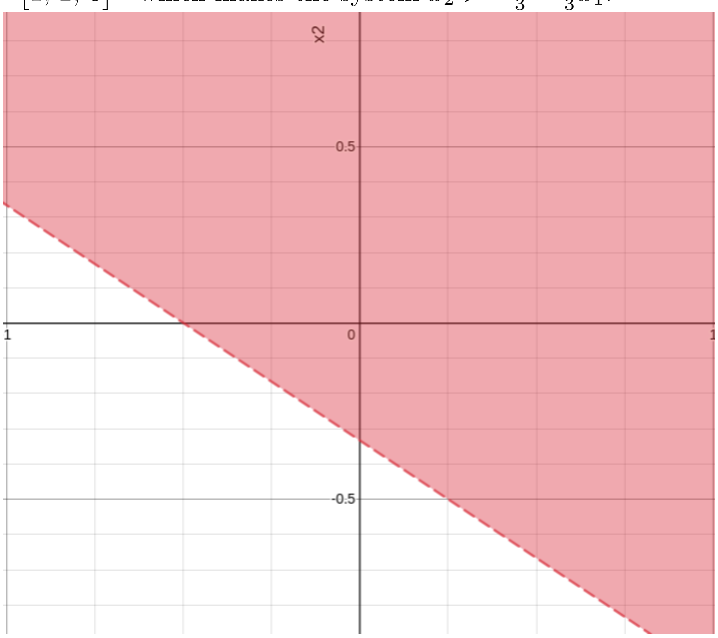
1) Consider the perceptron in two dimensions: $h(x) = sign(w^T x)$ where $w = \left[w_0,\ w_1,\ w_2\right]^T$ and $x = \left[1,\ x_1,\ x_2\right]^T$. Technically, x has three coordinates, but we call this perceptron two-dimensional because the first coordinate is fixed at 1.

(a) Show that the regions on the plane where $h(x) = +1$ and $h(x) = -1$ are separated by a line. If we express this line by the equation $x_2 = ax_1 + b$, what are the slope a and intercept b in terms of $w_0, w_{1,} w_2$

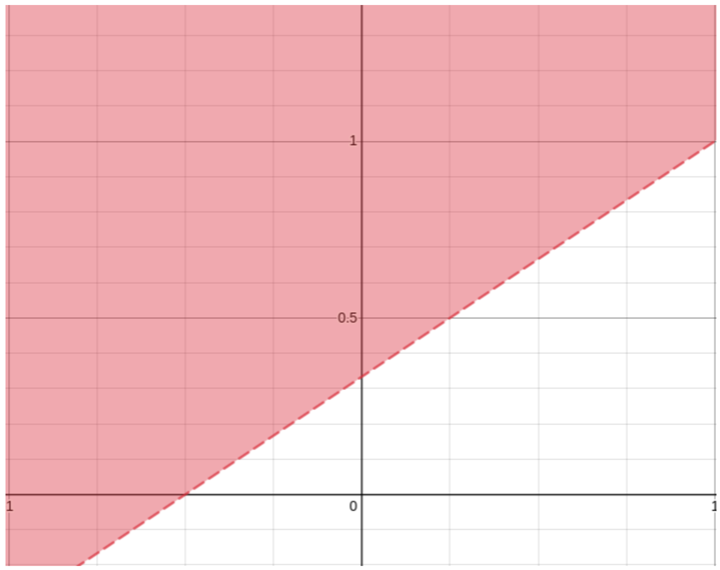
$sign(w^T x) = w_0 \cdot x_0 + w_1 \cdot x_1 + w_2 \cdot x_2 > 0$. $x_2 > -\frac{w_0}{w_2} - \frac{w_1}{w_2} \cdot x_1$where $w_2 \neq 0$. $a = -\frac{w_1}{w_2}$ and $b = -\frac{w_0}{w_2}$

(b) Draw a picture for the cases $w = \left[1,\ 2,\ 3\right]^T$ and $w = -\left[1,\ 2,\ 3\right]^T$.

$w = \left[1,\ 2,\ 3\right]^T$ which makes the system $x_2 > -\frac{1}{3} - \frac{2}{3}x_1$.



$w = -\left[1,\ 2,\ 3\right]$



In more than two dimensions, the $+1$ and -1 regions are separated by a hyperplane, the generalization of a line.

2) Given $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, show that the rank of a matrix xy^T is one.

$$\begin{aligned} xy^T &= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \begin{bmatrix} y_1^T & y_2^T & \dots & y_n^T \end{bmatrix} \\ &= \begin{bmatrix} x_1 \cdot y_1^T & x_1 \cdot y_2^T & \dots & x_1 \cdot y_n^T \\ x_2 \cdot y_1^T & x_2 \cdot y_2^T & \dots & x_2 \cdot y_n^T \\ \vdots & \vdots & \ddots & \vdots \\ x_m \cdot y_1^T & x_m \cdot y_2^T & \dots & x_m \cdot y_n^T \end{bmatrix} \end{aligned}$$

Columns linear combinations of x thus we conclude the rank is one

3) Given $X = \begin{bmatrix} x_1, x_2, \dots, x_n \end{bmatrix} \in \mathbb{R}^{m \times n}$ where $x_i \in \mathbb{R}^m$ for all i, and $Y^T = \begin{bmatrix} y^1, y^2, \dots, y^n \end{bmatrix} \in \mathbb{R}^{p \times n}$ where $y^i \in \mathbb{R}^p$ for all i. Show that

$$\begin{aligned} XY &= \sum_{i=1}^{n} x_i (y^i)^T \\ &= x_1 (y^1)^T + x_2 (y^2)^T + \ldots + x_n (y^n)^T \end{aligned}$$

$$
\begin{aligned}
\sum_{i=1}^{n} x_i (y^i)^T &= \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{m1} \end{bmatrix} \cdot \begin{bmatrix} y_{11}, \, y_{12}, \, \dots, \, y_{1p} \end{bmatrix} + \begin{bmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{m2} \end{bmatrix} \cdot \begin{bmatrix} y_{21}, \, y_{22}, \, \dots, y_{2p} \end{bmatrix} + \dots + \begin{bmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{mn} \end{bmatrix} \cdot \begin{bmatrix} y_{n1}, \, y_{n2}, \dots y_{np} \end{bmatrix} \\
&= \begin{bmatrix} x_{11} \cdot y_{11} & x_{11} \cdot y_{12} & \dots & x_{11} \cdot y_{1p} \\ x_{21} \cdot y_{11} & x_{21} \cdot y_{12} & \dots & x_{21} \cdot y_{1p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} \cdot y_{11} & x_{m1} \cdot y_{12} & \dots & x_{m1} \cdot y_{1p} \end{bmatrix} + \begin{bmatrix} x_{12} \cdot y_{21} & x_{12} \cdot y_{22} & \dots & x_{12} \cdot y_{2p} \\ x_{22} \cdot y_{21} & x_{22} \cdot y_{22} & \dots & x_{22} \cdot y_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m2} \cdot y_{21} & x_{m2} \cdot y_{22} & \dots & x_{m2} \cdot y_{2p} \end{bmatrix} + \dots + \\
&= \begin{bmatrix} \sum_{i=1}^{n} x_{1i} \cdot y_{i1} & \sum_{i=1}^{n} x_{1i} \cdot y_{i2} & \dots & \sum_{i=1}^{n} x_{1i} \cdot y_{ip} \\ \sum_{i=1}^{n} x_{2i} y_{i1} & \sum_{i=1}^{n} x_{2i} \cdot y_{i2} & \dots & \sum_{i=1}^{n} x_{2i} \cdot y_{ip} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{n} x_{mi} \cdot y_{i1} & \sum_{i=1}^{n} x_{mi} \cdot y_{i2} & \dots & \sum_{i=1}^{n} x_{mi} \cdot y_{ip} \end{bmatrix} \\
&= XY
\end{aligned}
$$

4) Given $X \in \mathbb{R}^{m \times n}$, show that the matrix $X^T X$ is symmetric and positive semi-definite. When is it positive definite?

We first show that $X^T X$ is symmetric

$$
\begin{aligned}
(X^T X)^T &= X^T X^{TT} \\
&= X^T X
\end{aligned}
$$

Thus $X^T X$ is symmetric

Next we show $X^T X$ is positive-semidefinite

$$
\begin{aligned}
v^T (X^T X) v &= \left\| Xv \right\|^2 \\
\left\| Xv \right\|^2 &\geq 0 \, \forall v \in \mathbb{R}^n
\end{aligned}
$$

Which shows that the matrix is positive-semidefinite

$X^T X$ is positive definite iff $\mathcal{N}(X) = \{\hat{0}\}$. Thus the matrix X must be full-column rank

5) Given $g(x, \, y) = e^x + e^{y^2} + e^{3xy}$, compute $\frac{\partial g}{\partial y}$.

$\frac{\partial g}{\partial y} g(x, y) = 2ye^{y^2} + 3xe^{3xy}$

6) Consider the matrix

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 3 & 2 & 5 \end{pmatrix}$$

(a) Compute the eigenvalues and corresponding eigenvectors of A. You are allowed to use Matlab to compute the eigenvectors

$\lambda_1 = 0$, $\lambda_2 = 4 + \sqrt{13}$, $\lambda_3 = 4 - \sqrt{13}$

$\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} \sqrt{13} - 3 \\ -\sqrt{13} + 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 - \sqrt{13} \\ \sqrt{13} + 4 \\ 1 \end{bmatrix}$

(b) What is the eigen-decomposition of A?

$$\begin{bmatrix} -1 & \sqrt{13} - 3 & -3 - \sqrt{13} \\ -1 & -\sqrt{13} + 4 & \sqrt{13} + 4 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 + \sqrt{13} & 0 \\ 0 & 0 & 4 - \sqrt{13} \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{5\sqrt{13}+13}{78} & \frac{13+2\sqrt{13}}{78} & \frac{7\sqrt{13}+26}{78} \\ -\frac{5\sqrt{13}-13}{78} & -\frac{2\sqrt{13}-13}{78} & \frac{26-7\sqrt{13}}{78} \end{bmatrix}$$

(c) What is the rank of A?

$$rref(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the matrix has two pivot columns the $Rank(A) = 2$

(d) Is A positive definite? Is A positive semi-definite?

A is not positive definite. A is positive semi-definite.

(e) Is A singular?

A is singular

7) What is cross-entropy error

$$p \cdot ln \frac{1}{1-q} \quad + \quad (1-p) \cdot ln \frac{1}{q}$$

$p = [y_n = 1]$ and $q = h(x_p)$