

1) Write down the three-by-three matrix with ones on the diagonal and zeros elsewhere.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2) Write down the three-by-four matrix with ones on the diagonal and zeros elsewhere

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

3) Write down the four-by-three matrix with ones on the diagonal and zeros elsewhere

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$4) A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & -2 & 1 \\ 2 & -4 & -2 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, D = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix},$$

$$E = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Compute if defined

$$B - 2A: \begin{bmatrix} 4 & -2 & 1 \\ 2 & -4 & -2 \end{bmatrix} - \begin{bmatrix} -4 & -2 & 2 \\ -2 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & -4 & -1 \\ 0 & -2 & -4 \end{bmatrix}$$

$3C - E$ : undefined

$AC$ : undefined

$$CD: \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 10 \\ 10 & 11 \end{bmatrix}$$

$$CB: \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & -2 & 1 \\ 2 & -4 & -2 \end{bmatrix} = \begin{bmatrix} 8 & -10 & -3 \\ 10 & -8 & 0 \end{bmatrix}$$

$$5) \text{ Let } A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, C = \begin{bmatrix} 4 & 3 \\ 0 & 2 \end{bmatrix}. \text{ Verify that } AB = AC \text{ and}$$

yet  $B \neq C$ .

$$A \cdot B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 8 & 14 \end{bmatrix}$$

$$A \cdot C = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 4 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 8 & 14 \end{bmatrix}$$

$$B \neq C$$

$$6) \text{ Let } A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix} \text{ and } D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}. \text{ Compute } AD \text{ and } DA.$$

$$A \cdot D = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 2 & 6 & 12 \\ 2 & 9 & 16 \end{bmatrix}$$

$$D \cdot A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 3 & 6 & 9 \\ 4 & 12 & 16 \end{bmatrix}$$

7) Prove the associative law for matrix multiplication. That is, let  $A$  be an  $m - by - n$  matrix,  $B$  an  $n - by - p$  matrix, and  $C$  a  $p - by - q$  matrix. Then prove that  $A(BC) = (AB)C$ .

$$A(BC)_{ij} = \sum_{k=1}^n a_{ik} \cdot (bc)_{kj} = \sum_{k=1}^n a_{ik} \sum_{r=1}^p b_{kr} \cdot c_{rk} = \sum_{r=1}^p \sum_{k=1}^n a_{ik} \cdot b_{kr} \cdot c_{rk} = \sum_{r=1}^p (AB)_{ir} C_{rj} = [(AB)C]_{ij}$$

8) Let  $A = \begin{bmatrix} -1 & 2 \\ 4 & -8 \end{bmatrix}$ . Construct a two-by-two matrix  $B$  such that  $AB$  is the zero matrix. Use two different nonzero columns for  $B$ .

$$\begin{bmatrix} -1 & 2 \\ 4 & -8 \end{bmatrix} \cdot \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 \\ 4 & -8 \end{bmatrix} \cdot \begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

9) Verify that  $\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \cdot \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 \cdot b_1 & 0 \\ 0 & a_2 \cdot b_2 \end{bmatrix}$ . Prove that in general that the product of the two diagonal matrices is a diagonal matrix, with elements given by the product of the diagonal matrix

$$\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \cdot \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 \cdot b_1 & 0 \\ 0 & a_2 \cdot b_2 \end{bmatrix}$$

In general

$$C_{ij} = \sum_{k=1}^n a_{ik} \cdot b_{kj} \text{ where the sum equals to zero when } i \neq j$$

10) Verify that  $\begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix} \cdot \begin{bmatrix} b_1 & b_2 \\ 0 & b_3 \end{bmatrix} = \begin{bmatrix} a_1 \cdot b_1 & a_1 \cdot b_2 + a_2 \cdot b_3 \\ 0 & a_3 \cdot b_3 \end{bmatrix}$ . Prove in general that the product of two upper triangular matrices is an upper triangular matrix, with the diagonal elements of the product given by the product of the diagonal elements.

$$\begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix} \cdot \begin{bmatrix} b_1 & b_2 \\ 0 & b_3 \end{bmatrix} = \begin{bmatrix} a_1 \cdot b_1 & a_1 \cdot b_2 + a_2 \cdot b_3 \\ 0 & a_3 \cdot b_3 \end{bmatrix}$$

$$C_{ij} = \sum_{k=1}^n a_{ik} \cdot b_{kj} \text{ where the sum equals to zero when } i > j$$

11) Identify the two-by-two matrix with matrix elements  $a_{ij} = i - j$ .

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$12) \text{ The matrix product } \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$$

13) Prove that  $(AB)^T = B^T A^T$ .

$$((AB)_{ij})^T = (\sum_{k=1}^n a_{ik} \cdot b_{kj})^T = \sum_{k=1}^n a_{k1} \cdot b_{jk} = B^T A^T$$

14) Show using the transpose operator that any square matrix  $A$  can be written as the sum of a symmetric matrix and a skew-symmetric matrix.

$$A = \frac{1}{2}(A + A^T) + (A - A^T)$$

15) Prove that  $A^T A$  is symmetric.

$$(A^T A)^T = A^T A^{TT} = A^T A$$

$$16) \text{ Let } A \text{ be a rectangular matrix given by } A = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

Compute  $A^T A$  and show that it is a symmetric matrix and that the sum of its diagonal elements is the sum of the square of all the elements of  $A$ .

$$A^T A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \cdot \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} = \text{trace} \left( \begin{bmatrix} a^2 + b^2 + c^2 & ad + be + cf \\ ad + be + cf & d^2 + e^2 + f^2 \end{bmatrix} \right) = a^2 + b^2 + c^2 + d^2 + e^2 + f^2$$

17) Let  $A$  be a rectangular matrix given by  $A = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$ .

$$\text{trace}(A^T A) = \sum_{i=1}^n a_{ii}^2$$

18) Find the inverses of the matrices  $\begin{bmatrix} 5 & 6 \\ 4 & 5 \end{bmatrix}$  and  $\begin{bmatrix} 6 & 4 \\ 3 & 3 \end{bmatrix}$

$$\frac{1}{1} \cdot \begin{bmatrix} 5 & -6 \\ -4 & 5 \end{bmatrix} \text{ and } \frac{1}{6} \cdot \begin{bmatrix} 3 & -4 \\ -3 & 6 \end{bmatrix}$$

19) Prove that if  $A$  and  $B$  are same-sized invertible matrices, then  $(AB)^{-1} = B^{-1}A^{-1}$ .  $B^{-1}A^{-1}AB = I$ ,

$$ABB^{-1}A^{-1} = I$$

20) Prove that if  $A$  is invertible then so is  $A^T$ , and  $(A^T)^{-1} = (A^{-1})^T$ .  
 $AA^{-1} = I$

$$(A^{-1})^T A^T = I$$

21) Prove that if a matrix is invertible, then its inverse is unique.

$$AB = I = AC$$

$$BAB = BAC$$

$$B = C$$

22) Show that the product of two orthogonal matrices is orthogonal

$$Q' = A \cdot B$$

$$Q'^T = B \cdot A = A \cdot B = Q'$$

$$|Q'| = I$$

23) Show that the n-by-n identity matrix is orthogonal

Norm of the identity is one

symmetric

24) Let  $R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ . Show that  $R(-\theta) = R(\theta)^{-1}$ .

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = R(\theta)^{-1}$$

$$\begin{bmatrix} R(\theta) & 0 \\ 0 & 1 \end{bmatrix}$$

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