1) Write down the three-by-three matrix with ones on the diagonal and zeros elsewhere.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2) Write down the three-by-four matrix with ones on the diagonal and zeros elsewhere

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

3) Write down the four-by-three matrix with ones on the diagonal and zeros elsewhere

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

4) $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -2 & 1 \\ 2 & -4 & -2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$, $E = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Compute if defined

$B - 2A$: $\begin{bmatrix} 4 & -2 & 1 \\ 2 & -4 & -2 \end{bmatrix} - \begin{bmatrix} -4 & -2 & 2 \\ -2 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & -4 & -1 \\ 0 & -2 & -4 \end{bmatrix}$

$3C - E$ : undefined

$AC$ : undefined

$CD : \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 10 \\ 10 & 11 \end{bmatrix}$

$CB : \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & -2 & 1 \\ 2 & -4 & -2 \end{bmatrix} = \begin{bmatrix} 8 & -10 & -3 \\ 10 & -8 & 0 \end{bmatrix}$

5) Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 4 & 3 \\ 0 & 2 \end{bmatrix}$. Verify that $AB = AC$ and yet $B \neq C$.

$A \cdot B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 8 & 14 \end{bmatrix}$

$A \cdot C = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 4 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 8 & 14 \end{bmatrix}$

$B \neq C$

6) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Compute $AD$ and $DA$.

$$A \cdot D = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 2 & 6 & 12 \\ 2 & 9 & 16 \end{bmatrix}$$

$$D \cdot A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 3 & 6 & 9 \\ 4 & 12 & 16 \end{bmatrix}$$

7) Prove the associative law for matrix multiplication. That is, let $A$ be an $m-by-n$ matrix, $B$ an $n-by-p$ matrix, and C a p-by-q matrix. Then prove that $A(BC) = (AB)C$.

$A(BC)_{ij} = \sum_{k=1}^{n} a_{ik} \cdot (bc)_{kj} = \sum_{k=1}^{n} a_{ik} \sum_{r=1}^{p} b_{kr} \cdot c_{rk} = \sum_{r=1}^{p} \sum_{k=1}^{n} a_{ik} \cdot b_{kr} \cdot c_{rk} = \sum_{r=1}^{p} (AB)_{ir} C_{rj} = [(AB)C]_{ij}$

8) Let $A = \begin{bmatrix} -1 & 2 \\ 4 & -8 \end{bmatrix}$. Construct a two-by-two matrix $B$ such that $AB$ is the zero matrix. Use two different nonzero columns for $B$.

$\begin{bmatrix} -1 & 2 \\ 4 & -8 \end{bmatrix} \cdot \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} -1 & 2 \\ 4 & -8 \end{bmatrix} \cdot \begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

9) Verify that $\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \cdot \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 \cdot b_1 & 0 \\ 0 & a_2 \cdot b_2 \end{bmatrix}$.Prove that in general that the product of the two diagonal matrics is a diagonal matrix, with elements given by the product of the diagonal matrix

$\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \cdot \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 \cdot b_1 & 0 \\ 0 & a_2 \cdot b_2 \end{bmatrix}$

In general

$C_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$ where the sum equals to zero when $i \neq j$

10) Verify that $\begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix} \cdot \begin{bmatrix} b_1 & b_2 \\ 0 & b_3 \end{bmatrix} = \begin{bmatrix} a_1 \cdot b_1 & a_1 \cdot b_2 + a_1 \cdot b_3 \\ 0 & a_3 \cdot b_3 \end{bmatrix}$. Prove in general that the product of two upper triangular matrics is an upper triangular matrix, with the diagonal elements of the product given bythe product of the diagonal elements.

$\begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix} \cdot \begin{bmatrix} b_1 & b_2 \\ 0 & b_3 \end{bmatrix} = \begin{bmatrix} a_1 \cdot b_1 & a_1 \cdot b_2 + a_1 \cdot b_3 \\ 0 & a_3 \cdot b_3 \end{bmatrix}$

$C_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$ where the sum equals to zero when $i > j$

11) Identify the two-by-two matrix with matrix elements $a_{ij} = i - j$.

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

12) The matrix product $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$

13) Prove that $(AB)^T = B^T A^T$.

$((AB)_{ij})^T = (\sum_{k=1}^{n} a_{ik} \cdot b_{kj})^T = \sum_{k=1}^{n} a_{k1} \cdot b_{jk} = B^T A^T$

14) Show using the transpose operator that any square matrix A can be written as the sum of a symmetric matrix and a skew-symmetric matrix.

$A = \frac{1}{2}(A + A^T) + (A - A^T)$

15) Prove that $A^T A$ is symmetric.

$(A^T A)^T = A^T A^{TT} = A^T A$

16) Let $A$ be a rectangular matrix given by $A = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$

Compute $A^T A$ and show that it is a symmetric matrix and that the sum of its diagonal elements is the sum of the square of all the elements of $A$.

$A^T A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \cdot \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$=trace($\begin{bmatrix} a^2+b^2+c^2 & ad+be+cf \\ ad+be+cf & d^2+e^2+f^2 \end{bmatrix}$)$= a^2+b^2+c^2+d^2+e^2+f^2$

17) Let A be an rectangular matrix given by $A = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$.

$trace(A^T A) = \sum_{i=1}^{n} a_{ii}^2$

18) Find the inverses of the matrices $\begin{bmatrix} 5 & 6 \\ 4 & 5 \end{bmatrix}$ and $\begin{bmatrix} 6 & 4 \\ 3 & 3 \end{bmatrix}$

$\frac{1}{1} \cdot \begin{bmatrix} 5 & -6 \\ -4 & 5 \end{bmatrix}$ and $\frac{1}{6} \cdot \begin{bmatrix} 3 & -4 \\ -3 & 6 \end{bmatrix}$

19) Prove that if $A$ and $B$ are same-sized invertible matrices, then $(AB)^{-1} = B^{-1}A^{-1}$. $B^{-1}A^{-1}AB = I$,

$ABB^{-1}A^{-1} = I$

20) Prove that if $A$ is invertible then so is $A^T$, and $(A^T)^{-1} = (A^{-1})^T$. $AA^{-1} = I$

$(A^{-1})^T A^T = I$

21) Prove that if a matrix is invertible, then its inverse is unique.

$AB = I$ $=AC$

$BAB = BAC$

$B = C$