IOE 516: Assignment 2

Due: 11:59 PM, February 23rd, 2024

1. An agent is used for decision-making. Let's suppose that this agent makes decisions randomly and with probability $\frac{1}{2}+\varepsilon$ where $\varepsilon > 0$ makes the right decision. In order to further examine and improve the performance of this agent, it has to make decisions N times and take the majority vote. Show that, for any $\alpha \in (0,1)$, the decision would be correct with probability $1-\alpha$ as long as

$$N \geq \frac{\ln(\frac{1}{\alpha})}{2\varepsilon^2}$$

[**Hint:** Apply Hoeffding's inequality]

Solution: Assume that $X_1, X_2, ...X_n$ are independent random variables where $X_i \in [r_i, R_i],\ \forall i$. Then, for any $t > 0$, we would have:

$$P\{\sum_{i=1}^{n}(X_i - \mathbb{E}[X_i]) \geq t\} \leq \exp(-\frac{2t^2}{\sum_{i=1}^{n}(r_i - R_i)^2})$$

However, in our case $X_i \in [0,1]$ and $\mathbb{E}[X_i] = \frac{1}{2}-\varepsilon$. Therefore, the above inequality simplifies to:

$$P\{\sum_{i=1}^{n}(X_i - \frac{1}{2}+\varepsilon) \geq t\} \leq \exp(-\frac{2t^2}{n})$$

For the majority of answers to be incorrect, we need $\sum X_i > \frac{n}{2}$. Thus,

$$\sum_{i=1}^{n}(X_i - \frac{1}{2}+\varepsilon) \geq \frac{n}{2} - n(\frac{1}{2}-\varepsilon) = n\varepsilon$$

Plugging this back in, the bound would become $\exp(-2\varepsilon^2 n)$.

2. Suppose X_i's are i.i.d. with mean 0 and finite variance, and N is a stopping time with finite mean. Show that

$$Var(\sum_{i=1}^{N} X_i) = E[N] \cdot Var(X_1).$$

Solution: By condition, $E[X_1^2] < \infty$. Now

$$\begin{aligned}
Var(\sum_{i=1}^{N} X_i) &= E[(\sum_{i=1}^{N} X_i)^2] \\
&= E[(\sum_{i=1}^{\infty} X_i 1[i \leq N])^2] \\
&= E[\sum_{i=1}^{\infty} X_i^2 1[i \leq N] + 2\sum_{1\leq i<j<\infty} X_i X_j 1[i \leq N]1[j \leq N]] \\
&= E[\sum_{i=1}^{\infty} X_i^2 1[i \leq N]] + 2E[\sum_{1\leq i<j<\infty} X_i X_j 1[i \leq N]1[j \leq N]]
\end{aligned}$$

We argue that the expectation and summation can be interchanged. Thus,

$$\begin{aligned}
& E\Big[\sum_{1\le i<j<\infty} E[X_iX_j1[i\le N]1[j\le N]]\Big] \\
= & \sum_{1\le i<j<\infty} E[X_iX_j1[i\le N]1[j\le N]] \\
= & \sum_{1\le i<j<\infty} E[E[X_iX_j1[i\le N]1[j\le N]|F_{j-1}]] \\
= & \sum_{1\le i<j<\infty} E[X_i1[i\le N]1[j\le N]E[X_j|F_{j-1}]] \\
= & \ 0,
\end{aligned}$$

where the last equality follows from $E[X_j|F_{j-1}] = E[X_j] = 0$ because $X_1, X_2, \ldots$ are independent. On the other hand,

$$\begin{aligned}
& E[\sum_{i=1}^{\infty} X_i^2 1[i\le N]] \\
= & \sum_{i=1}^{\infty} E[X_i^2 1[i\le N]] \\
= & \sum_{i=1}^{\infty} E[E[X_i^2 1[i\le N]|F_{i-1}]] \\
= & \sum_{i=1}^{\infty} E[1[i\le N]E[X_i^2|F_{i-1}]] \\
= & E[\sum_{i=1}^{\infty} E[1[i\le N]E[X_1^2]]] \\
= & E[X_1^2]\sum_{i=1}^{\infty} E[1[i\le N]] \\
= & E[X_1^2]E[\sum_{i=1}^{\infty} 1[i\le N]] \\
= & E[N]E[X_1^2]
\end{aligned}$$

3. Let $\varepsilon_j,\ j = 1, 2, \ldots$ be i.i.d random variables with common distribution

$$P(\varepsilon_j = +1) = p, \quad P(\varepsilon_j = -1) = q = 1 - p$$

Denote $S_n = \sum_{j=1}^{n} \varepsilon_j,\ \ n \ge 0$. Prove that $M_n = (q/p)^{S_n}$ is a martingale.

Solution:

$$\begin{aligned}
\mathbb{E}[M_{n+1}|\mathcal{F}_n] &= \mathbb{E}[M_n(\frac{q}{p})^{\varepsilon_{n+1}}|\mathcal{F}_n] \\
&= M_n\mathbb{E}[(\frac{q}{p})^{\varepsilon_{n+1}}|\mathcal{F}_n] \\
&= M_n[p(\frac{q}{p}) + q(\frac{p}{q})] \\
&= M_n
\end{aligned}$$

4. Let X_n be a sequence of independent but not identically distributed r.v.'s. Let $\phi_i(\theta) = E[e^{\theta X_i}]$ be the MGF of X_i (assuming it exists). Show that

$$Y_n = \frac{1}{\prod_{i=1}^n \phi_i(\theta)} \cdot e^{\theta \sum_{i=1}^n X_i}$$

with $Y_0 = 1$ is a martingale process.

Solution: It is clear that

$$Y_{n+1} = Y_n \cdot \frac{e^{\theta X_{n+1}}}{\phi_{n+1}(\theta)}$$

Hence

$$\begin{aligned} E[Y_{n+1}|F_n] &= E\left[Y_n \cdot \frac{e^{\theta X_{n+1}}}{\phi_{n+1}(\theta)}\right] \\ &= Y_n \frac{1}{\phi_{n+1}(\theta)} E[e^{\theta X_{n+1}}] \\ &= Y_n \end{aligned}$$

5. For a martingale $\{X_n, n \geq 0\}$ with $X_0 = 0$, let $Z_n = X_n - X_{n-1}, n = 1, 2,$ be the martingale difference. Thus

$$X_n = \sum_{i=1}^n Z_i.$$

Show that

$$Var(X_n) = \sum_{i=1}^n Var(Z_i)$$

Solution: First, by definition of martingale, we have $E[Z_n] = 0$ and $E[Z_j|F_{j-1}] = 0$. Thus $Var(Z_i) = E[Z_i^2]$. Now

$$\begin{aligned} Var(\sum_{i=1}^n Z_i) &= E[(\sum_{i=1}^n Z_i)^2] \\ &= \sum_{i=1}^n E[Z_i^2] + 2\sum_{i<j} E[Z_i Z_j] \end{aligned}$$

Then the result follows from

$$E[Z_i Z_j] = E[E[Z_i Z_j|F_{j-1}]] = E[Z_i E[Z_j|F_{j-1}]] = 0.$$

6. An urn initially contains one white and one black ball. At each stage, a ball is drawn and is then replaced in the urn along with another ball of the same color. Let X_n denote the fraction of balls in the urn that are white after the n-th replication.

(i) Show that $\{X_n; n \geq 1\}$ is a martingale.

(ii) Show that the probability that the fraction of white balls in the urn is ever as large is 3/4 is at most 2/3.

Solution:

Part (a) Given the information in the question, after n stages, we would have n+2 balls in the urn. Then, considering x as the number of white balls in the urn, the X_n, the fraction of white balls in the urn would be $X_n = \frac{x}{n+2}$. Then we would have:

$$\begin{aligned}\mathbb{E}[X_{n+1}|\mathcal{F}_n] &= \frac{x}{n+2}\frac{x+1}{n+3} + \frac{n+2-x}{n+2}\frac{x}{n+3}\\ &= \frac{x}{n+2}\\ &= X_n\end{aligned}$$

Therefore, we have shown that X_n is a martingale.

Part (b) We can use Doob's maximal inequality to solve this section because $P(X_i \geq \frac{3}{4} \text{ for some } i) = \lim_{n\to\infty} P(\max_{1\leq i\leq n} X_i \geq \frac{3}{4})$. Therefore,

$$P(\max_{1\leq i\leq n} X_i \geq \frac{3}{4}) \leq \frac{\mathbb{E}[X_i]}{\frac{3}{4}}$$

$$\longrightarrow X_n \text{ is a martingale: } \mathbb{E}[X_n] = \mathbb{E}[X_0] = \frac{1}{2}$$

$$\Longrightarrow P(\max_{1\leq i\leq n} X_i \geq \frac{3}{4}) \leq \frac{2}{3}$$

7. Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, P)$ be a filtered probability space and $Y_n,\ n \geq 0$, a sequence of absolutely integrable random variables adapted to the filtration $(\mathcal{F}_n)_{n\geq 0}$. Assume that for some real numbers $u_n,\ v_n,\ n \geq 0$, it holds that

$$\mathbb{E}[Y_{n+1}|\mathcal{F}_n] = u_n Y_n + v_n$$

Find two real sequences a_n and $b_n,\ n \geq 0$, so that the sequence of random variables $M_n := a_n Y_n + b_n,\ n > 1$, be martingale with respect to the same filtration.

Solution:

$$\begin{aligned}\mathbb{E}[M_{n+1}|\mathcal{F}_n] &= \mathbb{E}[a_{n+1}Y_{n+1} + b_{n+1}|\mathcal{F}_n]\\ &= a_{n+1}u_n Y_n + a_{n+1}v_n + b_{n+1}\end{aligned}$$

Through this, we get the recursion:

$$a_{n+1} = a_n u_n^{-1},\ \ b_{n+1} = b_n - a_{n+1}v_n$$

And therefore,

$$\begin{aligned}a_0 = 1, &\quad a_n = (\prod_{i=1}^{n-1} u_i)^{-1}\\ b_0 = 0, &\quad b_n = -\sum_{i=1}^{n} a_n v_{n-1}\end{aligned}$$

8. Assume $Y_i,\ i = 1, 2, ...$ be non-negative i.i.d. random variables where $\mathbb{E}[Y_n] = 1$ and $P(Y_n = 1) < 1$.

(a) Show that $X_n = \prod_{i=1}^{n} Y_i$ is a martingale.

(b) Using Strong Law of Large Numbers, show that

$$\frac{\log(X_n)}{n} \to c < 0, \quad a.s$$

Solution:
Part (a)

$$\begin{aligned}
\mathbb{E}[X_{n+1}|X_1, X_2, ..., X_n] &= \mathbb{E}[\prod_{i=1}^{n+1} Y_i|X_1, X_2, ..., X_n] \\
&= \mathbb{E}[Y_{n+1}X_n|X_1, X_2, ..., X_n] \\
&= \mathbb{E}[X_n|X_1, X_2, ..., X_n]\mathbb{E}[Y_{n+1}|X_1, X_2, ..., X_n] \\
(\mathbb{E}[Y_k] = 1 \implies) &= \mathbb{E}[X_n|X_1, X_2, ..., X_n] \\
&= X_n
\end{aligned}$$

Part (b) Based on SLLN:

$$\frac{\log(X_n)}{n} = \frac{1}{n}\sum_{i=1}^{n} \log(Y_k) \to \mathbb{E}[\log(Y_k)]$$

Now since $\log$ is a concave function, using Jensen's inequality and the fact that $\mathbb{E}[Y_k] = 1$, we can show that:

$$\begin{aligned}
&\mathbb{E}[\log Y_k] \leq \log(\mathbb{E}[Y_k]) \\
\longrightarrow\ &\mathbb{E}[\log Y_k] \leq 0 \\
\implies &\frac{\log(X_n)}{n} \to c < 0 \ \ a.s
\end{aligned}$$

9. Let $X_i,\ i = 0, 1, 2, ...$ be a sequence of random variables where $\mathbb{E}[|X_i|] < \infty$. Moreover, we have $\mathbb{E}[X_{n+1}|X_0, X_1, ..., X_n] = aX_n + bX_{n-1}$, for $n \geq 1,\ 0 < a, b < 1,$ and $a + b = 1$. Find the value of λ such that $S_n = \lambda X_n + X_{n-1},\ n \geq 1$ would be a martingale.

Solution: Certainly $\mathbb{E}[|S_n|] < \infty, \forall n$. Moreover, for $n \geq 1$, we have:

$$\begin{aligned}
\mathbb{E}[S_{n+1}|X_0, X_1, ..., X_n] &= \lambda\mathbb{E}[X_{n+1}|X_0, X_1, ..., X_n] + X_n \\
&= (\lambda a + 1)X_n + \lambda b X_{n-1} = S_n
\end{aligned}$$

$$S_n = \lambda X_n + X_{n-1} \implies \lambda = (1-a)^-1$$

Therefore, S_n would be a martingale if $\lambda = (1-a)^-1$.

10. Let X_n be the net profit of a gambler when betting a unit stake on the n$^{\text{th}}$ play in the casino. X_n can be dependent; however, $\mathbb{E}[X_{n+1}|X_1, X_2, ..., X_n] = 0,\ \forall n$ holds, meaning that the game is fair. In the first game, the gambler stakes Z and after that stakes $f_n(X_1, X_2, ..., X_n)$ on the (n+1)$^{\text{th}}$ play. Assuming that Z and these functions, f_i, are given, first **(a)** show that the gamblers profit after n plays would be

$$S_n = \sum_{i=1}^{n} X_i f_{i-1}(X_1, X_2, ..., X_{i-1})$$

and then **(b)** prove that the sequence S_i is a martingale, or in other words $\mathbb{E}[S_{n+1}|x_1, X_2, ..., X_n] = S_n, \;\; n \geq 1$.

Solution: The gambler stakes $Z_i = f_{i-1}(X_1, ..X_{i-1})$ on the i$^{\text{th}}$ play, at a return of X_i per unit. Consequently, $S_i = S_{i-1} + X_i Z_i,$ for $i \geq 2$ where $S_1 = X_1 Y$. Next, we can show that:

$$\begin{aligned}\mathbb{E}[S_{n+1} - S_n|X_1, ..., X_n] &= Z_{n+1}\mathbb{E}[X_{n+1}|X_1, ...X_n] \\ &= 0\end{aligned}$$

In which we have used the fact that Z_{n+1} depends only on $X_1, X_2, .., X_n$.