Divide and Conquer Algorithms

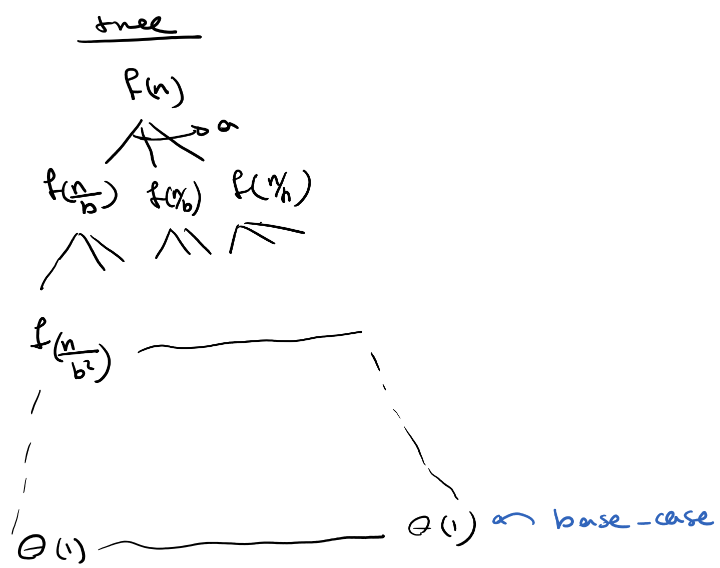
Thursday, October 15, 2020 5:00 PM

Reminder: HW5 is due this Sunday

$$T(n) = a\,T\left(\frac{n}{b}\right) + f(n)$$

- n : input size
- a : nr of sub-problems
- $\frac{n}{b}$: input size of EACH sub-problem
- $f(n)$: amt work (time) to divide & combine the problem into a sub-problems
- $T(n)$: time to run (solve) a problem of size n

step	size
0	n
1	$\frac{n}{b}$
2	$\frac{n}{b^2}$
k	$\frac{n}{b^k} = 1$



$n = b^k$

$k = \log_b n$

$$T(n) = a^0 f\left(\frac{n}{b^0}\right) + a\,f\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + \dots + a^{k-1} f\left(\frac{n}{b^{k-1}}\right) + a^k\, \underbrace{\cancel{\Theta(1)}}_{nr}$$

$$\approx \sum_{i=0}^{k-1} a^i f\left(\frac{n}{b^i}\right) + a^k$$

$$= \sum_{i=0}^{(\log_b n)-1} a^i f\left(\frac{n}{b^i}\right) + a^{\log_b n}$$

$$= \sum_{i=0}^{(\log_b n)-1} a^i f\left(\frac{n}{b^i}\right) + n^{\log_b a}$$

Example:

① $T(n) = 6T\left(\frac{n}{2}\right) + n^4$

$a = 6$

$b = 2$

$f(n) = n^4$

$k = \log_b n = \log n$

$$T(n) = \sum_{i=0}^{k-1} 6^i f\left(\frac{n}{2^i}\right) + n^{\log 6}$$

$$= \sum_{i=0}^{k-1} 6^i \left(\frac{n}{2^i}\right)^4 + n^{\log 6}$$

$$= n^4 \sum_{i=0}^{k-1} \left(\frac{6}{16}\right)^i + n^{\log 6}$$

$$= n^4 \sum_{i=0}^{k-1} \left(\frac{3}{8}\right)^i + n^{\log 6}$$

$$= n^4 \frac{\left(3/8\right)^{k \to 0} - 1}{3/8 - 1} + n^{\log 6}$$

+nv

$$\simeq n^4 + n^{\log 6} = n^4 + n^{2.\sim} = \boxed{\Theta(n^4)}$$

$\underbrace{\log 4}_{2} < \log 6 < \underbrace{\log 8}_{3}$

② $T(n) = 8T\left(\frac{n}{4}\right) + \sqrt{n}$

Selina

② $T(n) = 8T\left(\frac{n}{4}\right) + \sqrt{n}$

$a = 8$

$b = 4$

$f(n) = \sqrt{n}$

$k = \log_b a = \log_4 n$

$$T(n) = \sum_{i=0}^{k-1} 8^i f\left(\frac{n}{4^i}\right) + n^{\log_4 8}$$

$$= \sum_{i=0}^{k-1} 8^i \left(\frac{n}{4^i}\right)^{1/2} + n^{\log_4 8}$$

$$= \sqrt{n} \sum_{i=0}^{k-1} \left(\frac{8}{2}\right)^i + n^{\log_4 8}$$

$$= \sqrt{n} \sum_{i=0}^{k-1} 4^i + n^{\log_4 8}$$

$$= \sqrt{n} \left(\frac{4^k - 1}{4-1}\right) + n^{\log_4 8}$$

$$= \sqrt{n}\,(4^k) + n^{3/2}$$

$$= \theta(n^{3/2})$$

$n^{1/2}\, 4^{\log_4 n}$

$= n^{1/2}\, n^{\log_4 4} = n^{1.5}$

very nice :)

$\frac{\log_2 8}{\log_2 4} = \frac{3}{2}$

③ $T(n) = 4T\left(\frac{n}{10}\right) + 1$

$a = 4$

$b = 10$

$f(n) = 1$

$k = \log_b n = \log_b n$

$f\left(\frac{n}{10}\right) = 1$

$f(\text{pavage}) = 1$

$f(\sim) = 1$

$$T(n) = \sum_{i=0}^{k-1} a^i f\left(\frac{n}{b^i}\right) + a^{\log_b n}$$

$$= \sum_{i=0}^{k-1} 4^i f\left(\frac{n}{10^i}\right) + 4^{\log_{10} n}$$

$$= \sum_{i=0}^{k-1} 4^i + n^{\log_{10} 4}$$

$$= \frac{4^k - \cancel{1}}{\cancel{4} - 1} + n^{\log_{10} 4}$$

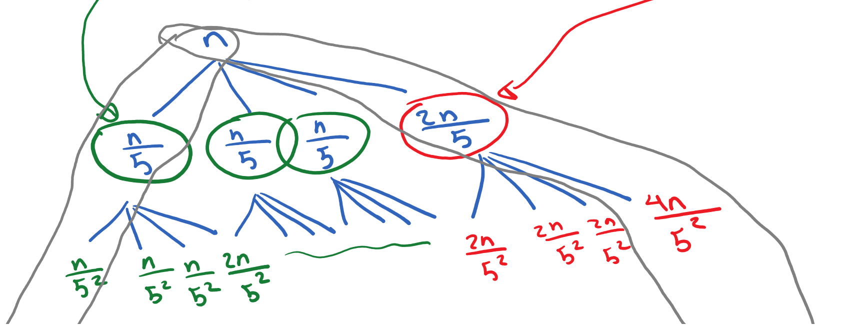
$$\simeq 4^k + n^{\log_{10} 4} = 4^{\log_{10} n} + n^{\log_{10} 4}$$

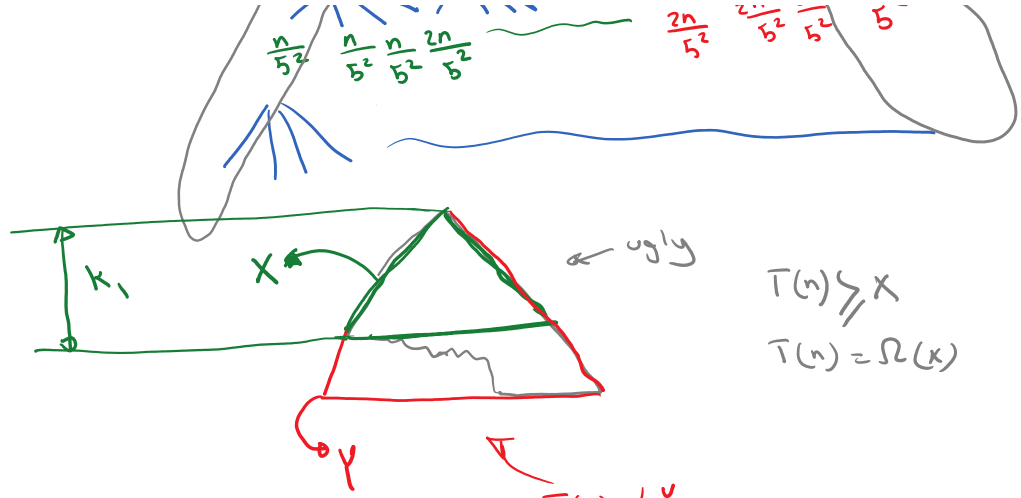
$$= n^{\log_{10} 4} + n^{\log_{10} 4} = \boxed{\theta(n^{\log_{10} 4})}$$

Example: $T(n) = 3T\left(\frac{n}{5}\right) + T\left(\frac{2n}{5}\right) + n \Rightarrow T\left(\boxed{\tfrac{n}{5}}\right) = 3T\left(\frac{\boxed{n/5}}{5}\right) + T\left(\frac{2\,\boxed{n/5}}{5}\right) + \boxed{\tfrac{n}{5}}$

$T\left(\boxed{\tfrac{2n}{5}}\right) = 3T\left(\frac{\boxed{2n/5}}{5}\right) + T\left(\frac{2\,\boxed{2n/5}}{5}\right) + \boxed{\tfrac{2n}{5}}$

step	size	tree
0	n	
1	?	





$T(n) \geq X$

$T(n) = \Omega(X)$

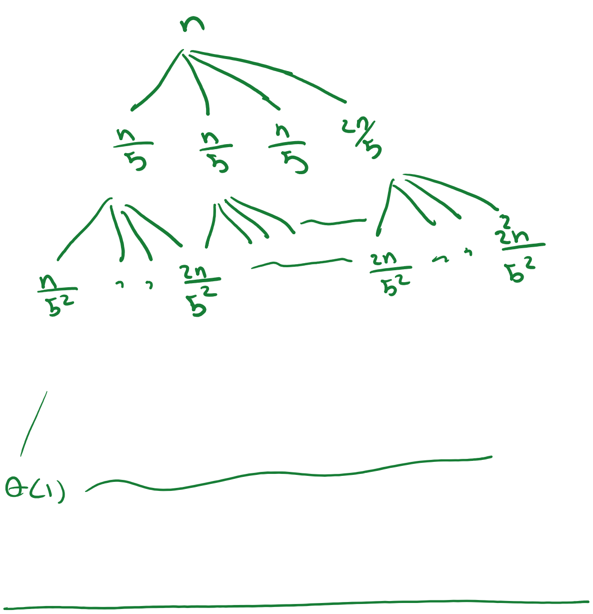
$T(n) \leq Y$

$T(n) = O(Y)$

Green Tree

$$T(n) = 3T\left(\frac{n}{5}\right) + T\left(\frac{2n}{5}\right) + n$$

Step	size of leftmost	tree
0	n	n
1	$\frac{n}{5}$	$\frac{n}{5}$ $\frac{n}{5}$ $\frac{n}{5}$ $\frac{2n}{5}$
2	$\frac{n}{5^2}$	$\frac{n}{5^2}$, , $\frac{2n}{5^2}$ —— $\frac{2n}{5^2}$, , $\frac{2n}{5^2}$
k_1	$\frac{n}{5^{k_1}} = 1$	$\Theta(1)$



$k_1 = \log_5 n$

Sol 1 $T(n) \geq n + n + n + \cdots + n$

$\geq (k_1+1)n \simeq n \log_5 n \Rightarrow T(n) = \Omega(n \log n)$

$\log_5 n = \frac{\log n}{\cancel{\log 5}}$

You only need to do one of them :)

Sol 2 $T(n) \geq n + n + n \cdots + n + 4^{k_1}$

$\geq \sum_{i=0}^{k_1-1} n + 4^{k_1}$

$$\geq \sum_{i=0}^{k_1-1} n \quad + 4^{k_1}$$

$$\geq nk_1 + 4^{k_1} = n\log_5 n + 4^{\log_5 n}$$

$$= n\log_5 n + n^{\cancel{\log_5 4}\ 0.\sim}$$

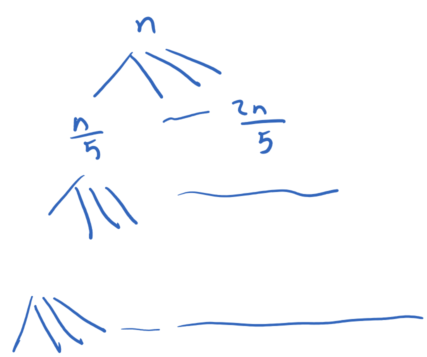
$$= \boxed{\Omega(n\log n)}$$

Rec Tree

$$T(n) = 3T\left(\frac{n}{5}\right) + T\left(\frac{2n}{5}\right) + n$$

step	size of right most
0	n
1	$\frac{2n}{5}$
2	$\frac{2^2 n}{5^2}$
$\vdots$	
k_2	$\frac{2^{k_2} n}{5^{k_2}} = 1$

tree



$$\left(\frac{2}{5}\right)^{k_2} n = 1$$

$$n = \left(\frac{5}{2}\right)^{k_2}$$

$$k_2 = \log_{5/2} n$$

Sol^n

$$T(n) \leq n + n + n + \cdots + n$$

$$\leq (k_2+1)n \simeq nk_2$$

$$T(n) = O(n\log_{5/2} n) = \boxed{O(n\log n)}$$

$K_2 = \log_{5/2} n$

$T(n) = O(n \log_{5/2} n) = O(n \log n)$

Sol2 $T(n) \le n + n + \cdots + n + 4^{K_2}$

$\le nK_2 + 4^{K_2}$

$\le n \log_{5/2} n + 4^{\log_{5/2} n}$

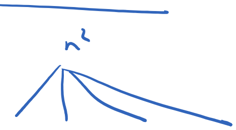
$\log_{2.5} 2.5 < \log_{5/2} 4 < \log_{2.5} (2.5)^2$

$\le n \log_{5/2} n + n^{\log_{5/2} 4}$

$= O\left(n^{\log_{5/2} 4}\right)$

HW5 Q8h

Example: $T(n) = 4T\left(\left\lfloor \frac{n}{2} \right\rfloor + 2\right) + n^2$

step	size	tree
0	n	
1	$\left\lfloor \frac{n}{2} \right\rfloor + 2$	
2	$\left\lfloor \frac{\boxed{\left\lfloor \frac{n}{2} \right\rfloor + 2}}{2} \right\rfloor + 2$	
	$\left\lfloor \frac{\frac{n}{2} + 2}{2} \right\rfloor + 2$ ← simplify	
3	?	
4	?	
$\vert$		
k		

EC question

$$T(n) = 2T\left(\left\lfloor \frac{n}{3} \right\rfloor + 1\right) + n^3$$