**1.** Use the definition of big-$\Omega$ to prove that $n \cdot log_2(n) = \Omega(n + n \cdot log_2(n^2))$. Provide appropriate $C$ and $k$ constants.

$$f(n) = \Omega(g(n)) \iff \exists c > 0, \exists k \geq 0 \, s.t \, f(n) \geq c \cdot g(n) \, \forall n \geq k$$

$$\begin{aligned} n + nlog_2(n^2) &= n + 2 \cdot n \cdot log_2(n) \\ &\leq n \cdot log_2(n) + 2 \cdot n \cdot log_2(n) \\ &= 3 \cdot n \cdot log_2(n), n \geq 1 \end{aligned}$$

Let $c = \frac{1}{3}$ and $k = 1$ then we have the equality

$$c \cdot (n + n \cdot log_2(n^2)) \leq n \cdot log_2(n), n \geq k$$

which by the definition of big-$\Omega$ notation $n \cdot log_2(n) = \Omega(n + n \cdot log_2(n^2))$.

**2.** Provide the big-$O$ relationshp between $f(n) = n \cdot log_2(n)$ and $g(n) = n + n \cdot log_2(n^2)$.

$$\begin{aligned} n + n \cdot log_2(n^2) &= n + 2 \cdot n \cdot log_2(n) \\ &\leq n \cdot log_2(n) + 2 \cdot n \cdot log_2(n) \\ &= 3 \cdot n \cdot log_2(n), n \geq 1 \end{aligned}$$

Let $c = 3$ and $k = 1$ then we have the equality

$$n + n \cdot log_2(n^2) \leq c \cdot n \cdot log_2(n), n \geq k$$

which by the definition of Big-$O$ notation, $n + n \cdot log_2(n^2)$ is $O(n \cdot log_2(n))$.

**3.** Prove that $f(n) = O(g(n)) \iff g(n) = \Omega(f(n))$.

Forward direction: $f(n) = O(g(n)) \implies g(n) = \Omega(f(n))$.

Since $f(n) = O(g(n))$ there exists constants $c$ and $k$ s.t $f(n) \leq c \cdot g(n), \forall n \geq k$ where $c > 0$ and $k \geq 0$. dividing by $c$ we obtain the equality

$$\frac{1}{c} \cdot f(n) \leq g(n), n \geq k$$

which by the definition of Big-$\Omega$ notation, $g(n) = \Omega(f(n))$.

Backward direction: $g(n) = \Omega(f(n)) \implies f(n) = O(g(n))$.

Since $g(n) = \Omega(f(n))$ there exists constants $c$ and $k$ s.t $g(n) \geq c \cdot f(n), \forall n \geq k$ where $c > 0$ and $k \geq 0$. Dividing by $c$ we obtain the equality

$$g(n) \cdot \frac{1}{c} \quad \geq \quad f(n), n \geq k$$

which by the definition of Big-$O$ notation, $f(n) = O(g(n))$.

We conclude $f(n) = O(g(n)) \iff g(n) = \Omega(f(n))$.

## 4. Use the definition of Big-$\Theta$ to prove that $f(n)+g(n) = \Theta(max(f(n), g(n)))$

$$f(n) = \Theta(g(n)) \quad \iff \quad \exists c_1, c_2 > 0, \exists k \geq 0 \, s.t \, c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n), \forall n \geq k$$

showing $f(n) + g(n) = O(max(f(n), g(n)))$.

$$f(n) + g(n) \quad \leq \quad 2 \cdot max(f(n), g(n))), n \geq 1$$

let $c = 1$ and $k = 1$ then equality translates to

$$f(n) + g(n) \quad \leq \quad c \cdot max(f(n), g(n))), n \geq k$$

which by the definition of Big-$O$ notation, $f(n) + g(n) = O(max(f(n), g(n)))$.

showing $f(n) + g(n) = \Omega(max(f(n), g(n)))$.

$$max(f(n), g(n))) \quad \leq \quad f(n) + g(n), n \geq 1$$

let $c = 1$ and $k = 1$, then the statement translates to

$$c \cdot max(f(n), g(n))) \quad \leq \quad (f(n) + g(n)), n \geq k$$

which by the definition of Big-$\Omega$ notation, $f(n) + g(n) = \Omega(max(f(n), g(n)))$. Since $f(n)+g(n) = O(max(f(n), g(n)))$ and $f(n)+g(n) = \Omega(max(f(n), g(n)))$ by the definition of Big-$\Theta$ notation, $f(n) + g(n) = \Theta(max(f(n), g(n)))$.

## 5. Prove that $(n + a)^b = \Theta(n^b)$.

$$f(n) = O(g(n)) \quad \iff \quad \exists c > 0, \exists k \geq 0 \, s.t \, f(n) \leq c \cdot g(n) \, \forall n \geq k$$

$$
\begin{aligned}
(n+a)^b &= \sum_{i=0}^{b} \binom{b}{i} \cdot n^{b-i} \cdot a^i \\
&\leq \sum_{i=0}^{b} \binom{b}{i} \cdot n^b, n \geq 1
\end{aligned}
$$

Let $c = \sum_{i=0}^{b} \binom{b}{i}$ and $k = 1$ then the statements translates to

$$
f(n) \quad \leq \quad c \cdot g(n), n \geq k
$$

which by the definition of $O$ notation $f(n) = O(g(n))$.

2. Showing $(n+a)^b = \Omega(n^b)$

$$
f(n) = \Omega(g(n)) \quad \iff \quad \exists c > 0, \exists k \geq 0 \, s.t \, f(n) \geq c \cdot g(n) \, \forall n \geq k
$$

$$
\begin{aligned}
(n+a)^b &= \sum_{i=0}^{b} \binom{n}{i} \cdot n^{b-i} \cdot a^i \\
&\geq n^{100}
\end{aligned}
$$

Let $c = 1$ and $k = 1$ then the statement translates to

$$
f(n) \quad \geq \quad c \cdot g(n), n \geq k
$$

which by the definition of $\Omega$ notation $f(n) = \Omega(g(n))$.

Since $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ by the definition of $\Theta$ notation $f(n) = \Theta(n^b)$.

## **6.** Prove that $lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, then $f(n) = O(g(n))$, but $g(n) \neq O(f(n))$.

$f(n) = o(g(n)) \implies f(n) = O(g(n))$. Thus there exists a constant $c > 0$ such that $f(n) \leq c \cdot g(n)$ for sufficiently large $n$. Now suppose $g(n) = O(f(n))$ then there must exist $c' > 0$ such that $g(n) \leq c' \cdot f(n)$ for sufficiently large $n$. $\frac{1}{c'} \leq \frac{f(n)}{g(n)} = 0$ as $n$ approaches 0. Thus $c' \leq 0$ which is a contradiction. We conclude that $lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, then $f(n) = O(g(n))$, but $g(n) \neq O(f(n))$.

**7.** Prove or disprove: $2^{n+1} = O(2^n)$.

$$2 \cdot 2^n \leq 3 \cdot 2^n, n \geq 1$$

Let $c = 3$ and $k = 1$ then the statement translate to

$$2^{n+1} \leq c \cdot 2^n, n \geq k$$

which by the definition of big-$O$ notation, $2^{n+1} = O(2^n)$.

**8.** Prove or disprove: $2^{2 \cdot n} = O(2^n)$.

$$\begin{aligned} 2^{2 \cdot n} &\leq c \cdot 2^n, n \geq k \\ 2^n &\leq c \end{aligned}$$

which is a contradiction, hence $2^{2 \cdot n} \neq O(2^n)$.

**9.** Use any technique to determine a succint big-$\Theta$ expression for the growth of the function $log^{50}(n) \cdot n^2 + log(n^4) \cdot n^{2.1} + 1000 \cdot n^2 + 100000000 \cdot n$

$\Theta(log(n) \cdot n^{2.1})$.

**10.** Prove or disprove: if $f(n) = O(g(n))$, then $2^{f(n)} = O(2^{g(n)})$.

$f(n) = 2 \cdot n$ and $g(n) = n$, then $f(n) = O(g(n))$ but $2^{2 \cdot n} \neq O(2^n)$.

**11.** Prove transitivity of big-$O$: if $f(n) = O(g(n))$ and $g(n) = O(h(n))$ then $f(n) = O(h(n))$.

Since $f(n) = O(g(n))$ and $g(n) = O(h(n))$ we have the equalities

$$\begin{aligned} f(n) &\leq c_1 \cdot g(n), n \geq k_1 \\ g(n) &\leq c_2 \cdot h(n), n \geq k_2 \end{aligned}$$

From this we obtain

$$f(n) \leq c_1 \cdot c_2 \cdot h(n), n \geq k'$$

where $k' = max(k_1, k_2)$. Let $c = c_1 \cdot c_2$ and $k = k'$ the statement then translate to

$$f(n) \leq c \cdot h(n), n \geq k$$

which by the definition of $O$ notation, $f(n) = O(h(n))$.

**12.** If $g(n) = o(f(n))$, then prove that $f(n) + g(n) = \Theta(f(n))$.

By theorem 2 $f(n)+g(n) = \Theta(max(f(n), g(n))) = \Theta(f(n))$ for sufficiently large $n$.

**13.** Use L'hopital's rule to prove Theorem 1. Assume $a \geq b$.

Let $p(n) = n^a$ and $q(n) = n^b$ and $a > b$.

$$\begin{aligned} lim_{n\to\infty}\frac{n^a}{n^b} &= lim_{n\to\infty}n^{a-b} \\ &= \infty \end{aligned}$$

$p(n) = \omega(q(n))$. $p(n) = \omega(q(n)) \implies p(n) = \Omega(q(n))$.

$$\begin{aligned} lim_{n\to\infty}\frac{n^b}{n^a} &= lim_{n\to\infty}\frac{1}{n^{a-b}} \\ &= 0 \end{aligned}$$

$q(n) = o(p(n)$. $q(n) = o(p(n)) \implies q(n) = O(p(n))$.
Now let $a = b$

$$lim_{n\to\infty}\frac{n^a}{n^b} = C$$

which implies $p(n) = \Theta(q(n))$.

**14.** Use L'hopital's rule to prove that $a^n = \omega(n^k)$, for every real $a > 1$ and integer $k \geq 1$.

The $kth$ deriative as a function of $n$ of $a^n$ is $ln^k a \cdot a^n$ while the $kth$ deriative of $n^k$ is $k!$. it follows the ratio of the limit is

$$lim_{n\to\infty}\frac{ln^k a \cdot a^n}{k!} = \infty$$

which by the definition of little-$\omega$ notation, $a^n = \omega(n^k)$. Since $a$ and $k$ were arbitrary this holds for all real $a > 1$ and all integer $k \geq 1$.

**15.** Prove that $log_a(n) = \Theta(log_b(n))$ for all $a, b > 0$.

$$\begin{aligned} log_a(n) &= \frac{log_b(n)}{log_a(b)} \\ \frac{log_b(n)}{log_a(b)} \leq log_a(n) &\leq \frac{log_b(n)}{log_a(b)} \end{aligned}$$

Therefore, $log_a(n) = \Theta(log_b(n))$.