Testing Gravity with Gravitational Waves

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Plan

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  - Ringdown in GR: black hole perturbation theory
- **Ringdown in Horndeski**
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    - Quasinormal mode calculation
    - Fisher forecast
- **Conclusions and future directions**

Motivation

- Probing gravity in strong field regime
- Dark energy

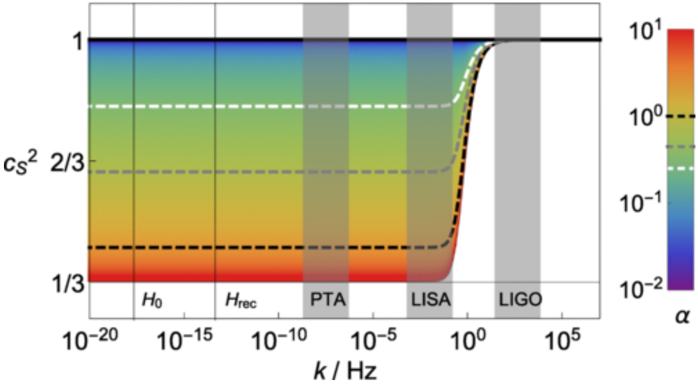
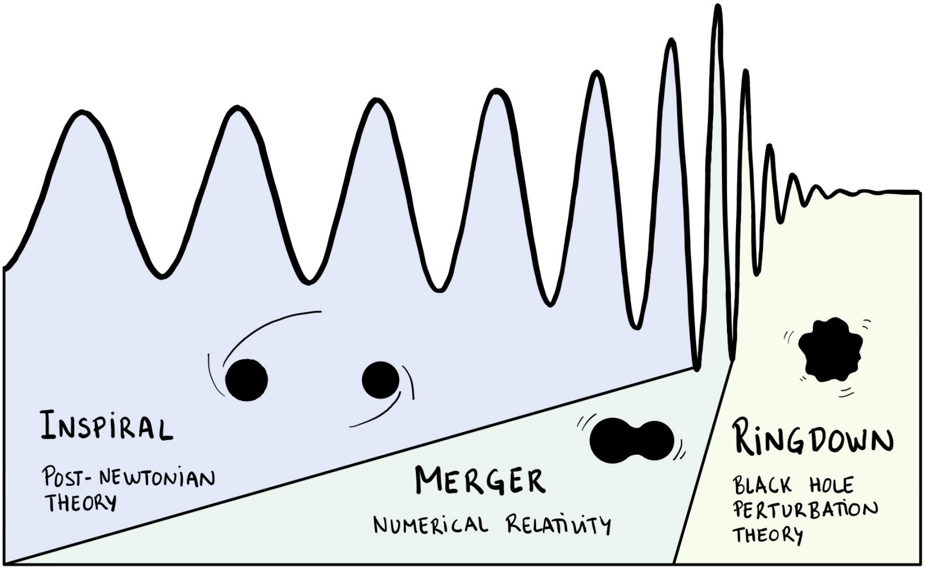


Figure: Gravitational Rainbows [C. de Rham + S. Melville, 1806.09417]

$$\alpha_T \equiv (c_{GW}^2 - c^2)/c^2. \tag{1}$$

Ringdown waves



Black hole perturbation theory

Black hole perturbation theory

Theory: GR

Most general action for $g_{\mu\nu}$ in $4D$ with 2^{nd} order equations of motion.

$$S = \int d^4x \sqrt{-g} R, \tag{2}$$

Background: Schwarzschild black hole

Most general spherically symmetric (static and asymptotically flat) solution.

$$ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -f(r)dt^2 + f(r)^{-1}dr^2 + d\Omega^2, \qquad f(r) = \left(1 - \frac{2M}{r}\right). \tag{3}$$

We expand the metric to linear order in small perturbations $h_{\mu\nu}$ around the background $\bar{g}_{\mu\nu}$.

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \tag{4}$$

Now we substitute this into the action and obtain the *quadratic action*.

$$S^{(2)} = \frac{1}{8}\int d^4x\sqrt{-g}\Big((\nabla_\mu h)^2 - 2\nabla^\mu h\nabla_\nu h_\mu^{\ \nu} + 2\nabla_\nu h_{\mu\sigma}\nabla^\sigma h^{\mu\nu} - (\nabla_\sigma h_{\mu\nu})^2\Big) \tag{5}$$

We have used the fact that the Schwarzschild geometry is Ricci-flat ($R_{\mu\nu}$=0=R).

Black hole perturbation theory

The next step is to substitute the components in the action. For this we expand $h_{\mu\nu}$ into *tensorial spherical harmonics*. Perturbations separate into two sectors

$$h_{\mu\nu} = h^{\mathrm{odd}}_{\mu\nu} + h^{\mathrm{even}}_{\mu\nu} \tag{6}$$

In the *Regge-Wheeler gauge* these look like

$$h^{odd}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & h_0 \\ 0 & 0 & 0 & h_1 \\ 0 & 0 & 0 & 0 \\ * & * & 0 & 0 \end{pmatrix} \sin\theta \partial_\theta Y_{lm} e^{-i\omega t}, \quad h^{even}_{ab} = \begin{pmatrix} fH_0 & H_1 & 0 & 0 \\ * & \frac{1}{f} H_2 & 0 & 0 \\ 0 & 0 & r^2 K & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta K \end{pmatrix} Y_{lm} e^{-i\omega t} \tag{7}$$

To linear order, the two sectors decouple and evolve independently, so we can write

$$S^{(2)} = S^{(2)}_{odd} + S^{(2)}_{even} \tag{8}$$

Odd sector: Regge-Wheeler equation

The odd quadratic action is given by

$$S^{(2)}_{odd} = \frac{\ell(\ell+1)\pi}{(1+2\ell)} \int dtdr \frac{1}{fr^2}\Big((\ell^2+\ell-2)({h_0}^2 - f^2{h_1}^2) + f(2{h_0}^2 + r^2({\dot{h}_0}^2 + {h_0'}^2) \tag{9}$$

$$-2r\dot{h}_1(-2h_0 + rh_0')) + 2f^3h_1(3h_1 - 2rh_1'))\Big) \tag{10}$$

It contains two functions $h_0(t,r)$ and $h_1(t,r)$, but only one physical degree of freedom.
Now we apply the variational principle

$$\frac{\delta S^{(2)}_{odd}}{\delta h_0} = 0 \Rightarrow \quad h_0'' + i\omega h_1' + 2i\omega\frac{h_1}{r} - \frac{h_0}{r^2}f^{-1}\left(\ell(\ell+1) - \frac{4M}{r}\right) = 0, \tag{11}$$

$$\frac{\delta S^{(2)}_{odd}}{\delta h_1} = 0 \Rightarrow \quad f^{-1}\left(2i\omega\frac{h_0}{r} + \omega^2 h_1 - i\omega h_0'\right) - \frac{h_1}{r^2}(\ell+2)(\ell-1) = 0. \tag{12}$$

Odd sector: Regge-Wheeler equation

Taking an r-derivative of the second equation gives the first one multiplied by an overall $i\omega/f$, which can be recast into

$$h_0 = -\frac{f(h_1 f)'}{i\omega}. \tag{13}$$

We substitute this into the second equation and, using the following definitions

$$Q_{odd} = \frac{f h_1}{r} \qquad r_* = r + 2M \log\frac{r-2M}{2M} \qquad f\partial_r = \partial_* \tag{14}$$

we finally obtain the *Regge-Wheeler equation* (1957)

$$\frac{d^2 Q_{\mathrm{odd}}}{dr_*^2} + \left[\omega^2 - f(r) V_{odd}\right] Q_{odd} = 0 \tag{15}$$

with the *Regge-Wheeler potential*

$$V_{odd} = \frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3}. \tag{16}$$

Even sector: Zerilli equation

The even sector contains 4 functions, $H_{0/1/2}$ and K, but only one degree of freedom. After similar (but more involved!) manipulations as in the odd case, we find the *Zerilli equation* (1970)

$$\frac{d^2 Q_{\rm even}}{dr_*^2} + \left[\omega^2 - f(r) V_{even}\right] Q_{even} = 0 \tag{17}$$

with the *Zerilli potential*

$$V_{even} = 2\frac{L^2 r^2[(L+1)r + 3M] + 9M^2(Lr+M)}{r^3(Lr+3M)^2}, \tag{18}$$
$$2L = (\ell+2)(\ell-1).$$

Black hole perturbation theory: summary

Expand metric to linear order in perturbation around black hole background:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \qquad h_{\mu\nu} = h^{\rm odd}_{\mu\nu} + h^{\rm even}_{\mu\nu}. \tag{19}$$

Each sector contains one degree of freedom which evolves independently:

$$\frac{d^2 Q_{\rm odd/even}}{dr_*^2} + \left[\omega^2 - f(r) V_{odd/even}\right] Q_{odd/even} = 0, \tag{20}$$

$$V_{odd} = \frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3},$$

$$V_{even} = 2\frac{L^2 r^2[(L+1)r + 3M] + 9M^2(Lr + M)}{r^3(Lr+3M)^2}, \qquad 2L = (\ell+2)(\ell-1).$$

Solutions are described by quasinormal modes (QNMs) with frequencies $\omega(M, a)$. The use of QNMs to test GR is known as *black hole spectroscopy*.

Ringdown in Horndeski

Simplest example

Theory: Horndeski scalar-tensor

Most general action for $g_{\mu\nu} + \phi$ in $4D$ with 2^{nd} order equations of motion.

$$\begin{aligned} S = \int d^4x\sqrt{-g}\Big[& G_2 + G_3\Box\phi + G_4 R + G_{4X}\left[(\Box\phi)^2 - \phi^{\mu\nu}\phi_{\mu\nu}\right] \\ & + G_5\, G_{\mu\nu}\phi^{\mu\nu} - \frac{1}{6}G_{5X}\left[(\Box\phi)^3 - 3\phi^{\mu\nu}\phi_{\mu\nu}\Box\phi + 2\phi_{\mu\nu}\phi^{\mu\sigma}\phi^{\nu}_{\sigma}\right]\Big], \end{aligned} \tag{21}$$

$$X \equiv -\frac{1}{2}\phi_\mu\phi^\mu, \qquad \phi_\mu \equiv \nabla_\mu\phi, \qquad \phi_{\mu\nu} \equiv \nabla_\nu\nabla_\mu\phi, \qquad G_{iX} \equiv \partial_X G_i..$$

Background: Schwarzschild black hole

$$\begin{aligned} ds^2 &= -\left(1 - \tfrac{2M}{r}\right)dt^2 + \frac{1}{\left(1 - \tfrac{2M}{r}\right)}dr^2 + d\Omega^2, \\ \bar{\phi} &= \text{const.} \end{aligned} \tag{22}$$

Simplest example

Now we have two types of perturbations, with the scalar belonging to the even sector

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \qquad \phi = \bar{\phi} + \delta\phi. \tag{23}$$

Substituting this into the action we obtain the quadratic action ($R_{\mu\nu} = 0 = R$)

$$S^{(2)} = \frac{1}{4}\int d^4x\sqrt{-g}\Big[\bar{G}_4\big((\nabla_\mu h)^2 + \nabla^\mu h^{\nu\sigma}(2\nabla_\sigma h_{\nu\mu} - \nabla_\mu h_{\nu\sigma}) - 2\nabla_\mu h^\mu_\nu \nabla^\nu h\big) + 2(\bar{G}_{2X} - 2\bar{G}_{3\phi})\delta\phi\Box\delta\phi + 4\bar{G}_{4\phi}\delta\phi(\nabla_\mu\nabla_\nu h^{\mu\nu} - \Box h) + 2\bar{G}_{2\phi\phi}\delta\phi^2\Big]. \tag{24}$$

Odd sector: same as GR, so we recover the same Regge-Wheeler equation.
Even sector: contains mixings of the two perturbations ($h_{\mu\nu}$ and $\delta\phi$).

Simplest example

The mixings can be removed with the transformation

$$h^{\mu\nu} \to h^{\mu\nu} - \frac{\bar{G}_{4\phi}}{\bar{G}_4}\bar{g}^{\mu\nu}\delta\phi, \qquad S^{(2)} = S_h^{(2)} + S_{\delta\phi}^{(2)}. \tag{25}$$

The action for metric perturbations becomes the same one as in GR, so we recover the Regge-Wheeler and Zerilli equations for odd and even sectors there.
The metric for the scalar perturbations becomes

$$S^{(2)} = \frac{1}{4}\int d^4x\sqrt{-g}\delta\phi\left[\left(\bar{G}_{2X} - 2\bar{G}_{3\phi} + 3\frac{\bar{G}_{4\phi}^2}{\bar{G}_4}\right)\Box\delta\phi + \bar{G}_{2\phi\phi}\delta\phi)\right]. \tag{26}$$

which gives a Klein-Gordon equation of motion for the scalar

$$(\Box - \mu^2)\delta\phi = 0, \qquad \mu^2 = -\frac{\bar{G}_{2\phi\phi}}{\bar{G}_{2X} - 2\bar{G}_{3\phi} + 3\frac{\bar{G}_{4\phi}^2}{\bar{G}_4}}. \tag{27}$$

So μ is a new Horndeski parameter acting as an effective mass for the scalar field.

Testing the Speed of Gravity with Black Hole Ringdown [2301.10272]

Set-up

Theory: Horndeski scalar-tensor

Most general action for $g_{\mu\nu} + \phi$ in $4D$ with 2^{nd} order equations of motion.

$$S = \int d^4x\sqrt{-g}\Big[G_2 + G_3\Box\phi + G_4 R + G_{4X}\left[(\Box\phi)^2 - \phi^{\mu\nu}\phi_{\mu\nu}\right] \\ + G_5 G_{\mu\nu}\phi^{\mu\nu} - \frac{1}{6}G_{5X}\left[(\Box\phi)^3 - 3\phi^{\mu\nu}\phi_{\mu\nu}\Box\phi + 2\phi_{\mu\nu}\phi^{\mu\sigma}\phi^{\nu}_{\sigma}\right]\Big], \tag{28}$$

$$X \equiv -\frac{1}{2}\phi_\mu\phi^\mu, \qquad \phi_\mu \equiv \nabla_\mu\phi, \qquad \phi_{\mu\nu} \equiv \nabla_\nu\nabla_\mu\phi, \qquad G_{iX} \equiv \partial_X G_i, \qquad \text{assumption: } G_{4\phi} = 0.$$

Background: Schwarzschild hairy black hole

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)}dr^2 + d\Omega^2, \\ \bar{\phi} = \widehat{\phi} + \epsilon\delta\phi. \tag{29}$$

Plan

1. Quasinormal mode calculation

Derive α_T-dependent expressions for QNM frequencies.

2. Fisher forecast

Estimate how precisely GW detectors will be able to 'see' α_T.

MATHEMATICA *notebook:* https://github.com/sergisl/ringdown-calculations

Quasinormal mode calculation

- We focus on odd metric perturbations

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \qquad h^{\rm odd}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & h_0 \\ 0 & 0 & 0 & h_1 \\ 0 & 0 & 0 & 0 \\ h_0 & h_1 & 0 & 0 \end{pmatrix} \sin\theta \partial_\theta Y_{\ell m}. \tag{30}$$

- Perturb Horndeski action to quadratic order in h and apply field redefinitions
- Obtain modified Regge-Wheeler equation
- Parameterize background functions
- Obtain corrections to quasinormal frequencies from the corrected potential

Quasinormal mode calculation

- We focus on odd metric perturbations
- Perturb Horndeski action to quadratic order in h and apply field redefinitions

$$S^{(2)} = \frac{\ell(\ell+1)}{4(\ell-1)(\ell+2)} \int dt dr_* \left[\frac{\mathcal{F}}{\mathcal{G}} \dot{Q}^2 - \left(\frac{dQ}{dr_*} \right)^2 - V(r) Q^2 \right] . \tag{31}$$

- Obtain modified Regge-Wheeler equation
- Parameterize background functions
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Quasinormal mode calculation

- We focus on odd metric perturbations
- Perturb Horndeski action to quadratic order in h and apply field redefinitions
- Obtain modified Regge-Wheeler equation

$$\frac{d^2Q}{dr_*^2} + \left[\omega^2 - f(V_{RW} + \epsilon^2 \alpha_T \delta V)\right] Q = 0, \tag{32}$$

$$\delta V = -\frac{1}{f}\omega_0^2 + \frac{M(2r-5M)}{r^3(r-2M)} + \frac{(\ell+2)(\ell-1)}{r^2} - \frac{r-2M}{2r}\left(\left(\frac{\delta\phi''}{\delta\phi'}\right)^2 - \frac{\delta\phi'''}{\delta\phi'}\right) + \frac{r-5M}{r^2}\frac{\delta\phi''}{\delta\phi'}.$$

- Parameterize background functions
- Obtain corrections to quasinormal frequencies from the corrected potential

Quasinormal mode calculation

- We focus on odd metric perturbations
- Perturb Horndeski action to quadratic order in h and apply field redefinitions
- Obtain modified Regge-Wheeler equation
- Parameterize background functions

$$\alpha_T = -f\frac{G_{4X} - G_{5\phi}}{G_4}\delta\phi'^2 = -\sum_{i=0}^{\infty} A_{Ti}\left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r}\right)^{i+4}, \tag{33}$$

$$A_{Ti} \equiv G_{Ti}\varphi_c^2, \qquad G_T \equiv \frac{1}{(2M)^2}\frac{G_{4X} - G_{5\phi}}{G_4} = \sum_i G_{Ti}\left(\frac{2M}{r}\right)^i, \qquad \delta\phi = \varphi_c\left(\frac{2M}{r}\right).$$

- Obtain corrections to quasinormal frequencies from the corrected potential

Quasinormal mode calculation

- We focus on odd metric perturbations
- Perturb Horndeski action to quadratic order in h and apply field redefinitions
- Obtain modified Regge-Wheeler equation
- Parameterize background functions
- Obtain corrections to quasinormal frequencies from the corrected potential

$$\omega = \omega_0 + \delta\omega, \tag{34}$$

$$\delta\omega = \sum_{i=0}^{\infty} A_{Ti}\left[(2M\omega_0)^2 e_{4+i} - (\ell(\ell+1) - 9)e_{6+i} + (\ell(\ell+1) - 20)e_{7+i} + \frac{45}{4}e_{8+i}\right].$$

Fisher forecast

Ringdown strain functions

$$h_{+/\times}(t) = \sum_{\ell m} A_{\ell m}^{+/\times} e^{-\frac{\pi t f_{\ell m}}{Q_{\ell m}}} S_{\ell m} \cos\left(\phi_{\ell m}^{+/\times} + 2\pi t f_{\ell m}\right), \tag{35}$$

$$\omega_{\ell m} = 2\pi f_{\ell m} + \frac{i}{\tau_{\ell m}}, \qquad Q_{\ell m} = \pi f_{\ell m} \tau_{\ell m}.$$

Noise-weighted product, SNR, Fisher matrix and parameter errors

$$(h_1|h_2) = 2\int_0^\infty d\nu \frac{\tilde{h}_1^* \tilde{h}_2 + \tilde{h}_2^* \tilde{h}_1}{S_h(\nu)}, \tag{36}$$

$$\rho^2 = (h|h) = \frac{QA^2}{\pi f S_h}, \qquad \Gamma_{ab} = \left(\frac{\delta h}{\delta \theta^a}\middle|\frac{\delta h}{\delta \theta^b}\right), \qquad \sigma_a = \sqrt{\Sigma_{aa}} = \sqrt{\Gamma_{aa}^{-1}}.$$

We perform the Fisher analysis on the most significant mode $(\ell, m) = (2, 2)$, assuming the only free parameters are the A_{Ti}'s.

For one parameter

$$\sigma^2_{A_{Ti}} \rho^2 = \frac{1}{2} \left(\frac{f}{Q f'} \right)^2 . \tag{37}$$

For two parameters

$$\sigma^2_{A_{Ti}} \rho^2 = \frac{\dot{f}^2}{2} \frac{(2Q)^2 + (1 - \frac{f \dot{Q}}{\dot{f} Q})^2}{(\dot{Q} f' - \dot{f} Q')^2}, \qquad \sigma^2_{A_{Tj}} \rho^2 = \frac{f'^2}{2} \frac{(2Q)^2 + (1 - \frac{f Q'}{f' Q})^2}{(\dot{Q} f' - \dot{f} Q')^2} . \tag{38}$$

Fisher forecasts - A_{T0}

$$\frac{\delta\omega_R}{\omega_{0R}} \approx -0.19 \cdot A_{T0}\%, \qquad \frac{\delta\omega_I}{\omega_{0I}} \approx 3.44 \cdot A_{T0}\%. \tag{39}$$

Detector(s)	ρ	Error on α_T
LVK	10	1
ET / CE	10^2	10^{-1}
pre-DECIGO	10^2	10^{-1}
DECIGO / AEDGE	10^3	10^{-2}
LISA	10^5	10^{-4}
TianQin	10^5	10^{-4}
AMIGO	10^5	10^{-4}

Table: Achievable order-of-magnitude ringdown SNRs for a single observed event for different GW detectors and the corresponding order-of-magnitude errors on α_T.

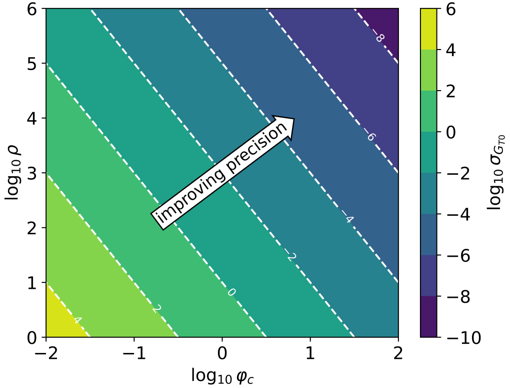

Figure: Forecasted errors on the strength of the interactions contributing to c_{GW} with G_{T0} as an example.

Conclusions

- For Schwarzschild black holes, the ringdown QNMs are sensitive to α_T only in the presence of a non-trivial scalar hair profile $\phi = \phi(r)$.
- A single supermassive black hole merger observed by LISA will constrain α_T to $\mathcal{O}(10^{-4})$ (with up to 2 orders of magnitude improvement for stacked observations).
- Ringdown observations can
  - provide novel constrains for interactions affecting c_{GW} for a sufficiently large scalar profile.
  - constrain the nature of scalar hair given complimentary information on the interactions.

Future directions

- Extend to (slowly) rotating and/or de Sitter black holes.
- Introduce frequency-dependence from UV physics.
- Implement parameters into LIGO data analysis.

Thank you!