IOE 516

# Stochastic Processes II

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# Recap

- Last week we discussed regular deviation, medium deviation, and large deviation theory, and started concentration inequality. In particular, we discussed an important concentration inequality, Hoeffding inequality, for the case of bounded random variables.

- We focused on i.i.d. case, but the result, and almost identical proof, extends to independent case: Suppose $X_i$'s are independent with mean $\mu_i$ and suppose $[a_i, b_i]$, and $E[X_i] = \mu_i$. Then, for any $\epsilon > 0$,

$$P\Big(S_n - \sum_{i=1}^n \mu_i > \epsilon\Big) \leq e^{-\frac{2\epsilon^2}{\sum_{i=1}^n (b_i - a_i)^2}}.$$

**Today, we first go over the homework problems**

# Drawback with Hoeffding inequality

- It requires the random variables to be bounded (and independent).

- **Question 1.** Can we extend the results to unbounded random variables?

- **Question 2.** Can we extend the results to dependent random variables?

- **Answer.** Yes, but we need some other, but relatively weak, conditions.

# Sub-Gaussian r.v.'s

- A random variable $X$ is said to be sub-Gaussian if its tail satisfies, for some $k > 0$,

$$P(|X| \geq t) \leq 2e^{-kt^2}, \quad \forall t \geq 0.$$

- **Claim.** It can be shown that $X$ is sub-Gaussian if and only if the MGF of $X^2$ is finite near 0.

# Examples

- **Examples.** Normal, and bounded random variables are sub-Gaussian.

- **Counter-examples.** Random variables, such as exponential, Poisson, etc., are not sub-Gaussian.

# Generalized Hoeffding inequality

- If $X_1, X_2, \ldots$ are i.i.d. sub-Gaussian, then there exists $c > 0$ such that

$$
\begin{aligned}
P(|S_n - n\mu| \geq \epsilon) &\leq 2e^{-\frac{c\epsilon^2}{n}}, \\
P(S_n \geq n\mu + \epsilon) &\leq e^{-\frac{c\epsilon^2}{n}}
\end{aligned}
$$

# Application of Generalized Hoeffding inequality

- Let $\epsilon = (\sqrt{\alpha/c})n$, then we obtain

$$P\left(S_n \geq n\mu + \frac{\alpha}{c}n\right) \;\leq\; e^{-\alpha n}$$

- Let $\epsilon = \sqrt{\frac{\alpha}{c} n \log n}$, then we obtain

$$P(S_n \geq n\mu + \epsilon) \;\leq\; \frac{1}{n^{\alpha}}$$

- These are "one-direction" result of large and medium deviation theory.

# Subexponential r.v.'s

- A random variable $X$ is said to be sub-exponential if its tail satisfies, for some $k > 0$,

$$P(|X| \geq t) \leq 2e^{-kt}, \quad \forall t \geq 0.$$

- **Claim.** It can be argued that $X$ is sub-exponential if and only if the MGF of $|X|$ is finite near 0. Also, $X$ is sub-exponential if and only if $X^2$ is sub-Gaussian.

# Bernstein inequality

- Let $X_1, X_2, \ldots$ be a sequence of i.i.d. sub-exponential random variable. Then there exist constants $c_1, c_2, c_3 > 0$, and $K > 0$, such that for any $\epsilon > 0$,

$$
\begin{aligned}
P|S_n - n\mu| \geq \epsilon) &\leq 2e^{-c_3 \min\left\{\frac{c_1\epsilon^2}{n},\, c_2\epsilon\right\}}, \quad \forall \epsilon \geq 0, \\
PS_n \geq n\mu + \epsilon) &\leq e^{-c_3 \min\left\{\frac{c_1\epsilon^2}{n},\, c_2\epsilon\right\}}, \quad \forall \epsilon \geq 0.
\end{aligned}
$$

# Discussion

- Typically we are interested in the case where $\epsilon$ grows slower than $O(n)$. Then the first term prevail for large $n$.

# McDiarmid inequality

- Concentration inequality can be extended to general function of independent random variables, say $g(x_1, \ldots, x_n)$.

- **McDiarmid inequality.** Suppose $X_1, X_2, \ldots$ are independent random variables. If $g$ satisfies,

$$\sup_{x_1,\ldots,x_n,x_i'} |g(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) - g(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n)| \leq c_i.$$

Then, we have

$$P(g(X_1, \ldots, X_n) - E[g(X_1, \ldots, X_n)] > \epsilon) \leq e^{-\frac{2\epsilon^2}{\sum_{i=1}^n c_i^2}}.$$

# Remark

- McDiarmid inequality is also known as Bounded Difference inequality.

- Proof is based on martingale, a subject we will discuss later.

# Operations applications

- Revenue management
- Multi-armed bandit problem

# Notation

- Notation commonly used in analyzing algorithms are
  - $O(\cdot)$
  - $\Omega(\cdot)$
  - $\Theta(\cdot)$.

# Revenue management

- A flight departs in period (day) $T$. The demand in period $t \leq T$, denoted by $D_t(p_t)$, is random and price sensitive that depends on selling price $p_t$.

- Let $d_t(p_t) = E[D_t(p_t)]$, and it is a decreasing in $p_t$.

- There is a total of $N$ seats available. How to set price to maximize expected revenue?

# Remark

- Standard formulation in revenue management is that, we divide time into $T$ small time intervals, each is short enough so that, in each time interval, we have either one potential demand or no demand.

- Then, $D_t(p_t)$ becomes Bernoulli random variables with success probability $d_t(p_t)$.

- For illustration we will assume $d_t(\cdot) = d(\cdot)$, but all results hold when it depends on $t$ as well. This uses precisely the remark we made at the beginning of today's lecture (Hoeffding inequality for independent but non-i.i.d. r.v.'s.)

# Formulation

- The dynamic optimization problem is

$$\begin{aligned}
\max_{p_t} \qquad & E\Big[\sum_{t=1}^{T} p_t D_t(p_t)\Big] \\
s.t. \quad & \sum_{t=1}^{T} D_t(p_t) \le N, \\
& p_t \ge 0,\ t = 1, \ldots T.
\end{aligned}$$

# Discussion

- This dynamic optimization problem can be formulated and solve using DP, and computation of optimal policy can be complex.

- Earlier research has investigated the structure of the optimal policy, which is ...

- There are simple approximate solutions. The result is that, using fixed price policy, the total loss of revenue is in the order $O(\sqrt{T \log T})$ (Gallego and van Ryzin, 1994). Or put in differently, the average loss per period is $O(T^{-1/2}(\log T)^{1/2})$, which is small when $T$ is large.

# Fluid approximation

- Fluid model is

$$\begin{aligned} \max_{p_t} \quad & \sum_{t=1}^{T} p_t d_t(p_t) \\ s.t. \quad & \sum_{t=1}^{T} d_t(p_t) \le N, \\ & p_t \ge 0,\ t = 1, \ldots T. \end{aligned}$$

# Reformulation

- As there is a one-to-one correspondence between $p_t$ and $d(p_t)$, we change the decision variable to $d_t$. Let $p(d_t)$ be the pricing corresponding to $d_t$:

$$
\begin{aligned}
\max_{p_t} \quad & \sum_{t=1}^{T} d_t p(d_t) \\
s.t. \quad & \sum_{t=1}^{T} d_t \leq N, \\
& d_t \geq 0,\ t = 1, \ldots T.
\end{aligned}
$$

- It is typical to assume revenue function $d_t p(d_t)$ is concave in $d_t$.

- This is a convex optimization problem. What's its optimal solution?

# Lemma

- Let $d^* = \arg\max_d dp(d)$ and $p^* = d(p^*)$.

- **Claim 1.** Optimal solution to the fluid model is

$$d^*(N/T) := \min\{d^*, N/T\}.$$

- **Claim 2.** The optimal objective value of fluid model,

$$Tp^*(N/T)d^*(N/T),$$

is an upper bound for the original problem.

# How do we evaluate a proposed policy?

- The loss, or regret is defined as the difference between the value function of the said policy and that of the true optimal solution.

- The smaller the loss, the better the policy.

- However, we do not know what is the true optimal value function.

- We use upper bound of the optimal revenue.

# Theorem

- Let

$$p^*(N/T) := \max\{p^*, p(N/T)\}.$$

- Then, using the static policy $p^*(N/T)$ in each period has a total loss bounded by

$$L(T) = O(\sqrt{T \log T}).$$

- BTW, what is the order of the maximum total revenue?