

1. What is the growth of the below functions?

1. $f(n) = 2 \cdot n^4 \cdot \log_2(n^4) + n^{4.0001} - 3 \cdot \log_2(n)$

$$f(n) = \Theta(n^{4.0001})$$

2. $f(n) = 3 \cdot n^3 \cdot \log(n^4 - n^2) + 100000$

$$f(n) = \Theta(n^3 \cdot \log_2(n))$$

3. $f(n) = \log_2^{100}(n^{50}) + n$

$$f(n) = \Theta(n)$$

4. $f(n) = n^4 \cdot \log_2^3(n) + 4$

$$f(n) = \Theta(n^4 \cdot \log_2^3(n))$$

5. $f(n) = 10000 \cdot n \cdot \log_2(n^7) + 3 \cdot \log_2(n) + 1000 \cdot \sqrt{n}$

$$f(n) = \Theta(n \cdot \log_2(n))$$

6. $f(n) = \sqrt[10]{n} + 10^{10} \cdot \log_2^{100}(n) + 8$

$$f(n) = \Theta(\sqrt[10]{n})$$

7. $f(n) = \sqrt{\sqrt{n}} + 9 \cdot \log_2(n)$

$$f(n) = \Theta(\sqrt{\sqrt{n}})$$

2. Discuss the growth of the below functions

1. $f(n) = \log_2(n)^{\log_2(n)}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{f(n)}{2^n} &= \lim_{n \rightarrow \infty} \frac{\log_2(n)^{\log_2(n)}}{2^n} \\ \log\left(\lim_{n \rightarrow \infty} \frac{\log_2(n)^{\log_2(n)}}{2^n}\right) &= \lim_{n \rightarrow \infty} \log_2(n) \cdot \log_2(\log_2(n)) - n \\ &= -\infty \\ 2^{-\infty} &= 0 \end{aligned}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{f(n)}{n^k} &= \lim_{n \rightarrow \infty} \frac{\log_2(n)^{\log_2(n)}}{n^k} \\ \log_2\left(\lim_{n \rightarrow \infty} \frac{\log_2(n)^{\log_2(n)}}{n^k}\right) &= \lim_{n \rightarrow \infty} \log_2(n) \cdot \log_2(\log_2(n)) - k \cdot \log_2(n) \\ &= \infty \\ 2^\infty &= \infty \end{aligned}$$

We conclude $f(n) = \omega(n^k)$.

2. $f(n) = 2^{\sqrt{2 \cdot \log_2(n)}}$

$$\begin{aligned} \log_2\left(\lim_{n \rightarrow \infty} \frac{2^{\sqrt{2 \cdot \log_2(n)}}}{2^n}\right) &= \lim_{n \rightarrow \infty} \sqrt{2 \cdot \log_2(n)} - n \\ &= -\infty \\ 2^{-\infty} &= 0 \end{aligned}$$

$$\begin{aligned} \log_2\left(\lim_{n \rightarrow \infty} \frac{2^{\sqrt{2 \cdot \log_2(n)}}}{n^k}\right) &= \lim_{n \rightarrow \infty} \sqrt{2 \cdot \log_2(n)} - k \cdot \log_2(n) \\ &= -\infty \\ 2^{-\infty} &= 0 \end{aligned}$$

$$\begin{aligned} \log_2\left(\lim_{n \rightarrow \infty} \frac{2^{\sqrt{2 \cdot \log_2(n)}}}{n^a}\right) &= \lim_{n \rightarrow \infty} \sqrt{2 \cdot \log_2(n)} - a \cdot \log_2(n) \\ &= -\infty \\ 2^{-\infty} &= 0 \end{aligned}$$

$$\begin{aligned}
\log_2(\lim_{n \rightarrow \infty} \frac{2^{\sqrt{2 \cdot \log_2(n)}}}{\log_2(n)}) &= \lim_{n \rightarrow \infty} \sqrt{2 \cdot \log_2(n)} - \log_2(\log_2(n)) \\
&= \infty \\
2^\infty &= \infty
\end{aligned}$$

We conclude $f(n) = \omega(\log_2(n))$.

$$3. f(n) = \sqrt{2}^{\log_2(n)}$$

$$\begin{aligned}
\sqrt{2}^{\log_2(n)} &= 2^{\frac{1}{2} \cdot \log_2(n)} \\
&= 2^{\log_2(n^{\frac{1}{2}})} \\
&= n^{\frac{1}{2}} \\
&= \Theta(n^{\frac{1}{2}})
\end{aligned}$$

We conclude $f(n) = \Theta(n^{\frac{1}{2}})$.

$$4. f(n) = n^{\frac{1}{\log_2(n)}}$$

$$\begin{aligned}
\log_2(\lim_{n \rightarrow \infty} \frac{n^{\frac{1}{\log_2(n)}}}{2^n}) &= \lim_{n \rightarrow \infty} 1 - n \\
&= -\infty \\
2^{-\infty} &= 0
\end{aligned}$$

$$\begin{aligned}
\log_2(\lim_{n \rightarrow \infty} \frac{n^{\frac{1}{\log_2(n)}}}{n^k}) &= \lim_{n \rightarrow \infty} 1 - k \cdot \log_2(n) \\
&= -\infty \\
2^{-\infty} &= 0
\end{aligned}$$

$$\begin{aligned}
\log_2(\lim_{n \rightarrow \infty} \frac{n^{\frac{1}{\log_2(n)}}}{n^a}) &= \lim_{n \rightarrow \infty} 1 - a \cdot \log_2(n) \\
&= -\infty \\
2^{-\infty} &= 0
\end{aligned}$$

$$\begin{aligned}
\log_2(\lim_{n \rightarrow \infty} \frac{n^{\frac{1}{\log_2(n)}}}{\log_2(n)}) &= \lim_{n \rightarrow \infty} 1 - \log_2(\log_2(n)) \\
&= -\infty \\
2^{-\infty} &= 0
\end{aligned}$$

$$\begin{aligned}
\log_2(\lim_{n \rightarrow \infty} \frac{n^{\frac{1}{\log_2(n)}}}{1}) &= \lim_{n \rightarrow \infty} 1 - \log_2(1) \\
&= 1 \\
2^1 &= 2
\end{aligned}$$

We conclude $f(n) = \Theta(1)$.

3. Suppose $g(n) \geq 1$ for all n , and that $f(n) \leq g(n) + L$, for some constant L and all n . Prove that $f(n) = O(g(n))$.

$$\begin{aligned}
f(n) &\leq g(n) + L \\
&\leq g(n) + g(n) \cdot L \\
&= g(n) \cdot (L + 1)
\end{aligned}$$

Hence $f(n) = O(g(n))$.

4. Prove or disprove: if $f(n) = O(g(n))$ and $f(n) \geq 1$ and $\log(g(n)) \geq 1$ for sufficiently large n then $\log(f(n)) = O(\log(g(n)))$.

$$\begin{aligned}
f(n) &\leq c \cdot g(n) \\
\log_2(f(n)) &\leq \log_2(c) + \log_2(g(n)) \\
&\leq \log_2(c) \cdot \log_2(g(n)) + \log_2(g(n)) \\
&= \log_2(g(n)) \cdot (\log_2(c) + 1)
\end{aligned}$$

Hence $\log_2(f(n)) = O(\log_2(g(n)))$.

5. Show that $\log_2(n!) = \Theta(n \cdot \log_2(n))$

$$\begin{aligned}
\log_2(n!) &= \log_2(1 \cdot 2 \cdot 3 \cdot \dots \cdot n) \\
&= \log_2(1) + \log_2(2) + \log_2(3) + \dots + \log_2(n) \\
&= \sum_{i=1}^n \log_2(i) \\
&= \Theta\left(\int_1^n \log_2(x) \cdot dx\right) \\
&= \Theta\left(\frac{x \cdot \ln(x)}{\ln(2)} \Big|_1^n - \frac{1}{\ln(2)} \cdot \int_1^n dx\right) \\
&= \Theta\left(\frac{n \cdot \ln(n)}{\ln(2)} - \frac{1}{\ln(2)} \cdot (n-1)\right) \\
&= \Theta(n \cdot \ln(n)) \\
&= \Theta\left(n \cdot \frac{\log_2(n)}{\log_2(e)}\right) \\
&= \Theta(n \cdot \log_2(n)).
\end{aligned}$$

Hence we conclude $\log_2(n!) = \Theta(n \cdot \log_2(n))$.

6. Prove that $n! = o(n^{n^2})$

$$\begin{aligned}
\lim_{n \rightarrow \infty} \log_2\left(\frac{n!}{n^{n^2}}\right) &= \lim_{n \rightarrow \infty} \log_2(n!) - n^2 \cdot \log_2(n) \\
&= \lim_{n \rightarrow \infty} \Theta(n \cdot \log_2(n)) - n^2 \cdot \log_2(n) \quad (\log_2(n!) = \Theta(n \cdot \log_2(n)) \text{ by problem (5)}) \\
&= -\infty \\
2^{-\infty} &= 0
\end{aligned}$$

Hence we conclude that $n! = o(n^{n^2})$

7. Prove that $n! = \omega(2^n)$

$$\begin{aligned}
\lim_{n \rightarrow \infty} \log_2\left(\frac{n!}{2^n}\right) &= \lim_{n \rightarrow \infty} \log_2(n!) - n \\
&= \lim_{n \rightarrow \infty} \Theta(n \cdot \log_2(n)) - n \quad (\log_2(n!) = \Theta(n \cdot \log_2(n)) \text{ by problem (5)}) \\
&= \infty \\
2^\infty &= \infty
\end{aligned}$$

Hence we conclude $n! = \omega(2^n)$.

8. Which one of the below functions grow faster?

$$\begin{aligned} f(n) &= 2^{2^n} \\ g(n) &= n! \end{aligned}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \log_2\left(\frac{2^{2^n}}{n!}\right) &= \lim_{n \rightarrow \infty} 2^n - \log_2(n!) \\ &= \lim_{n \rightarrow \infty} 2^n - \Theta(n \cdot \log_2(n)) \quad (\log_2(n!) = \Theta(n \cdot \log_2(n)) \text{ by problem (5)}) \\ &= \infty \\ 2^\infty &= \infty \end{aligned}$$

Hence we conclude $f(n)$ grows faster.

9. Provide a closed-form expression for the asymptotic growth of $n + \frac{n}{2} + \frac{n}{3} + \dots + 1$

$$\begin{aligned} n \cdot \sum_{i=1}^n \frac{1}{i} &= \Theta\left(n \cdot \int_1^n \frac{1}{x} dx\right) \\ &= \Theta\left(n \cdot (\ln(x)) \Big|_1^n\right) \\ &= \Theta(n \cdot \ln(n) - n \cdot \ln(1)) \\ &= \Theta(n \cdot \ln(n)) \end{aligned}$$

10. Use the integral theorem to calculate the growth of $1 + 2^k + 3^k + \dots + n^k$

$$\begin{aligned} \sum_{i=1}^n i^k &= \Theta\left(\int_1^n x^k dx\right) \\ &= \Theta\left(\frac{1}{k+1} \cdot x^{k+1} \Big|_1^n\right) \\ &= \Theta\left(\frac{n^{k+1}}{k+1} - \frac{1}{k+1}\right) \\ &= \Theta(n^{k+1}) \end{aligned}$$

11. Use the integral theorem to calculate the growth of $\log_2(1) + \log_2(2) + \log_2(3) + \dots + \log_2(n^4)$

$$\begin{aligned}
\sum_{i=1}^{n^4} \log_2(i) &= \Theta\left(\int_1^{n^4} \log_2(x) \cdot dx\right) \\
&= \Theta\left(\frac{\ln(x) \cdot x}{\ln(2)} \Big|_1^{n^4} - \frac{1}{\ln(2)} \cdot \int_1^{n^4} dx\right) \\
&= \Theta\left(\frac{1}{\ln(2)} \cdot (\ln(n^4) \cdot n^4 - \ln(1) \cdot 1) - \frac{1}{\ln(2)} \cdot (n^4 - 1)\right) \\
&= \Theta(\ln(n^4) \cdot n^4) \\
&= \Theta(4 \cdot \ln(n) \cdot n^4) \\
&= \Theta\left(4 \cdot \frac{\log_2(n)}{\log_2(e)} \cdot n^4\right) \\
&= \Theta(n^4 \cdot \log_2(n))
\end{aligned}$$

12. Use the integral theorem to calculate the growth of $\log_2(1) + 2 \cdot \log_2(2) + 3 \cdot \log_2(3) + \dots + n^2 \cdot \log_2(n^2)$

$$\begin{aligned}
\sum_{i=1}^{n^2} i \cdot \log_2(i) &= \Theta\left(\int_1^{n^2} x \cdot \log_2(x) \cdot dx\right) \\
&= \Theta\left(\frac{x^2 \cdot \ln(x)}{2 \cdot \ln(2)} \Big|_1^{n^2} - \frac{1}{2 \cdot \ln(2)} \int_1^{n^2} x \cdot dx\right) \\
&= \Theta\left(\frac{n^4 \cdot \ln(n^2)}{2 \cdot \ln(2)} - \frac{1}{4 \cdot \ln(2)} \cdot x^2 \Big|_1^{n^2}\right) \\
&= \Theta\left(\frac{n^4 \cdot \ln(n^2)}{2 \cdot \ln(2)} - \frac{1}{4 \cdot \ln(2)} (n^4 - 1)\right) \\
&= \Theta\left(\frac{n^4 \cdot \ln(n^2)}{2 \cdot \ln(2)}\right) \\
&= \Theta(n^4 \cdot \ln(n^2)) \\
&= \Theta(2 \cdot n^4 \cdot \ln(n)) \\
&= \Theta\left(2 \cdot n^4 \cdot \frac{\log_2(n)}{\log_2(e)}\right) \\
&= \Theta(n^4 \cdot \log_2(n))
\end{aligned}$$

13. Prove or disprove: if $f(n) = O(g(n)) \implies 2^{f(n)} = O(2^{g(n)})$

This is false, consider $f(n) = 2 \cdot n$ and $g(n) = n$. $f(n) = O(g(n))$ but $2^{f(n)} \neq O(2^{g(n)})$.