IOE 516: Assignment 1 Solutions

Due: 11:59 PM, February 2nd, 2024

1. Let A and B be independent events; show that A^c, B are independent, and deduce that A^c and B^c are independent as well.

Solution:

$$\begin{aligned} P(A^c \cap B) &= P(B \backslash \{A \cap B\}) \\ &= P(B) - P(A \cap B) \\ &= P(B) - P(A)P(B) \\ &= P(A^c)P(B) \end{aligned}$$

For the last part, you need to apply what you did for the first part but to the pair B, A^c.

2. Find the generating function $M(\theta) = \sum_{n=1}^{\infty} f(n)e^{\theta n}$ of the following mass functions. Then, calculate their means and variances.

(a) $f(m) = [m(m+1)]^{-1}, \text{ for } m \geq 1$

(b) $f(m) = \binom{n+m-1}{m} p^n (1-p)^m, \text{ for } m \geq 1$

Solution: **(a)** Assume $s = e^{\theta}$. Therefore, if $|s| < 1$,

$$\begin{aligned} G(s) &= \sum_{m=1}^{\infty} s^m \left(\frac{1}{m} - \frac{1}{m+1}\right) \\ &= \sum_{m=1}^{\infty} \frac{s^m}{m} - \frac{1}{s} \sum_{m=1}^{\infty} \frac{s^{m+1}}{m+1} \end{aligned}$$

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{s^m}{m} &= -\log(1-x) + C_1 \\ \sum_{m=1}^{\infty} \frac{s^{m+1}}{m+1} &= -\log(1-x) - x + C_2 \\ \longrightarrow G(s) &= \log(1-s)\left(\frac{1-s}{s}\right) + s + C_1 - C_2 \end{aligned}$$

To find $C_1 - C_2$, knowing that $M(0) = 1$, taking $\theta \to 0$ gives:

$$\begin{aligned} \lim_{s \to 1} \log(1-s)\left(\frac{1-s}{s}\right) &= 0 \\ \longrightarrow M(0) &= 1 + C_1 - C_2 \\ \Longrightarrow C_1 - C_2 &= 0 \end{aligned}$$

Therefore,

$$M(\theta) = e^{\theta} + \left(\frac{1-e^{\theta}}{e^{\theta}}\right) \log(1-e^{\theta})$$

Therefore, $M^{'}(0) = \infty$, and no moment of order 1 or greater exists.

(b) Taking $s = e^{\theta}$, if $|s| < (1-p)^{-1}$,

$$G(s) = \sum_{m=0}^{\infty} s^m \binom{n+m-1}{m} P^n (1-p)^m = [\frac{p}{1-s(1-p)}]^n$$

Therefore, the mean is $G^{'}(1) = \frac{n(1-p)}{p}$. The variance then would be:

$$G^{"}(1) + G^{'}(1) - G^{'}(1)^2 = \frac{n(1-p)}{p^2}$$

3. Let $X_1, ..., X_n$ be a random sample from a population with mean μ and variance $\sigma^2 < \infty$. Let

$$S^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2/(n-1)$$

be the sample variance. Show that $E[S] \leq \sigma$.

Solution: We know that $\mathbb{E}[S^2] = \sigma^2$ and since variance is always non-negative, we know that $Var(S) = \mathbb{E}[S^2] - (\mathbb{E}[S])^2 \geq 0$. Therefore,

$$[\mathbb{E}(s)]^2 = \sigma^2 - Var(S) \leq \sigma^2 \longrightarrow \mathbb{E}(s) \leq \sigma$$

4. Let $X_1, X_2, ..., X_n$ be a random sample from $Uniform(\theta)$ random variables. Define $X_{(n)} = \max\{X_1, \ldots, X_n\}$ to be the largest order statistic. Find the limiting probability distribution of $V_n = n(\theta - X_{(n)})$.

Solution: As a first step, we calculate the CDF of V_n as below

$$\begin{aligned}
P(V_n \leq x) &= P(n(\theta - X_{(n)}) \leq x) \\
&= P(X_{(n)} \geq \theta - \frac{x}{n}) \\
&= 1 - P(X_{(n)} \leq \theta - \frac{x}{n}) \\
&= 1 - [P(X_i \leq \theta - \frac{x}{n})]^n \\
&= 1 - (\frac{\theta - \frac{x}{n}}{\theta})^n \\
&= 1 - [1 - \frac{\frac{x}{\theta}}{n}]^n \to 1 - e^{\frac{-x}{\theta}}
\end{aligned}$$

As it can be seen, the cdf of V_n converges to the cdf of an exponential distribution with parameter θ. Therefore,

$$V_n \xrightarrow{d} \exp \theta$$

5. Let $X_1, X_2, ..., X_n$ be a random sample from exponential random variables with mean 1. Define $X_{(n)} = \max\{X_1, \ldots, X_n\}$ to be the largest order statistic. Find the limiting probability distribution of $V_n = X_{(n)} - \log n$.

Solution: For real number $x \in (-\infty, \infty)$, we have

$$P(V_n \leq x) = P(\max\{X_1, \ldots, X_n) \leq \log n + x)$$

$$= \prod_{i=1}^{n}(P(X_i < \log n + x))^n = (1 - e^{-(\log n + x)})^n = \left(1 - \frac{e^{-x}}{n}\right)^n.$$

Taking limit $n \to \infty$, we obtain the converging distribution as

$$P(V_n \leq x) \to e^{-e^{-x}}.$$

6. Suppose $X_1, X_2, ..., X_n$ be a random sample from a Poisson distribution with mean λ. Define $\bar{X}$ and S^2 to be the sample mean and sample variance based on the random sample.

(a) Show that S^2 converges to λ in probability $n \to \infty$.

(b) Show that $\frac{\sqrt{n}(\bar{X}-\lambda)}{S}$ converges to $N(0,1)$ in distribution as $n \to \infty$.

Solution:
(a) Notice that we can write S^2 as

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{n}{n-1}[\frac{1}{n}\sum_{i=1}^{n}X_i^2 - \bar{X}^2]$$

. Using WLLN and continuous mapping, we have

$$\frac{1}{n}\sum_{i=1}^{n}X_i^2 \xrightarrow{P} \mathbb{E}[X_i^2] = V(X_i) + \mathbb{E}[X_i]^2 = \lambda + \lambda^2$$

Moreover, $\bar{X}^2 \xrightarrow{P} \lambda^2$ and $\frac{n}{n-1} \to 1$. Therefore, using all these information and continuous mapping theorem, we would get the desired result,

$$S^2 \xrightarrow{P} \lambda$$

(b) OMITTED- Using CLT, we have

$$\sqrt{n}\frac{\bar{X}-\lambda}{\sqrt{\lambda}} \xrightarrow{d} N(0,1)$$

Based on continuous mapping theorem, we know that $\frac{1}{S} \xrightarrow{P} \frac{1}{\sqrt{\lambda}}$. Now using the Slutsky's theorem, we would have the desired result,

$$\sqrt{n}\frac{\bar{X}-\lambda}{S} = \sqrt{n}\frac{\bar{X}-\lambda}{\lambda}.\frac{\lambda}{s} \xrightarrow{d} N(0,1)$$

7. Let $X_1, X_2, ...$ be i.i.d. Poisson with mean 1, and let $S_n = X_1 + \cdots + X_n$. For $a > 1$ find

$$\lim_{n\to\infty}\frac{1}{n}\log \mathbb{P}(S_n \geq na)$$

.

Solution: In this question we have to find the large deviation exponent $-l(a)$. Since $M(\theta) = \mathbb{E}[e^{\theta X_1}] = \sum_{n=0}^{\infty} e^{-1}\frac{e^{\theta n}}{n!} = e^{e^{\theta}-1}$; then,

$$\begin{aligned} -l(a) &= -\sup_{\theta>0}(\theta a - \log M(\theta)) \\ &= -\sup_{\theta>0}(\theta a - e^{\theta} + 1) \end{aligned}$$

$$= -a \log a + a - 1$$

8. On the street of a large country, 10% of the vehicles have black color. When 6000 vehicles are randomly chosen, what is the probability the number of black vehicles is less than 660?

Solution: Let X_i be 1 if the i-th vehicle is black and 0 otherwise. Then $E[X_1] = p = 0.1$ and $Var(X_1) = p(1-p) = 0.09$. The total number of black vehicle is

$$S = \sum_{i=1}^{6000} X_i$$

Recall that

$$\frac{S - 6000p}{\sqrt{6000p(1-p)}} = \frac{S - 600}{23.24}$$

is approximately normal. Hence,

$$P(S < 660) = P\left(\frac{S - 600}{23.24} \leq \frac{60}{23.24}\right) \approx P(Z \leq 2.58) = 0.9951$$

9. Let Z be standard normal. Show that the tail function satisfies

$$\frac{x}{\sqrt{2\pi}(1 + x^2)} e^{-x^2/2} \leq P(Z > x) \leq \frac{1}{\sqrt{2\pi} x} e^{-x^2/2}$$

Solution: The second inequality has been proved in class. Refer to notes.
To prove the first inequality, we define

$$g(x) = P(Z > x) - \frac{x}{\sqrt{2\pi}(1 + x^2)} e^{-x^2/2} = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz - \frac{x}{\sqrt{2\pi}(1 + x^2)} e^{-x^2/2}.$$

It suffices to prove $g(x) > 0$ for all $x > 0$. It is easy to see $g(0) = 1/2 > 0$. Further, it can be verified that

$$g'(x) = -\frac{2}{\sqrt{2\pi}} \cdot \frac{e^{-x^2/2}}{(x^2 + 1)^2} < 0,$$

so $g(x)$ is strictly decreasing. Since $g(x)$ converges to 0 as $x \to \infty$, implies $g(x) > 0$ for all $x > 0$.

10. Use CLT to show that

$$e^{-n} \sum_{i=1}^{n} \frac{n^i}{i!} \to \frac{1}{2} \text{ as } n \to \infty.$$

Solution: Let X_i be i.i.d. Poisson with mean 1 and $S_n = \sum_{i=1}^n X_i$, then

$$P(S_n \leq n) = e^{-n} \sum_{k=1}^{n} \frac{n^k}{k!}.$$

Now, the LHS is

$$P(S_n \leq n) = P(S_n - n \leq 0) = P\Big(\frac{S_n - n}{\sqrt{n}} \leq 0\Big) \to P(Z < 0) = \frac{1}{2},$$

where the limit follows from CLT, and Z is standard normal.