PAPER NOTES

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Part I

urban economics

Chapter 1

Scale Economics, Product Differentiation, and the Pattern of Trade

1.1 Model

1.1.1 Assumptions of the Model

There are assumed to be a large number of potential goods, all of which enter systemtrically into demand. Specifically, we assume that all individuals in the economy have the same utility function:

$$U = \sum_i c_i^\theta, \quad 0 < \theta < 1 \tag{1.1}$$

where c_i is consumption of the ith good. The number of goods actually produced, n, will be assumed to be large, although smaller than the potential range of products.

There will be assumed to be only one factor of production, labor. All goods will be produced with the same cost function,

$$l_i = \alpha + \beta x_i, \quad \alpha, \beta > 0, i = 1, \ldots, n \tag{1.2}$$

where l_i is labor used in producing the ith good, and x_i is the output of that good. Average cost declines at all level of output, although at a diminishing rate.

Output of each good must equal the su of individual consumption. If we can identify individuals with workers, output must euqal consumption of a representative individualtimes the labor force:

$$x_i = Lc_i, \quad i = 1, \ldots, n \tag{1.3}$$

We assume full employment, so that the total labor force must just be exhausted by labor used in production:

$$L = \sum_{i=1}^{n} (\alpha + \beta x_i) \tag{1.4}$$

Finally, we assume that firms maximize profits, but that there is free entry and exit of firms, so that in equilibrium profits will always be zero.

1.1.2 Equilibrium in a Closed Economy

First, I analyze consumer behavior to derive demand functions. Then profit-maximizing behavior by firms is derived, treating the number of firms as given. Finally, the assumption of free entry is used to determine the equilibrium number of firms.

The reason that a Chamberlinian approach is useful here is that, in spite of imperfect competition, the equilibrium of the model is determinate in all essential respect because the special nature of demand rules out strategic interdependence among firms. Because firms can costlessly differentiate their products, and all products enter systemtrically into demand, two firms will never want to produce the same product; each good will be produced by only one firm.

COnsider, an individual maximizing utility subject to a budget constraint. The FOC from the maximum problem have the form:

$$\theta c_i^{\theta-1} = \lambda p_i, \quad i = 1, \ldots, n \tag{1.5}$$

where p_i is the price on the ith good and λ is the shadow price on the budget constraint, that is, the marginal utility of income. The single firm faces the demand curve by:

$$p_i = \theta \lambda^{-1} (x_i / L_i)^{\theta-1} \quad i = 1, \ldots, n \tag{1.6}$$

Each firm faces a demand curve with an elasticity $1/(1-\theta)$, and the profit-maximizing price is therefore,

$$p_i = \theta^{-1} \beta w \tag{1.7}$$

where w is the wage rate and prices and wages can be defined in terms of any unit. Note that since θ, β and w are the same for all firms, prices are the same for all godos and we can adopt the shorthand $p = p_i$ for all i.

The price p is independent of output given the special assumption about cost and utility. To determine profitability, however, we need to look at outputs. Profits of the firm producing good i are:

$$\pi_i = p x_i - \{\alpha + \beta x_i\} w \quad i = 1, \ldots, n \tag{1.8}$$

its profit are positive, new firms will enter, causing the marginal utility of income to rise and profits to fall until profits are driven to zero. In equilibrium, then $\pi = 0$, implying for the output of a representative firm:

$$x_i = \alpha / (p/w - \beta) = \alpha\theta / \beta(1-\theta) \quad i = 1, \ldots, n \tag{1.9}$$

Thus output per firm is determined by the zero-profit condition. Again, since α, β and θ are the same for all firms we can use the shorthand $x = x_i$ for all i.

Finally, we can determine the number of goods produced by using the condition of full employment. Then we have:

$$n = \frac{L}{\alpha + \beta x} = \frac{L(1-\theta)}{\alpha} \tag{1.10}$$

1.1.3 Effects of Trade

Trades can occur because in the presence of increasing returns, each good will be produced in only one country - for the same reasons that each good is produced by only one firm. The symmetry of the situation ensures that the two countries will have the same wage rate, and that the price of any good produced in either country will be the same. The number of goods produced in each country can be determined from the full employment condition:

$$n = L(1-\theta)/\alpha; \quad n^* = L^*(1-\theta)/\alpha \tag{1.11}$$

where L^* is the labor force of the second country and n^* is the number of goods produced there.

Individuals will maximize their utility but they will now distribute their expenditure over both the n goods produced in the home country and the n^* goods produced in the foreign country. Because of the extended range of choice, welfare will increase even though the "real wage" w/p remains unchanged. Also, the symmetry of the problem allows us to determine trade flows. It is apparent that individuals in the home country will spend a fraction $n^*/(n+n^*)$ of their income on foreign goods, while foreigners will spend $n/(n+n^*)$ of their income on home goods. Thus the value of home country imports measured in wage units is $Ln^*/(n+n^*) = LL^*/(L+L^*)$. This equals the value of foriegn country imports, confirming that with equal wage rates in the two conutries we will have balance-of-payments equilibrium.

1.2 Transport Costs

1.2.1 Individual Behavior

Transportation costs will be assumed to be of the "iceberg" type, that is, only a fraction g of any good shipped arrives, with $1-g$ lost in transit.

An individual in the home country will have a choice over n products produced at home and n^* products produced at home and n^* products produced abroad. The price of a domestic product will be the same as that received by the producer p. Foreign products, however, will cost more than the producer's price; if foreign firms charge p^*, home country consumers will have to pay the c.i.f. price $\hat{p}^* = p^*/g$. Similarly, foreign buyers of domestic products will pay $\hat{p} = p/g$.

Since the price to consumers of goods of differnet countries will in general not be the same, consumption of each imported good will differ from consumption of each domestic good.

To determine world equilibrium, we must also take into account the quantities of goods used up in transit. For determining total demand, then, we need to kow the ratio of total demand by domestic residents for each foreign product to demand for each domestic product. Letting σ denote this ratio, and σ^* the corresponding ratio for the other country, we can show that:

$$\begin{aligned} \sigma &= (p/p^*)^{1/(1-\theta)} g^{\theta/(1-\theta)} \\ \sigma^* &= (p^*/p)^{1/(1-\theta)} g^{\theta/(1-\theta)} \end{aligned} \tag{1.12}$$

The overall demand pattern of each individual can then be derived from the requirement that this spending just equal his wage; that is, in the home country we must have $(np+\sigma np^*)d = w$, where d is the consumption of a representative domestic good; and similarly in the foreign country. The behavior of individuals can now be used to analyze the behavior of firms. The important point to notice is that the elasticity of export demand facing any given firm is $1/(1-\theta)$, which is the same as the elasticity of domestic demand. Thus transportation costs have no effect on firm's policy/ Writing out these conditions again, we have:

$$\begin{aligned} &p = w\beta/\theta; p^* = w^*\beta/\theta \\ &n = L(1-\theta)/\alpha; n^* = L^*(1-\theta)/\alpha \end{aligned} \tag{1.13}$$

1.2.2 Determination of Equilibrium

The only variable which can be affected is the relative wage rate $w/w^* = w$, which no longer need be equal to one. We can determine w by looking at any one of three equivalent market-clearing conditions: (i) equality of demand and supply for home country labor; (ii) equality of demand and supply for foreign country labor;

(iii) balance-of-payments equilibrium. It can be shown that the home country's balance of payments measued in wage units of the other country, is:

$$\begin{aligned} B &= \frac{\sigma^* n w}{\sigma^* n + n^*} L^* - \frac{\sigma n^*}{n + \sigma n^*} w L \\ &= wLL^* \left[\frac{\sigma^*}{\sigma^* L + L^*} - \frac{\sigma}{L + \sigma L^*} \right] \end{aligned} \tag{1.14}$$

Since σ and σ^* are both functions of $p/p^* = w$, the condition $B = 0$ can be used to determine the relative wage. Since σ is an increasing function of w and σ^* a decreasing function of w, $B(w)$ will be negative (positive) if and only if w is greater (less) than $\overline{w}$, where $\overline{w}$ is the relative wage that the full employment condition holds, and the pattern shows that $\overline{w}$ is the unique equilibrium relative wage.

The simple proposition: the larger the country, other things equal, will have the higher wage. Suppose that we were to compute $B(w)$ for $w = 1$. In that case, we have $\sigma = \sigma^* < 1$. The expression for the balance of payments reduces to:

$$B = LL^* [\frac{1}{\sigma L + L^*} - \frac{1}{L + \sigma L^*}] \tag{1.15}$$

This equation will be positive if $L > L^*$, negative if $L < L^*$. This means that the equilibrium relative wage w must be greater than one if $L > L^*$, less than one if $L < L^*$.

1.3 "Home Market" Effects on the Pattern of Trade

1.3.1 Two-Industry Economy

Asssume that there are two classes of products, alpha and beta, with many potential products within each class. A tilde will distinguish beta products from alpha products. Demand for the two classes of products will be assumed to arise from the presence of two groups in the production. There will be one group with L members, which derives utility only from consumption of alpha products; and another group with $\tilde{L}$ memebrs, deriving utility only from beta products. The utility functions of representative members of the two classes may be written:

$$U = \sum_i c_i^\theta ; \tilde{U} = \sum_j \tilde{c}_j^\theta \quad 0 < \theta < 1. \tag{1.16}$$

On the cost side, the two kinds of products will be assumed to have idnetial cost

$$\begin{aligned} l_i &= \alpha + \beta x_i, \quad i = 1, \ldots, n \\ \tilde{l}_j &= \alpha + \beta \tilde{x}_j j = 1, \ldots, \tilde{n} \end{aligned} \tag{1.17}$$

where $l_i, \tilde{l}_j$ are labor used in production on typical goods in each class, and $x_i, \tilde{x}_j$ are total outputs of the goods.

The demand conditions now depend on the population shares. We have:

$$\begin{aligned} x_i &= L c_i \quad i = 1, \ldots, n \\ \tilde{x}_j &= \tilde{L} \tilde{c}_j \quad j = 1, \ldots, \tilde{n} \end{aligned} \tag{1.18}$$

The full-employment condition, however, applies to the economy as a whole:

$$\sum_{i=1}^{n} l_i + \sum_{j=1}^{\tilde{n}} \tilde{l}_j = L + \tilde{L} \tag{1.19}$$

Finally, we continue to assume free entry, driving profits to zero. The only modification we must make to the results of Section 1 is that we must divide the total production into two industries. A simple way of doing this is to note that the sales of each industry must equal the income of the appropriate group in the population:

$$npx = wL; \quad \tilde{n}\tilde{p}\tilde{x} = \tilde{w}\tilde{L} \tag{1.20}$$

But wages of the two groups must be equal, as must the prices and outputs of any products of either industry. So this reduces to the result $n/\tilde{n} = L/\tilde{L}$, the shares of the industries in the value of output equal the shares of the two demographic groups in the population.

1.3.2 Demand and the Trade Pattern: A simple Case

Suppose that there are two countries of the type just described, and that they can trade with trasport costs of the type analyzed in Section II.

In the home country, some fraction f of the population will be consumers of alpha products. The crucial simplification I will make is to assume that the other country is a mirror image of the home country. The labor forces will be assumed to be equal, so that:

$$L + \tilde{L} = L^* + \tilde{L}^* = \overline{L} \tag{1.21}$$

But in the foreign country, the population shares will be reserved, so that we have:

$$L = f\overline{L}; L^* = (1-f)\overline{L} \tag{1.22}$$

If f is greater than one-half, then the home country has the larger domestic market for the alpha industry's products; and conversely. In this case, there is a very simple home market proposition: that the home country will be a net exporter of the first industry's products if $f > 0.5$.

The first step in showing this is to notice that this is a wholly symmetrical world, so that wage rates will be equal, as will the output and prices of all goods. It follows that the ratio of demand for each imported product to the demand for each domestic product is the same in both countries.

$$\sigma = \sigma^* = g^{\theta/(1-\theta)} < 1 \tag{1.23}$$

The expenditure on goods in an industry is the sum of domestic residents' and foreigners' expenditures on that goods, so we can write the expression:

$$\begin{aligned} npx &= \frac{n}{n+\sigma n^*}wL + \frac{\sigma n}{\sigma n + n^*}wL^* \\ n^*px &= \frac{\sigma n^*}{n+\sigma n^*}wL + \frac{n^*}{n\sigma + n^*}wL^* \end{aligned} \tag{1.24}$$

where the price p of each product and the output x are the same in the two countries. We can use above to determine hte relative number of products produced in each country, n/n^*.

To see this, suppose provisionally that some products in the alpha industry are produced in both countries; i.e., $n > 0, n^* > 0$. We can then divide the equations through by n and n^* and rearrange it to get:

$$L/L^* = (n + \sigma n^*)/(\sigma n + n^*) \tag{1.25}$$

which can be arranged to give:

$$n/n^* = \frac{L/L^* - \sigma}{1 - \sigma L/L^*} \tag{1.26}$$

Suppose that the home country produced no alpha products, and that a firm attempted to start production of a single product. This firm's profit-maximizing f.o.b. price would be the same as that of the foreign firm's. But its sales would be less, in the ratio,

$$\frac{\sigma^{-1}L + \sigma L^*}{L + L^*} < 1$$

Thus such a firm could not compete. This gives us our first result on the effect of the home market. It says that if the two countries have sufficiently dissimilar tastes each will specialize in the industry for which it has the larger home market. Obviously, also, each will be a net exporter of the class of goods in which it specializes. Thus the idea that the pattern of exports is determined by the home market is quite nicely confirmed.

A final result we can take from this special case concerns the pattern of trade when specialization is incomplete. In this case each country will both import and export products in both classes (though not the same products). But it remains true that, if one country has the larger home market for alpha producers, it will be a net exporter in the alpha class and a net importer in the other. To see this, note that we can write the home country's trade balance in alpha products as

$$\begin{aligned} B_\alpha &= \frac{\sigma n}{\sigma n + n^*} wL^* - \frac{\sigma n^*}{n + \sigma n^*} wL \\ &= wL^*[\frac{\sigma n}{\sigma n + n^*} - \frac{\sigma n^*}{n + \sigma n^*}\frac{L}{L^*}] \\ &= \frac{\sigma w L^*}{\sigma n + n^*}[n - n^*] \end{aligned}$$

This says that the sign of the trade balance depends on whether the number of alpha products produced in the home country is more or less than the number produced abroad.

Chapter 2

Growth, speculation and sprawl in a monocentric city

2.1 The Model

A single composite commodity, Q, is produced by a competitive industry in the city in both periods. Using subscripts to designate time periods, the endogenous quantities produced are Q_1 and Q_2. Some Q is produced for export and the rest for local consumption. Regardless of whether it is bound for export or local use, every unit of Q must be transported to the central point at a cost of t per unit of Q, per unit of x. It is sold there at the (endogenous) price $P_i, i = 1, 2$. The Q production function has fixed factor proportions. It requires exactly λ units of land, μ of labor and v of capital to produce each unit of Q. Land rent and labor wages are endogenous to the model, but the cost of capital is exogenous; it is s per unit throughout the city in both periods.

Each provides a single unit of labor to the Q industry per period, receives (endogenous) wages of w per period, and has the same utility function. Because the city is open, migration will assure that every resident household attains the (exogenous) level of utility attained elsewhere in the naional economy. Arguments of the utility function are Q, residential land, and another composite commodity, Z, that is imported from outside the city and sold at the central point at unitary price in both periods. Nonland components of housing services are absorbed in Q and Z. Households must consume exactly q units of Q, z of Z, and 1 unit of residential land in each period. To ensure that local consumption of Q does not exhaust production, we assume that:

$$1/\mu > q$$

Households value residential proximity to the central point because workers must commute to work by passing through it and shoppers must shop at it. The combined cost of these trips is T per household, per unit of x, per period.

The exogenous growth mechanism in the model is an increase in export demand for Q between two periods. The export demand function is $f(P_i, \Gamma_i)$, where $\Gamma_i \in 1, 2,$ is an exogenous demand-shift variable that encodes all the information required to predict the usual price-quantity demand relation. Further,

$$\frac{\partial f}{\partial \Gamma_i} > 0, \frac{\partial f}{\partial P_i} < 0, i = 1, 2 \text{ and } \Gamma_1 < \Gamma_2$$

No other exogenous variables change between periods.

2.2 Perfect Foresight Planning

Suppose landowners have perfect foresight and know at the outset what their land rent will be in both periods for all possible development strategies. This requires that Γ_2 be known with certainty in period 1, and means that all development decisions - whether executed in the first or second period - are made simultanously in period 1. We call this perfect-foresight planning. It is to be distinguished from speculation, examined later, which occurs when landowners are uncertain about Γ_2 and future land rents.

In first-period equilibrium, households will reside in a Von-Thunen ring outside another where Q is produced if only:

$$t/\lambda > T$$

Similarly, second-hand residential development will lie more distant from the central point than second-period industrial development. Depending on the relative magnitudes of t/λ and T, it may be advantageous for landowners to withhold from development in period 1 a ring of land between the industrial and residential zones, and preserve it for second-period industrial use.

In particular, it will always occur if:

$$t/\lambda > (2+r)T,$$

where r is the (universal) discount rate between periods. Since $-t/\lambda$ and $-T$ are shown below to be the slope of industrial and residential bid-rent functions, this condition requires that industrial bid-rents decrease more than twice as rapidly with x as residential bid-rents.

2.2.1 First-Period Equations

equilibrium in period 1 is characterized by eight endogenous variables. Three of them are Q_1, P_1 and w_1. Two other indicate land rents in the residential and industrial zones. The remaining three are spatial boundaries: x_a, the outer edge of the industrial zone; and x_b and x_c, the inner and outer boundary of the residential zone. From previous assumptions, these will satisfy:

$$0 \leq x_a \leq x_b \leq x_c$$

To identify the industrial land rent variable, note that firms in the competitive Q industry must earn zero profit. Thus land rent and transportation charges must exhaust revenues after payments are made to capital and labor. Since the latter costs are the same for every firm, on a per-unit-of-Q basis, land rent and transportation charges per-unit-of-Q must be the same for every firm. And since each unit of Q requires exactly λ units of land, this sum is the same for every firm on a per-unit-of-land basis as well. In period 1, we call this amount R_1^Q per unit of land. Because total transportation charges per-unit-of-land are tx/λ at x, the rent on industrial land is indicated by the linear function $R_1^Q - tx/\lambda$.

To identify the residential land rent variable, note that each household has the same income, consumes the same consumption bundle, and faces the same prices for Q and Z. Their expenditure on land and transportation charges must therefore be the same at every location. In period 1, we call this amount R_1^H. Because household transportation charges are Tx at x, the rent on residential land is indicated by the linear function $R_1^H - Tx$.

First-period equilibrium is characterized by eight conditions. One of them equates the supply and demand for Q_1. This means local production of Q must equal the sum of quantities demanded for export and local consumption. Where Q_1 units are produced, μQ_1 households are required to supply the corresponding amount of labor. This means local consumption of Q_1 must be $\mu q Q_1$. Q-market equilibrium requires that:

$$Q_1(1-\mu q) = f(P_1, \Gamma_1) \tag{2.1}$$

A second condition is that household budgets balance:

$$w_1 = R_1^H + P_1 q + z. \tag{2.2}$$

A third is that competitive Q-firms earn zero profit:

$$P_1 = \mu w_1 + vs + \lambda R_1^Q \tag{2.3}$$

Each unit of Q requires λ unit of land, so Q_1 is related to x_a by:

$$Q_1 = \pi x_a^2 / \lambda \tag{2.4}$$

The last three conditions concern equilibrium in the land market. At x_a industrial land rent must be zero, since otherwise land beyond it would be offered for first-period industiral development or land inside it withheld from development until later. This requires that:

$$R_1^Q - t x_a / \lambda. \tag{2.5}$$

For similar reasons, residential land rent at x_c must be zero:

$$R_1^H - T x_c = 0. \tag{2.6}$$

The equilibrium condition concerning x_b is less easily stated. Owners of land near x_a must decide only whether to develop industrially in the first or second period. Those near x_c make a similar decision for residential development. The present value of the first option is:

$$R_1^H - Tx + \frac{1}{1+r}(R_2^H - Tx)$$

The present value of the second option is:

$$\frac{1}{1+r}(R_2^Q - tx/\lambda)$$

x_b is the location where these strategies are equally profitable:

$$R_1^H - T x_b = \frac{1}{1+r}[(R_2^Q - t x_b/\lambda) - (R_2^H - T x_b)]$$

Those eight equations are not a closed system since the last equation includes the second period, endogenous variables, R_2^H, R_2^Q. To solve this equalibrium, it is necessary to solve both periods' equations simultanously, as in a two-period dynamic problem.

2.2.2 Second-Period Equations

Equations in period 2 is characterized by six variables, five of which are $Q_2, P_2, w_2, R_2^Q, R_2^H$. The other one is x_d, the outer boundary of second-period, residential expansion. Of course,

$$x_d \geq x_c$$

The second period conditions corresponding to (1) - (3) are:

$$\begin{aligned} Q_2(1-\mu q) &= f(P_2, \Gamma_2) \\ w_2 &= R_2^H + P_2 q + z \\ P_2 &= \mu w_2 + vs + \lambda R_2^Q \end{aligned} \tag{2.7}$$

Since $\Gamma_2 > \Gamma_1$, Q-production will be greater in period 2 and the industrial zone will expand from x_a to x_b. Thus, Q_2 is related to x_b by:

$$Q_2 = \pi x_b^2/\lambda \tag{2.8}$$

The increase in production requires an expansion in the residential zone from x_c to x_d. In order that this expansion allow a total of Q_2 households to reside in the city:

$$Q_2 = \pi(x_d^2 - x_b^2)/\mu \tag{2.9}$$

The final condition is that land rent at x_d is zero:

$$R_2^H - Tx_d = 0 \tag{2.10}$$

condition (1) - (14) is a closed system of fourteen, independent conditions in fourteen endogenous variables.

2.2.3 The Equilibrium

The equilibrium land rent function in the first period is:

$$\begin{aligned} ER_1(x) &= R_1^Q - tx/\lambda \text{ for } x \in [0, x_a] \\ &= R_1^H - Tx \text{ for } x \in [x_a, x_c] \\ &= 0 \text{ for } x \in [x_c, \infty) \end{aligned}$$

It is continuous at every x except x_b. Land just beyond x_b earns a positive rent while land inside it earns none. The equilibrium land-rent in the second-period is:

$$\begin{aligned} ER_2(x) &= R_2^Q - tx/\lambda \text{ for } x \in [0, x_b] \\ &= R_2^H - Tx \text{ for } x \in [x_b, x_d] \\ &= 0 \text{ for } x \in [x_d, \infty) \end{aligned}$$

Note first that $ER_2(x) \geq ER_1(x), \forall x \geq 0$ and second that $ER_2(x)$ is also continuous at every x but x_b. In the second period, however, land just beyond x_b earns less than land just inside it. The function indicating equilibrium present value of both period's land rent,

$$PV(x) = ER_1(x) + \frac{1}{1+r}ER_2(x)$$

is continuous at every x.

2.2.4 Comparative Statics

Consider now the effect on both periods' spatial equilibrium of a change in Γ_2, all other exogenous variables remaining the same. We have:

$$\frac{\partial x_a}{\partial \Gamma_2} < 0, \frac{\partial x_b}{\partial \Gamma_2} > 0, \frac{\partial x_c}{\partial \Gamma_2} > 0, \frac{\partial x_d}{\partial \Gamma_2} > 0$$

All boundaries move outward as Γ_2 increases except x_a, which moves inward. Since x_b increases and x_a decreases with Γ_2, the second period industrial zone is larger to accomodate increased production of Q. Since the residential-to-individual land ratio is constant, the second-period residential zone must be greater. Thus the annulus of land added by the increase in x_d is greater than that lost by the increase in x_b.

2.3 Uncertainty and Speculation

We now drop the perfect foresight assumption and suppose instead that landowners are uncertain in period 1 about second-period land rent. In particular, they share the probability distribution $g(\Gamma_2)$ over Γ_2 in the first period where the true value of Γ_2 is resolved in period 2. We retain the assumption that $\Gamma_2 > \Gamma_1$, so:

$$\int_0^{\Gamma_1} g(\Gamma_2) d\Gamma_2 = 0$$

We assume landowners are risk neutral. In this environment, landowners' first-period decisions are speculative. This means the monocentric city model equilibrium is the result of a sequential decision proess. The smaller the value of Γ_2 resolved, the less will be needed. A related difference is that depending upon the value of Γ_2 resolved, the outer boundary of industrial expansion may not be x_b. Because of this, we introduce x_e as the boundary. While x_b is determined in period 1 on speculation, x_e is determined along with x_d in period 2 once Γ_2 is resolved. If Γ_2 is small, then $x_e < x_b$ indicating that more land than necessary was preserved. If Γ_2 is sufficiently large, then $x_e > x_b$, indicating all preserved land is used and that a second industiral ring occurs beyond the first-period residential zone.

A final difference is that if Γ_2 is sufficiently small, land rents can be negative at some locations in the second period where residential development occured in the first period.

2.3.1 Second-Period Equilibrium

The second-period equilibrium is characterized by seven variables, six from perfect-foresight planning and one from speculation. The six are $Q_2, P_2, w_2, R_2^Q, R_2^H, x_d$. The seventh is x_e.

Case 1:

We begin with the lower tail of $g(\Gamma_2)$. Suppose:

$$\Gamma_2 = \Gamma_1 + \delta$$

where δ is positive but very small, indicating miniscule growth in export demand for Q between periods. The amount of land needed for both industrial and residential expansion is much less than that preserved by speculation between x_a and x_b. It will occupy an annulus of land between x_a and x_e. The expanded workforce will be accomodated by an annulus of residential development between x_e and x_d where:

$$x_a < x_e < x_d < x_b < x_c.$$

The amount of land in the industrial zone must equal λQ_2, and that in the two residential zone μQ_2. This provides two of the remaining four equilibrium conditions:

$$Q_2 = \pi x_e^2 / \lambda \tag{2.11}$$

$$Q_2 = \pi(x_c^2 - x_b^2 + x_d^2 - x_e^2)/\mu \tag{2.12}$$

The other two conditions concern equilibrium in the land market. In other that land beyond x_d not be offered for residential development, or land inside it withheld, residential land rent must be zero at x_d:

$$R_2^H = T x_d. \tag{2.13}$$

In order that land beyond x_e not be developed industrially,or land inside it residentially, land rent must be the same for both uses at x_e:

$$R_2^Q - R_2^H = (t/\lambda - T) x_e. \tag{2.14}$$

An important feature of equilibrium in Case 1, and a direct implication of the above, is that second-period land rent is negative throughout the residential zone developed in the first period.

As Γ_2 increases, more land is required for both kinds of expansion, so x_e and x_d increase. Once x_d reaches x_b, there is no longer any vacant land in the leapfrog zone to keep residential land rent zero at x_d. Thus the above Equation (2.13) is no longer valid and Case 1 no longer applies. Let the value of Γ_2 for which:

$$R_2^H = Tx_b$$

be called Γ_2^1. That is the minimum value of Γ_2 for which

$$x_d = x_b \tag{2.15}$$

Case 1 and set of conditions above apply only when

$$\Gamma_1 < \Gamma_2 < \Gamma_2^1$$

Case 2:

Now consider:

$$\Gamma_2 = \Gamma_2^1$$

All land in the leapfrog will be developed here. The equilibrium shows that:

$$x_e = x_c\sqrt{\lambda/(\mu+\lambda)}$$

The case is distinguished from the previous one in two ways. First, as Γ_2 increases above Γ_2^1, x_d and x_e remain stationary. Second, land rent is negative in only part of the residential zone. As Γ_2 and consequently R_2^H increase, the point beyond which rent is negative becomes more remote. Let the value of Γ_2 for which:

$$R_2^H = Tx_c.$$

be called Γ_2^2. That is the minimum value of Γ_2 for which $x_d = x_b$ holds. Thus the result applies $\Gamma_2^1 \leq \Gamma_2 \leq \Gamma_2^2$.

Case 3:

Suppose

$$\Gamma_2 = \Gamma_2^2 + \delta.$$

We now have:

$$x_a < x_e < a_b < x_c < x_d.$$

Since the amount of land in the industrial zone must be λQ_2 and that in the residential zone must be μQ_2, we have:

$$Q_2 = \pi(x_d^2 - x_e^2)/\mu \tag{2.16}$$

As Γ_2 increases, both zones expand as x_e and x_d increase. but this expansion cannot continue indefinitely because x_e eventually reaches x_b. Once this happens, there is no longer any residential development in the leapfrog zone to keep both rents equal at x_e. Let the value of Γ_2 for which:

$$R_2^Q - R_2^H = (t/\lambda - T)x_b$$

be called Γ_2^3. That is the minimum value of Γ_2 for which

$$x_e = x_b \tag{2.17}$$

Therefore, $\Gamma_2^2 < \Gamma_2 < \Gamma_2^3$.
Case 4:

$$\Gamma_2 = \Gamma_2^3$$

so that all land in the leapforg zone is developed industrially. Taken together, we have x_d can be solved for as a function of x_b:

$$x_d = x_b\sqrt{1 + \mu/\lambda}$$

Let the value of Γ_2 for which:

$$R_2^Q - R_2^H = (t/\lambda - T)x_c$$

be called Γ_2^4. That is the minimum value of Γ_2 for which $x_e = x_b$.
Then $\Gamma_2^3 \le \Gamma_2 \le \Gamma_2^4$.
Case 5:

$$\Gamma_2 > \Gamma_2^4$$

Second period export demand for Q is especially strong here, and landowners find it advantageous to develop industrially not only the entire leapfrog zone, but another ring between x_c and x_e. Residential expansion occurs even further out between x_e and x_d. Thus there are four von Thunen rings of developed land - two of each kind, alternating as x increases.

Because the amount of land in the two industrial zones must be λQ_2,

$$Q_2 = \pi(x_b^2 + x_e^2 - x_c^2)/\lambda \tag{2.18}$$

must hold. The second period land rent function, $ER_2(x)$, differs among the cases. In case 1, it is

$$\begin{aligned} ER_2(x) &= R_2^Q - tx/\lambda \text{ for } x \in [0, x_e] \\ &= R_2^H - Tx \text{ for } x \in (x_e, x_d], x \in (x_b, x_c] \\ &= 0 \text{ for } x \in (x_d, \infty) \end{aligned}$$

For case 2, it is:

$$\begin{aligned} ER_2(x) &= R_2^Q - tx/\lambda \text{ for } x \in [0, x_e] \\ &= R_2^H - Tx \text{ for } x \in (x_c, x_e] \\ &= 0 \text{ for } x \in (x_e, \infty) \end{aligned}$$

For case 3 and 4, it is:

$$\begin{aligned} ER_2(x) &= R_2^Q - tx/\lambda \text{ for } x \in [0, x_e] \\ &= R_2^H - Tx \text{ for } x \in (x_e, x_d] \\ &= 0 \text{ for } x \in (x_d, \infty) \end{aligned}$$

For case 5, it is:

$$\begin{aligned} ER_2(x) &= R_2^Q - tx/\lambda \text{ for } x \in [0, x_b], x \in (x_c, x_e] \\ &= R_2^H - Tx \text{ for } x \in (x_b, x_c], x \in (x_e, x_d] \\ &= 0 \text{ for } x \in (x_d, \infty) \end{aligned}$$

2.3.2 First-Period Equilibrium

We designate $\phi_1 = (x_a, x_b, x_c, R_1^H, R_1^Q, Q_1, P_1, w_1)$ and the expected present value, then, of the strategy to develop residentially at x_c in period 1 is:

$$R_1^H - Tx_c + \frac{1}{1+r}[\int_{\Gamma_1}^{\infty} g(\Gamma_2)(R_2^H(\Gamma_2, \phi_1) - Tx_c)d\Gamma_2] \tag{2.19}$$

where the notation $R_2^H(\Gamma_2, \phi_1)$ indicates the dependence of R_2^H on ϕ_1 and the value Γ_2 that occurs in period 2. Under the strategy to wait until the second period to develop at x_c, and the values can be designated by different cases.

Under perfect-foresight planning, landowners may preserve a ring of undeveloped land between the first-period industrial and residential zones, providing an example of leapfrog development. In the second period this land is filled-in with industrial development to accomodate increased demand for industrial production.

Urban land conversion is a dynamic process and as such should be evaluated by dynamic rather than static criteria. It does not follow just because a land-use configuration is inefficient at one moment in time, that it is inefficient in the larger scheme of things where it is evolving. Indeed in a growing city, efficiency will require that interior parcels are sometimes withheld from early development and preserved for alternative future uses.

Chapter 3

THE ECONOMICS OF DENSITY: EVIDENCE FROM THE BERLIN WALL

3.1 THEORETICAL MODEL

The city consists of a set of discrete locations, index by $i = 1, \ldots, S$. Each block has an effective supply of floor space L_i. Floor space can be used commercially or residentially, and we denote the endogenous fractions of floor space allocated to commercial and residential use by θ_i and $1 - \theta_i$. City is populated by an endogenous measure of H workers, who are perfectly mobile within the city and the larger economy, which provides a reservation level of utility $\overline{U}$. Workers decide whether or not to move to the city before observing idiosyncratic utility shocks for each possible pair of residence and employment blocks within the city.

3.1.1 Workers

Workers are risk neutral and have preferences that are linear in a consumption index: $U_{ijo} = C_{ijo}$, where C_{ijo} denotes the consumption index for worker o residing in block i and working in block j. This consumption index depends on consumption of the single final good c_{ijo}; consumption of residential floor space (ℓ_{ijo}); residential amenities (B_i) that capture common characteristics that make a block a more or less attractive place to live; the disutility from commuting from residence block i to workplace block j ($d_{ij} \geq 1$); and an idiosyncratic shock that is specific to individual workers and varies with the worker's blocks of employment and residence (z_{ijo}). This idiosyncratic shock captures the idea that individual workers can have idiosyncratic reasons for living and working in different parts of the city. In particular, the aggregate consumption index is assumed to take the Cobb-Douglas form:

$$C_{ijo} = \frac{B_i z_{ijo}}{d_{ij}} (\frac{c_{ijo}}{\beta})^{\beta} (\frac{\ell_{ijo}}{1-\beta})^{1-\beta}, 0 < \beta < 1 \tag{3.1}$$

where the iceberg communiting cost $d_{ij} = e^{\kappa \tau_{ij}} \in [1, \infty)$ increases with the travel time (τ_{ij}) between blocks i and j. The parameter κ controls the size of commuting costs.

For each worker o living in block i and commuting to block j, the idiosyncratic component of utility (z_{ijo}) is drawn from an independent Frechet distribution:

$$F(z_{ijo}) = e^{-T_i E_j z_{ijo}^{-\varepsilon}}, T_i, E_j > 0, \varepsilon > 1 \tag{3.2}$$

where the scale parameter $T_i > 0$ determines the average utility derived from living in block i; the scale parameter E_j determines the average utility derived from working in block j; and the shape parameter $\varepsilon > 1$ controls the dispersion of idiosyncratic utility.

The indirect utility from residing in block i and working in block j can be expressed in terms of the wage paid at this workplace (w_j), commuting costs (d_{ij}), the residential floor price (Q_i), the common component of amenities (B_i), and the idiosyncratic shock (z_{ijo}):

$$u_{ijo} = \frac{z_{ijo} B_i w_j Q_i^{\beta-1}}{d_{ij}} \tag{3.3}$$

where we have used utility maximization and the choice of the final good as numeraire. Although we model commuting costs in terms of utility, there is an isomorphic formulation in terms of a reduction in effective units of labor, because the iceberg commuting cost $d_{ij} = e^{\kappa \tau_{ij}}$ enters the indirect utility function multiplicatively. As a result, commuting costs are proportional to wages, and this specification captures changes over time in the opportunity cost of travel time.

Since indirect utility is a monotonic function of the idiosyncratic shock (z_{ijo}), whch has a Frechet distribution, it follows that indirect utility for workers living in block i and working in block j also has a Frechet distribution. Each worker chooses the bilateral commute that offers her the maximum utility, where the maximum of Frechet distributed random variables is itself Frechet distributed. Using these distributions of utility, the probability that a worker choose to live in block i and work in block j is:

$$\pi_{ij} = \frac{T_i E_j (d_{ij} Q_i^{1-\beta})^{-\varepsilon} (B_i w_j)^{\varepsilon}}{\sum_{r=1}^{S} \sum_{s=1}^{S} T_r E_s (d_{rs} Q_r^{1-\beta})^{-\varepsilon} (B_r w_s)^{\varepsilon}} \equiv \frac{\Phi_{ij}}{\Phi}. \tag{3.4}$$

Summing these probabilities across workplaces for a given residence, we obtain the overall probability that a worker resides in block $i(\pi_{Ri})$, while summing these probabilities across residences for a given workplace, we obtain the overall probability that a worker works in block $j(\pi_{Mi})$:

$$\pi_{Ri} = \sum_{j=1}^{s} \pi_{ij} = \frac{\sum_{j=1}^{s} \Phi_{ij}}{\Phi}, \pi_{Mj} = \sum_{i=1}^{S} \pi_{ij} = \frac{\sum_{i=1}^{S} \Phi_{ij}}{\Phi} \tag{3.5}$$

These residential and workplace choice probabilities have an intuitive interpretation. The idiosyncratic shock to preferences z_{ijo} implies that individual workers choose different bilateral commutes when faced with the same prices $\{Q_i, w_j\}$, commuting costs $\{d_{ij}\}$ and location characteristics $\{B_i, T_i, E_j\}$. Other things equal, the more attractive its amenities B_i, the higher its average idiosyncratic utility determined by T_i, the lower its residential floor prices Q_i, and the lower its commuting costs d_{ij} to employment locations.

Conditional on living in block i, the probability that a worker commutes to block j is:

$$\pi_{ij|i} = \frac{E_j (w_j/d_{ij})^{\varepsilon}}{\sum_{s=1}^{S} E_s (w_s/d_{is})^{\varepsilon}} \tag{3.6}$$

where the terms in $\{Q_i, T_i, B_i\}$ have cancelled from the numerator and denominator. Therefore, the probability of commuting to block j conditional on living in block i depends on wage (w_j), average utility draw (E_j), and commuting costs (d_{ij}) of employment location j in the numerator as well as teh wage (w_s), average utility draw (E_s) and commuting costs (d_{is}) for all other possible employment locations s in the denominator.

Using the conditional commuting probabilities, we obtain the following commuting market clearing condition that equates the measure of worker employed in block $j(H_{M_j})$ with the measure of workers choosing to commute to block j:

$$H_{M_j} = \sum_{s=1}^{S} \frac{E_j (w_j/d_{ij})^{\varepsilon}}{\sum_{s=1}^{S} E_s (w_s/d_{is})^{\varepsilon}} H_{R_i} \tag{3.7}$$

where H_{R_i} is the measure of residents in block i. Since there is a continuous measure of workers residing in each location, there is no uncertainty in the supply of workers to each employment location.

Expected worker income conditional on living in block i is equal to the wages in all possible employment location weighted by the probabilities of commuting to choose those locations conditional on living in i:

$$E[w_j \mid i] = \sum_{j=1}^{S} \frac{E_j(w_j/d_{ij})^{\varepsilon}}{\sum_{s=1}^{S} E_s(w_s/d_{is})^{\varepsilon}} w_j \tag{3.8}$$

Finally, population mobility implies that the expected utility from moving to the city is equal to the reservation level of utility in the wider economy $(\tilde{U})$:

$$E[u] = \gamma[\sum_{r=1}^{S}\sum_{s=1}^{S} T_r E_s(d_{rs}Q_r^{1-\beta})^{-\varepsilon}(B_r w_s)^{\varepsilon}]^{1/\varepsilon} = \tilde{U} \tag{3.9}$$

where E is the expectation operator and expectation is taken over the distribution for the idiosyncratic component of utility: $\gamma = \Gamma(\frac{\varepsilon-1}{\varepsilon})$ and $\Gamma(\cdot)$ is the Gamma function.

3.1.2 Production

Production of the tradable final good occurs under conditions of perfectly competition and constant returns to scale. We can assume the production technology takes the CD-form

$$y_j = A_j H_{M_j}^{\alpha} L_{M_j}^{1-\alpha} \tag{3.10}$$

where A_j is final goods productivity and L_{M_j} is the measure of floor space used commercially. Firms choose their block of production and their inputs of workers and commercial floor space to maximize profits, taking as given final goods productivity A_j, the distribution of idiosyncratic utility, goods and factor prices, and the location decisions of other firms and workers. Profit maximization implies that equlibrium employment in block j is increasing in productivity (A_j), decreasing in wage (w_j), and increasing in commercial floor space (L_{M_j}).

$$H_{M_j} = (\frac{\alpha A_j}{w_j})^{1/(1-\alpha)} L_{M_j} \tag{3.11}$$

where the equilibrium wage is determined by the requirement that the demand for workers in each employment location equals the supply of workers to that location.

From the first order conditions for maximization and zero profit, equilibrium commercial floor prices (q_j) in each block with positive employment must satisfy:

$$q_j = (1-\alpha)(\frac{\alpha}{w_j})^{\alpha/(1-\alpha)} A_j^{1/(1-\alpha)} L_{M_j}^{-1/(1-\alpha)} \tag{3.12}$$

3.1.3 Land Market Cleaning

Land market equilibrium requires no-arbitrage conditions between the commercial and residential use of floor space after the tax equivalence of land use regulations. The share of floor space used commercially (θ_i) is:

$$\begin{aligned} &\theta_i = 1 \text{ if } q_i > \xi_i Q_i \\ &\theta_i \in [0,1] \text{ if } q_i = \xi_i Q_i \theta_i \quad = 0 \text{ if } q_i < \xi_i Q_i \end{aligned} \tag{3.13}$$

where $\xi_i \geq 1$ captures one plus the tax equivalent of land use regulations that restrict commercial land use relative to residential land use. We assume that the observed price of floor spaces in the data is the maximum of the commercial and residential price of floor space: $\mathbb{Q} = \max\{q_i, Q_i\}$. Hence the relationship between observed, commercial, and residential floor prices can be summarized as:

$$\begin{aligned} \mathbb{Q}_i &= q_i, q_i > \xi_i Q_i, \theta_i = 1 \\ \mathbb{Q}_i &= q_i, q_i = \xi_i Q_i, \theta_i \in [0, 1] \\ \mathbb{Q}_i &= Q_i, q_i < \xi_i Q_i, \theta_i = 0 \end{aligned} \tag{3.14}$$

We follow the standard approach in the urban literature of assuming that floor space L is supplied by a competitive construction sector that uses land K and capital M a inputs. We assume that the production function takes the CD-form: $L_i = M_i^{\mu} K_i^{1-\mu}$. Therefore, the corresponding dual cost function for floor space is $\mathbb{Q}_i = \mu^{-\mu}(1-\mu)^{-(1-\mu)}\mathbb{P}^{\mu}\mathbb{R}_i^{1-\mu}$, where $\mathbb{Q}_i = \max\{q_i, Q_i\}$ is the price for floor space, $\mathbb{P}$ is the common price for capital across all blocks, and $\mathbb{R}_i$ is the price for land. Since the price for capital is the same across all locations, the relationship between the quantities and prices of floor space and land can be summarized as:

$$L_i = \phi_i K_i^{1-\mu} \tag{3.15}$$

$$\mathbb{Q}_i = \chi \mathbb{R}_i^{1-\mu} \tag{3.16}$$

where we refer to $\phi_i = M_i^{\mu}$ as the density of development and χ is a constant. Residential land market clearing implies that the demand for residential floor space equals the supply of floor space allocated to residential use in each location: $(1-\theta_i)L_i$. Maximization for each worker and taking expectation over the distribution for idiosyncratic utility, the residential land market clearing condition can be expressed as:

$$\mathbb{E}[\ell_i]H_{R_i} = (1-\beta)\frac{\mathbb{E}[w_s \mid i]H_{R_i}}{Q_i} = (1-\theta_i)L_i. \tag{3.17}$$

Commercial land market cleaning requires that the demand for commercial floor space equals the supply of floor space allocated to commercial use in each location: $\theta_j L_j$. Using the FOC for profit maximization, the commercial land market clearing condition can be expressed as:

$$\left(\frac{(1-\alpha)A_j}{q_j}\right)^{1/\alpha} H_{M_j} = \theta_j L_j \tag{3.18}$$

We both residential and commercial land market clearing are satisfied, total demand for floor space equals the total supply of floor space:

$$(1-\theta_i)L_i + \theta_i L_i = L_i = \phi_i K_i^{1-\mu} \tag{3.19}$$

3.1.4 General Equilibrium With Exogenous Location characteristics

Given the model's parameters $\{\alpha, \beta, \mu, \varepsilon, \kappa\}$, the reservation level of utility in the wider economy $\overline{U}$, and vectors of exogenous location characteristics $\{T, E, A, B, \phi, K, \xi, \tau\}$, the general equilibrium is referred with six vectors $\{\pi_M, \pi_R, Q, q, w, \theta\}$ and total city population H.

Proposition 3.1. *Assuming exogenous finite, and strictly location characteristics* ($T_i \in (0,\infty)$, $E_i \in (0,\infty)$, $\phi_i \in (0,\infty)$, $K_i \in (0,\infty)$, $\xi_i \in (0,\infty)$, $\tau_{ij} \in (0,\infty) \times (0,\infty)$)*, and exogenous, finite and nonnegative final goods productivity* $A_i \in [0,\infty)$ *and residential amnetities* $B_i \in [0,\infty)$*, there exists a unique general equilibrium vector* $\{\pi_M, \pi_R, Q, q, w, \theta\}$

Proof. The proof can be decomposed into two parts: firstly, we show that under the assumptions that all blocks have strictly positive, finite, and exogenous location and we allow some blocks to be more attractive than others in terms of these characteristics. But workers draw idiosyncratic preferences from a Frechet distribution for pairs of residence and workplace locations, and therefore, since teh Frechet distribution is unbounded from above, any block with strictly positive characteristics has a positive measure of workers that prefer that location as a residence or workplace at a positive and finite price. Hence, all blocks with finite positive wages attract a positive measure of workers, and all blocks with finite positive floor prices attract a positive measure of residents.

Then We next show that blocks with strictly positive, finite, and exogenous location characteristics must have strictly positive and finite values of both wages and floor prices in equilibrium. □

3.1.5 Introducing Agglomenration Forces

We now introduce endogenous agglomenration forces. We allow final goods productivity to depend on production fundamentals (a_j) and production externalities (Y_j). Production fundamentals capture features of physical geography that make a location more or less productive independently of the surrounding density of economic activity. Production externalities impose structure on how the productivity of a given block is affected by the characteristics of other blocks. Specifically, we follow the standard approach in urban economics of modeling these externalities as depending on the travel-time weighted sum of workplace employment density in surrounding blocks:

$$A_j = a_j Y_j^{\lambda}, Y_j \equiv \sum_{s=1}^{S} e^{-\delta\tau_{js}}(\frac{H_{M_s}}{K_s}), \tag{3.20}$$

where H_{Ms}/K_s is workplace employment density per unit of land area; production externalities decline with travel time (τ_{js}) through the iceberg factor $e^{-\delta\tau_{js})} \in (0,1]$; δ determines their rate of spatial decay, and λ controls their relative importance in determining overall productivity.

We model the externalities in workers' residential choices analygously to the externalities in firms' production choices. We allow residential amenities to depend on residential fundamentals (b_i) and residential externalities (Ω_i). Residential fundamentals capture features of physical geography that make a location a more or less attractive place to live independently of the surrounding density of economic activity. Residential externalities again impose structure on how the amenities in a given block are affected by the characteristics of other blocks. Specifically, we adopt a symmetric specification as for production externalities, and model residential externalities as depending on the travel time weighted sum of residential employment density in surrounding blocks:

$$B_i = b_i \Omega_i^{\eta}, \Omega_i \equiv \sum_{r=1}^{S} e^{-\rho\tau_{ir}}(\frac{H_{Rr}}{K_r}) \tag{3.21}$$

where H_{Rr}/K_r is residence employment density per unit of land area; residential externalities decline with travel time (τ_{ir}) through the iceberg factor $e^{-\rho\tau_{ir}} \in (0,1]$; ρ determines their rate of spatial decay; and η controls their relative importance in overall residential amenities. The parameter η captures the net effect of residence employment density on amenities, including negative spillover such as air pollution and crime, and positive externalities through the availability of urban amenities. Although η captures the direct effect of higher residence employment density on utility through amenities, there are clearly other general equilibrium effects through floor prices, commuting times and wages.

3.1.6 Recovering Location Characteristics

We now show that there is a unique mapping from the observed variables to unobserved location characteristics. Since a number of these unobserved variables enter the model isomorphically, we define the following composites denoted bya tilde:

$$\begin{aligned}
\tilde{A}_i &= A_i E_i^{\alpha/\varepsilon}, \tilde{a}_i = a_i E_i^{\alpha/\varepsilon} \\
\tilde{B}_i &= B_i T_i^{1/\varepsilon} \zeta_{Ri}^{1-\beta}, \tilde{b}_i = b_i T_i^{1/\varepsilon} \zeta_{Ri}^{1-\beta} \\
\tilde{w}_i &= w_i E_i^{1/\varepsilon} \\
\tilde{\phi}_i &= \tilde{\phi}_i(\phi_i, E_i^{1/\varepsilon}, \xi_i)
\end{aligned}$$

where we use i to index all blocks, and the function $\tilde{\phi}_i(\cdot)$ is a defined function; $\zeta_{Ri} = 1$ for completely specialized residential blocks; and $\zeta_{Ri} = \zeta_i$ for residential blocks with some commercial land use.

In the labor market, the adjusted wage for each employment location $(\tilde{w}_i)$ captures the wage (w_i) and the Frechet scale parameter for the location $(E_i^{1/\varepsilon})$, because these both affect the relative attractiveness of an employment location to workers. On the production side, adjusted productivity for each employment location $(\tilde{A}_i)$ captures productivity (A_i) and the Frechet scale parameter for the location $(E_i^{\alpha/\varepsilon})$ because these both affect the adjusted wage consistent with zero profits. Adjusted production fundamentals are defined analogously. On the consumption side, adjusted amenities for each residence location $(\tilde{B}_i)$ capture amenities (B_i), the Frechet scale parameter for that location $(T_i^{1/]\varepsilon})$ and the relationship between observed and residential floor prices $(\zeta_{Ri} \in \{1, \zeta_i\})$, because these all affect the relative attractiveness of a location consistent with population mobility. Adjusted residential fundamentals are defined analogously. Finally, in the land market, the adjusted density of development $(\tilde{\phi}_i)$ includes the density of development (ϕ_i) and other production and residential parameters that affect land market clearing.

Proposition 3.2. 1. *Given known values for the parameters* $\{\alpha, \beta, \mu, \varepsilon, \kappa\}$ *and the observed data* $\{\mathbb{Q}, \mathbf{H_M}, \mathbf{H_R}, \mathbf{K}, \tau\}$ *there exists unique vectors of the unobserved location characteristics* $\{\tilde{\mathbf{A}}^*, \tilde{\mathbf{B}}^*, \tilde{\phi}^*\}$ *that are consistent with the data being an equilibrium of the model.*

2. *Given known values for the parameters* $\{\alpha, \beta, \mu, \varepsilon, \kappa, \lambda, \delta, \eta, \rho\}$ *and the observed data* $\{\mathbb{Q}, \mathbf{H_M}, \mathbf{H_R}, \mathbf{K}, \tau\}$ *there exists a unique vectors of the unobserved location characteristics* $\{\tilde{\mathbf{a}}^*, \tilde{\mathbf{b}}^*, \tilde{\phi}^*\}$ *that are consistent with the data being an equilibrium of the model.*

Chapter 4

THE MAKING OF THE MODERN METROPOLIS: EVIDENCE FROM LONDON

4.1 Theoretical Framework

We consider a city embedded in a wider economy (Great Britain). The economy as a whole consists of a discrete set of locations $\mathbb{M}$. Greater London is a subset of these locations $\mathbb{N} \subset \mathbb{M}$, Time is discrete and is indexed by t. The economy as a whole is populated by an exogenous continuous measure $L_{\mathbb{M}t}$ of workers, who are geographically mobile and wndowed with one unit of labor that is supplied inelastically. Workers simultaneously choose their preferred residence n and workplace i given their idiosyncratic draws. We denote the endogenous measure of workers who choose a residence-workplace pair in Greater London by $L_{\mathbb{N}t}$. We allow locations to differ from one another in terms of their attractiveness for production and residence, as determined by productivity, amenities, the supply of floor space, and transport connections, where each of these location characteristics can evolve over time.

4.1.1 Preferences

We assume that preferences take the CD-form, such that the indirect utility for a worker ω residing in n and working in i is:

$$U_{ni}(\omega) = \frac{B_{ni} b_{ni}(\omega) w_i}{\kappa_{ni} P_n^{\alpha} Q_n^{1-\alpha}}, 0 < \alpha < 1 \tag{4.1}$$

where we suppress the time subscript from now on; P_n is the price index for consumption goods, which may include both tradeable and nontradeable consumption goods; Q_n is the price of residential floor space; w_i is the wage, κ_{ni} is an iceberg commuting cost; B_{ni} captures amenities from the bilateral commute from residence n to workplace i that are common across all workers; and $b_{ni}(\omega)$ is an idiosyncratic amenity draw that captures all the idiosyncratic factors that can cause an individual to live and work in particular locations in the city.

We assume that idiosyncratic amenities $(b_{ni}(\omega))$ are drawn from an independent extreme value (Frechet) distribution for each residence-workplace pair and each worker:

$$G(b) = e^{-b^{-\varepsilon}}, \varepsilon > 1 \tag{4.2}$$

where we normalize the Frechet scale parameter in Equation (4.2) to 1 because it enters worker choice probabilities isomorphically to common bilateral amenities B_{ni}. The Frechet shape parameter ε regulates the dispersion of idiosyncratic amenities, which controls the sensitivity of worker location decisions to economic variables. The smaller the shape parameter ε, the greater the heterogeneity in idiosyncratic amenities, and the less sensitive are worker location decisions to economic variables.

We decompose the bilateral common amenities parameter (B_{ni}) into a residence component common across all workplaces $(B_n^{\mathcal{R}})$, a workplace component common across all residences (B_i^L), and an idiosyncratic component (B_{ni}^I) specific to an individual residence-workplace pair:

$$B_{ni} = B_n^{\mathcal{R}} B_i^L B_{ni}^I, \quad B_n^{\mathcal{R}}, B_i^L, B_{ni}^I > 0 \tag{4.3}$$

We allow the levels of $B_n^{\mathcal{R}}$, B_i^I and B_{ni}^I to differ across residences n and workplace i, although when we examine the impact of the construction of railway network, we assume that B_i^L and B_{ni}^I are time-invariant. In contrast, we allow $B_n^{\mathcal{R}}$ to change over time, and for those changes to be potentially endogenous to the evolution of the surrounding concentration of economic activity through agglomeration forces.

Conditional on choosing a residence-workplace pair in Greater London, we know that the probability a worker chooses to reside in location $n \in \mathbb{N}$ and work in location $i \in \mathbb{N}$ is given by:

$$\begin{aligned} \lambda_{ni} &= \frac{L_{ni}}{L_{\mathbb{M}}} \frac{L_{\mathbb{M}}}{L_{\mathbb{N}}} = \frac{L_{ni}}{L_{\mathbb{N}}} \\ &= \frac{(B_{ni} w_i)^{\varepsilon} (\kappa_{ni} P_n^{\alpha} Q_n^{1-\alpha})^{-\varepsilon}}{\sum_{k \in \mathbb{N}} \sum_{\ell \in \mathbb{N}} (B_{k\ell} w_{\ell})^{\varepsilon} (\kappa_{k\ell} P_k^{\alpha} Q_k^{1-\alpha})^{-\varepsilon}}, n, i \in \mathbb{N} \end{aligned} \tag{4.4}$$

where L_{ni} is the measure of commuters from n to i.

The probability of commuting between residence n and workplace i depends on the characteristics of that residence n, the attributes of that workplace i and bilateral commuting costs and amenities. Summing across workplaces $i \in \mathbb{N}$, we obtain the probability that a worker lives in residence $n \in \mathbb{N}$, conditional on choosing a residence-workplace pair in Greater London $(\lambda_n^R = \frac{R_n}{L_{\mathbb{N}}})$. Similarly, summing across residences $n \in \mathbb{N}$, we obtain the probability that a worker is employed in workplace $i \in \mathbb{N}$, conditional on choosing a residence-workplace pair in Greater London $(\lambda_i^L = \frac{L_i}{L_{\mathbb{N}}})$

$$\begin{aligned} \lambda_n^R &= \frac{\sum_{i \in \mathbb{N}} (B_{ni} w_i)^{\varepsilon} (\kappa_{ni} P_n^{\alpha} Q_n^{1-\alpha})^{-\varepsilon}}{\sum_{k \in \mathbb{N}} \sum_{\ell \in \mathbb{N}} (B_{k\ell} w_{\ell})^{\varepsilon} (\kappa_{k\ell} P_k^{\alpha} Q_k^{1-\alpha})^{-\varepsilon}} \\ \lambda_i^L &= \frac{\sum_{n \in \mathbb{N}} (B_{ni} w_i)^{\varepsilon} (\kappa_{ni} P_n^{\alpha} Q_n^{1-\alpha})^{-\varepsilon}}{\sum_{k \in \mathbb{N}} \sum_{\ell \in \mathbb{N}} (B_{k\ell} w_{\ell})^{\varepsilon} (\kappa_{k\ell} P_k^{\alpha} Q_k^{1-\alpha})^{-\varepsilon}} \end{aligned} \tag{4.5}$$

where R_n denotes employment by residence in location n and L_i denotes employment by workplace in location i. A second implication of our extreme value specification is that expected utility conditional on choosing a residence workplace pair $(\overline{U})$ is the same across all residence-workplace pairs in the economy:

$$\overline{U} = v \left[\sum_{k \in \mathbb{M}} \sum_{\ell \in \mathbb{M}} (B_{k\ell} w_{k\ell})^{\varepsilon} (\kappa_{k\ell} P_k^{\alpha} Q_k^{1-\alpha})^{-\varepsilon} \right]^{\frac{1}{\varepsilon}} \tag{4.6}$$

where the expectation is taken over the distribution for idiosyncratic amenities; $v \equiv \Gamma(\frac{\varepsilon-1}{\varepsilon})$; $\Gamma(\cdot)$ is the gamma function. Using the probability that a worker chooses a residence-workplace pair in Greater London $(\frac{L_{\mathbb{N}}}{L_{\mathbb{M}}})$, we can rewrite this probability mobility condition as:

$$\overline{U} (\frac{L_{\mathbb{N}}}{L_{\mathbb{M}}})^{\frac{1}{\varepsilon}} = v \left[\sum_{k \in \mathbb{N}} \sum_{\ell \in \mathbb{N}} (B_{k\ell} w_{k\ell})^{\varepsilon} (\kappa_{k\ell} P_k^{\alpha} Q_k^{1-\alpha})^{-\varepsilon} \right]^{\frac{1}{\varepsilon}} \tag{4.7}$$

where only the limits of the summations differ on the right hand sides of the equations.

Intuitively, for a given common level of expected utility in the economy $(\overline{U})$, locations in Greater London must offer higher real wages adjusted for common amenities (B_{ni}) and commuting costs (κ_{ni}) to attract workers with lower idiosyncratic draws with an elasticity determined by the parameter ε.

4.1.2 Production

We assume that consumption goods are produced according to a Cobb-Douglas technology using labor, machinery capital, and commercial floor space, where commercial floor space includes both building capital and land. Cost minimization and zero profits imply that payments for labor, commercial floor space, and machinery are constant shares of revenue (X_i):

$$w_i L_i = \beta^L X_i, q_i H_i^L = \beta^H X_i, r M_i = \beta^M X_i, \beta^L + \beta^H + \beta^M = 1 \tag{4.8}$$

where q_i is the price of commercial floor space; H_i^L is commercial floor space use; M_i is machinery use; and machinery is assumed to be perfectly mobile across locations with a common price r determined in the wider economy. We allow the price of commercial floor space (q_i) to potentially depart from the price of residential floor space (Q_i) in each location i through a location-specific wedge (ξ_i):

$$q_i = \xi_i Q_i. \tag{4.9}$$

From the relationship between factor payments and revenue in equation, payments for commercial floor space are proportional to workplace income ($w_i L_i$):

$$q_i H_i^L = \frac{\beta^H}{\beta^L} w_i L_i \tag{4.10}$$

4.1.3 Commuter Market Clearing

commuter market clearing implies that the measure of workers employed in each location (L_i) equals the measure of workers chooosing to commute to that location:

$$L_i = \sum_{n \in \mathbb{N}} \lambda_{ni|n}^R R_n \tag{4.11}$$

where $\lambda_{ni|n}^R$ is the probability of commuting to workplace i conditional on living in residence n:

$$\lambda_{ni|n}^R = \frac{\left(\frac{B_{ni} w_i}{\kappa_{ni}}\right)^{\varepsilon}}{\sum_{\ell \in \mathbb{N}} \left(\frac{B_{n\ell} w_\ell}{\kappa_{n\ell}}\right)^{\varepsilon}} \tag{4.12}$$

where all characteristics of residence n have canceled from the above equation because they do not vary across workplaces for a given residence.

Commuter market clearing also implies that per capita income by residence (v_n) is a weighted average of the wages in all locations, where the weights are these conditional commuting probabilities by residences ($\lambda_{ni|n}^R$):

$$v_n = \sum_{i \in \mathbb{N}} \lambda_{ni|n}^R w_i. \tag{4.13}$$

4.1.4 Land Market Clearing

We assume that floor space is owned by landlords, who receive payments from the residential and commercial use of floor space and consume only consumption goods. Land market clearing implies that total income from the ownership of floor space equals the sum of payments for residential and commercial floor space use:

$$\mathbb{Q}_n = Q_n H_n^R q_n H_n^L = (1-\alpha) \left[\sum_{i \in \mathbb{N}} \lambda_{ni|n}^R w_i\right] R_n + \frac{\beta^H}{\beta^L} w_n L_n \tag{4.14}$$

where H_n^R is residential floor space use; rateable values $(\mathbb{Q}_n)$ equals the sum of prices times quantities for residential floor space $(Q_n H_n^R)$ and commercial floor space $(q_n H_n^L)$; and we have used the expression for per capita income by residence (v_n) from commuter market clearing.

From the combined land and commuter market-clearing condition, payments for residential floor space are a constant multiple of residence income $(v_n R_n)$, and payments for commercial floor space are a constant multiple of workplace income $(w_n L_n)$. Importantly, we allow the supplies of residential floor space (H_n^R) and commercial floor space (H_n^L) to be endogenous, and we allow the prices of residential and commercial floor space to potentially differ from one another through the location-specific wedge $\xi_i (q_i = \xi_i Q_i)$. In our baseline quantitative analysis below, we are not required to make assumptions about these supplies of residential and commercial floor space or this wedge between commercial and residential floor prices. The reason is that we condition on the observed rateable values in the data $(\mathbb{Q}_n)$ and the supplies and prices for residential and commercial floor space (H_n^R, H_n^L, Q_n, q_n) only after the land market-clearing condition.

4.2 Quantitative Analysis

4.2.1 Combined Land and Commuter Market Clearing

We evaluate the effect of changes in the transport network by using an "exact hat algebra" approach. In particular, we rewrite our combined land and commuter market clearing condition for another year $\tau \neq t$ in terms of the values of variables in a baseline year t and the relative changes of variables between years τ and t:

$$\hat{\mathbb{Q}}_{nt} \mathbb{Q}_{nt} = (1-\alpha) \hat{v}_{nt} v_{nt} \hat{R}_{nt} R_{nt} + \frac{\beta^H}{\beta^L} \hat{w}_{nt} w_{nt} \hat{L}_{nt} L_{nt} \tag{4.15}$$

where $\hat{x}_{nt} = \frac{x_{n\tau}}{x_{nt}}$ for the variable x_{nt} and we now make explicit the time subscripts. The relative change in employment $(\hat{L}_{it})$ and the relative change in average per capita income by residence $(\hat{v}_{nt})$ for year τ can be expressed as:

$$\hat{L}_{it} L_{it} = \sum_{n \in \mathbb{N}} \frac{\lambda^R_{nit|n} \hat{w}^{\varepsilon}_{it} \hat{\kappa}^{-\varepsilon}_{nit}}{\sum_{\ell \in \mathbb{N}} \lambda^R_{n\ell t|n} \hat{w}^{\varepsilon}_{\ell t} \hat{\kappa}^{-\varepsilon}_{n\ell t}} \hat{R}_{nt} R_{nt} \tag{4.16}$$

$$\hat{v}_{nt} v_{nt} = \sum_{i \in \mathbb{N}} \frac{\lambda^R_{nit|n} \hat{w}_{it}}{\sum_{\ell \in \mathbb{N}} \lambda^R_{n\ell t|n} \hat{w}^{\varepsilon}_{\ell t} \hat{\kappa}^{-\varepsilon}_{n\ell t}} \hat{w}_{it} w_{it} \tag{4.17}$$

where these equations include terms in change in wages $(\hat{w}_n)$ and commuting costs $(\hat{\kappa}_{ni})$ but not in amenities, because we assume that the workplace and bilateral components of amenities are constant $(\hat{B}^L_{it} = 1$ and $\hat{B}^I_{nit} = 1)$, and changes in the residential component of amenities $(\hat{B}^R_{nt} \neq 1)$ cancel from the numerator and denominator of the fractions.

substituting the expressions to the market clearing conditions for year τ we can get the result.

Lemma 4.1. *Suppose that $(\hat{\mathbb{Q}}_{nt}, \hat{R}_{nt}, L_{nit}, \lambda^R_{nit|n}, \mathbb{Q}_{nt}, v_{nt}, R_{nt}, w_{nt}, L_{nt})$ are known. Given known values for model parameters $\{\alpha, \beta^L, \beta^H, \varepsilon\}$ and the change in bilateral commuting costs $(\hat{\kappa}^{-\varepsilon}_{nit})$, the combined land and commuter market clearing condition determines a unique vector of relative changes in wages $(\hat{w}_{it})$ in each location.*

Chapter 5

Urban Diversity, Process Innovation, and the Life Cycle of Products

5.1 The model

There are N cities in the economy, where N is endogenous, and a continuum L of infinitely lived workers, each of which has one of m possible discrete aptitudes. There are equal proportion of workers with each aptitude in the economy, but their distribution across cities is endogenously determined through migration. Let us index cities by i and worker aptitudes by subcript j so that l_i^j denotes the supply of labor with aptitude j in city i. Time is discrete and indexed by t.

5.1.1 Technology

The ideal production process is firm specific and randomly drawn from a set of m possible discrete processes, with equal probability for each. Each of the m possible processes for each firm requires process specific intermediate inputs from a local sector employing workers of a specific aptitude. We say that two production processes for different firms are of the same type if they require intermediates produced using workers with the same aptitude.

A newly created firm does not know its ideal production process, but it can find this by trying, one at a time, different processes in the production of prototypes. After producing a prototype with a certain process, the firm knows whether this process is its ideal one or not. Thus in order to switch from prototype to mass production a firm needs to have produced a prototype with its ideal process first, or to have tried all of its m possible processes except one. Furthermore, we allow for the possibility that a firm decides to stop searching before learning its ideal process. Firms have an exogenous probability δ of closing down each period. Firms also lose one period of production whenever they relocate from one city to another. Thus, the cost of firm relocation increases with the exogenous probability of closure δ.

THe intermediates specific to each type of process are produced by a monopolistically competitive intermediate sector, each such intermediate sector hires workers of aptitude j and sells process-specific nontradable intermediate services to final-good firms using a process of type j. These differentiated services enter the production function of final good producers with the same constant elasticity of substitution $\frac{\varepsilon+1}{\varepsilon}$. The production is:

$$\overset{?}{C}{}_i^j(h) = Q_i^j \overset{?}{x}{}_i^j(h) \tag{5.1}$$

$$\text{where } Q_i^j = (l_i^j)^{-\varepsilon} w_i^j, \varepsilon > 0 \tag{5.2}$$

We distinguish variables corresponding to prototypes from those corresponding to mass-produced goods by an accent in the form of a question mark, ?. INdexing the differentiated varieties of goods by h, we denote ouput of prototype h made with a process of type j in city i by $\overset{?}{x}{}_i^j(h)$. Q_i^j is the unit cost for firms producing prototypes using a process of type j in city i and w_i^j is the wage per unit of labor for the corresponding workers. Note that Q_i^j decreases as l_i^j increases: there are localization economies that reduce unit costs when there is a larger supply of labor with the relevant aptitude in the same city.

When a firm finds its ideal production process, it can engage in mass production at a fraction ρ of the cost of producing a prototype, where $0 < \rho < 1$. Thus the cost function for a firm engaged in mass production is:

$$C_i^j(h) = \rho Q_i^j x_i^j(h) \tag{5.3}$$

where $x_i^j(h)$ denotes the ouput of mass produced good h, made with a process of type j, in city i.

With respect to the internal structure of cities, there are congestion costs in each city incurred in labor time and parameterized by $\tau(> 0)$. Labor supply, l_i^j, and production, L_i^j, with aptitude j in city i are related by the following expression:

$$l_i^j = L_i^j(1 - \tau \sum_{j=1}^{m} L_i^j). \tag{5.4}$$

THe expected wage income of a worker with aptitude j in city i is then $(1 - \tau \sum_{j=1}^{m} L_i^j) w_i^j$, where the higher land rents pair by those living closer to the city center are offset by lower commuting costs.

5.1.2 Preferences

Each period consumers allocate a fraction μ of their expenditure to prototypes and a fraction of $1 - \mu$ to mass-produced goods. The instantaneous indirect utility of a consumer in city i is:

$$V_i = \overset{?}{P}{}^{-\mu} P^{-(1-\mu)} e_i^j \tag{5.5}$$

where e_i denotes individual expenditure.

$$\overset{?}{P} = \left\{ \sum_{j=1}^{m} \int \int [\overset{?}{P}{}_i^j(h)]^{1-\sigma} dh di \right\}^{1/(1-\sigma)} \tag{5.6}$$

$$P = \left\{ \sum_{j=1}^{m} \int \int [p_i^j(h)]^{1-\sigma} dh di \right\}^{1/(1-\sigma)} \tag{5.7}$$

and the appropriate price indices of prototypes and mass-produced goods respectively, and $\overset{?}{P}{}_i^j(h)$ and $p_i^j(h)$ denote the price of individual varieties of prototypes and mass-produced goods respectively. All prototypes enter consumer preferences with the same elasticity of substitution $\sigma(> 2)$, and so do all mass produced goods.

5.1.3 Income and Migration

National income, Y, is the sum of expenditure and investment:

$$Y = \sum_{j=1}^{m} \int L_i^j e_i^j di + \overset{?}{P}{}^{\mu} P^{1-\mu} F \overset{\circ}{n} \tag{5.8}$$

L_i^j denotes population with aptitude j in city i. Investment $\overset{?}{P}^{\mu} P^{1-\mu} F \overset{\circ}{n}$ comes from aggregation of the start-up costs incurred by newly created firms. To come up with a new product, the firm must spend F on market research, purchasing the same combination of goods bought by the representative consumer. Finally, $\overset{\circ}{n}$ denotes the total number of new firms.

Definition 5.1 (Specialized City)**.** A city is said to be fully specialzied if all its workers have the same aptitude, so that all local firms use the same type of production process.

Definition 5.2 (Diversified City)**.** A city is said to be (fully) diversified if it has the same proportion of workers with each of the m aptitudes, so that there are equal proportions of firms using each of the m types of production process.

5.1.4 City Formation

Each potential site for a city is controlled by a different land development company or land developer, not all of which will be active in equilibrium. Developers have the ability to tax local land rents and to make transfers to local workers. When active, each land developer commits to a contract with any potential worker in its city that specifies the size of the city, whether it has a dominant sector and if so which, and any transfers.

5.1.5 Equilibrium Definition

Finally, a steady state equilibrium in this model is a configuration such that all of the following are true. Each developer offers a contract designed so as to maximize its profits. Each firm chooses a location/production strategy and prices so as to maximize its expected lifetime profits. All profit opportunities are exploited and then urban structure is constant over time.

5.2 Equilibrium City Sizes

Lemma 5.1 (Output per Worker)**.** *In equilibrium, output per worker by firms using processes of type j in city i in a given period is:*

$$\frac{\overset{?}{n}{}_i^j \overset{?}{x}{}_i^j + \rho n_i^j x_i^j}{L_i^j} = (L_i^j)^{\varepsilon}(1 - \tau \sum_{j=1}^{m} L_i^j)^{\varepsilon+1}.$$

Chapter 6

Spatial Sorting and Inequality

6.1 Change in SKill Sourting: Framework

6.1.1 Setup

On the production side, rather than modelling imperfect trade between locations, we consider an economy that is more stylized spatially, with two types of goods: (a) a homogeneous manufactured good that is freely traded across space and (b) housing, a local nontraded good.

Preferences

Consider a spatial equilibrium framework with two skill groups $\theta = U, S$, who choose where to live among locations $i \in [1, \dots, N]$. Aggregate skill supply for each group, L^θ, is exogenously given, and each worker supplies one unit of labor for wage w_i^θ in location i. The utility of worker w, who is type θ and lives in location i, is:

$$u_i^\theta(w) = \max_{c,b} \log U^\theta(A_i, c, b) + \varepsilon_i^\theta(w), \text{ such that } c + r_i b = w_i^\theta \tag{6.1}$$

Here $\log U^\theta(\cdot)$ is the representative utility of a worker of type θ; c is the consumption of the freely traded good and is taken as the numeraire; b denote housing, with price r_i in location i; and A_i is a vector of amenities in location i. Finally, $\varepsilon_i^\theta(w)$ is a worker-specific preference shock for living in location i.

First, we make assumption of CD type preferences over traded and nontraded goods. Second, we assume that amenities are separable from consumption. We allow amenities in location i to be valued differently by the two groups, as caputured by a group-specific amenity level A_i^θ. Third, preference shocks are typically chosen to be extreme value (EV) distributed. Papers in the tradition of urban and labor economics or industrial organization tend to use logit shocks, with normalized variance $\frac{\pi^2}{6}$ shifted by a factor $\frac{1}{\kappa^\theta)}$, which together with CD utility lead to the following indirect utility of worker θ in location i:

$$v_i^\theta(w) = \log A_i^\theta + \log w_i^\theta - \alpha^\theta \log r_i + \frac{1}{\kappa^\theta} \varepsilon_i^\theta(w)$$

Equivalently, papers in the tradition of trade and economic geography typically choose Frechet shocks for $\varepsilon_i^\theta(w)$ with scale parameter $\kappa^\theta > 1$ that enter utility in a multiplicatively separable way. In that case, the indirect utility of worker θ in location i is:

$$v_i^\theta(w) = \frac{A_i^\theta w_i^\theta}{r_i^{\alpha^\theta}} \varepsilon_i^\theta(w).$$

In either case, location choices in group θ can be summarized with λ_i^θ, the share of θ workers who choose location i:

$$\lambda_i^\theta = \frac{(\frac{A_i^\theta w_i^\theta}{r_i^{\alpha^\theta}})^{\kappa^\theta}}{\sum_{j=1}^N (\frac{A_j^\theta w_j^\theta}{r_j^{\alpha^\theta}})^{\kappa^\theta}} \tag{6.2}$$

The parameter κ^θ captures the elasticity of population shares with respect to amenity-adjusted real wages and is therefore a measure of mobility of group θ, which we allow to be group specific. Expected utility for a worker in group θ across locations is:

$$W^\theta = \delta^\theta \left[\sum_{k=1}^N (\frac{A_i^\theta w_i^\theta}{r_i^{\alpha^\theta}})^{\kappa^\theta} \right]^{\frac{1}{\kappa^\theta}} \tag{6.3}$$

where $\delta^\theta = \Gamma(\frac{\kappa^\theta - 1}{\kappa^\theta})$ and $\Gamma(\cdot)$ is the gamma function in the Frechet case.

Supply of goods, amenities, and housing

We first write down the labor demand side of the economy. In location i, output is produced by perfectly competitive firms. They combine skilled and unskilled labor, who are imperfect substitutes in production:

$$Y_i = \left[(z_i^U)^{\frac{1}{\rho}} (L_i^U)^{\frac{\rho-1}{\rho}} + (z_i^S)^{\frac{1}{\rho}} (L_i^S)^{\frac{\rho-1}{\rho}} \right]^{\frac{\rho}{\rho-1}} \tag{6.4}$$

In the CES production function, $\rho \geq 1$ is the elasticity of substitution between skills and z_i^θ are location- and skill specific productivity shifters. The shifters can be in part exogenous and in part endogenous, reflecting externalities. We assume that, for $\theta = \{U, S\}$ and $\forall i$,

$$z_i^\theta = z^\theta(\overline{Z}_i, L_i^U, L_i^S) \tag{6.5}$$

where $\overline{Z}_i$ is the exogenous productivity component in city i. Local productivity spillovers are allowed here to depend not just on city size or density but also on its composition (L_i^U, L_i^S). Given equation, relative labor demand in location i is:

$$\log\left(\frac{L_i^S}{L_i^U}\right) = \log\left(\frac{z_i^S}{z_i^U}\right) - \rho \log\left(\frac{w_i^S}{w_i^U}\right) \tag{6.6}$$

Furthermore, competition across cities ensures that the unit cost of production in all cities is 1, the common price of the freely traded good:

$$\sum_\theta z_i^\theta (w_i^\theta)^{1-\rho} = 1, \forall i$$

Similar to productivity, amenities A_i^θ are assumed to be driven by both exogenous differences, and endogenous differences between cities, that is,

$$A_i^\theta = A^\theta(\overline{A}_i, L_i^U, L_i^S) \tag{6.7}$$

where A_i is the exogenous amenity component of city i. Endogenous amenities capture elements of quality of life that change when the size or composition of cities changes.

Finally, we assume that housing is supplied by atomistic absentee landowners and the aggregate housing supply function in city i is:

$$H_i = \overline{H}_i r_i^{\eta_i} \tag{6.8}$$

The housing supply elasticity η_i is allowed to be city specific.

A spatial equilibrium of this economy is a set of location choices $\{\lambda_i^\theta\}_{i,\theta}$ prices $\{w_i^\theta, r_i\}_{i,\theta}$ and amenities and productivity shifters $\{z_i^\theta, A_i^\theta\}_{i,\theta}$ such that workers and firms optimize, traded good firms make no profits, and markets clear. Since amenities and productivity shifters $\{z_i^\theta, A_i^\theta\}_{i,\theta}$ typically depend on the equilibrium distribution of economic activity, these local spillovers act as feedback loops that may amplify or dampen concentration and sorting.

We now discuss conditions under which sorting arises in the equilibrium. In using the term spatial sorting, we mean the fact that the skilled and unskilled groups make different location choices, i.e., there exist locations i and j such that, denoting $\Delta X = X_i - X_j$ for any variable X,

$$\Delta \log\left(\frac{L^S}{L^U}\right) \neq 0$$

Then relative spatial supply is combined with location choices, combining labor supply and labor demand:

$$\Delta \log\left(\frac{L^S}{L^U}\right) = \underbrace{\frac{\tilde{\kappa}^S}{\rho}\Delta \log\left(\frac{z^S}{z^U}\right)} \equiv \Delta z + \underbrace{\tilde{\kappa}^S \Delta \log\left(\frac{A^S}{A^U}\right)} \equiv \Delta A + \underbrace{\tilde{\kappa}^S(\alpha^U - \alpha^S)\Delta \log r} \equiv \Delta\alpha + \underbrace{\frac{\tilde{\kappa}^S}{\kappa^U}(1 - \frac{\kappa^U}{\kappa^S})\Delta \log L^U} \equiv \Delta\kappa \tag{6.9}$$

where we denote $\tilde{\kappa}^S = \frac{\kappa^S \rho}{\kappa^S + \rho}$. Conceptually, one can therefore distinguish four sources of sourcing in this framework: We say that soring is shaped by comparative advantage in production when $\Delta \log\left(\frac{z^S}{z^U}\right) \neq 0$, by amenities when $\Delta \log\left(\frac{A^S}{A^U}\right) \neq 0$ by housing price when $\alpha^S \neq \alpha^U$, and by heterogeneous across groups when $\kappa^U \neq \kappa^S$.

A first takeaway is that, when productivity is separable between a location shifter Z_i and nationwide group productivity z^θ so that $z_i^\theta = Z_i z^\theta$, the productivity advantage of a location is skill neutral and hence does not drive sorting directly. We now assume, in contrast, that some skill group has comparative advantage in production in some location over another so that $\Delta \log\left(\frac{z^S}{z^U}\right) \neq 0$. For simplicity, we assume that local productivity depends on population, which is the most classic way to parameterize agglomeration effects, but with a different intensity γ_P^θ for different skill groups, that is,

$$z_i^\theta = \overline{z}_i^\theta (L_i^U + L_i^S)^{\gamma_P^\theta} \tag{6.10}$$

In this expression, $\overline{z}_i^\theta$ are exogenous location-group productivity shifters. Equilibrium sorting is then pinned down by:

$$\Delta \log\left(\frac{L^S}{L^U}\right) = \frac{\tilde{\kappa}^S}{\rho}\Delta \log\left(\frac{\overline{z}^S}{\overline{z}^U}\right) + \frac{\tilde{\kappa}^S}{\rho}(\gamma_P^S - \gamma_P^U)\Delta \log L + \Delta_A + \Delta_\alpha + \Delta_\kappa$$

Changes in sorting due to productivity correspond to the first two terms on the right hand side of the above equation. First, such changes may occur because of changes in exogenous comparative advantage $\Delta \log\left(\frac{\overline{z}^S}{\overline{z}^U}\right)$. Second, changes in sorting may occur because of changes in relative city sizes $\Delta \log L$ or because of changes in relative agglomeration forces $\gamma_P^S - \gamma_P^U$.

We parameterize utility derived from amenities as a CD aggregator of a vectorof amenities $\{A_{ki}\}_k$ in location i, with skill group-specific preference parameters γ_{kA}^θ. This allow both skill groups to have different preferences over each city's amenity bundle:

$$A_i^\theta = \Pi_k (A_{ki})^{\gamma_{kA}^\theta} \tag{6.11}$$

We allow a component of each amenity in the amenity bundle to be endogenous. We model the endogenous component of amenity as responding to the skill ratio $\frac{L_i^S}{L_i^U}$ of that city; that is,

$$A_{ki} = \tilde{A}_{ki}(\frac{L_i^S}{L_i^U})^{\beta_k} \tag{6.12}$$

where $\tilde{A}_{ki}$ is the exogenous component of amenity k and β_k measures how elasticity the supply of amenity k is to the skill ratio. With these formulations, equilibrium sorting is:

$$\Delta \log\left(\frac{L^S}{L^U}\right) = \frac{\tilde{\kappa}^S}{1-\tilde{\kappa}^S(\tilde{\gamma}_A^S - \tilde{\gamma}_A^U)}\Delta \log\left(\frac{\tilde{A}^S}{\tilde{A}^U}\right) + \frac{1}{1-\tilde{\kappa}^S(\tilde{\gamma}_A^S - \tilde{\gamma}_A^U)}[\Delta_z + \Delta_\alpha + \Delta_\kappa]$$

where we denote $\tilde{\gamma}_A^\theta = \sum_k[\beta_k(\gamma_{kA}^\theta)]$ and $\frac{\tilde{A}_i^S}{\tilde{A}_i^U} = \Pi_k(\tilde{A}_{ki})^{\gamma_{kA}^S - \gamma_{kA}^U}$. First, amenities are a source of sorting in themselves only to the extent that the first term is nonzero. Second, the endogenous provision of amenities ($\beta_k \neq 0$) together with their heterogeneous valuation across skills ($\gamma_{kA}^S - \gamma_{kA}^U \neq 0$) serve only as an amplifier of other sorting forces.

To make clear the specific mechanisms at play here, we shut down the other sources of sorting by making the following assumptions:

Assumption 6.1. $\Delta \log\left(\frac{z^S}{z^U}\right) = 0, \Delta \log\left(\frac{A^S}{A^U}\right) = 0$ *and* $\kappa^U = \kappa^S$.

Under assumption 1, some cities may still be more productive or have hiigher amenities than others, but in a way that is skill neutral: there exist citywide shifters Z_i and A_i and nationwide group-specific shifters z^θ, A^θ such that $z_i^\theta = Z_i z^\theta$ and $A_i^\theta = A_i A^\theta$ for all location i. If housing is a necessary, then $\alpha^U - \alpha^S > 0$ and skilled workers are overrepresented in expensive cities. Given the housing supply equation, equilibrium housing prices are implicit solution to:

$$\frac{Z_i^{\frac{1+\kappa}{\rho-1}} A_i^\kappa}{\overline{H}_i} = r_i^{\eta_i}\left[\sum_\theta w^\theta f_i(r_i) r_i^{\kappa\alpha^\theta \frac{1-\rho}{\kappa+\rho}-1}\right]^{-1} \tag{6.13}$$

where $f_i(r_i)$ captures that, in equilibrium, wages depend on rents. This function can be shown to be equal to 1 when skills are perfect substitutes, and it is a decreasing function of r_i otherwise. Given that equation, rents r_i increase with $\frac{Z_i^{\frac{1+\kappa}{\rho-1}} A_i^\kappa}{\overline{H}_i}$: More productive cities and cities with higher amenities are more expensive in equilibrium. We denote $\mathcal{R}_i(\cdot)$ as the corresponding solution to that equation. Turning to the implication of housing rents for skill sorting, we obtain:

$$\Delta \log\left(\frac{L^S}{L^U}\right) = (\alpha^U - \alpha^S)\Delta \log \mathcal{R}(\frac{Z^{\frac{1+\kappa}{\rho-1}} A^\kappa}{\overline{H}}) \tag{6.14}$$

and high skill sorting into more productive and/or more attractive locations. First, when housing expenditure shares different across groups, Hicks-neutral city advantage is enough to drive sorting through its impact on housing prices. Second, the role of housing in mediating spatial sorting forces is stronger, all else equal, in locations with more inelastic housing supply (lower η). Inelastic supply directly leads to steeper $\mathcal{R}(\cdot)$, hence to a steeper response of housing price to productivity and amenities, and in turn to a steeper response of the skill ratio.

Heterogeneous migration elasticity

Finally, a last possible driver of sorting arises when $\kappa^U \neq \kappa^S$. Empirical findings find that higher-skill workers are more mobile than lower-skill workers, so, $\kappa^S > \kappa^U$.

Assumption 6.2. $\Delta \log\left(\frac{z^S}{z^U}\right) = 0, \Delta \log\left(\frac{A^S}{A^U}\right) = 0$, *and* $\alpha^U = \alpha^S$.

It is easy to see that:

$$\Delta \log\left(L^S\right) = \left[\underbrace{1 + \frac{\rho}{\kappa^S + \rho}\left(\frac{\kappa^S}{\kappa^U} - 1\right) > 0}\right] \Delta \log L^U \tag{6.15}$$

Under Assumption 2, there is no skill-biased amenity or productivity advantage of places. Still, the high-skill population increases faster than the low-skill population in attractive cities because members of the former group are more sensitive to city characteristics.

Urban skill premium and sorting

We now turn to considering how they shape the distribution of the skill wage premium in the cross section of cities, in equilibrium. Solving out for the equilibrium skill premium and its variation over space leads to

$$\Delta \log\left(\frac{w^S}{w^U}\right) = \frac{1}{\kappa^S}\Delta_z - \frac{1}{\rho}\Delta_A - \frac{1}{\rho}\Delta_\alpha - \frac{1}{\rho}\Delta_\kappa \tag{6.16}$$

Comparing this expression with the one that summarizes skill sorting we have:

$$\Delta \log\left(\frac{L^S}{L^U}\right) = \Delta_z + \Delta_A + \Delta_\alpha + \Delta_\kappa$$

Chapter 7

Urban Accounting and Welfare

7.1 The Model

7.1.1 Technology

Consider a model of a system of cities in an economy with N_t workers. Goods are produced in I monocentric circular cities. Cities have a local level of productivity. Production in city i in period t is given by:

$$Y_{it} = A_{it} K_{it}^{\theta} H_{it}^{1-\theta}$$

where A_{it} denotes city productivity, K_{it} denotes total capital, and H_{it} denotes total hours worked in the city. We denote the population size of city i by N_{it}. The standard first-order conditions of this problem are:

$$w_{it} = (1-\theta)\frac{Y_{it}}{H_{it}} = (1-\theta)\frac{y_{it}}{h_{it}} \text{ and } r_t = \theta \frac{Y_{it}}{K_{it}} = \theta \frac{y_{it}}{k_{it}} \tag{7.1}$$

where lowercase letters denote per capita variables. Note that capital is freely mobile across locations so there is a national interest rate r_t. Mobility patterns will not be determined solely by the wage, w_{it}, so there may be equilibrium differences in wages across cities at any point in time. We can then write down the "efficiency wedge", which is identical to the level of productivity, A_{it}, as:

$$A_{it} = \frac{Y_{it}}{K_{it}^{\theta} H_{it}^{1-\theta}} = \frac{y_{it}}{k_{it}^{\theta} h_{it}^{1-\theta}} \tag{7.2}$$

7.1.2 Preferences

Agents order consumption and hour sequenecs according to the following utility function:

$$\sum_{t=0}^{\infty} \beta^t \left[\log c_{it} + \psi \log(1-h_{it}) + \gamma_i\right],$$

where γ_i is a city-specific amenity and ψ is a parameter that governs the relative preference for leisure. Each agent lives on one unit of land and commutes from his home to work. Commuting is costly in terms of goods. The problem of an agent in city i_0 with cpaital k_0 is therefore,

$$\max_{\{c_{i_t,t}, h_{i_t,t}, k_{i_t,t}, i_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[\log c_{it} + \psi \log(1-h_{it}) + \gamma_i\right]$$

subject to the budget constraint:

$$c_{it} + x_{it} = r_t k_{it} + w_{it} h_{it}(1-\tau_{it}) - R_{it} - T_{it}$$
$$k_{it+1} = (1-\delta)k_{it} + x_{it}$$

where x_{it} is investment, τ_{it} is a labor tax or friction associated with the cost of building the commuting infrastructure, R_{it} are land rents, and T_{it} are commuting costs.

We assume that we are in steady state so $k_{it+1} = k_{it}$ and $x_{it} = \delta k_{it}$. Furthermore, we assume k_{it} is such that $r_t = \delta$. The simplified budget constraint of the agent becomes:

$$c_{it} = w_{it} h_{it}(1-\tau_{it}) - R_{it} - T_{it}. \tag{7.3}$$

The first-order conditions of this problem imply $\psi \frac{c_{it}}{1-h_{it}} = (1-\tau_{it})w_{it}$. Combining the expression with the first equation, we obtain:

$$(1-\tau_{it}) = \frac{\psi_{it}}{(1-\theta)} \frac{c_{it}}{1-h_{it}} \frac{h_{it}}{y_{it}} \tag{7.4}$$

We refer to τ_{it} as the "labor wedge". Although τ_{it} is modeled as a labor tax, it should be interpreted more broadly as anything that distorts an agent's optimal labor supply decision. Agents can move freely across cities so utility in each period has to be determined by:

$$\overline{u} = \log c_{it} + \psi \log(1-h_{it}) + \gamma_i \tag{7.5}$$

for all cities with $N_{it} > 0$, where $\overline{u}$ is the economy-wide per period utility of living in a city.

7.1.3 Commuting Cost, Land Rents, and City Equilibrium

Cities are monocentric, all production happens at the center, and people live in surrounding areas characterized by their distance to the center, d. Cities are surroundedby a vast amount of agricultural land that can be freely converted into urban land. We normalize the price of agricultural land to zero. Since land rents are continuous in equilibrium, this implies that at the boundary of a city, $\overline{d}_{it}$, land rents should be zero as well, namely, $R(\overline{d}_{id}) = 0$. Since, all agents in a city are identical, in equilibrium they must be indifferent between they live in a city, which implies that the total cost of rent plus commuting costs should be identical in all areas of a city. So,

$$R_{it}(d) + T(d) = T(\overline{d}_{it}) = \kappa \overline{d}_{it}, \forall d \in [0, \overline{d}_{it}]$$

since $T(d) = \kappa d$ where κ denotes commuting costs per mile.

Everyone lives on one unit of land, $N_{it} = \overline{d}_{it}^2 \pi$, and so $\overline{d}_{it} = (N_{it}/\pi)^{\frac{1}{2}}$. Thus $R_{it} + T(d) = \kappa (N_{it}/\pi)^{\frac{1}{2}}$. This implies that $R_{it}(d) = \kappa(\overline{d}_{it} - d)$ and so total land rents in a city of size N_{it} are given by $TR_{it} = \int_0^{\overline{d}_{it}} (\kappa(\overline{d}_{it} - d) d 2\pi) dd = \frac{\kappa}{3} \pi^{-\frac{1}{2}} N_{it}^{\frac{3}{2}}$. Hence, arrange land rents are equal to $AR_{it} = \frac{2\kappa}{3} (\frac{N_{it}}{\pi})^{\frac{1}{2}}$. Taking logs and rearranging terms, we obtain that:

$$\log(N_{it}) = o_1 + 2 \log AR_{it}. \tag{7.6}$$

where o_1 is a constant. We can also compute the total miles traveled by commuters in the city, which is given by:

$$TC_{it} = \int_0^{\overline{d}_{it}} (d^2 2\pi) dd = \frac{2}{3} \pi^{-\frac{1}{2}} N_{it}^{\frac{3}{2}} \tag{7.7}$$

7.1.4 Government Budget Constraint

The government levies a labor tax, τ_{it}, to pay for the transportation infrastructure. Let government expenditure be a function of total commuting costs and wages such that:

$$G(h_{it}w_{it}, TC_{it}) = g_{it}h_{it}w_{it}\kappa TC_{it} = g_{it}h_{it}w_{it}\kappa\frac{2}{3}\pi^{-\frac{1}{2}}N_{it}^{\frac{3}{2}}$$

where g_{it} is a measure of government inefficiency. That is, the government requires κg_{it} workers per mile commuted to build and maintain urban infrastructure. The government budget constraint is then given by:

$$\tau_{it}h_{it}N_{it}w_{it} = g_{it}h_{it}w_{it}\kappa\frac{2}{3}\pi^{-\frac{1}{2}}N_{it}^{\frac{3}{2}} \tag{7.8}$$

which implies that the labor wedge can be written as:

$$\tau_{it} = g_{it}\kappa\frac{2}{3}(\frac{N_{it}}{\pi})^{\frac{1}{2}} \tag{7.9}$$

or

$$\log\tau_{it} = o_2 + \frac{1}{2}\log N_{it} + \log g_{it} \tag{7.10}$$

7.1.5 Equilibrium

The consumer budget constraint is given by:

$$c_{it} = w_{it}h_{it}(1-\tau_{it}) - R_{it} - T_{it} = (1-\theta)(1-\tau_{it})y_{it} - \kappa(\frac{N_{it}}{\pi})^{\frac{1}{2}}$$

To determine output we know that the proudction function is given by $y_{it} = A_{it}k_{it}^{\theta}h_{it}^{1-\theta}$ and the decision of firms to rent capital implies that $r_t k_{it} = \theta y_{it}$. Hence,

$$y_{it} = A_{it}(\frac{\theta y_{it}}{r_t})^{\theta}h_{it}^{1-\theta} = A_{it}^{\frac{1}{1-\theta}}(\frac{\theta}{r_t})^{\frac{\theta}{1-\theta}}h_{it}.$$

Using the above result, we can derive

$$h_{it} = \frac{1}{1+\psi}(1 + \frac{\psi(R_{it}+T_{it})}{(1-\theta)(1-\tau_{it})}\frac{(\frac{r_t}{\theta})^{\frac{\theta}{1-\theta}}}{A_{it}^{\frac{1}{1-\theta}}})$$

and

$$c_{it} = \frac{1}{1+\psi}\left[(1-\theta)(1-\tau_{it})(\frac{\theta}{r_t})^{\frac{\theta}{1-\theta}}A_{it}^{\frac{1}{1-\theta}} - (R_{it}+T_{it})\right]$$

The free mobilility assumption in the result implies that $\overline{u}_t = \log c_{it} + \psi\log(1-h_{it}) + \gamma_{it}$ for some $\overline{u}_t$ determined in general equilibrium so:

$$\begin{aligned}
&\overline{u}_{it} + (1+\psi)\log(1+\psi) - \psi\log\psi \\
&= \log\left((1-\theta)\left(1-\kappa g_{it}\frac{2}{3}\left(\frac{N_{it}}{\pi}\right)^{\frac{1}{2}}\right)\frac{A_{it}^{\frac{1}{1-\theta}}}{(\frac{r_t}{\theta})^{\frac{\theta}{1-\theta}}} - \kappa\left(\frac{N_{it}}{\pi}\right)^{\frac{1}{2}}\right) \\
&+ \psi\log\left(1 - \frac{\kappa\left(\frac{N_{it}}{\pi}\right)^{\frac{1}{2}}}{(1-\theta)\left(1-\kappa g_{it}\frac{2}{3}\left(\frac{N_{it}}{\pi}\right)^{\frac{1}{2}}\right)}\frac{(\frac{r_t}{\theta})^{\frac{\theta}{1-\theta}}}{A_{it}^{\frac{1}{1-\theta}}}\right) + \gamma_{it}.
\end{aligned} \tag{7.11}$$

The last equation determines the size of the city N_{it} as implicit function of city productivity A_{it}, city amenities, γ_i, government inefficiency, g_{it}, and economy wide variables like r_t and $\overline{u}_{it}$. In the above euqation, the LHS is decreasing in N_{it}. THe LHS is also increasing in A_{it} and γ_i and decreasing in g_{it}. Hence, we can prove immediately that:

$$\frac{\partial N_{it}}{\partial A_{it}} > 0, \frac{\partial N_{it}}{\partial \gamma_i} > 0, \frac{\partial N_{it}}{\partial g_{it}} < 0 \tag{7.12}$$

The economy wide utility level $\overline{u}_t$ is determined by the labor market clearing conditions

$$\sum_{i=1}^{I} N_{it} = N_t \tag{7.13}$$

This last equation clarifies that our urban system is closed; we do not consider urban-rural migration.

7.2 Evidence of Efficiency, Amenities, and Frictions

7.2.1 Empirical Approach

We start by estimating the following equation:

$$\log N_{it} = \alpha_1 + \beta_1 \log A_{it} + \varepsilon_{it} \tag{7.14}$$

THe value of β_1 yields the effect of the "efficiency wedge" on city population. According to the model, $\beta_1 > 0$ by the Equation (7.12). Furthermore, $\log N_{it}(A_{it}) = \beta_1 \log A_{it}$ is the population size explained by the size of the "efficiency wedge". In contrast, ε_{1it} is the part of the observed population in the city that is unrelated to the productivity; according to the model it is related to both g_{it} and γ_i. We can then estimate the following equation: $\tilde{\varepsilon}_1(g_{it}, \gamma_{it}) \equiv \varepsilon_{1it}$.

Since the "efficiency wedge" increases population size, total commuting increases, which affects the "labor wedge". This is the standard urban trade-off between productivity and agglomeration. We can estimate the effect of producitvity on the labor wedge and the decomposition of $\log N_{it}$ into $\log \tilde{N}_{it}(A_{it})$ and ε_{1it}. Hence, we estiamte:

$$\log \tau_{it} = \alpha_2 + \beta_2 \log \tilde{N}_{it}(A_{it}) + \varepsilon_{2it} \tag{7.15}$$

We denote the effect of efficiency on distortions by $\log \tilde{\tau}_{it} = \beta_2 \log \tilde{N}_{it}(A_{it})$. We can then estimate the following equation: $\tilde{\varepsilon}_{2it} \equiv \varepsilon_{2it}$. The equation also implies that the error term ε_{2it} is related to g_{it} and to $\tilde{\varepsilon}_1(g_{it}, \gamma_{it})$. Hence, we define $\tilde{\varepsilon}_2(g_{it}, \tilde{\varepsilon}_1(g_{it}, \gamma_{it})) \equiv \varepsilon_{2it}$.

We now can decompose the effect from all three elements of $(A_{it}, \gamma_{it}, g_{it})$. To do so, we estimate:

$$\log(AR_{it}) = \alpha_3 + \beta_3 \log \tilde{\tau}_{it} + \beta_4 + \varepsilon_{1it} + \beta_5 \varepsilon_{2it} + \varepsilon_{3it} \tag{7.16}$$

using median rents for AR_{it}. The effect of γ_{it} and g_{it} are determined by the estimates of β_4 and β_5. Note that ε_{1it} and ε_{2it} depend on both γ_{it} and g_{it}. However, since $\varepsilon_{2it} = \tilde{\varepsilon}_2(g_{it}, \tilde{\varepsilon}_1(g_{it}, \gamma_{it}))$ depend only on γ_{it} through ε_{1it} and we are including ε_{1it} directly in the regression, β_5 capture only the effect of changes in g_{it} on land rents.

Note that we can then use the equation to relate average rents and population sizes. So we can estiamte the model using

$$\log(N_{it}) = \alpha_4 + \beta_6 \log(AR_{it}) + \varepsilon_{4it} \tag{7.17}$$

In a circular city $\beta_6 = 2 > 0$.

7.2.2 Effects of Efficiency, Amenities, and Frictions on City Size

We can decompose the labor wedge into taxes and other distortions such that

$$(1-\tau_{it}) = (1-\tau'_{it})(\frac{1-\tau_{ith}}{1+\tau_{ith}}) \tag{7.18}$$

where τ_{it} is our measure of the labor wedge, τ_{ith} is the labor tax rate, τ_{itc} is the consumption tax rate, and τ'_{it} are other distortions. Thus expect our measure of the total labor wedge, $(1-\tau_{it})$, to be correlated with $(1-\tau_{ith})/(1+\tau_{itc})$.

Chapter 8

Interacting Agents, Spatial Externalities and the Evolution of residential land use patterns

8.1 Optimal Timing of Development

Define $A(i,t)$ as the returns to the original, unsubdivided parcel (denoted i) in the undeveloped use in time period t. We will refer to this as agriculture, broadly defined to include any uses of the land in an undeveloped state. Conversion of parcel i at time T will require the agent to incur costs to reap expected one-time gross returns. Costs include the provision of subdivision infrastructure, as well as permitting and other administrative fees. We denote δ as the discount factor, defined as $\frac{1}{1+r}$ where r is the interest rate, and the one-time returns from development minus costs of conversion in time T as $V(i,T)$. Then the net returns from developing parcel i in time T equals the one time net returns minus the present value of foregone agricultural returns and is given by:

$$V(i,T) - \sum_{t=0}^{\infty} A(i,T+t)\delta^t \tag{8.1}$$

The net returns from keeping parcel i in an agricultural use in period T and developing in time period $T+1$, discounted to time T, are:

$$A(i,T) + \delta V(i,T+1) - \sum_{t=0}^{\infty} A(i,T+1)\delta^t \tag{8.2}$$

The optimal development time will occur in period T only if the Equation (8.1) is positive and if Equation (8.1) is greater than Equation (8.2). The optimal development time is the time period T if:

$$V(i,T) - \sum_{t=0}^{\infty} A(i,T+t)\delta^{T+t} > 0 \tag{8.3}$$

and

$$V(i,T) - A(i,T) \geq \delta V(i,T+1) \tag{8.4}$$

The agent develops in periop T only if (a) the net value of conversion is positive and (b)

$$\frac{V(i,T+1) - \{V(i,T) - A(i,T)\}}{V(i,T) - A(i,T)} < r$$

where r is the interest rate. Here we are interested in explaining the scattered pattern of exurban development. Such a pattern would result if net negative interactions were present, e.g., due to congestion externalities, and if these effects were sufficiently strong as to create a 'repelling' effect among residential development.

Let $\delta_s I_s(i,t)$ represent this spillover effect, where $I_s(i,t)$ is the proportion of neighboring parcels that are in a developed state at time the development decision is made, δ_s is the interaction parameter, and s indexes the order of the spaital lag, which increases with increasing distance from parcel i. Because neighboring developed lands could conceivably have positive and/or negative spillover effects, the parameter λ_s, which represents the net effect of these spillovers, could be either positive or negative at any given distance s.

Rewritting the Equation (8.1) to incorporate the effect of interactions, the net returns from developing parcel i in time period T equals:

$$V(i,T) + \sum_s \lambda_s I_s(i,T) - \sum_{t=0}^{\infty} A(i,T+t)\delta^{T+t} \tag{8.5}$$

Rewritting the conversion rule in Equation (8.4), development now occurs in the first period in which:

$$V(i,T) - \delta V(i,T+1) - A(i,T) + \sum_s (1-\delta)\lambda_s I_s(i,T) \geq 0 \tag{8.6}$$

where the value of $I(i,T)$ is assumed to be constant between periods T and $T+1$.

8.2 The Empirical Model

8.2.1 Hazard model of development

To take account of these differences across agents, define ε_i as these unobservable factors associated with the owner of parcel i. Given that ε is viewed by the research as a stochastic variable, the following gives the probability that parcel i, with surrounding land use pattern $\sum_s I_s(i,T)$ will be converted by period T:

$$\text{Prob}\{\varepsilon_i < \frac{1}{1-\delta}(V(i,T) - \delta V(i,T+1) - A(i,T)) + \sum_s \lambda_s I(i,T)\} \tag{8.7}$$

This implies that agents with large ε's, such as those who are particularly good farmers or those that place a particular high value on their undeveloped land as a source of direct utility, will convert later than agents with the same type of parcel but smalle values of ε.

The probability that a parcel with a given set of characteristics will be converted in period T is its hazard rate for period T, which is given by:

$$h(T) = \frac{F[\varepsilon^*(T+1)] - F[\varepsilon^*(T)]}{1 - F[\varepsilon^*(T)]} \tag{8.8}$$

where F is the cumulative distribution function for ε and define ε^* as the ε that makes the equation an equality. In this analysis, we choose Cox's partial likelihood method because it can accomodate time-varying covariates. Assuming that $V(i,t)$ is separable in these factors that are time varying, but spatially constant, we can apply Cox's model to our problem by defining a baseline hazard rate that is a function of time only. Let $\omega(T)$ represent the exponential of this baseline hazard rate and assume that the log of the hazard rate it linear in other arguments. Then the hazard rate for parcel i is given by:

$$h(i,T) = \omega(T) \times \exp(Z\beta). \tag{8.9}$$

where Z is a vector of parcel i's attributes, including $I_s(i,T)$, and β is a corresponding parameter vector. Cox's method is a semiparametric approach that relies on formulating the likelihood in such a way that the baseline hazard, $\omega(T)$, drops out and therefore specification of an error distribution is unnecessary. It is the product of N contributions to the likelihood function, where N is the number of developable parcels, and the form of the nth contribution is given by:

$$L_n = \frac{h(n, T_n)}{\sum_{j=1}^{J_n} h(j, T_n)} \tag{8.10}$$

By definition, T_n is the time at which the nth parcel is converted. In the above equation, $h(n, T_n)$ is the hazard rate for the nth parcel, $h(j, T_n)$ is the hazard rate for the jth parcel, but evaluated at time T_n, and J_n is the set of parcels that have 'survived' in the undeveloped state until time T_n.

8.2.2 The identification Problem

The vector of $Z(i)$ contains all attributes associated with parcel i and the stochastic term ε_i captures the existence of idiosyncratic factors associated with agent i.

Chapter 9

When is the economy monocentric

9.1 A formal model of a spatial economy

Consider a long-narrow country, in which area is represented by one-dimensional unbounded location space, $X = \mathbb{R}$. The quality of land is homogenous and density of land is equal to 1 everywhere. The country has a continuum of homogenous workers with a given size, N. Each worker is endowed with a unit of labor, and is free to chooose any location and job in the country. The consumers of the country consist of the workers and landlords.

Each consumer consumes a homogenous agricultural good (A-good) together with a continuum of differentiated manufacturing goods (M-goods) of size n. Here, n is to be determined endogenously. All consumers have the same utililty function given by:

$$u = \alpha_A \log z + \alpha_M \log \left\{ \int_0^n q(\omega)^\rho d\omega \right\}^{1/\rho} \tag{9.1}$$

where z represents the amount of A-good consumed, $q(\omega)$ is the consumption of each variety $\omega \in [0, n]$ of M-good; and α_A, α_M and ρ are positive constants such that $\alpha_A + \alpha_M = 1$ and $0 < \rho < 1$. Note that a smaller ρ means that consumers have a stronger preference for variety in M-goods.

Suppose that a consumer has an income, Y, and faces a set of prices p_A and $p_M(\omega)$. Then for choosing the consumption bundle that maximizes the equation above, subject to the budget constraint $p_A z + \int_0^n p_M(\omega) q(\omega) d\omega = Y$, demand functions of the consumer can be obtained as:

$$z = (\alpha_A Y)/p_A \tag{9.2}$$

$$q(\omega) = (\alpha_M Y / p_M(\omega)) \left(p_M(\omega)^{-\gamma} / \int_0^n p_M(\omega)^{-\gamma} d\omega \right) \tag{9.3}$$

for each $\omega \in [0, n]$, where $\gamma = \rho/(1-\rho)$. Note that the Equation (9.3) that the demand for any variety in M-good has the same price elasticity, E, given by:

$$E = 1/(1-\rho) = 1 + \gamma \tag{9.4}$$

Thus E increases as ρ increases. Substituting z and $q(\omega)$ into the utility function yields the following indirect utility function:

$$u = \log\{\alpha_A^{\alpha_A} \alpha_M^{\alpha_M} Y p_A^{\alpha_A}\} + \frac{\alpha_M}{\gamma} \log \left\{ \int_0^n (\omega)^{-\gamma} d\omega \right\} \tag{9.5}$$

Next, the A-good is assumed to be produced under constant returns, where each unit of A-good consumes a unit of land and a_A units of labor. Each M-good is produced with labor only. All types of M-good have the same production technology under increasing returns such that the total labor input, L, for production of quantity Q of any product is given by:

$$L = f + \alpha_M Q \tag{9.6}$$

where f is the fixed labor requirement and α_M the marginal labor input. We assume, for simplicity, that the transport cost of each good takes Samuelson's iceberg form: if a unit of good i ($i = A$ or M) is shapped over a distance d, only $e^{-t_i d}$ units actually arrive, where t_i is a positive constant.

Owing to scale economics in production, each variety of M-good is assumed to be produced by a single, specialized firm. If a firm locates at $x \in X$ and produces an M-product, it chooses an f.o.b. price $p_M(x)$ so as to maximize its profit at the Chamberlinian equilibrium. By the assumption of iceberg transport technology, the effective (delivered) price, $p_M(y \mid x)$, for consumers at location $y \in X$ of any M-product produced at location x is given by:

$$p_M(y \mid x) = p_M(x) e^{t_M |y-x|} \tag{9.7}$$

Given the delivered price function above, it can be readily verified that the price elasticity of the total demand for any M-product is independent of the spatial distribution of the demand, and equals the price elasticity, E, of each consumer's demand given by above demand function. Thus, given the equilibrium wage rate, $W(x)$, at x, by the equality of the marginal revenue and marginal cost, $p_M(x)(1-E)^{-1} = a_M W(x)$, the optimal f.o.b. price of the firm can be obtained by:

$$p_M(x) = a_M W(x)/\rho \tag{9.8}$$

which represents the familiar result that for each monopolistic firm will chare its f.o.b. price at a markup over the marginal cost $a_M W(x)$. Thus, if Q is the output of the firm, its profit equals:

$$\pi(x) = p_M(x)Q - W(x)(f + a_M Q) = a_M \gamma^{-1} W(x)(Q - \gamma f / a_M) \tag{9.9}$$

Therefore, given any equilibrium configuration of the economy, if an M-firm operates at x, then by the zero-profit condition, its output must be equal to

$$Q^* = \gamma f / a_M \tag{9.10}$$

which is a constant independent of location.

Consequently, the remaining unknown of the model are: (i) the price, $p_A(y)$, of the A-good at each y; (ii) the wage rate, $W(y)$, at each y; (iii) the land rent, $R(y)$, at each y; (iv) the equilibrium utility level, u, of workers; (v) the spatial distribution of workers; (vi) the spatial distribution of M-good production; and (vii) the trade pattern of each good. A spatial configuration is in equilibrium if all workers achieve the same highest utility, each active firm earns zero profit, and the equality of the demand and supply of each good is attained.

9.2 The monocentric equilibrium

In this section we determine all unknown assuming that all M-firms locate in the city. Then in the next section, we exaime the location equilibrium conditions under which indeed no M-firms would desire to deviate from the city.

Let the price curve, $p_A(y)$, of the A-good be normalized such that $p_A(0) = 1$. Then since all excess A-goods are to be transported to the city, at each location y in the agricultural hinterland, it must hold that:

$$p_A(y) = e^{-t_A|y|} \tag{9.11}$$

Next, let $p_M = p_M(0)$ be the f.o.b. price of each M-good produced in the city. Then the (delivered) price of each M-good at each location is given by:

$$p_M(y) \equiv p_M(y \mid 0) = p_M e^{t_M|y|} \tag{9.12}$$

Furthermore, let n be the size of the M-industry established in the city; by definition, n is then the number of firms producing M-goods in the city.

Now, let $W(y)$ be the equilibrium wage rate at each $y \in [-l, l]$. Then in equilibrium, since all workers must achieve the same utility level, say u, by the indirect utility function we have that:

$$W(y) = e^u \alpha_A^{-\alpha_A} \alpha_M^{-\alpha_M} n^{-\alpha_M/\gamma} p_M^{\alpha_M} e^{(\alpha_M t_M - \alpha_A t_A)|y|} \tag{9.13}$$

By the zero-profit condition in A-good production, the land rent at each location can be obtained as:

$$R(y) = p_A(y) - a_A W(y) = e^{-t_A|y|} - a_A W(y) \text{ for } y \in [-l, l], \tag{9.14}$$

since $R(l) = 0$ at the fringe location l, it holds by the above result that $e^{-t_A l} = a_A W(y)$, which together with the utility function yields the following wage curve:

$$\begin{aligned} W(y) &= \{a_A^{-1} e^{-t_A l}\} e^{(a_A t_A - \alpha_M t_M)(l - |y|)} \\ &= a_A^{-1} e^{-\alpha_M (t_A + t_M) l} e^{(\alpha_M t_M - \alpha_A t_A)|y|} \end{aligned}$$

Notice that $W(l) = a_A^{-1} e^{-t_A l}$, which determines the wage rate at the fringe location solely as a function of distance l. In turn, $W(y)$ gives the equilibrium wage rate at each y, which compensates for the prie differences between location y and the fringe location. Substituting the result back to the p_M yileds the following equilibrium price curve of the M-good:

$$p_M \equiv p_M(0) = a_M (a_A \rho)^{-1} e^{-\alpha_M (t_A + t_M) l}. \tag{9.15}$$

Next, to determine the number of firms, n, in the city, if we let N_A be the size of agricultural workers and N_M be that of manufacturing workers, then:

$$N_A = 2 a_A l \tag{9.16}$$

$$N_M = n(f + a_M Q^*) = nf(1 + \gamma) \tag{9.17}$$

Hence, by the full employment condition, $N_A + N_M = N$, we have:

$$n = \frac{N - 2a_A l}{f(1 + \gamma)} \tag{9.18}$$

Now, if we know l, all unknowns will be determined uniquely. The equilibrium value of l can be determined by the equality of demand and supply of the A-good as follows. By the excess supply of the A-good per unit distance at each $y \neq 0$ equals $1 - (\alpha_A Y(y)/p_A(y)) = 1 - \alpha_A = \alpha_M$ (where $Y(y) = \alpha_A W(y) + R(y) = p_A(y)$). Thus, considering the consumption of the A-good in transportation, the total net supply of the A-good to the city equals:

$$S_A(0) = \int_{-l}^{l} \alpha_M e^{-t_A|y|} dy = 2\alpha_M t_A^{-1}(1 - e^{-t_A l}) \tag{9.19}$$

where the total demand of the A-good at the city equals $\alpha_A Y(0)/p_A(0) = \alpha_A W(0) N_M$. Hence, the equality of the supply and demand requires that:

$$\frac{S_A(0)}{\alpha_A W(0)} = N_M, \tag{9.20}$$

or, since $N_M = N - N_A = N - 2a_A l$, if we define:

$$N_C(l) = \frac{S_A(0)}{\alpha_A W(0)} = \frac{2\alpha_M t_A^{-1}(1 - e^{-t_A l})}{\alpha_A a_A^{-1} e^{-\alpha_M (t_A + t_M) l}} \tag{9.21}$$

$$N_M(l) \equiv N - 2a_A l, \tag{9.22}$$

then the result can be restated as:

$$N_C(l) = N_M(l) \tag{9.23}$$

Here $N_C(l)$ represents the size of urban consumers, which is just sufficient to purchase all A-good units supplied from the hinterland when the wage rate at the city equals $W(0) \equiv a_A^{-1} e^{-\alpha_M (t_A + t_M) l}$, where $N_M(l)$ represents the size of urban workers when the agricultural fringe distance is l. We call $N_C(l)$ the A-good exhausting size of urban consumer, or AE urban consumer size, and call $N_M(l)$ the urban-labor force.

Notice that the function $N_C(l)$ is increasing in l, $N_C(0) = 0$ and $N_C(\infty) = \infty$. Therefore, $N_C(l)$ and $N_M(l)$ determines the equilibrium fringe distance l^* uniquely. Now substituting l^* into the results, all other unknowns can be determined uniquely. In particular, we have that:

$$n^* = \frac{N - 2a_A l^*}{f(1+\gamma)} \tag{9.24}$$

$$\begin{aligned} u^* &= -\alpha_A \alpha_M (t_A + t_M) l^* + \frac{\alpha_M}{\gamma} \log \frac{N - 2a_A l^*}{f(1+\gamma)} \\ &+ \log \alpha_A^{\alpha_A} \alpha_M^{\alpha_M} a_A^{-\alpha_A} a_M^{-\alpha_M} \rho^{\alpha_M} \end{aligned} \tag{9.25}$$

$$W^*(y) = a_A^{-1} e^{-\alpha_M (t_A + t_M) l^*} e^{(\alpha_M t_M - \alpha_A t_A)|y|} \tag{9.26}$$

9.3 The potential function and location-equilibrium of M-firms

To make sure the model is in equilibrium, we must make sure (i) no existing M-firms could increase their profit by moving away from the city; (ii) no new M-firm would enter the market.

Suppose an M-firm locates at $x \in \mathbb{R}$. Then given the market wage rate $W^*(y)$ at x, the firm will set its f.o.b. price at $a_M W^*(x) \rho^{-1}$. Hence, at each location $y \in \mathbb{R}$, the delivered price, $p_M(y \mid x)$, of the M-good produced by the firm is given by:

$$p_M(y \mid x) = a_M W^*(x) \rho^{-1} e^{t_M |y - x|}. \tag{9.27}$$

Using this equation, as a function of the market wage rate $W^*(x)$ there, the total demand for the firm located at x can be obtained as follows:

$$D(x, W^*(x)) = \frac{\alpha_A \gamma f}{2a_M} (\frac{W^*(0)}{W^*(x)})^{1+\gamma} \phi(x), \text{ for } x \geq 0 \tag{9.28}$$

where $\phi(x) \equiv$ a large equation by the original paper. The resulting firm's profit is:

$$\begin{aligned}\pi(x, W^*(x)) &= a_M W^*(x)\rho^{-1} D(x, W^*(x)) - W^*(x)[f + a_M D(x, W^*(x))] \\ &= a_M \gamma_M^{-1} W^*(x) \left\{ D(x, W^*(x)) - \gamma f / \alpha_M \right\}\end{aligned} \tag{9.29}$$

which implies that $\pi(x, W^*(x)) \geq\leq 0$ as $D(x, W^*(x)) \geq\leq \gamma f / a_M$. Define:

$$\Omega(x) \equiv \frac{D(x, W^*(x))}{\gamma f / a_M} \tag{9.30}$$

where $\gamma f / a_M$ represents the equilibrium output level determined by the zero-profit condition. we call $\Omega(x)$ the (market) potential function of the M-industry, which represents the relative profitability of each location for M-firms.

Chapter 10

The endogenous formation of a city: population agglomeration and marketplaces in a location-specific production economy

10.1 A general model

10.1.1 An overview of the economy

There are I (finite and integer) produced assumption commodities in the economy. There is a finite number of different locations in the economy. The location set is denoted by $J \subset \mathbb{R}^m$, where m is a positive integer representing the dimension of the location space and J is finite. Each location $j \in J$ is just a point, but it contains a positive amount of homogeneous land. Marketplaces can be established in a feasible marketplace location set $D \subset \mathbb{R}^m$. The set D is assumed to be compact. A consumer chooses her location j from J taking into account, for each location, the wage, rent, and commuting cost to the closest marketplace.

10.1.2 Marketplaces

The location of a marketplace can be anywhere in D. No marketplace requires land: a marketplace is a point $d \in D$. The locations of the marketplaces are denoted by $\{d_1, d_2, \dots, d_k\} = \mathbf{d} \subset \mathbf{D}$. Let K be the collection of finite sets in $D(\emptyset \in K)$. We assume that the transportation cost of commodities between marketplaces is negligible, while to access marketplaces consumers have to use their time endowments.

10.1.3 Consumers

There is a continuum of consumers. The set of consumers is denoted $A = [0, 1]$, and a representative element of A is a. The consumers from an atomless measure space $(A, \mathcal{A}, v)$, where $\mathcal{A}$ is the Borel sigma algebra of A and v is the Lebesgue measure on A. By definition, $v(A) = 1$.

10.1.4 Individual transportation

We assume that travel cost from location j to a marketplace d is represented by $\tilde{\delta}_j(d)$ where $\tilde{\delta}_j(d) : D \to \mathbb{R}_+$ is a continuous function. Travel cost is assumed to be independent of the quantity of goods transported. Since a consumer accesses a marketplace that is most convenient, time to travel to a marketplace in a marketplace structure $\mathbf{d}$ for a consumer residing at $j \in J$ is given by $\delta_j(\mathbf{d})$, where $\delta_j : K \to \mathbb{R}_+$ is such that $\delta_j(\mathbf{d}) = \min_{\mathbf{d} \in \mathbf{d}} \tilde{\delta}_{\mathbf{j}}(\mathbf{d})$. The Euclidean distance from location j to the closest marketplace given marketplace structure $\mathbf{d} \in \mathbf{K}$ is an example of $\delta_j(\mathbf{d})$, i.e., $\delta_j(\mathbf{d}) = \mathbf{2} \min_{\mathbf{d} \in \mathbf{d}} \|\mathbf{j} - \mathbf{d}\|$.

10.1.5 Mass transportation among marketplace

When there is more than one marketplace, it is necessary to build a mass transportation system to have commodity flows among them. One can imagine the situation where there is a railroad station in front of each marketplace.

10.1.6 The trading cost

It is assumed that each consumer must reside in exactly one location in J. Since consumers can choose their locations freely, consumption sets are non-convex, and even disconnected. This fact does not depend on the finitess of the location set J. Even if J is a convex subsset of $\mathbb{R}^m$, the trading cost is always non-convex since each consumer needs to choose one location as her residence. Since production is location-specific, we cannot treat labor as a homogeneous good, since it has a different effect on output at different locations. A trading set is a consumption set where the endowment point is normalized to the origin. Let $\Omega = \Omega_c \times \Omega_\ell \times \Omega_L \times \Omega_J$ be the potential trading set, which denote potential commodity, leisure, and land trading sets, while $\Omega_J = J$ denotes the potential location choice set, respectively. Consumers' trading sets with no transportation costs are represented by the closed-valued measurable correspondence $X : A \mapsto \Omega$. We let $X(a) = \bigcup_{j \in J} X_j(a)$, where we define $X_j(a) = \mathbb{R}^I_+ \times H_j(a) \times L(a) \times \{j\}$, and where $H_j(a)$ and $L(a)$ are the type a consumer's leisure trading set and land trading set, respectively, when she chooses location j. The jth axis of $H_j(a) \subset \mathbb{R}^J$ is $[-T(a), 0]$ and the other axes are all $\{0\}$, where $T(a)$ is the leisure endowment of a type a consumer. The land trading set is $L(a) = \{L \in \mathbb{R}^J : L \geq -b(a)\}$, where $b(a) \geq 0$ denotes the initial endowment of land belonging to consumer a.

THIS PAPER IS TOOOOOOO MANY MATHEMATICS.

Chapter 11

ON THE INTERNAL STRUCTURE OF CITIES

11.1 The Model

We consider a circular city of fixed radius S, located in a large economy. A single traded good is produced within the city, which is sold to the larger economy at a competitive price. Labor is supplied elastically at the reservation utility $\overline{u}$ that prevails in the larger economy. Workers have preferences over units of the produced good and the quantity of residential land that they consume. We would treat land as available at the boundary of the city at a price q_f, determined by its value in an agricultural use. We take the radius S of the city as given.

The total land area of the city, πS^2, is divided between production use and residential use. We describe locations within the city by their pollar coordinates (r, ϕ), but for most purposes we consider only symmetric equilibria, where nothing depends on ϕ, and refer simply to "location r". Let $\theta(r)$ be the fraction used for production. Let the employment density - employment per unit of production land - at location r be $n(r)$, implying that total employment at r is $2\pi r\theta(r)n(r)$. Let $N(r)$ be the number of workers housed at r, per unit of residential land. Then if each such person occupies $\ell(r)$ units of land, we have $\ell(r)N(r) = 1$.

Production technology involves: ordinary CRS that relates land, labor and the technology level to goods production. There is the external effect that relates technology level at any one location to the employment, weighted by distance, at other locations. Finally, there is a cost - in units of lost labor time - to commuting to and from work.

Production of the traded good at location r is assumed to be a CRS of land, $2\pi r\theta(r)$, and labor, $2\pi r\theta(r)n(r)$, at that location. Production per unit of land at location r can thus be written as:

$$x(r) = g(z(r))f(n(r)) \tag{11.1}$$

the functions g and f are taken to be CD:

$$g(z) = z^{\gamma} \tag{11.2}$$

and

$$f(n) = An^{\alpha} \tag{11.3}$$

The intercept term $g(z(r))$ is a productivity term that reflects an external effect on production at location $(r, 0)$ of employment at neighboring locations (s, ϕ). This production externality is assumed to be linear, and to decay exponentially at a rate δ with the distance between $(r, 0)$ and (s, ϕ):

$$z(r) = \delta \int_0^S \int_0^{2\pi} s\theta(s,\phi)n(s,\phi)e^{-\delta x(r,s,\phi)}d\phi ds,$$

where $x(r,s,\phi) = [r^2 - 2\cos(\phi)rs + s^2]^{1/2}$. Since allocations are assumed to be symmetric, we can write:

$$z(r) = \int_0^S \psi(r,s)s\theta(s)n(s)ds, \tag{11.4}$$

where

$$\psi(r,s) = \delta \int_0^{2\pi} e^{-\delta x(r,s,\phi)}d\phi. \tag{11.5}$$

Each worker is endowed with one unit of labor, which he supplies inelastically to the composite activity producing-and-commuting. The third aspect of the technology is a commuting cost that takes the form of a loss of labor time that depends on the distance traveled to and from work each day. Specifically, if a worker lives at location s and works at location r, he delivers:

$$e^{-\kappa|r-s|}$$

hours of labor at location r.

Workers have identical preferences $U(c,\ell)$ over consumption of the produced good c and residential land ℓ. The function U is CD:

$$U(c,\ell) = c^\beta \ell^{1-\beta} \tag{11.6}$$

Let $c(r)$ and $\ell(r)$ denote the goods and land consumption of everyone housed at r. Every consumer-worker at every location must receive the reservation utility level:

$$U(c(r),\ell(r)) = \overline{u} \tag{11.7}$$

In this setting an allocation will mean a collection of functions (z,θ,n,N,c,ℓ) on $[0,S]$ that describe productivity, land use, employment, and consumption at each location $r \in [0,S]$. To be feasible, an allocation must satisfy $\theta(r) \leq 1$, $N(r)\ell(r) = 1$. We need a test to determine whether any given triple $(\theta(r),n(r),N(r))$ of functions on $[0,S]$ describes an internally consistent pattern of land use, employement, and residential housing. Think of filling up the city, proceeding from the center, $r = 0$, outward to the edge, $r = S$. We define a state variable $H(r)$ with the interpretation as the stock of workers that remain unhoused at r, after employment and housing have been determined for locations $s \in [0,r)$. Let

$$y(r) = 2\pi r[\theta(r)n(r) - (1-\theta(r))N(r)] \tag{11.8}$$

be the excess of people employed at location r over people housed at r. Thus positive $y(r)$ values add to the stock $H(r)$ of unhoused workers and negative $y(r)$ values reduce $H(r)$. In addition, even if $y(r) = 0$, if the stock $H(r)$ is positive it will increase by the amount $\kappa H(r)\varepsilon$ over the interval $[r, r+\varepsilon)$ because housing is moved further away from employment: To bring $H(r)$ units of full time equivalent labor to r requires that $e^{\kappa\varepsilon}H(r)$ units be brought to $r+\varepsilon$, provided we are bringing labor toward the center. Combining these two forces, we have that:

$$\frac{dH(r)}{dr} = \begin{cases} y(r) - \kappa H(r) \text{ for } H(r) < 0 \\ y(r) + \kappa H(r) \text{ for } H(r) > 0 \end{cases} \tag{11.9}$$

The opposite logic applies when $H(r) < 0$. In that case, there are people who are housed at locations $s < r$ who can be employed at locations $s \geq r$: These workers are traveling away from the center to get to work, so carrying the stock outward brings people farther from home on their way to work.

For an assignment of jobs and residences to be feasible, it must be the case that every worker be housed on $[0, S]$, or that:

$$H(S) \leq 0 \tag{11.10}$$

The price of goods, set equal to one, and the utility level of workers, $\overline{u}$, are both determined by forces outside the city. It remains to determine the wage paid at location r per unit of labor employed there, and the earnings received at r per person housed at that location.

We will use the same notation, $w(r)$, to denote both the wage rate paid at location r and the earnings of a worker housed at location r.

Chapter 12

Market thickness and the impact of unemployment on housing market outcomes

12.1 The Model

12.1.1 The basic setup

In our model, the number of households in a city, denoted by M, is given. A household either lives in her own house or rents an apartment. A house cannot become an apartment. The total number of houses ^{H} in the city is fixed. All houses have the same quality but they differ in characteristics such as design, yard, etc. We use a unit circle to model the characteristics space of houses. Each point on the circle represents a unique characteristic. All households differ in their preferences regarding housing characteristics. They are uniformly distributed around the circle. A buyer's location on the circle means that any house at the location would be a perfect match for the buyer. The buyer's location is private information.

At the beginning of each period, sellers post advertisements and announce the characteristics of their house to the public. We assume each buyer can visit at most one house per period. It is thus optimal for each buyer to choose to visit the house that best matches her. A seller may have multiple visitors. We assume each seller can negotiate with at most one buyer per period; it is then optimal for the seller to choose to negotiate with the visitor who best matches the seller's house and hence shows the strongest interest in the house.

During each period, every hosuehold who lives in her own house may be hit by a shock. When hit by such a shock, the household may choose whether or not to move to a different house. If she moves, she will need to sell her current house and buy a new one. We assume that when a household moves, she will move out of her current house and rent a place to live during the transition if she cannot immediately find a new house to move into. This assumption allows us not to consider the situation in which a household continues living in her current house while it is for sale.

First, at the beginning of time t, the number of owner-occupied houses in the city N_t^H plus the number of renters, is equal to the number of households in the city; namely $M = N_t^R + N_t^H$. Second, during time t, the number of houses for sale on the market N_t^S, reduced by the number of sales made N_t^{sales}, is equal to the number of houses left unsold at the end of this period: $U_t = N_t^S - N_t^{sales}$. Third, the number of houses for sale during this period is euqal to the number of unsold houses from the previous period U_{t-1}, plus those from homeowners who move out of their houses in this period $\mu_t N_t^H$, where μ_t is a homeowner's probability of moving in this period. Namely, $N_t^S = U_{t-1} + \mu_t N_t^H$. Finally, the sum of the number of owner-occupied houses at the beginning of this period, N_t^H, and the number of unsold houses for sale from last period, U_{t-1}, is equal to the total number of houses in the city; that is, $T^H = N_t^H + U_{t-1}$.

Next, we introduce the unemployment rate, denoted $urate$, into the model. Both renters and home occupiers are assumed to have the same probability of being unemployed in each period. We assume that unemployed people are not in the market to buy houses because it is difficult for them to obtain mortgages.

Therefore, the probability that a potential buyer will actually enter the market as a buyer, denoted as γ, cannot exceed the employment rate. Moreover, because of the financial constraint, only those households whose income is above a certain fraction of the expected house price will enter the housing market because they expect to be able to afford to buy a house.

$$\gamma_t = (1 - urate) \times prob(y_t > (\tau_0 + \tau_1 urate) \times E(P_t) \mid employed). \tag{12.1}$$

where y_t is a household income, which is assumed to be i.i.d. drawn from a Pareto distribution conditional on being employed; that is, the cdf of income is $F(y) = 1 - (y/y_{min})^{-1/\sigma}$ if employed. $E(P_t)$ is the expected house price; $y > (\tau_0 + \tau_1 urate) \times E(P)$ reflects the financial constraint. A positive τ_1 means the unemployment rate has an additional discouraging effect on the household's probability of entering the market as a buyer.

The total number of buyers in the market during time t, therefore, is γ_t times the sum of those homeowners who move out of their current houses, $\mu_t N_t^H$, and those people who are currently renters, N_t^R; namely, $N_t^B = \gamma_t \mu_t N_t^H + \gamma_t N_t^R$.

12.1.2 The seller's problem

Next, we study the decision of sellers in the search-and-matching process. Suppose at time t, seller i meets with buyer j and they negotiate the sale price. The seller's value function is as follows:

$$J_R^S \tag{12.2}$$