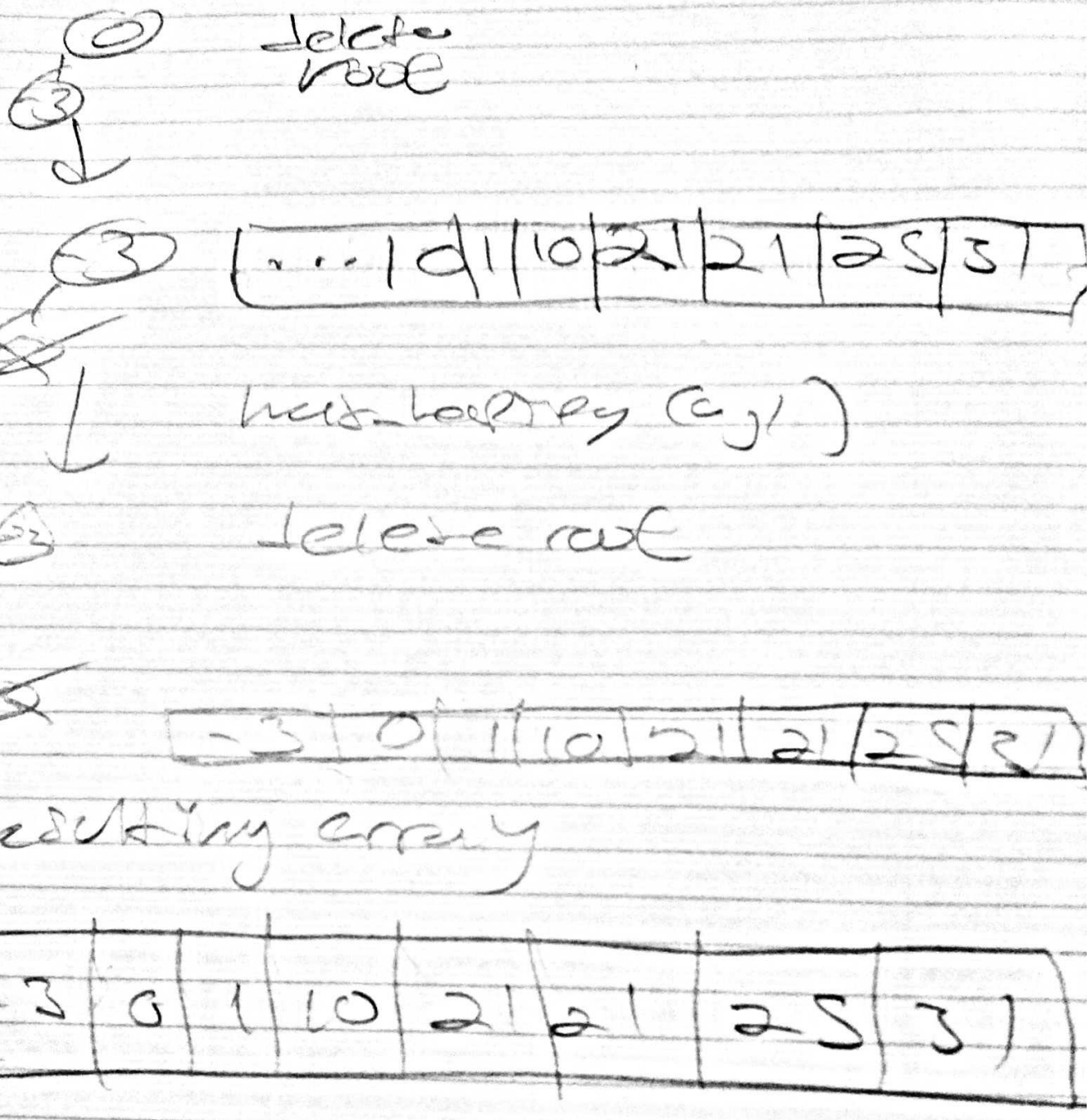




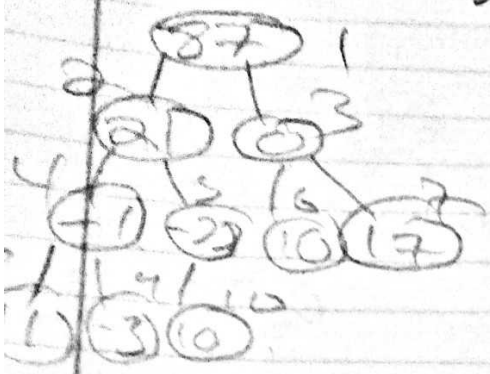

delete root
31
max_comp(key)
delete root
max_left(key)


delete root
max_heapify(arr, 0)
delete root
max_heapify(arr, 0)

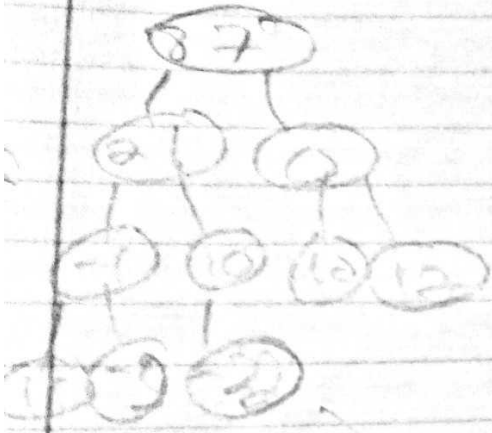

delete root
next_leaf(cur)
delete root
next_leaf(cur)


delete root
-3 0 1 10 21 21 25 31
next_heapify(arr)
delete root
-3 0 1 10 21 21 25 31
resulting array
-3 0 1 10 21 21 25 31

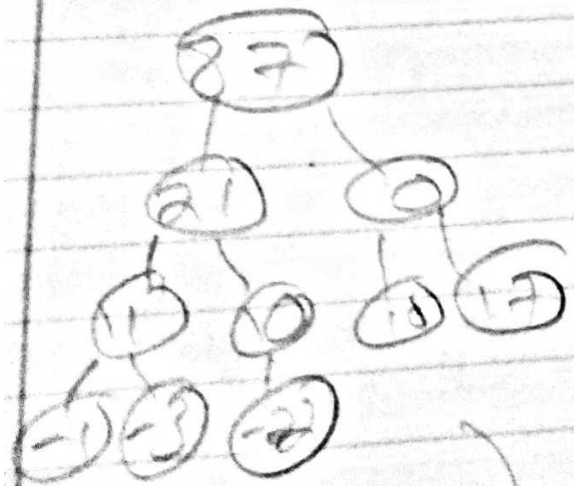
8.2 87, 21, 0, -1, -22, 10, 17, 1, -3, 0



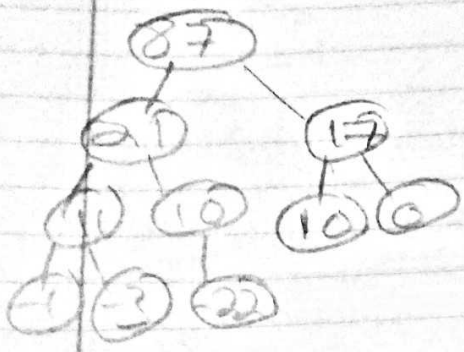
max_heapify(A, 5)



max_heapify(A, 4)



max_heapify(A, 3)

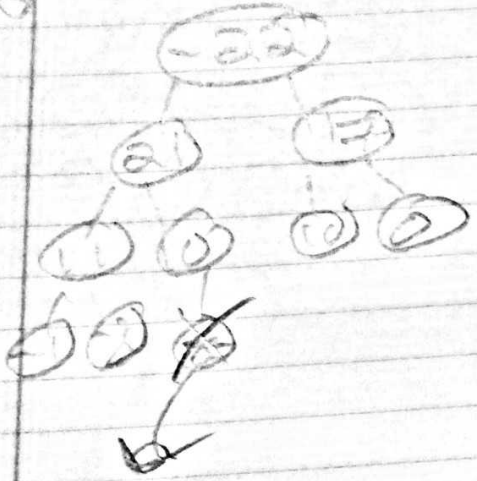


max_heapify(a, 2) ✓

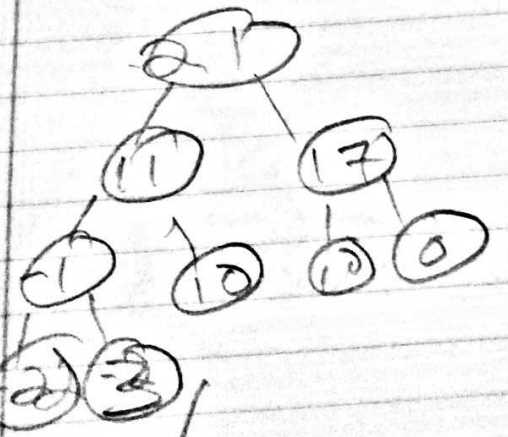
max_heapify(a, 1) ✓

delete root

heap	87

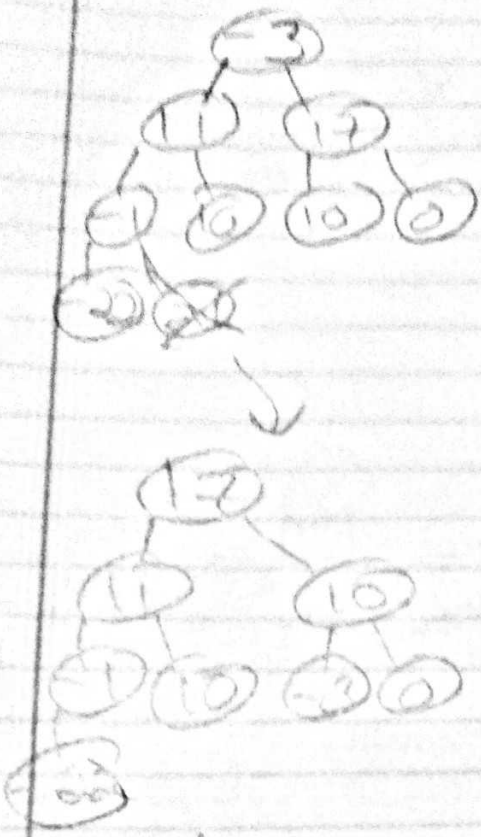


max-heapify(a, 1)



delete root

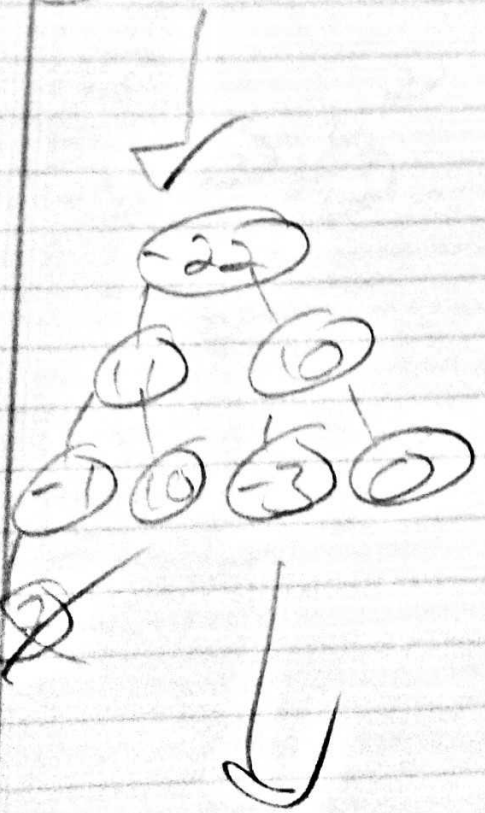
heap	21	87



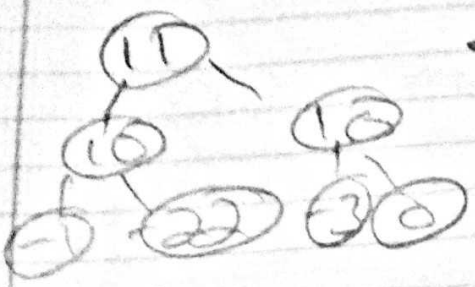
max-heapify(A, 1)

delete root

heap | 17 | 10 | 8 | 7



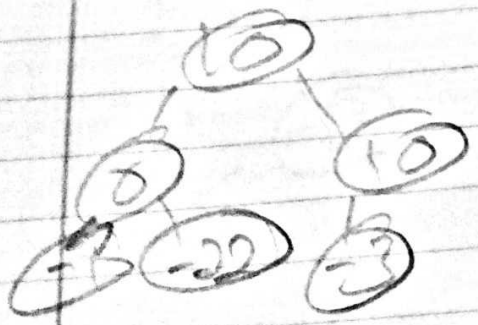
max-heapify(A, 2)



delete root

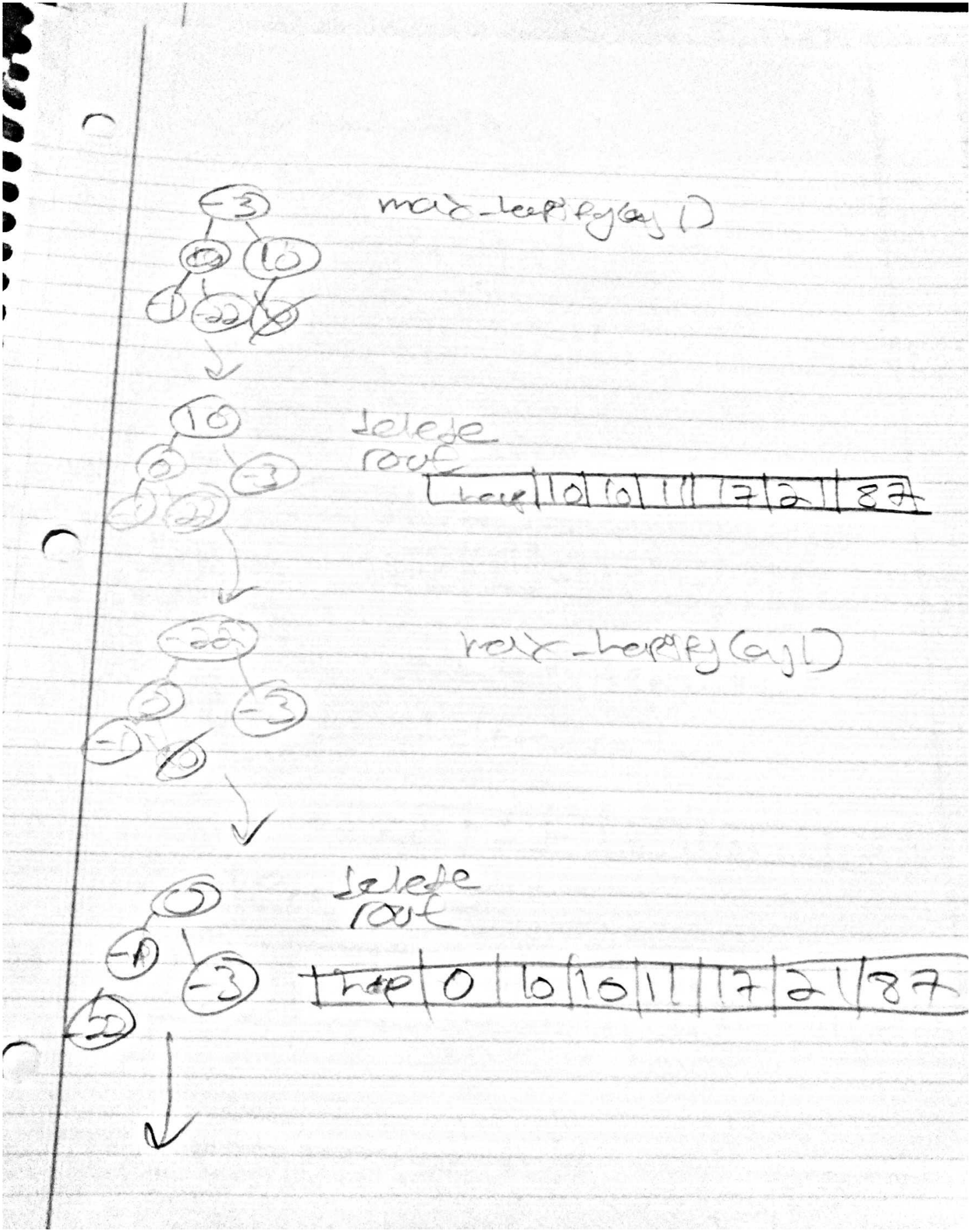
heap	11	17	21	87

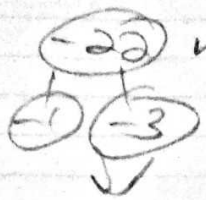
max-heapify(0)



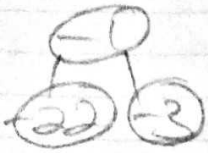
dele. root

heap	10	11	17	21	87


max_heapify(a,1)
delete root
max_heapify(a,1)
delete root



max_heapify(arr, 1)



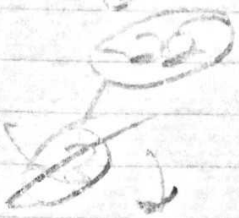
delete root

Leaf	-1	0	10	10	11	17	21	87

max_heapify(arr, 1) ✓

delete root

Leaf -3	-1	0	10	10	11	17	21	87



max_heapify(arr, 1) ✓

delete root

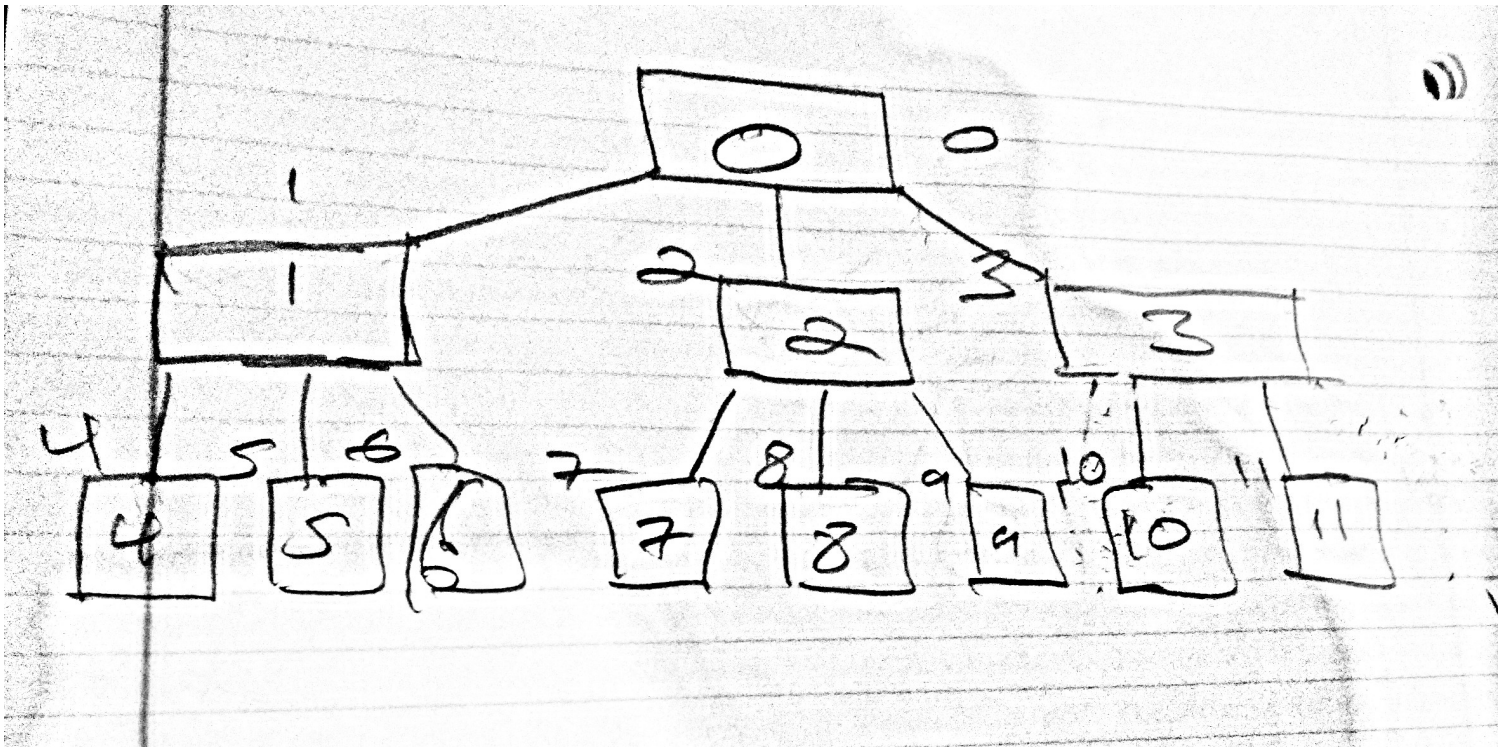
-22	-3	-1	0	10	10	11	17	21	87

resulting array

-22	-3	-1	0	10	10	11	17	21	87

9. Suppose that instead of binary heaps, we wanted to work with ternary heaps. Suggest an appropriate indexing scheme so that a complete tree will yield a contigious sequence.

$$\begin{aligned} left\_child &= 3 \cdot i + 1 \\ middle\_child &= 3 \cdot i + 2 \\ right\_child &= 3 \cdot i + 3 \\ parent &= \lfloor \frac{i-1}{3} \rfloor \end{aligned}$$


0
1
2
3
4
5
6
7
8
9
10
11

10. Use induction to prove that $1 + 2 + 4 + \ldots + 2^h = 2^{h+1} - 1$.

Proof. We prove by induction

Let $P(h) = 1 + 2 + 4 + \ldots + 2^h = 2^{h+1} - 1$

Base case: $h = 0$

$1 = 1 = 2^{0+1} - 1$.

Thus $P(0)$ holds.

Inductive step: Let $P(k)$ be true we show that $P(k+1)$ is true, that is $1 + 2 + 4 + \ldots + 2^k = 2^{(k+1)+1} - 1$

$$\begin{aligned} 1 + 2 + 4 + \ldots + 2^k + 2^{k+1} &= \sum_{i=0}^{k+1} 2^i \\ &= \sum_{i=0}^{k} 2^i + 2^{k+1} \\ &= 2^{k+1} - 1 + 2^{k+1} \\ &= 2 \cdot 2^{k+1} - 1 \\ &= 2^{k+1+1} - 1 \\ &= 2^{(k+1)+1} - 1 \end{aligned}$$

Thus $P(k+1)$ holds. By the principle of mathematical induction, $P(h)$ holds for for all integers $h \geq 0$. □

11. What is the minimum and maximum number of leaves in a binary heap that has height h. Explain.

When $h = 0$ then the minimum equal the maximum namely 1. Suppose that $h \geq 1$. The minimum case is where there is only one left child in the level h. The rest of the leaves are on the $h-1$ level where the leftmost node is the only parent. Thus there are a total of $2^{h-1} - 1 + 1 = 2^{h-1}$ minimum number of leafs for height h. In a perfect tree where all levels are filled we have $\sum_{i=0}^{h} 2^i = 2^{h+1} - 1$ total nodes where the leafs are the last summation term and thus contribute 2^h nodes. Thus the minimum and maximum number of leaves in a binary heap that has height h is 2^{h-1} and 2^h leaves for $h \geq 1$.

12. Prove that a binary heap with n elements has height $\lfloor log_2(n) \rfloor$.

$$\begin{aligned} 2^h \leq \quad & n \quad && \leq 2^{h+1} - 1 \\ 2^h \leq \quad & n \quad && < 2^{h+1} \\ log_2(2^h) \leq \quad & log_2(n) \quad && < log_2(2^{h+1}) \\ h \leq \quad & log_2(n) \quad && < h + 1 \end{aligned}$$

which by definition of the floor $h = \lfloor log_2(n) \rfloor$. Hence we conclude a binary heap with n elements has height $\lfloor log_2(n) \rfloor$.

13. Prove that a binary heap with n nodes has exactly $\lceil \frac{n}{2} \rceil$ leaves.

Version 1:

The nth element of the heap is the last node. If the nth element is a left child then it can be expressed as $n = 2 \cdot i$ where i is the index value of the parent node. solving for i we obtain $\frac{n}{2} = \lfloor \frac{n}{2} \rfloor = i$. If the nth element is a right child then it can be expressed as $n = 2 \cdot i + 1$. Solving for i we obtain $\frac{n-1}{2} = \lfloor \frac{n}{2} \rfloor = i$. We conclude the parent index of the last node can be expressed as $\lfloor \frac{n}{2} \rfloor$. The parent is also the last internal node and thus the last internal node has index $\lfloor \frac{n}{2} \rfloor$. n can be expressed as the sum of $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$ which when solved for $\lceil \frac{n}{2} \rceil$ gives

$$\begin{aligned} n &= \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil \\ n - \lfloor \frac{n}{2} \rfloor &= \lceil \frac{n}{2} \rceil. \end{aligned}$$

Hence the number of elements n subtracted by the last internal node index tells us the amount of leaves in the heap. We conclude that binary heap with n nodes has exactly $\lceil \frac{n}{2} \rceil$ leaves.

Version 2:

We prove by induction. Let $P(n)$ be the statement that a binary heap with n nodes has exactly $\lceil \frac{n}{2} \rceil$ leaves. $P(1)$ holds because $\lceil \frac{1}{2} \rceil = 1$ leaf, which is namely the root. We assume $P(k)$ holds for some integer k and show that is must hold for the $k+1$ integer. Case 1: the $\lfloor \frac{k}{2} \rfloor$ internal node has one child, then the next node will be its right child.

$$\begin{aligned} \lceil \frac{k}{2} \rceil + 1 &= \lceil \frac{k+2}{2} \rceil \\ &= \frac{k+1}{2} \ (since \ \ k+2 \, is \, odd) \\ &= \lceil \frac{k+1}{2} \rceil \end{aligned}$$

which proves the 1^{st} case.

Case 2: The $\lfloor \frac{k}{2} \rfloor$ internal node has two children, then the next node will be the left child of the $\lfloor \frac{k+1}{2} \rfloor$. Prior to the insertion, by the inductive hypothesis the binary heap has $\lceil \frac{k}{2} \rceil$ leaves. Adding the $k+1$ node will add one leaf but will make the parent of the node no longer a leaf so the tree still has $\lceil \frac{k}{2} \rceil$ leaves. Since k is odd this can be expressed as $\frac{k+1}{2}$ which is equivalent to $\lceil \frac{k+1}{2} \rceil$ which proves the 2^{nd} case.

Thus $P(k+1)$ holds for both cases. By the principle of mathematical induction, $P(n)$ holds for all integers $n \geq 1$.

14. Show that there are at most $\lceil \frac{n}{2^{h+1}} \rceil$ nodes of height h in any $n-$element heap.

We prove by induction. Let $P(h)$ be the statement that there are at most $\lceil \frac{n}{2^{h+1}} \rceil$ nodes of height h in any n-element heap. $P(0)$ holds since $\lceil \frac{n}{2^1} \rceil$ are the number of leaves in the heap which by definition have height 0. We assume $P(k)$ holds for some integer k and show that it must hold for the $k+1$ integer. We have that there at most $\lceil \frac{n}{2^{k+1}} \rceil$ nodes of height k in any $n-$element heap. We make a new heap H' which haves all the elements of the original heap H except the leaves, hence it has $\lfloor \frac{n}{2} \rfloor$ total elements. The nodes of height k in this heap will be of height $k+1$ in the original heap. $\lceil \frac{\lfloor \frac{n}{2} \rfloor}{2^{k+1}} \rceil \leq \lceil \frac{\frac{n}{2}}{2^{k+1}} \rceil = \lceil \frac{n}{2^{(k+1)+1}} \rceil$. Hence $P(k+1)$ holds. By the principle of mathematical induction, $P(h)$ holds for all integers $h \geq 0$.