Mid Term Review

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The exam will have 4 questions. Three of them will be selected from the homework assignments (HW6-HW7) and any of the problems below that are from the textbook by Axler's.

Chapters covered: *Sections 4A-4B, 5A-5C*. You are only responsible to review only the materials that were discussed during lectures (*Lecture 13 - Lecture 16*). Anything that is written in the book but not included in the lecture notes will **NOT** be tested.

Problems:

- Page 106-107: 3, 11, 12
- Page 115: 5

Page 106. 3: Suppose (X, S, μ) is a measure space. Suppose $h \in \mathcal{L}^1(\mu)$ and $||h||_1 > 0$. Prove that there is at most one number $c \in (0, \infty)$ such that

$$\mu(\{x \in X : |h(x)| \geq c\}) = \frac{1}{c}||h||_1$$

Solutions. Assume that there are two distinct positive numbers $c_1 \leq c_2$ such that it satisfies the given condition.

$$\mu(\{x \in X : |h(x)| \geq c_1\}) = \frac{1}{c_1}||h||_1 \text{ and } \mu(\{x \in X : |h(x)| \geq c_2\}) = \frac{1}{c_2}||h||_1$$

Because $c_1 \leq c_2$ then we have denote set $A_{c_1} = \{x \in X : |h(x)| \geq c_2\} \subseteq A_{c_2} = \{x \in X : |h(x)| \geq c_1\}$. Therefore: $\mu(A_{c_2}) \leq \mu(A_{c_2})$ which also implies that: $\mu(A_{c_1}) - \mu(A_{c_2}) = \frac{1}{c_1}||h||_1 - \frac{1}{c_2}||h||_2$ with $||h||_1 > 0$ and $c_1 \leq c_2$ and therefore the right hand side is non-negative. We can consider the set $c_1 \leq |h(x)| \leq c_2$:
We have $\mu(A_{c_1} \backslash A_{c_2})$ then:

$$\int_{A_{c_1} \backslash A_{c_2}} |h| d\mu > \int_{A_{c_1} \backslash A_{c_2}} c_1 d\mu = c_1 \mu(A_{c_1} \backslash A_{c_2})$$

However, because we have:

$$\int_{A_{c_2}} |h| d\mu = c_2 \mu(A_{c_2})$$

and therefore, we have $\int_{A_{c_1}} |h| d\mu \geq c_1 \mu(A_{c_1} \backslash A_{c_2}) + c_2 \mu(A_{c_2}) > c_1 \mu(A_{c_1})$ which violates the given hypothesis. Therefore, there is a contradiction and we at most have one value holds.

□

Page 107. 11: Give an example of a Borel measurable function $h : \mathbb{R} \to [0, \infty)$ such that $h^*(b) < \infty$ for all $b \in \mathbb{R}$ but $\sup\{h^*(b) : b \in \mathbb{R}\} = \infty$.

Solutions.

□

Page 107. 12: Show that $|\{b \in \mathbb{R} : h^*(b) = \infty\}| = 0$ for every $h \in \mathcal{L}^1(\mathbb{R})$

Solutions. □

Page 115. 5: Suppose $f : \mathbb{R} \to \mathbb{R}$ is a Lebesgue measurable function. Prove that

$$|f(b)| \leq f^*(b)$$

for almost every $b \in \mathbb{R}$.

Solutions. □