UNIVERSITY OF MICHIGAN - MATH 551

Mid Term Review

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1 Definition Problem

Definition 1 (outer-measure). *The outer measure $\mu^*(A)$ of a set $A \subseteq \mathbb{R}$ is definede by:*

$$\mu^*(A) = \inf\{\sum_{k=1}^{\infty} \ell(I_k), I_1, I_2, \dots \text{ are open intervals such that } A \subseteq \bigcup_{k=1}^{\infty} I_k\}.$$

Definition 2 (σ-algebra). *Suppose X is a set and S is a set of subsets of X. Then S is called a σ-algebra on X if the following three conditions are satisfied:*

- $\emptyset \in S$;
- *if* $E \in S$, *then* $X \backslash E \in S$;
- *if* $E_1, E_2, \dots$ *is a sequence of elements of* S, *then* $\bigcup_{k=1}^{\infty} E_k \in S$.

Definition 3 (Borel σ-algebra). *The smallest σ-algebra on $\mathbb{R}$ containing all open subsets of $\mathbb{R}$ is called the collection of Borel subsets of $\mathbb{R}$ (Borel σ-algebra). An element of this σ-algebra is called a Borel set.*

Definition 4 (measurable function: Alternative 1). *Suppose (X, S) is a measurable space. A function $f : X \to [-\infty, \infty]$ is called S-measurable if:*

$$f^{-1}(B) \in S$$

for every Borel set $B \subseteq [-\infty, \infty]$.

Definition 5 (Condition for measurable function: Alternative 2). *Suppose (X, S) is a measurable function, and $f : X \to \mathbb{R}$ is a function such that:*

$$f^{-1}((a, \infty)) \in S)$$

for all $a \in \mathbb{R}$. Then f is an S-measurable function.

Definition 6 (measurable function: Alternative 3). *suppose (X, S) is a measurable space. A function $f : X \to [-\infty, \infty]$ is called S-measurable if:*

$$f^{-1}(B) \in S$$

for every Borel set of $B \subseteq [-\infty, \infty]$.

Definition 7 (measure). *Suppose X is a set and S is σ-algebra on X. A measure on (X, S) is a function $\mu : S \to [0, \infty]$ such that $\mu(\emptyset) = 0$ and:*

$$\mu(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu(E_k)$$

for every disjoint sequence $E_1, E_2, \dots$ of sets in S.

Definition 8 (Lebesgue measurable set). *A set $A \subset \mathbb{R}$ is called Lebesgue measurable if:*

- *For each $\varepsilon > 0$, there exists a closed set $F \subseteq A$ such that $|A \backslash F| < \varepsilon$.*
- *There exists closed set $F_1, F_2, \ldots$ contained in A such that $|A \backslash \bigcup_{k=1}^{\infty} F_k| = 0$.*
- *There exists a Borel set $B \subseteq A$ such that $|A \backslash B| = 0$.*
- *For each $\varepsilon > 0$, there exists an open set $G \supseteq A$ such that $|G \backslash A| < \varepsilon$.*
- *There exists open sets $G_1, G_2, \ldots$ containing A such that $|(\bigcap_{k=1}^{\infty} G_k) \backslash A| = 0$.*
- *There exists a Borel set $B \supseteq A$ such that $|B \backslash A| = 0$.*

Theorem 1 (Egorov's Theorem)**.** *Suppose (X, S, μ) is a measure space with $\mu(X) < \infty$. Suppose $f_1, f_2, \ldots$ is a sequence of S-measurable functions from X to $\mathbb{R}$ that converges pointwise on X to a function $f : X \to \mathbb{R}$. Then for every $\varepsilon > 0$, there exists a set $E \in S$ such that $\mu(X \backslash E) < \varepsilon$ and $f_1, f_2, \ldots$ converges uniformly to f on E.*

Theorem 2 (Luzin's theorem)**.** *Suppose $g : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function. Then for every $\varepsilon > 0$, there exists a closed set $F \subseteq \mathbb{R}$ such that $|\mathbb{R} \backslash F| < \varepsilon$ and $g|_F$ is a continuous function on F.*

Theorem 3 (Monotone Convergence Theorem)**.** *Suppose (X, S, μ) is a measure space and $0 \leq f_1 \leq f_2 \leq \ldots$ is an increasing sequence of S-measureble functions. Define $f : X \to [0, \infty]$ by:*

$$f(x) = \lim_{k \to \infty} f_k(x)$$

Then:

$$\lim_{k \to \infty} \int f_k d\mu = \int f d\mu.$$

2 Problem-solving questions

> Give an example of a measurable space (X, S) and a function $f : X \to \mathbb{R}$ such that $|f|$ is S-measurable but f is not S-measurable.

Solutions. A function $f : X \to \mathbb{R}$ is said to be S-measurable if, for every Borel set $B \subseteq \mathbb{R}$, the preimage $f^{-1}(B) = x \in X : f(x) \in B$ belongs to the σ-algebra S over X.
Let $X = \mathbb{R}$ and $S = \{\emptyset, \mathbb{R}, A, A^c\}$ with $A \subset \mathbb{R}$ such that A is not a Borel set
Define the function $f : X \to \mathbb{R}$ by:

$$f = \begin{cases} 1 & \text{if } x \in A \\ -1 & \text{if } x \notin A. \end{cases}$$

Therefore, by our definition the pre-image of 1 is exactly A not in S; however, the function $|f|$ is the constant function, and all the sets are contained in S, and then the pre-image of $|f|$ is either $\emptyset, A \cup A^c = X$ all of which are in X. □

> Suppose X is a Borel subset of $\mathbb{R}$ and $f : X \to \mathbb{R}$ is a function such that $\{x \in X : f \text{ is not continuous at } x\}$ is a countable set. Prove that f is a Borel measurable function.

Solutions. Denote the countable set as $\mathcal{B}$ and we can enumerate the elements as: $(b_1, b_2, \ldots)$. Consider the function f restricted to $X \setminus \mathcal{B}$ is continuous then from the continuity $\implies$ measurability, we have f is Borel measurable at $X \setminus \mathcal{B}$. Now we prove that despite f is not continuous at set $\mathcal{B}$, f is still measurable over the entire set X.
Since we already know that the pre-image of U under the restriction of f to $X \setminus \mathcal{B}$, denoted as $f^{-1}(U) \cap (X \setminus \mathcal{B})$ is open (hence Borel) because f restricted to $X \setminus \mathcal{B}$ is continuous.
The preimage of U that intersects with $\mathcal{B}$ denoted as $f^{-1}(U) \cap \mathcal{B}$ is countable since the set is a subset of a countable set $\mathcal{B}$. We already know that countable set is Borel set then we can take the union so that we have shown that the pre image of U is Borel set and we have concluded the proof. □

Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable at every element of $\mathbb{R}$. Prove that f' is a Borel measurable function from $\mathbb{R} \to \mathbb{R}$.

Solutions. Given that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at every element of $\mathbb{R}$, the derivative f' exists at every $x \in \mathbb{R}$. For each $n \in \mathbb{N}$ define the function $g_n : \mathbb{R} \to \mathbb{R}$ as:

$$g_n(x) = n(f(x + \frac{1}{n}) - f(x))$$

Each g_n is measurable because it is a composition and linear combination of measurable functions (differentiable $\implies$ continuous $\implies$ measurable). The $f'(x)$ at point x can be understood as the limit of difference quotients as $n \to \infty$ then:

$$f'(x) = \lim_{n\to\infty} g_n(x)$$

Therefore, $f'(x)$ is the pointwise limit of the sequence of measurable function $\{g_n(x)\}$. Therefore, since each g_n is measurable, the limit f' is also measurable. [For open intervals B, which form a basis for the Borel σ-algebra, we can use the property of pointwise convergence to show that $f^{-1}(B)$ is the countable union or intersection of $g_n^{-1}(B)$] $\square$

Prove that if $A \subseteq \mathbb{R}$ is Lebesgue measurable, then there exists an increasing sequence $F_1 \subseteq F_2 \subseteq \dots$ of closed sets contained in A such that:

$$|A \backslash \bigcup_{k=1}^{\infty} F_k| = 0.$$

Solutions.
Given A is Lebesgue measurable, for any $\epsilon > 0$, we can find an open set O containing A such that $|O \backslash A| < \epsilon$. Open sets in $\mathbb{R}$ can be expressed as countable unions of disjoint open intervals, i.e., $O = \bigcup_{i=1}^{\infty}(a_i, b_i)$.
Since each open interval (a_i, b_i) contains a closed interval $[a_i + \frac{\epsilon}{2^{i+1}}, b_i - \frac{\epsilon}{2^{i+1}}]$ (assuming $b_i - a_i > \frac{\epsilon}{2^i}$ to make this meaningful), we can construct a sequence of closed sets F_k within A by selecting closed intervals within each (a_i, b_i) such that these closed sets approach the open set O as closely as desired, by adjusting the ϵ parameter.
Define F_k as the union of closed intervals $[a_i + \frac{\epsilon}{2^{i+1}}, b_i - \frac{\epsilon}{2^{i+1}}]$ for i up to k. This sequence $F_1 \subseteq F_2 \subseteq \dots$ is increasing because each F_k includes all intervals of the previous F_{k-1} and possibly more, getting closer to filling the entire open set O as k increases.
As $k \to \infty$, the union $\bigcup_{k=1}^{\infty} F_k$ approaches A in measure. Since $|O \backslash A| < \epsilon$ for arbitrarily small ϵ, and our construction of F_k's approaches O, it follows that the measure of $A \backslash \bigcup_{k=1}^{\infty} F_k$ is less than any $\epsilon > 0$. Thus, by the definition of Lebesgue measure, $|A \backslash \bigcup_{k=1}^{\infty} F_k| = 0$.

$\square$

Prove that if $A \subseteq \mathbb{R}$ is Lebesgue measurable, then there exists a decreasing sequence $G_1 \supseteq G_2 \supseteq \dots$ of open sets containing A such that:

$$|(\bigcap_{k=1}^{\infty} G_k) \backslash A| = 0$$

Solutions.
Since A is Lebesgue measurable, for every $\epsilon > 0$, there exists an open set O containing A such that the measure of the set $O \backslash A$ is less than ϵ.
Specifically, for each $n \in \mathbb{N}$, choose $\epsilon = \frac{1}{n}$ and find an open set G_n containing A such that $|G_n \backslash A| < \frac{1}{n}$.
Each G_n is chosen to be an open set containing A with increasingly stringent conditions on the measure of the excess part $(G_n \backslash A)$, ensuring that $G_{n+1} \subseteq G_n$. This is because, as $\frac{1}{n+1} < \frac{1}{n}$, the condition for G_{n+1} to contain A with a smaller excess part naturally leads to G_{n+1} being "tighter" around A than G_n.
Consider the intersection of all sets in the sequence, $\bigcap_{k=1}^{\infty} G_k$. This intersection contains A, and every point not in A but in any G_k is in a set whose measure decreases to zero as $k \to \infty$. Therefore, the measure of $(\bigcap_{k=1}^{\infty} G_k) \backslash A$ is the limit of the measures of the excess parts, which is less than $\frac{1}{n}$ for all n, and thus goes to 0 as $n \to \infty$.

$\square$

Suppose μ is the measure on $(\mathbb{Z}^+, 2^{\mathbb{Z}^+})$ defined by:

$$\mu(E) = \sum_{n \in E} \frac{1}{2^n}$$

Prove that for every $\varepsilon > 0$ there exists a set $E \subseteq \mathbb{Z}^+$ with $\mu(\mathbb{Z}^+ \backslash E) < \varepsilon$ such that $f_1, f_2, \ldots$ converges uniformly on E for every sequence of functions $f_1, f_2, \ldots$ from $\mathbb{Z}^+$ to $\mathbb{R}$ that converges pointwise on $\mathbb{Z}^+$ [This result does not follow from Egorov's Theorem because here we are asking for E to depend only on ε. In Egorov's Theorem, E depends on ε and on the sequence $f_1, f_2, \ldots$]

Solutions. We note that as n increases, the $\frac{1}{2^n}$ decreases geometrically. Given any $\varepsilon > 0$, we can choose N where $N \in \mathbb{Z}^+$ large enough such that $\sum_{i=N+1}^{\infty} \frac{1}{2^n} = \frac{1}{2^N} < \varepsilon$. This is possible because the LHS can be arbitrarily small. Let $E = 1, 2, \ldots, N$ Thus $\mathbb{Z}^+ \backslash = \{N+1, N+2, \ldots\}$ and we have $\mu(\mathbb{Z}^+ \backslash E) = \sum_{n=N+1}^{\infty} \frac{1}{2^n} < \varepsilon$.
Given any sequence of functions $f_1, f_2, \ldots$ that converges pointwise on $\mathbb{Z}^+$ we need to show that it converges uniformly on E. Since E is a finite set (consisting of the first N positive integers), the pointwise convergence of the sequence on $\mathbb{Z}^+$ implies that it will also converge uniformly on E. This is because on a finite set, the maximum difference between the functions in the sequence and their pointwise limit can be made as small as desired by choosing a sufficiently large index in the sequence. This follows from the definition of pointwise convergence and the fact that a finite set has a bounded number of elements over which the maximum is taken.
We can prove it mathematically: pointwise convergence $\Longrightarrow$ uniform convergence on finite set: if $D = \{x_1, x_2, \ldots, x_n\}$ then for $\varepsilon > 0$, $\exists N_1, N_2, \ldots, N_k \in \mathbb{N}$ such that:

$$|f_n(x_i) - f(x_i)| < \varepsilon$$

So just by take $N = \max\{N_1, N_2, \ldots, N_k\}$ and we have:

$$\sup_{x \in D} |f_n(x) - f(x)| < \varepsilon, \forall n \geq N$$

Therefore, $f_n(x) = f(x), \forall x \in D$. $\square$

Suppose (X, S, μ) is a measure space and $f : X \to [0, \infty]$ is an S-measurable function such that $\int f d\mu < \infty$. Explain why:

$$\inf_E f = 0$$

for each set $E \in S$ with $\mu(E) = \infty$

Solutions. Given that $f : X \to [0, \infty]$ is an S-measurable function with $\int f d\mu < \infty$.
We already know that $\int_E f d\mu = \int \chi_E f d\mu$ Since we already know that $\chi_E f \leq f$ and $\chi_E f \geq 0$ then we have: $\int_E f d\mu \leq \int f d\mu < \infty$. Assuming $\int_E f > 0$: If the infimum of f over E were positive, say $\int_E f = \delta > 0$ then for every $x \in E, f(x) \geq \delta$. This would imply that the integral of f over E would be at least $\delta\mu(E)$ which is $\delta \cdot \infty = \infty$, which contradicts the fact that $\int f d\mu < \infty$. Therefore, the infimum of f over E must be 0. $\square$

Suppose (X, S, μ) is a measure space and $f : X \to [0, \infty]$ is an S-measurable function Prove that:

$$\int f d\mu > 0 \text{ if and only if } \mu(\{x \in X : f(x) > 0\}) > 0.$$

Solutions. We first prove the forward direction: $\int f d\mu > 0 \implies \mu(\{x \in X : f(x) > 0\}) > 0$.
If $\mu(\{x \in X : f(x) > 0\}) = 0$, then this would result $f(x) = 0$ almost everywhere on X with respect to the measure μ, because the set of points where $f(x) > 0$ would have measure zero. Consequently, $\int f d\mu = 0$ because f would be zero almost everywhere, contradiction.
Now we prove the reverse direction: $\mu(\{x \in X : f(x) > 0\}) > 0 \implies \int f d\mu > 0$.
Consider the set $\mathcal{E} = \{x \in X : f(x) > 0\}$. Since f is S-measurable and $\mathcal{E}$ is measurable, and $\mu(\mathcal{E}) > 0$. On this set we have $f(x) > 0$ meaning that $\int_{\mathcal{E}} f d\mu > 0$. Simiarly, since $f \geq 0$ we have $\int_E f d\mu = \int_{\mathcal{E}} f d\mu + \int_{X \backslash \mathcal{E}} f d\mu \geq \int_{\mathcal{E}} f d\mu > 0$. Therefore, $\int f d\mu > 0$. $\square$

Given an example to show that the Monotone Convergence Theorem can fail if the hypothesis of an increasing sequence of functions is replaced by a hypothesis of a decreasing sequence of functions. [This exercise shows that the Monotone Convergence Theorem should be called the increasing Convergence Theorem]

Solutions. We can think about the function:

$$f(x) = \frac{x}{n}, \text{ for } x \in [0, \infty]$$

It is easy to know that $f(x)$ is a decreasing function. We know that $f(x) = \lim_{n\to\infty} f_n(x) = 0$ for all $x \in [0, \infty]$ (because for fixed x the $\frac{x}{n}$ would tend to 0.). However, we have:

$$\int f_n(x)dx = \int \frac{x}{n}dx = \frac{x^2}{2n}\bigg|_0^\infty = \infty$$

Therefore, this counter-example shows that the Monotone Convergence Theorem can fail if the hypothesis of an increasing sequence of functions is replaced by a hypothesis of a decreasing sequence of functions. □

Give an example of a sequence $x_1, x_2, \ldots$ of real numbers such that:

$$\lim_{n\to\infty} \sum_{k=1}^{n} x_k \text{ exists in } \mathbb{R}$$

but $\int x d\mu$ is not defined, where μ is counting measure on $\mathbb{Z}^+$ and x is the function from $\mathbb{Z}^+$ to $\mathbb{R}$ defined by $x(k) = x_k$.

Solutions. With the counting measure, the integral of a function $x(k)$ over $\mathbb{Z}^+$ is equivalent to summing up the values of $x(k)$ over all $k \in \mathbb{Z}^+$, which is $\int x d\mu = \sum_{k=1}^{\infty} |x_k|$.
We can think of $x_k = (-1)^k/k$. Then we have: $\sum_{k=1}^{\infty} x_k = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$. The sum converges to $\ln(2)$, which is a real number. Therefore, the limit of the sequence exists in $\mathbb{R}$.
However, we also have that the sum $\sum_{k=1}^{\infty} \frac{1}{k}$ does not converge because the sequence diverges. Therefore, we note that the integral $\int x d\mu$ is not defined. □

UNIVERSITY OF MICHIGAN - MATH 551

Homework 1

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Problem 1. *(25 points)* The goal of this problem is to prove the following theorem regarding convergence of Fourier series.

Theorem. Let $\{a_n\}$ be a decreasing sequence of real numbers that converges to 0. Then the (Fourier) series $\sum_{n=1}^{\infty} a_n \cos nx$ converges pointwise for all $x \in \mathbb{R}$ such that $\cos x \neq 1$.

a) *(10 points)* Let $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$, let $b_1, \ldots, b_n \in \mathbb{R}$, and let $S_k = b_1 + \cdots + b_k$ for each k. Prove that
$$|a_1 b_1 + \cdots + a_n b_n| \leq a_1 \max(|S_1|, \ldots, |S_n|)$$

b) *(5 points)* Let $\{a_n\}$ be a decreasing sequence of real numbers converging to 0, and let $\{b_n\}$ be a sequence of real numbers for which $\sup_{n\in\mathbb{N}} |b_1 + \cdots + b_n| < \infty$. Prove that the series $\sum_{n=1}^{\infty} a_n b_n$ converges.

c) *(5 points)* Prove that
$$\sum_{n=1}^{N} \cos nx = \frac{\cos(N+1)x - \cos Nx - \cos x + 1}{2(\cos x - 1)}$$
for all $N \in \mathbb{N}$ and all $x \in \mathbb{R}$ for which $\cos x \neq 1$.

d) *(5 points)* Prove the *Theorem.*

Solutions. (a):
We need to prove that: $|a_1 b_1 + \cdots + a_n b_n| \leq a_1 \max(|S_1|, \ldots, |S_n|)$ where $S_k = b_1 + \cdots + b_k$. Note that we have $\sum_{i=1}^{n} a_i b_i = a_1 S_1 + a_2(S_2 - S_1) + \cdots + a_n(S_n - S_{n-1}) = (a_1 - a_2) \cdot S_1 + (a_2 - a_3) \cdot S_2 + \cdots + (a_{n-1} - a_n) \cdot S_{n-1} + a_n S_n$. Then we have:

$$\begin{aligned}
|\sum_{i=1}^{n} a_i b_i| &\leq |(a_1 - a_2)S_1| + |(a_2 - a_3)S_2| + \cdots + |(a_{n-1} - a_n)S_{n-1}| + |a_n S_n| \\
&= (a_1 - a_2)|S_1| + (a_2 - a_3)|S_2| + \cdots + (a_{n-1} - a_n)|S_{n-1}| + |a_n S_n| \\
&\leq (a_1 - a_2 + a_2 - a_3 + \cdots + a_{n-1} - a_n + a_n) \max\{|S_1|, \ldots, |S_n|\} \\
&= a_1 \max\{|S_1|, \ldots, |S_n|\}
\end{aligned}$$

where the first follows from triangle inequality and the second follows from the property $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$.
(b):
We already know that the sum of the sequence $\{b_n\}$, denoted as $\{S_n\}$, is bounded in the sense that $\sup_{n\in\mathbb{N}} |b_1 + \cdots + b_n| = \sup_{n\in\mathbb{N}} |S_n| < \infty$, and we can denote the upper bound as M, i.e., $\forall n \in \mathbb{N}$, we have: $|S_n| \leq M$.
To prove the sequence $\sum_{n=1}^{\infty} a_n b_n$, $\forall \varepsilon > 0$, we need to find an N such that for all $m > n \geq N$, the inequality $|\sum_{k=n+1}^{m} a_k b_k| < \varepsilon$.
Consider the partial sum from $n+1$ to m as: $|\sum_{k=n+1}^{m} a_k b_k| = \sum_{k=n+1}^{m} a_k |b_k| \leq a_{n+1} \sum_{k=n+1}^{m} |b_k|$, for $\{a_n\}$ is a decreasing sequence. Using the bounding property that $\sum_{k=n+1}^{m} |b_k| \leq |b_1 + \cdots + b_m| + |b_1 + \cdots + b_n| \leq 2M$ (triangle inequality). Therefore, we have:

$$|\sum_{k=n+1}^{m} a_k b_k| \leq a_{n+1} \cdot 2M$$

since $\{a_n\}$ converges to 0, there exists N such that $a_{n+1} < \frac{\varepsilon}{2M}$ for all $n \geq N$. This implies: $|\sum_{k=n+1}^{m} a_k b_k| < \varepsilon$ for all $m > n \geq N$. Therefore, it is convergent.
(c):

$$\begin{aligned}
\sum_{n=1}^{N} \cos(nx) &= \Re(\sum_{i=1}^{N} e^{inx}) = \Re(e^{ix}\frac{1-e^{iNx}}{1-e^{ix}}) \\
&= \Re((\cos(x)+i\sin(x))\frac{1-\cos(Nx)-i\sin(Nx)}{1-\cos(x)-i\sin(x)}) \\
&= \Re((\cos(x)+i\sin(x))\cdot(1-\cos(x)+i\sin(x))\frac{1-\cos(Nx)-i\sin(Nx)}{2-2\cos(x)}) \\
&= [(1-\cos(Nx))(1-\cos(x))\cos(x)+\sin(x)\sin(Nx)\cos(x)+ \\
&\sin(x)(1-\cos(x))\sin(Nx)-\sin(x)\sin(x)(1-\cos(Nx))]/(2-2\cos(x) \\
&= [\cos(x)(1-\cos(Nx)-\cos(x)+\cos((N-1)x))+\sin(x)[\sin(Nx)- \\
&\sin(x)-\sin((N-1)x)]]/(2-2\cos(x)) \\
&= \frac{\cos(x)-1-\cos((N+1)x)+\cos(Nx)}{2-2\cos(x)} \\
&= \frac{\cos((N+1)x)-\cos(Nx)-\cos(x)+1}{2(\cos(x)-1)}
\end{aligned}$$

(d):
Consider the sequence $\{b_n\}$ as $\{\cos(nx)\}$ and denote $\{S_n\}$ as the summation of $\sum_{k=1}^{n} b_k$. Therefore, from part (c) we have the partial sums $\{S_n\}$ is a bounded sequence. $\sum_{n=1}^{N} \cos nx = \frac{\cos(N+1)x-\cos Nx-\cos x+1}{2(\cos x-1)}$ and we know that these sums are bounded for all x such that $\cos(x) \neq 1$. Besides, we know that the sequence $\{a_n\}$ is given to be decreasing and converging to 0. Thus, it satiesfies the condition for Part (b). Therefore, the fourier series $\sum_{n=1}^{\infty} a_n \cos(nx)$ converges pointwise for all $x \in \mathbb{R}$.

□

Problem 2. *(10 points)* Let $f : [0, \pi] \to \mathbb{R}$ be a continuous function satisfying $f(0) = f(\pi) = 0$, define real numbers $a_1, a_2 \ldots, a_n$ by the formula

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin(nx)f(x)dx.$$

(*note that since f is continuous, the above integral is well-defined in any kind of integration*)
Show that the infinite sum $\sum_{n>0} a_n^2$ converges.

(*Hint*: *consider the expression* $\int_0^{\pi}\left(f(x) - \sum_{k=1}^{n} a_k \sin(kx)\right)^2 dx$.)

Solutions.
We first consider the fact of the orthogonality of the sine function on $[0, \pi]$. The integral $\sin(kx)\sin(mx)$ over $[0, \pi]$ is 0 for $k \neq m$ and $\pi/2$ if $k = m$. This is because:

$$\begin{aligned}
\int_0^{\pi} \sin(kx)\sin(mx) &= \frac{1}{2}\int_0^{\pi}[\cos((k-m)x) - \cos((k+m)x)]dx \\
&= \begin{cases} \frac{1}{2}\left[\frac{\sin((k-m)x)}{k-m} - \frac{\sin((k+m)x)}{k+m}\right]_0^{\pi} = 0 & \text{if } k \neq m \\ \frac{1}{2}\int_0^{\pi} 1dx - \frac{1}{2}\int_0^{\pi}\cos(2kx)dx = \frac{\pi}{2} & \text{if } k = m. \end{cases}
\end{aligned}$$

Therefore, we can write the expression of $\int_0^{\pi}(f(x) - \sum_{k=1}^{n} a_k \sin(kx))^2 dx$ as:

$$\int_0^\pi (f(x) - \sum_{k=1}^n a_k \sin(kx))^2 dx = \int_0^\pi f(x)^2 dx - 2\int_0^\pi f(x) \sum_{k=1}^n a_k \sin(kx) dx + \int_0^\pi (\sum_{k=1}^n a_k \sin(kx))^2 dx.$$
$$= \int_0^\pi f(x)^2 dx - \pi \sum_{k=1}^n a_k^2 + \frac{\pi}{2} \sum_{k=1}^n a_k^2$$
$$= \int_0^\pi f(x)^2 dx - \frac{\pi}{2} \sum_{k=1}^n a_k^2.$$

We know that the left hand side should be non-negative, so we have: $\frac{\pi}{2}\sum_{k=1}^n a_k^2 \leq \int_0^\pi f(x)^2 dx = C$, where C is a fixed value. Thus we have: $\sum_{k=1}^n a_k^2 \leq \frac{2C}{\pi}$. Therefore, it is bouneded above. Furthermore, we have each a_k^2 is non-negative, so the sequence $\{a_n^2\}$ is non-decreasing. Therefore, the series $\sum_{n>0} a_n^2$ is convergent. □

Problem 3. (*10 points - Axler's 1B, 1*) Define $f : [0,1] \to \mathbb{R}$ as follows:

$$f(a) = \begin{cases} 0 & \text{if } a \text{ is irrational} \\ 1/n & \text{if } a \text{ is rational and } n \text{ is the smallest positive} \\ & \text{integer such that } a = m/n \text{ for some integer } m. \end{cases}$$

Show that f is Riemann integrable and compute $\int_0^1 f$.

Solutions. We can first calculate the lower Riemann Sum as: $L(f,P) = \sum_{i=1}^n (x_i - x_{i-1}) \cdot \inf\{f(x) : x \in [x_{i-1}, x_i]\}$. From the density of Real Number, we can always find at least one irrational number in any subinterval $([x_{i-1}, x_i]) \subseteq ([0,1])$ and for any irrational number $a \in [0,1]$ the function is equal to $f(a) = 0$. Given this, there is at least one irrational number in subinterval $[x_{i-1}, x_i]$ and that the function $f(\cdot) = 0$ at every irrational number, then the infimum of $f(\cdot)$ over any subinterval is 0, which implies the Lower Riemann Sum is equal to 0.
Now the upper Riemann Sum for the partition P of $[0,1]$ is defined as: $U(f,P) = \sum_{i=1}^n (x_i - x_{i-1}) \cdot \sup\{f(x) : x \in [x_{i-1}, x_i]\}$. For any rational number $\frac{m}{n}$, $f(\frac{m}{n}) = \frac{1}{n}$. Besides, Rational numbers are dense in the real numbers, which means between any two real numbers, no matter how close, there are infinitely many rational numbers. For any $\varepsilon > 0$, we want to find a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[0,1]$ such that in each subinterval $[x_{i-1}, x_i]$, the supremum of f is less than ε:
As n becomes larger, the value $\frac{1}{n}$ is decreasing, and $\forall \varepsilon$ we have a natural number N such that $n > N, \frac{1}{n} < \varepsilon$. In any interval of real numbers, while there are infinitely many rational numbers, only finitely many of them have denominator smaller than or equal than N. Therefore, we have a finite many of them or finite integers m such that $\frac{m}{n}$ lies in the interval.
The partition can then be created via these finitely many rationals with small denominators and refining the rest of the interval such that each $[x_{i-1}, x_i]$ does not contain any rationals with denominator less than or equal to N. In subintervals without these denominator less than or equal to N, we have the supremum of f will be taken over rationals with denominators larger than N, and hence, $\sup\{f(x) : x \in [x_{i-1}, x_i]\} < \varepsilon$. Therefore, we have: $U(f,P) = \sum_{i=1}^n (x_i - x_{i-1}) \cdot \varepsilon = \varepsilon \cdot \sum_{i=1}^n (x_i - x_{i-1}) = \varepsilon \cdot (1-0) < \varepsilon$ for any $\varepsilon > 0$.
Therefore, we have the upper Riemann Sums is also equal to 0, which implies that the result is integrable. Besides, $\int_0^1 f = 0$ □

Problem 4. *(5 points)* State (without proving) the Heine-Borel theorem and Bolzano-Weierstrass theorem. *There may be several versions of these theorems, state the ones that make most sense to you!*

Solutions.

Theorem 1 (Heine-Borel Theorem)**.** *Every closed bounded subset of $\mathbb{R}$ is compact (has a finite subcover).*

Theorem 2 (Bolzano-Weierstrass Theorem)**.** *Every bounded sequence of real numbers has a convergent subsequence.*

□

UNIVERSITY OF MICHIGAN - MATH 551

Homework 2

Xuyuan Zhang, Uniqname: zxuyuan

February 18, 2024

Problem 1. *(30 points)*

i) *(5 points)* Prove that if A and B are subsets of $\mathbb{R}$ such that $\mu^*(B) = 0$ (B is a *null set*), then $\mu^*(A \cup B) = \mu^*(A)$.

ii) *(10 points)* Prove that if $A \subset \mathbb{R}$ and $t > 0$, then

$$\mu^*(A) = \mu^*(A \cap (-t,t)) + \mu^*(A \cap (\mathbb{R} \backslash (-t,t)).$$

iii) *(15 points)* Prove that $\mu^*(A) = \lim_{t \to \infty} \mu^*(A \cap (-t,t))$ for all $A \subset \mathbb{R}$.

Solutions. (i):
From the definition of outer measure, we have $\mu^*(E) \geq 0 \ \forall E \subset \mathbb{R}$ and for any countable collection of sets $\{E_i\}, \mu^*(\bigcup_i E_i) \leq \sum_i \mu^*(E_i)$.
Using this property, and $\mu^*(B) = 0$, we have $\mu^*(A \cup B \cup \emptyset \cup \ldots \cup \emptyset) \leq \mu^*(A) + \mu^*(B) + 0 + \ldots + 0 = \mu^*(A)$.
To make the expression clear, we would just write $\mu^*(A \cup B) \leq \mu^*(A) + \mu^*(B)$ for the below questions
To prove the other side, Since $A \subseteq A \cup B$, by the property of outer measure, we have $\mu^*(A) \leq \mu^*(A \cup B)$, which is because if $E \subseteq F, \mu^*(E) \leq \mu^*(F)$.
Combining the results, we have $\mu^*(A \cup B) = \mu^*(A)$.
(ii):
To prove the results, notice that $A = (A \cap (-t,t)) \cup (A \cap (\mathbb{R} \backslash (-t,t)))$. By the countable subadditivity property of outer measure, we have $\mu^*(A) = \mu^*(A \cap (-t,t)) \cup (A \cap (\mathbb{R} \backslash (-t,t))) \leq \mu^*(A \cap (-t,t)) + \mu^*(A \cap (\mathbb{R} \backslash (-t,t)))$
It is obvious that: $(A \cap (\mathbb{R} \backslash (-t,t))) \backslash (A \cap (\mathbb{R} \backslash [-t,t]))$ can only be $\emptyset, \{t\}, \{-t\}, \{t,-t\}$ and therefore, $\mu^*(A \cap (\mathbb{R} \backslash (-t,t))) \leq \mu^*(A \cap (\mathbb{R} \backslash [-t,t]))$ by the countable subadditivity property and $\mu^*(A \cap (\mathbb{R} \backslash (-t,t))) \geq \mu^*(A \cap (\mathbb{R} \backslash [-t,t]))$ holds because the outer measure preseves order. Therefore we have: $\mu^*(A \cap (\mathbb{R} \backslash (-t,t))) = \mu^*(A \cap (\mathbb{R} \backslash [-t,t]))$.
We prove the other side by definition: Consider $I_1, I_2, \ldots$ be a sequence of open intervals such that $\sum_{k=1}^{\infty} \ell^*(I_k) < \mu^*(A) + \varepsilon$ where ε is an arbitrary positive real number (this is from the definition of greatest lower bound). The $\ell(\cdot)$ is the length function. Then we have: $A \subset \bigcup_{k=1}^{\infty} I_k$. We can decompose each I_k as: $\ell(I_k) = \ell(I_k \cap (-\infty, -t)) + \ell(I_k \cap (-t,t)) + \ell(I_k \cap (t, \infty))$ (we already have $\ell(I_k \cap [-t,t]) = \ell(I_k \cap (-t,t))$ from above). This results holds because of the previous proof and then using the countable subadditivity property, we have:

$$\begin{aligned}
\mu^*(A \cap (-t,t)) + \mu^*(A \cap (\mathbb{R} \backslash (-t,t))) &\leq \sum_{k=1}^{\infty} [\ell(I_k \cap (-\infty, -t)) + \ell(I_k \cap (-t,t)) + \ell(I_k \cap (t, \infty))] \\
&= \sum_{k=1}^{\infty} \ell(I_k) < \mu^*(A) + \varepsilon
\end{aligned}$$

Since ε is an arbitrary positive number, it follows that:

$$\mu^*(A \cap (-t,t)) + \mu^*(A \cap (\mathbb{R} \backslash (-t,t))) \leq \mu^*(A)$$

Therefore, combining together, the proof holds.
(iii):

For any $A \subset \mathbb{R}$, the measure $\mu^*(A \cap (-t,t))$ is non-decreasing as t increases. Therefore, the upper bound of $\mu^*(A \cap (-t,t))$ is $\mu^*(A)$ because $A \cap (-t,t) \subseteq A$.
To prove the other side, suppose for an arbitrary $\varepsilon > 0$, there exists a countable collection of open intervals $\{I_k\}$ such that $A \subset \bigcup_{k=0}^{\infty} I_k$. Then, as n increases, $A \cap (-n,n)$ eventually contains all fixed interval I_k because $(-n,n)$ will expand to cover $\mathbb{R}$ when $n \to \infty$. In this case, for sufficiently large T_k, with $t > T_k$, $I_k \subset (-t,t)$ and denote $T = \max\{T_k\}$ we have $n > T$, $\bigcup_{n=0}^{\infty} I_k \subset (-n,n)$ and thus $A \subset (-n,n)$.
In this case we can write $\mu^*(A) = \mu^*(A \cap (-n,n)) = \mu^*(A \cap (\bigcup_{n=0}^{\infty}(-n-1,n+1)\backslash(-n,n))) \leq \sum_{n=0}^{\infty} \mu^*(A \cap ((-n-1,n+1)\backslash(-n,n)))$

$$\begin{aligned}
\mu(A) &\leq \sum_{n=0}^{\infty} \mu^*(A \cap ((-n-1,n+1)\backslash(-n,n))) \\
&= \sum_{n=0}^{\infty} (\mu^*(A \cap (-n-1,n+1)) - \mu^*(A \cap (-n,n))) \quad \text{(by using part b)} \\
&= \lim_{n\to\infty} \mu^*(A \cap (-n-1,n+1)) \quad \text{(here n is an integer)} \\
&= \lim_{t\to\infty} \mu^*(A \cap (-t,t)) \quad \text{(here t is a real number.)}
\end{aligned}$$

$\square$

Problem 2. *(20 points, Axler's 1B, 5)* Show an example of a sequence of continuous real-valued functions $f_1, f_2, \ldots$ on $[0,1]$ and a continuous real-valued functions f on $[0,1]$ such that

$$f(x) = \lim_{n\to\infty} f_n(x)$$

for each $x \in [0,1]$ but

$$\int_0^1 f(x)dx \neq \lim_{n\to\infty} \int_0^1 f_n(x)dx.$$

Solutions.
We can define the function $f_n(\cdot)$ as:

$$f_n(x) = \begin{cases} n^2 x & \text{if } 0 \le x \le \frac{1}{n} \\ 2n - n^2 x & \text{if } \frac{1}{n} < x \le \frac{2}{n} \\ 0 & \text{if } \frac{2}{n} < x \le 1 \end{cases}$$

We can check that the function is continuous since $\lim_{x\to\frac{1}{n}-} = n^2\frac{1}{n} = n = \lim_{x\to\frac{1}{n}+}$ and $\lim_{x\to\frac{2}{n}-} = 0 = \lim_{x\to\frac{2}{n}+}$. Because as $n \to \infty$, $\exists$ a N such that $n \ge N$, $x > \frac{2}{n}$, and thus $f_n(x) = 0, \forall n \ge N$. Therefore, at any point $x \in [0,1]$, all subsequent functions $f_n(x)$ will be zero, which means that the $\lim_{n\to\infty} f_n(x) = 0 = f$.
Now we first calculate the integral of $f_n(x)$ over $[0,1]$, which is equal to $\int_0^{\frac{1}{n}} f_n(x) + \int_{\frac{1}{n}}^{\frac{2}{n}} f_n(x) + 0 = \frac{1}{2} + \frac{1}{2} = 1$. However, $\int_0^1 f(x)dx = 0$ which satisfies the condition of the problem.

□

Problem 3. (*20 points*) Let $A \subset \mathbb{R}$ with $\mu^*(A) > 0$. Show that for every $\alpha \in (0,1)$ there exists an open interval I such that

$$\mu^*(A \cap I) \geq \alpha \mu^*(I).$$

(This problem loosely shows that any positive measure set almost contains an interval. Hint: start with the definition of $\mu^*(A)$).

Solutions. From the definition of the outer measure, we have that for any $\varepsilon > 0$ there exists a countable collection of open intervals $\{I_k\}$ such that $A \subseteq \bigcup_{k=1}^{\infty} I_k$ and $\mu^*(A) > \sum_{k=1}^{\infty} \mu^*(I_k) - \varepsilon$ (from the definition of greatest lower bound). Therefore, by the property of outer measure we have:

$$\mu^*(A) = \mu^*(A \cap (\bigcup_{k=1}^{\infty} I_k)) = \mu^*(\bigcup_{k=1}^{\infty}(A \cap I_k)) \leq \sum_{k=1}^{\infty} \mu^*(A \cap I_k)$$

We prove the result by contradiction. Now suppose for every $\alpha \in (0,1)$ we have every open interval I satisfying this condition: $\mu^*(A \cap I) < \alpha\mu^*(I)$. Then:

$$\sum_{k=1}^{\infty} \mu^*(A \cap I_k) < \alpha \sum_{k=1}^{\infty} \mu^*(I_k) < \alpha\mu^*(A) + \alpha\varepsilon$$

In this case, we can choose $\varepsilon < \mu^*(A)\frac{1-\alpha}{\alpha}$, then we have the RHS of the equation: $\alpha(\mu^*(A) + \varepsilon) < \alpha \cdot \frac{1}{\alpha}\mu^*(A) = \mu^*(A)$ and then we have: $\mu^*(A) < \mu^*(A)$ which is clearly a contradiction.
Therefore, our assumption is incorrect and the there must exist some open interval I such that

$$\mu^*(A \cap I) \geq \alpha\mu^*(I)$$

$\square$

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Homework 3

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February 1, 2024

Problem 1. *(20 points, Axler's 2X, 10)* Give an example of a measure space $(X, \mathcal{S}, \mu)$ and a decreasing sequence $E_1 \supset E_2 \supset \cdots$ of sets in $\mathcal{S}$ such that

$$\mu\left(\bigcap_{k=1}^{\infty} E_k\right) \neq \lim_{k\to\infty} \mu(E_k).$$

Solutions.
We consider the measure space $(\bar{\mathbb{R}}, \mathcal{S}, \mu)$ where $\bar{\mathbb{R}}$ denotes the extended real number, and we select open interval $E_k = (k, \infty)$. In this case, it satisfies the condition that $E_1 \supset E_2 \supset \ldots$.
We know that E_k is a Borel set. Each set E_k has an infinite measure, the limit of the measures as k approaches infinity remains infinite. Therefore, we have $\lim_{k\to\infty} \mu(E_k) = \infty$. Similarly, the intersection $\bigcap_{k=1}^{\infty} E_k$ is the set of all points that are greater than every natural number k. However, no such point exists, and $\mu(\bigcap_{k=1}^{\infty} E_k) = 0$
Combining together we find an example that does not satisfy:

$$\mu\left(\bigcap_{k=1}^{\infty} E_k\right) \neq \lim_{k\to\infty} \mu(E_k).$$

□

Problem 2. (*20 points*) Let $(X, \mathcal{S}, \mu)$ be a measure space such that there is $B \in \mathcal{S}$ such that $0 < \mu(B) < \infty$. Fix such a set B, and define a function $\mu_B : \mathcal{S} \to (-\infty, \infty)$ by the formula $\mu_B(A) := \mu(A \cap B)/\mu(B)$.

a) (*10 points*) Show that $(X, \mathcal{S}, \mu_B)$ is a measure space.

b) (*10 points*) Define the collection $S_B := \{A \cap B : A \in S\}$. Show that S_B is a σ-algebra on B.

(*This is how we define conditional probability in the language of measure theory.*)

Solutions.
(a):
To prove that (X, S, μ_B) is a measure space we need to prove that it satisfies the following properties:

- For every $A \in S, \mu_B(A) \geq 0$
- $\mu_B(\emptyset) = 0$
- For any countable collection $\{A_i\}$ of pairwise disjoint sets in S, $\mu_B(\bigcup_i A_i) = \sum_i \mu_B(A_i)$

Since μ is a measure on S, $\mu(A \cap B) \geq 0$ for every $A \in S$. Also $\mu(B) > 0$ and hence, $\mu_B(A) = \frac{\mu(A\cap B)}{\mu(B)} \geq 0$ for every $A \in S$.
$\mu_B(\emptyset) = \frac{\mu(\emptyset \cap B)}{\mu(B)} = \frac{\mu(\emptyset)}{\mu(B)} = 0$ Since $\mu(\emptyset) = 0$ by the properties of μ.
Lastly, we consider any disjoint sequences $A_1, A_2, \ldots$ of sets in S:

$$\begin{aligned}
\mu_B(\bigcup_i A_i) &= \frac{\mu((\bigcup_i A_i) \cap B)}{\mu(B)} = \frac{\mu(\bigcup_i (A_i \cap B))}{\mu(B)} \\
&= \frac{\sum_i \mu(A_i \cap B)}{\mu(B)} = \sum_i \frac{\mu(A_i \cap B)}{\mu(B)} = \sum_i \mu_B(A_i).
\end{aligned}$$

Therefore, it satisfies the three conditions and we prove that it is a measure space.
(b):
To show that $S_B = \{A \cap B : A \in S\}$ is a σ-algebra, we need to show that it satisfies the following properties of σ-algebra:

- $\emptyset \in S_B$;
- If $A \cap B \in S_B$ then $B \backslash (A \cap B) \in S_B$;
- If $\{A_i \cap B\} \subseteq S$ for a countable number of sequence $\{A_i\}$ then $\bigcup_i (A_i \cap B) \in S_B$;

Since S is a σ-algebra on X, it contains the empty set. Therefore, $\emptyset \in S$, now consider the intersection with B: $\emptyset \cap B = \emptyset$, and hence $\emptyset \in S_B$.
Suppose $A \cap B \in S_B$ where $A \in S$ we need to show that the complement also in S_B: We can express it as: $B \backslash (A \cap B)$ as B's intersection with the full set without that of A, i.e., $B \backslash (A \cap B) = B \cap (X \backslash A)$ and therefore, since $X \backslash A \in S$ (S is a σ-algebra, which is closed under complements), then we have: $B \cap (X \backslash A) \in S_B$.
Let $\{A_i\}$ be a countable collection of sets in S, so that $\{A_i \cap B\} \subseteq S_B$. We need to show that $\bigcup_i (A_i \cap B) \in S_B$.
Since S is a σ-algebra, it has the property that it is closed under countable unions. Hence, $\bigcup_i A_i \in S$. Then we have $\bigcup_i (A_i \cap B) = (\bigcup_i A_i) \cap B \in S_B$.
Combining the above, we conclude that it is a σ-algebra on B. $\square$

Problem 3. (*30 points*) Let (X, S, μ) be a measure space with $\mu(X) = 1$. Let $A_1, A_2, \ldots$ be a countable collection of sets in S such that $\mu(A_k) = 1$ for all $k > 0$. Prove that

$$\mu\left(\bigcap_{k=1}^{\infty} A_k\right) = 1.$$

(*Note that in this problem we* ***don't*** *assume A_k's are either decreasing or increasing.*)

Solutions.
Just consider the complement of the sets A_k in X. The complement $A_k \in X$ is A_k^c and since we have $\mu(A_k) = 1$ and $\mu(X) = 1$ then have $\mu(A_k^c) = \mu(X) - \mu(A_k) = 1 - 1 = 0$ Since the countable number sequence $\{A_k^c\}$ have measure of zero, we use the countable subadditivity property and:

$$0 \leq \mu(\bigcup_{k=1}^{\infty} A_k^c) \leq \sum_{k=1}^{\infty} \mu(A_k^c) = 0 \implies \mu(\bigcup_{k=1}^{\infty} A_k^c) = 0$$

Now using the complement property and De Morgan's law $((\bigcap_{k=1}^{\infty} A_k)^c = (\bigcup_{k=1}^{\infty} A_k^c))$ we find:

$$\mu(\bigcap_{k=1}^{\infty} A_k) = \mu((\bigcup_{k=1}^{\infty} A_k^c)^c) = 1 - \mu(\bigcup_{k=1}^{\infty} A_k^c) = 1 - 0 = 1$$

where $\mu((\bigcup_{k=1}^{\infty} A_k^c)^c) = \mu(X) - \mu(\bigcup_{k=1}^{\infty} A_k^c)$

$\square$

Homework 4

Xuyuan Zhang, Uniqname: zxuyuan

February 8, 2024

Problem 1. (*40 points*) For this problem, to prove Lebesgue measurability of a set, you need to show that the set satisfies one of the *equivalent conditions* of Lebesgue measurability.

a) (*20 points*) Suppose $A \subset \mathbb{R}$ with finite outer measure $\mu^*(A) < \infty$. Prove that A is Lebesgue measurable **if and only if** for every $\varepsilon > 0$ there exists a set G that is the union of *finitely* many bounded open intervals such that

$$\mu^*(A \backslash G) + \mu^*(G \backslash A) \leq \varepsilon.$$

b) (*20 points*) Suppose $A \subset \mathbb{R}$ and $A \subset (b, c)$ for some $b < c$. Prove that A is Lebesgue measurable **if and only if**

$$\mu^*(A) + \mu^*((b, c) \backslash A) = c - b.$$

Solutions.
(a):
To prove the result, we first prove that A is Lebesgue measurable $\implies$ Existence of G.
If A is Lebesgue measurable, by definition, for any $\varepsilon > 0$, there exists an open set $\mathcal{O}$ containing A ($A \subseteq \mathcal{O}$) that satisfies $\mu(\mathcal{O} \backslash A) < \frac{\varepsilon}{2}$. Since $\mathcal{O}$ is an open set in $\mathbb{R}$, it can be expressed as a countable union of disjoint open intervals such that $\mathcal{O} \supseteq \bigcup_{i=1}^{\infty} I_i$, and given that $\mu^*(A) < \infty$ and $\mathcal{O}$ covers A with only a very small number, we have $\mu^*(\mathcal{O}) < \infty$ and then we can find finite subcover $\bigcup_{i=1}^{\infty} I_i$ of $\mathcal{O}$ such that $\mu^*(\mathcal{O} \backslash \bigcup_{i=1}^{\infty} I_i) < \frac{\varepsilon}{2}$.
Now let $G = \bigcup_{i=1}^{\infty} I_i$ and it follows that G is a finite union of bounded open intervals. Therefore, from $A \subseteq \mathcal{O}$ we have $\mu^*(A \backslash G) \leq \mu^*(\mathcal{O} \backslash G) < \frac{\varepsilon}{2}$ and since $G \subseteq \mathcal{O}$ and then we have: $G \backslash A \subseteq \mathcal{O} \backslash A$ and thus $\mu^*(G \backslash A) \leq \mu^*(\mathcal{O} \backslash A) < \frac{\varepsilon}{2}$.
Therefore,

$$\mu^*(A \backslash G) + \mu^*(G \backslash A) < \varepsilon.$$

Now we prove the other direction $\Longleftarrow$.
From the definition: *A set $A \subset \mathbb{R}$ is Lebesgue measurable if for every $\varepsilon > 0$, there exists an open set $\mathcal{O}$ such that $A \subseteq \mathcal{O}$ and $\mu^*(\mathcal{O} \backslash A) < \varepsilon$.*
Given $\varepsilon > 0$ let G be $\mu^*(A \backslash G) + \mu^*(G \backslash A) \leq \varepsilon$. We can choose $\mathcal{O} = G \cup (A \backslash G)$ and in this case we would get $A \subseteq \mathcal{O}$ and we aim to prove that $\mu^*(\mathcal{O} \backslash A) < \varepsilon$.
The set $\mathcal{O} \backslash A = (G \cup (A \backslash G)) \backslash A = (G \backslash A) \cup ((A \backslash G) \backslash A) = G \backslash A$ because $((A \backslash G) \backslash A) = \emptyset$.
Since we have $\mu^*(\cdot) \geq 0$ and from the condition we have $\mu^*(\mathcal{O} \backslash A) = \mu^*(G \backslash A) \leq \varepsilon - \mu^*(A \backslash G) < \varepsilon$, which proves the result.
(b):
We first prove $\implies$
If A is Lebesgue measure, then by the definition we have the outer measure μ^* of any set $E \subset \mathbb{R}$ is additive, i.e., for any E_1, E_2 are disjoint measurable sets, then $\mu^*(E_1 \cup E_2) = \mu^*(E_1) + \mu^*(E_2)$. Since we already know that $A \subset (b, c)$, so $((b, c) \backslash A)$ and A are disjoint sets in $\mathbb{R}$ and therefore, we can apply the above claim and get $\mu^*(A) + \mu^*((b, c) \backslash A) = \mu^*(((b, c) \backslash A) \cup A) = \mu^*((b, c))$. For the case where $b = -\infty, c = \infty$ we also have the above claim because the left hand side and right hand side are all ∞ and therefore, they are equal.
Now we derive $\Longleftarrow$
From the definition of Lebesgue measure, a set $A \subset \mathbb{R}$ is Lebesgue measurable if for every $\varepsilon > 0$, there exists an open set $\mathcal{O}$ containing A such that $\mu^*(\mathcal{O} \backslash A) < \varepsilon$ since the outer measure of (b, c) is the length then we have $\mu^*((b, c)) = c - b, (c > b)$. Given the condition $\mu^*(A) + \mu^*((b, c) \backslash A) = \mu^*((b, c)) = c - b$.

We can consider $\mathcal{O} \subseteq (b,c)$ and $\mathcal{O}$ is a Borel set, with $A \subseteq \mathcal{O}, \mu^*(A) = \mu^*(\mathcal{O})$. Since $\mathcal{O} \subseteq (b,c)$ then we have $\mu^*((b,c)\backslash\mathcal{O}) = \mu^*((b,c)) - \mu^*(\mathcal{O}) = \mu^*((b,c)) - \mu^*(A) = \mu^*((b,c)\backslash A)$.
Similarly, we also have a Borel set $\mathcal{E} \subseteq (b,c)$ and $\mathcal{E}$ is a Borel set, with $((b,c)\backslash A) \subseteq \mathcal{E}, \mu^*((b,c)\backslash A)) = \mu^*(\mathcal{E})$. Since $\mathcal{E}$ is a Borel set with $\mathcal{E} \subseteq (b,c)$ then we have $\mu^*(A) = \mu^*((b,c)) - \mu^*((b,c)\backslash A)) = \mu^((b,c)) - \mu^*(\mathcal{E}) = \mu^*((b,c)\backslash\mathcal{E})$
We note that $((b,c)\backslash A) \subseteq \mathcal{E} \subseteq (b,c)$ then we would get $((b,c)\backslash\mathcal{E}) \subseteq A$ (De Morgan's law) and therefore, combining together we have: $((b,c)\backslash\mathcal{E}) \subseteq A \subseteq \mathcal{O}$ Then for Borel set $\mathcal{O}, \mathcal{E}$, we have: $\mu^*(\mathcal{O}) = \mu^*((b,c)\backslash\mathcal{E}) + \mu^*(\mathcal{O}\backslash((b,c)\backslash\mathcal{E}))$ (meausrable additivity, given the union of right hand side is the left hand side). However, from previous analysis we have:

$$\begin{aligned}\mu^*(\mathcal{O}) &= \mu^*((b,c)) - \mu^*((b,c)\backslash\mathcal{O}) = \mu^*((b,c)) - \mu^*((b,c)\backslash A) \\ &= \mu^*(A) = \mu^*((b,c)\backslash\mathcal{E})\end{aligned}$$

Then we can conclude that $\mu^*(\mathcal{O}\backslash((b,c)\backslash\mathcal{E})) = 0$ Since $A \subseteq \mathcal{O}$ then we have: $\mu^*(A\backslash((b,c)\backslash\mathcal{E})) = 0$. However, since $((b,c)\backslash\mathcal{E})$ is a Borel set and $\mu^*(A\backslash((b,c)\backslash\mathcal{E})) = 0$, from the definition of Lebesgue measure, we conclude that A is Lebesgue measurable.

$\square$

Problem 2. (*10 points*) Show that if a measurable set $A \subset [0,1]$ has positive Lebesgue measure $\mu(A) > 0$, then there are two elements x and y in A such that $|x-y|$ is an irrational number.

Solutions.
We first prove the following lemma:

Lemma 1. *The Lebesgue measure of rational numbers at the interval* $[0,1]$ *is* 0. *(in the textbook there is a more generalized proposition: Every countable subset of* $\mathbb{R}$ *has outer measure* 0.*)*

Given the dense property of rational numbers in the real interval $[0,1]$, We can enumerate all the rational numbers in $[0,1]$ as a sequence of $\frac{m}{n}, 0 \leq m \leq n$ and $gcd(m,n) = 1$. For given $\varepsilon > 0$ we can make an interval $[\frac{m}{n} - \frac{\varepsilon}{2^{n+1}n}, \frac{m}{n} + \frac{\varepsilon}{2^{n+1}n}$ which covers the single point $\frac{m}{n}$. The Lebesgue measure of the interval is $\frac{\varepsilon}{2^n n}$
The total measure can be formulated as: $\sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{\varepsilon}{n 2^n} = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \frac{\varepsilon}{2} \frac{1-\frac{1}{2^\infty}}{1-\frac{1}{2}} = \varepsilon$. Given the arbitrary small ε we can conclude that the Lebesgue measure is 0.
Because $A \subset [0,1]$ and $\mu(A) > 0$ then using this lemma, we can say that A must contain irrational numbers.
We first suppose A contains rational and irrational numbers. Then Given that a rational number - an irrational number is irrational, we would directly complete the proof. (Think about $a = \frac{p}{q}, r = \frac{s}{v}$ and irrational number t with $r = t + a$ then we can express $t = \frac{sq-pv}{vq}$ is a rational number, contradiction.)
Now suppose A only contains irrational numbers. We can prove the result by contradiction. Consider x and y in A such that $|x-y|$ is always a rational number. Given the dense property of rational numbers in the real interval $[0,1]$, we can enumerate all the rational numbers in $[0,1]$ as a sequence $\{q_n\}_{n=0}^{\infty}$.
Given the dense and uncountable nature of irrational numbers in $[0,1]$ and the countable nature of rational numbers, assuming we can cover A with intervals of rational lengths derived from assumed rational differences between elements of A leads to an inconsistency. This approach implies a mapping between the uncountably infinite set of points in A and a countable set of rational differences, which contradicts the reality that a set of positive measure consisting solely of irrationals cannot be comprehensively described by a countable collection of intervals without omitting uncountably many points.
Suppose, for contradiction, that an uncountable set U can be fully covered or described by a countable set C, assuming this coverage involves a countable collection of elements or intervals from C. Attempting to establish a bijection between U and a subset of $\mathbb{N}$, by associating each element of U with the index of C covering it, directly contradicts U's uncountability.
Therefore, our assumption must be false and the original claim holds.

$\square$

Problem 3. (*20 points*) Let A be a Lebesgue measurable subset of $\mathbb{R}$ with Lebesgue measure $\mu(A) > 0$. Define the set $A - A := \{x - y : x, y \in A\}$, the set of all of the differences of elements of A (for example, $[0,1] - [0,2] = [-2,1]$.) Show that $A - A$ contains an open symmetric interval around 0, i.e there exists $c > 0$ such that $(-c, c) \subset A - A$.
(*Hint: Use Problem 3 of HW2*)

Solutions.
Given A is a Lebesgue measurable subset of $\mathbb{R}$ with $\mu(A) > 0$, we aim to prove that there exists $c > 0$ such that $(-c, c) \subset A - A$.
From $A \subset \mathbb{R}$ and $\mu(A) > 0$. We already have for every $\alpha \in (0,1)$, there exists an open interval I such that $\mu(A \cap I) > \alpha\mu(I)$ (Problem 3 of HW2). We can consider the interval $I = (x - \varepsilon, x + \varepsilon)$ and $A \subseteq I$. Therefore, for $\alpha > \frac{1}{2}$, we have more than half of the interval I are occpuied by A. Then $\mu(A \cap I) > 2\alpha\varepsilon$.
Consider a transformation $|x| > \delta$ where $\delta = (4\alpha - 2)\varepsilon$ then we would have the claim that there are overlapping between the transformed set $x + A$ and the original set A. Consider prove by contradiction, if the claim does not hold, we would have:

$$\begin{aligned}
2\mu(A) &= \mu(\{x + A\} \cup A) \leq \mu(\{x + I\} \cup I) \\
&= \mu(I) + \mu(x) = 2\varepsilon + x < (4\alpha - 2)\varepsilon = 4\alpha\varepsilon \\
&\implies \mu(A) < 2\alpha\varepsilon
\end{aligned}$$

Therefore, there is a contradiction, and we would have $(x + A) \cap A \neq \emptyset$ when $|x| < \delta$. Therefore, we would have $\{x : |x| < \delta\} \subset A - A$, and we can let $c = \delta$ and conclude that $(-c, c) \subset A - A$.

$\square$

Homework 5

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Problem 1. (*20 points*) Give an example to show that Egorov's Theorem can fail if the measure of the whole space $\mu(X) = \infty$.

Solutions.
We can consider the case where $x \in X$ and $\mu(X) = \infty$

$$f_n(x) = \begin{cases} 1 & \text{if } x > |n| \\ 0 & \text{otherwise} \end{cases}$$

Correspondingly, we can define the $f(x)$ as: $f(x) = 0$. We first use definition to prove that $f_n(x) \to f(x)$ pointwise:
For any fixed x, choose N such that $|N| > x$ then we would have $f_n(x) = 0 = f(x)$ Therefore, we have $\lim_{n\to\infty} f_n(x) = 0 = f(x)$ for all x. Given this, we show that the Egorov's Theorem can not hold:
Egorov's Theorem requires that we can find an arbitrarily small set $\mathcal{E}$ such that $f_n \to f$ uniformly on $X \backslash \mathcal{E}$. Suppose this holds, we can always find a x such that $|x| > n$ where $f_n(x) = 1$ and $f(x) = 0$ leading to $|f_n(x) - f(x)| = 1$, which means no matter how we choose $\mathcal{E}$ as long as it has a measure less than ∞, we would always have a set with $\mu(X\backslash\mathcal{E})$ such that $f_n \not\to f$ uniformly, which violates the Egorov's Theorem when $\mu(X) = \infty$.

□

Problem 2. (*20 points*) Suppose $f_1, f_2, \ldots$ is a sequence of $\mathcal{S}$-measurable functions on a measure space $(X, \mathcal{S}, \mu)$ and that

$$\sum_{n=1}^{\infty} \mu(\{x \in X : |f_n(x)| > 1/n\}) < \infty.$$

Prove that

$$\mu(\{x \in X : f_n(x) \text{ does not converge to } 0\}) = 0.$$

Solutions.

We can denote $A = \{x \in X : f_n(x) \text{ does not converge to } 0\}$ and we aim to prove that $\mu(A) = 0$. We can denote $\mathcal{E}_n = \{x \in X : |f_n(x)| > 1/n\}$. From the condition, we have $\sum_{i=1}^{n} \mu(\mathcal{E}_n) < \infty$. Since each $\mu(\cdot) \geq 0$ and from the condition of the summation is finite, we know that total measure of points where the function $f_n(x)$ exceeds $1/n$ in absolute value is finite (Suppose f_n does not converge to 0, then for any given $\varepsilon > 0$, there are infinitely many n for which $|f_n(x)| \geq \varepsilon$. By choosing $\varepsilon = \frac{1}{n}$, we can check that the sequence would sum to ∞.) This also implies that for large n, the event $|f_n(x)| > \frac{1}{n}$ becomes arbitrarily small. (we can directly implement the Cauchy criterion of convergence and get that $\lim_{n\to\infty} \mu(\{x \in X : |f_n(x)| > 1/n\}) = 0$)

For each $k \in \mathbb{N}$ we can let $\mathcal{O}_k = \bigcup_{n=k}^{\infty} \mathcal{E}_n$. Since $\mathcal{E}_n \subseteq \mathcal{O}_k$ for all $n \geq k$, we know from the sub-additive property of Lebesgue measure that:

$$\mu(\mathcal{O}_k) \leq \sum_{n=k}^{\infty} \mu(\mathcal{E}_n)$$

By the condition, we know that the right hand side approaches 0 as $k \to \infty$ implying that $\lim_{k\to\infty} \mu(\mathcal{O}_k) = 0$. Note that for any point $x \in X$ belongs to A if and only if for every $N > 0$, there exists $n > N$ such that $|f(x)| > 1/n$, i.e., $x \in \bigcap_{k=1}^{\infty} \mathcal{O}_k$ (from the definition of not convergence to 0 and let $\varepsilon = \frac{1}{n}$.). Since $\mathcal{O}_k$ is a decreasing sequence and $\mu(\mathcal{O}_1) < \infty$, then we know that $\mu(A) = \mu(\bigcap_{k=1}^{\infty} \mathcal{O}_k) = \lim_{k\to\infty} \mu(\mathcal{O}_k) = 0$.

$\square$

Problem 3. (*20 points*) Let μ be the Lebesgue measure on $\mathbb{R}$. Let $b_1, b_2, \ldots$ be a sequence of real numbers. Define $f : \mathbb{R} \to [0, \infty]$ by :

$$f(x) = \begin{cases} \sum_{k=1}^{\infty} \dfrac{1}{4^k |x - b_k|} & \text{if } x \notin \{b_1, b_2, \ldots\} \\ \infty & \text{if } x \in \{b_1, b_2, \ldots\}. \end{cases}$$

Show that $\mu(\{x \in \mathbb{R} : f(x) < 1\}) = \infty$.

(*Informally, this means $f(x) < 1$ (or, in fact, any number) on an infinite measure set.)*

Solutions.

To prove the claim, we focus on the contribution of each term in the series that defines $f(x)$, and we aim to construct a small enough intervals around each b_k where $f(x)$ can be shown to be greater than 1 and outside these intervals we have $f(x) < 1$.

For each b_k, let's construct such an interval $B_k = (b_k - \varepsilon_k, b_k + \varepsilon_k) \setminus \{b_k\}$ where ε_k is chosen so that for the boundary condition $b_k - \delta_k$ and $b_k + \delta_k$ we have: $\frac{1}{4^k |b_k \pm \delta_k - b_k|} = \frac{1}{2^k}$. We can choose $\varepsilon_k = \frac{1}{2^k}$. Note that this interval does not fully imply that $f(x) > 1$ but it is a necessary condition for $f(x) > 1$.

Outside these intervals $B_k \cup \{b_k\}$, the contribution of the term $\frac{1}{4^k |x - b_k|}$ to $f(x)$ decreases significantly because $|x - b_k|$ increases, making the denominator larger. Thus, as k increases, each term's contribution to the sum becomes progressively smaller, especially for x far from each b_k. As a result, outside this small interval, we have:

$$\sum_{k=1}^{\infty} \frac{1}{4^k |x - b_k|} < \sum_{k=1}^{\infty} \frac{1}{4^k |b_k - \delta_k - b_k|} = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$$

Since the length of the interval B_k is $2\varepsilon_k = \frac{1}{2^{k-1}}$, we know that total length of all B_k is: $\sum_{i=1}^{\infty} \frac{1}{2^{k-1}} = 1 \cdot \frac{1 - \frac{1}{2^\infty}}{1 - \frac{1}{2}} = 2$. Therefore, the real line consists of all the $\mathbb{R}$ but only countable number of intervals whose total length is no more than 2 are extracted so that $f(x) > 1$ then we have $\mu(\{x \in \mathbb{R} : f(x) < 1\}) = \infty$ (removing of a set of finite measure from $\mathbb{R}$ does not affect the infiniteness of $\mathbb{R}$). $\square$