**1** Use $\Theta$ notation to express the statement

$$4n^6 \leq 17n^6 - 45n^3 + 2n + 8 \leq 30n^6, n \geq 3$$

Let $A = 4$, $B = 30$ and $k = 3$ then the statement translates to

$$An^6 \leq 17n^6 - 45n^3 + 2n + 8 \leq Bn^6, n \geq k$$

hence by the definition of $\Theta$ notation $17n^6 - 45n^3 + 2n + 8$ is $\Theta(n^6)$.

**2** Use $\Omega$ notation to express the statement

1. Use $\Omega$ notation to expres the statement

$$\frac{11}{4}n^2 \leq 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2, n \geq 2$$

Let $A = \frac{11}{4}$ and $k = 2$ then $An^2 \leq 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2, n \geq 2$ then the statement translates to

$$An^2 \leq 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2, n \geq k$$

which by the definition of $\Omega$ notation, $3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2$ is $\Omega(n^2)$.

2. Use $O$ notation to express the statement

$$0 \leq 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2 \leq 6n^2, n \geq 1$$

Let $A = 6$ and $k = 1$ then the statement translates to

$$0 \leq 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2 \leq An^2, n \geq k$$

which by the defnition of $O$ notation, $3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2$ is $O(n^2)$.

3. Justify the statement: $3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2$ is $\Theta(n^2)$.

Let $A = \frac{11}{4}, B = 6$ and $k = 2$ then $A \cdot n^2 \leq 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2 \leq Bn^2, n \geq k$ which by the definition of $\Theta$ notation, $3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2$ is $\Theta(n^2)$.

**3.** Given the function $15n^3 + 11n^2 + 9$

1. Show that the function is $\Omega(n^3)$.

$$15n^3 \leq 15n^3 + 11n^2 + 9, n \geq 1$$

Let $A = 15$ and $k = 1$ then the statements translates to $An^3 \leq 15n^3 + 11n^2 + 9, n \geq k$ which by the definition of $\Omega$ notation, $15n^3 + 11n^2 + 9$ is $\Omega(n^3)$.

2. Show that the function is $O(n^3)$.

$$15n^3 + 11n^2 + 9 \leq 15n^3 + 11n^3 + 9n^3$$
$$\leq 35n^3, n \geq 1$$

Let $A = 35$ and $k = 1$ then the statement translates to $15n^3 + 11n^2 + 9 \leq An^3, n \geq k$ which by the definition of $O$ notation, $15n^3 + 11n^2 + 9$ is $O(n^3)$.

**4.** Given the function $n^4 - 5n - 8$

1. Show that the function is $\Omega(n^4)$.

Let $A = \frac{1}{2}$ and $a = (|-5| + |-8|)$

$$n \geq \frac{2}{1} \cdot (|-5| + |-8|)$$
$$\frac{1}{2}n^4 \geq 5n^3 + 8n^3$$
$$\frac{1}{2}n^4 \geq 5n + 8$$
$$n^4 - 5n - 8 \geq \frac{1}{2}n^4, n \geq a$$

Hence by the definition of $\Omega$ notation, $n^4 - 5n - 8$ is $\Omega(n^4)$.

2. Show that the function is $O(n^4)$.

$$n^4 - 5n - 8 \leq n^4 + 5n + 8$$
$$\leq n^4 + 5n^4 + 8n4$$
$$= 14n^4, n \geq 1$$

Let $A = 14$ and $k = 1$ then the statement translates to $n^4 - 5n - 8 \leq An^4, n \geq k$ which by teh definition of $O$ notation translates, $n^4 - 5n - 8$ is $O(n^4)$.

**5.** Show that $15n^3 + 11n^2 + 9$ is $\Theta(n^3)$.

Since we have $\Omega(n^3)$ and $O(n^3)$ we have that there exists real positive number constants $A$ and $B$ such that $Ag(n) \leq f(n) \leq Bg(n), k \geq n$ where $k = max(k\prime, k\prime\prime)$ obtained from the previous inequalities. By definition of $\Theta$, $15n^3 + 11n^2 + 9$ is $\Theta(n^3)$.

**6.** Show that $n^4 - 5n - 8$ is $\Theta(n^4)$.

Since we have shown tha the function is $\Omega(n^4)$ and $O(n^4)$ we have that there exists real positive number constants $A$ and $B$ such that $Ag(n) \leq f(n) \leq Bg(n), k \geq n$ where $k = max(k\prime, k\prime\prime)$ obtained from the previous inequaliies. by definition of $\Theta$, $n^4 - 5n - 8$ is $\Theta(n^4)$.

**7.** Let $g(n) = n^4 - 5n - 8$, show that $g(n)$ is not $O(n^r)$ for any positive real number $r < 4$.

We prove this by contradiction. Suppose that $g(n)$ is $O(n^r)$ for any positive real number $r < 4$. then

$$g(n) \leq An^r, n \geq a$$

where $A$ and $a$ are real positive numbers.

$$
\begin{aligned}
g(n) &\leq n^4 \\
&\leq An^r \\
n^{4-r} &\leq A \\
n &\leq \sqrt[4-r]{A}
\end{aligned}
$$

which is a contradiction. We conclude that $g(n)$ is not $O(n^r)$ for any positive real number $r < 4$.

**8.** Use theorem on polynomial orders to find orders for the function given by the following formulas.

1. $f(n) = 7n^5 + 5n^3 - n + 4$ for each positive integer $n$.

   By direct application of theorem on polynomial orders, $7n^5 + 5n^3 - n + 4$ is $\Theta(n^5)$.

2. $g(n) = \frac{(n-1)(n+1)}{4}$ for each positive integer $n$.

3

$$\frac{(n-1)\cdot(n+1)}{4} = \frac{n^2+n-n+1}{4}$$
$$= \frac{n^2+1}{4}$$
$$= \frac{n^2}{4}+\frac{1}{4}$$

Thus $g(n)$ is $\Theta(n^2)$.

**9.** Show that for a positive integer variable $n$,

$$1+2+3\ldots+n \quad is \quad \Theta(n^2)$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$
$$= \frac{n^2}{2}+\frac{n}{2}$$

**10.** Express $5x^8 - 9x^7 + 2x^5 + 3x - 1 \le 6x^8, x > 3$ using $O$ notation

Let $A = 6$ and $a = 3$ then $5x^8 - 9x^7 + 2x^5 + 3x - 1 \le Ax^8, x > a$ and by definition of $O$ notation, $5x^8 - 9x^7 + 2x^5 + 3x - 1$ is $O(x^8)$.

**11.** Express $x^{\frac{7}{2}} \le \frac{(x^2-7)^2(10x^{\frac{1}{2}}+3)}{x+1}, x > 4$ using $\Omega$ notation

Let $A = 1$ and $k = 4$ then the statement translates to

$$Ax^{\frac{7}{2}} \le \frac{(x^2-7)^2(10x^{\frac{1}{2}}+3)}{x+1}, x > k$$

which by the definition of $\Omega$ notation, $\frac{(x^2-7)^2(10x^{\frac{1}{2}}+3)}{x+1}$ is $\Omega(x^{\frac{7}{2}})$.

**12.** Express $3x^6 + 5x^4 - x^3 \le 9x^6, x > 1$ using $O$ notation. Let $A = 9$ and $k = 1$ then the statement translate to

$$3x^6 + 5x^4 - x^3 \le Ax^6, x > k$$

which by the definition of $\Omega$ notation, $3x^6 + 5x^4 - x^3$ is $O(x^6)$.

**13.** Express $\frac{1}{2}x^4 \le x^4 - 50x^3 + 1$ for all real numbers $x > 101$ using $\Omega$ notation.

Let $A = \frac{1}{2}$ and $k = 101$ this statement translates to $Ax^4 \le x^4 - 50x^3 + 1, n > k$ which by the definition of $\Omega$ notation, $x^4 - 50x^3 + 1$ is $\Omega(x^4)$.

**14.** Express $\frac{1}{2}x^2 \le 3x^2 - 80x + 7 \le 3x^2, x > 25$

Let $A = \frac{1}{2}, B = 3$ and $k = 25$ then the statement translates to

$$Ax^2 \le \quad 3x^2 - 80x + 7 \quad \le Bx^2, x > k$$

which by the definition of $\Theta$ notation $3x^2 - 80x + 7$ is $\Theta(x^2)$.

**15.** Suppose $g(x)$ is $O(f(x))$ show $f(x)$ is then $\Omega(g(x))$

Since $g(x)$ is $O(f(x))$ then there exists a real positive numbers $B$ and $k$ such that $g(x) \le B \cdot f(x), n > k$. We obtain $\frac{g(x)}{B} \le f(x), n > k$ which by the definition of $\Omega$ notation $f(x)$ is $\Omega(g(x))$.

**16.** Prove that if $f(x)$ is $O(g(x))$ and $c$ is any nonzero real number, then $c \cdot f(x)$ is $O(g(x))$.

Since $f(x)$ is $O(g(x))$ then there exists real positive numbers $B$ and $k$ such that $g(x) \le B \cdot f(x), n > k$. Multiplying by the constant $c$ we obtain $c \cdot g(x) \le c \cdot B \cdot f(x), n > k$ which by the definition of $O$ notation $c \cdot g(x)$ is $O(g(x))$.

**17.** Prove that if $f(x)$ is $O(h(x))$ and $g(x)$ is $O(l(x))$, then $f(x) + g(x)$ is $O(G(x))$ where $G(x) = max(h(x), l(x))$.

$$
\begin{aligned}
f(x) + g(x) &\le \quad 2 \cdot max(h(x), l(x)) \\
&= \quad 2 \cdot G(x), n \ge k
\end{aligned}
$$

where $k = max(k\prime, k\prime\prime)$ where $k\prime$ and $k\prime\prime$ are terms that satisfy the previous $O$ notations.

**18.** Prove that $f(x)$ is $\Theta(f(x))$

Let $A = 1, B = 1$ and $k = 1$ the $Af(x) \le f(x) \le Bf(x), n \ge k$ which by the definition of $\Theta$ notation $f(x)$ is $\Theta(f(x))$.

**19.** Prove that if $f(x)$ is $O(h(x))$ and $g(x)$ is $O(k(x))$ then $f(x)g(x)$ is $O(h(x)k(x))$.

we have that $f(x)$ is $O(h(x))$ and $g(x)$ is $O(k(x))$ hence we have the we have constants $B, B', b$ and $b'$ such that

$$\begin{aligned} f(x) &\leq B \cdot h(x), x > b \\ g(x) &\leq B' \cdot k(x), x > b' \end{aligned}$$

Let $b_1 = max(b, b')$ then

$$\begin{aligned} f(x) \cdot g(x) &\leq B \cdot h(x) \cdot g(x) \\ &\leq B \cdot h(x) \cdot g(x) \\ &\leq B \cdot B' \cdot h(x) \cdot k(x), x > b_1 \end{aligned}$$

then by the definition of $O$ notation, $f(x) \cdot g(x)$ is $O(h(x) \cdot g(x))$.

**20.** Prove that if $x$ is a real number with $x > 1$, then $x^n > 1$ for all integers $n \geq 1$.

We prove this by mathematical induction, Let $P(n)$ be the statement $x^n > 1$.
   $P(1)$ is trivially true since $x^1 > 1$. Let $P(k)$ be true, thus $x^k > 1$ we now show that $P(k+1)$ is true.

$$\begin{aligned} x^{k+1} &= x^k x^1 \\ &> x^k \\ &> 1 \end{aligned}$$

Thus $x^{k+1} > 1$, which implies that $P(k+1)$ is true. By the principle of mathematical induction, $P(n)$ holds for all positive integers $n$.

**21.** Prove that if $x > 1$ then $x^m < x^n$ for any integers $m$ and $n$ with $m < n$.

$$\begin{aligned} x^m &= \underbrace{x \cdot x \cdot \ldots \cdot x}_{m\,times} \\ &< \underbrace{x \cdot x \cdot \ldots \cdot x}_{m\,times} \cdot \underbrace{1 \cdot 1 \ldots \cdot 1}_{n-m\,times} \\ &< \underbrace{x \cdot x \cdot \ldots \cdot x}_{m\,times} \cdot \underbrace{x \cdot x \cdot \ldots \cdot x}_{n-m\,times} \\ &= x^n \end{aligned}$$

**22.** Given the function $g(x) = 2x^2 + 15x + 4$

   1. Show that $x^2 \leq 2x^2 + 15x + 4$

$$
\begin{aligned}
x^2 \quad &< \quad x^2 + 15 + 4 \\
&< \quad x^2 + 15x + 4 \\
&< \quad 2x^2 + 15x + 4 \\
&\leq \quad 2x^2 + 15x + 4
\end{aligned}
$$

2. Show that $2x^2 + 15x + 4 \leq 21x^2$

$$
\begin{aligned}
2x^2 + 15x + 4 \quad &\leq \quad x^2 + 15x^2 + 14x^2 \\
&\leq \quad 21x^2
\end{aligned}
$$

3. Rewrite statement (1) using $\Omega$ notation

Let $A = 1$ and $a = 1$ then $A \cdot f(x) \leq g(x)$ and by the definition of $\Omega$ notation $g(x)$ is $\Omega(n^2)$.

4. Rewrite statement (2) using $O$ notation

Let $A = 21$ and $a = 1$ then $g(x) \leq A \cdot x^2$ and by the definition of $O$ notation $g(x)$ is $O(n^2)$.

**23.** Given the function $g(x) = x^4 \leq 23x^4 + 8x^2 + 4x$.

1. Show that if $x > 1$ then $x^4 \leq 23x^4 + 8x^2 + 4x$

Let $x$ be a real number such that $x > 1$.

$$
\begin{aligned}
x^4 \quad &\leq \quad 23 \cdot x^4 \\
&\leq \quad 23 \cdot x^4 + 8x^2 \\
&\leq \quad 23 \cdot x^4 + 8x^2 + 4x
\end{aligned}
$$

2. Show that if $x > 1$ then $23x^4 + 8x^2 + 4x \leq 23x^4$

$$
\begin{aligned}
23x^4 + 8x^2 + 4x \quad &\leq \quad 23x^4 + 8x^4 + 4x^4 \\
&\leq \quad 35x^4
\end{aligned}
$$

3. What can you conclude by part (1) and part (2)

the function $g(x)$ is $\Omega(x^4)$ and $O(x^4)$.

**24.** Prove $5x^3 + 65x + 30$ is $\Theta(x^3)$ by proving that $5x^3 + 65x + 30$ is $\Omega(x^3)$ and $5x^3 + 65x + 30$ is $O(x^3)$

$$
\begin{aligned}
5x^3 &\leq 5x^3 + 65x^3 + 30x^3 \\
&\leq 100x^3, x > 1
\end{aligned}
$$

$$
x^3 \leq 5x^3
$$

From these two inequalities we obtain that $x^3 \leq 5x^3 \leq 5x^3 + 65x^3 + 30x^3, x \geq 1$ hence by the definition of $\Theta$ notation $5x^3$ is $\Theta(x^3)$.