

1 Use  $\Theta$  notation to express the statement

$$4n^6 \leq 17n^6 - 45n^3 + 2n + 8 \leq 30n^6, n \geq 3$$

Let  $A = 4$ ,  $B = 30$  and  $k = 3$  then the statement translates to

$$An^6 \leq 17n^6 - 45n^3 + 2n + 8 \leq Bn^6, n \geq k$$

hence by the definition of  $\Theta$  notation  $17n^6 - 45n^3 + 2n + 8$  is  $\Theta(n^6)$ .

2 Use  $\Omega$  notation to express the statement

1. Use  $\Omega$  notation to express the statement

$$\frac{11}{4}n^2 \leq 3 \cdot \left(\left\lfloor \frac{n}{4} \right\rfloor\right)^2 + 5n^2, n \geq 2$$

Let  $A = \frac{11}{4}$  and  $k = 2$  then  $An^2 \leq 3 \cdot \left(\left\lfloor \frac{n}{4} \right\rfloor\right)^2 + 5n^2, n \geq 2$  then the statement translates to

$$An^2 \leq 3 \cdot \left(\left\lfloor \frac{n}{4} \right\rfloor\right)^2 + 5n^2, n \geq k$$

which by the definition of  $\Omega$  notation,  $3 \cdot \left(\left\lfloor \frac{n}{4} \right\rfloor\right)^2 + 5n^2$  is  $\Omega(n^2)$ .

2. Use  $O$  notation to express the statement

$$0 \leq 3 \cdot \left(\left\lfloor \frac{n}{4} \right\rfloor\right)^2 + 5n^2 \leq 6n^2, n \geq 1$$

Let  $A = 6$  and  $k = 1$  then the statement translates to

$$0 \leq 3 \cdot \left(\left\lfloor \frac{n}{4} \right\rfloor\right)^2 + 5n^2 \leq An^2, n \geq k$$

which by the definition of  $O$  notation,  $3 \cdot \left(\left\lfloor \frac{n}{4} \right\rfloor\right)^2 + 5n^2$  is  $O(n^2)$ .

3. Justify the statement:  $3 \cdot \left(\left\lfloor \frac{n}{4} \right\rfloor\right)^2 + 5n^2$  is  $\Theta(n^2)$ .

Let  $A = \frac{11}{4}$ ,  $B = 6$  and  $k = 2$  then  $A \cdot n^2 \leq 3 \cdot \left(\left\lfloor \frac{n}{4} \right\rfloor\right)^2 + 5n^2 \leq Bn^2, n \geq k$  which by the definition of  $\Theta$  notation,  $3 \cdot \left(\left\lfloor \frac{n}{4} \right\rfloor\right)^2 + 5n^2$  is  $\Theta(n^2)$ .

3. Given the function  $15n^3 + 11n^2 + 9$

1. Show that the function is  $\Omega(n^3)$ .

$$15n^3 \leq 15n^3 + 11n^2 + 9, n \geq 1$$

Let  $A = 15$  and  $k = 1$  then the statements translates to  $An^3 \leq 15n^3 + 11n^2 + 9, n \geq k$  which by the definition of  $\Omega$  notation,  $15n^3 + 11n^2 + 9$  is  $\Omega(n^3)$ .

2. Show that the function is  $O(n^3)$ .

$$\begin{aligned} 15n^3 + 11n^2 + 9 &\leq 15n^3 + 11n^3 + 9n^3 \\ &\leq 35n^3, n \geq 1 \end{aligned}$$

Let  $A = 35$  and  $k = 1$  then the statement translates to  $15n^3 + 11n^2 + 9 \leq An^3, n \geq k$  which by the definition of  $O$  notation,  $15n^3 + 11n^2 + 9$  is  $O(n^3)$ .

4. Given the function  $n^4 - 5n - 8$

1. Show that the function is  $\Omega(n^4)$ .

Let  $A = \frac{1}{2}$  and  $a = (|-5| + |-8|)$

$$\begin{aligned} n &\geq \frac{2}{1} \cdot (|-5| + |-8|) \\ \frac{1}{2}n^4 &\geq 5n^3 + 8n^3 \\ \frac{1}{2}n^4 &\geq 5n + 8 \\ n^4 - 5n - 8 &\geq \frac{1}{2}n^4, n \geq a \end{aligned}$$

Hence by the definition of  $\Omega$  notation,  $n^4 - 5n - 8$  is  $\Omega(n^4)$ .

2. Show that the function is  $O(n^4)$ .

$$\begin{aligned} n^4 - 5n - 8 &\leq n^4 + 5n + 8 \\ &\leq n^4 + 5n^4 + 8n^4 \\ &= 14n^4, n \geq 1 \end{aligned}$$

Let  $A = 14$  and  $k = 1$  then the statement translates to  $n^4 - 5n - 8 \leq An^4, n \geq k$  which by the definition of  $O$  notation translates,  $n^4 - 5n - 8$  is  $O(n^4)$ .

5. Show that  $15n^3 + 11n^2 + 9$  is  $\Theta(n^3)$ .

Since we have  $\Omega(n^3)$  and  $O(n^3)$  we have that there exists real positive number constants  $A$  and  $B$  such that  $Ag(n) \leq f(n) \leq Bg(n), k \geq n$  where  $k = \max(k', k'')$  obtained from the previous inequalities. By definition of  $\Theta$ ,  $15n^3 + 11n^2 + 9$  is  $\Theta(n^3)$ .

6. Show that  $n^4 - 5n - 8$  is  $\Theta(n^4)$ .

Since we have shown that the function is  $\Omega(n^4)$  and  $O(n^4)$  we have that there exists real positive number constants  $A$  and  $B$  such that  $Ag(n) \leq f(n) \leq Bg(n), k \geq n$  where  $k = \max(k', k'')$  obtained from the previous inequalities. by definition of  $\Theta$ ,  $n^4 - 5n - 8$  is  $\Theta(n^4)$ .

7. Let  $g(n) = n^4 - 5n - 8$ , show that  $g(n)$  is not  $O(n^r)$  for any positive real number  $r < 4$ .

We prove this by contradiction. Suppose that  $g(n)$  is  $O(n^r)$  for any positive real number  $r < 4$ . then

$$g(n) \leq An^r, n \geq a$$

where  $A$  and  $a$  are real positive numbers.

$$\begin{aligned} g(n) &\leq n^4 \\ &\leq An^r \\ n^{4-r} &\leq A \\ n &\leq \sqrt[4-r]{A} \end{aligned}$$

which is a contradiction. We conclude that  $g(n)$  is not  $O(n^r)$  for any positive real number  $r < 4$ .

8. Use theorem on polynomial orders to find orders for the function given by the following formulas.

1.  $f(n) = 7n^5 + 5n^3 - n + 4$  for each positive integer  $n$ .

By direct application of theorem on polynomial orders,  $7n^5 + 5n^3 - n + 4$  is  $\Theta(n^5)$ .

2.  $g(n) = \frac{(n-1)(n+1)}{4}$  for each positive integer  $n$ .

$$\begin{aligned}
\frac{(n-1) \cdot (n+1)}{4} &= \frac{n^2 + n - n + 1}{4} \\
&= \frac{n^2 + 1}{4} \\
&= \frac{n^2}{4} + \frac{1}{4}
\end{aligned}$$

Thus  $g(n)$  is  $\Theta(n^2)$ .

9. Show that for a positive integer variable  $n$ ,

$$1 + 2 + 3 \dots + n \text{ is } \Theta(n^2)$$

$$\begin{aligned}
\sum_{i=1}^n i &= \frac{n(n+1)}{2} \\
&= \frac{n^2}{2} + \frac{n}{2}
\end{aligned}$$

10. Express  $5x^8 - 9x^7 + 2x^5 + 3x - 1 \leq 6x^8, x > 3$  using  $O$  notation

Let  $A = 6$  and  $a = 3$  then  $5x^8 - 9x^7 + 2x^5 + 3x - 1 \leq Ax^8, x > a$  and by definition of  $O$  notation,  $5x^8 - 9x^7 + 2x^5 + 3x - 1$  is  $O(x^8)$ .

11. Express  $x^{\frac{7}{2}} \leq \frac{(x^2-7)^2(10x^{\frac{1}{2}}+3)}{x+1}, x > 4$  using  $\Omega$  notation

Let  $A = 1$  and  $k = 4$  then the statement translates to

$$Ax^{\frac{7}{2}} \leq \frac{(x^2-7)^2(10x^{\frac{1}{2}}+3)}{x+1}, x > k$$

which by the definition of  $\Omega$  notation,  $\frac{(x^2-7)^2(10x^{\frac{1}{2}}+3)}{x+1}$  is  $\Omega(x^{\frac{7}{2}})$ .

12. Express  $3x^6 + 5x^4 - x^3 \leq 9x^6, x > 1$  using  $O$  notation. Let  $A = 9$  and  $k = 1$  then the statement translate to

$$3x^6 + 5x^4 - x^3 \leq Ax^6, x > k$$

which by the definition of  $\Omega$  notation,  $3x^6 + 5x^4 - x^3$  is  $O(x^6)$ .

**13.** Express  $\frac{1}{2}x^4 \leq x^4 - 50x^3 + 1$  for all real numbers  $x > 101$  using  $\Omega$  notation.

Let  $A = \frac{1}{2}$  and  $k = 101$  this statement translates to  $Ax^4 \leq x^4 - 50x^3 + 1, n > k$  which by the definition of  $\Omega$  notation,  $x^4 - 50x^3 + 1$  is  $\Omega(x^4)$ .

**14.** Express  $\frac{1}{2}x^2 \leq 3x^2 - 80x + 7 \leq 3x^2, x > 25$

Let  $A = \frac{1}{2}, B = 3$  and  $k = 25$  then the statement translates to

$$Ax^2 \leq 3x^2 - 80x + 7 \leq Bx^2, x > k$$

which by the definition of  $\Theta$  notation  $3x^2 - 80x + 7$  is  $\Theta(x^2)$ .

**15.** Suppose  $g(x)$  is  $O(f(x))$  show  $f(x)$  is then  $\Omega(g(x))$

Since  $g(x)$  is  $O(f(x))$  then there exists a real positive numbers  $B$  and  $k$  such that  $g(x) \leq B \cdot f(x), n > k$ . We obtain  $\frac{g(x)}{B} \leq f(x), n > k$  which by the definition of  $\Omega$  notation  $f(x)$  is  $\Omega(g(x))$ .

**16.** Prove that if  $f(x)$  is  $O(g(x))$  and  $c$  is any nonzero real number, then  $c \cdot f(x)$  is  $O(g(x))$ .

Since  $f(x)$  is  $O(g(x))$  then there exists real positive numbers  $B$  and  $k$  such that  $g(x) \leq B \cdot f(x), n > k$ . Multiplying by the constant  $c$  we obtain  $c \cdot g(x) \leq c \cdot B \cdot f(x), n > k$  which by the definition of  $O$  notation  $c \cdot g(x)$  is  $O(g(x))$ .

**17.** Prove that if  $f(x)$  is  $O(h(x))$  and  $g(x)$  is  $O(l(x))$ , then  $f(x) + g(x)$  is  $O(G(x))$  where  $G(x) = \max(h(x), l(x))$ .

$$\begin{aligned} f(x) + g(x) &\leq 2 \cdot \max(h(x), l(x)) \\ &= 2 \cdot G(x), n \geq k \end{aligned}$$

where  $k = \max(k', k'')$  where  $k'$  and  $k''$  are terms that satisfy the previous  $O$  notations.

**18.** Prove that  $f(x)$  is  $\Theta(f(x))$

Let  $A = 1, B = 1$  and  $k = 1$  the  $Af(x) \leq f(x) \leq Bf(x), n \geq k$  which by the definition of  $\Theta$  notation  $f(x)$  is  $\Theta(f(x))$ .

**19.** Prove that if  $f(x)$  is  $O(h(x))$  and  $g(x)$  is  $O(k(x))$  then  $f(x)g(x)$  is  $O(h(x)k(x))$ .

we have that  $f(x)$  is  $O(h(x))$  and  $g(x)$  is  $O(k(x))$  hence we have the we have constants  $B, B', b$  and  $b'$  such that

$$\begin{aligned} f(x) &\leq B \cdot h(x), x > b \\ g(x) &\leq B' \cdot k(x), x > b' \end{aligned}$$

Let  $b_1 = \max(b, b')$  then

$$\begin{aligned} f(x) \cdot g(x) &\leq B \cdot h(x) \cdot g(x) \\ &\leq B \cdot h(x) \cdot g(x) \\ &\leq B \cdot B' \cdot h(x) \cdot k(x), x > b_1 \end{aligned}$$

then by the definition of  $O$  notation,  $f(x) \cdot g(x)$  is  $O(h(x) \cdot g(x))$ .

**20.** Prove that if  $x$  is a real number with  $x > 1$ , then  $x^n > 1$  for all integers  $n \geq 1$ .

We prove this by mathematical induction, Let  $P(n)$  be the statement  $x^n > 1$ .

$P(1)$  is trivially true since  $x^1 > 1$ . Let  $P(k)$  be true, thus  $x^k > 1$  we now show that  $P(k+1)$  is true.

$$\begin{aligned} x^{k+1} &= x^k x^1 \\ &> x^k \\ &> 1 \end{aligned}$$

Thus  $x^{k+1} > 1$ , which implies that  $P(k+1)$  is true. By the principle of mathematical induction,  $P(n)$  holds for all positive integers  $n$ .

**21.** Prove that if  $x > 1$  then  $x^m < x^n$  for any integers  $m$  and  $n$  with  $m < n$ .

$$\begin{aligned} x^m &= \underbrace{x \cdot x \cdot \dots \cdot x}_{m \text{ times}} \\ &< \underbrace{x \cdot x \cdot \dots \cdot x}_{m \text{ times}} \cdot \underbrace{1 \cdot 1 \cdot \dots \cdot 1}_{n-m \text{ times}} \\ &< \underbrace{x \cdot x \cdot \dots \cdot x}_{m \text{ times}} \cdot \underbrace{x \cdot x \cdot \dots \cdot x}_{n-m \text{ times}} \\ &= x^n \end{aligned}$$

**22.** Given the function  $g(x) = 2x^2 + 15x + 4$

1. Show that  $x^2 \leq 2x^2 + 15x + 4$

$$\begin{aligned}
x^2 &< x^2 + 15 + 4 \\
&< x^2 + 15x + 4 \\
&< 2x^2 + 15x + 4 \\
&\leq 2x^2 + 15x + 4
\end{aligned}$$

2. Show that  $2x^2 + 15x + 4 \leq 21x^2$

$$\begin{aligned}
2x^2 + 15x + 4 &\leq x^2 + 15x^2 + 14x^2 \\
&\leq 21x^2
\end{aligned}$$

3. Rewrite statement (1) using  $\Omega$  notation

Let  $A = 1$  and  $a = 1$  then  $A \cdot f(x) \leq g(x)$  and by the definition of  $\Omega$  notation  $g(x)$  is  $\Omega(n^2)$ .

4. Rewrite statement (2) using  $O$  notation

Let  $A = 21$  and  $a = 1$  then  $g(x) \leq A \cdot x^2$  and by the definition of  $O$  notation  $g(x)$  is  $O(n^2)$ .

**23.** Given the function  $g(x) = x^4 \leq 23x^4 + 8x^2 + 4x$ .

1. Show that if  $x > 1$  then  $x^4 \leq 23x^4 + 8x^2 + 4x$

Let  $x$  be a real number such that  $x > 1$ .

$$\begin{aligned}
x^4 &\leq 23 \cdot x^4 \\
&\leq 23 \cdot x^4 + 8x^2 \\
&\leq 23 \cdot x^4 + 8x^2 + 4x
\end{aligned}$$

2. Show that if  $x > 1$  then  $23x^4 + 8x^2 + 4x \leq 23x^4$

$$\begin{aligned}
23x^4 + 8x^2 + 4x &\leq 23x^4 + 8x^4 + 4x^4 \\
&\leq 35x^4
\end{aligned}$$

3. What can you conclude by part (1) and part (2)

the function  $g(x)$  is  $\Omega(x^4)$  and  $O(x^4)$ .

**24.** Prove  $5x^3 + 65x + 30$  is  $\Theta(x^3)$  by proving that  $5x^3 + 65x + 30$  is  $\Omega(x^3)$  and  $5x^3 + 65x + 30$  is  $O(x^3)$

$$\begin{aligned} 5x^3 &\leq 5x^3 + 65x^3 + 30x^3 \\ &\leq 100x^3, x > 1 \end{aligned}$$

$$x^3 \leq 5x^3$$

From these two inequalities we obtain that  $x^3 \leq 5x^3 \leq 5x^3 + 65x^3 + 30x^3, x \geq 1$  hence by the definition of  $\Theta$  notation  $5x^3$  is  $\Theta(x^3)$ .

**25.** Prove that  $x^2 + 100x + 88$  is  $\Theta(x^2)$  by proving that  $x^2 + 100x + 88$  is  $\Omega(x^2)$  and  $O(x^2)$ .

$$\begin{aligned} x^2 + 100x + 88 &< x^2 + 100x \\ &< x^2 + 100x + 8, x > 1 \end{aligned}$$

$$\begin{aligned} x^2 + 100x + 88 &< x^2 + 100x^2 + 88x^2 \\ &< 189x^2 \end{aligned}$$

From these two inequalities we obtain that  $x^2 \leq x^2 + 100x + 88 < 189x^2$ , which by the definition of  $\Theta$  notation is  $\Theta(x^2)$ .

**26.** Show that for any real number  $x$ , If  $x > 1$  then  $7x^4 - 95x^3 + 3 \leq 105x^4$  and use  $O$  notation to express the result.

$$\begin{aligned} 7x^4 - 95x^3 + 3 &\leq 7x^4 + 95x^4 + 3x^4 \\ &\leq 105x^4 \end{aligned}$$

Which by the definition of  $O$  notation,  $7x^4 - 95x^3 + 3$  is  $O(x^4)$ .

**27.** Show that for any real number  $x$ , if  $x > 1$  then  $\frac{1}{5}x^2 - 42x - 8 \leq 51x^2$  and use  $O$  notation to express the result.

$$\begin{aligned} \frac{1}{5}x^2 - 42x - 8 &\leq \frac{1}{5}x^2 + 42x^2 + 8x^2 \\ &\leq x^2 + 42x^2 + 8x^2 \\ &\leq 51x^2 \end{aligned}$$

Which by the definition of  $O$  notation,  $\frac{1}{5}x^2 - 42x - 8$  is  $O(x^2)$ .



**28.** Show that for any real number  $x$ , if  $x > 1$  then  $\frac{1}{4}x^5 - 50x^3 + 3x + 12 \leq 66x^5$ .

$$\begin{aligned}\frac{1}{4}x^5 - 50x^3 + 3x + 12 &\leq x^5 + 50x^5 + 3x^5 + 12x^5 \\ &\leq 66x^5\end{aligned}$$

which by the definition of  $O$  notation,  $\frac{1}{4}x^5 - 50x^3 + 3x + 12$  is  $O(x^5)$ .

**29.** Show that  $x^5$  is not  $O(x^2)$ .

Suppose that  $x^5$  is in fact  $O(x^2)$  then by definition there exists positive real numbers  $B$  and  $k$  such that

$$\begin{aligned}x^5 &\leq Bx^2, x > k \\ x^3 &\leq B, x > k\end{aligned}$$

which is absurd hence  $x^5$  is not  $O(x^2)$ .

**30.** Suppose  $a_0, a_1, a_2, \dots, a_n$  are real numbers and  $a_n \neq 0$ . Use the generalization of the triangle inequality to  $n$  integers to show that  $a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$  is  $O(x^n)$ .

$$\begin{aligned}|a_n x^n + a_{n-1} x^{n-1} + \dots + a_0| &\leq |a_n x^n| + |a_{n-1} x^{n-1}| + \dots + |a_0| \\ &\leq |a_n| x^n + |a_{n-1}| x^{n-1} + \dots + |a_0| \\ &\leq |a_n| x^n + |a_{n-1}| x^n + \dots + |a_0| x^n \\ &\leq x^n \sum_{i=0}^n |a_i|\end{aligned}$$

Which by the definition of  $O$  notation  $a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$  is  $O(x^n)$ .

**31.** Use the theorem on polynomial orders to prove  $\frac{(x+1)(x-2)}{4}$  is  $\Theta(x^2)$

$$\begin{aligned}\frac{(x+1)(x-2)}{4} &= \frac{x^2 - 2x - x - 2}{4} \\ &= \frac{x^2}{4} - \frac{2x}{4} - \frac{x}{4} - \frac{2}{4}\end{aligned}$$

which by the theorem on polynomial orders  $\frac{(x+1)(x-2)}{4}$  is  $\Theta(x^2)$ .

**32.** Use the theorem on polynomial orders to prove that  $\frac{x}{3}(4x^2 - 1)$  is  $\Theta(x^3)$ .

$$\frac{x}{3}(4x^2 - 1) = \frac{4x^3}{3} - \frac{x}{3}$$

which by the theorem on polynomial orders is  $\Theta(x^3)$ .

**33.** Use the theorem on polynomial orders to prove that  $\frac{x(x-1)}{2} + 3x$  is  $\Theta(x^2)$ .

$$\frac{x(x-1)}{2} + 3x = \frac{x^2 - x}{2} + 3x$$

which by the order of polynomial orders is  $\Theta(x^3)$ .

**34.** Use the theorem on polynomial orders to prove that  $\frac{n(n+1)(2n+1)}{6}$  is  $\Theta(n^3)$ .

$$\begin{aligned} \frac{n(n+1)(2n+1)}{6} &= \frac{(n^2 + n)(2n+1)}{6} \\ &= \frac{2n^3 + n^2 + 2n^2 + n}{6} \\ &= \frac{2n^3}{6} + \frac{3n^2}{6} + \frac{n}{6} \end{aligned}$$

which by the order of polynomial orders is  $\Theta(n^3)$ .