1. Prove that $f(n) = 10 \cdot n^4 + 2 \cdot n^2 + 3$ is $O(n^4)$, provide the appropriate C and k constants.

$$f(n) = O(g(n)) \iff \exists c > 0, \exists k \geq 0 \; s.t \; f(n) \leq c \cdot g(n) \, \forall n \geq k$$

$$\begin{aligned} 10 \cdot n^4 + 2 \cdot n^2 + 3 &\leq 10 \cdot n^4 + 2 \cdot n^4 + 3 \\ &\leq 10 \cdot n^4 + 2 \cdot n^4 + 3 \cdot n^4 \\ &= 15 \cdot n^4, n \geq 1 \end{aligned}$$

Let $c = 15$ and $k = 1$ then the statement translates to

$$f(n) \leq c \cdot g(n), n \geq k$$

By the definition of O notation $f(n) = O(g(n))$

2. Prove that $f(n) = 2 \cdot n^2 - n \cdot log_2(n) + 3 \cdot log_2(n)$ is $O(n^2)$, provide the appropriate C and k constants.

$$f(n) = O(g(n)) \iff \exists c > 0, \exists k \geq 0 \; s.t \; f(n) \leq c \cdot g(n) \, \forall n \geq k$$

$$\begin{aligned} 2 \cdot n^2 - n \cdot log_2(n) + 3 \cdot log_2(n) &\leq 2 \cdot n^2 + n^2 + 3 \cdot log_2(n) \\ &\leq 2 \cdot n^2 + n^2 + 3 \cdot n^2 \\ &= 6 \cdot n^2, n \geq 1 \end{aligned}$$

Let $c = 6$ and $k = 1$, then the statement translates to

$$f(n) \leq c \cdot g(n), n \geq k$$

which by the definition of O notation $f(n) = O(g(n))$.

3. Prove that $f(n) = 2 \cdot n^4 \cdot log_2(n^4) - n^2 + 3 \cdot log_2(n)$ is $O(n^4 log_2(n))$, provide the appropriate C and k constants.

$$f(n) = O(g(n)) \iff \exists c > 0, \exists k \geq 0 \; s.t \; f(n) \leq c \cdot g(n) \, \forall n \geq k$$

$$
\begin{aligned}
2 \cdot n^4 \cdot log_2(n^4) - n^2 + 3 \cdot log_2(n) &= 8 \cdot n^4 \cdot log_2(n) - n^2 + 3 \cdot log_2(n) \\
&\leq 8 \cdot n^4 \cdot log_2(n) + n^4 \cdot log_2(n) + 3 \cdot log_2(n) \\
&\leq 8 \cdot n^4 \cdot log_2(n) + n^4 \cdot log_2(n) + 3 \cdot n \cdot log_2(n) \\
&= 12 \cdot n^4 \cdot log_2(n), n \geq 1
\end{aligned}
$$

Let $c = 12$ and $k = 1$ then the statement translates to

$$2 \cdot n^4 \cdot log_2(n^4) - n^2 + 3 \cdot log_2(n) \quad \leq \quad c \cdot n^4 \cdot log_2(n), n \geq 1$$

which by the definition O notation $f(n) = O(g(n))$.

4. Prove or disprove $f(n) = 5 \cdot n^3 - n + 3$

1. $f(n) = O(n^2)$
2. $f(n) = \Omega(n)$
3. $f(n) = \Theta(n^3)$
4. $f(n) = \omega(n)$
5. $f(n) = o(n^2)$

Provide the appropriate C and k constants if possible

5. Prove that $(n+5)^{100} = \Theta(n^{100})$

$$f(n) = \Theta(g(n)) \quad \iff \quad f(n) = O(g(n))\, and\, f(n) = \Omega(g(n))$$

1. Showing $(n+5)^{100} = O(n^{100})$

$$f(n) = O(g(n)) \quad \iff \quad \exists c > 0, \exists k \geq 0\, s.t\, f(n) \leq c \cdot g(n)\, \forall n \geq k$$

$$
\begin{aligned}
(n+5)^{100} &= \sum_{i=0}^{100} \binom{100}{i} \cdot n^{100-i} \cdot 5^i \\
&\leq \sum_{i=0}^{100} \binom{100}{i} \cdot n^{100}, n \geq 1
\end{aligned}
$$

Let $c = \sum_{i=0}^{100} \binom{100}{i}$ and $k = 1$ then the statements translates to

$$f(n) \quad \leq \quad c \cdot g(n), n \geq k$$

which by the definition of O notation $f(n) = O(g(n))$.

2. Showing $(n+5)^{100} = \Omega(n^{100})$

$$f(n) = \Omega(g(n)) \quad \iff \quad \exists c > 0, \exists k \geq 0 \, s.t \, f(n) \geq c \cdot g(n) \, \forall n \geq k$$

$$\begin{aligned} (n+5)^{100} &= \sum_{i=0}^{100} \binom{n}{i} \cdot n^{100-i} \cdot 5^i \\ &\geq n^{100} \end{aligned}$$

Let $c = 1$ and $k = 1$ then the statement translates to

$$f(n) \quad \geq \quad c \cdot g(n), n \geq k$$

which by the definition of Ω notation $f(n) = \Omega(g(n))$.

6. Prove transitivity of big-O: if $f(n) = O(g(n))$, and $g(n) = O(h(n))$, then $f(n) = O(h(n))$.

7. Prove that $f(n) = O(g(n)) \iff g(n) = \Omega(f(n))$.

Forward direction: $f(n) = O(g(n)) \implies g(n) = \Omega(f(n))$.

Since $f(n) = O(g(n))$ there exists a number c and a number k such that $f(n) \leq c \cdot g(n), n \geq k$ where $c > 0$ and $k \geq 0$. From this we obtain $\frac{1}{c} \cdot f(n) \leq g(n), n \geq k$. Which by the definition of Ω notation, $g(n) = \Omega(f(n))$.

Backward direction: $g(n) = \Omega(f(n)) \implies f(n) = O(g(n))$

Since $g(n) = \Omega(f(n))$ there exists a number c and a number k such that $g(n) \geq c \cdot f(n), n \geq k$ where $c > 0$ and $k \geq 0$. From this we obtain $g(n) \cdot \frac{1}{c} \geq f(n), n \geq k$. Which by the definition of O notation, $f(n) = O(g(n))$.

We conclude $f(n) = O(g(n)) \iff g(n) = \Omega(f(n))$.

8. Compare the growth of

9. Prove or disprove $2^{n+1} = O(2^n)$

$$f(n) = O(g(n)) \iff \exists c > 0, \exists k \geq 0 \, s.t \, f(n) = c \cdot g(n) \, \forall n \geq k$$

$$\begin{aligned} 2^{n+1} &= 2 \cdot 2^n \\ &\leq 3 \cdot 2^n, n \geq 1 \end{aligned}$$

Let $c = 3$ and $k = 1$ then the statement translates to

$$2^{n+1} \leq c \cdot 2^n, n \geq k$$

which by the definition of O notation, $2^{n+1} = O(2^n)$.

10. Prove or disprove $2^{2 \cdot n} = O(2^n)$

$$f(n) = O(g(n)) \iff \exists c > 0, \exists k \geq 0 \, s.t \, f(n) = c \cdot g(n) \, \forall n \geq k$$

$$\begin{aligned} 2^{2 \cdot n} &\leq c \cdot 2^n, n \geq k \\ 2^n &\leq c \end{aligned}$$

which is a contradiction, hence $2^{2 \cdot n} \neq O(2^n)$.

11. Prove that if $lim_{n \to \infty} \frac{f(n)}{g(n)} = C$, for some constant $C > 0$ then $f(n) = \Theta(g(n))$.

Since $lim_{n \to \infty} \frac{f(n)}{g(n)} = C$, for every $\epsilon > 0$, there exists $k \geq 0$ such that, for all $n \geq k$, $|\frac{f(n)}{g(n)} - C| < \epsilon$. From this we obtain

$$\begin{aligned} -\epsilon < \quad & \frac{f(n)}{g(n)} - C \quad < \epsilon \\ C - \epsilon < \quad & \frac{f(n)}{g(n)} < \quad C + \epsilon \\ g(n) \cdot (C - \epsilon) < \quad & f(n) < \quad g(n) \cdot (C + \epsilon), n \geq k \end{aligned}$$

Let $c_1 = (C - \epsilon)$ and $c_2 = C + \epsilon$ then the statement translates to

$$c_1 \cdot g(n) < \quad f(n) < \quad c_2 \cdot g(n)$$

which by the definition of Θ notation, $f(n) = \Theta(g(n))$.