1 Use Θ notation to express the statement

$$4n^6 \quad \leq \quad 17n^6 - 45n^3 + 2n + 8 \leq 30n^6, n \geq 3$$

Let $A = 4$, $B = 30$ and $k = 3$ then the statement translates to

$$An^6 \quad \leq 17n^6 - 45n^3 + 2n + 8 \quad \leq Bn^6, n \geq k$$

hence by the definition of Θ notation $17n^6 - 45n^3 + 2n + 8$ is $\Theta(n^6)$.

2 Use Ω notation to express the statement

1. Use Ω notation to expres the statement

   $$\frac{11}{4}n^2 \quad \leq \quad 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2, n \geq 2$$

   Let $A = \frac{11}{4}$ and $k = 2$ then $An^2 \leq 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2, n \geq 2$ then the statement translates to

   $$An^2 \quad \leq \quad 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2, n \geq k$$

   which by the definition of Ω notation, $3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2$ is $\Omega(n^2)$.

2. Use O notation to express the statement

   $$0 \quad \leq \quad 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2 \leq 6n^2, n \geq 1$$

   Let $A = 6$ and $k = 1$ then the statement translates to

   $$0 \quad \leq 3 \cdot (\lfloor \tfrac{n}{4} \rfloor)^2 + 5n^2 \quad \leq An^2, n \geq k$$

   which by the defnition of O notation, $3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2$ is $O(n^2)$.

3. Justify the statement: $3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2$ is $\Theta(n^2)$.
   Let $A = \frac{11}{4}, B = 6$ and $k = 2$ then $A \cdot n^2 \leq 3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2 \leq Bn^2, n \geq k$ which by the definition of Θ notation, $3 \cdot (\lfloor \frac{n}{4} \rfloor)^2 + 5n^2$ is $\Theta(n^2)$.

3. Given the function $15n^3 + 11n^2 + 9$

1. Show that the function is $\Omega(n^3)$.

$$15n^3 \leq 15n^3 + 11n^2 + 9, n \geq 1$$

   Let $A = 15$ and $k = 1$ then the statements translates to $An^3 \leq 15n^3 + 11n^2 + 9, n \geq k$ which by the definition of Ω notation, $15n^3 + 11n^2 + 9$ is $\Omega(n^3)$.

2. Show that the function is $O(n^3)$.

$$\begin{aligned} 15n^3 + 11n^2 + 9 &\leq 15n^3 + 11n^3 + 9n^3 \\ &\leq 35n^3, n \geq 1 \end{aligned}$$

   Let $A = 35$ and $k = 1$ then the statement translates to $15n^3 + 11n^2 + 9 \leq An^3, n \geq k$ which by the definition of O notation, $15n^3+11n^2+9$ is $O(n^3)$.

4. Given the function $n^4 - 5n - 8$

1. Show that the function is $\Omega(n^4)$.
   Let $A = \frac{1}{2}$ and $a = (|-5|+|-8|)$

$$\begin{aligned} n &\geq \frac{2}{1} \cdot (|-5|+|-8|) \\ \frac{1}{2}n^4 &\geq 5n^3 + 8n^3 \\ \frac{1}{2}n^4 &\geq 5n + 8 \\ n^4 - 5n - 8 &\geq \frac{1}{2}n^4, n \geq a \end{aligned}$$

   Hence by the definition of Ω notation, $n^4 - 5n - 8$ is $\Omega(n^4)$.

2. Show that the function is $O(n^4)$.

$$\begin{aligned} n^4 - 5n - 8 &\leq n^4 + 5n + 8 \\ &\leq n^4 + 5n^4 + 8n4 \\ &= 14n^4, n \geq 1 \end{aligned}$$

   Let $A = 14$ and $k = 1$ then the statement translates to $n^4 - 5n - 8 \leq An^4, n \geq k$ which by teh definition of O notation translates, $n^4 - 5n - 8$ is $O(n^4)$.

5. Show that $15n^3 + 11n^2 + 9$ is $\Theta(n^3)$.

Since we have $\Omega(n^3)$ and $O(n^3)$ we have that there exists real positive number constants A and B such that $Ag(n) \leq f(n) \leq Bg(n), k \geq n$ where $k = max(k\prime, k\prime\prime)$ obtained from the previous inequalities. By definition of Θ, $15n^3 + 11n^2 + 9$ is $\Theta(n^3)$.

6. Show that $n^4 - 5n - 8$ is $\Theta(n^4)$.

Since we have shown tha the function is $\Omega(n^4)$ and $O(n^4)$ we have that there exists real positive number constants A and B such that $Ag(n) \leq f(n) \leq Bg(n), k \geq n$ where $k = max(k\prime, k\prime\prime)$ obtained from the previous inequaliies. by definition of Θ, $n^4 - 5n - 8$ is $\Theta(n^4)$.

7. Let $g(n) = n^4 - 5n - 8$, show that $g(n)$ is not $O(n^r)$ for any positive real number $r < 4$.

We prove this by contradiction. Suppose that $g(n)$ is $O(n^r)$ for any positive real number $r < 4$. then

$$g(n) \leq An^r, n \geq a$$

where A and a are real positive numbers.

$$\begin{aligned} g(n) &\leq n^4 \\ &\leq An^r \\ n^{4-r} &\leq A \\ n &\leq \sqrt[4-r]{A} \end{aligned}$$

which is a contradiction. We conclude that $g(n)$ is not $O(n^r)$ for any positive real number $r < 4$.

8. Use theorem on polynomial orders to find orders for the function given by the following formulas.

1. $f(n) = 7n^5 + 5n^3 - n + 4$ for each positive integer n.

   By direct application of theorem on polynomial orders, $7n^5 + 5n^3 - n + 4$ is $\Theta(n^5)$.

2. $g(n) = \frac{(n-1)(n+1)}{4}$ for each positive integer n.

$$\begin{aligned}\frac{(n-1)\cdot(n+1)}{4} &= \frac{n^2+n-n+1}{4} \\ &= \frac{n^2+1}{4} \\ &= \frac{n^2}{4}+\frac{1}{4}\end{aligned}$$

Thus $g(n)$ is $\Theta(n^2)$.

9. Show that for a positive integer variable n,

$$1+2+3\ldots+n \quad is \quad \Theta(n^2)$$

$$\begin{aligned}\sum_{i=1}^{n} i &= \frac{n(n+1)}{2} \\ &= \frac{n^2}{2}+\frac{n}{2}\end{aligned}$$

10. Express $5x^8-9x^7+2x^5+3x-1 \le 6x^8, x>3$ using O notation

Let $A=6$ and $a=3$ then $5x^8-9x^7+2x^5+3x-1 \le Ax^8, x>a$ and by definition of O notation, $5x^8-9x^7+2x^5+3x-1$ is $O(x^8)$.

11. Express $x^{\frac{7}{2}} \le \frac{(x^2-7)^2(10x^{\frac{1}{2}}+3)}{x+1}, x>4$ using Ω notation

Let $A=1$ and $k=4$ then the statement translates to

$$Ax^{\frac{7}{2}} \quad \le \quad \frac{(x^2-7)^2(10x^{\frac{1}{2}}+3)}{x+1}, x>k$$

which by the definition of Ω notation, $\frac{(x^2-7)^2(10x^{\frac{1}{2}}+3)}{x+1}$ is $\Omega(x^{\frac{7}{2}})$.