Lecture 7 (Integral theorem)

Tuesday, September 15, 2020 5:00 PM

(Reminder: HW 3 is due this Sunday)

Example: What is the growth?

① $n\log(n^2+\sqrt{n}) + 4n + 6\sqrt{n} = \Theta(n\log(n^2+\cancel{\sqrt{n}})) = \boxed{\Theta(n\log n)}$

② $8n^2\sqrt{n+1} + \underbrace{2^{\log n\sqrt{n}}}_{n\sqrt{n}} + \underbrace{\log^{10} n^6}_{(\log n)^{10}} = \boxed{\Theta(n^2\sqrt{n})}$

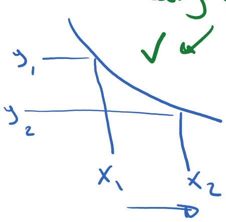
③ $n^{2.01} + \log^3(n^5+n^{10}) = \boxed{\Theta(n^{2.01})}$

Theorem:

① If $f(n)$ is monotonically decreasing $\Rightarrow \sum_{i=1}^{n} f(i) = \Theta(\int_1^n f(x)\,dx)$

② " " " " increasing and $f(n) = O(\int_1^n f(x)\,dx) \Rightarrow \sum_{i=1}^{n} f(i) = \Theta(\int_1^n f(x)\,dx)$

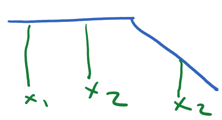
monotonically decreasing ✓



$x_2 > x_1 \Rightarrow y_2 < y_1$

$f(x_2) < f(x_1)$

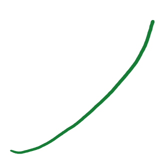
decreasing



$\Rightarrow x_2 > x_1 \Rightarrow f(x_2) \leq f(x_1)$

increasing

increasing or decreasing



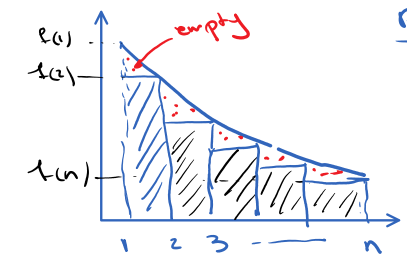
monotonically increasing

① $\sum_{i=1}^{n} f(i) = \Theta(\int_1^n f(x)\,dx) \iff$ ⓐ $\sum f(i) = O(\int_1^n f(x)\,dx)$ ✓

and

ⓑ $\sum f(i) = \Omega(\int_1^n f(x)\,dx)$

ⓐ $\sum f(i) = O\left(\int_1^n f(x)\,dx\right) \iff \exists c>0, \exists k\geq 0$ s.t. $\sum_{i=1}^n f(i) \leq c \int_1^n f(x)\,dx \quad \forall n\geq k$



Riemann Sum

area 1 area 2

$1\times f(2) + 1\times f(3) + 1\times f(4) + \sim + f(n)$

$f(2)+f(3)+ \sim + f(n) \leq \int_1^n f(x)\,dx$

$+f(1)$

$\underbrace{f(1)+f(2)+ \sim + f(n)} \leq \int_1^n f(x)\,dx + f(1)$

$\sum_{i=1}^n f(i) \leq \int_1^n f(x)\,d + f(1) \leq \int_1^n f(x)\,dx + f(1)\int_1^n f(x)\,dx$

$\int f(x)\,dx \geq 1$

note1: $a < b + 5 \leftarrow$:^

my goal: $\underline{a < bc}$

$b \geq 1$ $\quad a < \underbrace{b + 5b}_{6b}$

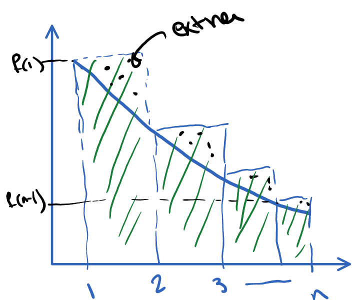
Note2: $2<7 \Rightarrow 2<10$

$\Rightarrow \sum_{i=1}^n f(i) \leq \overbrace{\int_1^n f(x)\,dx}^{a} + f(1)\overbrace{\int_1^n f(x)\,dx}^{a}$

$\Rightarrow \sum_{i=1}^n f(i) \leq \int_1^n f(x)\,dx\,(1+f(1))$

$c \geq 1+f(1) \Rightarrow \boxed{c = 1+f(1)}$ ✓

ⓑ $\sum_{i=1}^n f(i) = \Omega\left(\int_1^n f(x)\,dx\right) \iff \exists c_1>0, \exists k_1\geq 0$, s.t. $\sum_{i=1}^n f(i) \geq c_1 \int_1^n f(x)\,dx \quad \forall n\geq k_1$



$f(1) + f(2) + \sim + f(n-1) \geq \int_1^n f(x)\,dx$ $\quad$ note: $5\geq 2 \Rightarrow 7\geq 2$

$+$ $\quad f(n) \geq 0$

running time

$\sum_{i=1}^n f(i) \geq \int_1^n f(x)\,dx$

I know $a \geq b$ ✓
my goal $a \geq c_1 b$
c_1?

$\Rightarrow 0 < c_1 \leq 1 \Rightarrow \boxed{c_1 = 1}$

Example:

Example:

① for i=1:n
    for j=1:i
        sum++
    end
end

$$\Rightarrow \sum_{i=1}^{n} \sum_{j=1}^{i} c = \sum_{i=1}^{n} ci$$

ci

$$= \Theta\left(\int_1^n x\,dx\right)$$

$$= \Theta\left(\frac{x^2}{2}\Big|_1^n\right)$$

$$= \boxed{\Theta(n^2)}$$

② for i=1:n
    for j=1:n^2
        for k=1:i^4
            sum++
        end
    end
end

$$\Rightarrow \sum_{i=1}^{n} \sum_{j=1}^{n^2} \sum_{k=1}^{i} 1$$

$$= \sum \sum_{j=1}^{n^2} i^4$$

$$= \sum_{i=1}^{n} i^4 n^2 = n^2 \sum_{i=1}^{n} i^4$$

$$= n^2\, \Theta\left(\int_1^n x^4\,dx\right)$$

$$= n^2\, \Theta\left(\frac{x^5}{5}\Big|_1^n\right)$$

$$= n^2\, \Theta(n^5)$$

$$= \Theta(n^2 n^5)$$

$$= \boxed{\Theta(n^7)}$$

③ for i=1:logn^2
    pauge(i)
end

pauge(n) = Θ(log n)

function call

$$\Rightarrow \sum_{i=1}^{\log n^2} \text{pauge}(i) \simeq \sum_{i=1}^{\log n^2} \log i$$

$$= \Theta\left(\int_1^{\log n^2} \log x\,dx\right)$$

$$= \Theta\left(x\log x - x \Big|_1^{\log n^2}\right)$$

$$= \Theta\left(\log n^2 \log\log n^2 - \cancel{\log n^2}\right)$$

$= \Theta(\log n^2 \log\log n^2)$

$\boxed{= \Theta(\log n \log\log n)}$ ✓

④ $1+2+3+\dots+n^5 = \sum_{i=1}^{n^5} i = \Theta\left(\int_1^{n^5} x\,dx\right) = \Theta\left(x^2 \Big|_1^{n^5}\right)$

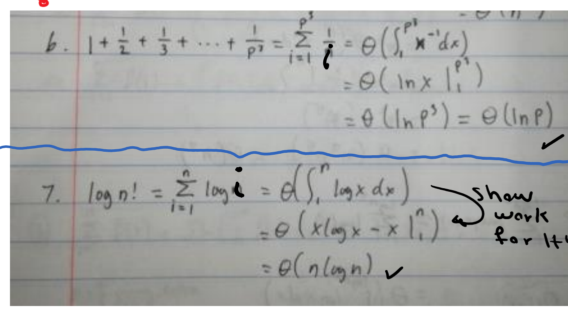
$= \Theta((n^5)^2) = \boxed{\Theta(n^{10})}$

⑤ $1^5+2^5+3^5+\dots+n^{15} = \sum_{i=1}^{n^3} i^5 = \Theta\left(\int_1^{n^3} x^5\,dx\right) = \Theta\left(x^6 \Big|_1^{n^3}\right)$

$\boxed{= \Theta(n^{18})}$

Khai :)

⑥ $1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{p^3} =$



6. $1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{p^3} = \sum_{i=1}^{p^3} \frac{1}{i} = \Theta\left(\int_1^{p^3} x^{-1}\,dx\right)$

$= \Theta\left(\ln x \Big|_1^{p^3}\right)$

$= \Theta(\ln p^3) = \Theta(\ln p)$ ✓

7. $\log n! = \sum_{i=1}^{n} \log i = \Theta\left(\int_1^{n} \log x\,dx\right)$ show work for HW

$= \Theta\left(x\log x - x \Big|_1^{n}\right)$

$= \Theta(n\log n)$ ✓

⑦ $\log \underbrace{n!}_{1\times2\times3\times\dots\times n} =$

$\log(1\times2\times\dots n) = \log1+\log2+\dots+\log n$

⑧ $e^1+e^2+\dots+e^{n^2} = \sum_{i=1}^{n^2} e^i = \Theta\left(\int_1^{n^2} e^x\,dx\right) = \Theta\left(e^x \Big|_1^{n^2}\right)$

$\boxed{= \Theta(e^{n^2})}$ ✓

⑨ $e^1+2e^2+\dots+k^3e^{k^3} = \sum_{i=1}^{k^3} ie^i = \Theta\left(\int_1^{k^3} x e^x\,dx\right)$

$\boxed{\begin{array}{l} P' = e^x \to P = e^x \\ S = x \to S' = 1 \end{array}} \Rightarrow \Theta\left(xe^x - e^x \Big|_1^{k^3}\right)$

$\boxed{= \Theta(k^3 e^{k^3})}$