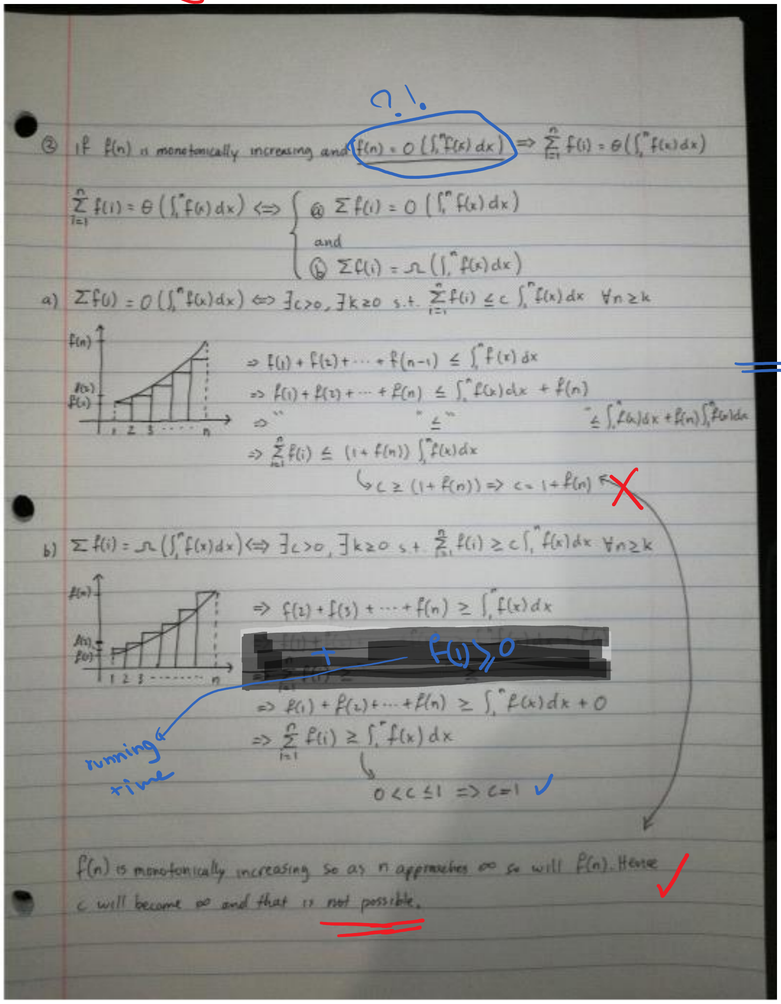
# Lecture 8 (Growth of functions)

Thursday, September 17, 2020 5:00 PM

(Reminder: HW 3 is due this Sunday)

Question that I asked on Tue { Prove the integral theorem when $f(n)$ is monotonically increasing. (Why do we need this extra condition in our proof? $\Rightarrow f(n) = O(\int_1^n f(x)\,dx)$)

Khai's answer



? !

③ If $f(n)$ is monotonically increasing and $f(n) = O(\int_1^n f(x)\,dx) \Rightarrow \sum_{i=1}^{n} f(i) = \Theta(\int_1^n f(x)\,dx)$

$\sum_{i=1}^{n} f(i) = \Theta(\int_1^n f(x)\,dx) \Leftrightarrow \begin{cases} ⓐ\ \sum f(i) = O(\int_1^n f(x)\,dx) \\ \text{and} \\ ⓑ\ \sum f(i) = \Omega(\int_1^n f(x)\,dx) \end{cases}$

a) $\sum f(i) = O(\int_1^n f(x)\,dx) \Leftrightarrow \exists c>0, \exists k\geq 0$ s.t. $\sum_{i=1}^{n} f(i) \leq c\int_1^n f(x)\,dx \quad \forall n \geq k$

$\Rightarrow f(1)+f(2)+\cdots+f(n-1) \leq \int_1^n f(x)\,dx$

$\Rightarrow f(1)+f(2)+\cdots+f(n) \leq \int_1^n f(x)\,dx + f(n)$

$\Rightarrow$ " " $\leq$ " $\leq \int_1^n f(x)\,dx + f(n)\int_1^n f(x)\,dx$

$\Rightarrow \sum_{i=1}^{n} f(i) \leq (1+f(n))\int_1^n f(x)\,dx$

$\hookrightarrow c \geq (1+f(n)) \Rightarrow c = 1+f(n)$ ✗

b) $\sum f(i) = \Omega(\int_1^n f(x)\,dx) \Leftrightarrow \exists c>0, \exists k\geq 0$ s.t. $\sum_{i=1}^{n} f(i) \geq c\int_1^n f(x)\,dx \quad \forall n \geq k$

$\Rightarrow f(2)+f(3)+\cdots+f(n) \geq \int_1^n f(x)\,dx$

[redacted] $+$ $f(1) \geqslant 0$

$\Rightarrow f(1)+f(2)+\cdots+f(n) \geq \int_1^n f(x)\,dx + 0$

$\Rightarrow \sum_{i=1}^{n} f(i) \geq \int_1^n f(x)\,dx$

$\hookrightarrow 0 < c \leq 1 \Rightarrow c = 1$ ✓

running time

$f(n)$ is monotonically increasing so as $n$ approaches $\infty$ so will $f(n)$. Hence $c$ will become $\infty$ and that is not possible. ✓

①

$f(1)+\cdots+f(n-1) \leq \int_1^n f(x)\,dx$

②

$f(n) = O(\int_1^n f(x)\,dx) \Leftrightarrow f(n) \leq c_1\int_1^n f(x)\,dx \quad \exists c>0$

① + ②

$$\underbrace{f(1)+f(2)+\cdots+f(n)}_{\downarrow\ \sum_{i=1}^{n} f(i)} \leq \underbrace{\int_1^n f(x)\,dx}_{a} + c_1\underbrace{\int_1^n f(x)\,dx}_{a}$$

$$\sum_{i=1}^{n} f(i) \leq \underbrace{(1+c_1)}_{\downarrow}\int_1^n f(x)\,dx$$

$$\boxed{c = 1+c_1}$$

**Example**: Discuss the growth.

→ Ⓐ Start comparing with {
① $2^n$
② $n^k \quad \forall k$
③ $n^a$ choose an $a$
④ $\log n$
⑤ $c$ constant

Ⓑ Stop when you find the closest lower bound: $\lim \frac{f(n)}{\ } = \begin{cases} \infty \\ \text{or} \end{cases}$

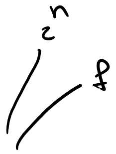
Ⓑ Stop when you find the closest lower bound : $\lim \frac{f(n)}{Ⓐ} = \begin{cases} \infty \\ \text{or} \\ nr \neq 0 \end{cases}$

① $n^{\log^4 n}$

$$\lim_{n\to\infty} \boxed{\frac{n^{\log^4 n}}{2^n}}_{P} \Rightarrow \lim_{n\to\infty} \log\left(\frac{n^{\log^4 n}}{2^n}\right) = \lim_{n\to\infty} \log(n^{\log^4 n}) - \log 2^n$$

$$= \lim_{n\to\infty} \log^4 n \log n - n \overset{1}{\cancel{\log}2} = \lim_{n\to\infty} \log^5 n - n = \boxed{-\infty}$$

$$\lim \log P = -\infty \Rightarrow \lim P = 2^{-\infty} = 0$$



$a^5 - ka$

$$\lim_{n\to\infty} \boxed{\frac{n^{\log^4 n}}{n^k}}_{P} \Rightarrow \lim_{n\to\infty} \log^5 n - k\log n = +\infty$$

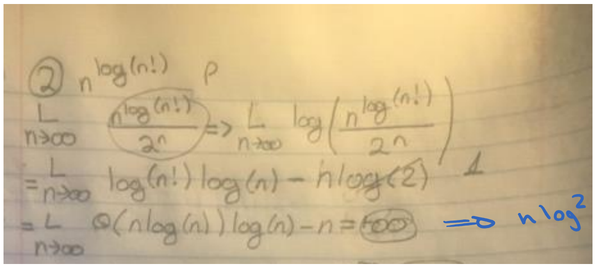
$$\Rightarrow \lim \log P = +\infty \Rightarrow \lim P = +\infty$$

$$\Rightarrow \boxed{f(n) = \Omega(n^k)} \checkmark$$

② $n^{\log(n!)}$

$$\log n! = \log(1 \times 2 \times \cdots \times n)$$
$$= \sum_{i=1}^{n} \log i = \Theta\left(\int_1^n \log x\, dx\right)$$
$$= \Theta(n\log n - n)$$

Johnathon :)



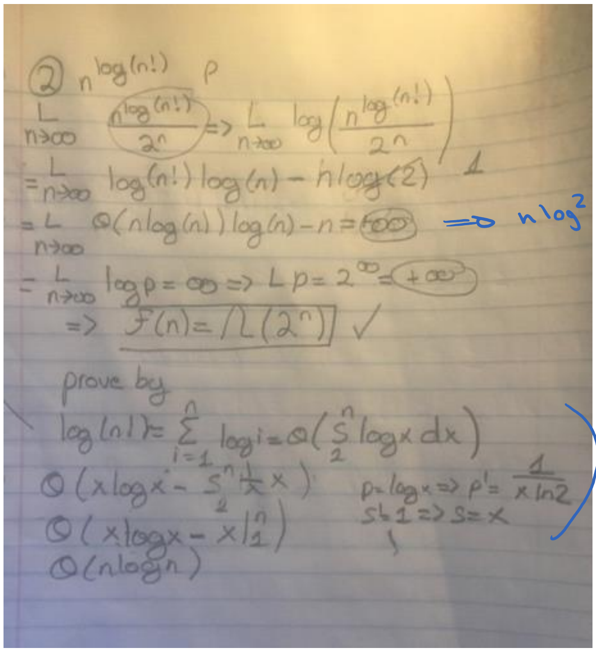
② $n^{\log(n!)}$ P

$$\lim_{n\to\infty} \frac{n^{\log(n!)}}{2^n} \Rightarrow \lim_{n\to\infty} \log\left(\frac{n^{\log(n!)}}{2^n}\right)$$

$$= \lim_{n\to\infty} \log(n!)\log(n) - n\overset{1}{\cancel{\log}}(2)$$

$$= \lim_{n\to\infty} \Theta(n\log(n))\log(n) - n = \boxed{+\infty}$$

$$\Rightarrow n\log^2 - n = +\infty$$



② $n^{\log(n!)}$ $p$

$\lim\limits_{n\to\infty} \boxed{\frac{n^{\log(n!)}}{2^n}} \Rightarrow \lim\limits_{n\to\infty} \log\left(\frac{n^{\log(n!)}}{2^n}\right)$ 1

$= \lim\limits_{n\to\infty} \log(n!)\log(n) - n\log(2)$

$= \lim\limits_{n\to\infty} \Theta(n\log(n))\log(n) - n = \boxed{+\infty}$ $\Rightarrow n\log^2 - n = +\infty$

$= \lim\limits_{n\to\infty} \log p = \infty \Rightarrow \lim p = 2^{\infty} = \boxed{+\infty}$

$\Rightarrow \boxed{f(n) = \Omega(2^n)}$ ✓

prove by

$\log(n!) = \sum_{i=1}^{n} \log i = \Theta\left(\int_2^n \log x \, dx\right)$

$\Theta\left(x\log x - \int_2^n \frac{1}{x} x\right)$ $p = \log x \Rightarrow p' = \frac{1}{x\ln 2}$

$\Theta\left(x\log x - x\Big|_1^n\right)$ $s' = 1 \Rightarrow s = x$

$\Theta(n\log n)$

③ $n\left(2^{\log\sqrt{n}}\right) = n\sqrt{n} = \boxed{\Theta(n^{3/2})}$

④ $(\log n)^{\log^{1/2} n} =$

$\lim\limits_{n\to\infty} \frac{(\log n)^{\log^{1/2} n}}{2^n} \Rightarrow \lim\limits_{n\to\infty} \left(\log^{1/2} n \,\log\log n - n\right) = -\infty \Rightarrow \lim p = 0$

$f(n) = o(2^n)$; in~

$\sqrt{a}\,\log a - ka$

$\lim\limits_{n\to\infty} \frac{(\log n)^{\log^{1/2} n}}{n^k} \Rightarrow \lim\limits_{n\to\infty} \left(\sqrt{\log n}\,\log\log n - k\log n\right) = -\infty \Rightarrow \lim p = 0$

$\Rightarrow f(n) = o(n^k)$

$\lim\limits_{n\to\infty} \frac{(\log n)^{\log^{1/2} n}}{n^a} \Rightarrow \lim\limits_{n\to\infty} \left(\sqrt{\log n}\,\log\log n - a\log n\right) = -\infty$

what no 2 or 3

SKIP

$$\lim_{n\to\infty} p = \frac{(\log n)^{\log^{\frac{1}{2}} n}}{\log n} \Rightarrow \lim_{n\to\infty}\left(\sqrt{\log n}\,\log\log n - \log\log n\right) = +\infty$$

$$\Rightarrow \lim \log p = +\infty \Rightarrow \lim p = +\infty$$

$$(\log n)^{\log^{\frac{1}{2}} n} = \Omega(\log n)$$

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⑤ $16^{\log n} = n^{\log 16} = n^4 = \Theta(n^4)$

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⑥ $\log(\overset{a}{n^2}!) = \log(1\times 2\times \text{———} \times n^2)$

$$= \sum_{i=1}^{n^2} \log i = \Theta\left(\int_1^{n^2} \log x \, dx\right)$$

$$= \Theta\left(x\log x - x \Big|_1^{n^2}\right)$$

$$= \Theta(n^2 \log n^2) = \boxed{\Theta(n^2 \log n)}$$

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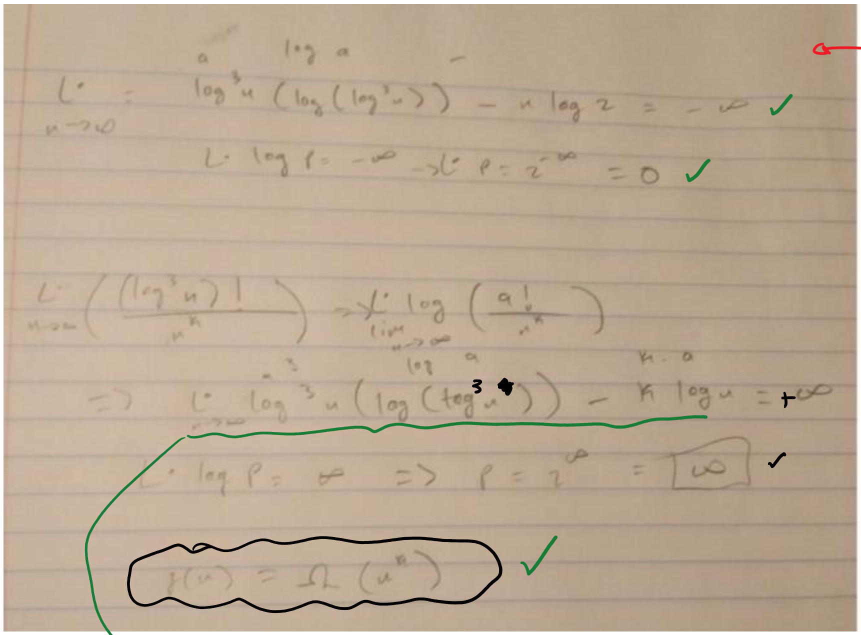
⑦ $\overset{a}{(\log^3 n)}! = a! = (1\times 2\times 3\times \text{———} \times a)$

① $$\lim_{n\to\infty} \frac{\overset{a}{(\log^3 n)}!}{2^n} \Rightarrow \lim_{n\to\infty} \log a! - n\overset{1}{\cancel{\log 2}} = \lim \sum_{i=1}^{a} \log i - n = \lim_{n\to\infty} \Theta\left(\int_1^a \log x\,dx - n\right)$$

$$\Rightarrow \simeq \lim_{n\to\infty} a\log a - a - n = \lim \log^3 n \log\log^3 n - \cancel{\log^3 n} - n$$

$$= \lim 3\log^3 n \log\log n - n$$



$$\lim_{n\to\infty} \overset{a}{\log^3 n}\left(\overset{\log a}{\log(\log^3 n)}\right) - n\log 2 = -\infty \checkmark$$

$$\lim \log p = -\infty \to \lim p = 2^{-\infty} = 0 \checkmark$$

$$\lim_{n\to\infty}\left(\frac{(\log^3 n)!}{n^k}\right) \to \lim_{n\to\infty} \log\left(\frac{a!}{n^k}\right)$$

$$\Rightarrow \lim_{n\to\infty} \overset{a}{\log^3 n}\left(\overset{\log a}{\log(\log^3 n)}\right) - \overset{k \cdot a}{k\log n} = +\infty$$

$$\lim \log p = \infty \Rightarrow p = 2^{\infty} = \boxed{\infty} \checkmark$$

$$f(n) = \Omega(n^k) \checkmark$$

Gideon

$\log n = S \rightarrow \lim \underline{\underline{S^3}}\log S - kS = +\infty$

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⑧ $$\log n^{\log n!} = (\log n!)\log n = \left(\sum_{i=1}^{n}\log i\right)\log n$$

$$= \Theta(n\log n)\log n = \boxed{\Theta(n\log^2 n)}$$

<u>hint</u> on Q3/4 of HW3 look at the proof of integral theorem (one of the tricks I used :D)