IOE 516

Stochastic Processes II

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Recap

- Renewal process is a counting process with interarrival times being i.i.d. random variables that are not identically equal to 0. Let $T_1, T_2, \ldots$ denote the interarrival times and $S_1, S_2, \ldots$ the arrival times, then

$$N(t) = \max\{t : S_n \leq t\} = \sum_{n=1}^{\infty} 1[S_n \leq t].$$

- Let $E[T_1] = \mu$. We have shown that, almost surely,

$$\frac{N(t)}{t} \to \frac{1}{\mu}.$$

- An important questions is: Does it mean that $E[N(t)]/t$ also converges to $1/\mu$? This is what we will study next.

Renewal function

- The average number of renewals,

$$m(t) = E[N(t)]$$

  is called the renewal function.

- Note the following

$$\{N(t) \geq n\} \longleftrightarrow S_n \leq t.$$

- Why?

Claim

- For any $t \geq 0$,

$$m(t) < \infty$$

- Why?

Recall

- $N(t)$ is the number of renewals by time t:

$$N(s) = \sum_{n=1}^{\infty} 1[S_n \leq t].$$

- The average number of renewals,

$$m(t) = E[N(t)] = \sum_{n=1}^{\infty} P(S_n \leq t)$$

is called the renewal function.

Renewal of equation

- The renewal function $m(t)$ satisfies the so-called renewal equation

$$m(t) = F(t) + \int_0^t m(t-s) dF(s).$$

- Why? Always prove by conditioning on the first renewal event.

Example

- The inter-arrival times of a renewal process is uniform [0,1]. Compute $m(t)$ for $0 \leq t \leq 1$.

- The renewal equation is ...

- The solution is ...

However,

- Very few renewal processes have closed form solution of $m(t)$?

- Can you give another example?

- When we do not know $m(t)$, we try to understand its other properties.

Elementary Renewal Theorem

- We have

$$\frac{m(t)}{t} \to 1/\mu.$$

- As discussed earlier, this is L_1 convergence, we cannot directly expect it is true from

$$\frac{N(t)}{t} \to 1/\mu, \qquad a.s.$$

Proof

- Is $N(t)$ a stopping time with respect to $T_1, T_2, \ldots$?

- How about $N(t) + 1$?

- Recall that $m(t) < \infty$ for any $t \geq 0$?

- Thus, by Wald's equation,

$$E[S_{N(t)+1}] = E\Big[\sum_{i=1}^{N(t)+1} X_i\Big] = (m(t)+1)E[X_1].$$

- On the other hand, $S_{N(t+1)} > t$. Hence $E[S_{N(t)+1}] \geq t$.

- We obtain $(m(t)+1)\mu \geq t$, or

$$m(t)/t \geq \frac{1}{\mu} - \frac{1}{t}.$$

- Letting $t \to \infty$, we obtain

$$\liminf_{t\to\infty} \frac{m(t)}{t} \geq \frac{1}{\mu}.$$

- To prove the other direction, consider another renewal process with

$$\bar{X}_i = \min\{X_i, M\}.$$

- The corresponding renewal process clearly satisfies $\bar{N}(t) \geq N(t)$, $\bar{m}(t) \geq m(t)$, and $\bar{S}_n(t) \leq S(t)$. In addition

$$\bar{S}_{N(t)+1} \leq t + M.$$

- Thus, for any $t \geq 0$,

$$(\bar{m}(t) + 1)\bar{\mu}_M \leq t + M.$$

- Letting $t \to \infty$, we obtain

$$\limsup_{t\to\infty} \frac{\bar{m}(t)}{t} \leq \frac{1}{\bar{\mu}_M}.$$

- This implies
$$\limsup_{t\to\infty}\frac{m(t)}{t} \le \frac{1}{\bar{\mu}_M}.$$

- Since this is true for any $M > 0$, letting $M \to \infty$, we obtain (why?)
$$\limsup_{t\to\infty}\frac{m(t)}{t} \le \frac{1}{\mu}.$$

- Putting the two inequalities together, we prove the **Elementary Renewal Theorem**.

Renewal reward process

- An important extension of renewal process is the so-called renewal reward process.

- There is a renewal process, with inter-arrival times $T_1, T_2, \ldots$. Associated with renewal cycle n is an reward R_n, which may depend on T_n, but $(T_1, R_1), (T_2, R_2), \ldots$, are i.i.d.

- Let $R(t)$ denote the total rewards received by time t. That is

$$R(t) = \sum_{n=1}^{N(t)} R_n.$$

Remark

- In the definition above, we assume that the reward is received at the end of the cycle. All the results presented below hold when the reward is received gradually during the cycle.

Renewal reward theorem

- Suppose $E[T_1] < \infty$ and $E[R_1] < \infty$. Then

  (i) With probability 1,

$$\frac{R(t)}{t} \to \frac{E[R_1]}{E[T_1]}$$

  (ii)] Its mean converges as well

$$\frac{E[R(t)]}{t} \to \frac{E[R_1]}{E[T_1]}.$$

Proof

- The first follows from

$$\frac{R(t)}{t} = \frac{\sum_{n=1}^{N(t)} R_n}{N(t)} \cdot \frac{N(t)}{t}$$

  and SLLN.

- The proof of the second is similar to that of Elementary Renewal theorem, that uses Wald's equation.

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Example 1:
Car buying model

- Car lifetime is random with cdf H ad pdf h.

- New car costs C_1, and displacement of a broken car costs C_2.

- Policy is to buy a new car either when the current one fails or it reaches T.

- What is the average cost per period?

Example 2:
Average age of renewal

- $A(t) = t - S_{N(t)}$.

- Time average of the age of renewal process by time T is
$$\frac{\int_0^T A(t)dt}{T}$$

- What is the long run time average of the age of renewal process?

Analysis

- We are interested in the limit of
$$\lim_{T\to\infty} \frac{\int_0^T A(t)dt}{T}$$

- By Renewal Reward theorem, the limit is equal to the ration of average reward during a cycle divided by the average length of a cycle, which is
$$\frac{E\left[\int_0^{T_1} tdt\right]}{E[T_1]} = \frac{E[T_1^2]}{2E[T_1]}.$$

Example 3:
Average excess of renewal

- $Y(t) = S_{N(t)+1} - t$.

- Time average of the residual life of the renewal process is

$$\frac{\int_0^T Y(t)dt}{T}$$

- What is the long run time average of the residual life of renewal process?

Analysis

- We are interested in the limit of

$$\lim_{T\to\infty} \frac{\int_0^T Y(t)dt}{T}$$

- By Renewal Reward theorem, the limit is equal to the ration of average reward during a cycle divided by the average length of a cycle, which is

$$\frac{E\Big[\int_0^{T_1}(T_1 - t)dt\Big]}{E[T_1]} = \frac{E[T_1^2]}{2E[T_1]}.$$

Average inter-arrival time containing t

- The average inter-arrival time containing the time is average age plus average residual life time

$$\frac{E[T_1^2]}{2E[T_1]} + \frac{E[T_1^2]}{2E[T_1]} = \frac{E[T_1^2]}{E[T_1]}.$$

- Note that

$$\frac{E[T_1^2]}{E[T_1]} \geq E[T_1]$$

- The LHS is actually twice the RHS when T_1 is exponentially distributed.

Renewal paradox

- Hence, the inter-arrival time containing t is longer than a standard inter-arrival time!

- Your intuition?