Homework 7

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Problem 1. (*15 points*) Give an example of a doubly indexed collection $\{x_{m,n} : m, n \in \mathbb{Z}\}$ of real numbers such that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{m,n} = 0 \quad \text{and} \quad \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} x_{m,n} = \infty.$$

Solutions. We consider the following example:

$$x_{m,n} = \begin{cases} m & \text{if } n = m \\ -m & \text{if } n = m+1 \\ 0 & \text{otherwise.} \end{cases}$$

In this case $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{m,n} = (1-1+0+\dots) + (0+2-2+\dots) + \dots = 0$ and $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} x_{m,n} = (1+0+\dots) + (-1+2+0+\dots) + (0-2+3+\dots) = 1+1+1+\dots = \infty$ □

Problem 2. (*15 points*) Let λ be the Lebesgue measure on $[0,1]$. Compute

$$\int_{[0,1]}\int_{[0,1]}\frac{x^2-y^2}{(x^2+y^2)^2}d\lambda(x)d\lambda(y)$$

and

$$\int_{[0,1]}\int_{[0,1]}\frac{x^2-y^2}{(x^2+y^2)^2}d\lambda(y)d\lambda(x)$$

Explain this violates neither Tonelli's Theorem nor Fubini's Theorem.

Solutions.
We can think about the function $f(y)=[E]_a=\frac{x^2-y^2}{(x^2+y^2)^2}$ with fixed x and this function is bounded when fixed x and the function is continuous except for $x=y=0$ but this point has measure 0, so we have this function is Riemann integrable. Therefore, we can directly evaluate the Riemann integral.

$$F(y)=\int_{[0,1]}\frac{x^2-y^2}{(x^2+y^2)^2}d\lambda(x)=\int_0^1\frac{x^2-y^2}{(x^2+y^2)^2}dx=\frac{-x}{x^2+y^2}\Big|_0^1=-\frac{1}{1+y^2}$$

since this function $F(y)$ is also integrable over $[0,1]$ then we have:

$$\int_{[0,1]}F(y)dy=\int_0^1-\frac{1}{1+y^2}dy=-\arctan(1)+\arctan(0)=-\frac{\pi}{4}$$

For the second one, we similarly have:

$$F(x)=\int_{[0,1]}\frac{x^2-y^2}{(x^2+y^2)^2}dy=\frac{y}{y^2+x^2}\Big|_0^1=\frac{1}{1+x^2}$$

and therefore, $\int_{[0,1]}F(x)dx=\arctan(x)\big|_0^1=\frac{\pi}{4}$
The interval $[0,1]$ is definitely σ-finite measure spaces. The Tonelli's Theorem assumes that the function is non-negative, but this function is definitely not positive in some intervals, so we can not directly use Tonelli's Theorem. Besides, we do not have this function integrable on $[0,1]\times[0,1]$ so doesn't satisfy the assumption of Fubini's theorem (for $x\to 0, y\to 0$ this function is definitely infinite, violating the absolutely integrability condition). $\square$

Problem 3. (*40 points*) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a Lebesgue measurable function such that $f^2 \in L^1(\mathbb{R})$. Prove that

a) For almost every $b \in \mathbb{R}$, we have

$$\lim_{t\to 0} \frac{1}{2t} \int_{b-t}^{b+t} |f - f(b)| = 0.$$

b) For almost every $b \in \mathbb{R}$, we have

$$\lim_{t\to 0} \frac{1}{2t} \int_{b-t}^{b+t} |f - f(b)|^2 = 0.$$

(*Note that in part a), Lebesgue Differentiation Theorem cannot be applied directly since f is not known to be in $L^1(\mathbb{R})$, which is one of the requirements for LDT.*)

Solutions.
We start by considering the sequence of functions $f_n = f\chi_{[-n,n]}$ where $\chi_{[-n,n]}$ is the characteristic function defined on the interval $[-n, n]$ The sequence f_n converges pointwise to f as $n \to \infty$ because for any $x \in \mathbb{R}$, x will belong to $[-n, n]$ and thus $\lim_{n\to\infty} f_n = f$.
Given that $f^2 \in \mathcal{L}^1(\mathbb{R})$ we know that $\int_{\mathbb{R}} |f^2| < \infty$ and therefore, $f \in \mathcal{L}^2(\mathbb{R})$, which implies that f is also in $\mathcal{L}^1_{loc}(\mathbb{R})$ (square integrability ensures local integrability). To show that $f \in \mathcal{L}^2(\mathbb{R})$ implies $\mathcal{L}^1_{loc}(\mathbb{R})$. We can consider a compact set $K \subseteq \mathbb{R}$ and from Cauchy-Schwarz ineuqality we have:

$$(\int_K f)^2 \leq \mu(K) \int_K |f|^2$$

Therefore, since $\int_K |f|^2$ is finite and the length of the K is also finite, we have f in $\mathcal{L}^1_{loc}(\mathbb{R})$. Then we can apply Lebesgue Differentiation Theorem for f_n.
Since f^2 is integrable we have f^2 can serve as a dominating function for f_n and by the dominated convergence theorem, we can conclude that $\lim_{n\to\infty} \int_{\mathbb{R}} f_n = \int_{\mathbb{R}} f$.
Combining together we conclude that f also satisfies the Lebesgue Differentiation Theorem and we conclude the proof.
(b)
Since $f^2 \in \mathcal{L}(\mathbb{R})$ we can directly use Lebesgue Differentiation Theorem and for almost every $b \in \mathbb{R}$ we have:

$$\lim_{t\to 0} \frac{1}{2t} \int_{b-t}^{b+t} |f^2(x) - f^2(b)| dx = 0$$

we can write

$$\frac{1}{2t} \int_{b-t}^{b+t} |f - f(b)|^2 = \frac{1}{2t} \int_{b-t}^{b+t} f^2 - 2f(b) \frac{1}{2t} \int_{b-t}^{b+t} f + f^2(b)$$

then from part (a) we have $\lim_{t\to\infty} \frac{1}{2t} \int_{b-t}^{b+t} f = f(b)$ and from $f \in \mathcal{L}^2(\mathbb{R})$ we have: $\lim_{t\to 0} \frac{1}{2t} \int_{b-t}^{b+t} f^2(x) = f^2(b)$ then we conclude that $\lim_{t\to\infty} \frac{1}{2t} \int_{b-t}^{b+t} |f - f(b)|^2 = 2f^2(b) - 2f(b)f(b) = 0.$ $\square$