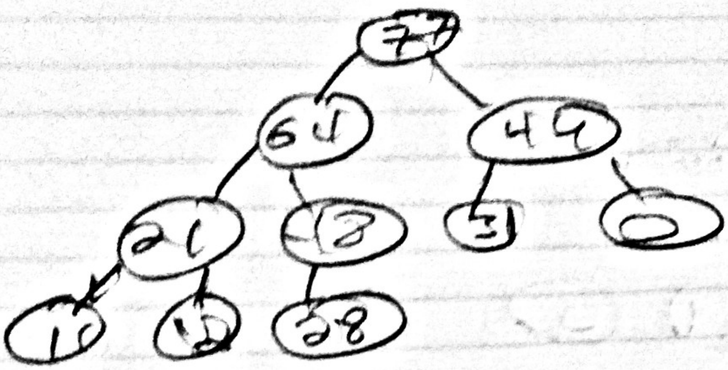
1. Where in a min heap the largest element resides? (Assume all elements are distinct) Explain.

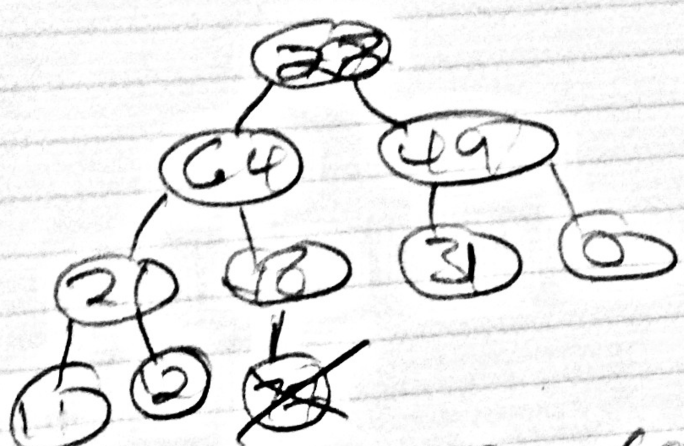
Suppose that we have a min heap with n elements where $n > 1$. Suppose further that the largest element was an internal node, then that means it has atleast one child. This would be a contradiction since the largest element would be larger than its children which contradicts the min-heap property. we conclude that the largest element must be a leaf.

7. Delete the root of the below max-heap

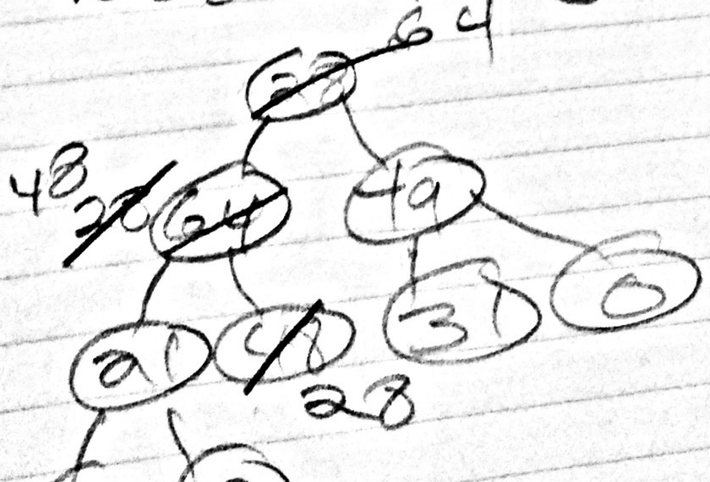
$$[77, 64, 49, 21, 48, 31, 0, 1, 12, 28]$$



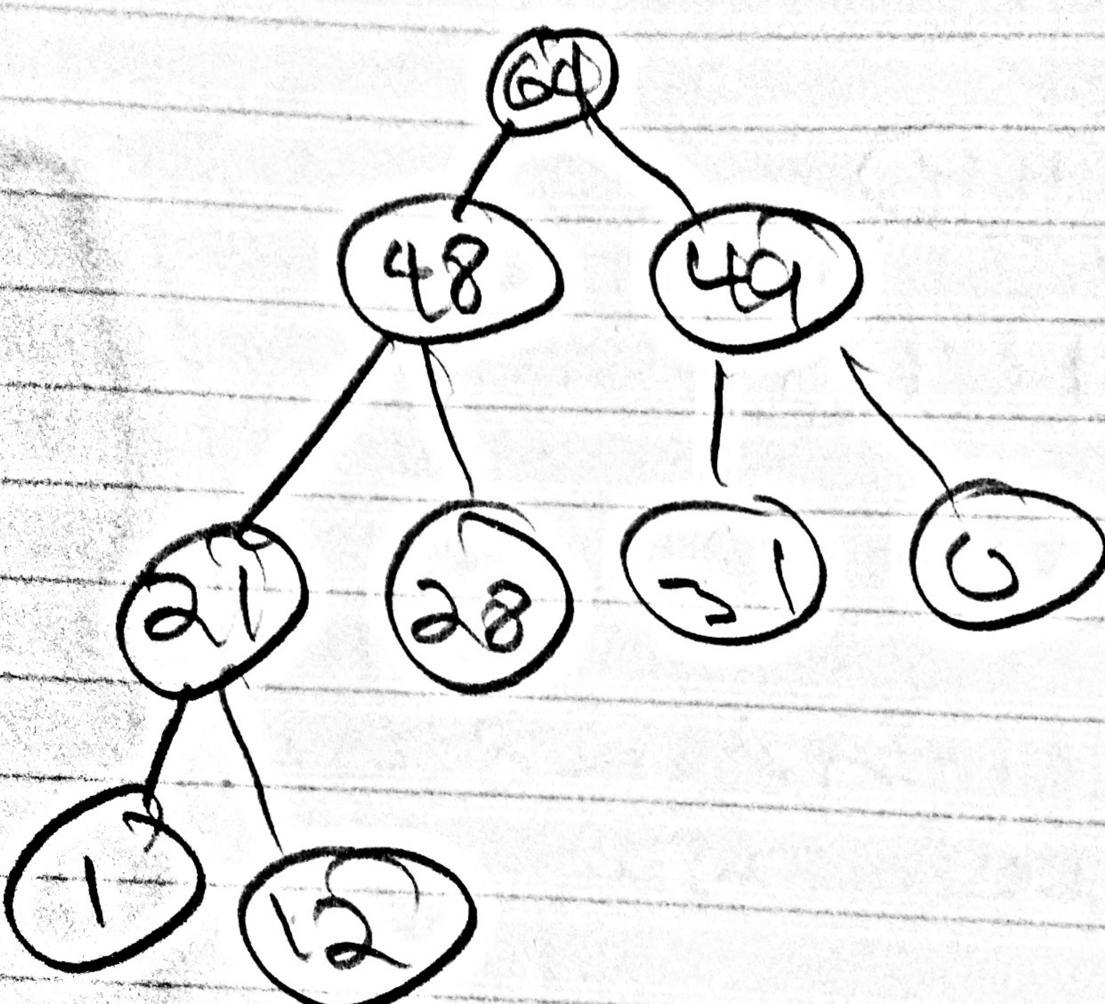
Swap 77 and 28
then remove 77



max heapify (a,0)

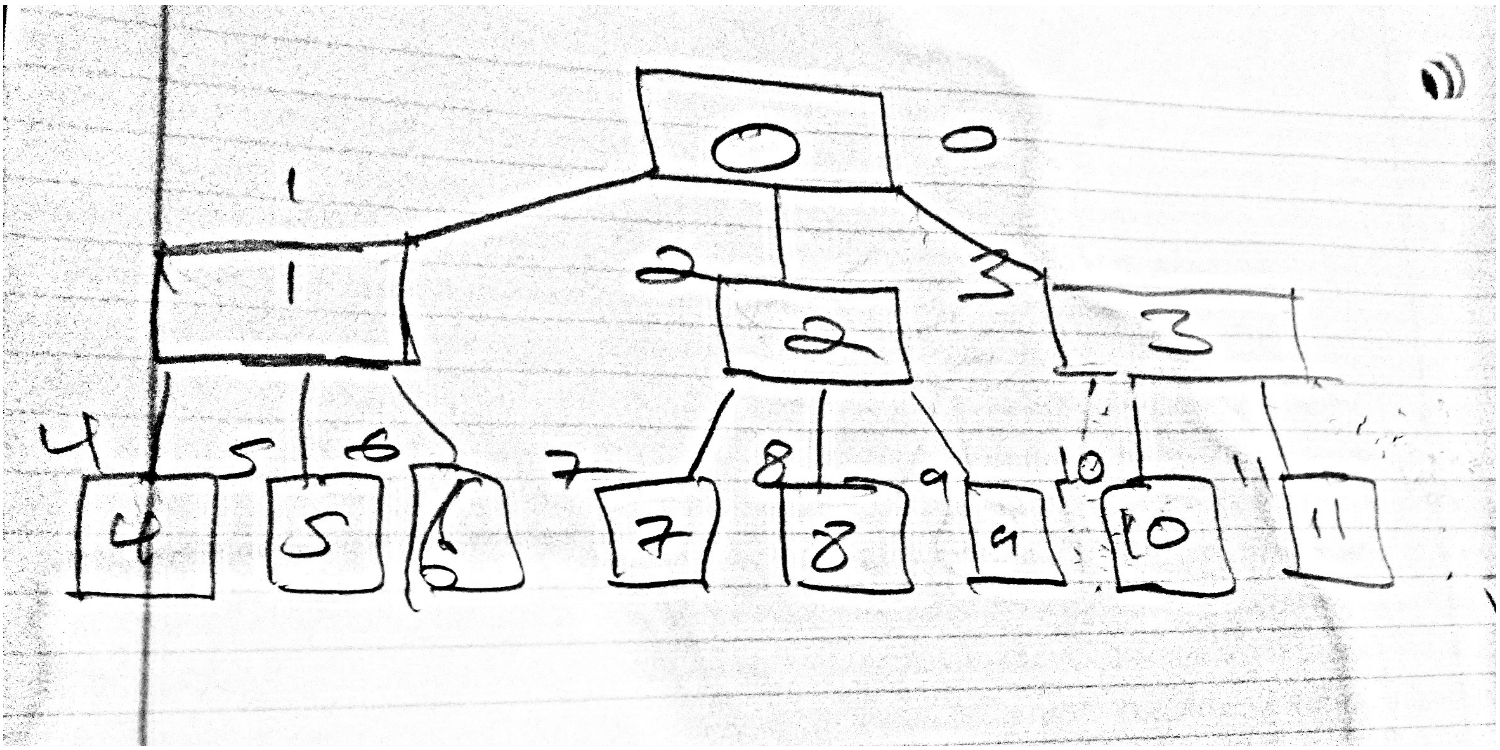


result



9. Suppose that instead of binary heaps, we wanted to work with ternary heaps. Suggest an appropriate indexing scheme so that a complete tree will yield a contigious sequence.

$$\begin{aligned} left\_child &= 3 \cdot i + 1 \\ middle\_child &= 3 \cdot i + 2 \\ right\_child &= 3 \cdot i + 3 \\ parent &= \lfloor \frac{i-1}{3} \rfloor \end{aligned}$$



0
0
1
1
2
2
3
3
4
4
5
5
6
6
7
7
8
8
9
9
10
10
11
11

10. Use induction to prove that $1 + 2 + 4 + \ldots + 2^h = 2^{h+1} - 1$.

Proof. We prove by induction

Let $P(h) = 1 + 2 + 4 + \ldots + 2^h = 2^{h+1} - 1$

Base case: $h = 0$

$1 = 1 = 2^{0+1} - 1$.

Thus $P(0)$ holds.

Inductive step: Let $P(k)$ be true we show that $P(k+1)$ is true, that is $1 + 2 + 4 + \ldots + 2^k = 2^{(k+1)+1} - 1$

$$\begin{aligned}
1 + 2 + 4 + \ldots + 2^k &= \sum_{i=0}^{k+1} 2^i \\
&= \sum_{i=0}^{k} 2^i + 2^{k+1} \\
&= 2^{k+1} - 1 + 2^{k+1} \\
&= 2 \cdot 2^{k+1} - 1 \\
&= 2^{k+1+1} - 1 \\
&= 2^{(k+1)+1} - 1
\end{aligned}$$

Thus $P(k+1)$ holds. By the principle of mathematical induction, $P(h)$ holds for for all integers $h \geq 0$. □

11. What is the minimum and maximum number of leaves in a binary heap that has height h. Explain.

When $h = 0$ then the minimum equal the maximum namely 1. Suppose that $h \geq 1$. A tree with height h must have at least one leaf at level h. The rest of the leaves are on the $h-1$ level where the leftmost node is the only parent. Thus there are a total of $2^{h-1} - 1 + 1 = 2^{h-1}$ minimum number of leafs for height h. In a perfect tree where all levels are filled we have $\sum_{i=0}^{h} 2^i = 2^{h+1} - 1$ total nodes where the leafs are the last summation term and thus contribute 2^h nodes. Thus the minimum and maximum number of leaves in a binary heap that height h is 2^{h-1} and 2^h leaves for $h \geq 1$.

12. Prove that a binary heap with n elements has height $\lfloor log_2(n) \rfloor$.

$$\begin{array}{rcl}
2^h \leq & n & \leq 2^{h+1} - 1 \\
2^h \leq & n & < 2^{h+1} \\
log_2(2^h) \leq & log_2(n) & < log_2(2^{h+1}) \\
h \leq & log_2(n) & < h + 1
\end{array}$$

which by definition of the floor $h = \lfloor log_2(n) \rfloor$. Hence we conclude a binary heap with n elements has height $\lfloor log_2(n) \rfloor$.

13. Prove that a binary heap with n nodes has exactly $\lceil \frac{n}{2} \rceil$ leaves.