1. What is the growth of the below functions?

1. $f(n) = 2 \cdot n^4 \cdot log_2(n^4) + n^{4.0001} - 3 \cdot log_2(n)$

$$f(n) = \Theta(n^{4.0001})$$

2. $f(n) = 3 \cdot n^3 \cdot log(n^4 - n^2) + 100000$

$$f(n) = \Theta(n^3 \cdot log_2(n))$$

3. $f(n) = log_2^{100}(n^{50}) + n$

$$f(n) = \Theta(n)$$

4. $f(n) = n^4 \cdot log_2^3(n) + 4$

$$f(n) = \Theta(n^4 \cdot log_2^3(n))$$

5. $f(n) = 10000 \cdot n \cdot log_2(n^7) + 3 \cdot log_2(n) + 1000 \cdot \sqrt{n}$

$$f(n) = \Theta(n \cdot log_2(n))$$

6. $f(n) = \sqrt[10]{n} + 10^{10} \cdot log_2^{100}(n) + 8$

$$f(n) = \Theta(\sqrt[10]{n})$$

7. $f(n) = \sqrt{\sqrt{n}} + 9 \cdot log_2(n)$

$$f(n) = \Theta(\sqrt{\sqrt{n}})$$

2. Discuss the growth of the below functions

1. $f_1(n) = log_2(n)^{log_2(n)}$

$$\begin{aligned} lim_{n\to\infty}\frac{f_1(n)}{f_2(n)} &= lim_{n\to\infty}\frac{log_2(n)^{log_2(n)}}{2^{\sqrt{2\cdot log_2(n)}}} \\ &= lim_{n\to\infty}\frac{log_2(n)^{log_2(n)}}{2^{\sqrt{2\cdot log_2(n)}}} \\ &= \infty \end{aligned}$$

$f_1(n) = \omega(f_2(n))$

$$\begin{aligned} lim_{n\to\infty}\frac{f_1(n)}{f_3(n)} &= lim_{n\to\infty}\frac{log_2(n)^{log_2(n)}}{\sqrt{2}^{log_2(n)}} \\ &= lim_{n\to\infty}\frac{log_2(n)^{log_2(n)}}{n^{\frac{1}{2}}} \\ &= \infty \end{aligned}$$

$f_1(n) = \omega(f_3(n))$

$$\begin{aligned} lim_{n\to\infty}\frac{f_1(n)}{f_4(n)} &= \frac{log_2(n)^{log_2(n)}}{n^{\frac{1}{log_2(n)}}} \\ &= \infty \end{aligned}$$

$f_1(n) = \omega(f_4(n))$

2. $f_2(n) = 2^{\sqrt{2\cdot log_2(n)}}$

$$\begin{aligned} lim_{n\to\infty}\frac{f_2(n)}{f_3(n)} &= lim_{n\to\infty}\frac{2^{\sqrt{2\cdot log_2(n)}}}{\sqrt{2}^{log_2(n)}} \\ &= lim_{n\to\infty}\frac{2^{\sqrt{2\cdot log_2(n)}}}{n^{\frac{1}{2}}} \\ &= 0 \end{aligned}$$

$f_2(n) = o(f_3(n))$

$$\begin{aligned} lim_{n\to\infty}\frac{f_2(n)}{f_4(n)} &= lim_{n\to\infty}\frac{2^{\sqrt{2\cdot log_2(n)}}}{n^{\frac{1}{log_2(n)}}} \\ &= \infty \end{aligned}$$

$f_2(n) = \omega(f_4(n))$

3. $f_3(n) = \sqrt{2}^{log_2(n)}$

$$\begin{aligned} lim_{n\to\infty}\frac{f_3(n)}{f_4(n)} &= \frac{\sqrt{2}^{log_2(n)}}{n^{\frac{1}{log_2(n)}}} \\ &= \infty \end{aligned}$$

$f_3(n) = \omega(f_4(n))$

4. $f_4(n) = n^{\frac{1}{log_2(n)}}$

3. Suppose $g(n) \geq 1$ for all n, and that $f(n) \leq g(n) + L$, for some constant L and all n. Prove that $f(n) = O(g(n))$.

$$\begin{aligned} f(n) &\leq g(n) + L \\ &\leq g(n) + g(n) \cdot L \\ &\leq g(n) \cdot (L + 1) \end{aligned}$$

Hence $f(n) = O(g(n))$.

4. Prove or disprove: if $f(n) = O(g(n))$ and $f(n) \geq 1$ and $log(g(n)) \geq 1$ for sufficiently large n then $log(f(n)) = O(log(g(n))$.

$$\begin{aligned} f(n) &\leq c \cdot g(n) \\ log_2(f(n)) &\leq log_2(c) + log_2(g(n)) \\ &\leq log_2(c) \cdot log_2(g(n)) + log_2(g(n)) \\ &= log_2(g(n)) \cdot (log_2(c) + 1) \end{aligned}$$

Hence $log_2(f(n)) = O(log_2(g(n))$.

5. Show that $log_2(n!) = \Theta(n \cdot log_2(n))$

$log_2(n!) = O(n \cdot log_2(n))$

$$\begin{aligned} log_2(n!) &= log_2(\Pi_{i=1}^{n} i) \\ &= \sum_{i=1}^{n} log_2(i) \\ &\leq \sum_{i=1}^{n} log_2(n) \\ &= n \cdot log_2(n), n \geq 1 \end{aligned}$$

Hence $log_2(n!) = O(n \cdot log_2(n))$.

$log_2(n!) = \Omega(n \cdot log_2(n))$

$$
\begin{aligned}
log_2(n!) &= log_2(\Pi_{i=1}^{n} i) \\
&= \sum_{i=1}^{n} log_2(i) \\
&\geq \sum_{i=\frac{n}{2}}^{n} log_2(i) \\
&\geq \sum_{i=\frac{n}{2}}^{n} log_2(i) \\
&\geq \sum_{i=\frac{n}{2}}^{n} log_2(\frac{n}{2}) \\
&= \sum_{i=\frac{n}{2}}^{n} log_2(n) - \sum_{i=\frac{n}{2}}^{n} 1 \\
&= (n - \frac{n}{2} + 1) \cdot log_2(n) - (n - \frac{n}{2} + 1) \\
&= n \cdot log_2(n) - \frac{log_2(n) \cdot n}{2} + log_2(n) + n + \frac{n}{2} - 1 \\
&= \frac{n \cdot log_2(n)}{2} + log_2(n) + n + \frac{n}{2} - 1 \\
&\geq \frac{n \cdot log_2(n)}{2}, n \geq 1
\end{aligned}
$$

Hence $log_2(n!) = \Omega(n \cdot log_2(n))$.

Since $log_2(n!) = O(n \cdot log_2(n))$ and $log_2(n!) = \Omega(n \cdot log_2(n))$ we conclude $log_2(n!) = \Theta(n \cdot log_2(n))$.

6. Prove that $n! = o(n^{n^2})$

$$
\begin{aligned}
lim_{n\to\infty} log_2(\frac{n!}{n^{n^2}}) &= lim_{n\to\infty} log_2(n!) - n^2 \cdot log_2(n) \\
&= -\infty \\
2^{-\infty} &= 0
\end{aligned}
$$

Hence we conclude that $n! = o(n^{n^2})$

7. Prove that $n! = \omega(2^n)$

$$\begin{aligned} lim_{n\to\infty} log_2(\frac{n!}{2^n}) &= lim_{n\to\infty} log_2(n!) - n \\ &= \infty \\ 2^{\infty} &= \infty \end{aligned}$$

Hence we conclude $n! = \omega(2^n)$.

8. Which one of the below functions grow faster?

$$\begin{aligned} f(n) &= 2^{2^n} \\ g(n) &= n! \end{aligned}$$

$$\begin{aligned} lim_{n\to\infty} log_2(\frac{2^{2^n}}{n!}) &= lim_{n\to\infty} 2^n - log_2(n!) \\ &= \infty \\ 2^{\infty} &= \infty \end{aligned}$$

Hence we conclude $f(n)$ grows faster.

9. Provide a closed-form expression for the asymptotic growth of $n + \frac{n}{2} + \frac{n}{3} + \ldots + 1$

$$\begin{aligned} n \cdot \sum_{i=1}^{n} \frac{1}{n} &= \Theta(n \cdot \int_1^n \frac{1}{n} dx) \\ &= \Theta(n \cdot ln(x)) \end{aligned}$$

10. Use the integral theorem to calculate the growth of $1 + 2^k + 3^k + \ldots + n^k$

$$\begin{aligned} \sum_{i=1}^{n} i^k &= \Theta(\int_1^n x^k dx) \\ &= \Theta(x^{n+1}) \end{aligned}$$

11. Use the integral theorem to calculate the growth of $log_2(1) + log_2(2) + log_2(3) + \ldots + log_2(n^4)$

$$\begin{aligned} \sum_{i=1}^{n^4} log_2(i) &= \Theta(\int_1^{n^4} log_2(x) \cdot dx) \\ &= \Theta(n^4 \cdot log_2(n)) \end{aligned}$$

12. Use the integral theorem to calculate the growth of $log_2(1) + 2 \cdot log_2(2) + 3 \cdot log_2(3) + \ldots + n^2 \cdot log_2(n^2)$

$$\begin{aligned}\sum_{i=1}^{n^2} i \cdot log_2(i) &= \Theta(\int_1^{n^2} x \cdot log_2(x) \cdot dx) \\ &= \Theta(n^4 \cdot log_2(n))\end{aligned}$$

13. Prove or disprove: if $f(n) = O(g(n)) \implies 2^{f(n)} = O(2^{g(n)})$

This is false, consider $f(n) = 2 \cdot n$ and $g(n) = n$. $f(n) = O(g(n))$ but $2^{f(n)} \neq O(2^{g(n)})$.