On the Internal Structure of Cities

Author(s): Robert E. Lucas Jr. and Esteban Rossi-Hansberg

Source: *Econometrica*, Jul., 2002, Vol. 70, No. 4 (Jul., 2002), pp. 1445-1476

Published by: The Econometric Society

Stable URL: https://www.jstor.org/stable/3082004

ON THE INTERNAL STRUCTURE OF CITIES

By Robert E. Lucas, Jr. and Esteban Rossi-Hansberg[1]

We prove the existence of a symmetric equilibrium in a circular city in which businesses and housing can both be located anywhere in the city. In this equilibrium, firms balance the external benefits from locating near other producers against the costs of longer commutes for workers. An equilibrium city need not take the form of a central business district surrounded by a residential area. We propose a general algorithm for constructing equilibria, and use it to study the way land use is affected by changes in the model's underlying parameters.

Keywords: Land use, rent gradient, urban economics, commuting costs, externalities.

1. INTRODUCTION

The concentration of much of the economy's production activity in cities reflects the existence of production externalities: Benefits to any one producer from the existence of other producers nearby. Without such external benefits producers would disperse from cities to areas where land for production and residential use is cheaper. In view of these external effects, there is no reason to believe that market prices—land rents and location-specific wage rates—give firms and households the right incentives for making land use decisions.

These observations underlie many different kinds of government policies intended to influence economic activity in cities: zoning policies to restrict land use, location-specific tax incentives, and transportation subsidies to foster concentration. To design such policies well and to assess their potential, one needs a theory of equilibrium land use and a theory of optimal land use. Most existing spatial models focus on the competition among firms for high productivity sites, as in Lucas (2001), or on the competition among workers for housing near jobs, as in Mills (1967). Models of both types lead to land price gradients that capture elements of reality. But models of both types take a "map" of the city, a map that designates some areas for business use and others for residential use, as a given. Thus Mills's classic paper *assumes* a central business district surrounded by a ring of residences. The competition between firms and households for land anywhere in the city is left unstudied, and central features of the internal structure of cities are thus resolved by assumption rather than deduced from economic principles.

An important exception is Fujita and Ogawa's (1982) theory of land use in a linear city. These authors study a linear city with a fixed population, where firms and households compete for space at the different locations. The productivity of firms is determined by an external effect that depends on the distance at which other firms locate. Production requires both land and labor, in a fixed proportion, and each worker-consumer requires a fixed quantity of residential land. Thus residential and employment densities do not vary across locations. Under these assumptions, Fujita and Ogawa construct a series of examples to illustrate some possible kinds of equilibria. In their examples, low commuting costs are consistent with a Mills map, with a specialized production sector bordered by areas specialized to residential use. As commuting costs increase, areas of mixed use can emerge in equilibrium.[2]

This paper provides a competitive market theory of land use in cities that shares many features with the Fujita and Ogawa (1982) analysis. We analyze a spatial model of a city in which a single good is produced using land and labor, and in which people consume goods and residential land. Production takes place in the city and not in outlying areas because of a production externality: Productivity at any location is higher the higher is employment in neighboring locations. Moreover, workers who do not live next to their workplaces lose a part of their labor endowment getting to and from work. These two forces draw both employment and residential housing together, closer to the city's center, but the needs for land in production and for residential housing combine to keep the city from collapsing on a point.

We formulate an explicit model of the interaction of these forces that relaxes many of the assumptions imposed by Fujita and Ogawa. We assume a production technology that permits substitution between land and labor, so that changes in productivity affect the density of workers at different locations. We permit consumers to choose any quantities of both land and goods. Residential densities will thus also vary across areas of the city. Under these assumptions, we define an equilibrium and prove that one exists, and develop and apply an algorithm for calculating equilibria. We construct land-use, employment, and residential density maps of a hypothetical city, and show numerically how these maps are altered by changes in parameters.

We study a circular city in the plane, considering *only* symmetric equilibria: Resource use and prices at any location are assumed to depend only on that location's distance from the city center. Aside from symmetry in this sense, we impose *no* assumptions on land use within the city. Consumers and producers compete for land at *all* locations, subject to no constraints other than the ability to pay. A fixed area is thus divided between residential and business use in a way

---

[2] Other precursors to our work include Stull (1974) and Helpman and Pines (1977) who study land allocations that maximize aggregate land rent. These authors restrict attention to cities with the "map" of the Mills (1967) city and study land use only at the margin between business and residential use. See also Borukhov and Hochman (1977) and Anas and Kim (1996).

Anas, Arnott, and Small (1998) provide a very useful recent survey that discusses these and other related papers.

that is fully derived from standard, neoclassical assumptions on preferences and the technology for goods production, and on commuting costs. The mathematical structure of the problem of determining equilibrium in such a setting is novel, and most of the paper is devoted to developing suitable methods of analysis.

In return, the results are surprisingly simple and easy to interpret economically. The main departure from the Mills city is a mixed sector at the center, where land is used for both business and residential purposes. The size of this sector, in which commuting costs are zero, is larger the more time-consuming the commuting technology is assumed to be. A remarkable feature of the examples we compute is an extreme sensitivity of the nature of equilibria to small changes in assumed travel costs. One wonders whether this feature may carry over to other spatial models of trade.

We will set out the model in the next section, and study its mathematical structure in Sections 3 and 4. In Section 3 we take the productivity at each location in the city as given, and study the determination of equilibrium land use, production, consumption of goods and residential land, wage rates, and land rents. We prove the existence of an equilibrium in this restricted sense, and derive some of its main properties. In Section 3, the spatial pattern of productivity determines the pattern of employment and other variables. Appendix 3 of the paper shows that this equilibrium, given productivity, solves a control problem.

In Section 4, we study an operator that describes the way the pattern of employment in turn influences productivity: an external effect that is the keystone of the theory. This simultaneity is captured in a functional equation, on which an algorithm for calculating equilibria is based. We prove the existence of a solution to this equation, and thus of an equilibrium of the model. Section 5 presents the results of numerical solutions designed to illustrate the possibilities of the theory and the ways its predictions change as key parameters are varied.

It is evident that the equilibria we calculate in this paper will not be economically efficient. The location and employment decisions taken by firms in equilibrium reflect private returns that are quite different from social returns. The analysis of optimal allocations and the study of policies that might bring equilibrium and optimal allocations closer together is addressed in Rossi-Hansberg (2001).

2. THE MODEL

We consider a circular city of fixed radius S, located in a large economy. A single traded good is produced within the city, which is sold to (or purchased from) the larger economy at a competitive price. This good is produced with land and labor, under a technology we describe in a moment. The city land is owned by agents who play no role in the theory: absentee landlords. Labor is supplied elastically at the reservation utility $\bar{u}$ that prevails in the larger economy. Workers have preferences over units of the produced good and the quantity of residential land that they consume. Symmetrically with our treatment of labor input, we could treat land as available at the boundary of the city at a price q_f

determined by its value in an agricultural use, say. Instead, we simply take the radius S of the city as given.

Our objective is to understand equilibrium land use in the city, and the determination of equilibrium goods production, employment, and consumption. We first set up a notation for describing land use, then describe the production technology, then describe worker preferences, and finally describe the way employment at various locations and housing at other locations can be reconciled, given a commuting technology.

The total land area of the city, πS^2, is divided between production use and residential use. We describe locations within the city by their polar coordinates, (r, ϕ), but for most purposes we consider only symmetric equilibria, where nothing depends on ϕ, and refer simply to "location r." For any location r, then, let $\theta(r)$ be the fraction of land used for production, so that the fraction $1 - \theta(r)$ is residential land. Let the employment density—employment per unit of production land—at location r be $n(r)$, implying that total employment at r is $2\pi r\theta(r)n(r)$. Let $N(r)$ be the number of workers housed at r, per unit of residential land. Then if each such person occupies $\ell(r)$ units of land, we have $\ell(r)N(r) = 1$.

There are three aspects to the production technology. There is an ordinary, constant returns production function that relates land, labor, and the technology level to goods production. There is the external effect that relates the technology level at any one location to the employment, weighted by distance, at other locations. Finally, there is a cost—in units of lost labor time—to commuting to and from work. We describe each in turn.

Production of the traded good at location r is assumed to be a constant returns to scale function of land, $2\pi r\theta(r)$, and labor, $2\pi r\theta(r)n(r)$, at that location. Production per unit of land at location r can thus be written as

$$(2.1) \qquad x(r) = g(z(r))f(n(r)).$$

In all the numerical work reported below, the functions g and f are taken to be Cobb-Douglas:

$$(2.2) \qquad g(z) = z^\gamma$$

and

$$(2.3) \qquad f(n) = An^\alpha.$$

The intercept term $g(z(r))$ is a productivity term that reflects an external effect on production at location $(r, 0)$ of employment at neighboring locations (s, ϕ). This production externality is assumed to be linear, and to decay exponentially at a rate δ with the distance between $(r, 0)$ and (s, ϕ):[3]

$$z(r) = \delta \int_0^S \int_0^{2\pi} s\theta(s, \phi)n(s, \phi)e^{-\delta x(r, s, \phi)}d\phi\,ds,$$

---

[3] Changes in δ affect the rate at which the externality decays with distance as well as the level of the external effect. The average external effect is independent of δ.

where

$$x(r, s, \phi) = [r^2 - 2\cos(\phi)rs + s^2]^{1/2}.$$

Since allocations are assumed to be symmetric, we can write

$$(2.4) \qquad z(r) = \int_0^S \psi(r, s)s\theta(s)n(s)ds,$$

where

$$(2.5) \qquad \psi(r, s) = \delta \int_0^{2\pi} e^{-\delta x(r, s, \phi)}d\phi.$$

Each worker is endowed with one unit of labor, which he supplies inelastically to the composite activity producing-and-commuting. The third aspect of the technology is a commuting cost that takes the form of a loss of labor time that depends on the distance traveled to and from work each day. Specifically, if a worker lives at location s and works at location r, he delivers

$$e^{-\kappa|r-s|}$$

hours of labor at location r. (Clearly the restriction to symmetric allocations implies that people will commute only along rays.)

Workers have identical preferences $U(c, \ell)$ over consumption of the produced good c and residential land ℓ. In the numerical work reported below, the function U is assumed to be Cobb-Douglas:

$$(2.6) \qquad U(c, \ell) = c^\beta \ell^{1-\beta}.$$

Let $c(r)$ and $\ell(r)$ denote the goods and land consumption of everyone housed at r. Every consumer-worker at every location must receive the reservation utility level:

$$(2.7) \qquad U(c(r), \ell(r)) = \bar{u}.$$

In this setting an *allocation* will mean a collection of functions $(z, \theta, n, N, c, \ell)$ on $[0, S]$ that describe productivity, land use, employment, and consumption at each location $r \in [0, S]$. To be *feasible*, an allocation must satisfy $\theta(r) \leq 1$, $N(r)\ell(r) = 1$, (2.4), and (2.7). In addition, we need a constraint that expresses the idea that all workers must be housed somewhere in the city. We develop this constraint next.

We need a test to determine whether any given triple $(\theta(r), n(r), N(r))$ of functions on $[0, S]$ describes an internally consistent pattern of land use, employment and residential housing. Think of filling up the city, proceeding from the center, $r = 0$, outward to the edge, $r = S$. We define a state variable $H(r)$ with the interpretation as the stock of workers that remain unhoused at r, after employment and housing have been determined for locations $s \in [0, r)$.

Let

(2.8) $y(r) = 2\pi r[\theta(r)n(r) - (1 - \theta(r))N(r)]$

be the excess of people employed at location r over people housed at r. Thus positive $y(r)$ values add to the stock $H(r)$ of unhoused workers and negative $y(r)$ values reduce $H(r)$. In addition, even if $y(r) = 0$, if the stock $H(r)$ is positive it will increase by the amount $\kappa H(r)\varepsilon$ over an interval $[r, r + \varepsilon)$ because housing is moved further away from employment: To bring $H(r)$ units of full time equivalent labor to r requires that $e^{\kappa\varepsilon}H(r)$ units be brought to $r + \varepsilon$, *provided* we are bringing labor toward the center. Combining these two forces, we have that

(2.9) $\dfrac{dH(r)}{dr} = y(r) + \kappa H(r)$ if $H(r) > 0$.

The opposite logic applies when $H(r) < 0$. In that case, there are people who are housed at locations $s < r$ who can be employed at locations $s \geq r$: These workers are traveling *away from* the center to get to work, so carrying the stock outward brings people farther from home on their way to work. In this case, we have

(2.10) $\dfrac{dH(r)}{dr} = y(r) - \kappa H(r)$ if $H(r) < 0$.

Given a function $y(r)$ and the initial condition $H(0) = 0$, it is clear that (2.9) and (2.10) define a continuous function $H(r)$ on $[0, S]$. For an assignment of jobs and residences to be feasible, then, it must be the case that every worker be housed on $[0, S]$, or that

(2.11) $H(S) \leq 0$.

Then equations (2.9), (2.10), and (2.11), together with the initial condition $H(0) = 0$, complete the definition of a feasible allocation.

Now that we have described the feasible allocations, we turn to the economics of the problem. In an equilibrium, land must be allocated to firms and households, and workers must be allocated over firms. The price of goods, set equal to one, and the utility level of workers, $\bar{u}$, are both determined by forces outside the city. It remains to determine the wage paid at location r per unit of labor employed there, and the earnings received at r per person housed at that location.

We will use the *same* notation, $w(r)$, to denote both the wage rate paid at location r and the earnings of a worker housed at location r. If r is a purely business location, $w(r)$ denotes the wage paid by firms operating there, and a worker who commutes to r from s has earnings $w(s) = e^{-\kappa|r-s|}w(r)$ available to spend at his place of residence. If r is a purely residential location, then $w(r)$ denotes the earnings of people who live there, and a resident at r who works at s must receive $w(s) = e^{\kappa|r-s|}w(r)$ per unit of labor supplied at s. Finally, if r is

a mixed-use location, people who live there also work there, and $w(r)$ denotes *both* the wage rate and net earnings at r.

We let $q(r)$ be firm profit per unit of land (land rent per unit of land) at location r; let $Q(r)$ be the rent per unit of residential land at r. A firm located at r chooses employment to maximize profit:

$$(2.12) \quad q(r) = g(z(r))f(n(r)) - w(r)n(r) = \max_{n}\{g(z(r))f(n) - w(r)n\}.$$

The maximized value is $q(r)$. Since the decision problem at location r is completely determined by the technology level and the wage rate at r, we can solve the first order condition for the maximum problem in (2.12) to obtain $n = \hat{n}(w, z)$. The maximized value can be written $q = \hat{q}(w, z)$. This is the business *bid rent*, given (w, z): the rent per unit of land that a firm would be willing to pay to operate with these cost and productivity parameters. Under the Cobb-Douglas assumptions (2.2) and (2.3) we can solve explicitly for

$$\hat{n}(w, z) = \left(\frac{\alpha A z^{\gamma}}{w}\right)^{1/(1-\alpha)}$$

and

$$\hat{q}(w, z) = (1 - \alpha)\left(\frac{\alpha}{w}\right)^{\alpha/(1-\alpha)} A^{1/(1-\alpha)} z^{\gamma/(1-\alpha)}.$$

A consumer who lives at r divides his earnings $w(r)$ over goods consumption c and residential land ℓ, at the prices 1 and $Q(r)$. His earnings must be enough to yield him utility $\bar{u}$. Thus

$$(2.13) \quad w(r) = c(r) + Q(r)\ell(r) = \min_{c, \ell}[c + Q(r)\ell],$$

subject to

$$U(c, \ell) \geq \bar{u}.$$

The minimizing values depend on Q and $\bar{u}$. Suppressing the latter, we write them $\tilde{c}(Q)$ and $\tilde{\ell}(Q)$. Given these functions, we can solve

$$w = \tilde{c}(Q) + Q\tilde{\ell}(Q)$$

for $Q = \widehat{Q}(w)$, say, and then define $\hat{c}(w) = \tilde{c}(Q(w))$ and $\hat{\ell}(w) = \tilde{\ell}(\widehat{Q}(w))$. Since the market clearing condition for residential land is $N\ell = 1$, we also have $\widehat{N}(w) \equiv 1/\hat{\ell}(w)$. Under the assumption (2.6) of Cobb-Douglas utility, the functions $\widehat{N}(w)$ and $\widehat{Q}(w)$ are given by:

$$\widehat{N}(w) = \beta^{\beta/(1-\beta)} \bar{u}^{-1/(1-\beta)} w^{\beta/(1-\beta)}$$

and

$$\widehat{Q}(w) = \beta^{\beta/(1-\beta)}(1 - \beta)\left(\frac{w}{\bar{u}}\right)^{1/(1-\beta)}.$$

It will be assumed that land is allocated to its highest-value use. In context, this means that

$$(2.14) \qquad \theta(r) > 0 \quad \text{implies } q(r) \geq Q(r),$$

and

$$(2.15) \qquad \theta(r) < 1 \quad \text{implies } q(r) \leq Q(r).$$

Finally, free mobility of labor implies a sharp restriction on equilibrium wages $w(r)$:

$$(2.16) \qquad e^{-\kappa|r-s|}w(s) \leq w(r) \leq e^{\kappa|r-s|}w(s) \quad \text{for all } r, s \in [0, S].$$

This *wage arbitrage* condition says that no one can gain by changing his job location, incurring the loss in labor endowment that such a change may entail.

We conclude this section with a formal definition of an equilibrium:

DEFINITION: Let $\hat{n}(w, z)$ and $\hat{q}(w, z)$ be the employment and bid-rent functions defined by the firm's problem (2.12), and let $\widehat{N}(w)$ and $\widehat{Q}(w)$ be the residential density and bid-rent functions defined by the household's problem (2.13). Then an *equilibrium* is a pair of piecewise continuous functions θ and y, and a collection (z, n, N, w, q, Q, H) of continuous functions, all on $[0, S]$, such that for all r,

(i) $w(r)$ satisfies (2.16),
(ii) $n(r) = \hat{n}(w(r), z(r))$ and $q(r) = \hat{q}(w(r), z(r))$,
(iii) $N(r) = \widehat{N}(w(r))$ and $Q(r) = \widehat{Q}(w(r))$,
(iv) $\theta(r), q(r),$ and $Q(r)$ satisfy $0 \leq \theta(r) \leq 1$, (2.14), and (2.15),
(v) $y(r), n(r), N(r), \theta(r),$ and $H(r)$ satisfy $H(0) = 0$, (2.8)–(2.10), and
(vi) $H(S) = 0$, and
(vii) $z, \theta,$ and n satisfy (2.4).

An aerial map of an equilibrium city must thus look like a family of concentric circles, as in Figure 1. The solid circles on Figure 1 represent locations r where $H(r) = 0$. The dashed circles represent boundaries between all-business and all-residential areas. In the area label "mixed" on the figure, every location contains both businesses and housing. As we will see, no one ever moves across a solid line travelling to or from work. But people are free to move across them, so land rents and wage rates must be continuous at these boundaries. The production externalities, moreover, do drift across all boundaries, as one can see in the formula (2.4).

For the Cobb-Douglas case, we have translated our assumptions on technology and preferences into formulas for the four functions $\hat{n}(w, z), \hat{q}(w, z), \widehat{N}(w),$ and $\widehat{Q}(w)$. The proof of the existence of an equilibrium and much of the characterization of equilibria will be carried out under the more general Assumption (A).
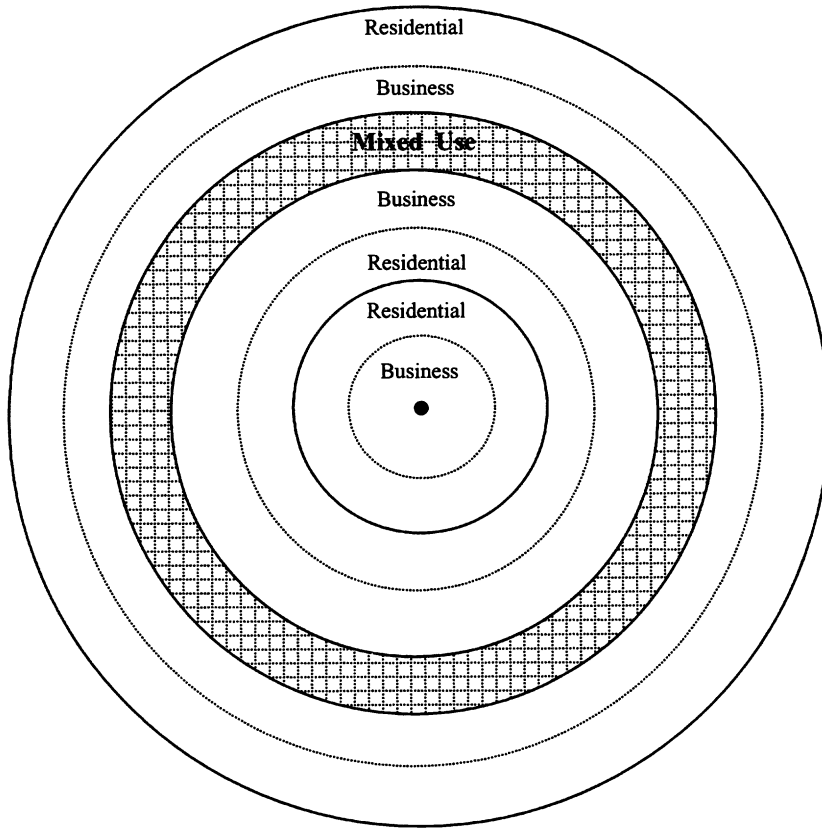
FIGURE 1.—Sample land-use map.

ASSUMPTION (A): (i) *The functions* $\hat{n}, \hat{q}$: $R_+^2 \to R_+$ *and* $\widehat{N}, \widehat{Q}$: $R_+ \to R_+$, *are continuously differentiable. Both* $\hat{n}$ *and* $\hat{q}$ *are decreasing in* w *and increasing in* z; *both* $\widehat{N}$ *and* $\widehat{Q}$ *are increasing in* w. *The first derivative* $\hat{n}_z(w, z)$ *satisfies* (ii)

$$\lim_{z \to \infty} \hat{n}_z(w, z) = 0,$$

for all $w > 0$. (iii) *The function* $U(c, \ell)$ *is strictly increasing in* c *and* ℓ.

3. EQUILIBRIUM WITH PRODUCTIVITY FIXED

The construction of an equilibrium as defined in the last section falls naturally into three steps.

Step 1: Given a continuous productivity function z on $[0, S]$ and an initial wage $\omega > 0$, find the set of functions $(n, N, w, q, Q, \theta, y, H)$ that satisfy conditions (i)–(v) of the definition of equilibrium.
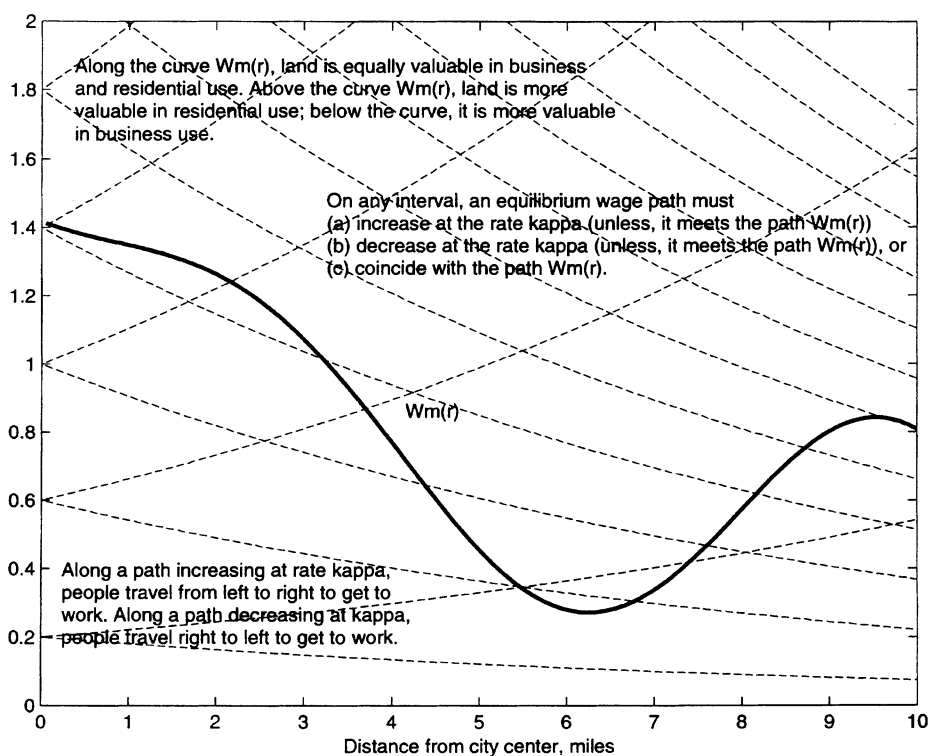
FIGURE 2.—Equilibrium wage determination.

This step will define a correspondence $\varphi(\cdot; z)$ mapping initial wages ω into terminal stocks $H(S; \omega, z)$. The construction and characterization of this set of functions, and the definition of the correspondence $\varphi(\cdot; z)$ occupies most of the rest of this section.

Step 2: Given a continuous productivity function z on $[0, S]$, show that there is a unique value ω^* such that $0 \in \varphi(\omega^*; z)$, and exactly one set of equilibrium functions $(n, N, w, q, Q, \theta, y, H)$ such that $H(S; \omega^*, z) = 0$.

That is, we show that for each z, there is exactly one allocation satisfying conditions (i)–(vi) in the definition of equilibrium. This conclusion is stated at the end of this section, as Theorem 1.

In light of Theorem 1, land use and employment functions $\theta(r; z)$ and $n(r; z)$ are uniquely defined for any productivity function z. Using (2.4), these functions define an operator T (say) on functions z.

Step 3: Show that this operator T satisfies the hypotheses of the Schauder fixed point theorem, and hence that an equilibrium in the sense of (i)–(vii) exists.

This result is stated as Theorem 2, in Section 4 of the paper.

Figure 2 is a useful diagram for thinking about Step 1: the construction of an allocation satisfying (i)–(v) with the wage rate at $r = 0$ set at $\omega > 0$. The

figure plots various paths for wages $w(r)$ against distance r from the city center. The solid curve $w_m(r)$—which we call the "mixed wage" curve—is the function defined by

(3.1) $\hat{q}(w_m, z(r)) = \widehat{Q}(w_m).$

That is, the mixed wage at any location is the wage that just equates the business and residential bid rents. It is the only wage rate consistent with a mixed land use. Clearly, $w_m(r)$ depends on r only through the level of productivity $z(r)$ at location r. For the case of Cobb-Douglas technology and preferences, the mixed wage is given by

(3.2) $w_m(r) = Kz(r)^{\gamma(1-\beta)/(1-\alpha\beta)},$

where K depends on the production intercept A and the other parameters of the theory. In general, $w_m(r)$ is higher the higher is the technology level $z(r)$ at r.

The remaining curves on Figure 2—the dashed lines—are members of the families of curves of the form $Ke^{\kappa r}$ and $Ke^{-\kappa r}$. Of course, some member of each family passes through every point in the plane. These curves are the wage paths implied under some circumstances by the wage arbitrage condition (2.16). This condition involves four inequalities, depending on whether $|r - s| = r - s$ or $s - r$. Which of these constraints binds depends on the direction in which people are moving to get to work. Consider an interval (r_1, r_2) on which $H(r) > 0$, implying that people are moving from right to left (toward the center) to get to work. Someone living at a location r on this interval has the option to travel to r_1, reducing his labor endowment by the factor $e^{-\kappa(r-r_1)}$ and earning the wage $w(r_1)$ for each unit he supplies from that point. This option puts a lower bound on his earnings:

$$w(r) \geq w(r_1)e^{-\kappa(r-r_1)}.$$

By the same reasoning, a firm located at r_1 that hires $e^{\kappa(r-r_1)}$ units of labor at $w(r)$ from someone living at r and moves these units to r_1 has the option of hiring one unit at $w(r_1)$. This option puts an upper bound on the wage it will pay at r:

$$w(r_1) \leq w(r)e^{\kappa(r-r_1)}.$$

Combining these two inequalities, we conclude that for all $r \in (r_1, r_2)$,

(3.3) $w(r) = w(r_1)e^{-\kappa(r-r_1)}.$

When $H(r) < 0$, commuting flows are in the opposite direction: Moving to the right (away from the center) means getting closer to work. In this case, reasoning analogous to the last paragraph implies

(3.4) $w(r) = w(r_1)e^{\kappa(r-r_1)}.$

As long as $H(r) \neq 0$, then, (2.16) implies that either (3.3) or (3.4) must hold, which is to say that wages must move along one of the dashed curves in Figure 2.

We describe a "shooting" algorithm for constructing equilibria, given $z(r)$, based on the observation that wages must always vary by location along one of the three kinds of paths shown on Figure 2. Before getting into the details of this construction, some features of Figure 2 should be noted. First, if the wage at a location r lies above the curve $w_m(r)$, that location must be in an exclusively residential area: By Assumption (A) and (3.1), $w(r) > w_m(r)$ implies that $\hat{q}(w(r), z(r)) < \widehat{Q}(w(r))$ at r. By the same reasoning, a location where the wage is below the mixed wage must be a purely business location.

Second, note that if the wage at r, $w(r)$, is not equal to $w_m(r)$ and if $H(r) > 0$, then the wage rate at locations near r must vary along one of the decreasing exponential paths $Ke^{-\kappa r}$, satisfying (3.3). This is because $H(r) > 0$ implies that people are travelling from right to left to get to work, and wages must vary so as to compensate those who travel furthest. By the same reasoning, $H(r) < 0$ implies that the wage must vary along one of the increasing exponential paths $Ke^{\kappa r}$, satisfying (3.4).

With these principles in mind, we turn to constructing a wage path $w(r; \omega, z)$, from a given initial value $w(0) = \omega$. Figure 3 will be helpful at this point. On the figure $\omega < w_m(0)$, so we know that location 0 is an exclusively business location. People must travel from right to left to get to work at $r = 0$—they cannot get there any other way—which implies that the wage path $w(r; \omega, z)$ beginning at ω must follow the exponentially decreasing path through the point $(0, \omega)$. As long as this path remains below the mixed path, the corresponding land use $\theta(r; \omega, z)$ remains at one, employment is given by $n(r; \omega, z) = \hat{n}(w(r; \omega, z), z(r))$, and the stock $H(r)$ of unhoused workers increases along the path given by (2.9).

If the wage path so constructed always remains below the mixed path (this would require a lower initial wage than the one shown on Figure 3), then all of the features just noted will continue to hold on the entire interval $[0, S]$, and the stock of unhoused workers at S, $H(S; \omega, z)$, will be positive. The value of the correspondence $\varphi(\cdot; z)$ will then be the singleton $\varphi(\omega; z) = H(S; \omega, z)$. It is evident that $\varphi(\cdot; z)$ will be single-valued, continuous, and strictly decreasing in a neighborhood of such an initial value ω.

The path shown on Figure 3 in fact meets the mixed path. At the point of intersection, $H(r; \omega, z) > 0$, so the path $w(r; \omega, z)$ continues to the right along the same decreasing exponential, passing into a purely residential area. Land use switches, residential density is now given by the function $\widehat{N}$, and the stock H continues to follow (2.9) but with $y(r) < 0$. The associated value of the stock $H(r; \omega, z)$ as this evolution occurs cannot be seen on Figure 3: a third axis would be needed. But suppose after a time in the residential area this stock should hit zero. Then further residents must travel to the right to get to work—all the jobs to the left have been filled—so the wage changes direction and begins to follow one of the increasing exponentials. This is the possibility illustrated on the figure.

Beginning from any initial ω, the equilibrium principles illustrated on Figure 3 can be applied in the way we have just described to derive a unique path

A shooting algorithm for constructing an equilibrium wage path:
(1) Pick a wage at r = 0: w0 (say).
(2) Continue to the right in the only possible way;
(3) Keep track of the implied path H(r,w0) of the stock of unhoused workers.
(4) When the path crosses Wm(r) into a residential area just keep going.
(5) When H(r,w0) = 0, rate of change of wages changes sign
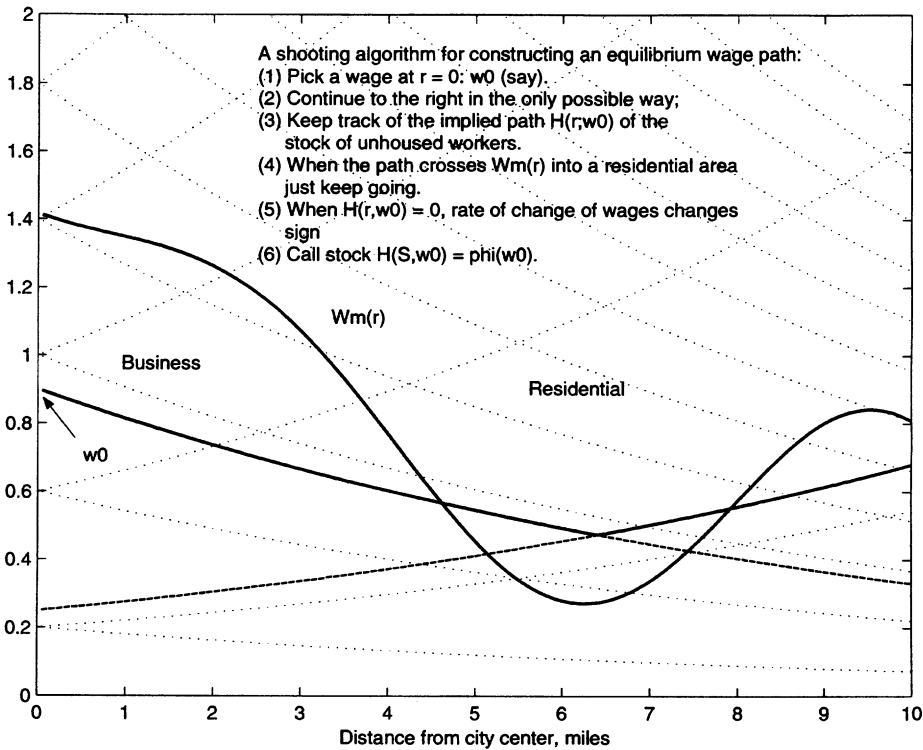(6) Call stock H(S,w0) = phi(w0).

FIGURE 3.—Equilibrium wage determination.

$w(r; \omega, z)$ and a unique associated terminal stock $\varphi(\omega; z) = H(S; \omega, z)$ *provided* that the path never meets the mixed path at a point r when it has a stock $H(r; \omega, z) = 0$. If a path meets the mixed path with a nonzero stock, it simply crosses over into a different land use area. If the stock $H(r)$ reaches zero at a point off the mixed path, the wage simply switches from a decreasing exponential to an increasing one (or the other way around). For any such path that may cross the mixed path but never coincides with it for an interval of positive length, the correspondence $\varphi(\omega; z)$ will be single valued, continuous, and strictly decreasing in a neighborhood of ω.

In our construction of the set of equilibrium paths and the correspondence φ it remains only to consider paths that meet the mixed path with a zero stock. Suppose, then, that a wage path $w(r; \omega, z)$ has reached the location r_1 with $w(r_1; \omega, z) = w_m(r)$ and $H(r_1; \omega, z) = 0$. One such point occurs at $r = 0$ along the path starting at $\omega = w_m(0)$. There may be others. One possibility is shown on Figure 4: Here the slope of the mixed wage, $w'_m(r_1)$ (where $r_1 = 4.5$ in Figure 4), is between $\kappa w_m(r_1)$ and $-\kappa w_m(r_1)$ so the mixed path satisfies the wage arbitrage condition (2.16) to the right of r_1. In this situation, the path $w(r; \omega, z)$ can be continued to the right of r_1 in three different ways. It can follow the increasing
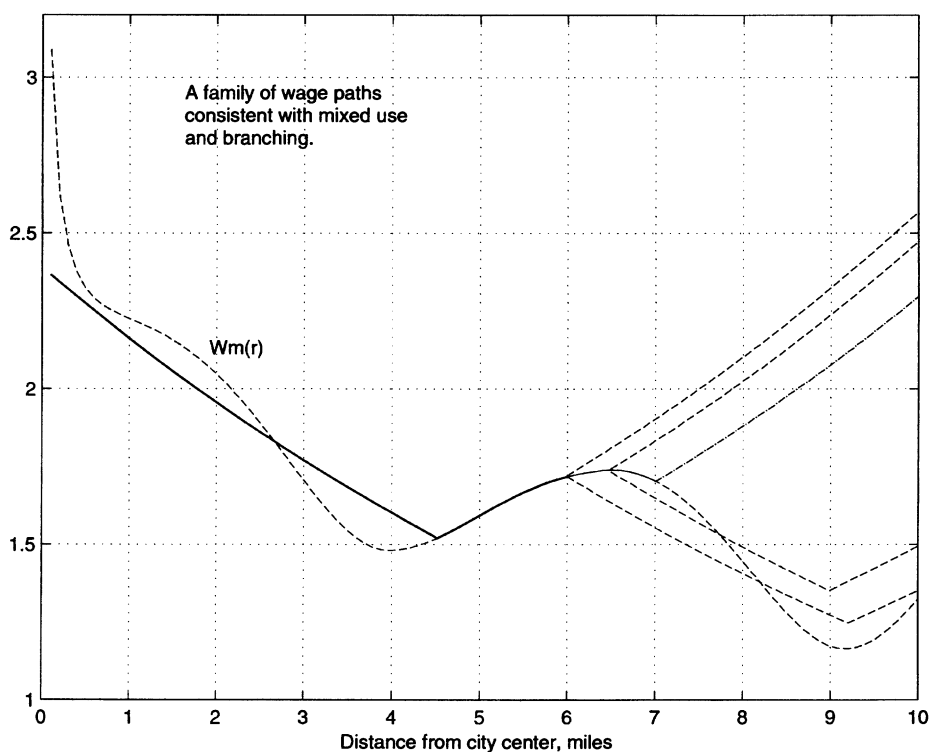
FIGURE 4.—Equilibrium wage determination.

exponential path, or the decreasing exponential path, or it can continue along the mixed path. These three possibilities continue to hold at points $r > r_1$ as long as the inequality

$$-\kappa w_m(r) \leq w'_m(r) \leq \kappa w_m(r)$$

continues to hold.

All of the branches so constructed can be continued using the principles we have already applied, possibly branching again later on, until they reach the point S. Since all of them have the initial value ω, all of the terminal values $H(S; \omega, z)$ are elements of the *set* $\varphi(\omega; z)$.

Figure 4 illustrates a possible situation in which a path that coincides with the mixed path at r_2 (where $r_2 = 7$ in Figure 4) with a stock of zero cannot be continued along $w_m(r)$. On the figure, $w'_m(r_2) < -\kappa w_m(r_2)$. Then continuing on $w_m(r)$ cannot be equilibrium behavior—wages are declining at a rate exceeding κ, so no one would be willing to work at r_2 if $w_m(r)$ were an option to the right of r_2. Since both the other continuations are above $w_m(r)$, they both require moving into a residential area. But since the stock $H(r_2) = 0$, workers must

commute to the right, ruling out the decreasing exponential curve. This leaves only the increasing exponential as an equilibrium continuation. Clearly there is an analogous possibility if both branches lie below $w_m(r)$.

In the construction just described, the wage rate uniquely determines the employment density in a business area or the residential density in a residential area, whichever may be the case, using the functions $\hat{n}$, $\widehat{N}$, and $\hat{\ell}$ defined in Section 2. On an interval where the wage coincides with $w_m(r)$, the equality of $\hat{q}(w_m(r), z(r))$ and $\widehat{Q}(w_m(r))$ again determines the allocation uniquely. An ambiguity arises, however, if the mixed path should happen to coincide with a path of the form (3.3) or (3.4) for some interval of positive length. In such a case, both the mixed allocation and a specialized allocation would satisfy (ii)–(iv), and multiple allocations would be consistent with the same wage. This observation does not affect our construction of the family of potential equilibrium wage paths, but as noted in the statement of Theorem 1, below, it does affect the way this construction is interpreted.

The next six results collect some features of the construction we have just described that we will draw on below. All the proofs of the lemmas can be found in Appendix 1.

LEMMA 1: *The correspondence* φ: $R_{++} \to R$ *is strictly decreasing,*

$$\omega' > \omega, H' \in \varphi(\omega'), \text{ and } H \in \varphi(\omega) \text{ imply } H' < H,$$

and for every $\omega > 0$, *every point in* $\varphi(\omega)$ *is reached by a unique equilibrium wage path* $w(r, \omega)$ *on* $[0, S]$.

LEMMA 2: *For all* ω, *if* $a, b \in \varphi(\omega)$ *and* $a < b$, *then there is a point* $c \in \varphi(\omega)$ *with* $a < c < b$.

LEMMA 3: *For all* ω, $\varphi(\omega)$ *is closed.*

LEMMA 4: *For all* ω, $\varphi(\omega)$ *is convex.*

Lemmas 2, 3, and 4 then imply the following.

LEMMA 5: *The correspondence* φ *is compact valued and upper-hemicontinuous at all* $\omega > 0$.

We sum up this analysis in Theorem 1.

THEOREM 1: *Under Assumption* (A), *for any continuous productivity function* z *there is an allocation that satisfies conditions* (i)–(vi). *Any such allocation is associated with a uniquely determined wage path* $w(r)$. *Except for intervals on which* $w(r)$ *coincides with the mixed path and with either* (3.3) *or* (3.4), *the allocation is uniquely determined.*
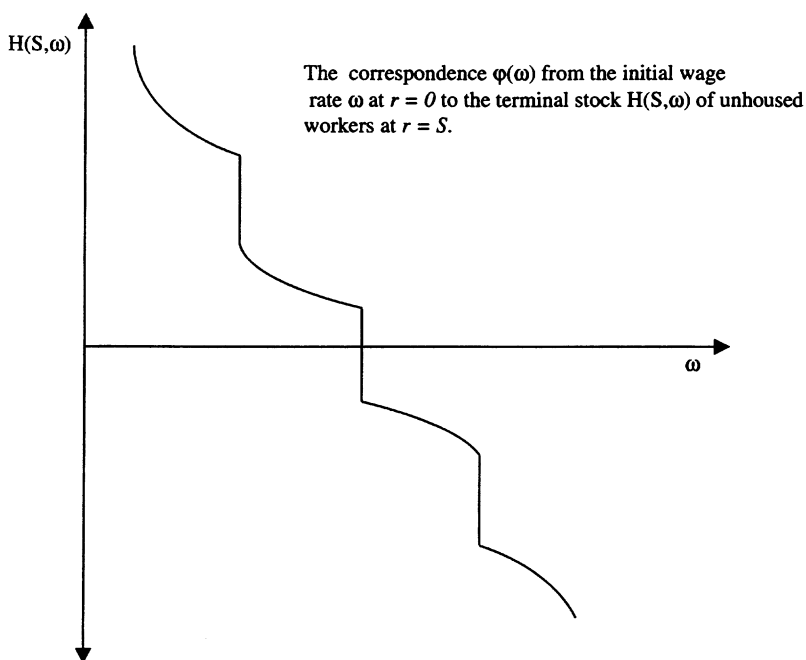
FIGURE 5.

PROOF: The correspondence φ satisfies $\varphi(\omega) > 0$ if ω is close enough to 0, and $\varphi(\omega) < 0$ if ω is large enough. It is upper-hemicontinuous, convex valued, and strictly decreasing in the sense of Lemma 1: All the features displayed in Figure 5 have been verified. Hence there is a unique ω^* such that $0 \in \varphi(\omega^*)$. By Lemma 1, there is a unique wage path that reaches the terminal stock 0 from ω^*. We have shown in the first part of this section that a unique allocation satisfying (i)–(vi) is consistent with a given wage path.   Q.E.D.

The external effect in this model involves the effects of the employment distribution on the productivity function z. For the analysis of this section, in which z is simply taken as a given, the external effect plays no role. It should not be surprising, then, that the unique equilibrium allocation solves a maximum problem. This problem is formulated and an analogue to the first theorem of welfare economics is proved, in Appendix 3.

We conclude this section with three illustrative applications of the construction described above, all for given functions $z(r)$, and all for the Cobb-Douglas case described in (2.2), (2.3), and (2.6). The first and simplest is the case of a constant z. In this case, the function $w_m(r)$ is also constant, at the value given in (3.2). Obviously this path satisfies the wage arbitrage condition (2.16), and so is an equilibrium wage path. The corresponding mixed allocation is the unique
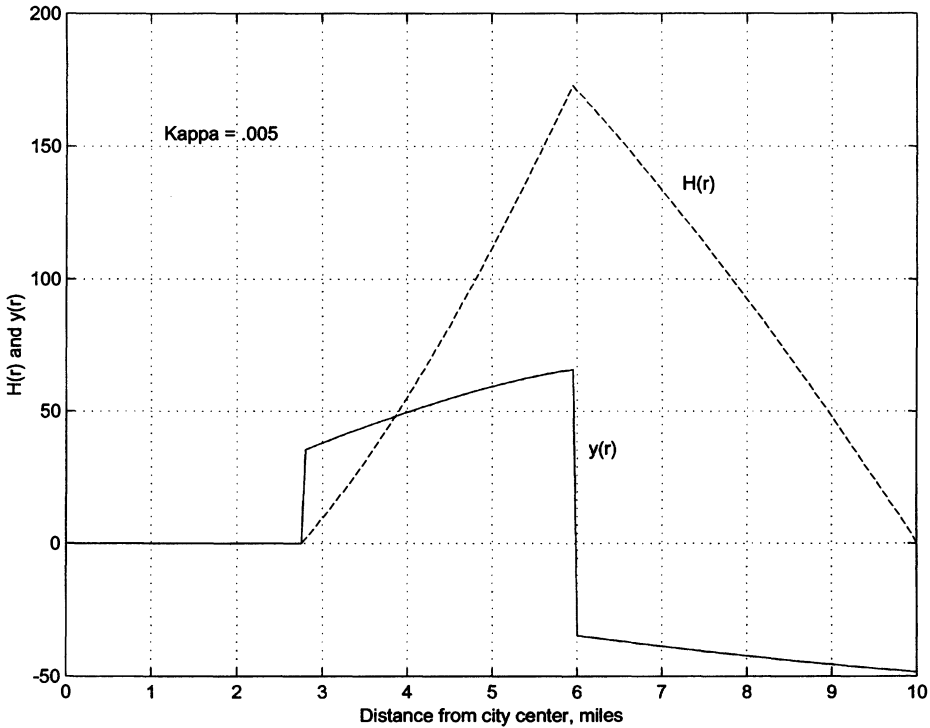
FIGURE 6.—Linear productivity example.

equilibrium. One can think of this example as an equilibrium with positive transportation costs and no comparative advantage at any location. In this situation, autarchy is the natural—and optimal—allocation.

The other two examples were solved numerically, using an algorithm based on the above construction, which is described in more detail in Section 5. Both are based on the parameter values $\kappa = .005$, $\delta = 5$, $\gamma = 0.04$, $A = u = 1$, $\alpha = 0.95$, and $\beta = 0.9$. (See Section 5 for a more detailed discussion of calibration.)

The first illustration, shown in Figure 6, is based on an assumed productivity function $z(r)$ that declines linearly from a peak at $r = 0$ to the value 0 at $r = S$. The figure plots the stock of unhoused workers $H(r)$ and the total flow of unhoused workers accumulated at r, $y(r)$. For this case, there is a mixed area for $r \in [0, 2.7]$, a business area for $r \in [2.7, 6]$, and a residential area for $r \in [6, 10]$.

The second illustration, shown in Figure 7, is based on the productivity function

$$z(r) = \begin{cases} 0 & \text{if } r \in [0, \frac{10}{3}) \cup (\frac{20}{3}, 10] \\ 20 & \text{if } r \in [\frac{10}{3}, \frac{20}{3}] \end{cases}.$$

In this example, people commute both to and away from the center, and some do not commute at all. Since productivity is zero at the center and at the boundaries,
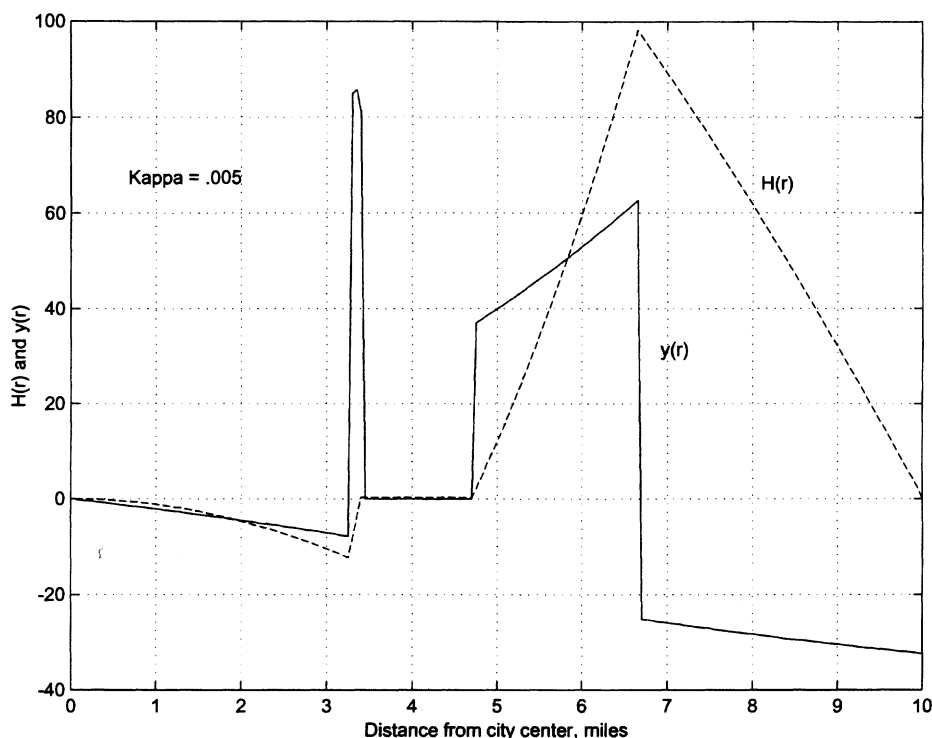
FIGURE 7.—Step function productivity example.

the city has residential areas in those locations. Between these residential areas land use is given by two business sectors with a mixed sector in the middle.

4. EXISTENCE OF EQUILIBRIUM

In the last section we proved that any continuous, nonnegative productivity function z on $[0, S]$ implies the existence of an allocation satisfying conditions (i)–(vi). Here we combine this result with equation (2.4), which describes the way productivity at each location is, through an external effect, influenced by employment at neighboring locations. We show that there is an allocation satisfying all the conditions (i)–(vii) in the definition of an equilibrium.

We denote equilibrium employment and land use at location r by $n(r; z)$ and $\theta(r; z)$, where the notation is chosen to emphasize the dependence of both variables on the entire productivity *function* z: $[0, S] \to \mathbf{R}_+$. On any interval on which there is more than one allocation consistent with the unique equilibrium wage path $w(r; z)$, we use the notation $n(r; z)$ and $\theta(r; z)$ to designate the mixed allocation *only*. Then (2.4) can be restated as a fixed point problem

(4.1) $z = Tz,$

where the operator T is defined by

(4.2) $\qquad (Tz)(r) = \int_0^S \psi(r, s)\theta(s; z)n(s; z)\,s\,ds.$

Let M be the space of continuous functions on $[0, S]$, normed by

(4.3) $\qquad \|f\| = \max_{r \in [0, S]} |f(x)|.$

Let M_+ be the subset of nonnegative valued functions. The proofs of Lemmas 6–12 can be found in Appendix 2.

LEMMA 6: $T: M_+ \to M_+$.

For the continuity of T in the norm $\|\cdot\|$, we need to study the way the functions $\theta(s; z)$ and $n(s; z)$ vary with changes in the entire function z. Since these functions are determined by the wage path, we begin by studying the continuity of the equilibrium wage paths constructed in Section 3, the existence and uniqueness of which was established in Theorem 1. Denote these paths $w(r; z)$.

LEMMA 7: *For each r, $w(r; z)$ is continuous in z.*

The next Lemma uses the continuity of the wage function to prove the continuity of the operator T in the sup norm $\|\cdot\|$.

LEMMA 8: *The operator $T: M_+ \to M_+$ is continuous in the sup norm $\|\cdot\|$.*

We will use the Schauder fixed point theorem to prove existence of an equilibrium. This theorem requires that T map a certain set $\overline{M}_+$, to be defined below, into an equicontinuous set of functions. The next three lemmas help us define and establish this property.

LEMMA 9: *For each r, $w(r; z)$ is an increasing function of z, in the sense that $z(r) \geq z'(r)$ for all r implies $w(r; z) \geq w(r; z')$ for all r.*

LEMMA 10: *There exists a positive number $\bar{z}$ such that if $z(r) \leq \bar{z}$ for all $r \in [0, S]$, then $(Tz)(r) \leq \bar{z}$ for all $r \in [0, S]$.*

LEMMA 11: *The operator T maps the set of uniformly continuous functions $z: [0, S] \to [0, \bar{z}]$ into itself.*

LEMMA 12: *Let $\overline{M}_+$ be the set of uniformly continuous functions $z: [0, S] \to [0, \bar{z}]$. Then $T(\overline{M}_+)$ is equicontinuous.*

We sum up this analysis in Theorem 2.
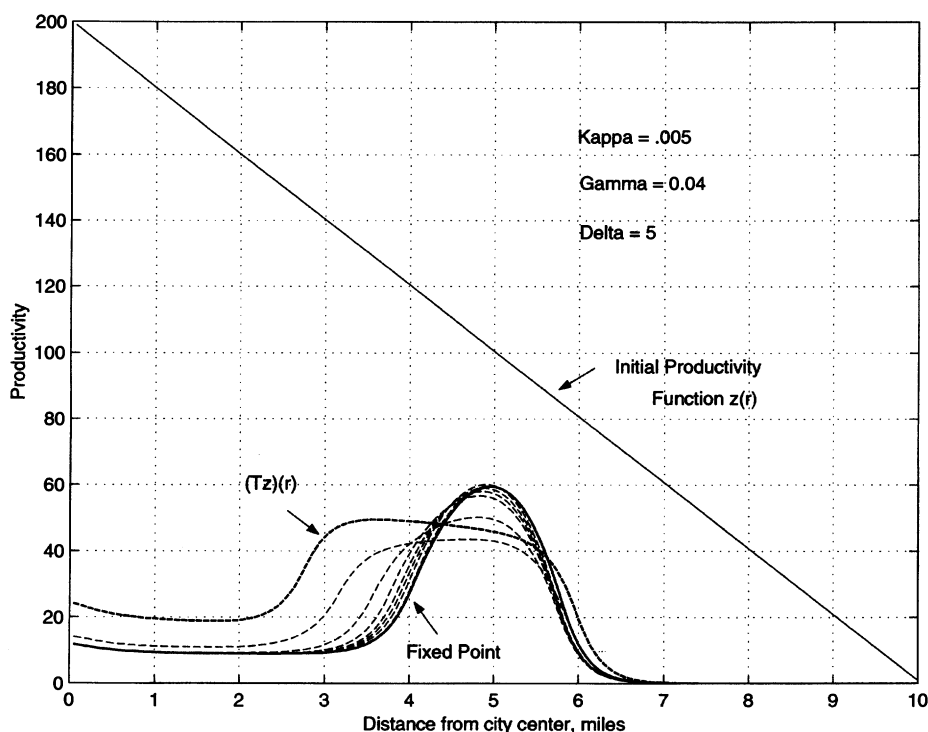
FIGURE 8.—Iterates of the operator T.

THEOREM 2: *Under Assumption* (A), *there exists an equilibrium allocation that satisfies* (i)–(vii). *That is,* $T: \overline{M}_+ \to \overline{M}_+$ *has a fixed point in* $\overline{M}_+$.

PROOF: By Lemmas 8 and 12, T is a continuous operator and $T(\overline{M}_+)$ is equicontinuous. The result then follows from Schauder's fixed point theorem.

$$Q.E.D.$$

The algorithm based on the construction of Section 3 was used as a subroutine to calculate the value Tz of the operator T, given a productivity function z. This subroutine in turn was used to calculate a sequence

$$T^{n+1}z_0 = T(T^n z_0) \qquad\qquad\qquad (n = 0, 1, 2, \dots)$$

from an initial function z_0. Figure 8 shows one such sequence, based on the parameter values given at the end of Section 3. The initial function z_0 is linear, the straight line on the figure. The equilibrium obtained by applying T to this function is shown on Figure 8: It generated the dashed curve that is highest at low r-values. Successive iterates $T^n z_0$ are also shown. One can see that this sequence converges, and that the limit function is a fixed point of T. The figure

shows that the operator T in this example is not monotone. This property is the one that forces us to use Schauder's Theorem instead of a fixed point theorem for monotone operators.

5. NUMERICAL EXPERIMENTS

In this section we report the results of computational experiments designed to illustrate the implications of the theory of Sections 3 and 4 for equilibrium land use and land prices within a city. We will describe the numerical algorithm we use to compute the examples, describe the calibration of parameters and discuss the results.

In all the computations presented we use a Cobb-Douglas functional form for preferences and technology, as given in (2.2), (2.3), and (2.6). At the end of Section 2 we presented the functions $\hat{n}$, $\hat{q}$, $\widehat{N}$, and $\widehat{Q}$ implied by this specification. In order for these functions to satisfy Assumption (A) we need to impose suitable restrictions on the parameters. In particular we need

$$(5.1) \qquad 0 < \gamma < 1 - \alpha.$$

(See Lucas (2001) or Fujita, Krugman, and Venables (1999) for a discussion of this condition.)

The mixed wage function, $w_m(r)$, is given in (3.2). Hence the wage arbitrage condition (2.16) holds at a location r if the externality function z satisfies

$$-\kappa \le \frac{\gamma(1-\beta)z'(r)}{(1-\alpha\beta)\, z(r)} \le \kappa.$$

In a mixed area, $y(r) = 0$, which implies that

$$\theta(r)\hat{n}(w_m(r), z(r)) = (1 - \theta(r))\widehat{N}(w_m(r)).$$

Using the Cobb-Douglas formulas for $\hat{n}$ and $\widehat{N}$, we conclude that

$$(5.2) \qquad \theta(r) = \frac{1-\alpha}{1-\alpha\beta}$$

holds at any mixed-use location. (The constancy of $\theta(r)$ on mixed areas is specific to the Cobb-Douglas specification.) Notice that by construction $0 \le \theta(r) \le 1$.

The numerical exercises that we describe in a moment were carried out with an algorithm based on the constructions in Sections 3 and 4. We start with two initial wages $\omega_1 > \omega_2$ and compute the corresponding values $\varphi(\omega_1)$ and $\varphi(\omega_2)$. If $0 > \varphi(\omega_2) > \varphi(\omega_1)$ or $\varphi(\omega_2) > \varphi(\omega_1) > 0$ we decrease ω_2 or increase ω_1, until $\varphi(\omega_2) > 0 > \varphi(\omega_1)$. We proceed using a bracketing algorithm until we reach ω^* such that $\varphi(\omega^*) = 0$. Notice that if we reach such an ω^*, it must be the case that $\varphi(\omega^*)$ is a singleton. It may also be the case that we converge to an ω such that $\varphi(\omega + \varepsilon) < 0 < \varphi(\omega - \varepsilon)$ for all $\varepsilon > 0$. In that case we know that $\varphi(\omega^*)$ is not a singleton. We then fix ω and find the location r at which $H(r, \omega) = 0$ and

$w(r) = w_m(r)$. At this location we set $y(r) = 0$, and proceed to location $r + dr$ (where dr is the grid size). At location $r + dr$ we let $w(r)$ grow at rate κ and calculate the resulting stock of unhoused workers at S. If this stock is equal to zero, we have found the equilibrium given the function z. If it is not zero, we repeat the calculations but letting $w(r)$ grow at rate $-\kappa$. Again we compare the resulting stock of unhoused workers with zero. If it is not equal to zero, we set $y(r + dr) = 0$ and proceed in the same fashion until either a deviation leads to a stock of unhoused workers equal to zero, or we reach S.

Once such an allocation is found we use equation (4.2) to calculate Tz. Using Tz as our new externality function we repeat the algorithm above to find the new allocation. This procedure goes on until we find a function z^* such that $Tz^* = z^*$.

Notice that in Section 4 we proved that the operator T has a fixed point, but it may not be unique. Hence the fixed point may depend on the initial function z. In all the results presented below we use a linear function z with intercept $z(0) = 200$, and slope $z'(r) = -20$ as the initial productivity function. This choice is of course arbitrary. We calculated some of the exercises below using other initial conditions and obtained the same results.

In the theory developed in this paper, there are 6 parameters describing preferences and technology: land's share in consumption expenditures, $1 - \beta$, land's share in production, $1 - \alpha$, the intercept of the production function, A, the two externality parameters γ and δ, and the travel cost parameter κ. The radius S of the city is also taken as given. The theory is a partial equilibrium model of a single city, situated in a larger economy and taking certain prices as given. These prices are the price of goods, unity, and the reservation utility level of workers, $\bar{u}$.

Taking the eight parameters $(\alpha, \beta, \gamma, \delta, A, \kappa, \bar{u}, S)$ as given, the theory determines equilibrium land use $\theta(r)$, goods production $x(r)$, employment density $n(r)$, residential density $N(r)$, and land rents $q(r)$, as well as the equilibrium radius S of the city. Given equilibrium behavior at each location r, aggregates such as total production and total employment are also determined.

We calibrate the two share parameters as $\alpha = .95$ and $\beta = .9$, following the evidence in Casselli and Coleman (2001) and Roback (1982) respectively. The externality parameter γ is set at $\gamma = .04$. These numbers will not be varied in any of the experiments reported here.

We fix the size of the city so that $S = 10$: A city always has a radius of 10 miles. In all the calculations presented below, we also set $A = 1$ and $\bar{u} = 1$.

The remaining parameters δ and κ will be varied to obtain different structures of land use. Specifically, Figures 9, 10, and 11 all show three land rent functions, corresponding to the δ values 5, 10, and 15. Figure 9 uses a κ value of .001, Figure 10 uses $\kappa = .005$, and Figure 11 uses $\kappa = .07$.

In Figure 9, all three cities portrayed have the Mills map. There is a pure business district with a radius just over 4 miles; the rest of the city is purely residential. We know (Lucas (2001)) that the limiting city in which $\kappa = 0$ takes this form. Perhaps it is not surprising that it holds for $\kappa = .001$ (where commuting time is one tenth of one percent of the work day per mile) as well. Note too that
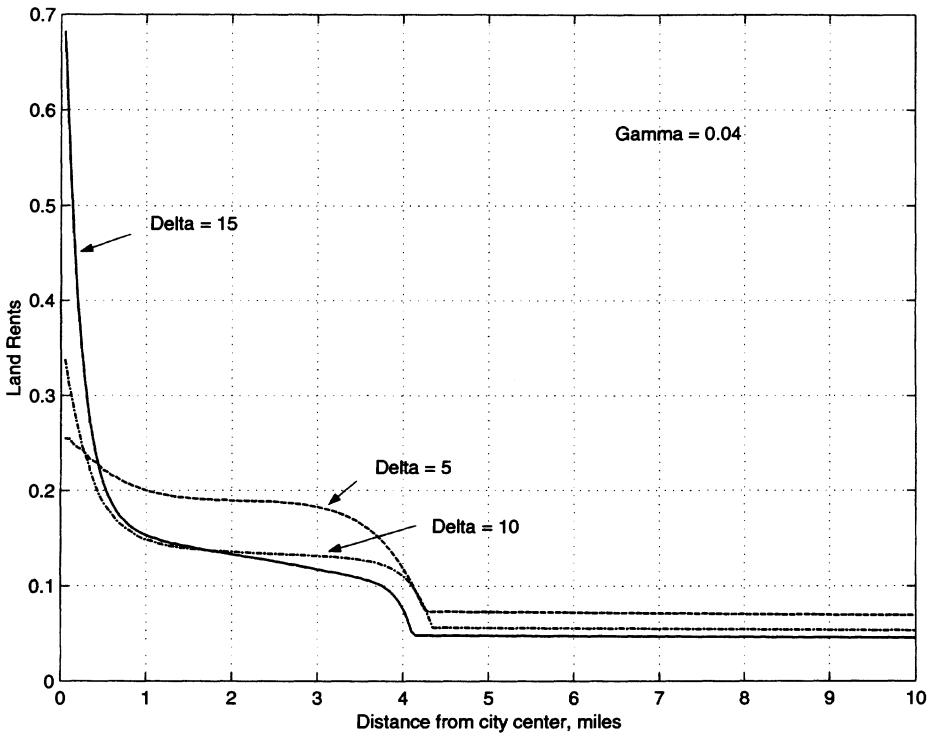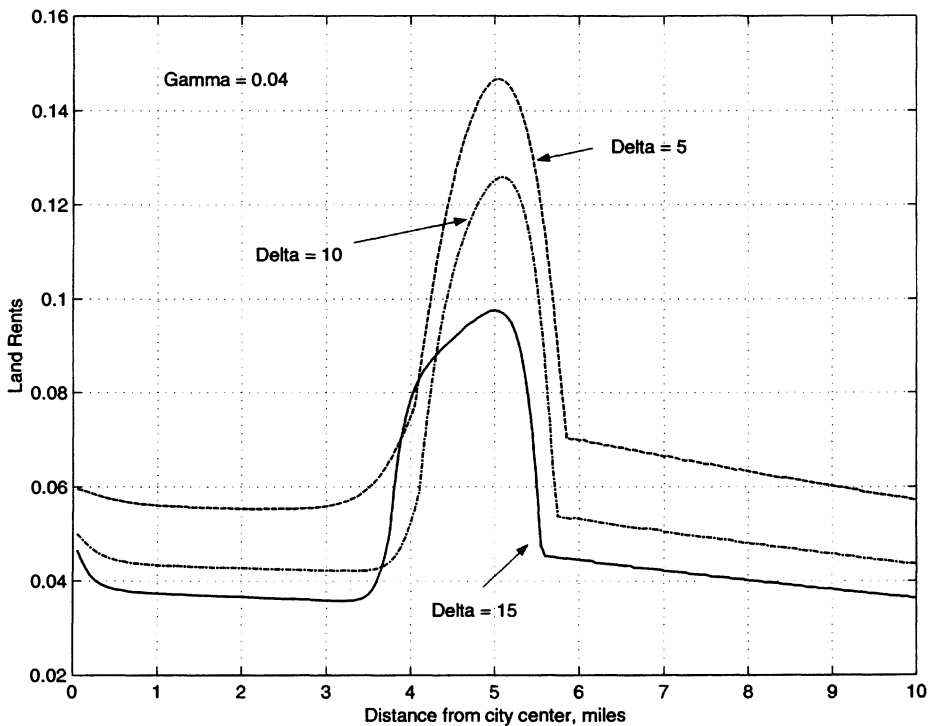
FIGURE 9.—Land rents, various delta values, kappa = 0.001.

on this figure the rent gradient within the business sector can be made as steep as desired by increasing the value of δ.

In Figure 10, κ has been increased by a factor of 5, to .005. Now a mixed area takes over the center, to a radius of about four miles (depending on δ). A purely business district then occupies the ring from $r = 4$ to almost $r = 6$. A purely residential district occupies the outer ring of the city. For this city, the highest land rent peak is associated with the lowest δ value, and rents at this peak are lower, relative to rents at the edge, than in the $\kappa = .001$ city.

In Figure 11, κ has been increased by another factor of 14, to .07. The mixed area now extends to the entire city, for $\delta = 5$, and to $r = 9.5$ miles for $\delta = 10$ and 15. In the latter two cities the pure business region has shrunk to a narrow spike, as has the outer residential ring. Land rents throughout the city are very flat at all δ values.

The main finding of these three experiments, taken together, is that the larger is κ the more land is used as a mixed area. Very low commuting costs lead to a Mills-like configuration with specialized land use. For the highest levels of commuting costs, the city reverts to something like local autarchy, with everyone

FIGURE 10.—Land rents, various delta values, kappa = 0.005.

living where he works, interacting with the world outside the city (with which transportation costs are zero) but less so with other areas within the city.

Figure 12 shows the result of another experiment, in which $\kappa = .005$ is combined with δ values of 25, 30, and 40 (a δ value of 40 means that the force of the external effect drops by one-half every 90 feet of distance). For $\delta = 25$ we obtain the same pattern of land use structure as the one shown in Figure 10. That is, there is a mixed sector in the center for $r \in [0, 2.9]$, a business sector for $r \in [2.9, 5.3]$ and a residential sector for $r \in [5.3, 10]$. As we increase δ to 30, the land use structure changes, and we obtain a small business sector in the middle with its corresponding residential sector surrounding it. Then for $r \in [0.9, 2.8]$ we again obtain a mixed sector and, surrounding it, another business area with a residential sector at the boundary of the city. Increasing δ further, to $\delta = 40$, results in two pure business districts, one within one mile of the center and a second occupying a ring between 2.5 and 4 miles. The rest of the city is residential. Here the business districts are small and concentrated, due to the high δ, and people live close to—but not at—their workplace. This figure then illustrates how a higher δ concentrates production in one or many business sectors and how mixed areas tend to disappear.
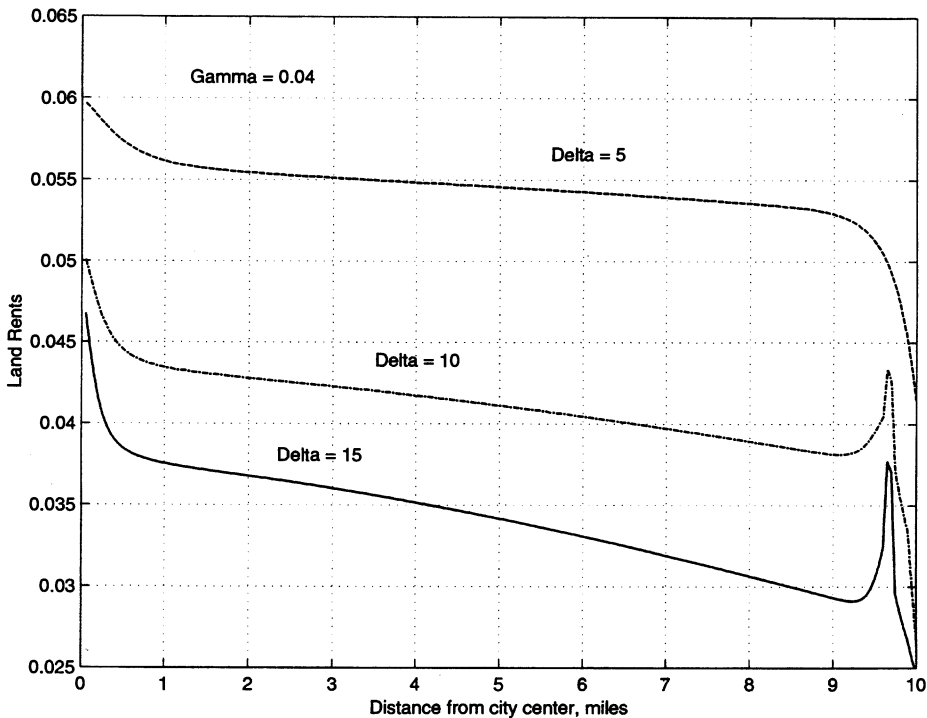
FIGURE 11.—Land rents, various delta values, kappa = 0.07.

The examples in this section illustrate that by changing the commuting cost parameter κ we can obtain a broad variety of land use patterns that go from the classic Mills city to an all mixed city. The intuition for these results is clear: The higher is κ the more costly it is to commute and so the larger are the mixed areas in the city. The effects of δ are more difficult to interpret. Increasing the value of δ decreases the external effect between firms that are far away from each other. Hence by increasing δ we can obtain multiple pure business areas in a city, as in Figure 12.

6. CONCLUSION

Let us try to summarize as simply as possible what we have learned about the spatial structure of cities. For us, a city is contained in a circle, and by virtue of symmetry the economics of the city can be studied along any radius: an interval $[0, S]$. The city has an internal structure because of a production externality under which employment at any site is more productive the higher is employment at neighboring sites, and because producers at the center, $r = 0$, have neighbors to the left while producers at the edge, $r = S$, do not have neighbors to the right.
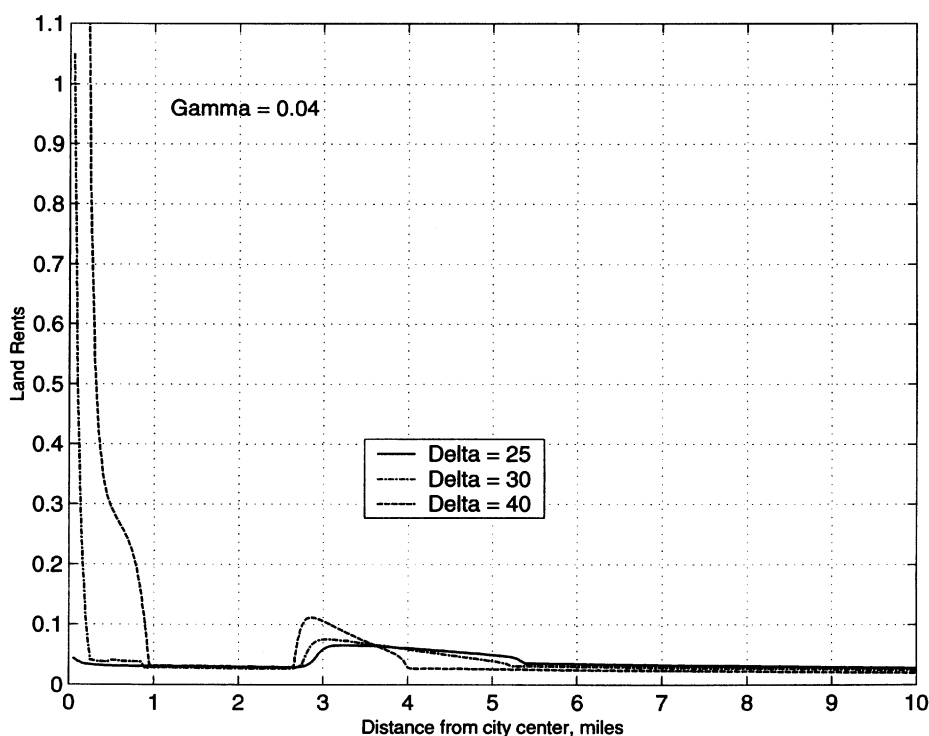
FIGURE 12.—Land rents, various delta values, kappa = 0.005.

This simple fact of geometry, together with familiar assumptions on preferences and technology, generates all the results in the paper.

Consider, as an initial example, a city in which employment is uniformly distributed over the circle and in which everyone lives where he works. In such a city, productivity differs from one site to another only because sites near the center have less empty space nearby than do sites near the edge. As we have defined the production externality, then, productivity in such a uniform city will be highest at $r = 0$ and will decline monotonically to a minimum at $r = S$. This fact will draw employment toward the city center and away from the edge, which will in turn intensify the productive advantage of the center. Only the higher land prices at the center and the necessity of land in production keep the productive activity in the city from concentrating at the point $r = 0$.

If we maintain the assumption that people live where they work, these forces can be shown to produce an equilibrium in which employment and land rents decline from the center outward. Apart from symmetry, here we impose *no* assumptions on land use, and let producers and consumers compete for land at all locations in the city, just as they would in actual cities in the absence of zoning restrictions. Many new possibilities then emerge.

The simplest of these arises when commuting costs are high enough that in a market equilibrium people live next to their jobs, but by choice, not by assumption. As commuting costs are lowered, specialized production and residential areas emerge, sometimes along side a mixed area, sometimes eliminating the latter entirely. In the limit, as commuting costs approach zero, the equilibrium takes the familiar form of the Mills city, with production in the center surrounded by a residential ring. These effects are illustrated in Figures 9–12 and coincide with the results in Fujita and Ogawa (1982).

The production externality is a second force that works against the desire to economize on commuting costs. The more localized is the external effect (the higher is the parameter δ) the higher is the value to a firm from locating near other producers and the more likely are firms to outbid residential users for land near production centers. The force is also shown in Figures 9–12.

The theory we have developed in this paper provides a basis for a theory of zoning. Rossi-Hansberg (2001) characterizes the efficient allocation in this framework. He shows numerically that production in market equilibrium is less concentrated than is efficient, because a firm deciding where to locate has no incentive to take its effect on other producers into account. He also studies policies that can increase the efficiency of land use.

Dept. of Economics, The University of Chicago, 1126 East 59th Street, Chicago, IL 60637, U.S.A.; relucas@midway.uchicago.edu,

and

Dept. of Economics, The University of Chicago, 1126 East 59th Street, Chicago, IL 60637, U.S.A.; earossih@midway.uchicago.edu; http://home.uchicago.edu/~earossih.

APPENDIX 1: PROOFS OF LEMMAS 1–5

We repeat Lemmas 1–5 from the text, and prove them.

LEMMA 1: *The correspondence φ: $R_{++} \to R$ is strictly decreasing,*

$$\omega' > \omega, H' \in \varphi(\omega'), \text{ and } H \in \varphi(\omega) \text{ imply } H' < H,$$

and for every $\omega > 0$, every point in $\varphi(\omega)$ is reached by a unique equilibrium wage path $w(r, \omega)$ on $[0, S]$.

PROOF: It is clear from the construction described above that the wage paths satisfy $w(r, \omega') > w(r, \omega)$ at all r if $\omega' > \omega$. We have shown that a larger wage strictly reduces employment at a business location and strictly increases residential density at a residential location. Thus if $w(r, \omega') > w(r, \omega)$ for any r, the corresponding stocks satisfy $H(s, \omega') < H(s, \omega)$ for all $s > r$. In the same way, if $w(r, \omega)$ and $\tilde{w}(r, \omega)$ are two paths both starting from ω that satisfy $w(r, \omega) > \tilde{w}(r, \omega)$ for any r, the corresponding stocks satisfy $H(s, \omega) < \tilde{H}(s, \omega)$ for all $s > r$. Conversely, no two terminal stocks can differ unless their associated wage paths differ at some r. Q.E.D.

LEMMA 2: *For all ω, if $a, b \in \varphi(\omega)$ and $a < b$, then there is a point $c \in \varphi(\omega)$ with $a < c < b$.*

PROOF: By Lemma 1, there are distinct wage paths, $w_a(r)$ and $w_b(r)$, say, that reach a and b, with $w_a(0) = w_b(0) = \omega$. Since $a \neq b$, these paths must diverge at some point, which is to say that there is a point r_1 such that $w_a(r) = w_b(r)$ for $r \in [0, r_1]$ and $w_a(r) > w_b(r)$ for $r \in (r_1, S]$. At this point r_1, the

corresponding values of the state variable implied by these two paths are $H_a(r_1, \omega) = H_b(r_1, \omega) = 0$ and the wage rates satisfy $w_a(r_1) = w_b(r_1) = w_m(r_1)$. To the right of r_1, there are several ways in which $w_a(r)$ and $w_b(r)$ can diverge. We consider these in turn.

Suppose $w_a(r)$ begins to grow at the rate κ, while $w_b(r)$ begins to decline at the rate κ. Then define the path $w_c(r)$ by $w_c(r) = w_a(r)$ for $r \in [0, r_1]$, $w_c(r) = w_m(r)$ for $r \in (r_1, r_1 + \varepsilon]$, and $w_c(r) = w_m(r_1 + \varepsilon)e^{\kappa(r - r_1 - \varepsilon)}$ to the right of $r_1 + \varepsilon$, where $\varepsilon > 0$ is chosen small enough so that $w_m(r)$ satisfies (2.16) on $(r_1, r_1 + \varepsilon)$. Assume that the path $w_c(r)$ is then continued as described above. Then the terminal stock c associated with the wage path $w_c(r)$ is between a and b, and since $w_c(0) = \omega$, $c \in \varphi(\omega)$ as was to be shown.

Suppose instead that $w_a(r)$ begins to grow at the rate κ while $w_b(r) = w_m(r)$ for $r \in (r_1, r_1 + \varepsilon]$, for some $\varepsilon > 0$, chosen as above. Then let $w_c(r) = w_b(r)$ for $r \in [0, r_1 + \varepsilon/2]$ and $w_c(r) = w_m(r_1 + \varepsilon/2)e^{\kappa(r - r_1 - \varepsilon/2)}$ to the right of $r_1 + \varepsilon/2$. Continue this path as described above. Then the terminal stock c associated with $w_c(r)$ is between a and b, and $c \in \varphi(\omega)$.

The final possibility, that $w_a(r) = w_m(r)$ for $r \in (r_1, r_1 + \varepsilon]$ while $w_b(r)$ begins to decline at the rate κ, can be treated in the same way.   Q.E.D.

LEMMA 3: *For all ω, $\varphi(\omega)$ is closed.*

PROOF: We show that $\{a_n\} \in \varphi(\omega)$ and $a_n \to \bar{a}$ imply $\bar{a} \in \varphi(\omega)$. If there is some N such that $n \geq N$ implies $a_n = \bar{a}$, then $a_n \in \varphi(\omega)$ implies $\bar{a} \in \varphi(\omega)$. Then without loss of generality we can take $a_n \neq a_m$ for $n \neq m$.

Since $a_n \in \varphi(\omega)$ for all n, the wage functions $w_{a_n}(r)$ and $w_{a_{n+1}}(r)$ associated with a_n and a_{n+1} must be identical on an interval $[0, r_n]$ and different on $(r_n, S]$, where r_n has the properties $H_n(r_n, \omega) = 0$ and $w_{a_n}(r_n) = w_m(r_n)$. Since the sequence $\{r_n\}$ so defined lies in the compact $[0, S]$, it has a subsequence converging to a point $\bar{r} \in [0, S]$.

All of the wage paths $w_{a_n}(r)$ depart from the mixed path, either growing or declining at the rate κ. One of these two possibilities must occur infinitely often. To be specific, we take this to be growth at the rate κ. Thus we can choose a subsequence of $\{a_n\}$ with associated wage paths $\{w_{a_n}(r)\}$ and departure locations $\{r_n\}$ such that $r_n \to \bar{r}$ and such that every $w_{a_n}(r)$ departs from the mixed path at r_n by growing at the rate κ. We can take $\{a_n\}$ to be this subsequence.

Now we define the wage path $\bar{w}(r)$ by $\bar{w}(r) = \lim_{n \to \infty} w_{a_n}(r)$ for $r \in [0, \bar{r})$, $\bar{w}(r) = w_m(\bar{r})$, and $\bar{w}(r) = w_m(\bar{r})e^{\kappa(r - \bar{r})}$ to the right of $\bar{r}$. Let $\hat{a}$ be the terminal stock associated with the path $\bar{w}(r)$. Since $\bar{w}(0) = \lim_{n \to \infty} w_{a_n}(0) = \omega$, $\hat{a} \in \varphi(\omega)$. We have constructed $\bar{w}(r)$ in such a way that $a_n \to \hat{a}$. Hence $\bar{a} = \hat{a}$, which implies $\bar{a} \in \varphi(\omega)$.   Q.E.D.

LEMMA 4: *For all ω, $\varphi(\omega)$ is convex.*

PROOF: Suppose $a, b \in \varphi(\omega)$ and $a < c < b$. By Lemma 2, we can construct a monotonic sequence $\{a_n\}$ of points in $\varphi(\omega)$, either increasing toward c or decreasing toward c. Considering the first case only, suppose the limit of any such increasing sequence is less than or equal to a point $d < c$. By Lemma 3, $d \in \varphi(\omega)$. But then by Lemma 2 there is a point $e \in \varphi(\omega)$ with $e > d$, a contradiction.   Q.E.D.

LEMMA 5: *The correspondence φ is compact valued and upper-hemicontinuous at all $\omega > 0$.*

PROOF: The sets $\varphi(\omega)$ are closed by Lemma 3 and each element of $\varphi(\omega)$ is associated with a wage path contained in the interval $[\omega e^{-\kappa S}, \omega e^{\kappa S}]$. Thus φ is compact valued. Suppose $\{\omega_n\}$ is a sequence of positive numbers with $\omega_n \to \omega > 0$, and that $\{a_n\}$ is a sequence of numbers with $a_n \in \varphi(\omega_n)$. If $\omega_n = \omega$ for $n > N$, for some N, then $\{a_n\}$ has a subsequence converging to a point $a \in \varphi(\omega)$ by the compactness of the set $\varphi(\omega)$. To show that $\{a_n\}$ has a subsequence converging to a point $a \in \varphi(\omega)$ for all sequences $\{\omega_n\}$, then, we need only show that on the intervals on the interior of which $\varphi(\omega)$ is a function, it is continuous.

By Lemma 1, each point a_n is reached by a distinct wage path $w(r, \omega_n)$, and by the construction above it is clear that $\omega_n \to \omega$ implies $w(r, \omega_n) \to w(r, \omega)$ for all r. If $\varphi(\omega)$ is a singleton, then the path $w(r, \omega)$ is uniquely defined by Lemma 1. If $\varphi(\omega)$ is not a singleton then it is an interval, and

we take the path $w(r, \omega)$ to be the uniquely defined path that reaches the appropriate endpoint $a \in \varphi(\omega)$. The continuity of the wage paths then ensures that $\omega_n \to \omega$ implies $H(S, \omega_n) \to H(S, \omega)$, or that $a_n \to a$.   Q.E.D.

APPENDIX 2: PROOFS OF LEMMAS 6–12

We repeat Lemmas 6–12 from the text and prove them.

LEMMA 6: $T: M_+ \to M_+$.

PROOF: The function ψ is continuous in r, so Tz is continuous. It is nonnegative, because the functions ψ, $\theta(\cdot; z)$, and $n(\cdot; z)$ are nonnegative.   Q.E.D.

LEMMA 7: *For each r, $w(r; z)$ is continuous in z.*

PROOF: In Figure 2, z appears only in the mixed path wage function $w_m(r) \equiv \hat{w}_m(z(r))$. If a sequence $\{z_n\}$ converges to z in the norm (4.3), the sequence $\{\hat{w}_m(z_n(r))\}$ will converge to $\hat{w}_m(z(r))$ in this norm, too, since the function $\hat{w}_m(z)$, as are all the functions defined in Sections 2 and 3, is continuous. Then we follow a construction of a wage path from a given initial ω, as described in Section 3. The same exponentials are followed, and their meeting times with the mixed path vary continuously with z. Hence, the implied stocks $H(r, \omega; z)$ vary continuously with z.

Suppose that the equilibrium wage path satisfies $w(0) = \omega^*(z)$ where $\varphi(\omega^*(z))$ is a singleton. By Lemmas 1 and 5, φ is a continuous and decreasing function for ω close to ω^*. Since $\omega^*(z) = \{\omega : H(S, \omega; z) = 0\}$, $\omega^*(z)$ is continuous in z. Then $w(r; z)$ is piecewise exponential with intercept continuous in z, and slope sign changes at locations that vary continuously with z. Hence $w(r; z)$ is continuous in z.

Suppose that the equilibrium wage path satisfies $w(0) = \omega^*(z)$ where $\varphi(\omega^*(z))$ is not a singleton and $0 \in \text{int}(\varphi(\omega^*(z)))$. Assume further that for locations such that $w(r; z) = \hat{w}_m(z(r))$, $\hat{w}_m(z(r))$ satisfies the wage arbitrage condition (2.16) with strict inequality. Then there exists an $r^m(z)$ such that $H(r^m(z), \omega^*(z); z) = 0$ and $w(r^m(z); z) = \hat{w}_m(z(r^m))$. Notice that under the above assumptions, for ω close enough to ω^*, $H(r, \omega; z)$ is continuous in z, and $H(r, \omega; z) = 0$ defines ω as a continuous function of z. Hence $r^m(z)$ and $\omega^*(z)$ are continuous functions of z. The argument above and $\hat{w}_m(z)$ continuous then imply that $w(r; z)$ is continuous in z.

Assume that either $0 \notin \text{int}(\varphi(\omega^*(z)))$ but $0 \in \varphi(\omega^*(z))$ or $\hat{w}_m(z(r))$ satisfies condition (2.16) with equality for some location such that $w(r; z) = \hat{w}_m(z(r))$. It was proven in Lemma 5 that if $\varphi(\bar{\omega})$ is not a singleton $\lim_{\omega \to \bar{\omega}} \varphi(\omega) \in \varphi(\bar{\omega})$, where $\varphi(\omega)$ is a singleton for ω close to $\bar{\omega}$. Hence since in this case $0 \in \lim_{\omega \to \omega^*(z)} \varphi(\omega)$, by both proofs above $\omega^*(z)$ is continuous in z and so $w(r; z)$ is continuous in z.   Q.E.D.

LEMMA 8: *The operator $T: M_+ \to M_+$ is continuous in the sup norm $\| \cdot \|$.*

PROOF: We need to show that for every $z \in M_+$ and any $\varepsilon > 0$ there is an $\eta > 0$ such that $z' \in M_+$ and $\|z - z'\| < \eta$ implies $\|Tz - Tz'\| < \varepsilon$. For any $z, z' \in M_+$ we have

$$\|Tz - Tz'\| = \left\| \int_0^S \psi(r, s) s \theta(s; z) \hat{n}(w(s; z), z) \, ds - \int_0^S \psi(r, s) s \theta(s; z') \hat{n}(w(s; z'), z') \, ds \right\|$$

$$\leq \left\| \int_0^S \psi(r, s) s (\theta(s; z) - \theta(s; z')) \hat{n}(w(s; z), z) \, ds \right\|$$

$$+ \left\| \int_0^S \psi(r, s) s \theta(s; z') (\hat{n}(w(s; z), z) - \hat{n}(w(s; z'), z')) \, ds \right\|$$

$$\leq \int_0^S \psi(r, s) s \, | \, \theta(s; z) - \theta(s; z') \, | \, \hat{n}(w(s; z), z) \, ds$$

$$+ \int_0^S \psi(r, s) s \theta(s; z') \, | \, \hat{n}(w(s; z), z) - \hat{n}(w(s; z'), z') | \, ds.$$

By Lemma 7, $w(s; \cdot)$ is continuous in z for all $s \in [0, S]$. Since $\hat{n}$ is a continuous function in both arguments, the second term can be made arbitrarily small by choice of z and z'.

For the first term, recall that the definition of equilibrium implies that $\theta(s; z) = 1$ if $w_m(s; z) > w(s; z)$, and $\theta(s; z) = 0$ if $w_m(r; z) < w(r; z)$. When $w_m(r; z) = w(r; z)$, $\theta(s; z) \in [0, 1]$ varies continuously with $w_m(r; z)$. From Lemma 7, both $w_m(s; z)$ and $w(s; z)$ are continuous functions of z. Hence, the points at which $\theta(\cdot; z)$ jumps are also continuous in z. Since we are integrating over $s \in [0, S]$ this implies that the first term can be made arbitrarily small by choice of z and z'.   Q.E.D.

LEMMA 9: *For each r, $w(r; z)$ is an increasing function of z, in the sense that $z(r) \geq z'(r)$ for all r implies $w(r; z) \geq w(r; z')$ for all r.*

PROOF: The function $w_m(r; z)$ is increasing in z. This implies (see Figure 3) that if $z'(r) \geq z(r)$ for all r, then for any given ω, the associated terminal stocks satisfy $H(S, \omega; z) \leq H(S, \omega; z')$. Since by Lemma 1 $H(S, \omega; z)$ is strictly decreasing in ω, this implies that $\omega^*(z) \geq \omega^*(z')$. It follows that $w(r; z) \geq w(r; z')$ for all r.   Q.E.D.

LEMMA 10: *There exists a positive number $\bar{z}$ such that if $z(r) \leq \bar{z}$ for all $r \in [0, S]$, then $(Tz)(r) \leq \bar{z}$ for all $r \in [0, S]$.*

PROOF: Let z be any function on $[0, S]$ that takes values $z(r) \in [0, \bar{z}]$ for some $\bar{z} > 0$. We find a lower bound for $w(r; z)$ and then an upper bound for $\hat{n}[w(r; z), z(r)]$.

An agent who lives at r divides his earnings $w(r; z)$ into goods consumption $c(r; z)$ and residential land $\ell(r; z)$ in a way that satisfies (2.7):

$$\bar{u} = U(c(r; z), \ell(r; z)).$$

Clearly, $w(r; z) \geq c(r; z)$ and $2\pi S^2 \geq \ell(r; z)$, so part (iii) of Assumption (A) implies

$$\bar{u} \leq U(w(r; z), 2\pi S^2).$$

For an agent who works at r and lives at r', we have similarly

$$\bar{u} = U(c(r'; z), \ell(r'; z)) \leq U(w(r'; z), 2\pi S^2) \leq U(w(r; z), 2\pi S^2).$$

Then part (iii) of Assumption (A) implies

$$w(r; z) \geq U^{-1}(\bar{u}, 2\pi S^2) \equiv \underline{w},$$

where U^{-1} denotes the inverse function of U given a value of ℓ.

Part (i) of Assumption (A) implies that

$$\hat{n}[w(r; z), z(r)] \leq \hat{n}[\underline{w}, z(r)] \qquad \text{for all } r \in [0, S].$$

Since $z(r) \leq \bar{z}$ for all r, part (i) of Assumption (A) also implies

$$\hat{n}[\underline{w}, z(r)] \leq \hat{n}[\underline{w}, \bar{z}] \qquad \text{for all } r \in [0, S].$$

The shares $\theta(s; z)$ are bounded by one, and $\psi(r, s) \in [0, 2\pi\delta]$. Thus

$$(Tz)(r) = \int_0^S \psi(r, s)\theta(s; z)n(s; z)s \, ds$$

$$\leq 2\pi\delta \int_0^S \hat{n}(w(s; z), z(s))s \, ds$$

$$\leq \pi\delta S^2 \hat{n}[\underline{w}, \bar{z}].$$

By part (ii) of Assumption (A), $\bar{z}$ can be chosen so that $z(r) \leq \bar{z}$ for all r implies

$$\pi\delta S^2 \hat{n}[\underline{w}, \bar{z}] \leq \bar{z}.$$

Hence if $z(r) \leq \bar{z}$ for all r, $(Tz)(r) \leq \bar{z}$ for all r.   Q.E.D.

LEMMA 11: *The operator T maps the set of uniformly continuous functions z: $[0, S] \to [0, \bar{z}]$ into itself.*

PROOF: By Lemma 6, Tz is continuous and since $[0, S]$ is compact, Tz is uniformly continuous. The result then follows by Lemma 10.   *Q.E.D.*

LEMMA 12: *Let $\bar{M}_+$ be the set of uniformly continuous functions z: $[0, S] \to [0, \bar{z}]$. Then $T(\bar{M}_+)$ is equicontinuous.*

PROOF: By Lemma 11, for any $\varepsilon > 0$ there is an $\eta_z > 0$ such that $|r - r'| < \eta_z$ implies $|z(r) - z'(r)| < \varepsilon$. Since $z \in [0, \bar{z}]$, we can define $\bar{\eta} = \inf[\eta_z] > 0$, where the inequality comes from the fact that the space of functions with compact range and domain is closed in the sup norm. Hence $T(\bar{M}_+)$ is equicontinuous.   *Q.E.D.*

APPENDIX 3: A CONTROL PROBLEM RELATED TO SECTION 3

We state a control problem for allocating resources, given a productivity function z, and prove that the equilibrium of Section 3 satisfies a version of the first welfare theorem.

PROBLEM (P): Given z: $[0, S] \to \mathbf{R}_+$, choose functions n, N, θ, c, ℓ, and H on $[0, S]$ so as to maximize

(A.1) $$\int_0^S 2\pi r[\theta(r)g(z(r))f(n(r)) - (1 - \theta(r))N(r)c(r)]\,dr,$$

subject to

(A.2) $1 \geq \theta(r) \geq 0,$

(A.3) $U(c(r), \ell(r)) \geq \bar{u},$

(A.4) $n(r), N(r) \geq 0,$

(A.5) $\ell(r) = \dfrac{1}{N(r)},$

(A.6) $H(0) = 0, H(S) \leq 0,$

where the evolution of H is given by equations (2.9) and (2.10).

THEOREM 3: *Under Assumption (A), if the set of functions $\{\theta, y, n, N, w, q, Q, H\}$ satisfies conditions (i)–(vi) in the definition of equilibrium given z, the functions $\{n, N, \theta, c, \ell, H\}$ solve Problem (P). That is, given z, the equilibrium allocation is Pareto-optimal.*

PROOF: We need to show that the functions $\{n, N, \theta, c, \ell, H\}$ solve Problem (P). By Assumption (A) the solution to Problem (P) is given by the conditions of the maximum principle; that is,

$$g(z(r))f'(n(r)) = \lambda(r),$$

$$c(r) + \frac{U_\ell \xi(r)}{N(r)^2 2\pi r(1 - \theta(r))} = \lambda(r),$$

$$2\pi r(1 - \theta(r))N(r) = U_c \xi(r),$$

(A.7) $\dfrac{\partial \lambda(r)}{\partial r} = \begin{cases} -\kappa\lambda(r) & \text{if } H(r) > 0 \\ \kappa\lambda(r) & \text{if } H(r) < 0 \end{cases}$,

and constraints (A.3), (A.5), and (A.6), where λ is the co-state and $\xi(r)$ is the Lagrange multiplier associated with constraint (A.3).

Given w, conditions (ii) and (iii) in the definition of equilibrium imply that

$$g(z(r))f'(n(r)) = w(r),$$

$$\xi(r) = U_c,$$

$$\xi(r)Q(r) = U_\ell,$$

$$w(r) = c(r) + \frac{Q(r)}{N(r)},$$

plus constraint (A.3) and (A.5). These conditions are equivalent to the ones derived for Problem (P) if $\lambda(r) = w(r)$. Notice that if $\lambda(r) = w(r)$, λ satisfies condition (A.7) and that the n, N, and θ implied by λ then satisfy condition (A.6).   *Q.E.D.*

REFERENCES

ANAS, A., R. ARNOTT, AND K. A. SMALL (1998): "Urban Spatial Structure," *Journal of Economic Literature*, 36, 1426–1464.

ANAS, A., AND I. KIM (1996): "General Equilibrium Models of Polycentric Urban Land Use with Endogenous Congestion and Job Agglomeration," *Journal of Urban Economics*, 40, 232–156.

BORUKHOV, E., AND O. HOCHMAN (1977): "Optimum and Market Equilibrium in a Model of a City without a Predetermined Center," *Environment and Planning*, A, 9, 849–856.

CASELLI, F., AND W. J. COLEMAN II (2001): "The U.S. Structural Transformation and Regional Convergence: A Reinterpretation," *Journal of Political Economy*, 109, 584–616.

FUJITA, M., P. KRUGMAN, AND A. J. VENABLES: (1999): *The Spatial Economy*. Cambridge: MIT Press.

FUJITA, M., AND H. OGAWA (1982): "Multiple Equilibria and Structural Transition of Non-monocentric Urban Configurations," *Regional Science and Urban Economics*, 12, 161–196.

HELPMAN, E., AND D. PINES (1977): "Land and Zoning in an Urban Economy: Further Results," *American Economic Review*, 67, 982–986.

LUCAS, R. E., JR. (2001): "Externalities and Cities," *Review of Economic Dynamics*, 4, 245–274.

MILLS, E. S. (1967): "An Aggregate Model of Resource Allocation in a Metropolitan Area," *American Economic Review*, 57, 197–210.

ROBACK, J. (1982): "Wages, Rents, and the Quality of Life," *Journal of Political Economy*, 90, 1257–1278.

ROSSI-HANSBERG, E. A. (2001): "Optimal Urban Land Use and Zoning," University of Chicago Working Paper.

STULL, W. J. (1974): "Land Use and Zoning in an Urban Economy," *American Economic Review*, 64, 337–347.