Homework 2

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An agent is used for decision-making. Let's suppose that this agent makes decisions randomly and with probability $\frac{1}{2}+\varepsilon$ where $\varepsilon > 0$ makes the right decision. In order to further examine and improve the performance of this agent, it has to make decisions N times and take the majority vote. Show that, for any $\alpha \in (0,1)$, the decision would be correct with probability $1-\alpha$ as long as

$$N \geq \frac{\ln(\frac{1}{\alpha})}{2\varepsilon^2}$$

(hint: apply hoeffding's inequality)

Solutions. The strategy is to implement N times and use a majority vote to determine the result. Therefore, each $X_1, X_2, \ldots$ would be i.i.d. Bernoulli random variables on a bounded support $\{0,1\}$ where 1 denotes success and 0 denotes failure. Therefore, from Hoeffding Inequality, we would have:

$$P(S_n - n\mu \geq \xi) \leq e^{-\frac{2\xi^2}{n(b-a)^2}}$$
$$P(S_n - n\mu \leq -\xi) \leq e^{-\frac{2\xi^2}{n(b-a)^2}}$$

From the upper bound - lower bound, we have: $b-a=1$. The mean would be $\frac{1}{2}+\varepsilon$ and therefore, $N\mu = \frac{N}{2}+N\varepsilon$. We are interested in the probability that the majority of decisions are correct and we look at the deviation ξ from the expected value $E[X] = N(\frac{1}{2}+\varepsilon)$ to the point $N/2$, which represents the threshold for a majority. Then we can select $\xi = -N\varepsilon$ and we can simplify the equation as:

$$P(S_n \geq N/2) = P(S_n - N\mu \geq -N\varepsilon) = 1 - P(S_n - N\mu < -N\varepsilon) = 1 - \exp(-\frac{2(-N\varepsilon)^2}{N}) \geq 1-\alpha$$

and then: $-2\frac{(N\varepsilon)^2}{N} \leq \ln\alpha \implies N \geq \frac{\ln(\frac{1}{\alpha})}{2\varepsilon^2}$ which is what we want to prove. □

Suppose X_i's are i.i.d. with mean 0 and finite variance, and N is a stopping time with finite mean. Show that:

$$Var(\sum_{i=1}^{N} X_i) = E[N]\cdot Var(X_1)$$

Solutions. Given that $E[X_i]=0$ and we start with the definition:

$$Var(\sum_{i=1}^{N} X_i) = E[(\sum_{i=1}^{N} X_i)^2] - (E[\sum_{i=1}^{N} X_i]^2)$$

From the first Wald's equality, we have $E[\sum_{i=1}^{N} X_i] = E[N]\mu = E[N]0 = 0$. So the second item simplifies to 0. Now, for the first term, we apply a similar method as for the expectation:

$$E[(\sum_{i=1}^{N} X_i)^2] = E[\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} X_i X_j 1[i \leq N, j \leq N]]$$

This double sum can be split into two parts where $i = j$ and $i \neq j$. For $i = j$ we have $\sum_{i=1}^{\infty} E[X_i^2 1[i \leq N]] = \sum_{i=1}^{\infty} E[1[i \leq N]]E[X_i^2] = E[N]E[X_1^2] = E[N]Var(X_1)$ since $E[X_1] = 0$. Similarly, for the case where $i \neq j$. We can take the condition on

$$\begin{aligned} E[E[X_i X_j 1\{i \leq N\} 1\{j \leq N\} | X_1, \ldots, X_{j-1}]] &= E[X_i 1\{i \leq N\} 1\{j \leq N\} E[X_j | X_1, \ldots, X_{j-1}]] \\ &= E[X_i 1\{i \leq N\} 1\{j \leq N\} \cdot E[X_j]] = 0 \end{aligned}$$

□

Let $\varepsilon_j : j = 1, 2, \ldots$ be i.i.d. random variables with common distribution:

$$P(\varepsilon_j = 1) = p, P(\varepsilon_j = -1) = q = 1 - p$$

Denote $S_n = \sum_{j=1}^{n} \varepsilon_j, n \geq 0$. Prove that $M_n = (q/p)^{S_n}$ is a martingale.

Solutions. To prove that the sequence is a martingale, we first need to prove that $E[|M_n|] < \infty$ and then we prove that $E[M_{n+1}|M_0, M_1, \ldots, M_n] = M_n$.
First, we note that $M_n = (q/p)^{S_n}$, and we know that S_n can take integer value between $-n$ to n and $p > 0, q > 0$ so M_n is always positive and finite for any finite n, which implies $E[|M_n|] < \infty$.
We have $M_{n+1} = (q/p)^{S_{n+1}} = (q/p)^{S_n} \cdot (q/p)^{\varepsilon_{n+1}}$. Given this, we have:

$$E[M_{n+1}|M_0, M_1, \ldots, M_n] = (q/p)^{S_n} E[(q/p)^{\varepsilon_{n+1}}|M_0, M_1, \ldots, M_n]$$

Given ε_{n+1} is independent of $M_0, M_1, \ldots, M_{n+1}$ and it can only take integer value -1 and 1 with probability p and q Therefore we have:

$$E[(q/p)^{\varepsilon_{n+1}}|M_0, M_1, \ldots, M_n] = E[(q/p)^{\varepsilon_{n+1}}] = p \cdot (q/p) + q(q/p)^{-1} = q + p = 1$$

Therefore, we have $E[M_{n+1}|M_0, M_1, \ldots, M_n] = M_n$ and the sequence is a martingale. □

Let X_n be a sequence of independent but not identically distributed r.v.'s. Let $\phi_i(\theta) = E[e^{\theta X_i}]$ be the MGF of X_i (assuming it exists). Show that:

$$Y_n = \frac{1}{\Pi_{i=1}^{n} \phi_i(\theta)} \cdot e^{\theta \sum_{i=1}^{n} X_i}$$

with $Y_0 = 1$ is a martingale process.

Solutions. To prove that the process is a martingale process we need to show that $E[|Y_n|] < \infty$ and $E[Y_{n+1}|Y_0, \ldots, Y_n] = Y_n$.
Since $\phi_i(\theta) = E[e^{\theta X_i}]$ exists for all i , it follows that $E[|Y_n|]$ is finite, assuming X_i have finite expected values and their MGFs are finite for all θ.
We now prove that $E[Y_{n+1}|Y_0, Y_1, \ldots, Y_n] = Y_n$. We have:

$$Y_{n+1} = \frac{1}{\Pi_{i=1}^{n+1} \phi_i(\theta)} \cdot e^{\theta \sum_{i=1}^{n+1} X_i} = \frac{1}{\Pi_{i=1}^{n} \phi_i(\theta)} \cdot \frac{1}{\phi_{n+1}(\theta)} \cdot e^{\theta \sum_{i=1}^{n} X_i} \cdot e^{\theta X_{n+1}}$$

taking the conditional expectation given $Y_0, Y_1, \ldots, Y_n$ we have:

$$E[Y_{n+1}|Y_0, Y_1, \ldots, Y_n] = \frac{e^{\theta \sum_{i=1}^{n} X_i}}{\Pi_{i=1}^{n} \phi_i(\theta)} \cdot E[\frac{e^{\theta X_{n+1}}}{\phi_{n+1}(\theta)}|Y_0, Y_1, \ldots, Y_n]$$

Since X_{n+1} is independent of $Y_0, Y_1, \ldots, Y_n$, and using the definition of MGF $\phi_{n+1}(\theta) = E[e^{\theta X_{n+1}}]$, we have:

$$E[Y_{n+1}|Y_0, Y_1, \ldots, Y_n] = \frac{\phi_{n+1}(\theta)}{\phi_{n+1}(\theta)} \cdot \frac{e^{\theta \sum_{i=1}^{n} X_i}}{\Pi_{i=1}^{n} \phi_i(\theta)} = Y_n$$

This shows that Y_n satisfies the martingale property. □

For a martingale $\{X_n, n \geq 0\}$ with $X_0 = 0$ and let $Z_n = X_n - X_{n-1}, n = 1, 2, \ldots$, be the martingale difference sequence. Thus:

$$X_n = \sum_{i=1}^{n} Z_i$$

Show that:

$$Var(X_n) = \sum_{i=1}^{n} Var(Z_i)$$

Solutions. We already know that $X_n = \sum_{i=1}^{n} Z_i$ and then we have $Var(X_n) = Var(\sum_{i=1}^{n} Z_i) = \sum_{i=1}^{n} Var(Z_i) + 2\sum_{i<j} Cov(Z_i, Z_j)$ (exapnd it). We now justify $Cov(Z_i, Z_j) = 0$ for all $i \neq j$.
Recall that $Cov(Z_i, Z_j) = E[Z_i Z_j] - E[Z_i]E[Z_j]$. $Z_i = X_i - X_{i-1}$, we know that since $i < j$ and the law of total expectation we have $E[Z_j|X_0, \ldots, X_{i-1}] = E[X_j - X_{j-1}|X_0, \ldots, X_{i-1}] = E[E[(X_j - X_{j-1})|X_0, \ldots, X_{j-1}]|X_0, \ldots, X_{i-1}] = E[0|X_0, \ldots, X_{i-1}] = 0$. Similarly, we have $E[Z_i|X_0, \ldots, X_{i-1}] = 0$. Therefore, using law of total expectation we have:

$$E[Z_i] = E[E[Z_i|X_0, \ldots, X_{i-1}]] = 0, E[Z_j] = E[E[Z_j|X_0, \ldots, X_{j-1}]] = 0$$

Furthermore, $E[Z_i Z_j] = E[E[Z_i Z_j|X_0, \ldots, X_i]] = E[Z_i E[Z_j|X_0, \ldots, X_i]] = E[Z_i \cdot 0] = 0$. Therefore, we have $Cov(Z_i, Z_j) = 0$ for all $i \neq j$ and the result follows.
Combining together we have: $Var(X_n) = \sum_{i=1}^{n} Var(Z_i)$. □

An urn initially contains one white and one black ball. At each stage, a ball is drawn and is then replaced in the urn along with another ball of the same color. Let Xn denote the fraction of balls in the urn that are white after the n-th replication.

- Show that $\{X_n; n \geq 1\}$ is a martingale.
- Show that the probability that the fraction of white balls in the urn is ever as large is $3/4$ is at most $2/3$.

Solutions.
(a):
$E[|X_n|]$ is definitely finite because there is just $n+2$ balls in total in the urn. Now we prove the result:

$$E[X_{n+1}|X_0, X_1, \ldots, X_n] = X_n$$

Denote the number of white balls in the urn as W_n and the number of black balls in the urn as B_n. We have $W_n + B_n = n + 2$ and $X_n = \frac{W_n}{n+2}$. The probability of drawing a white ball in round n is: $\frac{W_n}{n+2}$ and the probability of drawing a black ball is: $\frac{n+2-W_n}{n+2} = 1 - \frac{W_n}{n+2}$. Given that $X_n = \frac{W_n}{n+2}$ We have:

$$\begin{aligned} E[X_{n+1}|X_0, \ldots, X_n] &= \frac{W_n}{n+2} \cdot \frac{W_n + 2}{n+3} + (1 - \frac{W_n}{n+2})\frac{W_n}{n+3} = \frac{W_n}{(n+2)(n+3)} + \frac{W_n}{n+3} \\ &= \frac{W_n(n+3)}{(n+2)(n+3)} = \frac{W_n}{n+2} = X_n \end{aligned}$$

(b):
We want to know that $P(X_i \geq \frac{3}{4}$, for some $i \geq 1) \leq \frac{2}{3}$ which $P(X_i \geq \frac{3}{4}, 1 \leq i \leq n) = P(\max_{1\leq i \leq n} X_i \geq \frac{3}{4}) = \frac{E[X_i]}{\frac{3}{4}} = \frac{E[X_0]}{\frac{3}{4}} = \frac{2}{3}$ (from Doob's Kolmogov's inequality).
not finished □

Let $(W, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, P)$ be the filtered probability space and $Y_n, n \geq 0$, a sequence of absolutely integrable random variables adapted to the filtration $(\mathcal{F}_n)_{n\geq 0}$. Assume that for some real numbers $u_n, v_n, n \geq 0$, it holds that:

$$E[Y_{n+1}|\mathcal{F}_n] = u_n Y_n + v_n$$

Find two real sequences a_n and $b_n, n \geq 0$, so that the sequence of random variables $M_n := a_n Y_n + b_n, n > 1$

be martingale with repect to the same filtration.

Solutions. To find a_n and b_n such that the sequence of random variables $M_n := a_n Y_n + b_n, n > 1$ is a martingale with respect to $(\mathcal{F}_n)_{n\geq 0}$, we need to ensure it satisfies: $E[M_{n+1}|\mathcal{F}_n] = M_n$. We have:

$$E[M_{n+1}|\mathcal{F}_n] = a_{n+1}E[Y_{n+1}|\mathcal{F}_n] + b_{n+1} = a_{n+1}(u_n Y_n + v_n) + b_{n+1} = M_n = a_n Y_n + b_n$$

Therefore, we have: $a_{n+1}u_n = a_n$ and $a_{n+1}v_n + b_{n+1} = b_n$. We can solve this system of equations to find a_n and b_n. We first recursively solve the equation for a_n:

$$a_n = \frac{1}{\Pi_{i=0}^{n-1} u_i}, n > 1$$

For b_n, we have $b_{n+1} = b_n - a_{n+1}v_n$ Then we have $b_n = b_0 - \sum_{i=0}^{n-1} a_{i+1}v_i$. Therefore, the sequences are established with given initial vallues. □

Assume $Y_i : i = 1, 2, \ldots$ be non-negative i.i.d. random variables where $E[Y_n] = 1$.

- Show that $X_n = \Pi_{i=1}^n Y_i$ is a martingale.
- Using Strong law of large numbers, show that:

$$\frac{\log(X_n)}{n} \to c < 0, a.s.$$

Solutions.
(a):
To prove that it is a martingale, we first show that $[|X_n|] < \infty$ then we show that $E[X_{n+1}|Y_0, Y_1, \ldots, Y_n] = X_n$. We have $E[|X_n|] = E[|\Pi_{i=1}^n Y_i|] = \Pi_{i=1}^n E[|Y_i|] = 1 < \infty$.

$$E[Y_{n+1}|Y_0, Y_1, \ldots, Y_n] = X_n \cdot E[Y_{n+1}|Y_0, Y_1, \ldots, Y_n] = X_n \cdot E[Y_{n+1}] = X_n \cdot 1 = X_n.$$

(b):
Given that $X_n = \Pi_{i=1}^n Y_i$, we can take natural logarithm (because each $Y_i > 0$) to get: $\log(X_n) = \log(\Pi_{i=1}^n Y_i) = \sum_{i=1}^n \log(Y_i)$.
The SLLN states for i.i.d. random variables Z_i with finite expectation $E[Z_i]$, $\frac{1}{n}\sum_{i=1}^n Z_i$ converges almost surely to $E[Z_i]$ as $n \to \infty$. We can apply this to the sequence $\log(Y_i)$ and we have:

$$\frac{1}{n}\log(Y_i) \xrightarrow{a.s.} E[\log(Y_1)](\text{ i.i.d. })$$

The function $\log(y)$ is strictly increasing for $y > 0$ and because $\log(\cdot)$ is a concave function and we can apply Jensen's inequality to show that $E[\log(Y_1)] < \log(E[Y_1]) = 0$. Therefore, we have:

$$\frac{1}{n}\log(X_n) = \frac{1}{n}\sum_{i=1}^n \log(Y_i) \xrightarrow{a.s.} E[\log(Y_1)] = c < 0$$

□

Let $X_i, i = 0, 1, 2, \ldots$ be a sequence of random variables where $E[X_i] < \infty$. Moreover, we have:

$$E[X_{n+1}|X_0, X_1, \ldots, X_n] = aX_n + bX_{n-1}, n \geq 1, 0 < a, b < 1, \text{ and } a + b = 1$$

Find the value of λ such that $S_n = \lambda X_n + X_{n-1}, n \geq 1$ would be a martingale.

Solutions. To find the value of λ such that $S_n = \lambda X_n + X_{n-1}, n \geq 1$ is a martingale, we need to satisfy: $E[S_n|X_0, X_1, \ldots, X_{n-1}] = S_{n-1}$. Given $S_{n+1} = \lambda X_{n+1} + X_n$ and according to the martingale property, we have: $E[X_{n+1}|X_0, X_1, \ldots, X_n] = aX_n + bX_{n-1}$. We have:

$$E[S_{n+1}|X_0, X_1, \ldots, X_n] = E[\lambda X_{n+1} + X_n|X_0, \ldots, X_n] = (a\lambda + 1)X_n + b\lambda X_{n-1}$$

since we want to have $E[S_{n+1}|X_0, X_1, \ldots, X_n] = S_n = \lambda X_n + X_{n-1}$, we have:

$$a\lambda + 1 = \lambda \implies \lambda = \frac{1}{1-a}$$
$$b\lambda = 1 \implies \lambda = \frac{1}{b}$$

Since $a + b = 1$ we know that the number is exactly the same, and therefore, $\lambda = \frac{1}{1-a}$ would be number we are looking for. Furthermore, we can verify the number would satisfy the fininess condition, i.e., $E[|S_n|] < \infty$.

$$E[|S_n|] = E[|\lambda X_n + X_{n-1}|] = E[|\lambda X_n|] + E[|X_{n-1}|] = |\lambda|E[|X_n|] + E[|X_{n-1}|] < \infty$$

Since λ is finite, and $E[X_i] < \infty$ by assumption, we have $E[|S_n|] < \infty$. Therefore, it is indeed an martingale. □

Let X_n be the net profit of a gambler when betting a unit stake on the n-th play in the casino. X_n can be dependent; however, $E[X_{n+1}|X_1, X_2, \ldots, X_n] = 0, \forall n$ holds, meaing that the game is fair. In the first game, the gambler stakes Z and after that stakes $f_n(X_1, \ldots, X_n)$ on the $(n+1)$-th game. Assuming that Z and these functions, f_i, are given, first (a) show that the gamblers profit after n plays would be

$$S_n = \sum_{i=1}^{n} X_i f_{i-1}(X_1, X_2, \ldots, X_{i-1})$$

and then (b) prove that the sequence S_i is a martingale or in other words $E[S_{n+1}|X_1, X_2, \ldots, X_n] = S_n, n \geq 1$.

Solutions.
(a):
The gambler starts with Z in the initial play and for each $i > 1$, the gambler stakes $f_{i-1}(X_1, \ldots, X_{i-1})$, and the net profit from the ith game is then X_i. Thus the contribution to the net profit from the ith game is $X_i \cdot f_{i-1}(X_1, \ldots, X_{i-1})$. Therefore, the net profit after n plays would be:

$$S_n = \sum_{i=1}^{n} X_i f_{i-1}(X_1, X_2, \ldots, X_{i-1})$$

(b):
Since each play only gives finite profit, then we would have $E[|S_n|] < \infty$. We now prove that S_n is a martingale. We have:

$$S_{n+1} = \sum_{i=1}^{n+1} X_i f_{i-1}(X_1, X_2, \ldots, X_{i-1}) = S_n + X_{n+1} f_n(X_1, X_2, \ldots, X_n)$$

Taking the conditional expectation with respect to $X_1, X_2, \ldots, X_n$, we have:

$$E[S_{n+1}|X_0, X_1, \ldots, X_n] = S_n + E[X_{n+1} f_n(X_1, X_2, \ldots, X_n)|X_0, X_1, \ldots, X_n]$$

Given that $E[X_{n+1}|X_0, X_1, \ldots, X_n] = 0$, and $f_n(X_1, X_2, \ldots, X_n)$ is a function of $X_1, X_2, \ldots, X_n$ (and thus a constant with respect to the conditional expectation of X_{n+1}), we have: $E[X_{n+1} f_n(X_1, X_2, \ldots, X_n)|X_0, X_1, \ldots, X_n] = f_n(X_1, X_2, \ldots, X_n) E[X_{n+1}|X_0, X_1, \ldots, X_n] = 0$. Therefore, we have $E[S_{n+1}|X_0, X_1, \ldots, X_n] = S_n$ and the sequence is a martingale. □