University of Michigan
Econ 502: Applied Macroeconomics
Instructor: Oleg Zamulin

Notes on
The General Equilibrium model

General equilibrium means that all agents in the economy make optimal decisions given their objectives and constraints. Specifically, in our model, producers maximize profits, while households maximize utility with respect to consumption and labor.

We make some standard assumptions in our model:

- Perfect competition in all markets: the goods market, the labor market, and the market for capital
- Representative agents: we model all firms as identical and all households as identical, and ignore differences between them

The firm's problem

To model the firm's problem, we need to introduce a production function:

$$Y = F(K, L, A), \tag{1}$$

where Y is output, K is capital stock, L is labor input, and A is exogenously given total factor productivity. There are some standard classical assumptions about this function:

- Constant returns to scale with respect to K and L: $F(aK, aL, A) = aF(K, L, A)$,
- Decreasing marginal product: $F_K > 0$, $F_L > 0$, $F_A > 0$, but $F_{KK} < 0$ and $F_{LL} < 0$, where F_X is the partial derivative with respect to variable X
- Complementarity of factors: $F_{KL} > 0$, $F_{KA} > 0$, $F_{LA} > 0$
- Inada conditions: $\lim_{K\to 0} F_K = \lim_{L\to 0} F_L = \infty$ and $\lim_{K\to\infty} F_K = \lim_{L\to\infty} F_L = 0$

This function is true for the whole economy and is also true for each individual firm, since all firms are the same. The total output in the economy is the average of individual firms' output, and the same goes for each firm's labor and capital. Therefore, according to our perfect competition assumption, each firm hires labor L at nominal wage W and rents capital K at price R to produce the final product and sells it at price P. The firm takes all prices, P, W, and R, as given, but decides on L and K. The profit maximization problem is then

$$\max_{K,L} PF(K, L, A) - RK - WL. \tag{2}$$

Performing this maximization then gives us the following first order conditions:

$$\begin{aligned} F_K &= \frac{R}{P} \\ F_L &= \frac{W}{P} \end{aligned} \tag{3}$$

These conditions represent the demand for the two factors of production: labor and capital. The terms F_K and F_L are the marginal product of capital and labor, and the equations (3) represent the standard result that each factor is paid their marginal product. The right-hand sides of the two equations are the real rental rate and the real wage rent. This makes sense: if the marginal product of labor were above the real wage, the firm would keep hiring workers, and if it were below, the firm would be firing them, until the two numbers become equal to each other.

Since the marginal product is diminishing in both factors, the demand curves found in (3) are downward sloping.

The household's problem

Households maximize their utility with respect to consumption and labor. The one-period utility function is given by

$$U(C, L), \qquad (4)$$

where C is consumption and L is labor. The standard assumptions are that $U_C > 0$, $U_L < 0$, while $U_{CC} < 0$ and $U_{LL} < 0$. The last two assumptions indicate diminishing marginal utility of consumption and increasing marginal disutility of labor.

Each period, households earn the real wage W/P per unit of labor input supplied and make two decisions. The intratemporal (within period) decision is how much labor L to supply (this is the labor-leisure trade-off). The intertemporal (between periods) decision is how to split the income between consumption in different periods (consumption-saving trade-off). For now, we will concentrate on the intratemporal choice and leave the intertemporal choice for later.

Intratemporal choice

Households need to decide how much labor to supply. To determine that amount, technically we need to maximize the life-time utility function (i.e., the sum of all expected utility from today into the future) subject to the life-long budget constraint (lifetime income should cover lifetime consumption). This is possible but requires some complicated mathematics. Instead, we will use the perturbation argument to solve the problem.

The perturbation argument goes like this. Suppose the household is currently maximizing utility. Then, we perturb their choice slightly by increasing both L and C in such a way as to stay at the budget constraint. By design, this perturbation cannot increase utility, since it was maximal already.

The perturbation we consider is increasing L by ΔL and consumption by $\Delta C = \frac{W}{P}\Delta L$. The change of utility relative to optimum is then

$$U_L\Delta L + U_C\Delta C \leq 0. \qquad (5)$$

Correspondingly, if we change labor and consumption by -ΔL and consumption by $-\Delta C = \frac{W}{P}(-\Delta L)$ we get a similar result

$$-U_L\Delta L - U_C\Delta C \leq 0. \qquad (6)$$

Combining (5) and (6) we get

$$U_L\Delta L + U_C\Delta C = 0 \qquad (7)$$

And then we apply the budget constraint condition $\Delta C = \frac{W}{P}\Delta L$ to get

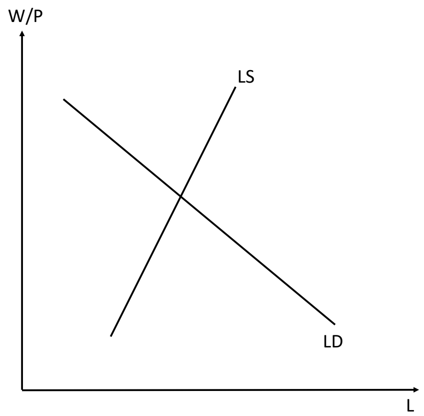
$$\frac{W}{P} = -\frac{U_L}{U_C}. \qquad (8)$$

Equation (8) is a standard optimality condition from microeconomics known as "marginal rate of substitution (MRS) equals to marginal rate of transformation (MRT)". MRT here is the wage rate W/P – this is the rate at which one unit of leisure can be transformed into one unit of consumption. The right-hand side in the equation is the marginal rate of substitution and shows the ratio of marginal utilities for the two goods in the utility function. In the optimal basket, all goods should yield the same marginal utility adjusted for their relative price.

In our case, however, equation (8) is the labor supply function. Note that given our assumptions about the derivatives of the utility function, L and W/P are positively related, which means that people will choose to supply more labor when real wage is high. This is the substitution effect.

But also note than L is negatively related to C, so the more people consume, the less they are willing to work. The reason is that when people consume more, the marginal utility of consumption falls, and people are less willing to sacrifice leisure for the sake of more consumption. This is the income effect.

The labor demand (second equation in (3)) and labor supply (8) give us the standard graph of the labor market:



The position of the LD curve is determined by the marginal product of labor (which, in turn, depends on the total factor productivity and the amount of available capital), while the labor supply curve is determined by preferences and the current level of consumption.

The equilibrium in the labor market then determines the amount of output produced in the economy given the production function (1), the current accumulated amount of capital, and the current level of productivity.

A specific utility function

It is easier to see the derivation of the labor supply function using a particular utility function:

$$U(C,L) = lnC - \frac{L^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}} \qquad (9)$$

You can check that all derivatives are as we assumed in (4): $U_C = 1/C > 0$, $U_L = -L^{1/\nu} < 0$, $U_{cc} = -\frac{1}{C^2} < 0$, and $U_{LL} = -\frac{1}{\nu}L^{\frac{1}{\nu}-1} < 0$. Plugging these into (8), we get the labor supply function

$$L = \left(\frac{1}{C}\frac{W}{P}\right)^{\nu} \qquad (10)$$

This labor supply function shows that parameter ν is the elasticity of labor supply with respect to the real wage, and labor supply depends negatively on C as discussed above.

A static case

Finally, it is useful to look at a static case, that is, a one-period model in which households cannot save and therefore choose to consume their whole income. Then, the budget constraint in that one period is

$$PC = WL \qquad (11)$$

Such a budget constraint can also be thought of a lifetime or long-run budget constraint: over the lifetime, households will choose to consume all of their income, as they cannot die in debt and there is no reason to leave any savings after death (setting aside any bequest motives or uncertainty about time of death).

With such a budget constraint, we no longer need to use the perturbation argument, we can simply maximize the utility function subject to this budget constraint. Thus, take the utility function (9), plug in C expressed from (11) and maximize over L by taking the derivative with respect to L. We leave this exercise to you, but instead we will use the result we already obtained by perturbation argument and simply plug (11) into (10) and get

$$L = 1 \qquad (12)$$

We see that labor supply is constant and does not depend on either the real wage or consumption. In other words, the labor supply curve in our graph is vertical. Why is that? The reason is that with our utility function, the substitution effect and the income effect exactly cancel each other out. The substitution effect says that as wage goes up, people find leisure more expensive and therefore have less of it and supply more labor. The income effect says that as wage goes up, people can afford more consumption and leisure at the same time, and therefore supply less labor. If income effect dominates, the labor supply curve can actually slope down. But with our utility function, the two effects exactly cancel each other out. For the long run, this is actually a desirable feature, since we do not see any marked trend in the labor input per person over decades despite the growth in the real wage.

Expenditure side

So far we looked at the production side of the economy. But we also know that GDP equals the total expenditure. In a closed economy, the correspond equation is

$$Y = C + I + G \qquad (13)$$

We found Y above from the production side and we treat government expenditure G as exogenously given. Hence, the remaining task is to determine how the rest of the production is split between consumption and investment.

Consumption

We model consumption in very general terms as

$$C = C(Y\text{-}T, r) \qquad (14)$$

Here, T is the taxes, (Y – T) is disposable income, and r is the real interest rate. Hence, the function in (14) says that consumption depends positively on current disposable income and negatively on the real interest rate.

This consumption function is a blend of the traditional Keynesian function which posits that consumption is a linear function of current disposable income alone, and the Friedman-Modigliani Permanent Income (PIH) and Life Cycle (LCH) hypotheses which posit that consumption is influenced by the lifetime income and the interest rate. In (14) we make consumption dependent on current disposable income and the interest rate, but will acknowledge that life-cycle considerations also matter.

In terms of the interest rate, an argument can be made that dependence of C on r can be both positive and negative. Again, the substitution effect says that high r makes current consumption more expensive relative to future consumption, hence households reduce C in response to increase in r. The income effect says that with higher r households have a higher lifetime income and can afford more consumption both today and in the future, so dependence will be positive. Empirical evidence suggests that there is a small negative dependance, so the substitution effect likely dominates in real life.

Investment, capital stock, and the real interest rate

Now, let us see what determines investment. First, let us link investment and capital stock from our production function (1):

$$K_{t+1} = K_t + I_t - \delta K_t \quad (15)$$

Subscript t here denotes time periods. This is the capital accumulation equation that shows that capital stock is determined by previous investment decisions. Capital is added by means of investment, while share δ depreciates away every period. Hence, you can think of investment as adjustment of the capital stock to the desired level.

So let us see how firms decide how much to invest. Consider a firm that owns a unit of capital. It can do two things with it:

1) Sell it at the real price P_K/P and invest at interest rate r. P_K is the market price of the capital good.
2) Use it in production, earn the marginal product $MPK = F_K = R/P$ and lose $\delta \frac{P_K}{K}$ on depreciation.

The *no arbitrage condition* says that a firm should be indifferent between these two strategies, which implies the condition

$$\frac{P_K}{P} r = MPK - \delta \frac{P_K}{P}$$

Or

$$\frac{P_K}{P}(r + \delta) = MPK = \frac{R}{P} \quad (16)$$

There are some important conclusions we can make from (16). First, suppose that $P_K/P=1$, which means that capital goods cost the same as consumer goods in the market, then we get the condition that the real interest rate simply equals the real rental price less depreciation rate:

$$r = \frac{R}{P} - \delta \quad (17)$$

This is a good approximation for the very long run, and we will use it later on in this course. In the long run, we can assume that the market will drive P_K to equal to P: if $P_K > P$, then firms will switch to producing investment goods, and if $P_K < P$, then they will prefer to produce consumer goods, but if $P_K = P$, then they will be indifferent between producing either.

Equation (17) thus links the long-run real interest rate to the marginal product of capital and allows to estimate its exact value.

Now suppose the more general case when $P_K \neq P$. Again, it is logical to assume that the higher the P_K/P, the more firms will be willing to invest, since they can sell the capital good at a higher price. In other words,

$$I = I(P_K/P) \quad (18)$$

where the dependance is positive. From the equation (16) we see that this implies that desired investment depends positively on MPK and negatively on r and δ:

$$I = I(MPK, r, \delta) \quad (19)$$

Thus, we formally obtained the standard result that investment depends negatively on the real interest rate.

The market for loanable funds

Now we are ready to connect our consumption and investment functions and find the equilibrium real interest rate. For that, we are going to present the model of the market for loanable funds, defined by the equilibrium of desired saving and desired investment.

Desired investment is given by (19). National saving is defined as

$$S \equiv Y - C - G \tag{20}$$

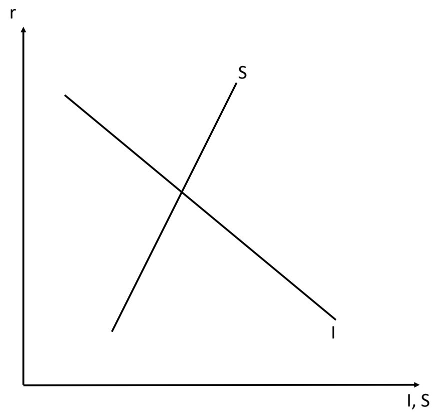
Combining this with (13), we get the standard identity

$$S = I \tag{20'}$$

We know that consumption in general depends negatively on the real interest rate r, which means that desired saving depends positively on r and disposable income, and negatively on government expenditure:

$$S = S(Y\text{-}T, G, r) \tag{21}$$

The market for loanable funds can be drawn in the space with S and I on horizonal axis and the real interest rate on the vertical axis:



This graph determines the equilibrium amount of investment and the equilibrium real interest rate. The desired investment curve can be shifted by MPK, which in turns depends on productivity A, the amount of labor L, and the amount of capital K. The desired saving curve can be shifted by income, life-cycle considerations, and government expenditure.

How does the interest rate determined in this graph compare to the one given by (17)? Note that the expression in (17) is true only for the case when $P_K/P=1$, which only has to be true in the very long run when capital adjusts to its desired level. In general, condition (17) does not hold and the real interest rate is instead determined by the market for loanable funds. Yet, the interest rate is probably not very far from the value in (17) unless there is a very big discrepancy between the current and the desired level of capital.

The general equilibrium

Now we are ready to present the whole model as a system of equations defining the model within period:

$$Y = F(K, L, A),$$

$$\frac{W}{P} = F_L,$$

$$\frac{W}{P} = -\frac{U_L}{U_C},$$

$$Y = C + I + G,$$

$C = (Y\text{-}T, r),$

$I = I(F_K, r, \delta).$

Note that these six equations have six endogenous variables, whose value we are trying to find: Y, L, W/P, C, I, and r. Other variables are either exogenous (A and G) or are pre-determined (K).

The income side

Finally, we know that GDP also must identically equal to income. We can demonstrate this in our model as well. There are two types of income in our model: labor income, which is the wage rate times labor input, and capital income, which is the rental price of capital times the amount of capital. We can demonstrate that their sum indeed equals the GDP.

Our production function (1) exhibits constant returns to scale, which means, in mathematical terms, that it is homogeneous of degree one. The Euler theorem for such functions states that (dropping A, which we just treat as a fixed parameter here)

$$Y(K,L) = F_K K + F_L L \tag{22}$$

Substituting (3) into this equation, we get

$$Y = \frac{R}{P}K + \frac{W}{P}L \tag{23}$$

Thus, indeed, we see that GDP equals the total income in the economy. The usual question here is "where is the profit?" It seems like companies function without earning any profit for themselves, which is a glaring contradiction to reality. The answer is that we assumed perfect competition, which implies zero *economic* profits. But even with perfect competition, we can still have *accounting* profit in this economy as long as firms own their own capital instead of renting it from the market. In this case, the accounting profit is $Y - \frac{W}{P}L$. But an economist does not consider this to be profit, an economist would call this capital income – the amount of money the company would earn if it simply rented its capital out. Rent is the opportunity cost of using your own capital.

If we deviated from the assumption of perfect competition and assumed that firms have some monopoly power, then the right-hand side in (23) would indeed be smaller than the left-hand side, so firms would earn some economic profit as well. This would probably be a somewhat more realistic assumption.

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Notes on
The open economy model
(Based on Obstfeld and Rogoff, 1996, Chapter 1)

The basic identities

In an open economy, we have

Y = C + I + G + NX

Let us define national saving in an open economy as

S ≡ Y + NFP – C – G

This definition is very similar to the one we had in a close economy but has an additional term NFP, which stands for net factor payments. These are the earnings from abroad: capital income, such as profits from owned factories or dividends from holdings of stocks and bonds, and labor income from abroad. Y + NFP is then the Gross National Product, as opposed to Y, which is Gross Domestic Product. Thus, definition of saving in open economy is the same as in closed (income minus spending) but income has this extra term NFP in it.

Combining these together, we get

S – I = NX + NFP (0)

This is the central identity of an open economy and it serves as the basis for Balance of Payments (BoP). BoP is the record of al international transactions of a country. The Current Account NX + NFP shows a country's earning from international transactions (trade and work). Financial Account S – I show the capital flows: foreign direct investment, foreign portfolio investment, lending, etc. Lending can also take the form of foreign reserves accumulation by the Central Bank, but this item is frequently presented separately as ΔR:

S – I + ΔR = NX + NFP

The meaning of identity (0) is the following: if a county earns through trade and work more than it pays abroad, so that NX + NFP > 0, this amount has to turn into a capital outflow, S – I > 0. Conversely, if the trade balance is negative, and the country imports more than it exports, it has to borrow, so capital flows in: S – I < 0.

Now let us turn to the model of current account: how the equilibrium current account is determined?

The budget constraint

The general budget constraint looks like this:

$$B_{t+1} = Y_t + (1+r)B_t - C_t - I_t - G_t \quad (1)$$

where r is the exogenous world interest rate and B_t is holdings of an international financial asset.

We can re-write this as

$$CA_t \equiv \Delta B_{t+1} = Y_t + rB_t - C_t - I_t - G_t = rB_t - NX_t, \quad (2)$$

where CA_t is the current account and rB_t is the net factor payments from abroad.

The two-period model with endowment

Now let us look at a two-period model where economy has endowments of income Y_1 and Y_2 in the two periods. We assume that economy starts and finishes with no assets, so that $B_1 = B_3 = 0$. The condition $B_3 = 0$ is a combination of the No-Ponzi-Game condition (assets at the end cannot be negative) and Transversality condition (does not make sense to leave positive assets at the end).

Also assume that $G = 0$ in all periods. Then the budget constraints in the first and the second periods are:

$$B_2 = Y_1 - C_1 \tag{3}$$

$$0 = (1+r)B_2 + Y_2 - C_2 \tag{4}$$

Note that $Y_1 - C_1 = NX_1$ and $Y_2 - C_2 = NX_2$.

Combining (3) and (4) and eliminating B_2, we get the budget constraint

$$C_1 + \frac{1}{1+r}C_2 = Y_1 + \frac{1}{1+r}Y_2, \tag{5}$$

which means that the present value of consumption equals the present value of income.

Consumers maximize their utility with respect to consumption in the two periods:

$$\max_{C_1,C_2} U(C_1) + \frac{1}{1+\rho}U(C_2), \tag{5'}$$

where ρ is the subjective discount factor. Maximization of (5') with respect to budget constraint (5) gives us the Euler equation

$$\frac{u\prime(C_1)}{u\prime(C_2)} = \frac{1+r}{1+\rho}. \tag{6}$$

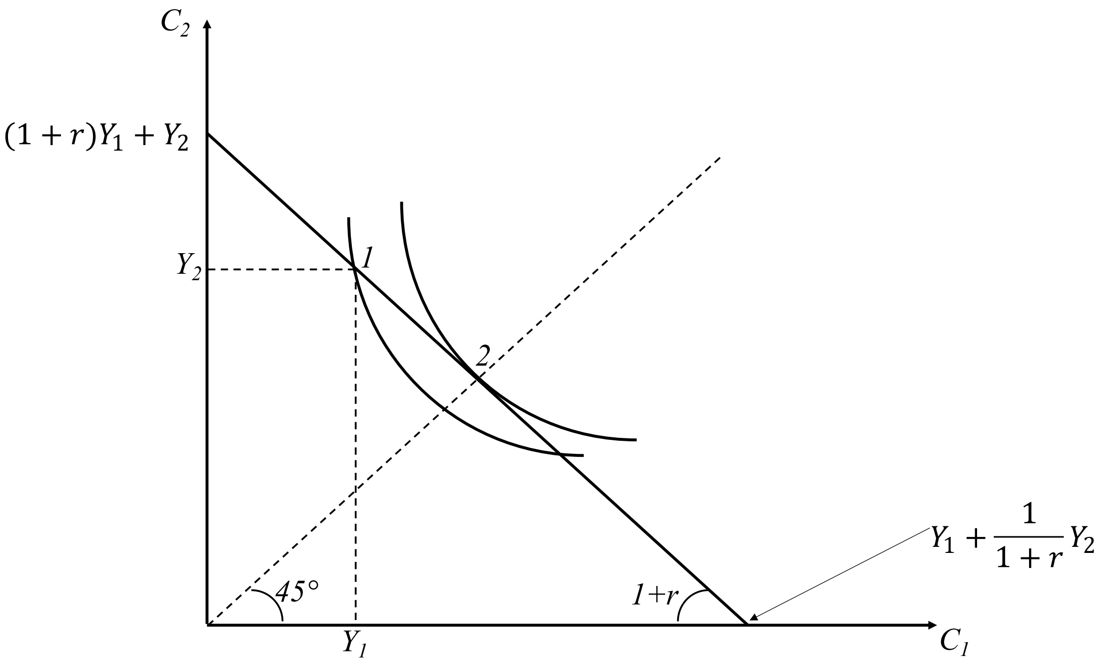
It is sensible to assume that r = ρ. This is a condition for the steady state (when consumption is constant), so if we assume that the world is in the steady state, then the world interest rate must have converged to the rate of intertemporal preferences. Then we get a simple optimality condition

$$C_1 = C_2 = C \tag{7}$$

Plugging this into budget constraint (5) we get consumption in both periods

$$C = \frac{1+r}{2+r}Y_1 + \frac{1}{2+r}Y_2 \tag{8}$$

This is easiest seen on the following graph:



In a closed economy, we are stuck at point 1, where consumption has to equal production. However, if we open the economy, then we can trade our endowment for future consumption according to budget constraint (5). This constraint is depicted on the graph as the straight line with slope -(1+r). We can pick any point on it, and as we know, utility is maximized at the point 2, where $C_1 = C_2 = C$. The indifference curve at point 2 is tangent to the budget constraint, which shows the highest attainable utility. It is strictly better than the indifference curve at point 1.

The first period current account in this setting is simply $CA_1 = Y_1 - C_1$, which is the horizontal difference between points 2 and 1 (with a negative sign).

Two-period model with production

Now let us assume that we do not have endowments, but instead we produce according to production function

$$Y_t = F(K_t) \tag{9}$$

We assume that there is no depreciation, so

$$I_t = K_{t+1} - K_t \tag{10}$$

The general budget constraint now looks like this (again assuming *G = 0*):

$$B_{t+1} = (1+r)B_t + F(K_t) - C_t - I_t \tag{11}$$

Or, plugging in (10),

$$B_{t+1} + K_{t+1} = (1+r)B_t + F(K_t) + K_t - C_t \tag{12}$$

This means that we can accumulate debt $B<0$ to borrow not only for consumption purposes, but also for production purposes, buying capital in the rest of the world.

In a two-period setting, we assume that we start with a certain initial capital stock K_1, but we end the second period with no capital and no financial assets: $K_3 = B_3 = B_1 = 0$.

Therefore, our budget constraints in the two periods are:

$$B_2 + K_2 = F(K_1) + K_1 - C_1 \tag{13}$$

$$0 = (1+r)B_2 + F(K_2) + K_2 - C_2 \tag{14}$$

The second equation says that our consumption in the second period equals the sum of three things: international assets with interest $(1+r)B_2$, own production $F(K_2)$, and left-over capital K_2. The last term is very unrealistic of course: it says that we can easily turn capital goods into consumption goods. In international setting this makes some sense, since some equipment we can sell abroad in exchange for consumption goods.

Combining (13) and (14) and eliminating B_2, we get the unified budget constraint

$$C_2 = (1+r)\big(F(K_1) - C_1 - (K_2 - K_1)\big) + F(K_2) + K_2 \,. \tag{15}$$

International markets allow us once again to equalize consumption between periods as in (7), so plugging this into (15) we get

$$(2+r)C = (1+r)\big(F(K_1) - (K_2 - K_1)\big) + F(K_2) + K_2 \tag{16}$$

Now we need to find such K_2 that maximizes this consumption. In other words, we need to figure out how much we need to invest to maximize our consumption in both periods. Taking derivative of right-hand side of (16) with respect to K_2, we get the familiar condition:

$$F'(K_2) = r \tag{17}$$

This means that we borrow/lend so much capital so that its marginal product becomes exactly equal to the world interest rate. If our MPK is lower than *r*, then it makes more sense to lend to the rest of the world, so we reduce our capital and thus raise its MPK. If our MPK is high, then we prefer to borrow at *r* and produce in our country, which gives us a higher return. This is the principle determinant of the current account in this setting.

Now, let us see this logic on out graph. Now we need to substitute the straight budget constraint line with a production possibility frontier (PPF). To get the production possibility frontier, we take the autarky versions of (13) and (14), where we set $B_2 = 0$. Combining the two with these equations and eliminating K_2, we get

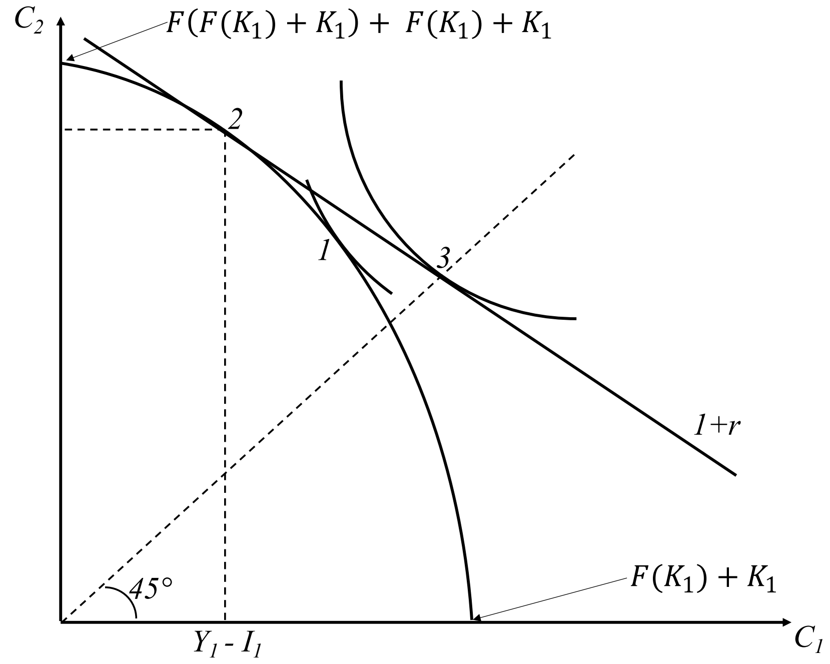
$$C_2 = F(F(K_1) + K_1 - C_1) + F(K_1) + K_1 - C_1 \tag{18}$$

Since *F(K)* is concave, then dependency of C_2 on C_1 is also concave, as shown on the graph below. Observe that the intercept of the PPF with the horizontal axis is given by formula (13) with $K_2 = 0$ (we leave nothing for the second period capital), and with the vertical axis by formula (18) with C_1=0 (we consume nothing in the first period).

In autarky, we are forced to choose a point on this PPF, and the best one is represented by point 1. This is where the indifference curve is tangent to the PPF. Here, consumptions in two periods are not necessarily equal to each other since we do not have the option of lending and borrowing at the rate r. Instead, the marginal rate of transformation is $F'(K_2)$ which can equal something else in a closed economy.

But if we open the economy, then we can choose a different allocation of both production and consumption in two periods, since we can borrow not only for purposes of consumption, but also investment. Observe that on our graph at point 1 the slope of the PPF is steeper than (1+r), also shown on the graph as the straight line. This means that at this point MPK > *r*, and it makes sense to borrow and invest in more capital until condition (17) is satisfied. Notice that the derivative of (18) with respect to C_1 is exactly – (1+$F'(K_2)$), so condition (17) implies that in the optimal production point the interest rate line should be tangent to PPF. The interest rate line is then the budget constraint for our choice of consumption in two periods, directly analogous to the case of endowment economy. So we consume at point 3.

What does point 2 represent? This is not exactly output, this is the consumption we would have if we produced at that point but did not borrow extra from the rest of the world, which is output less investment. So by equation (13), setting $B_2 = 0$ this is $C_1 = F(K_1) - (K_2 - K_1) = Y_1 - I_1$. Then $B_2 < 0$ is what we borrow to achieve the level C_1 at point 3, and this borrowing is exactly the current account deficit. Thus, horizontal difference between 3 and 2 is -B_2 also known as the current account deficit.



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Winter 2024
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Problem set 1
Suggested answers

1. Consider the Cobb-Douglas production function $Y = AK^{\alpha}L^{1-\alpha}$, where A is the productivity parameter, K is capital, L is labor input.

    a. Demonstrate that this function exhibits constant returns to scale with respect to K and L.

    Answer: $A(zK)^{\alpha}(zL)^{1-\alpha} = A\, z^{\alpha}z^{1-\alpha}\, K^{\alpha}L^{1-\alpha} = zAK^{\alpha}L^{1-\alpha}$

    b. Demonstrate that the Cob-Douglass production function has the standard classic features: diminishing marginal product of each factor, complementarity of factors, and the Inada conditions.

    Answer:

    MPL, derivative with respect to L, is $(1-\alpha)AK^{\alpha}L^{-\alpha} = (1-\alpha)A(K/L)^{\alpha}$. *This is a positive number, but we clearly see that it is decreasing in L, which is in the denominator, so marginal product of labor is diminishing. Likewise, the MPL is increasing in K, which shows complementarity of the two factors.*

    MPK is correspondingly $\alpha AK^{\alpha-1}L^{1-\alpha} = \alpha A(L/K)^{1-\alpha}$. *We see that this value is decreasing in K (diminishing marginal product) and increasing in L (complementarity).*

    Finally, the Inada condition. Derivative with respect to labor $(1-\alpha)A(K/L)^{\alpha}$ *goes to infinity when L goes to zero, and goes to zero when L approaches infinity. Same for derivative with respect to capital. Thus, Inada conditions are satisfied.*

    c. Suppose the economy is competitive. Derive expressions for the real wage and the real rental price of capital.

    Answer: They are derived above. They are equal to MPL and MPK.

    d. Calculate the shares of labor and capital income in the total income in the economy.

    Answer: Labor income is $MPL\times L = (1-\alpha)AK^{\alpha}L^{-\alpha}\times L = (1-\alpha)AK^{\alpha}L^{1-\alpha} = (1-\alpha)Y$. *Hence, labor share is* $(1-\alpha)Y/Y = (1-\alpha)$.
    Likewise, capital income is $\alpha AK^{\alpha-1}L^{1-\alpha}\times K = \alpha Y$, *so capital share is* α.
    The two shares sum up to 1, which means there is no economic profit in the economy.

2. Consider an economy where $Y = AK^{\alpha}L^{1-\alpha}$, capital is fixed at K = 1, and labor is also supplied inelastically so that L = 1. All markets are competitive. Suppose that consumption function is $C = bY/(1+r)$, while investment function is $I = MPK/(1+r)$. The balance condition is $Y = C + I + G$.

a. Derive expressions for the wage rate, interest rate, investment, and labor as functions of exogenous variables.

Answer:

W/P *= (1-α)A(K/L)*$^{\alpha}$ *=* ***(1-α)A***

L = 1

MPK = αA(K/L)$^{\alpha-1}$ *= αA, so I = αA/(1+r). This should equal to Y – C – G, and observe that Y = A and C = bA/(1+r)*

From here we get ***1+r = (α + b)A/(A-G)***

And

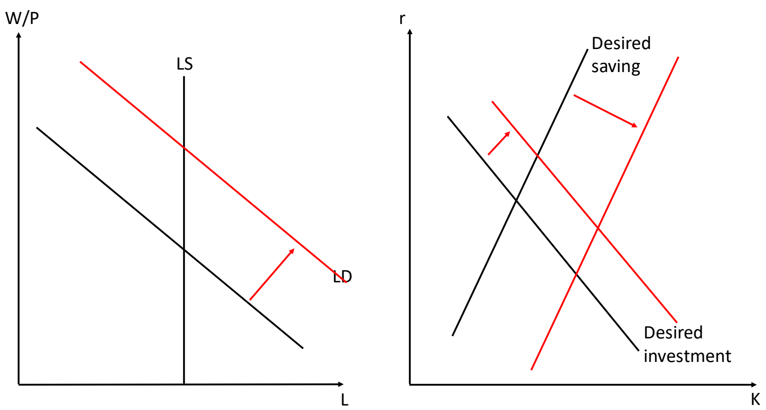
I = α(A-G)/(α + b)

b. Demonstrate on the graphs of the labor market and the market for loanable funds what happens to these variables if productivity A goes up.

Answer:

On the labor market graph, labor demand shifts to the right, so wage goes up. L stays at 1. Indeed, we saw that that wage depends positively on A.

On the graph for the marker for loanable funds, desired saving shifts right because Y goes up (C also increases but less than Y, so S still increases for each r). Desired investment also shifts to the right because MPK increases. Investment unambiguously goes up. You cannot tell from the graph what happens to the interest rate, but you can tell from the expression you got in (a). Differentiating the expression ***(α + b)A/(A-G)*** *with respect to A you see that the derivative is negative, so interest rate goes down. That means that the effect of the increase in saving unambiguously dominates in this case.*

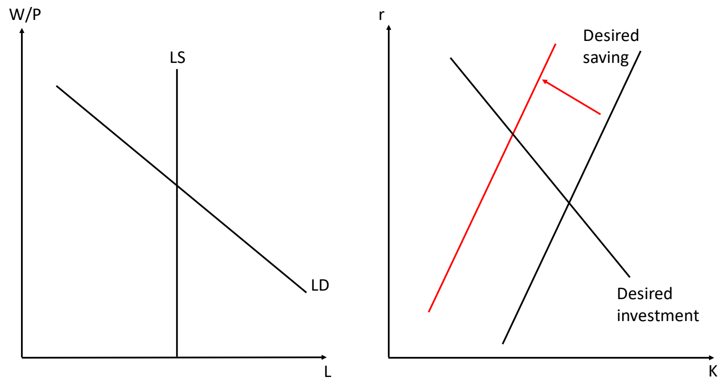


c. Demonstrate on the graphs of the labor market and the market for loanable funds what happens to these variables if government expenditure G goes up.

Answer:

Nothing happens on the labor market graph, so neither wage nor labor change.

In the market for loanable funds, desired saving curve shifts to the left, so investment falls and interest rates goes up. This corresponds to the expressions found in (a).



3. Consider an economy where the only factor of production is labor:

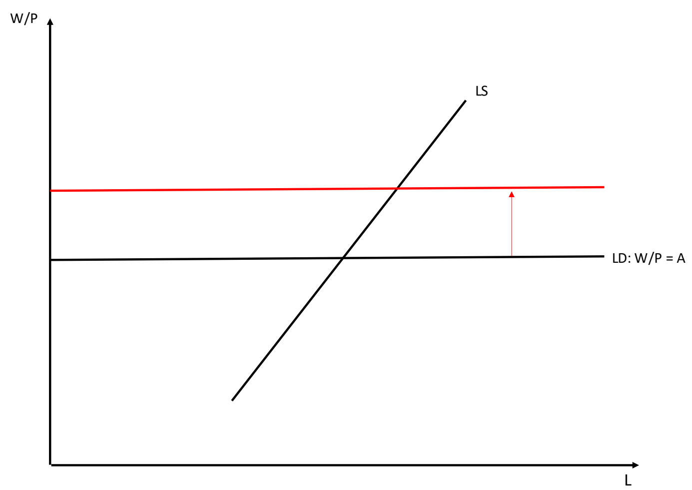
   Production function: Y = AL
   Labor supply: L = W/P

   a. What does the marginal product of labor equal to in this case? Does this production function have the feature of diminishing marginal product?

      Answer: The marginal product of labor is the derivative of production function with respect to labor. Here, it is simply MPK = A. With such a production function, the MPL is not diminishing. Every unit of labor increases production by A.

   b. Draw the labor market graph for this. Suppose the level of technology A goes up. How does that affect equilibrium real wage and labor input?

      Answer: Labor demand becomes W/P = A, a horizontal line. Since marginal product of labor is constant, firms are willing to hire any amount of labor at this wage. The labor supply is a line with 45° slope. If A goes up, then MPL goes up and firms are willing to pay more for labor, so wage goes up, and equilibrium labor input goes up proportionally.



c. What is the labor income in this economy? How does it compare to the GDP?

Answer: The labor income is W/P times L, which is AL. This is the total income (GDP) in the economy. Since this is the only input of production, it gets all of the income. Economy is competitive, so we do not have any economic profit here, just like in class.

d. We know that technological level grows over time. We also know from statistics that the average real wage also grows over time, while labor input per person stays roughly the same. Is the model considered in this exercise consistent with these facts?

Answer: The model is consistent with growth of the real wage but not consistent with constancy of the labor input. In this model, as wage grows, there is a trend for labor input to grow as well, which we do not observe in real life.

4. Consider utility function U = $\sqrt{C} - \frac{L^{1-\frac{1}{v}}}{1-\frac{1}{v}}$, where C is consumption and L is labor input.

Note: unfortunately, the problem contains a typo, and the utility function was supposed to be U = $\sqrt{C} - \frac{L^{1+\frac{1}{v}}}{1+\frac{1}{v}}$*. The problem can still be solved with the provided utility function, but the results are ambiguous and not easily interpretable, see below. Results with the correct utility function are also provided below.*

a. Does this function satisfy the standard assumptions of diminishing marginal utility with respect to consumption and leisure?

Answer:

Derivative with respect to C is $U_C = 1/2C^{-1/2}$*, which is decreasing in C, so yes, marginal utility is diminishing.*

Derivative with respect to L is $U_L = -L^{-1/v}$*, which is increasing in L, so no, marginal disutility from labor is decreasing, which is not the assumption we made in class. To see that the marginal utility of leisure is not decreasing, you can also denote leisure by H, substitute in the function H = I -L and see that the second derivative with respect to H is positive, not negative.*

If you take utility U = $\sqrt{C} - \frac{L^{1+\frac{1}{v}}}{1+\frac{1}{v}}$*, then* $U_L = -L^{1/v}$*, which satisfies our assumption from class.*

b. Derive the labor supply function for the general case using the perturbation argument.

Answer:

Perturbation argument gives us the condition $W/P = -U_L/U_C$. Plugging in the marginal utilities from (a) we get $L^{1/v} = 2C^{1/2}(W/P)^{-1}$. Thus, we get a result that labor depends negatively on the wage rate even holding consumption constant. This means that income effect dominates substitution effect with this utility function.

With the utility function $U = \sqrt{C} - \frac{L^{1+\frac{1}{v}}}{1+\frac{1}{v}}$, the same result is $L^{1/v} = C^{-1/2}(W/P)/2$. Here we get the same dependence as in class, labor supply depends positively on the wage and negatively on consumption.

c. Now suppose that the problem is static, so that WL = PC. What is the labor supply function in this case? Do income and substitution effect cancel each other out, as with the logarithmic case?

Answer: we want to eliminate consumption from our labor supply to see how it depends on wage alone. So observe that the budget constraint here can be re-written as $C^{1/2} = (W/P)^{1/2}L^{1/2}$, which we can plug directly into either labor supply function obtained above.

With the utility function provided in the formulation of the problem, we get that $L^{(2-v)/2v} = 2(W/P)^{-1/2}$. We cannot tell if this dependence is negative or positive as it depends on whether v is less or bigger than 2.

With the utility function $U = \sqrt{C} - \frac{L^{1+\frac{1}{v}}}{1+\frac{1}{v}}$, the answer is $L^{(2+v)/2v} = (1/2)(W/P)^{1/2}$. Here, the dependence is unambiguously positive (as long as v>0), which means that the substitution effect dominates. With higher wage, people prefer to work more.

5. One of the important trends in recent decades has been a marked decline in interest rates. Read the provided article by N. Gregory Mankiw published in New York Times. Comments on how the reasons for low interest rates that he lists correspond to our general equilibrium model.

   Answer: Mankiw identifies six reasons for the interest rate to fall. Three of them (rising inequality, growth of Chinese economy, and precautionary saving) shift the desired saving curve to the right. The other three (growth slowdown, decline in capital intensity of production, and increase in monopoly power of firms) shift the desired investment curve to the left. As a result, a much lower interest rate is needed to equalize saving and investment. Note that the whole world is a closed economy (until we start trading with Mars), so global saving must equal global investment.

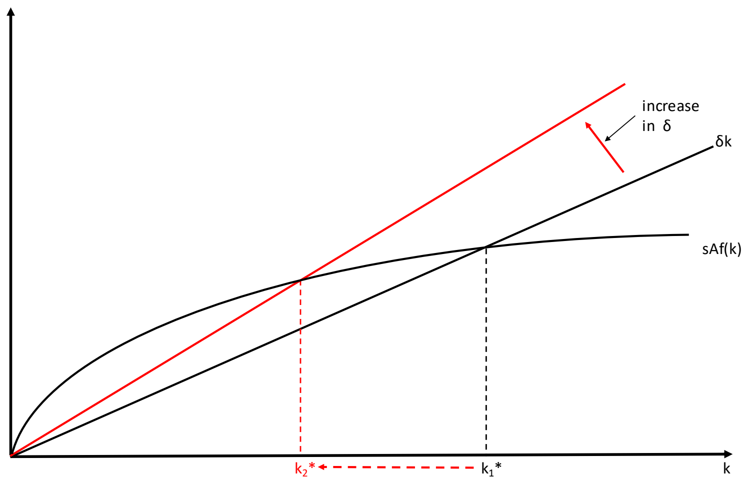
University of Michigan
Econ 502: Applied Macroeconomics
Winter 2024
Instructor: Oleg Zamulin

Problem set 2
Suggested answers

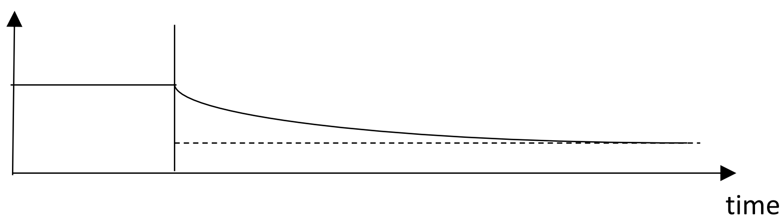
1. Consider the Solow model with the production function Y = AF(K,L). Labor force L and productivity factor A are constant unless specifically stated otherwise. Define k ≡ K/L. Suppose that originally the economy is in the steady state. Demonstrate on the graph and explain intuitively what happens to output per capita in the following situations. Make sure you indicate what happens on impact, in the transition period, and in the new steady state. Draw the dynamic path for income per capita in each case and explain intuition.

   a. An increase in the depreciation rate of capital

   Answer: Note that now the production function per capita is y = Af(k), which, however, changes nothing in this question, as A is just a constant. As a result of an increase in δ, the line δk shifts up, and the new steady state ends up being below the original one. This makes sense: with a higher depreciation rate, the same amount of investment will be able to maintain only a smaller amount of capital stock. However, the move towards the new steady state will be gradual. On impact, no variable will change, but capital will eventually go down as it now depreciates faster, and so will the output per capita, since y = Af(k).

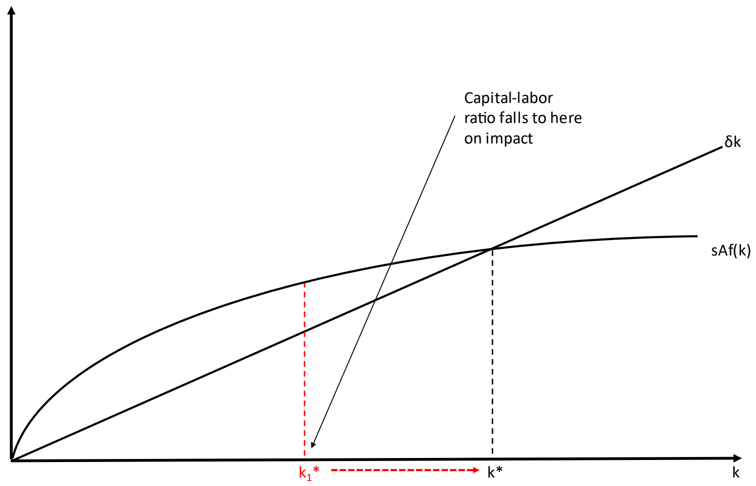


Output per capita:

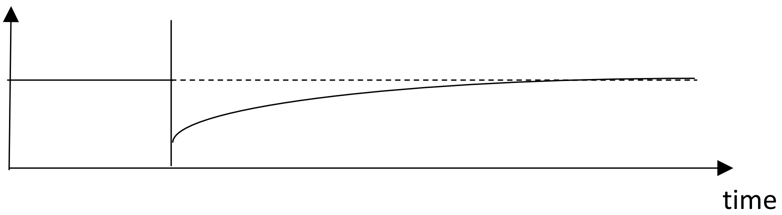


b. Destruction of a share of capital stock due to a natural disaster

Answer: Destruction of capital stock makes k fall on impact. However, this is not a shift in any of the curves, since k is an endogenous variable on the axis, while no parameter of any curve changed. Hence, steady state k does not change, but k falls below steady state on impact. At this point, capital is more productive (MPK, slope of f(k), is bigger), but depreciation is smaller, so capital grows until it reached the (old and only) steady state.*

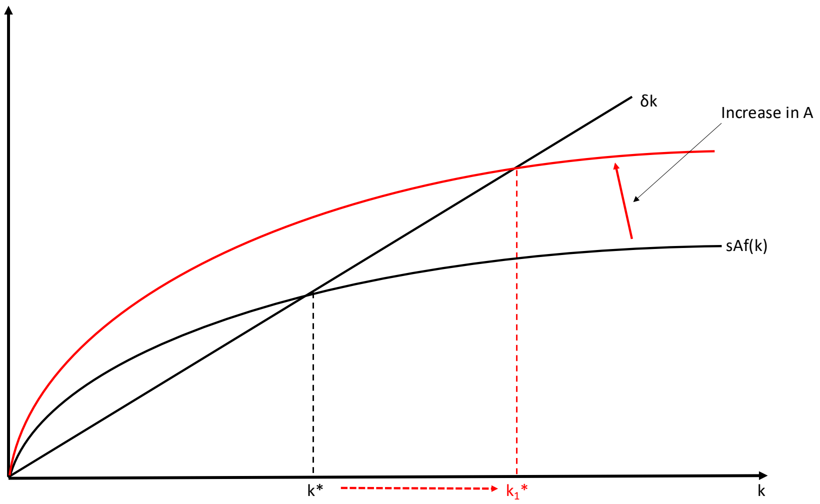


Output per capita:

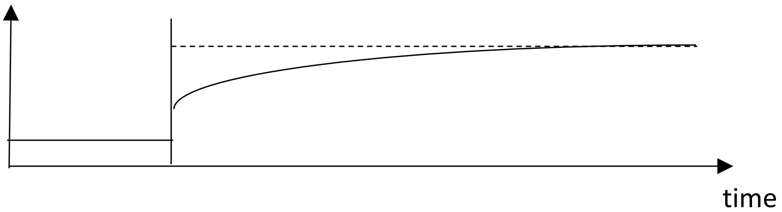


c. A technological breakthrough (increase in A)

Answer: Now A changes, which makes the production function and the saving function move up, just like an increase in the saving rate would. That moves the steady state to a higher level. On impact, capital per worker does not change, but its productivity is higher, so it starts moving towards the new steady state. The output per worker, on the other hand, jumps up on impact because of increase in A, and then grows even more together with k.



Output per capita:



Note that if we had a specification Y = F(K,AL) and k = K/AL, we could also analyze such a shock as an increase in A. The resulting dynamics of capital and output would be exactly the same, but the graph would be different. It would not be a shift of the saving curve, but rather a drop in k, as in question (b). Capital would then grow to reach the old steady state. But this is not the model's specification given in this problem.

2. Consider the Solow model with constant population and no technological progress. The production function is $Y = K^{\alpha}L^{1-\alpha}$. The saving rate is denoted by s, the depreciation rate by δ.

   a. Derive the expression for production function in per capita terms, y = f(k), where y ≡ Y/L and k ≡ K/L

      Answer: $\boldsymbol{y = k^{\alpha}}$

b. Find the expression for the steady-state level of k as a function of exogenous parameters s, δ, and α.

Answer: The steady state condition is $sk^{\alpha} = \delta k$. *From here we get* $\mathbf{k^* = (s/\delta)^{1/(1-\alpha)}}$.

c. Now find the Golden Rule level of steady state k (i.e., the level that maximizes steady state consumption). Combine it with the result in (b) to find the Golden Rule saving rate as a function of exogenous parameters.

Answer: the Golden rule condition is $MPK = \delta$. *MPK is derivative of F with respect to K, which is* $\alpha K^{\alpha-1}L^{1-\alpha}$ *or* $\alpha k^{\alpha-1}$. *So Golden Rule condition for k is*

$$\alpha k^{\alpha-1} = \delta$$

To derive the saving rate, re-write the general steady-state condition in (b), which relates k and s, as

$$k^{\alpha-1} = \delta/s$$

Combining them, we get

$$\mathbf{s^{GR} = \alpha}$$

Note that the Golden Rule saving rate does not depend on the depreciation rate, and equals the capital share of income. This is true specifically for the Cobb-Douglas production function and will not hold in general.

d. It is normally believed that the share of capital in total income in the US economy is 1/3, so $\alpha = 1/3$. The standard estimate for the depreciation rate is $\delta = 0.05$. Find the Golden Rule saving rate for these numbers. The actual investment rate in the US economy has been about 21%. Based on this exercise, does it look like the US economy is investing too much, too little, or just enough?

Answer: This example suggests that the saving/investment rate should be 33%, while in reality it is only 21% This strongly suggests that the US economy is not investing enough. Note that we are using the investment rate here, not the saving rate, because that is what matters for capital accumulation. In a closed economy they should be the same, but modern economies are open and can borrow/lend to each other. Thus, investment in the US is higher than saving and is partially financed by borrowing from abroad.

e. Now find the expression for steady state income as a function of exogenous parameters s, δ, and α. Suppose again that $\alpha = 1/3$ and $\delta = 0.05$. Compare income for the case when saving rate is 5% and 50%. Based on your finding, do you think that differences in the saving rate can possibly explain the differences in levels of income that we observe between countries?

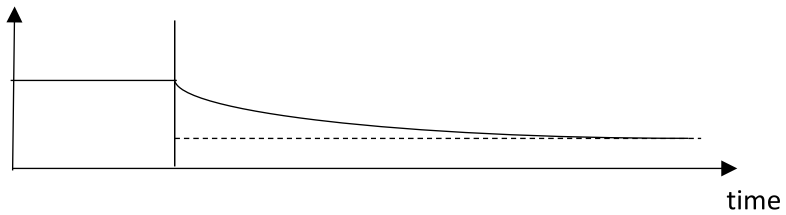
Answer: Since $y = k^{\alpha}$ *and* $k^* = (s/\delta)^{1/(1-\alpha)}$, *the expression for steady state output (per capita) is* $y^* = (s/\delta)^{\alpha/(1-\alpha)}$. *Plugging parameters* $\alpha = 1/3$ *and* $\delta = 0.05$ *we get* $y^* = (s/0.05)^{1/2}$. *For s = 0.05, this number is 1. For s = 0.5, this number is 3.3. Thus, even for two such different saving rates (which are both way outside the standard range for most* *countries) we get only a threefold difference in income per capita, while in reality we observe much bigger gaps between countries. Hence, the saving rate alone cannot explain the observed differences in income.*

3. Now consider the Solow model with population growth and technological progress.

   a. Suppose the rate of population growth increases. Demonstrate what happens on the graph of the model and draw a dynamic path for income per capita. If the new steady state is below the original one, does that mean that people are worse off in terms of their income? Why?

   Answer: The (n+g+δ)k line shifts up, and the new steady state is below the old one. But it is not because K falls, but rather because it gets dispersed among the faster growing population. K actually starts to grow faster as a result of this shift, but not enough to offset the growth in population between the two steady states. So this does mean a reduction in capital per worker and, therefore, in income per worker, since y = f(k). So people are worse off on average.

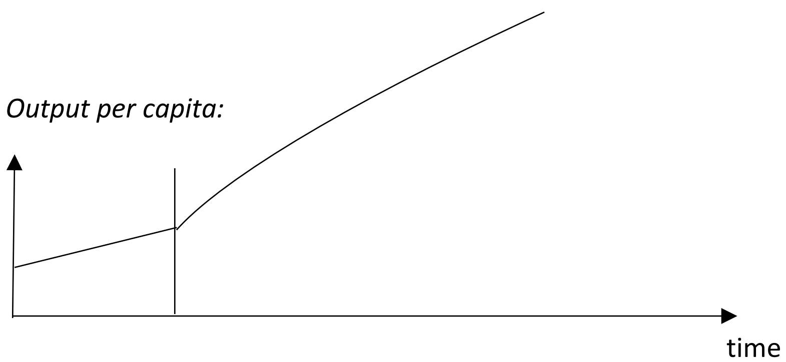
   Output per capita (for the case when g=0):



   b. Repeat the same exercise with an increase in the rate of technological progress. Again, if the new steady state is below the old one, does that mean that people are worse off?

   Answer: the shift on the graph is the same as above. New steady state level of k = K/EL is again lower than the old one. Yet, this time people do not lose income. Capital per effective worker goes down, because it gets dispersed among the faster growing effective workforce EL. But total capital K actually starts growing faster because faster growing E eventually makes MPK higher, while labor L does not grow faster. So capital per worker K/L and, therefore, output per worker Y/L end up above the growth path they followed before, despite the fact that output per effective worker decreases.

   Output per capita:

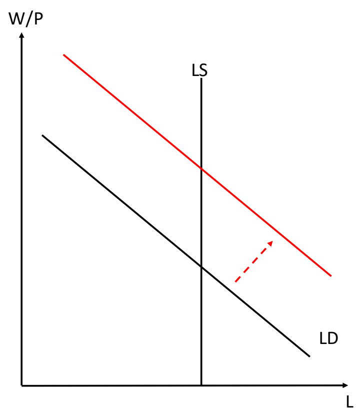


4. Now let us link the Solow model with the labor market.

   a. Suppose there is no population growth and no technological progress. Also suppose that economy starts from below steady state level of capital and approaches the steady state. Demonstrate what happens in the labor market during this process. What happens to the real wage? Show on graph and explain intuition.

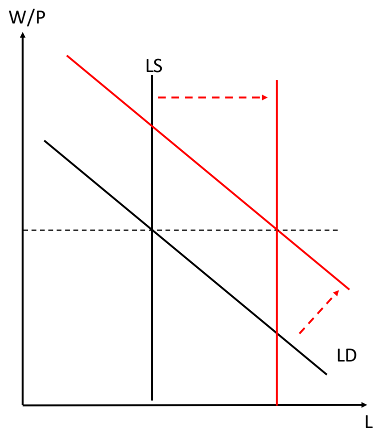
   Answer: As the economy approaches the steady state, labor supply stays constant. Labor demand, on the other hand, increases, since MPL depends positively on K, and K grows. So the wage rate grows.

   Note that the dashed arrow represents slow movement rather than one-time shifts of the curve.



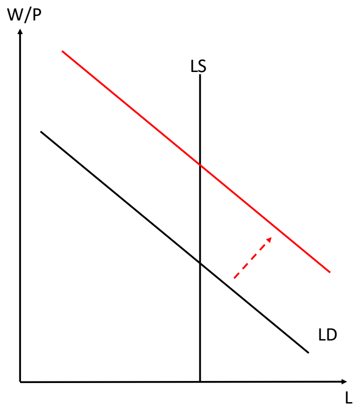
   b. Now suppose economy is in the steady state, and there is population growth, but no technological progress. Show what happens on the graph now.

   Answer: LS moves to the right because of population growth, but so does LD, since K grows and increases MPL. We know that W/P grows at the rate of technological progress in the steady state, which is zero here. Thus, these two curves also move at the same speed, leaving W/P unchanged.



c. Finally, repeat the same exercise for the situation when the economy is in the steady state, in which there is technological progress but no population growth.

Answer: Here, A increases productivity of labor, so labor demand moves to the right. However, LS stays unchanged, so W/P grows at the rate of g.



5. Let us present growth decomposition with the Cobb-Douglas production function $Y_t = K_t^{\alpha}(A_tL_t)^{1-\alpha}$.

a. Growth rate of a variable in continuous time is defined as $\frac{\dot{X}_t}{X_t}$. Derive expression for $\frac{\dot{Y}_t}{Y_t}$ as a function of $\frac{\dot{K}_t}{K_t}$, $\frac{\dot{A}_t}{A_t}$, and $\frac{\dot{L}_t}{L_t}$. Explain the intuition for the result. If the Solow residual is defined as growth of economy unexplained by growth in factors of production, what is the expression for the Solow residual in this case?

Answer: We need to differentiate the production function with respect to time. The production function is a function of two variables, K and AL, of which the second is a function of two variables, A and L, and all three of these variables are functions of time. Hence, we need to use the chain rule to compute derivative of a composite function. For a general formula Y = F(K,AL) the derivative would be

$$\dot{Y} = F_K\dot{K} + F_{AL}A\dot{L} + F_{AL}L\dot{A}$$

Where F_K and F_{AL} are partial derivative with respect to the first and the second variables in the function.

Observing that $F_K = \alpha\frac{Y}{K}, and\ F_{AL} = (1-\alpha)\frac{Y}{AL},$ *we get*

$$\dot{Y} = \alpha\frac{Y}{K}\dot{K} + (1-\alpha)\frac{Y}{AL}A\dot{L} + (1-\alpha)\frac{Y}{AL}L\dot{A}$$

Dividing both sides by Y, we get

$$\frac{\dot{Y}}{Y} = \alpha\frac{\dot{K}}{K} + (1-\alpha)\frac{\dot{L}}{L} + (1-\alpha)\frac{\dot{A}}{A}$$

In this formulation, growth unexplained by capital and labor is $(1-\alpha)\frac{\dot{A}}{A}$ which becomes the expression for the Solow residual in this setting.

b. We normally hope that the measured Solow residual represents growth in total factor productivity (TFP). Can you name some problems with such an interpretation of the Solow residual?

Answer: Solow residual is all of the growth in Y unexplained by growth of factors of production, in our case, K and L. We like to think about it as change in total factor productivity, or technological progress, which in the example here is the case. However, there are a number of reasons why this may not be so.

1) *Suppose we assumed a wrong production function. For example, the true production function may not have constant returns to scale. If they are increasing, then change in K and L would produce a bigger effect on Y, which would mistakenly take for TFP.*
2) *Suppose there are other factors of production, such as human capital (or "quality of labor" in Jorgenson's terms). Then we would confuse technological progress with better education.*
3) *Finally, what if factors are used with different intensity over time? Suppose the same capital is used three shifts a day instead of two. We would mistakenly take this for growth in TFP. This problem emerges especially when we look at short-term fluctuations.*

6. Now let us see what the Solow model implies about steady state levels of the real wage and the real interest rate. Suppose that the production function is again $Y = K^{\alpha}(AL)^{1-\alpha}$, L grows at the rate n and E grows at the rate g.

a. Derive expressions for the steady state real wage and real interest rate. Can you express them as functions of exogenous parameters?

Answer: We did this in class for the general function. Interest rate r is MPK – δ, so here it is

$$r = \alpha k^{\alpha-1} - \delta$$

We also know that in steady state $k^{\alpha-1} = (\delta+n+g)/s$, so

$$r = \alpha\frac{\delta+n+g}{s} - \delta$$

The wage rate is MPL, which is here

$$\frac{W}{P} = A(1-\alpha)k^{\alpha} = A(1-\alpha)\left(\frac{s}{\delta+n+g}\right)^{\frac{\alpha}{1-\alpha}}$$

As we see, the wage is a function of A, which makes sense – real wages should grow over time. Unlike the real interest rate, wage is a positive function of s. This also makes sense: the more we save, the higher is steady state capital stock, the higher is the MPL.

b. Let us take parameter values $\alpha = 1/3$, $\delta = 0.05$, $n = 0.01$, $g = 0.02$, $s=0.21$. What is the real interest rate predicted by the model? How well does it correspond to the actual interest rate observed in the US economy?

Answer: Plugging in these numbers in the formula above, we get $r = 0.077$ or 7.7%. This is clearly well above of what we observe in the US.

7. We know a number of facts about growth, among which are:
   I. Growth in developed countries remains roughly constant over decades at about 2-3% per year
   II. Growth in some developing counties is much higher than 2-3% per year
   III. The richest countries are more than 100 times richer than the poorest countries in terms of their GDP per capita.

   a. Which of these facts can the Solow model explain, and where does it fail?

   Answer: The Solow model can explain the first two, but not the third. The first fact is consistent with model's prediction that countries at the steady state grow at rate n+g. The second is consistent with prediction that below steady state, countries get additional growth from high MPK. But we saw above that no parameters can explain the large differences in income we observe in real life.

   b. Where the model fails, what are alternative explanations/theories can you propose to explain the observed facts (read section 7.2 of the book Advanced Macroeconomics by Campante, Sturzenegger, and Velasco for this question)?

   Answer: The theories that try to explain this generally introduce things like institutions or human capital into the picture. Poor countries are those with poor protection of property rights and inefficient bureaucracy, so they get little investment, little demand for education, and therefore their productivity is very low. Again, read the book chapter for a more comprehensive answer.