

1 Compute the values for

1. $\sum_{i=-1}^4 3$

$$\begin{aligned}\sum_{i=-1}^4 3 &= 3 + 3 + 3 + 3 + 3 + 3 \\ &= 3 \cdot 6 \\ &= 18\end{aligned}$$

2. $\sum_{i=1}^5 (\frac{1}{3})^i$

$$\begin{aligned}\sum_{i=1}^5 (\frac{1}{3})^i &= (\frac{1}{3})^1 + (\frac{1}{3})^2 + (\frac{1}{3})^3 + (\frac{1}{3})^4 + (\frac{1}{3})^5 \\ &= \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} \\ &= \frac{121}{243}\end{aligned}$$

3. $\sum_{i=1}^n 3$

$$\begin{aligned}\sum_{i=1}^n 3 &= 3 \cdot \sum_{i=1}^n 1 \\ &= \frac{3 \cdot n \cdot (n+1)}{2}\end{aligned}$$

4. $\sum_{i=-3}^n 3$

$$\begin{aligned}\sum_{i=-3}^n 3 &= \sum_{i=-3}^0 3 + \sum_{i=1}^n 3 \\ &= 3 \cdot 4 + \frac{3 \cdot n \cdot (n+1)}{2} \\ &= \frac{3 \cdot n^2 + 3 \cdot n + 24}{2}\end{aligned}$$

5. $\sum_{k=0}^n 2^k + \sum_{k=5}^n 2^k$

$$\begin{aligned}
\sum_{k=0}^n 2^k + \sum_{k=5}^n 2^k &= 2^{n+1} - 1 + \sum_{k=5}^n 2^k \\
&= 2^{n+1} - 1 + \sum_{k=0}^n 2^k - \sum_{k=0}^4 2^k \\
&= 2^{n+1} - 1 + 2^{n+1} - 1 - (2^5 - 1) \\
&= 2 \cdot 2^{n+1} - 2 - (31) \\
&= 2 \cdot 2^{n+1} - 33 \\
&= 2^{n+2} - 33
\end{aligned}$$

$$6. \sum_{i=0}^n \left(\frac{2}{3}\right)^i + \sum_{i=-4}^n \left(\frac{2}{3}\right)^i$$

$$\begin{aligned}
\sum_{i=0}^n \left(\frac{2}{3}\right)^i + \sum_{i=-4}^n \left(\frac{2}{3}\right)^i &= \sum_{i=0}^n \left(\frac{2}{3}\right)^i + \sum_{i=-4}^{-1} \left(\frac{2}{3}\right)^i + \sum_{i=0}^n \left(\frac{2}{3}\right)^i \\
&= 2 \cdot \sum_{i=0}^n \left(\frac{2}{3}\right)^i + \sum_{i=-4}^{-1} \left(\frac{2}{3}\right)^i \\
&= 2 \cdot \frac{1}{1 - \frac{2}{3}} + \sum_{i=-4}^{-1} \left(\frac{2}{3}\right)^i \\
&= \frac{2}{\frac{1}{3}} + \sum_{i=-4}^{-1} \left(\frac{2}{3}\right)^i \\
&= 6 + \left(\frac{2}{3}\right)^{-4} + \left(\frac{2}{3}\right)^{-3} + \left(\frac{2}{3}\right)^{-2} + \left(\frac{2}{3}\right)^{-1} \\
&= \frac{291}{16}
\end{aligned}$$

$$7. \sum_{i=1}^n (i^3 + 2 \cdot i^2 - i + 1)$$

$$\begin{aligned}
\sum_{i=1}^n (i^3 + 2 \cdot i^2 - i + 1) &= \sum_{i=1}^n i^3 + \sum_{i=1}^n 2 \cdot i^2 - \sum_{i=1}^n i + \sum_{i=1}^n 1 \\
&= \left(\frac{n(n+1)}{2}\right)^2 + 2 \cdot \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + n \\
&= \left(\frac{n(n+1)}{2}\right)^2 + \frac{n(n+1)(2n+1)}{3} - \frac{n(n+1)}{2} + n
\end{aligned}$$

$$8. \sum_{i=5}^n (-4 \cdot i + \frac{i}{5})$$

$$\begin{aligned}
\sum_{i=5}^n (-4 \cdot i + \frac{i}{5}) &= \sum_{i=5}^n -4 \cdot i + \sum_{i=5}^n \frac{i}{5} \\
&= -4 \cdot \sum_{i=5}^n i + \frac{1}{5} \cdot \sum_{i=5}^n i \\
&= -4 \cdot \sum_{i=1}^n i + \frac{1}{5} \cdot \sum_{i=1}^n i + 4 \cdot \sum_{i=1}^4 i - \frac{1}{5} \cdot \sum_{i=1}^4 i \\
&= -4 \cdot \frac{n(n+1)}{2} + \frac{1}{5} \cdot \frac{n(n+1)}{2} + 4 \cdot \frac{4(5)}{2} - \frac{1}{5} \cdot \frac{4(5)}{2} \\
&= -4 \cdot \frac{n(n+1)}{2} + \frac{n(n+1)}{10} + 40 - 2 \\
&= -4 \cdot \frac{n(n+1)}{2} + \frac{n(n+1)}{10} + 38
\end{aligned}$$

$$9. \sum_{j=0}^k \sum_{i=1}^j (i - j^2 - 2)$$

$$\begin{aligned}
\sum_{j=0}^k \sum_{i=1}^j (i - j^2 - 2) &= \sum_{j=0}^k \sum_{i=1}^j i - \sum_{j=0}^k \sum_{i=1}^j j^2 - \sum_{j=0}^k \sum_{i=1}^j 2 \\
&= \sum_{j=0}^k \frac{j(j+1)}{2} - \sum_{j=0}^k j^3 - \sum_{j=0}^k 2 \cdot j \\
&= \sum_{j=0}^k \frac{j^2 + j}{2} - \sum_{j=0}^k j^3 - \sum_{j=0}^k 2 \cdot j \\
&= \sum_{j=0}^k \frac{j^2}{2} + \sum_{j=0}^k \frac{j}{2} - \sum_{j=0}^k j^3 - \sum_{j=0}^k 2 \cdot j \\
&= \frac{1}{2} \cdot \sum_{j=0}^k j^2 + \frac{1}{2} \cdot \sum_{j=0}^k j - \sum_{j=0}^k j^3 - 2 \cdot \sum_{j=0}^k j \\
&= \frac{n(n+1)(2n+1)}{12} + \frac{n(n+1)}{4} - (\frac{n(n+1)}{2})^2 - n(n+1)
\end{aligned}$$

$$10. \sum_{j=1}^m \sum_{k=1}^j (3 \cdot C + k - 3 \cdot j + i)$$

$$\begin{aligned}
\sum_{j=1}^m \sum_{k=1}^j (3 \cdot C + k - 3 \cdot j + i) &= \sum_{j=1}^m \sum_{k=1}^j 3 \cdot C + \sum_{j=1}^m \sum_{k=1}^j k - \sum_{j=1}^m \sum_{k=1}^j 3 \cdot j + \sum_{j=1}^m \sum_{k=1}^j i \\
&= \sum_{j=1}^m 3 \cdot C \cdot j + \sum_{j=1}^m \frac{j \cdot (j+1)}{2} - \sum_{j=1}^m 3 \cdot j^2 + \sum_{j=1}^m i \cdot j \\
&= 3 \cdot C \frac{m \cdot (m+1)}{2} + \frac{1}{2} \sum_{j=1}^m j^2 + \frac{1}{2} \sum_{j=1}^m j - 3 \cdot \sum_{j=1}^m j^2 + i \cdot \sum_{j=1}^m j \\
&= 3 \cdot C \cdot \frac{m \cdot (m+1)}{2} + \frac{1}{2} \cdot \frac{m \cdot (m+1)(2m+1)}{6} + \frac{1}{2} \cdot \frac{m \cdot (m+1)}{2} - \\
&\quad 3 \cdot \frac{m \cdot (m+1) \cdot (2m+1)}{6} + i \cdot \frac{m(m+1)}{2}
\end{aligned}$$

$$11. \sum_{l=-4}^n \sum_{j=1}^k \sum_{i=1}^j (i-4)$$

$$\begin{aligned}
\sum_{l=-4}^n \sum_{j=1}^k \sum_{i=1}^j (i-4) &= \sum_{l=-4}^n \sum_{j=1}^k \sum_{i=1}^j i - 4 \sum_{l=-4}^n \sum_{j=1}^k \sum_{i=1}^j 1 \\
&= \sum_{l=-4}^n \sum_{j=1}^k \frac{j \cdot (j+1)}{2} - 4 \cdot \sum_{l=-4}^n \cdot \sum_{j=1}^k j \\
&= \frac{1}{2} \cdot \sum_{l=-4}^n \sum_{j=1}^k j^2 + \frac{1}{2} \cdot \sum_{l=-4}^n \sum_{j=1}^k j - 4 \cdot \sum_{l=-4}^n \sum_{j=1}^k j \\
&= \frac{1}{2} \cdot \sum_{l=-4}^n \frac{k(k+1)(2k+1)}{6} + \frac{1}{2} \cdot \sum_{l=-4}^n \frac{k(k+1)}{2} - 4 \cdot \sum_{l=-4}^n \frac{k(k+1)}{2} \\
&= \frac{1}{2} \cdot (n+5) \cdot \frac{k \cdot (k+1) \cdot (2k+1)}{6} + \frac{1}{2} \cdot (n+5) \cdot \frac{k \cdot (k+1)}{2} \\
&\quad - 4 \cdot (n+5) \cdot \frac{k \cdot (k+1)}{2}
\end{aligned}$$

2. Calculate the answer

$$1. \log_4 x = 5 \rightarrow x = ?$$

$$\begin{aligned}
\log_4 x &= 5 \\
x &= 4^5
\end{aligned}$$

$$2. \log_3 y = 4 \rightarrow y = ?$$

$$\begin{aligned}
\log_3 y &= 4 \\
y &= 3^4
\end{aligned}$$

$$3. \ x = 7^2 \rightarrow \log_7 x = ?$$

$$\begin{aligned} x &= 7^2 \\ \log_7 x &= \log_7 7^2 \\ \log_7 x &= 2 \cdot \log_7 7 \\ \log_7 x &= 2 \end{aligned}$$

$$4. \ x = 32 \rightarrow \log_2 x = ?$$

$$\begin{aligned} x &= 32 \\ \log_2 x &= \log_2 32 \\ \log_2 x &= 5 \end{aligned}$$

$$5. \ 2^{\log 5} + 4^{\log 6} - 27^{\log 3 5}$$

$$\begin{aligned} 2^{\log 5} + 4^{\log 6} - 27^{\log 3 5} &= 5 + 2^{2^{\log 2 6}} - 3^{3^{\log 3 5}} \\ &= 5 + 2^{\log 2 6^2} - 3^{\log 3 5^3} \\ &= 5 + 6^2 - 5^3 \\ &= 5 + 36 - 125 \\ &= -84 \end{aligned}$$

$$6. \ 9^{\log 3 2} - 25^{\log 5 4} - 36^{\log 6 7} + 8^{\log 8 6}$$

$$\begin{aligned} 9^{\log 3 2} - 25^{\log 5 4} - 36^{\log 6 7} + 8^{\log 8 6} &= 3^{3^{\log 3 2}} - 5^{2^{\log 2 4}} - 6^{2^{\log 6 7}} + 6 \\ &= 3^{\log 3 8} - 5^{\log 2 16} - 6^{\log 6 49} + 6 \\ &= 8 - 5^4 - 49 + 6 \\ &= 660 \end{aligned}$$

$$7. \ \log(4^5 \times 8^3) - \log(16 - 8) + \log\left(\frac{2^{10}}{4 \times 3^2}\right)$$

$$\begin{aligned} \log_2(4^5 \times 8^3) - \log_2(16 - 8) + \log_2\left(\frac{2^{10}}{4 \times 3^2}\right) &= \log_2(4^5) + \log_2(8^3) - \log_2(8) + \log_2(2^{10}) - \log_2(4 \times 3^2) \\ &= 5 \cdot \log_2(4) + 3 \cdot \log_2(8) - \log_2(8) + 10 \cdot \log_2(2) - \log_2(4) \\ &= 5 \cdot 2 + 3 \cdot 3 - 3 + 10 - 2 - 2 \cdot 1.5 \\ &= 21 \end{aligned}$$

$$8. \log(3^2 \times 64^3) - \log\left(\frac{2^{10} \times 128^3}{9 \times 8^2}\right)$$

$$\begin{aligned} \log_2(3^2 \times 64^3) - \log_2\left(\frac{2^{10} \times 128^3}{9 \times 8^2}\right) &= \log_2(3^2) + \log_2(64^3) - \log_2(2^{10} \times 128^3) + \log_2(9) + \log_2 8^2 \\ &= 2 \cdot \log_2(3) + 3 \cdot \log_2(64) - 10 \cdot \log_2 2 - 3 \cdot \log_2 128 + 2 \cdot \log_2 3 + 2 \cdot \\ &= 2 \cdot 1.5 + 3 \cdot 6 - 10 - 3 \cdot 7 + 2 \cdot 1.5 + 2 \cdot 3 \\ &= -1 \end{aligned}$$

$$9. \log \log 16$$

$$\begin{aligned} \log_2 \log_2 16 &= \log_2 4 \\ &= 2 \end{aligned}$$

$$10. \log 16 \times \log 16$$

$$\begin{aligned} \log_2 16 \times \log_2 16 &= 4 \times 4 \\ &= 16 \end{aligned}$$

$$11. \log^2 16$$

$$\begin{aligned} \log_2^2 16 &= 4^2 \\ &= 16 \end{aligned}$$

$$12. \log_2 \log_5 625 - \log_3 \log_4 2^{3^9} + \log^4 2^5 - \frac{\log^2(4^3 \times 3^5)}{\log_5 125}$$

$$\begin{aligned} \log_2 \log_5 625 - \log_3 \log_4 2^{3^9} + \log^4 2^5 - \frac{\log^2(4^3 \times 3^5)}{\log_5 125} &= \log_2 4 - \log_3(27 \cdot \log_4 2) + \log_2^4 2^5 - \frac{\log_2^2(4^3 \times 3^5)}{\log_5 125} \\ &= 2 - \log_3\left(\frac{27}{2}\right) + 5^4 - \frac{\log_2^2(4^3) + \log_2(3^5)}{2} \\ &= 2 - \log_3(27) + \log_3(2) + 5^4 - \frac{(2 \cdot 3)^2 + 5 \cdot \log_2 3}{2} \\ &= 2 - 3 + 1.5 + 625 - \frac{36 + 5 \cdot 1.5}{2} \\ &= 603.75 \end{aligned}$$

$$13. \log \log_8 \log 256 + \log^5(3^2) \times 4^{\log 7}$$

$$\begin{aligned}
\log_2 \log_8 \log_2 256 + \log_2^5(3^2) \cdot 4^{\log_2 7} &= \log_2 \log_8 8 + \log_2^5(3^2) \cdot 4^{\log_2 7} \\
&= \log_2 1 + \log_2^5(3^2) \cdot 4^{\log_2 7} \\
&= \log_2^5(3^2) \cdot 4^{\log_2 7} \\
&= (2 \cdot 1.5)^5 \cdot 2^{\log_2 49} \\
&= (2 \cdot 1.5)^5 \cdot 49 \\
&= 11907
\end{aligned}$$

14. $\log_6 x = 5 \rightarrow \log_x 6 = ?$

$$\begin{aligned}
\log_6 x &= 5 \\
x &= 6^5 \\
\log_x x &= 5 \cdot \log_x 6 \\
1 &= 5 \cdot \log_x 6 \\
\frac{1}{5} &= \log_x 6
\end{aligned}$$

15. $\log_y x = 10 \rightarrow \log_x y = ?$

$$\begin{aligned}
\log_y x &= 10 \\
x &= y^{10} \\
\log_x x &= 10 \cdot \log_x y \\
\frac{1}{10} &= \log_x y
\end{aligned}$$

16. $\log_4 32 - \log_8^2 4$

$$\begin{aligned}
\log_4 32 - \log_8^2 4 &= \log_4 2 \cdot 16 - \log_8^2 4 \\
&= \log_4 2 + \log_4 16 - \log_8^2 4 \\
&= \frac{1}{2} + 2 - \left(\frac{1}{2}\right)^2 \\
&= \frac{1}{2} + 2 - \frac{1}{4} \\
&= 2.25
\end{aligned}$$

17. $\log_4 8 + \log_9 27 - \log_{25}^2 125 - \log_8^3 16 + \log_4 \log_2 256$

$$\begin{aligned}
\log_4 8 + \log_9 27 - \log_{25}^2 125 - \log_8^3 16 + \log_4 \log_2 256 &= \log_4 4 \cdot 2 + \log_9 9 \cdot 3 - \log_{25}^2 25 \cdot 5 - \log_8^3 8 \cdot 2 + \log_4 4 \cdot \log_2 256 \\
&= \log_4 4 + \log_4 2 + \log_9 9 + \log_9 3 - \log_{25}^2 25 - \log_{25}^2 5 \\
&\quad - \log_8^3 8 + \log_8^3 2 + \log_4 \log_2 256 \\
&= 1 + \frac{1}{2} + 1 + \frac{1}{2} - 1 - \frac{1}{4} - 1 + \frac{1}{27} + 2 \\
&= \frac{301}{108}
\end{aligned}$$

3. Compute the derivative of

$$1. -5 \cdot x^3 + 2 \cdot x - 1$$

$$\frac{d}{dx}(-5 \cdot x^3 + 2 \cdot x - 1) = -15 \cdot x^2 + 2$$

$$2. 3 \cdot x^4 - 2\sqrt{x} + x^{\frac{1}{2}} - 6x^{-\frac{2}{3}} - 5$$

$$\begin{aligned}
\frac{d}{dx}(3 \cdot x^4 - 2\sqrt{x} + x^{\frac{1}{2}} - 6x^{-\frac{2}{3}} - 5) &= 12 \cdot x^3 - x^{-\frac{1}{2}} + \frac{1}{2} \cdot x^{-\frac{1}{2}} + 4 \cdot x^{-\frac{5}{3}} \\
&= 12 \cdot x^3 - \frac{1}{2}x^{-\frac{1}{2}} + 4 \cdot x^{-\frac{5}{3}}
\end{aligned}$$

$$3. x \cdot \sqrt{x} + \sqrt{\sqrt{x}}$$

$$\frac{d}{dx}(x^{\frac{3}{2}} + x^{\frac{1}{4}}) = \frac{3}{2}x^{\frac{1}{2}} + \frac{1}{4} \cdot x^{-\frac{3}{4}}$$

$$4. \log x - x^2 \ln x + \ln x^4$$

$$\frac{d}{dx}(\log_2 x - x^2 \cdot \ln x + \ln x^4) = \frac{1}{x \cdot \ln 2} - 2 \cdot x \cdot \ln x - x + \frac{4}{x}$$

$$5. \ln^3(x\sqrt{2x-3}) + \sqrt{\ln x^2}$$

$$\begin{aligned}
\frac{d}{dx}(\ln^3(x \cdot \sqrt{2x-3}) + \sqrt{\ln x^2}) &= \frac{3 \cdot (\ln(x\sqrt{2x-3}))^2 \cdot (\sqrt{2x-3} + x \cdot (2 \cdot x - 3)^{-\frac{1}{2}})}{x\sqrt{2x-3}} \\
&\quad + \frac{1}{x \cdot \sqrt{\ln(x^2)}}
\end{aligned}$$

6. $\frac{\sqrt[4]{x+5}-\ln x}{(x-1)^3}$

$$\frac{d}{dx}\left(\frac{(x+5)^{\frac{1}{4}}-\ln x}{(x-1)^3}\right) = \frac{(x-1)^3 \cdot \left(\frac{1}{4} \cdot (x+5)^{-\frac{3}{4}} - \frac{1}{x}\right) - 3 \cdot ((x+5)^{\frac{1}{4}} - \ln x) \cdot (x-1)^2}{(x-1)^6}$$

4. Determine the limit of

1. $\lim_{x \rightarrow \infty} \frac{3x+2}{-5x-6}$

2. $\lim_{x \rightarrow \infty} \left(\frac{1}{x} + 3\right)$

3. $\lim_{x \rightarrow \infty} \frac{x^3+x-\sqrt{3x}}{\sqrt{x}}$

4. $\lim_{x \rightarrow \infty} \frac{x^3+x-\sqrt{3x}}{5 \cdot x^{2.25} \cdot \sqrt{\sqrt{x}}}$

5. $\lim_{x \rightarrow \infty} \frac{x^{0.1}-\sqrt{3}}{\sqrt{\sqrt{x}}}$

6. $\lim_{x \rightarrow \infty} \frac{x^x}{2^x}$

7. $\lim_{x \rightarrow \infty} \frac{x^x}{x(2^x)}$

8. $\lim_{x \rightarrow \infty} \frac{\sqrt{2}^{\log^4 x^3}}{\log(2 \cdot x + 7)}$

9. $\lim_{x \rightarrow \infty} \frac{x+1}{\frac{3 \cdot x \ln x}{2x^2}}$

10. $\lim_{x \rightarrow \infty} \frac{\sqrt{2}^{\log x^3}}{\log^{\ln x}(2x)}$

5. Compute the exact values for

1. $\int_1^n (2 \cdot x^4 + 5\sqrt{x}) \cdot dx$

$$\begin{aligned} \int_1^n (2 \cdot x^4 + 5\sqrt{x}) \cdot dx &= 2 \int_1^n x^4 dx + 5 \int_1^n \sqrt{x} dx \\ &= 2 \cdot \left(\frac{x^5}{5} \Big|_1^n\right) + 5 \left(\frac{2}{3} x^{\frac{3}{2}} \Big|_1^n\right) \\ &= 2 \cdot \left(\frac{n^5}{5} - \frac{1}{5}\right) + 5 \left(\frac{2}{3} n^{\frac{3}{2}} - \frac{2}{3}\right) \\ &= \frac{2 \cdot n^5}{5} - \frac{2}{5} + \frac{10}{3} n^{\frac{3}{2}} - \frac{10}{3} \end{aligned}$$

$$2. \int_1^n (x^4 - 3 \cdot x^2 + \frac{1}{x} - \frac{1}{x^2}) dx$$

$$\begin{aligned} \int_1^n (x^4 - 3 \cdot x^2 + \frac{1}{x} - \frac{1}{x^2}) dx &= \int_1^n x^4 dx - 3 \int_1^n x^2 dx + \int_1^n \frac{1}{x} dx - \int_1^n \frac{1}{x^2} dx \\ &= \frac{x^5}{5} \Big|_1^n - 3 \cdot \frac{x^3}{3} \Big|_1^n + \ln x \Big|_1^n + \frac{1}{x} \Big|_1^n \\ &= (\frac{n^5}{5} - \frac{1}{5}) - n^3 + 1 + \ln n + \frac{1}{n} - 1 \\ &= \frac{n^5}{5} - n^3 + \ln n + \frac{1}{n} - \frac{1}{5} \end{aligned}$$

$$3. \int_1^n (\frac{3}{\sqrt{x}} + \ln x + e^x) dx$$

$$4. \int_1^n x \cdot \sin x dx$$

6. Use mathematical induction to prove that

$$1 + 2 + \dots + n = \frac{n(n+1)}{2}$$

Proof. $1 + 2 + \dots + n = \frac{n(n+1)}{2}$

Base case $n = 1$: If $n = 1$ then the left hand side and the right hand side is $1 = 1 = \frac{1(2)}{2}$.

Inductive hypothesis: Suppose the theorem holds for all values of n up to some $k, k \geq 1$.

Inductive step: let $n = k + 1$ then our left hand side is

$$\begin{aligned} \sum_{i=1}^{k+1} i &= \sum_{i=1}^k i + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1) + 2 \cdot k + 2}{2} \\ &= \frac{(k+1) \cdot (k+2)}{2} \end{aligned}$$

which is equal to our right hand side. By the principle of mathematical induction, the theorem holds for all integers $n \geq 1$. \square

7. Use mathematical induction to prove that

$$1 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1 + 2 + \dots + n = \frac{n(n+1)}{2}$$

Proof. Base case $n = 1$: If $n = 1$ then the left hand side and the right hand side is $1^2 = 1 = \frac{1(2)(3)}{6}$.

Inductive hypothesis: Suppose the theorem holds for all values of n up to some $k, k \geq 1$.

Inductive step: let $n = k + 1$ then our left hand side is

$$\begin{aligned} \sum_{i=1}^{k+1} i^2 &= \sum_{i=1}^k i^2 + (k+1)^2 \\ &= \frac{k \cdot (k+1) \cdot (2 \cdot k + 1)}{6} + (k+1)^2 \\ &= \frac{k \cdot (k+1) \cdot (2 \cdot k + 1) + 6 \cdot (k+1)^2}{6} \\ &= \frac{(6 \cdot (k+1) + k \cdot (2 \cdot k + 1)) \cdot (k+1)}{6} \\ &= \frac{(6 \cdot k + 6 + 2 \cdot k^2 + k) \cdot (k+1)}{6} \\ &= \frac{(2 \cdot k^2 + 7 \cdot k + 6) \cdot (k+1)}{6} \\ &= \frac{(2 \cdot k^2 + 4 \cdot k + 3 \cdot k + 6) \cdot (k+1)}{6} \\ &= \frac{(2 \cdot k \cdot (k+2) + 3 \cdot (k+2)) \cdot (k+1)}{6} \\ &= \frac{(2 \cdot k + 3) \cdot (k+2) \cdot (k+1)}{6} \end{aligned}$$

which is equal to our right hand side. By the principle of mathematical induction, the theorem holds for all integers $n \geq 1$. \square