1. Compare the growth of $f(n) = 2^{log_2(n^2)} + 4^{log_2(\sqrt{n})} \cdot log_2(n)$ and $g(n) = 6 \cdot n^2 + log_2(n) + n^2$

$$\begin{aligned} lim_{n\to\infty} \frac{f(n)}{g(n)} &= lim_{n\to\infty} \frac{2^{log_2(n^2)} + 4^{log_2(\sqrt{n})} \cdot log_2(n)}{6 \cdot n^2 + log_2(n) \cdot n^2} \\ &= lim_{n\to\infty} \frac{n^2 \cdot log_2(n)}{6 \cdot n^2 + log_2(n) \cdot n^2} \\ &= c \end{aligned}$$

where $c \in \mathbb{R}$, hence $f(n) = \Theta(g(n))$.

2. Compare the growth of $f(n) = n^{1.06} + 6 \cdot n \cdot log^5(n)$ and $g(n) = 2 \cdot n \cdot log^{10}(n)$

$$f(n) = \omega(g(n))$$

3. Compare the growth of $f(n) = n^{2.01}$ and $g(n) = n^2 \cdot log^{50}(n)$

$$f(n) = \omega(g(n))$$

4. Compare the growth of

$$\begin{aligned} f(n) &= n^{log_2^2(n)} \\ g(n) &= log_2^n(n) \end{aligned}$$

$$f(n) = o(g(n))$$

5. Compare the growth of

$$\begin{aligned} f(n) &= 8 \cdot n^{2+sin(n)} \\ g(n) &= n \end{aligned}$$

$$f(n) = \Omega(g(n))$$

6. Prove $\forall k > 0, \epsilon > 0 \implies log^k(n) = o(n^\epsilon)$.

We prove this by induction on k, let ϵ be a real number greater than 0. When $k = 1$ we have

$$\begin{aligned} lim_{n\to\infty}\frac{log(n)}{n^\epsilon} &= lim_{n\to\infty}\frac{\frac{1}{n\cdot ln(2)}}{\epsilon \cdot n^{\epsilon-1}} \\ &= lim_{n\to\infty}\frac{1}{\epsilon \cdot ln(2) \cdot n^\epsilon} \\ &= 0 \end{aligned}$$

Assume that the result holds for all integers n less than or equal to some integer k, we show it must also hold for the $k+1$ integer.

$$\begin{aligned} lim_{n\to\infty}\frac{log^{k+1}(n)}{n^\epsilon} &= lim_{n\to\infty}\frac{(k+1)\cdot log^k(n)\cdot \frac{1}{n\cdot ln(2)}}{\epsilon \cdot n^{\epsilon-1}} \\ &= lim_{n\to\infty}\frac{(k+1)\cdot log^k(n)}{\epsilon \cdot ln(2) \cdot n^\epsilon} \\ &= 0 \end{aligned}$$

which is what we wanted to show. Hence by the principle of mathematical induction the result holds for all $n > 0$ and $\epsilon > 0$.

7. Prove $f(n) + g(n) = \Theta(max(f(n), g(n)))$

$$max(f(n), g(n)) \quad \leq f(n) + g(n) \quad \leq 2 \cdot max(f(n), g(n))$$

We conclude $f(n) + g(n) = \Theta(max(f(n), g(n)))$.