Lecture 6 (Growth of Functions)

Thursday, September 10, 2020 3:01 PM

(Reminder: HW2 is due this Saturday)

Example: Compare the growth of f(n) and g(n)

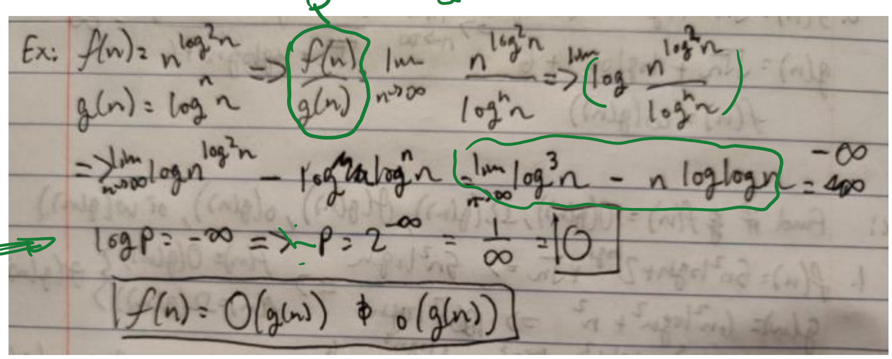
f(n)	O Ω Θ o ω	g(n)
$\underbrace{2^{\log n^2}}_{n^2} + \underbrace{4^{\log\sqrt{n}}}_{n} + n^2\log n$	O Ω Θ	$6n^2 + \underbrace{\log\log^{20} n}_{20\log\log n} + \underbrace{n^2\log n^2}_{2n^2\log n}$
$n^{1.06} + 6n\log^5 n$; $\cancel{n}\, n^{0.06}$, $\cancel{n}\log^5 n$	Ω ω	$2n\log^{10} n + 20n \Rightarrow \lim_{n\to\infty} \frac{n^{1.06}}{2\cancel{n}\log^{10} n} = \lim_{n\to\infty}\frac{n^{0.06}}{\log^{10} n} = \infty$
$n^{2.01}$; $\cancel{n^2}\, n^{0.01}$	Ω ω	$n^2\log^{50} n^{10}$; $\cancel{n^2}(\log n^{10})^{50}$

Example: Compare the growth of f(n) and g(n).

$f(n) = n^{\log^2 n}$

$g(n) = \log^n n$

Michael


Ex: $f(n) = n^{\log^2 n}$ ⇒ $\frac{f(n)}{g(n)}$ $\lim_{n\to\infty} \frac{n^{\log^2 n}}{\log^n n}$ ⇒ $\left(\log \frac{n^{\log^2 n}}{\log^n n}\right)$

$g(n) = \log^n n$

⇒ $\lim_{n\to\infty} \log n^{\log^2 n} - \log \log^n n$ $\lim_{n\to\infty} \log^3 n - n\log\log n = -\infty$

$\log P = -\infty \Rightarrow P = 2^{-\infty} = \frac{1}{\infty} = \boxed{0}$

$\boxed{f(n) = O(g(n))\ \&\ o(g(n))}$

$\log_a x = c \iff a^c = x$

$\Rightarrow \log\left(\frac{G^n}{\log^n n}\right) = \log G^n = n\log G$

Example: Compare the growth

$f(n) = 8n^{2+\sin n}$

$g(n) = n$ $\quad 1 \le 2+\sin n \le 3$

$\lim_{n\to\infty} \frac{8n^{2+\sin n}}{n} = \infty$ nr

① $\sin n = -1$ $\quad \lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{8n^{2-1}}{n^1} = \lim_{n\to\infty} \frac{8\cancel{n}}{\cancel{n}} = \boxed{8} \Rightarrow f = \Theta(g(n))$, O, Ω

② $\sin n = 0$ $\quad \lim_{n\to\infty} \frac{8n^2}{n} = \infty \Longrightarrow f(n) = \Omega(g(n))$, ω

③ $\sin n = 1 \Rightarrow \lim_{n\to\infty} \frac{8n^3}{n} = \infty \Longrightarrow$ " "

④ ... $2+\sin n$ ...

$n \to \infty$

④ $-1 < \sin n < 0 \Rightarrow \lim_{n\to\infty} \frac{8 n^{2+\sin n}}{n} = \lim_{n\to\infty} \frac{8 n^{1.\sim}}{n} = \infty \Rightarrow$

⑤ $0 < \sin n < 1 \Rightarrow \lim_{n\to\infty} \frac{8 n^{2.\sim}}{n} = \infty \Rightarrow$

$\Rightarrow \boxed{f(n) = \Omega(g(n))}$

$f(n) = n^{2+\sin n}$

$g(n) = n^2$

I cannot compare :(

$1 < 2+\sin n < 3$

① $\sin n = -1 \Rightarrow \lim \frac{n^1}{n^2} = 0 \leftarrow o$

② $\sin n = +1 \Rightarrow \lim \frac{n^3}{n^2} = \infty \leftarrow \omega\ \Omega$

③ X

Theorem: $\varepsilon > 0 \Rightarrow \log^k n = o(n^\varepsilon)$

① basis step: $k = 1 \Rightarrow \log n \overset{?}{=} o(n^\varepsilon)$

$$\Rightarrow \lim_{n\to\infty} \frac{\log n}{n^\varepsilon} \overset{HL}{=} \lim_{n\to\infty} \frac{\frac{1}{n \ln 2}}{\varepsilon n^{\varepsilon-1}} = \lim_{n\to\infty} \frac{1}{\underbrace{\varepsilon \ln 2}_{nr}\, n^1 n^{\varepsilon-1}}$$

$$= \lim_{n\to\infty} \frac{1}{\varepsilon \ln 2\, n^1} = \boxed{0} \Rightarrow \boxed{\log n = o(n^\varepsilon)}$$

② Inductive Step:

ⓐ assumption: when $k = p \Rightarrow \log^p n = o(n^\varepsilon) \Leftrightarrow \lim_{n\to\infty} \frac{\log^p n}{n^\varepsilon} = 0$

$(u^a)' = a u^{a-1} u'$ ⓑ proving $\log^{p+1} n \overset{?}{=} o(n^\varepsilon)$

nr

ⓐ

$(u^a)' = a u^{a-1} u'$

$((\log n)^{p+1})'$

(b) proving $\log^{p+1} n \stackrel{?}{=} o(n^{\varepsilon})$

$$\Rightarrow \lim_{n\to\infty} \frac{\log^{p+1} n}{n^{\varepsilon}} \stackrel{?L}{=} \lim_{n\to\infty} \frac{(p+1)\log^p n \, \frac{1}{n \ln 2}}{\varepsilon n^{\varepsilon-1}}$$

nr

$$= \lim_{n\to\infty} nr \frac{\log^p n}{n^{\varepsilon-1}\, n} = \lim_{n\to\infty} nr \frac{\log^p n}{n^{\varepsilon}} = 0 \checkmark \Rightarrow \boxed{\log^{p+1} n = o(n^{\varepsilon})}$$

based on (a)

Induction:

any $k=1, 2, 3, 4, 5, \sim$

(I) $k=1$ ✓

(II) $\underbrace{k=p}_{\text{if}} \Rightarrow \underbrace{k=p+1}_{\text{then}}$ ✓

Example (Sergio asked this)

2^n and n^k $\forall k$

$$\lim_{n\to\infty} \frac{2^n}{n^k} = p \Rightarrow \lim_{n\to\infty} (n \log 2 - k \log n) = \lim (n - k \log n) = +\infty$$

nr

$$\Rightarrow \lim_{n\to\infty} \log p = \infty \Leftrightarrow \lim p = 2^{\infty} = \infty$$

$$\Rightarrow \boxed{2^n = \omega(n^k)}$$

Example: Compare the growth:

$f(n)$	o / ω	$g(n)$
$n^{0.001}$	ω	$(\log n)^{1000}$
$n \log n$	ω	n
$n^{2.1}$	ω	$n^{2.01}$ $\log^{200} n^5$

$n^{2.1}$	ω	$n^{2.01}$	$\log^{200} n^5$
$n^2 \log^2 n$	O	$2^{\log n^2} \times \log^3 n$	

$n^2 \log^3 n$

Hints Q6 7 11

play with

? ?

$f(n) = O(g(n)) \iff \exists c > 0, \exists k \geq 0$ s.t. $f(n) \leq c\, g(n)\ \forall n \geq k$

$g(n) = O(h(n)) \iff \exists c_1 > 0, \exists k_1 \geq 0$ s.t. $g(n) \leq c_1 h(n)\ \forall n \geq k_1$

$\Rightarrow$

Q11) third def for $\Theta \Rightarrow f(n) = \Theta(g(n)) \iff c_1 g(n) \leq f(n) \leq c_2 g(n)$

$\hookrightarrow$ find c_1 and c_2

Theorem: $f(n) + g(n) = \Theta(\max(f(n), g(n)))$

$\Leftarrow$ ① $f(n) + g(n) = O(\max(f(n), g(n)))$

② $f(n) + g(n) = \Omega(\max(f(n), g(n)))$

① $\iff \exists c > 0, \exists k \geq 0$ s.t. $f(n) + g(n) \overset{?}{\leq} c \max(f(n) + g(n)), \forall n \geq k$

$\max(4, 8) = 8$

$4 \leq \max(4, 8)$

$8 \leq \max(4, 8)$

$\Rightarrow$ $\oplus$

$f(n) \leq \max(f(n), g(n))$

$g(n) \leq \max(f(n), g(n))$

$f(n) + g(n) \leq 2 \max(f(n), g(n))$

$\hookrightarrow c \geq 2$

$\Rightarrow \boxed{c = 2}$

② $\Leftarrow \exists c_1 > 0, \exists k_1 \geq 0$ s.t. $f(n) + g(n) \overset{?}{\geq} c_1 \max(f(n), g(n))\ \forall n \geq k_1$

$\max(4, 8) = 8$

$8 \geq \max(4, 8)$

WLOG (without loss of generality): $f(n) = \max(f(n), g(n))$

$\Rightarrow$ $f(n) \geq \max(f(n), g(n))$

$\oplus$

$8 \geq \max(4,8) \Longrightarrow$ ④ $f(n) \geq \max(f(n), g(n))$

runtime $\to g(n) \geq 0$

✓ $\to f(n)+g(n) \geq \max(f(n), g(n))$

$5 \geq \frac{1}{2} \cdot 3 \to 0 < c_1 \leq 1$

$\to c_1 = \frac{1}{2}$ ✓

$\deg(f(n)) = a$

$\deg(g(n)) = b$

① $f(n) = O(g(n)) \iff a \leq b$

② $f(n) = \Omega(g(n)) \iff a \geq b$

③ $f(n) = \Theta(g(n)) \iff a = b$

④ $f(n) = o(g(n)) \iff a < b$

⑤ $f(n) = \omega(g(n)) \iff a > b$

$\Rightarrow$ ④ $\Rightarrow$ ①

$\Rightarrow$ ⑤ $\Rightarrow$ ②

Today very first example

Example: What is the growth of $f(n)$?

① $f(n) = n^3 + n^2 \log n + \log^3 n = \boxed{\Theta(n^3)}$

② $n^2 \log^2 n + n^2 \log^3(n+6) + n^2 = \Theta(n^2 \log^3(n+6))$

$= \boxed{\Theta(n^2 \log^3 n)}$

③ $n^{0.1} + \log^3 n + n^{.1} \log n^5 = \Theta(n^{.1} \log n^5) = \Theta(n^{.1} \log n)$

(3) $n^{0.1} + \log^3 n + n^{.1} \log n^5 = \Theta(n^{.1} \log n^5) = \boxed{\Theta(n^{.1} \log n)}$

(4) $\underbrace{4^{\log n^2}}_{n^4} + n^3 \log^2(n) + \underbrace{n^{\log_n n^2}}_{n^2} = \boxed{\Theta(n^4)}$

(5) $\underbrace{6(n+5)^3}_{n^3} + \underbrace{n^2 \log^5 n^2}_{n^2 \log^5 n} + \underbrace{n\sqrt{n}} = \boxed{\Theta(n^3)}$