Homework 6

Xuyuan Zhang, Uniqname: zxuyuan

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Problem 1. (*15 points*) Compute and prove a formula for the Hardy–Littlewood maximal function for the characteristic function $I_{[0,1]}$.

Solutions.
The Hardy-Littlewood maximal function is defined as:

$$h^*(x) = \sup_{t>0} \frac{1}{2t} \int_{x-t}^{x+t} |h|$$

For the characteristic function it is 1 for $x \in [0,1]$ then we can write it as:

$$h^*(x) = \sup_{t>0} \frac{1}{2t} \int_{x-t}^{x+t} \chi_{[0,1]} dt$$

If $x \geq 1$ then the largest average of f over all interval $(x-t, x+t)$ happens when the interval maximally overlap the $[0,1]$, which is when $t = x$. Then we have: $h^*(x) = \frac{1}{2x}$.
Similarly, we have for $x \leq 0$ the maximal achieves when $t = -x + 1$. Therefore, $h^*(x) = \frac{1}{2(1-x)}$
For the case where $x \in [0,1]$ then the maximal interval is exactly one because we can always select $t < x$ such that $(t-x, t+x) \subseteq [0,1]$ then the maximal function is 1.
therefore,

$$h^*(x) = \begin{cases} \frac{1}{2(1-x)} & \text{if } x \leq 0 \\ 1 & \text{if } x \in (0,1) \\ \frac{1}{2x} & \text{if } x \geq 1 \end{cases}$$

□

Problem 2. (*20 points*) Prove that if $h : \mathbb{R} \to [0, \infty)$ is an increasing function, then the Hardy-Littlewood maximal function h^* is an increasing function.

Solutions. We need to show that for $x < y$ we have $h^*(x) < h^*(y)$ then the result holds, where $h^*(x) = \sup_{t>0} \frac{1}{2t} \int_{x-t}^{x+t} |h|, h^*(y) = \sup_{t>0} \frac{1}{2t} \int_{y-t}^{y+t} |h|$

We can consider the shift of $y - x > 0$, i.e., define $g(a) = h(a + y - x)$ then we have: $h^*(y) = \sup_{t>0} \frac{1}{2t} \int_{y-t}^{y+t} |h| = \sup_{t>0} \frac{1}{2t} \int_{x-t}^{x+t} |g|$ and since $y > x$ then we have $g(a) > h(a)$ then we have $\frac{1}{2t} \int_{x-t}^{x+t} |g| > \frac{1}{2t} \int_{x-t}^{x+t} |h|$. Since taking supremum would not change the direction of the inequality, we have $\sup_{t>0} \frac{1}{2t} \int_{x-t}^{x+t} |g| > \sup_{t>0} \frac{1}{2t} \int_{x-t}^{x+t} |h| \implies h^*(y) > h^*(x)$ and this implies that $h^*(\cdot)$ is an increasing function.

$\square$

Problem 3. (*15 points*) Suppose $h : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable. Prove that

$$\{b \in \mathbb{R} : h^*(b) > c\}$$

is an open set of $\mathbb{R}$ for every $c \in \mathbb{R}$.

(*This implies that h^* is a measurable function.*)

Solutions. To prove that $\{b \in \mathbb{R} : h^*(b) > c\}$ is an open set of $\mathbb{R}$ for every $c \in \mathbb{R}$, we need to show that for every $b_0 \in \{b \in \mathbb{R} : h^*(b) > c\}$ there exists an $\varepsilon > 0$ such that the interval $(b_0 - \varepsilon, b_0 + \varepsilon)$ is entirely contained in $\{b \in \mathbb{R} : h^*(b) > c\}$

We first fix c, and from the definition of Hardy-Littlewood maximal function, we can find an interval $(b - r, b + r)$ such that with $\alpha > 1$ we have: $\frac{1}{2r} \int_{b-r}^{b+r} |h| > \alpha c$. Then consider a point $b_0 \in (b - \varepsilon, b + \varepsilon)$ and an interval $(b_0 - r - \varepsilon, b_0 + r + \varepsilon)$. Then we have $(b - r, b + r) \subseteq (b_0 - r - \varepsilon, b_0 + r + \varepsilon)$ we have:

$$\alpha c < \frac{1}{2r} \int_{b-r}^{b+r} |h| \leq \frac{1}{2r} \int_{b_0-r-\varepsilon}^{b_0+r+\varepsilon} |h| = \frac{2r + 2\varepsilon}{2r} \cdot \frac{1}{2r + 2\varepsilon} \int_{b_0-r-\varepsilon}^{b_0+r+\varepsilon} |h|$$

Since as $\varepsilon \to 0$ we have: $\frac{2r+2\varepsilon}{2r} \to 1$ then there exists an ε sufficiently small so that $b_0 \in (b - \varepsilon, b + \varepsilon)$ such that:

$$\alpha c < \frac{1}{2r + 2\varepsilon} \int_{b_0-r-\varepsilon}^{b_0+r+\varepsilon} |h|$$

and $c < h^*(b_0)$. Therefore, we can see that every point $b_0 \in \{b \in \mathbb{R} : h^*(b) > c\}$ can be covered by the open intervals and therefore it is an open set of $\mathbb{R}$ for every $c \in \mathbb{R}$. $\square$

Problem 4. (*20 points*) Let $|I|$ denote the length of a finite open interval I. Suppose $f \in L^1(\mathbb{R})$. Prove that

$$\lim_{t\to 0}\left(\sup_I \left\{\frac{1}{|I|}\int_I |f - f_I| : I \text{ is an interval of length } t \text{ containing b}\right\}\right) = 0$$

for almost every $b \in \mathbb{R}$. Here we use the notation f_I to denote the average of f over I:

$$f_I := \frac{1}{|I|}\int_I f.$$

Solutions.
We can decompose $\frac{1}{|I|}\int_I |f - f_I|$ as follows:

$$\begin{aligned}\frac{1}{|I|}\int_I |f - f_I| &\leq \frac{1}{|I|}\int_I |f - f(b)| + |f(b) - f_I| \text{ (triangle inequality)}\\ &= \frac{1}{|I|}\int_I |f - f(b)| + \frac{1}{|I|}\int_I |f(b) - f_I| = \frac{1}{|I|}\int_I |f - f(b)| + |f(b) - f_I|\end{aligned}$$

The last equality holds because the term $|f(b) - f_I|$ is constant so we can take it out. From the proposition of $L^1(\mathbb{R})$ function equals its local average almost everywhere, we have: for almost every $b \in \mathbb{R}$: $f(b) = \lim_{t\to 0}\frac{1}{|I|}\int_I f = \lim_{t\to 0} f_I$. Therefore, $\lim_{t\to 0}|f(b) - f_I| = |f(b) - f(b)| = 0$.
By Lebesgue Differentiation theorem, we have for almost every $b \in \mathbb{R}$: $\lim_{t\to 0}\frac{1}{|I|}\int_I |f - f(b)| = 0$ then for interval containing b with $|I| = t$, we have the first term equals to 0.
Simultaneously, we also have the absolute value is greater than 0, then combining the result, we have proved that for almost every $b \in \mathbb{R}$ the supremum over all intervals I containing b with $|I| = t$ as $t \to 0$, the average abslute difference between f and f_I is 0:

$$\lim_{t\to 0}(\sup_{|I|=t, b\in I}\frac{1}{|I|}\int |f - f_I|) = 0$$

$\square$