

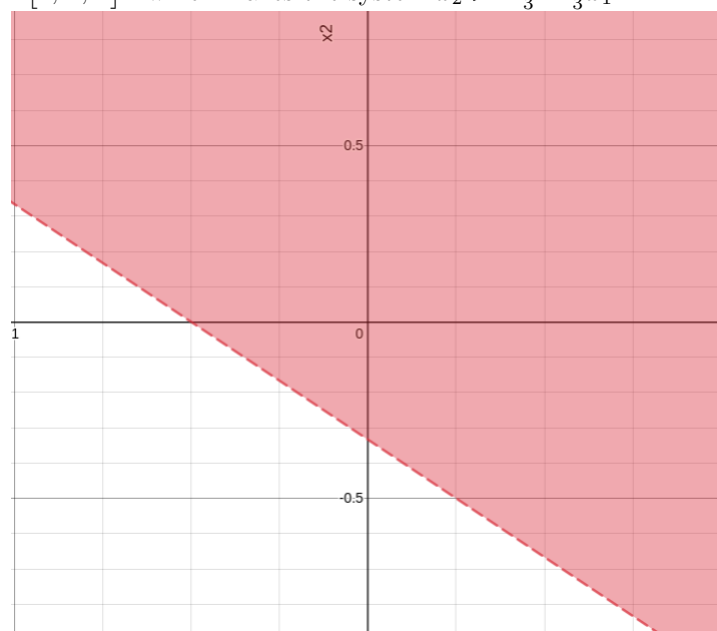
1) Consider the perceptron in two dimensions:  $h(x) = \text{sign}(w^T x)$  where  $w = [w_0, w_1, w_2]^T$  and  $x = [1, x_1, x_2]^T$ . Technically,  $x$  has three coordinates, but we call this perceptron two-dimensional because the first coordinate is fixed at 1.

(a) Show that the regions on the plane where  $h(x) = +1$  and  $h(x) = -1$  are separated by a line. If we express this line by the equation  $x_2 = ax_1 + b$ , what are the slope  $a$  and intercept  $b$  in terms of  $w_0, w_1, w_2$

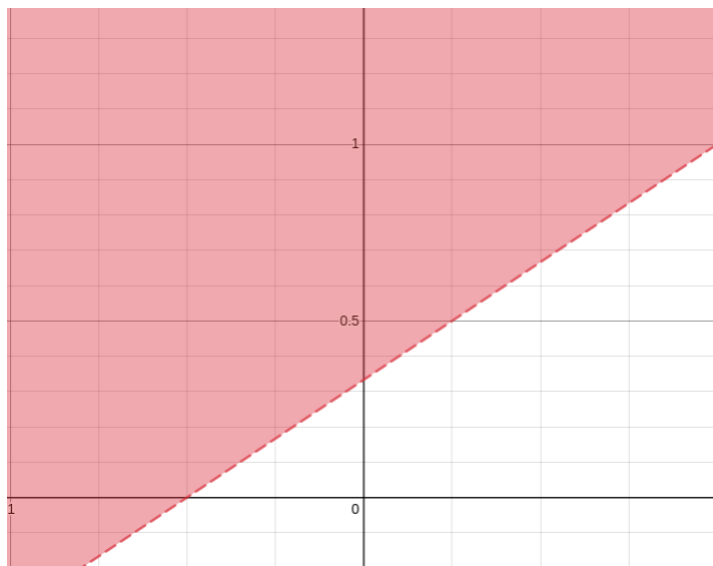
$\text{sign}(w^T x) = w_0 \cdot x_0 + w_1 \cdot x_1 + w_2 \cdot x_2 > 0$ .  $x_2 > -\frac{w_0}{w_2} - \frac{w_1}{w_2} \cdot x_1$  where  $w_2 \neq 0$ .  
 $a = -\frac{w_1}{w_2}$  and  $b = -\frac{w_0}{w_2}$

(b) Draw a picture for the cases  $w = [1, 2, 3]^T$  and  $w = -[1, 2, 3]^T$ .

$w = [1, 2, 3]^T$  which makes the system  $x_2 > -\frac{1}{3} - \frac{2}{3}x_1$ .



$w = -[1, 2, 3]$



In more than two dimensions, the  $+1$  and  $-1$  regions are separated by a hyper-plane, the generalization of a line.

2) Given  $x \in \mathbb{R}^m$ ,  $y \in \mathbb{R}^n$ , show that the rank of a matrix  $xy^T$  is one.

$$\begin{aligned}
 xy^T &= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \begin{bmatrix} y_1^T & y_2^T & \dots & y_n^T \end{bmatrix} \\
 &= \begin{bmatrix} x_1 \cdot y_1^T & x_1 \cdot y_2^T & \dots & x_1 \cdot y_n^T \\ x_2 \cdot y_1^T & x_2 \cdot y_2^T & \dots & x_2 \cdot y_n^T \\ \vdots & \vdots & \ddots & \vdots \\ x_m \cdot y_1^T & x_m \cdot y_2^T & \dots & x_m \cdot y_n^T \end{bmatrix}
 \end{aligned}$$

Columns linear combinations of  $x$  thus we conclude the rank is one

3) Given  $X = [x_1, x_2, \dots, x_n] \in \mathbb{R}^{m \times n}$  where  $x_i \in \mathbb{R}^m$  for all  $i$ , and  $Y^T = [y^1, y^2, \dots, y^n] \in \mathbb{R}^{p \times n}$  where  $y^i \in \mathbb{R}^p$  for all  $i$ . Show that

$$XY = \sum_{i=1}^N x_i (y^i)^T$$

$$\sum_{i=1}^N x_i (y^i)^T =$$

4) Given  $X \in \mathbb{R}^{m \times n}$ , show that the matrix  $X^T X$  is symmetric and positive semi-definite. When is it positive definite?

We first show that  $X^T X$  is symmetric

$$\begin{aligned} (X^T X)^T &= X^T X^{TT} \\ &= X^T X \end{aligned}$$

Thus  $X^T X$  is symmetric

Next we show  $X^T X$  is positive-semidefinite

$$\begin{aligned} v^T (X^T X) v &= \|Xv\|^2 \\ \|Xv\|^2 &\geq 0 \quad \forall v \in \mathbb{R}^n \end{aligned}$$

Which shows that the matrix is positive-semidefinite

$X^T X$  is positive definite iff  $\mathcal{N}(X) = \{\hat{0}\}$ . Thus the matrix  $X$  must be full-column rank

5) Given  $g(x, y) = e^x + e^{y^2} + e^{3xy}$ , compute  $\frac{\partial g}{\partial y}$ .

$$\frac{\partial g}{\partial y} g(x, y) = 2ye^{y^2} + 3xe^{3xy}$$

6) Consider the matrix

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 3 & 2 & 5 \end{pmatrix}$$

(a) Compute the eigenvalues and corresponding eigenvectors of A. You are allowed to use Matlab to compute the eigenvectors

$$\lambda_1 = 0, \lambda_2 = 4 + \sqrt{13}, \lambda_3 = 4 - \sqrt{13}$$

$$\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} \sqrt{13} - 3 \\ -\sqrt{13} + 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 - \sqrt{13} \\ \sqrt{13} + 4 \\ 1 \end{bmatrix}$$

(b) What is the eigen-decomposition of A?

$$\begin{bmatrix} -1 & \sqrt{13}-3 & -3-\sqrt{13} \\ -1 & -\sqrt{13}+4 & \sqrt{13}+4 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4+\sqrt{13} & 0 \\ 0 & 0 & 4-\sqrt{13} \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{5\sqrt{13}+13}{78} & \frac{13+2\sqrt{13}}{78} & \frac{7\sqrt{13}+26}{78} \\ -\frac{5\sqrt{13}-13}{78} & -\frac{2\sqrt{13}-13}{78} & \frac{26-7\sqrt{13}}{78} \end{bmatrix}$$

(c) What is the rank of A?

$$rref(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the matrix has two pivot columns the  $Rank(A) = 2$

(d) Is A positive definite? Is A positive semi-definite?

A is not positive definite. A is positive semi-definite.

(e) Is A singular?

A is singular

7) What is cross-entropy error

$$p \cdot \ln \frac{1}{1-q} + (1-p) \cdot \ln \frac{1}{q}$$

$$p = [y_n = 1] \text{ and } q = h(x_p)$$