1. What is the growth of the below functions?

1. $f(n) = 2 \cdot n^4 \cdot log_2(n^4) + n^{4.0001} - 3 \cdot log_2(n)$

$$f(n) \quad = \quad \Theta(n^{4.0001})$$

2. $f(n) = 3 \cdot n^3 \cdot log(n^4 - n^2) + 100000$

$$f(n) \quad = \quad \Theta(n^3 \cdot log_2(n))$$

3. $f(n) = log_2^{100}(n^{50}) + n$

$$f(n) \quad = \quad \Theta(n)$$

4. $f(n) = n^4 \cdot log_2^3(n) + 4$

$$f(n) \quad = \quad \Theta(n^4 \cdot log_2^3(n))$$

5. $f(n) = 10000 \cdot n \cdot log_2(n^7) + 3 \cdot log_2(n) + 1000 \cdot \sqrt{n}$

$$f(n) \quad = \quad \Theta(n \cdot log_2(n))$$

6. $f(n) = \sqrt[10]{n} + 10^{10} \cdot log_2^{100}(n) + 8$

$$f(n) \quad = \quad \Theta(\sqrt[10]{n})$$

7. $f(n) = \sqrt{\sqrt{n}} + 9 \cdot log_2(n)$

$$f(n) \quad = \quad \Theta(\sqrt{\sqrt{n}})$$

2. Discuss the growth of the below functions

1. $f(n) = log_2(n)^{log_2(n)}$

$$\begin{aligned} lim_{n\to\infty}\frac{f(n)}{2^n} &= lim_{n\to\infty}\frac{log_2(n)^{log_2(n)}}{2^n} \\ log(lim_{n\to\infty}\frac{log_2(n)^{log_2(n)}}{2^n}) &= lim_{n\to\infty}log_2(n)\cdot log_2(log_2(n)) - n \\ &= -\infty \\ 2^{-\infty} &= 0 \end{aligned}$$

$$\begin{aligned} lim_{n\to\infty}\frac{f(n)}{n^k} &= lim_{n\to\infty}\frac{log_2(n)^{log_2(n)}}{n^k} \\ log_2(lim_{n\to\infty}\frac{log_2(n)^{log_2(n)}}{n^k}) &= lim_{n\to\infty}log_2(n)\cdot log_2(log_2(n)) - k\cdot log_2(n) \\ &= \infty \\ 2^{\infty} &= \infty \end{aligned}$$

We conclude $f(n) = \omega(n^k)$.

2. $f(n) = 2^{\sqrt{2\cdot log_2(n)}}$

$$\begin{aligned} log_2(lim_{n\to\infty}\frac{2^{\sqrt{2\cdot log_2(n)}}}{2^n}) &= lim_{n\to\infty}\sqrt{2\cdot log_2(n)} - n \\ &= -\infty \\ 2^{-\infty} &= 0 \end{aligned}$$

$$\begin{aligned} log_2(lim_{n\to\infty}\frac{2^{\sqrt{2\cdot log_2(n)}}}{n^k}) &= lim_{n\to\infty}\sqrt{2\cdot log_2(n)} - k\cdot log_2(n) \\ &= -\infty \\ 2^{-\infty} &= 0 \end{aligned}$$

$$\begin{aligned} log_2(lim_{n\to\infty}\frac{2^{\sqrt{2\cdot log_2(n)}}}{n^a}) &= lim_{n\to\infty}\sqrt{2\cdot log_2(n)} - a\cdot log_2(n) \\ &= -\infty \\ 2^{-\infty} &= 0 \end{aligned}$$

$$
\begin{aligned}
log_2(lim_{n\to\infty}\frac{2^{\sqrt{2\cdot log_2(n)}}}{log_2(n)}) &= lim_{n\to\infty}\sqrt{2\cdot log_2(n)} - log_2(log_2(n)) \\
&= \infty \\
2^{\infty} &= \infty
\end{aligned}
$$

We conclude $f(n) = \omega(log_2(n))$.

3. $f(n) = \sqrt{2}^{log_2(n)}$

$$
\begin{aligned}
\sqrt{2}^{log_2(n)} &= 2^{\frac{1}{2}\cdot log_2(n)} \\
&= 2^{log_2(n^{\frac{1}{2}})} \\
&= n^{\frac{1}{2}} \\
&= \Theta(n^{\frac{1}{2}})
\end{aligned}
$$

We conclude $f(n) = \Theta(n^{\frac{1}{2}})$.

4. $f(n) = n^{\frac{1}{log_2(n)}}$

$$
\begin{aligned}
log_2(lim_{n\to\infty}\frac{n^{\frac{1}{log_2(n)}}}{2^n}) &= lim_{n\to\infty}1 - n \\
&= -\infty \\
2^{-\infty} &= 0
\end{aligned}
$$

$$
\begin{aligned}
log_2(lim_{n\to\infty}\frac{n^{\frac{1}{log_2(n)}}}{n^k}) &= lim_{n\to\infty}1 - k\cdot log_2(n) \\
&= -\infty \\
2^{-\infty} &= 0
\end{aligned}
$$

$$
\begin{aligned}
log_2(lim_{n\to\infty}\frac{n^{\frac{1}{log_2(n)}}}{n^a}) &= lim_{n\to\infty}1 - a\cdot log_2(n) \\
&= -\infty \\
2^{-\infty} &= 0
\end{aligned}
$$

$$
\begin{aligned}
log_2(lim_{n\to\infty}\frac{n^{\frac{1}{log_2(n)}}}{log_2(n)}) &= lim_{n\to\infty}1 - log_2(log_2(n)) \\
&= -\infty \\
2^{-\infty} &= 0
\end{aligned}
$$

$$
\begin{aligned}
log_2(lim_{n\to\infty}\frac{n^{\frac{1}{log_2(n)}}}{1}) &= lim_{n\to\infty}1 - log_2(1) \\
&= 1
\end{aligned}
$$

We conclue $f(n) = \Theta(1)$.

3. Suppose $g(n) \geq 1$ for all n, and that $f(n) \leq g(n) + L$, for some constant L and all n. Prove that $f(n) = O(g(n))$.

$$
\begin{aligned}
f(n) &\leq g(n) + L \\
&\leq g(n) + g(n) \cdot L \\
&= g(n) \cdot (L + 1)
\end{aligned}
$$

Hence $f(n) = O(g(n))$.

4. Prove or disprove: if $f(n) = O(g(n))$ and $f(n) \geq 1$ and $log(g(n)) \geq 1$ for sufficiently large n then $log(f(n)) = O(log(g(n))$.

$$
\begin{aligned}
f(n) &\leq c \cdot g(n) \\
log_2(f(n)) &\leq log_2(c) + log_2(g(n)) \\
&\leq log_2(c) \cdot log_2(g(n)) + log_2(g(n)) \\
&= log_2(g(n)) \cdot (log_2(c) + 1)
\end{aligned}
$$

Hence $log_2(f(n)) = O(log_2(g(n))$.

5. Show that $log_2(n!) = \Theta(n \cdot log_2(n))$

$$\begin{aligned}
log_2(n!) &= log_2(1 \cdot 2 \cdot 3 \cdot \ldots \cdot n) \\
&= log_2(1) + log_2(2) + log_2(3) + \ldots + log_2(n) \\
&= \sum_{i=1}^{n} log_2(i) \\
&= \Theta(\int_1^n log_2(x) \cdot dx) \\
&= \Theta(\frac{x \cdot ln(x)}{ln(2)}|_1^n - \frac{1}{ln(2)} \cdot \int_1^n dx) \\
&= \Theta(\frac{n \cdot ln(n)}{ln(2)} - \frac{1}{ln(2)} \cdot (n-1)) \\
&= \Theta(n \cdot ln(n)) \\
&= \Theta(n \cdot \frac{log_2(n)}{log_2(e)}) \\
&= \Theta(n \cdot log_2(n)).
\end{aligned}$$

Hence we conclude $log_2(n!) = \Theta(n \cdot log_2(n))$.

6. Prove that $n! = o(n^{n^2})$

$$\begin{aligned}
lim_{n\to\infty} log_2(\frac{n!}{n^{n^2}}) &= lim_{n\to\infty} log_2(n!) - n^2 \cdot log_2(n) \\
&= lim_{n\to\infty} \Theta(n \cdot log_2(n)) - n^2 \cdot log_2(n) \quad (log_2(n!) = (\Theta(n \cdot log_2(n)) \;\; by \;\; problem \;(5)) \\
&= -\infty \\
2^{-\infty} &= 0
\end{aligned}$$

Hence we conclude that $n! = o(n^{n^2})$

7. Prove that $n! = \omega(2^n)$

$$\begin{aligned}
lim_{n\to\infty} log_2(\frac{n!}{2^n}) &= lim_{n\to\infty} log_2(n!) - n \\
&= lim_{n\to\infty} \Theta(n \cdot log_2(n)) - n \quad (log_2(n!) = (\Theta(n \cdot log_2(n)) \;\; by \;\; problem \;(5)) \\
&= \infty \\
2^{\infty} &= \infty
\end{aligned}$$

Hence we conclude $n! = \omega(2^n)$.

8. Which one of the below functions grow faster?

$$\begin{aligned} f(n) &= 2^{2^n} \\ g(n) &= n! \end{aligned}$$

$$\begin{aligned} lim_{n\to\infty} log_2(\frac{2^{2^n}}{n!}) &= lim_{n\to\infty} 2^n - log_2(n!) \\ &= lim_{n\to\infty} 2^n - \Theta(n \cdot log_2(n)) \quad (log_2(n!) = (\Theta(n \cdot log_2(n)) \;\; by \;\; problem\ (5)) \\ &= \infty \\ 2^{\infty} &= \infty \end{aligned}$$

Hence we conclude $f(n)$ grows faster.

9. Provide a closed-form expression for the asymptotic growth of $n + \frac{n}{2} + \frac{n}{3} + \ldots + 1$

$$\begin{aligned} n \cdot \sum_{i=1}^{n} \frac{1}{n} &= \Theta(n \cdot \int_1^n \frac{1}{n} dx) \\ &= \Theta(n \cdot (ln(x)|_1^n)) \\ &= \Theta(n \cdot ln(n) - n \cdot ln(1)) \\ &= \Theta(n \cdot ln(n)) \end{aligned}$$

10. Use the integral theorem to calculate the growth of $1 + 2^k + 3^k + \ldots + n^k$

$$\begin{aligned} \sum_{i=1}^{n} i^k &= \Theta(\int_1^n x^k dx) \\ &= \Theta(\frac{1}{k+1} \cdot x^{k+1}|_1^n) \\ &= \Theta(\frac{n^{k+1}}{k+1} - \frac{1}{k+1}) \\ &= \Theta(n^{k+1}) \end{aligned}$$

11. Use the integral theorem to calculate the growth of $log_2(1) + log_2(2) + log_2(3) + \ldots + log_2(n^4)$

$$\begin{aligned}
\sum_{i=1}^{n^4} log_2(i) &= \Theta(\int_1^{n^4} log_2(x) \cdot dx) \\
&= \Theta(\frac{ln(x) \cdot x}{ln(2)}|_1^{n^4} - \frac{1}{ln(2)} \cdot \int_1^{n^4} dx) \\
&= \Theta(\frac{1}{ln(2)} \cdot (ln(n^4) \cdot n^4 - ln(1) \cdot 1) - \frac{1}{ln(2)} \cdot (n^4 - 1)) \\
&= \Theta(ln(n^4) \cdot n^4) \\
&= \Theta(4 \cdot ln(n) \cdot n^4) \\
&= \Theta(4 \cdot \frac{log_2(n)}{log_2(e)} \cdot n^4) \\
&= \Theta(n^4 \cdot log_2(n))
\end{aligned}$$

12. Use the integral theorem to calculate the growth of $log_2(1) + 2 \cdot log_2(2) + 3 \cdot log_2(3) + \ldots + n^2 \cdot log_2(n^2)$

$$\begin{aligned}
\sum_{i=1}^{n^2} i \cdot log_2(i) &= \Theta(\int_1^{n^2} x \cdot log_2(x) \cdot dx) \\
&= \Theta(\frac{x^2 \cdot ln(x)}{2 \cdot ln(2)}|_1^{n^2} - \frac{1}{2 \cdot ln(2)} \int_1^{n^2} x \cdot dx) \\
&= \Theta(\frac{n^4 \cdot ln(n^2)}{2 \cdot ln(2)} - \frac{1}{4 \cdot ln(2)} \cdot x^2|_1^{n^2}) \\
&= \Theta(\frac{n^4 \cdot ln(n^2)}{2 \cdot ln(2)} - \frac{1}{4 \cdot ln(2)}(n^4 - 1)) \\
&= \Theta(\frac{n^4 \cdot ln(n^2)}{2 \cdot ln(2)}) \\
&= \Theta(n^4 \cdot ln(n^2)) \\
&= \Theta(2 \cdot n^4 \cdot ln(n)) \\
&= \Theta(2 \cdot n^4 \cdot \frac{log_2(n)}{log_2(e)}) \\
&= \Theta(n^4 \cdot log_2(n))
\end{aligned}$$

13. Prove or disprove: if $f(n) = O(g(n)) \implies 2^{f(n)} = O(2^{g(n)})$

This is false, consider $f(n) = 2 \cdot n$ and $g(n) = n$. $f(n) = O(g(n))$ but $2^{f(n)} \neq O(2^{g(n)})$.