Lecture 13 (Divide & Conquer Algorithms)

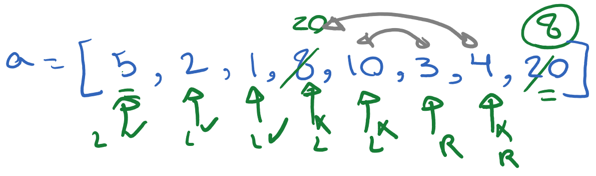
Tuesday, October 6, 2020 5:00 PM

Reminder: * Exam 1 is this Thursday
* Start HW 5

Example: Apply partitioning step to a.

Θ(n) ① find pivot
② divide a to 2 sub arrays

a = [5, 2, 1, ~~8~~, 10, 3, 4, ~~20~~]



pivot = m(5, 8, 20) = $\boxed{8}$

[5 | 2 | 1 | 4 | 3] a_L [10 | 20] a_R

a = [0, 5, 7, 1, 2, 3, 10, 20, 6]

```
L = 0, R = a.length-1      your choice
while  L <= R
    L++     <- keep increasing
    R--     <- keep decreasing
    swap
end
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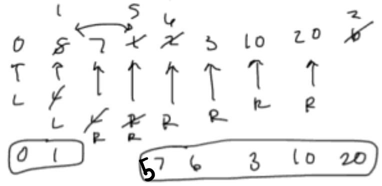
Kristinaann

pivot = m(0, 2, 4) = 2



Example:

① $T(\boxed{n}) = 3T(\frac{\boxed{n}}{4}) + \sqrt{\boxed{n}}$ ⇒ $T(\boxed{\ }) = 3T(\frac{\boxed{\ }}{4}) + \sqrt{\boxed{\ }}$

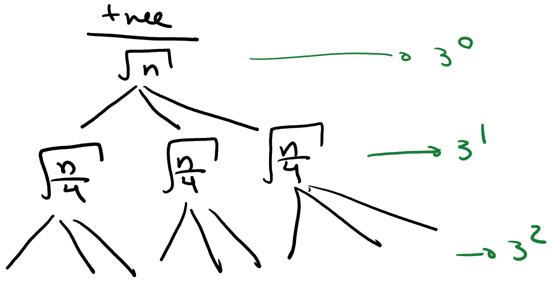
input size

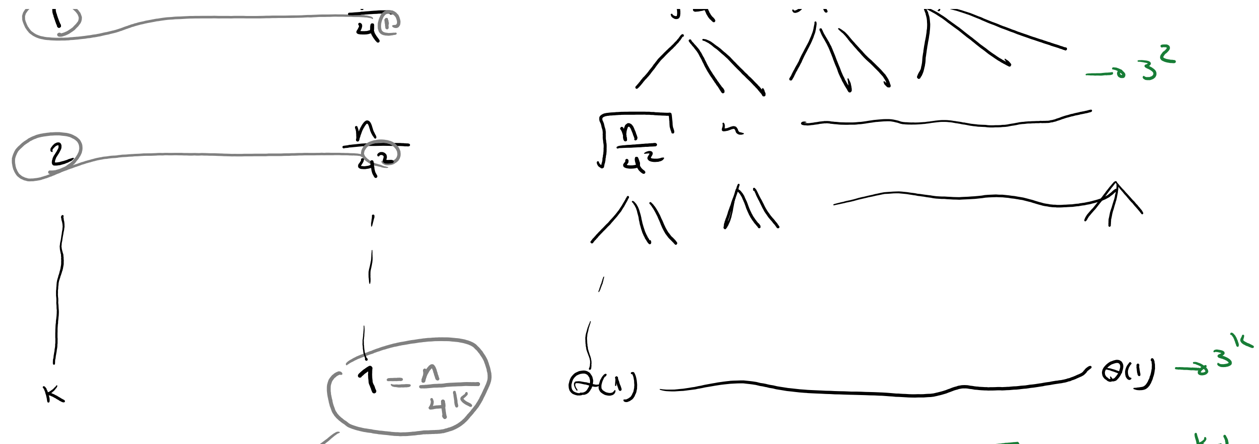
subproblems

time to divide & combine

step	size
0	$\frac{n}{4^{0}}$
1	$\frac{n}{4^{1}}$

tree





$n = 4^k$

$k = \log_4 n$

$$T(n) = 3^{0}\sqrt{\frac{n}{4^0}} + 3\sqrt{\frac{n}{4}} + 3^2\sqrt{\frac{n}{4^2}} + \sim + 3^{k-1}\sqrt{\frac{n}{4^{k-1}}} + 3^k\cancel{\Theta}(1)$$

$$\simeq \left(\sum_{i=0}^{k-1} 3^i\sqrt{\frac{n}{4^i}}\right) + \left(3^k\right)$$

$$= \sum 3^i \frac{\sqrt{n}}{\sqrt{4^i}} + 3^k$$

$$= \sum 3^i \frac{\sqrt{n}}{2^{2i/2}} + 3^k$$

$$= \left(\sum_{i=0}^{k-1} 3^i \frac{\sqrt{n}}{2^i}\right) + 3^k$$

$$= \sqrt{n} \sum_{i=0}^{k-1} \frac{3^i}{2^i} + 3^k$$

$$= \sqrt{n} \sum_{i=0}^{k-1} \left(\frac{3}{2}\right)^i + 3^k$$

$$= \sqrt{n}\left(\frac{\left(\frac{3}{2}\right)^k \cancel{-1}}{\underset{+}{\boxed{\frac{3}{\cancel{2}}-1}}}\right) + 3^k$$

$$\boxed{\frac{3}{2}}^k \xrightarrow{} \infty \gg 1$$

is alone

$k = \log_4 n$

$$\sim \sqrt{n}\left(\frac{3}{2}\right)^k + 3^k$$

$$= \underbrace{\sqrt{n}}_{n^{1/2}} \underbrace{\left(\frac{3}{2}\right)^{\log_4 n}}_{\log 3/2} + 3^{\log_4 n}$$

$$n^{\log_4 3}$$

$a^x a^y = a^{x+y}$

$\log p + \log s = \log ps$

$$= \overbrace{n^{1/2}}\ \overbrace{n^{\log_4 3/2}} + n^{\log_4 3}$$

$$= \overbrace{n^{\log_4 2}}\ n^{\log_4 3/2} + n^{\log_4 3}$$

$$= n^{\log_4 2 + \log_4 3/2} + n^{\log_4 3}$$

$$= n^{\log_4 \cancel{2}\times \frac{3}{\cancel{2}}} + n^{\log_4 3}$$

$$= n^{\log_4 3} + n^{\log_4 3} = \Theta\left(n^{\log_4 3}\right) = \Theta\left(n^{0.\sim}\right)$$

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② $T(n) = 2T\left(\frac{n}{3}\right) + n^2$

③ $T(n) = 4T\left(\frac{n}{2}\right) + n$

④ $T(n) = 4\,T\left(\frac{n}{3}\right) + n^5$

⑤ $T(n) = 6T(n-1) + 1$

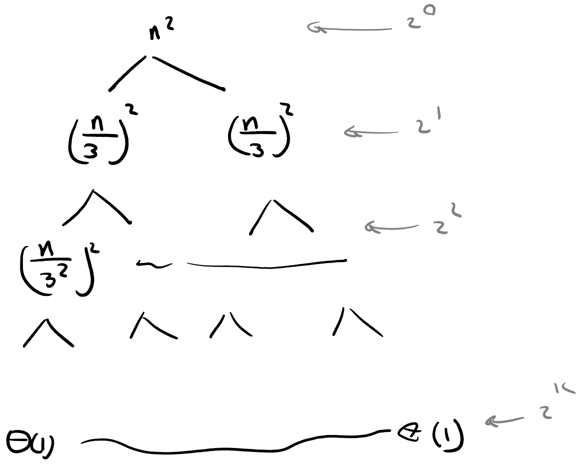
⑥ $T(n) = T\left(\frac{n}{2}\right) + 1$

② $T(n) = 2\,T\left(\frac{n}{3}\right) + n^2$

Step	size
⓪	$\frac{n}{3^0}$
①	$\frac{n}{3^1}$
②	$\frac{n}{3^2}$
|	|
Ⓚ	$1 = \frac{n}{3^K}$

$n = 3^K$

tree



$1 = \frac{n}{3^K}$

$n = 3^K$

$K = \log_3 n$

$\Theta(1)$ —— $\Theta(1)$

$$T(n) = \overset{0}{2}\left(\frac{n}{3^0}\right)^2 + 2\left(\frac{n}{3}\right)^2 + 2^2\left(\frac{n}{3^2}\right)^2 + \sim + 2^{K-1}\left(\frac{n}{3^{K-1}}\right)^2 + 2^K\,\Theta(1)$$

nv

$$= \sum_{i=0}^{K-1} 2^i\left(\frac{n}{3^i}\right)^2 + 2^K$$

$$= n^2 \sum_{i=0}^{K-1} \frac{2^i}{(3^i)^2} + 2^K$$

Note: $(3^i)^2 = 3^{2i} = (3^2)^i$

$$= n^2 \sum \frac{2^i}{9^i} + 2^K$$

$$= n^2 \sum_{i=0}^{K-1} \left(\frac{2}{9}\right)^i + 2^K$$

$$= n^2 \frac{\left(\frac{2}{9}\right)^K \xrightarrow{} 0 \;\; - 1}{\frac{2}{9} - 1} + 2^K$$

because $0 < \frac{2}{9} < 1$

$\left(\frac{2}{9}\right)^K \to 0$

$$\sim n^2 + 2^K = n^2 + 2^{\log_3 n}$$

$K = \log_3 n$

$$= n^2 + n^{\overset{0.\sim}{\log_3 2}} = \boxed{\Theta(n^2)} \checkmark$$

$$\underset{0}{\log_3 1} < \log_3 2 < \underset{1}{\log_3 3}$$

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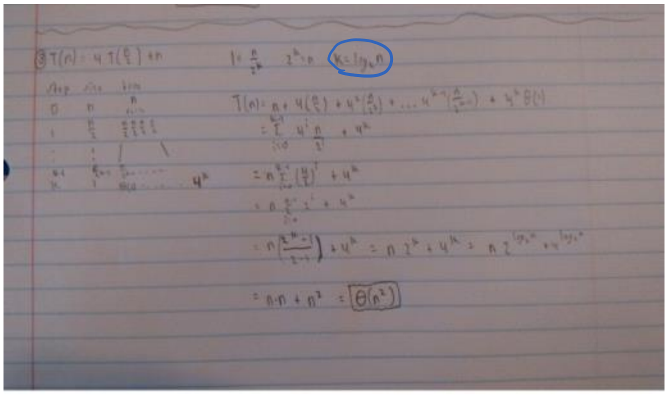
③

Matthew's answer :)



③ $T(n) = 4T(\frac{n}{2}) + n$

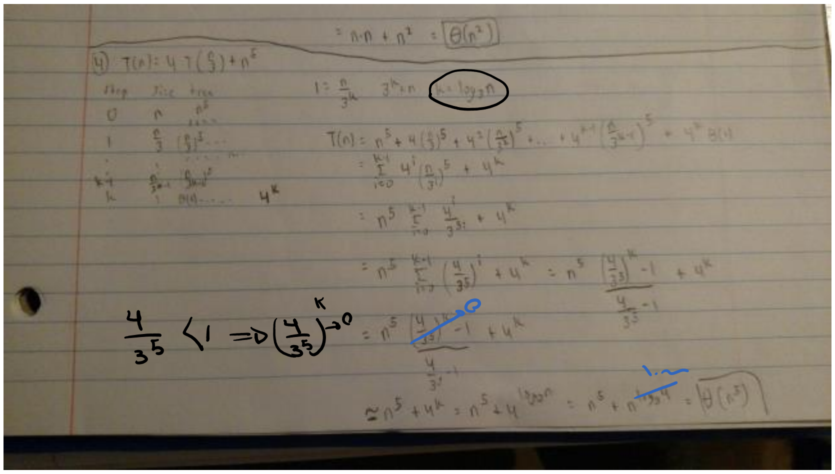
$1 = \frac{n}{2^k}$ $2^k = n$ $k = \log_2 n$

step	size	tree
0	n	n
1	$\frac{n}{2}$	$\frac{n}{2}\ \frac{n}{2}\ \frac{n}{2}\ \frac{n}{2}$
$\vdots$	$\vdots$	
$k-1$	$\frac{n}{2^{k-1}}$	$\frac{n}{2^{k-1}} \cdots$
k	1	$\Theta(1) \cdots\ 4^k$

$$T(n) = n + 4(\tfrac{n}{2}) + 4^2(\tfrac{n}{2^2}) + \dots 4^{k-1}(\tfrac{n}{2^{k-1}}) + 4^k\Theta(1)$$
$$= \sum_{i=0}^{k-1} 4^i \frac{n}{2^i} + 4^k$$
$$= n\sum_{i=0}^{k-1} \left(\frac{4}{2}\right)^i + 4^k$$
$$= n\sum_{i=0}^{k-1} 2^i + 4^k$$
$$= n\left(\frac{2^k-1}{2-1}\right) + 4^k = n\,2^k + 4^k = n\,2^{\log_2 n} + 4^{\log_2 n}$$
$$= n \cdot n + n^2 = \boxed{\Theta(n^2)}$$

④

Matthew's ans :D



$= n \cdot n + n^2 = \boxed{\Theta(n^2)}$

④ $T(n) = 4T(\frac{n}{3}) + n^5$

$1 = \frac{n}{3^k}$ $3^k = n$ $k = \log_3 n$

step	size	tree
0	n	n^5
1	$\frac{n}{3}$	$(\frac{n}{3})^5 \cdots$
$\vdots$	$\vdots$	
$k-1$	$\frac{n}{3^{k-1}}$	$(\frac{n}{3^{k-1}})^5$
k	1	$\Theta(1) \cdots\ 4^k$

$$T(n) = n^5 + 4(\tfrac{n}{3})^5 + 4^2(\tfrac{n}{3^2})^5 + \dots + 4^{k-1}(\tfrac{n}{3^{k-1}})^5 + 4^k\Theta(1)$$
$$= \sum_{i=0}^{k-1} 4^i \left(\frac{n}{3^i}\right)^5 + 4^k$$
$$= n^5 \sum_{i=0}^{k-1} \frac{4^i}{3^{5i}} + 4^k$$
$$= n^5 \sum_{i=0}^{k-1} \left(\frac{4}{3^5}\right)^i + 4^k = n^5 \frac{\left(\frac{4}{3^5}\right)^k - 1}{\frac{4}{3^5} - 1} + 4^k$$

$$\frac{4}{3^5} < 1 \Rightarrow \left(\frac{4}{3^5}\right)^k \to 0$$

$$= n^5 \frac{\cancel{\left(\frac{4}{3^5}\right)^k}^{\,0} - 1}{\frac{4}{3^5} - 1} + 4^k$$

$$\approx n^5 + 4^k = n^5 + 4^{\log_3 n} = n^5 + n^{\overset{1.\sim}{\cancel{\log_3 4}}} = \boxed{\Theta(n^5)}$$