

1) Consider the perceptron in two dimensions: $h(x) = \text{sign}(w^T x)$ where $w = [w_0, w_1, w_2]^T$ and $x = [1, x_1, x_2]^T$. Technically, x has three coordinates, but we call this perceptron two-dimensional because the first coordinate is fixed at 1.

(a) Show that the regions on the plane where $h(x) = +1$ and $h(x) = -1$ are separated by a line. If we express this line by the equation $x_2 = ax_1 + b$, what are the slope a and intercept b in terms of w_0, w_1, w_2

(b) Draw a picture for the cases $w = [1, 2, 3]^T$ and $w = -[1, 2, 3]^T$.

In more than two dimensions, the $+1$ and -1 regions are separated by a hyperplane, the generalization of a line.

2) Given $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, show that the rank of a matrix xy^T is one.

$$\begin{aligned} xy^T &= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \begin{bmatrix} y_1^T & y_2^T & \dots & y_n^T \end{bmatrix} \\ &= \begin{bmatrix} x_1 \cdot y_1^T & x_1 \cdot y_2^T & \dots & x_1 \cdot y_n^T \\ x_2 \cdot y_1^T & x_2 \cdot y_2^T & \dots & x_2 \cdot y_n^T \\ \vdots & \vdots & \ddots & \vdots \\ x_m \cdot y_1^T & x_m \cdot y_2^T & \dots & x_m \cdot y_n^T \end{bmatrix} \end{aligned}$$

Columns linear combinations of x thus we conclude the rank is one

3) Given $X = [x_1, x_2, \dots, x_n] \in \mathbb{R}^{m \times n}$ where $x_i \in \mathbb{R}^m$ for all i , and $Y^T = [y^1, y^2, \dots, y^n] \in \mathbb{R}^{p \times n}$ where $y^i \in \mathbb{R}^p$ for all i . Show that

$$XY = \sum_{i=1}^N x_i (y^i)^T$$

$$\sum_{i=1}^N x_i (y^i)^T =$$

4) Given $X \in \mathbb{R}^{m \times n}$, show that the matrix $X^T X$ is symmetric and positive semi-definite. When is it positive definite?

We first show that $X^T X$ is symmetric

$$\begin{aligned}(X^T X)^T &= X^T X^{TT} \\ &= X^T X\end{aligned}$$

Thus $X^T X$ is symmetric

Next we show $X^T X$ is positive-semidefinite

$$\begin{aligned}v^T (X^T X) v &= \|Xv\|^2 \\ \|Xv\|^2 &\geq 0 \quad \forall v \in \mathbb{R}^n\end{aligned}$$

Which shows that the matrix is positive-semidefinite

$X^T X$ is positive definite iff $\mathcal{N}(X) = \{\hat{0}\}$. Thus the matrix X must be full-column rank

5) Given $g(x, y) = e^x + e^{y^2} + e^{3xy}$, compute $\frac{\partial g}{\partial y}$.

$$\frac{\partial g}{\partial y} g(x, y) = 2ye^{y^2} + 3xe^{3xy}$$

6) Consider the matrix

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 3 & 2 & 5 \end{pmatrix}$$

- (a) Compute the eigenvalues and corresponding eigenvectors of A. You are allowed to use Matlab to compute the eigenvectors
- (b) What is the eigen-decomposition of A?
- (c) What is the rank of A?
- (d) Is A positive definite? Is A positive semi-definite?
- (e) Is A singular?