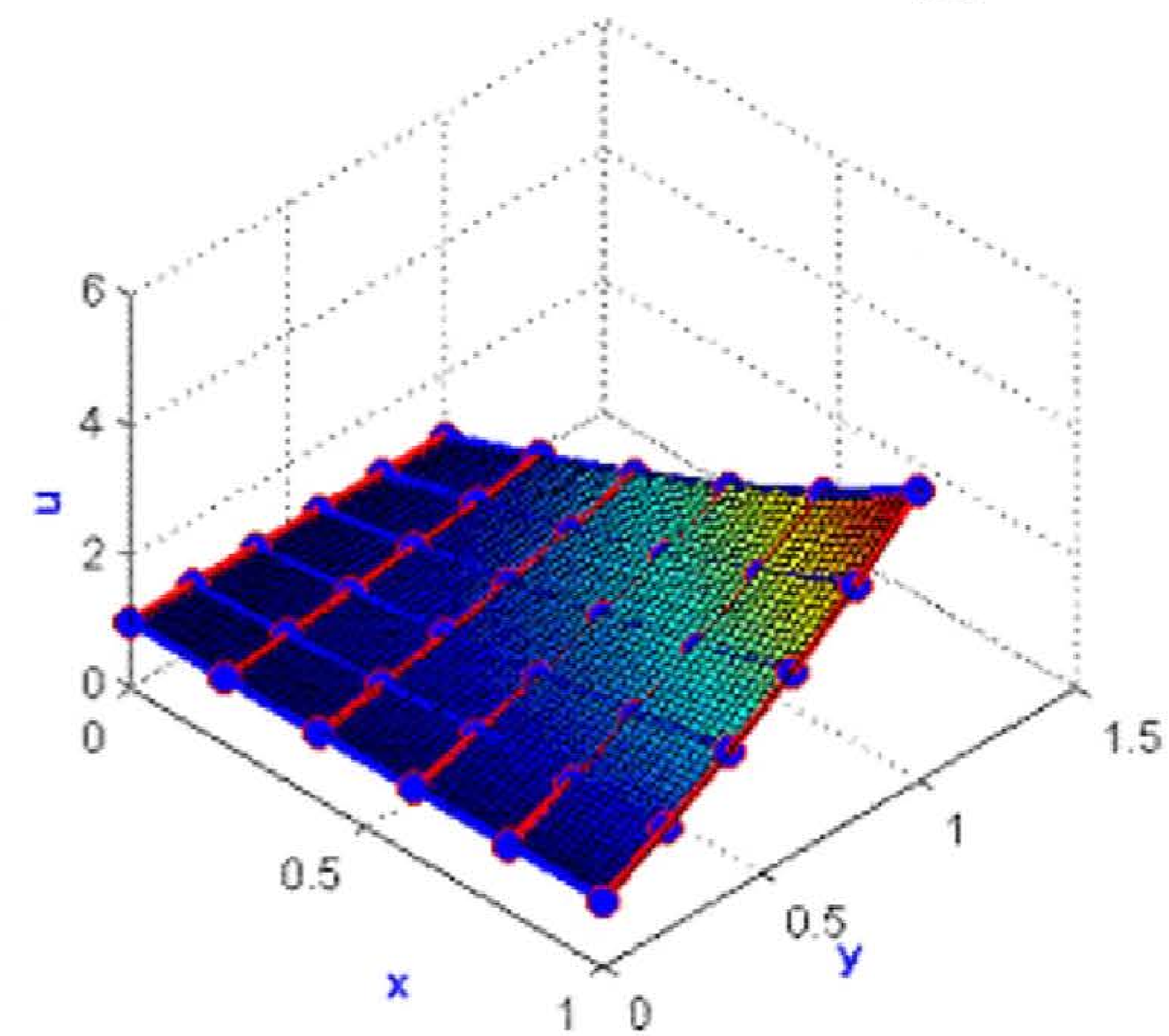
Mathematical modeling with classes of functions as classes of solutions

Basic idea

- Many phenomena or processes can be modeled by <u>a set of algebraic or differential equations</u>
- The quantities involved are modeled by <u>functions</u>
- The solutions – the set of functions that obey the equations and the <u>conditions</u>

The conditions

- Additional <u>conditions</u> are generated by imposing some limit values for the functions:
- Specific values in given points
- Specific values along the border of a given domain



The solutions

Several classes of functions are of importance:

- Exact solutions
- Exact *solutions of a given type*, for example polynomial solutions
- Approximate solutions, like:
  - Functions that *approximate the exact solutions over a given domain*
  - Functions that *asymptotically approximate* the exact solutions

Example -The Biharmonic Equation

$$\frac{\partial^4 \Phi}{\partial x^4} + 2\frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \Phi}{\partial y^4} = f(x,y)$$

This equation appears in many models

Some important models will follow

1. Approximate the Solutions within a given class of functions

The Biharmonic Equation

$$\frac{\partial^4 \Phi}{\partial x^4} + 2\frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \Phi}{\partial y^4} = ye^x + xe^y$$

A solution of the differential equation can be viewed
as a surface $\varphi(x,y)$

$0 \leq x \leq 1, 0 \leq y \leq 1$

A square domain in the plane

$$\Phi(x=0,y)=1$$
$$\Phi(x,y=0)=1$$
$$\Phi(x=1,y)=-y+e^y+ye$$
$$\Phi(x,y=1)=-x+xe+e^x$$

Along the four segments of the domain frontier, the shape of the surface is imposed by these curves

As one can expect, there are multiple surfaces $\varphi(x,y)$ obeying these restrictions.

Additional conditions can be imposed by selecting a given class of surfaces and a fitness criteria might be defined for further selecting the best candidate from such a family.

The Bezier Function Formulation

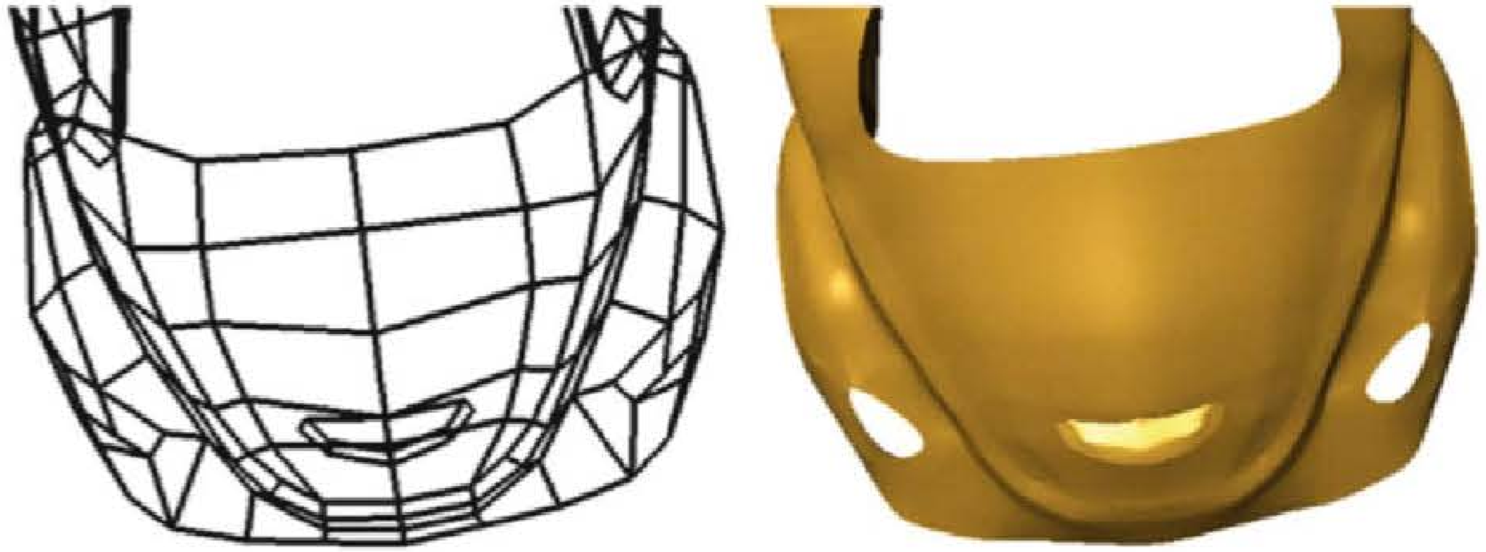
Find the Bezier surface $\varphi(x,y)$ that minimizes the square of the residuals over the domain

$$Minimize\ F(X) = \sum_{i=1}^{p} \sum_{j=1}^{q} \left[\left.\frac{\partial^4 \Phi}{\partial x^4}\right|_{i,j} + 2 \left.\frac{\partial^4 \Phi}{\partial x^2 \partial y^2}\right|_{i,j} + \left.\frac{\partial^4 \Phi}{\partial y^4}\right|_{i,j} - ye^x|_{i,j} - xe^y|_{i,j} \right]^2$$

Bézier curves and surfaces

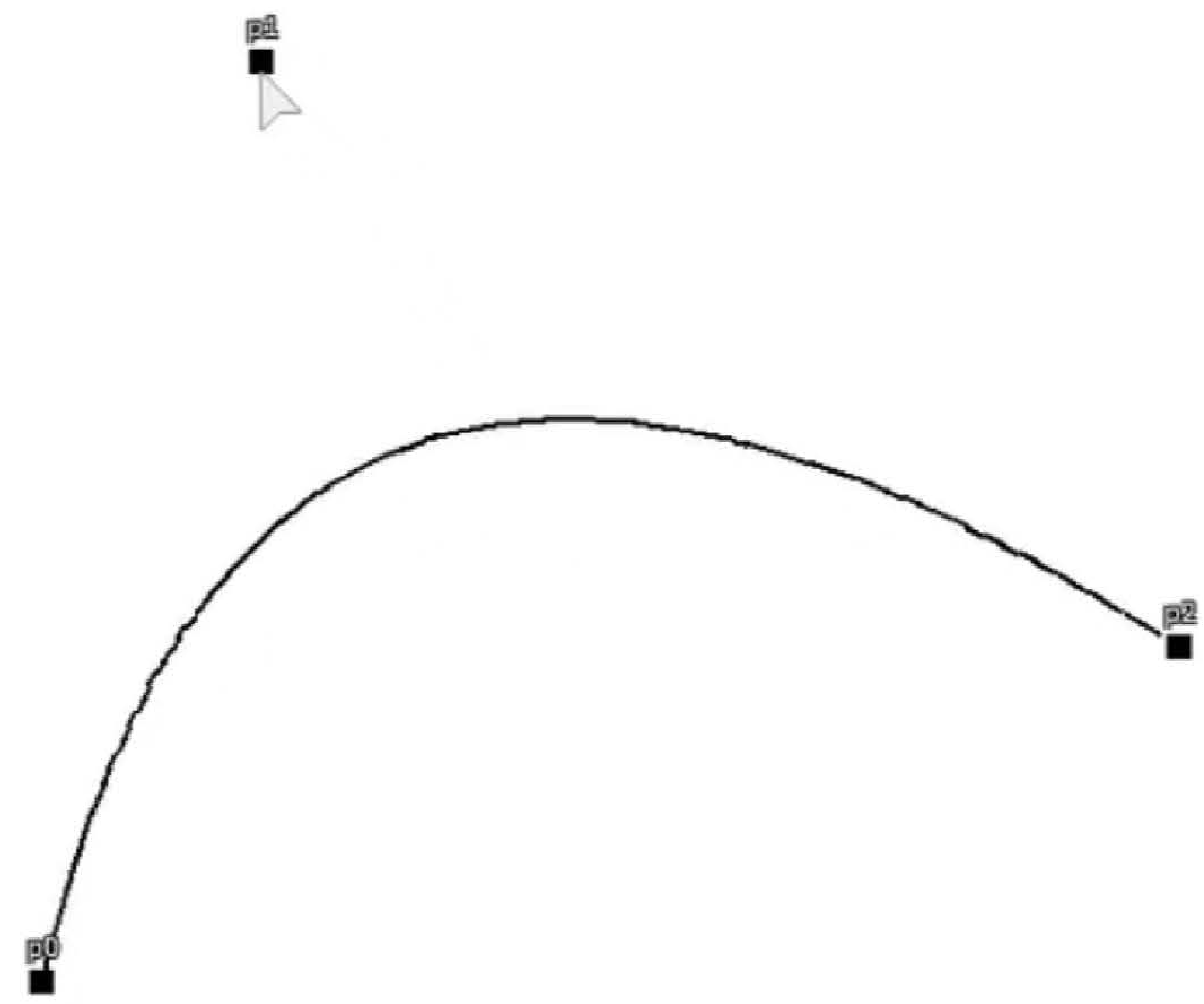
- P. E. Bézier (1910-1999), a French designer in the automobile industry for Rénault, suggested a concept for the interactive design of curves and surfaces using the Bernstein basis and some control points in his design and developed the UNISURF CAD software.
- This concept of control points and their positioning play the most significant and vital role in his interactive design
- This concept helps to design and draw smooth curves and surfaces of different shapes and sizes, corresponding to different arbitrary objects, based on a set of control points
- Bézier curves and surfaces are driven by Bernstein basis
- Bézier spline-based drawing technique starts from the zeroth order Bernstein approximation (which is exactly the line drawing between control points) of the data points and goes to some higher order (quadratic or cubic) approximation, until it mimics the shape of the object.

The Peugeot 204 was the first car who's bodywork was completely build using UNISURF CAD in 1968

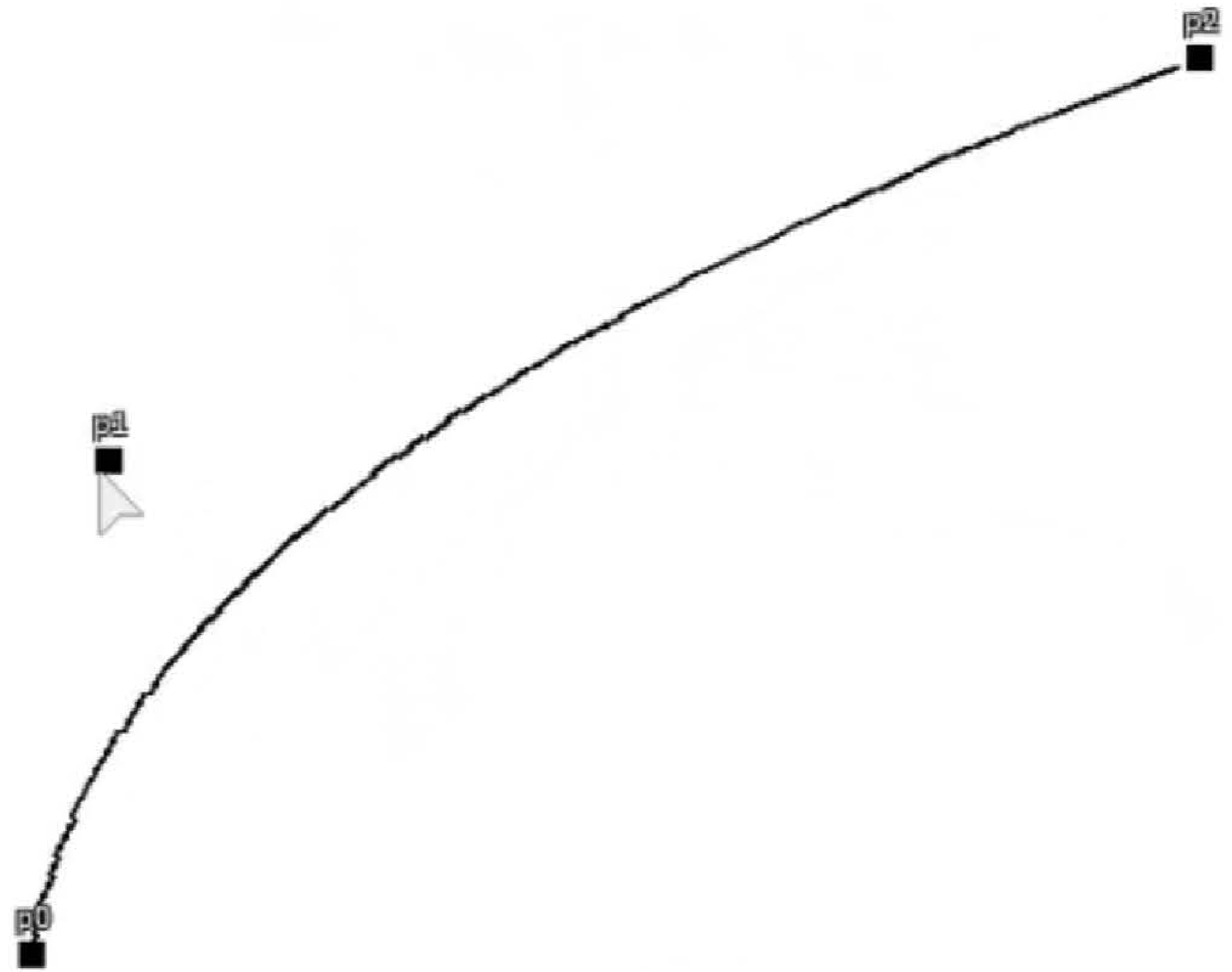


Back of a Beatle, Mesh and Surface

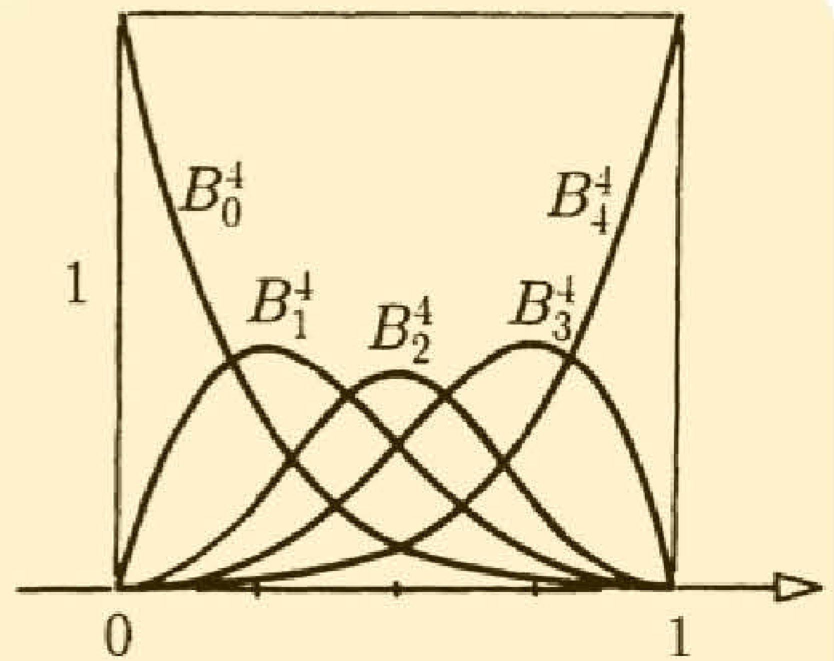
Bezier curve in computer assisted design



Bezier curve in computer assisted design



Bernstein Polynomial



Bernstein polynomials of degree 4 over [0,1]

- The Bernstein basis functions:

$$\phi_{ip}(t) = \binom{p}{i} t^i (1-t)^{p-i}, \qquad i \in [0, p]$$

- Bernstein *polynomial approximation* of degree p to an arbitrary real valued function *f(t)* are converging uniformly (with increasing p) to the function they approximate.

$$B_p[f(t)] = \sum_{i=0}^{p} f(\frac{i}{p}) \phi_{ip}(t) \qquad 0 \le t \le 1$$

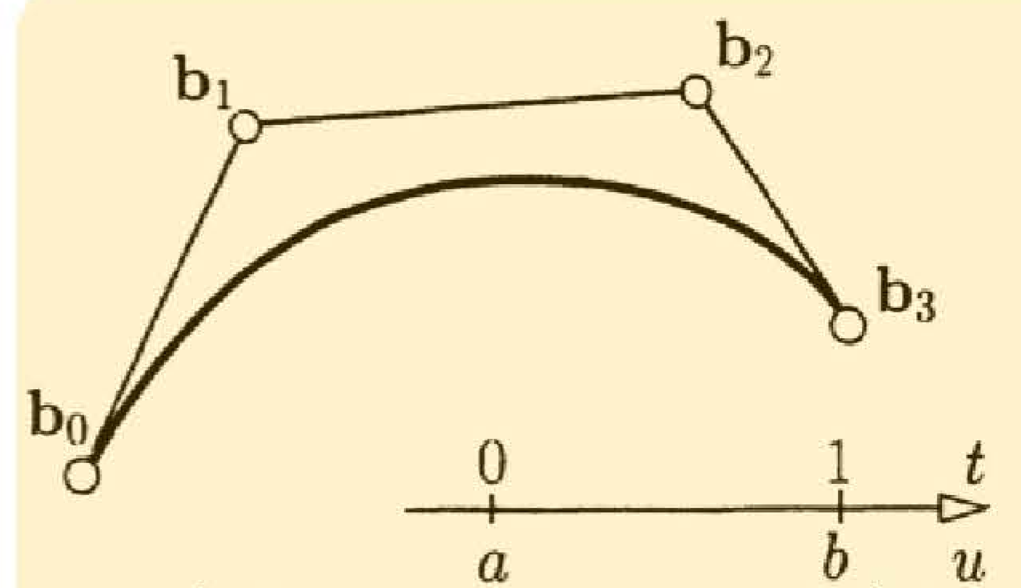
- Remark: The binomial expansion $1 = (u + (1-u))^n = \sum_{i=0}^{n} \binom{n}{i} u^i (1-u)^{n-i}$

leads to the Bernstein polynomials of degree n $B_i^n(u) = \binom{n}{i} u^i (1-u)^{n-i}$

Bernstein Polynomial

- n + 1 (linearly independent) Bernstein polynomials form a basis for all polynomials of degree ≤ n.
- Every polynomial curve b(u) of degree ≤ n has a unique nth degree Bezier representation

$$\mathbf{b}(u) = \sum_{i=0}^{n} \mathbf{c}_i B_i^n(u)$$



A cubic curve segment with its Bezier polygon over [a, b]

- The linear transformation $u = a(1 - t) + bt$

leaves the degree of the curve **b** unchanged,

making that $\mathbf{b}(\mathrm{u}(\mathrm{t}))$ has also an nth degree Bezier representation:

$$\mathbf{b}(u(t)) = \sum_{i=0}^{n} \mathbf{b}_i B_i^n(t)$$

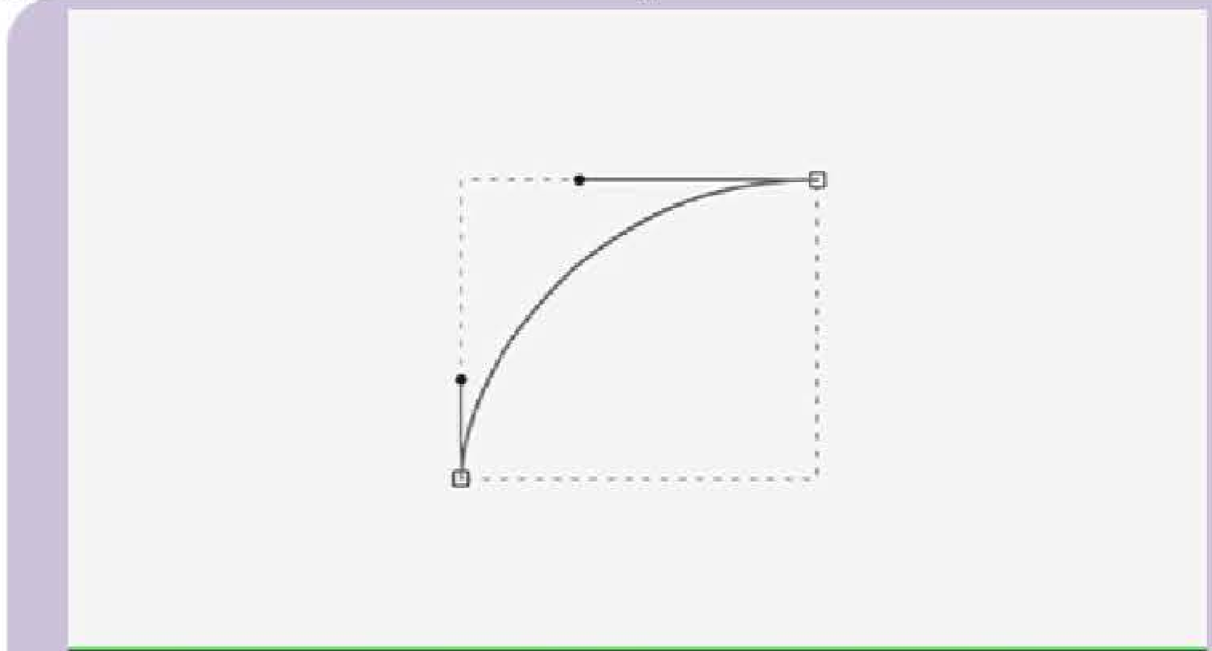
- The coefficients $\mathbf{b}_\mathbf{i}$ are called Bezier points. They are the vertices of the Bezier polygon of $\mathbf{b}(\mathrm{u})$ over the interval [a, b].
- The parameter **t** is called the **local** parameter of **b**, and **u** the global parameter of **b**
- One dimensional Bezier-Bernstein polynomial represents a curve that can be

generated from a set of ordered representative points (the control points or the guiding points). The line joining these control points is called the control line of the polynomial.

Bezier-Bernstein Curves

- The parametric form of the curves: $X = P_x(t) \quad Y = P_y(t)$
- Let $(x_0, y_0), (x_1, y_1) \cdots (x_p, y_p)$ be $(p + 1)$ ordered points in a plane. The Bezier curve associated with the polygon through these control points is:

$$P_x(t) = \sum_{i=0}^{p} \phi_{ip}(t)\ x_i \quad P_y(t) = \sum_{i=0}^{p} \phi_{ip}(t)\ y_i$$



Curve shape adjusted by the control points position

- The vector valued Bernstein polynomial:

$$P(t) = \sum_{i=0}^{p} \phi_{ip}(t)\ V_i$$

where $V_0, V_1, \cdots, V_p$ are the guiding points (the control points)

Remark: $P(0) = V_0 \ \ and \ \ P(1) = V_p$

Example: For cubic Bezier curve (p = 3) the control polygon consists of four control vertices (V_0, V_1, V_2, V_3) and the Bezier curve is:

$$P(t) = (1-t)^3 V_0 + 3t(1-t)^2 V_1 + 3t^2(1-t) V_2 + t^3 V_3$$

The Bernstein basis functions in this case are:

$\varphi_{03}(t) = (1 - t)^3 = 1 - 3t^2 + 3t - t^3$,

$\varphi_{13}(t) = 3t(1 - t)^2 = 3t - 6t^2 + 3t^3$,

$\varphi_{23}(t) = 3t^2(1 - t) = 3t^2 - 3t^3$, $\varphi_{33}(t) = t^3$.

$$P(t) = \left((1-t)^3 \ 3t(1-t)^2 \ 3t^2(1-t) \ t^3 \right) \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

$$P(t) = \left(t^3 \ t^2 \ t \ 1 \right) \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{pmatrix}$$
$$= (T)(C)(V).$$

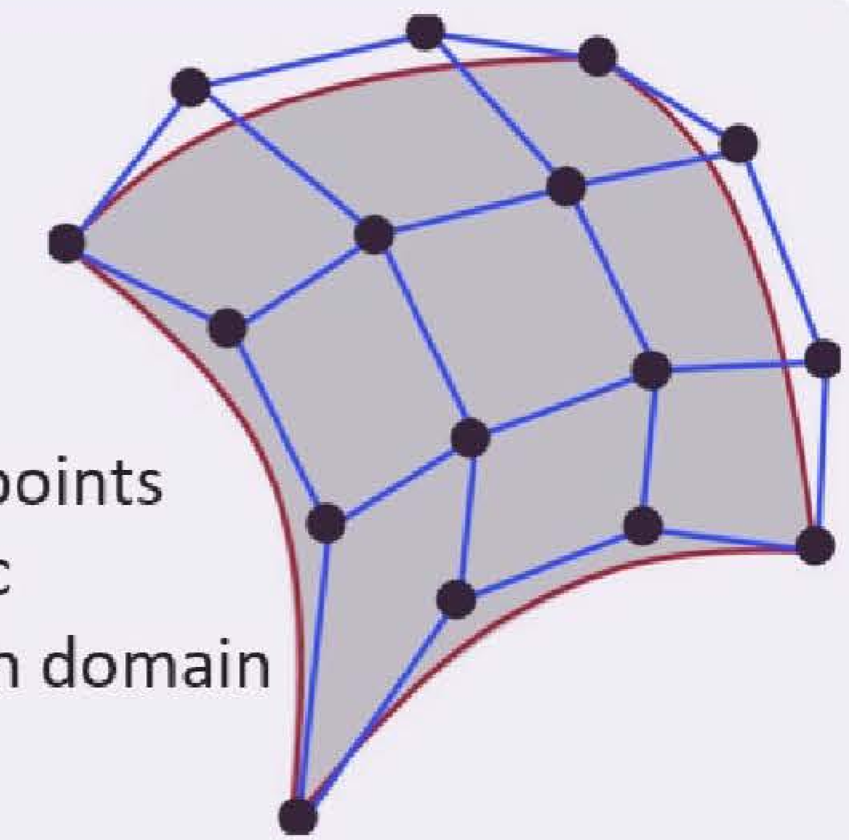
Bezier-Bernstein Surfaces

A Bezier-Bernstein surface is a tensor product surface and is represented by a two-dimensional Bezier-Bernstein polynomial.

The surface patch $S(u,v)$:

$$S(u,v) = \sum_{i=0}^{p} \sum_{j=0}^{q} \phi_{ip}(u)\phi_{jq}(v)V_{ij}$$

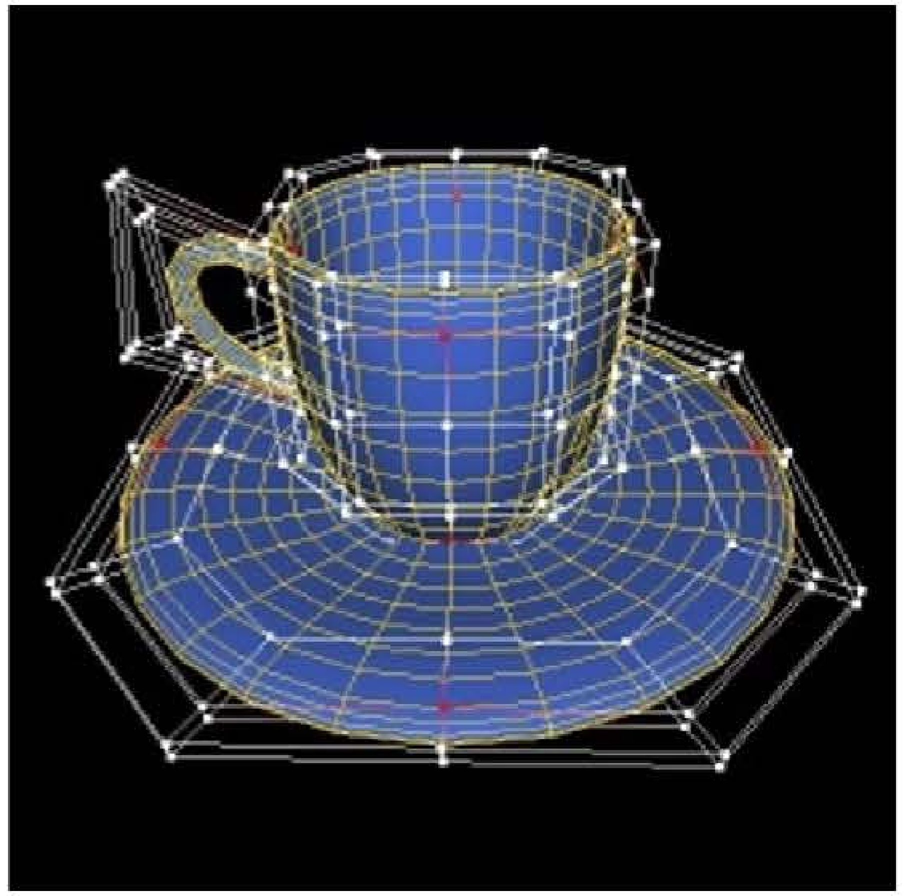
where $0 \leq u \leq 1$ and $0 \leq v \leq 1$.



16 control points
of a bicubic
Bezier patch domain

V_{ij} is the (i,j)th control point

ϕ_{ip} is the ith basis Bernstein basis function of order p and ϕ_{jq} is the Bernstein basis of order q.



Remark: When $p \neq q$, the Bezier-Bernstein surface is defined on a rectangular support. This support becomes a square for $p = q$.

For example, $p = 3$ and $q = 3$, are producing a bicubic surface on a square support.

- Explicit polynomial solution method for surface generation:
  - the resulting surface conforms to a Partial Differential Equation (PDE)

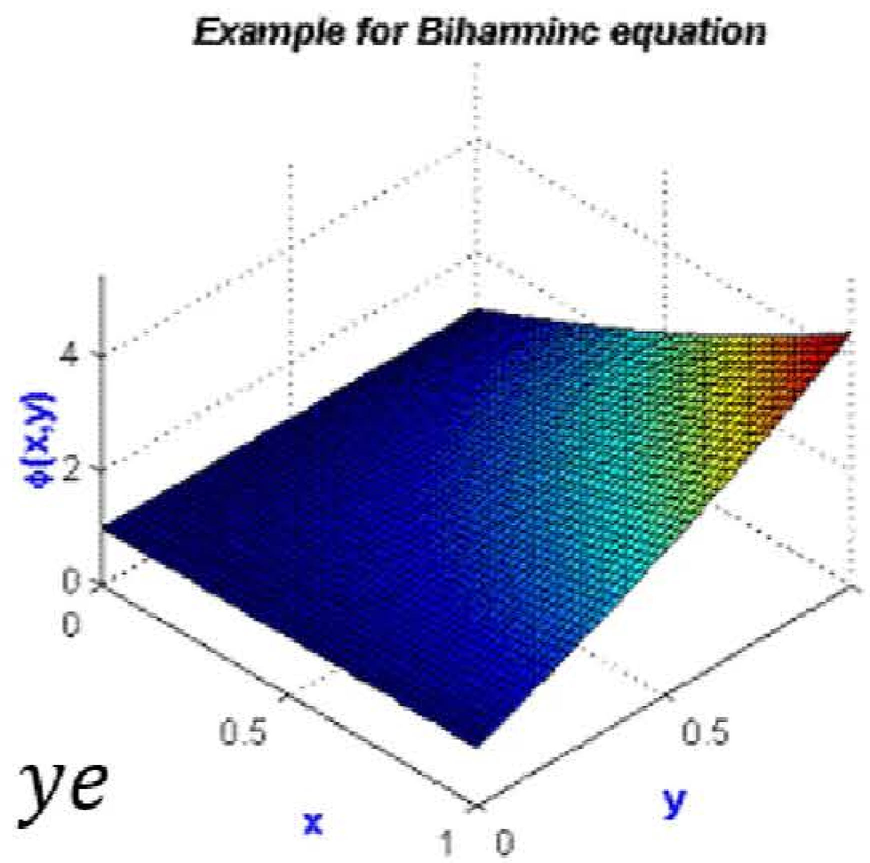
$$\frac{\partial^4 \Phi}{\partial x^4} + 2\frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \Phi}{\partial y^4} = ye^x + xe^y$$

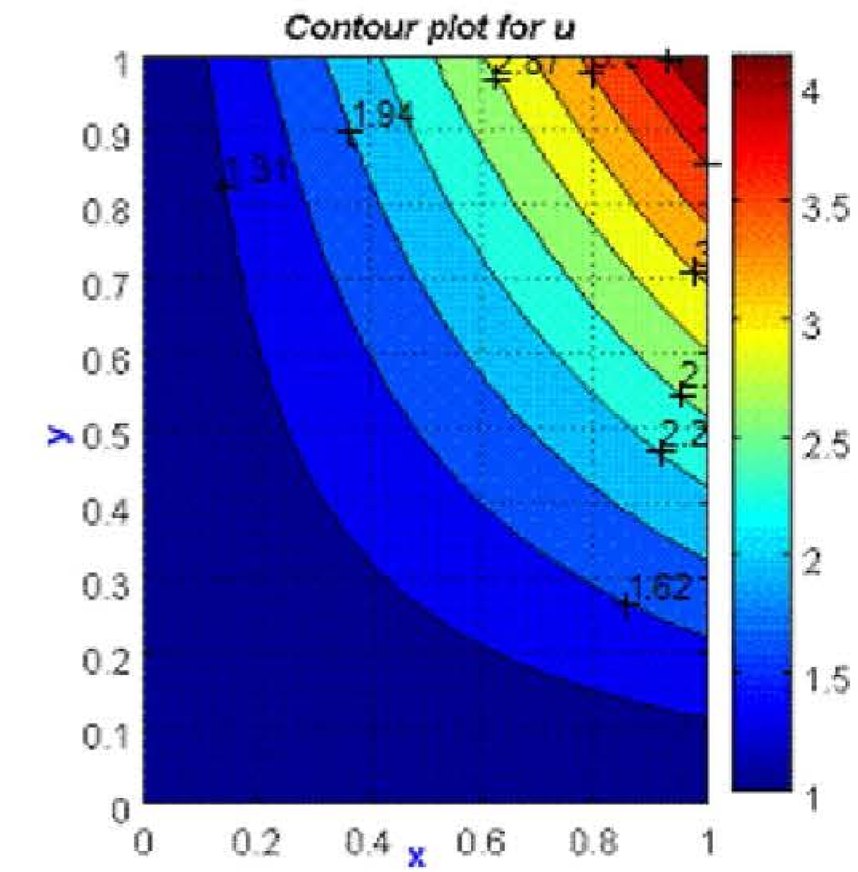
The domain is the unit square

$0 \le x \le 1, 0 \le y \le 1$

Boundary conditions expressed as boundary curves:

$$\Phi(x = 0, y) = 1$$
$$\Phi(x, y = 0) = 1$$
$$\Phi(x = 1, y) = -y + e^y + ye$$
$$\Phi(x, y = 1) = -x + xe + e^x$$





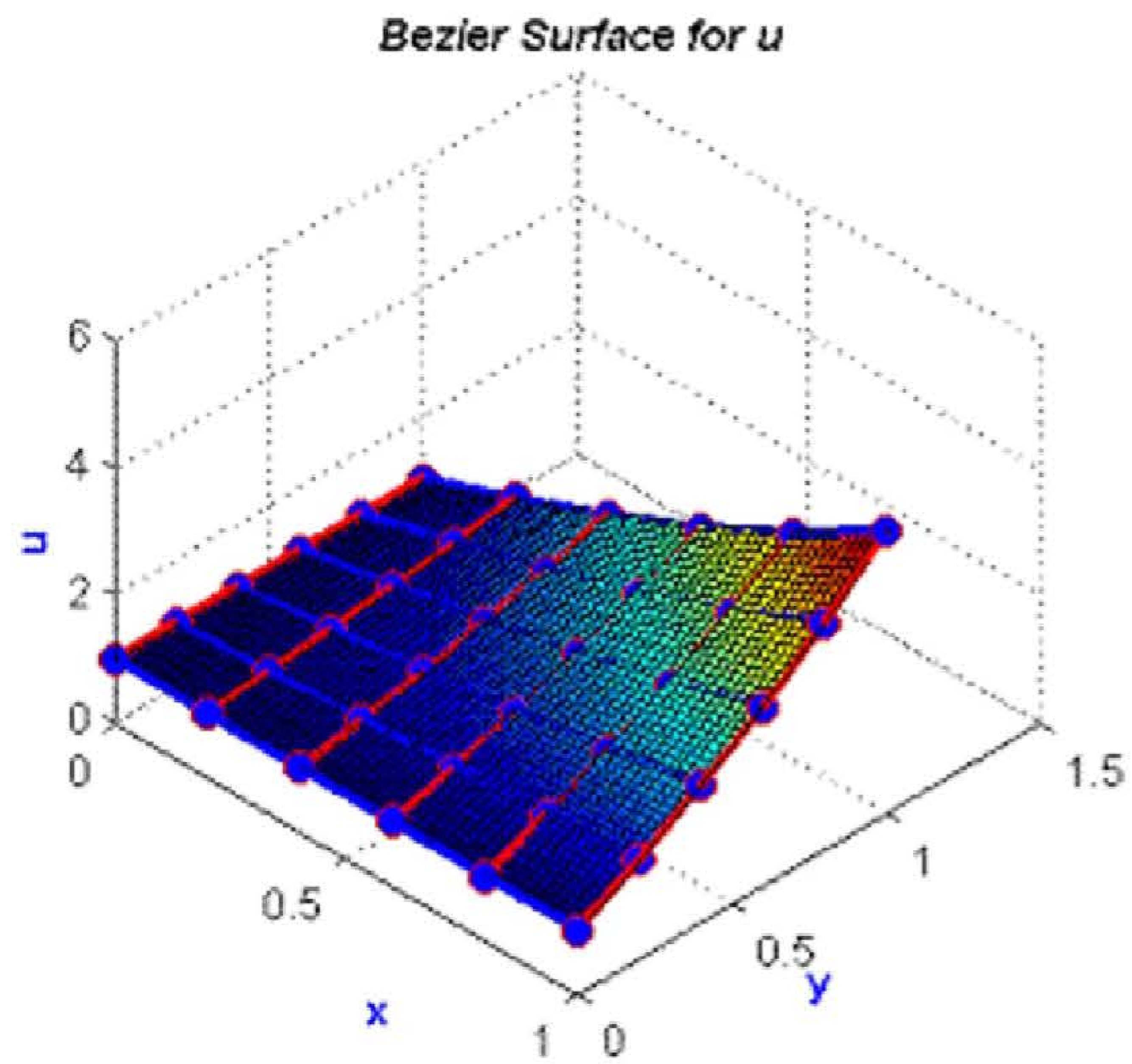
Given these boundary curves find a parametric surface patch $\vec{s}$ such that $\vec{s} : [r, s] \to R^3$ whereby the surface patch $\vec{s}$ smoothly interpolates the four curves:

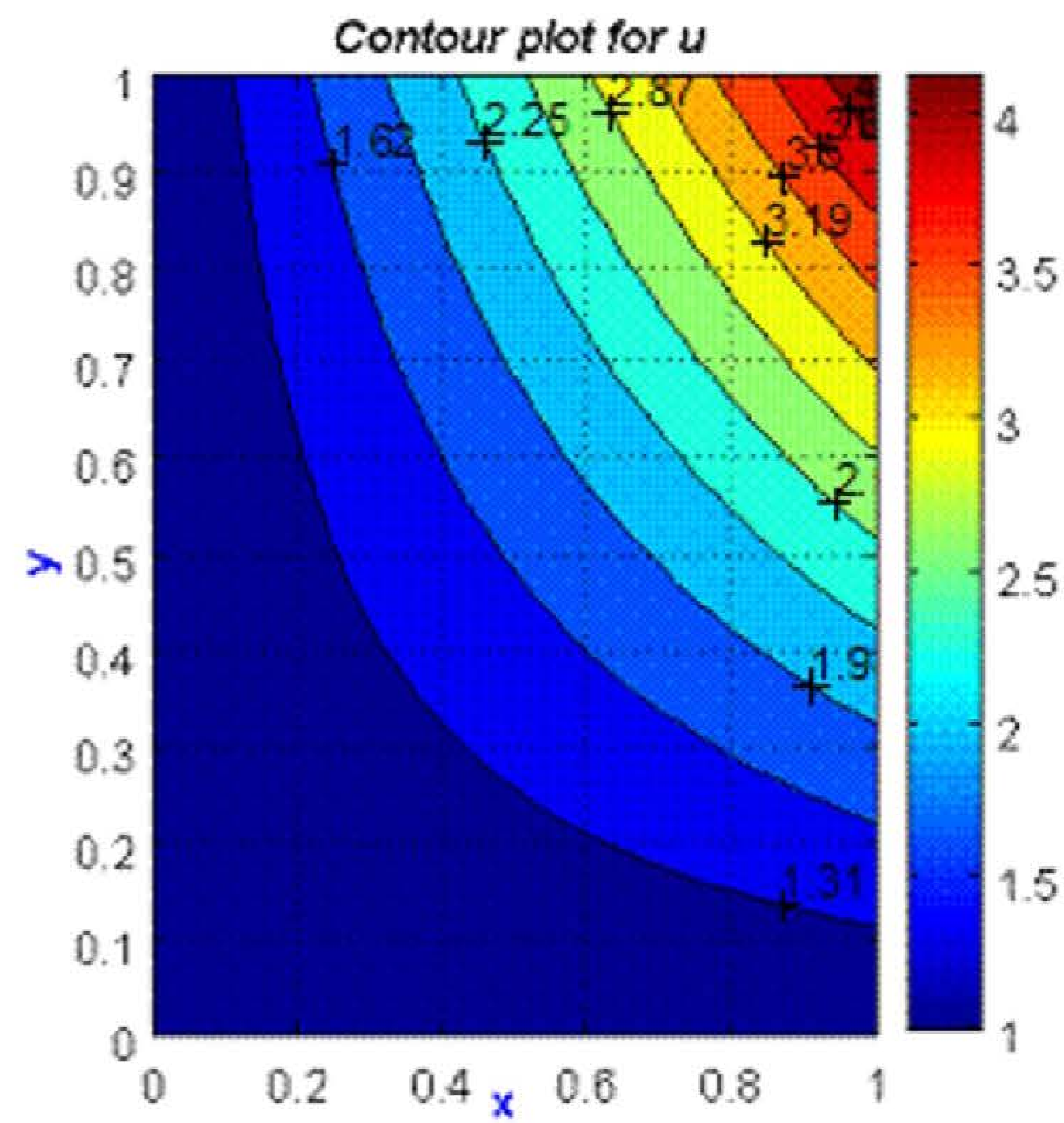
1. Start with an initial approximation (for example a plane)
2. Build a new Bezier candidate (modify the control points) and sample its value for the q x q grid points.
3. If an acceptable error margin $\boldsymbol{\varepsilon}$ of the square of the residuals over the domain is reached:

$F(X) = \sum_{i=1}^{p} \sum_{j=1}^{q} \left[\left.\frac{\partial^4 \Phi}{\partial x^4}\right|_{i,j} + 2\left.\frac{\partial^4 \Phi}{\partial x^2 \partial y^2}\right|_{i,j} + \left.\frac{\partial^4 \Phi}{\partial y^4}\right|_{i,j} - ye^x|_{i,j} - xe^y|_{i,j} \right]^2 \le \varepsilon$, then stop. Otherwise go to step 2.

The Bezier Solution

Final Bezier solution





The Bezier Solution

Explicit Bezier Solution

$x(r,s) = r$
$y(r,s) = s$
$\varphi(r,s) =$
$(0.033067r - 0.0011573r^2 + 0.0000229r^4 + 0.00070456r^3 + 0.81159\ \ 10^{-5}\, r^5)s^5$
$+(-0.014571r - 0.000070879r^2 - 0.00013698r^4 + 0.0041345r^3 + 0.000023601\, r^5)s^4$
$+(0.00070308r^5 + 0.0041405r^4 - 0.02592r^3 - 0.026085r^2 + 0.25570\, r)s^3$
$+(-0.0011527r^5 - 0.000088229r^4 + 0.056283r^3 - 0.000025391r^2 + 0.43115\, r)s^2$
$+(-0.00069555r^5 + 0.067790r^4 + 0.10382r^3 + 0.53704r^2 + 2.0119\, r)s + 1$
$0 \le r \le 1, 0 \le s \le 1$

2. Applications of the biharmonic equation in elasticity problems

- Under specific assumptions, some mathematical models for elastic deformation of elastic bodies can be defined based on the biharmonic equation
- The class of exact polynomial solutions will be exemplified for some typical applications

Elasticity

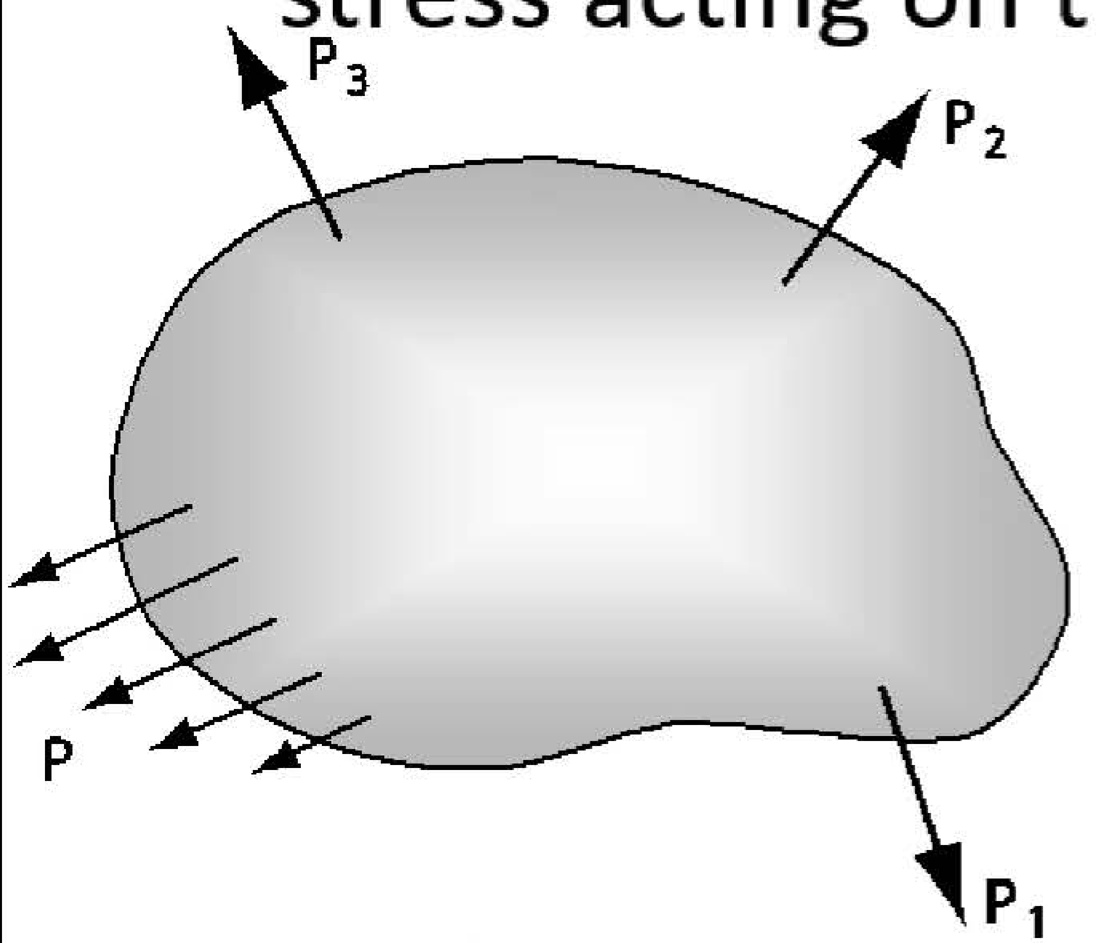
- The property of *elasticity*: if external forces, producing *deformation* of a structure, do not exceed a certain limit, the deformation disappears with the removal of the forces
- *Perfectly elastic* bodies: they resume their initial form completely after the removal of forces

Elasticity

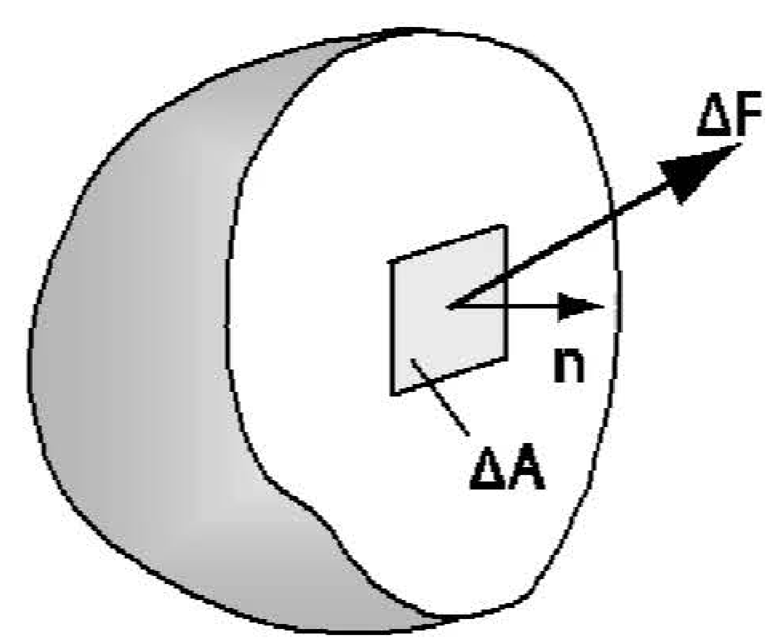
- Assumptions:
  - The matter of an elastic body is *homogeneous* and continuously distributed over its volume so that the smallest element cut from the body possesses the same specific physical properties as the body
  - The body is *isotropic*, i.e., that the elastic properties are the same in all directions

Elasticity

- Let us imagine a body divided into two parts *A* and *B* by a cross section *mm* through this point
- To obtain the magnitude of stress acting on a small area ΔA, cut out from the cross section *mm* at any point *O*, we assume that the forces acting across this element area, due to the action of material of the part *B* on the material of the part *A*, can be reduced to a resultant ΔF
- If we now continuously contract the elemental area ΔA, the limiting value of the ratio $\delta F/\delta A$ gives the magnitude of the stress acting on the cross section *mm* at the point *O*



(Externally Loaded Body)



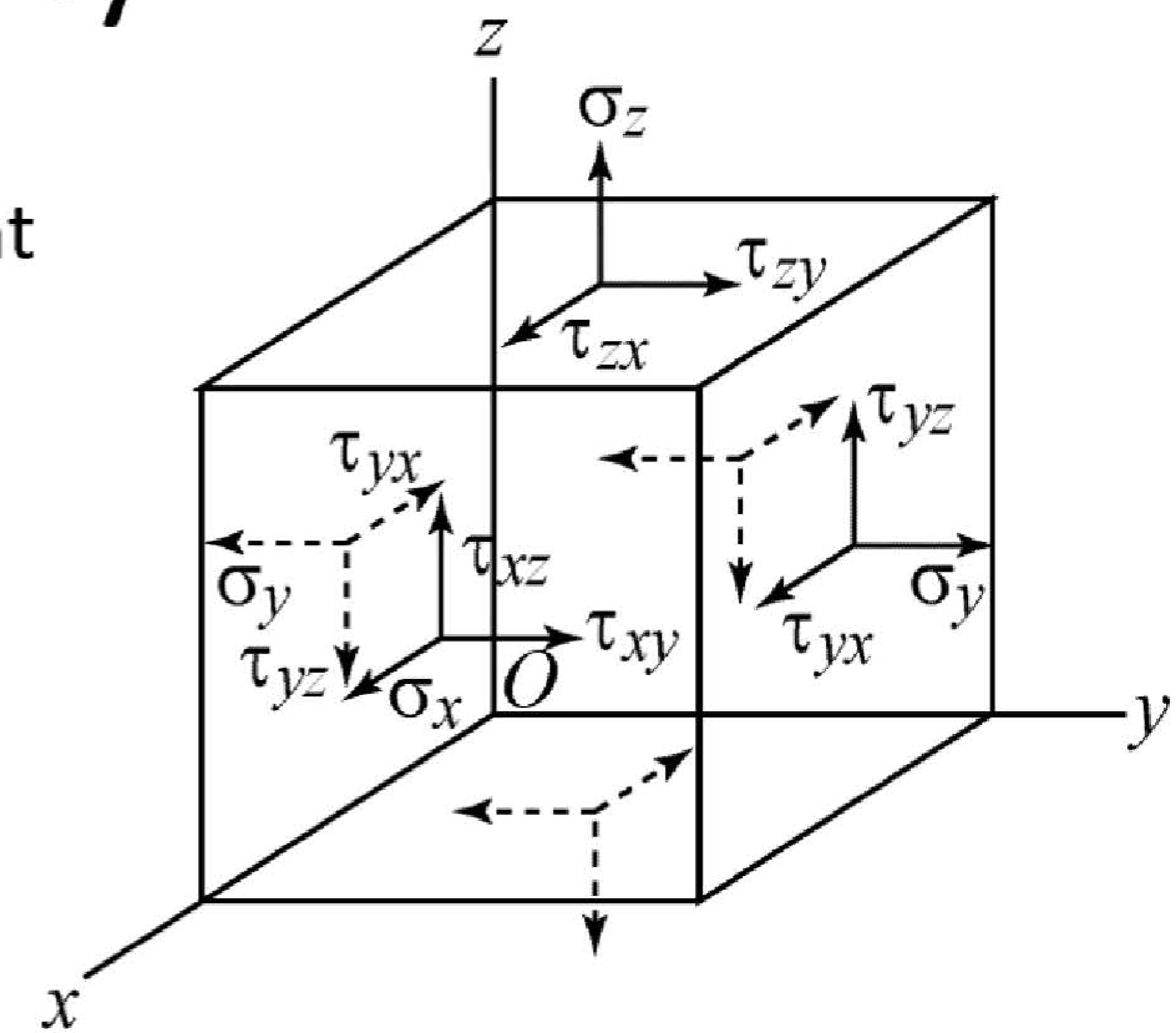
(Sectioned Body)

Elasticity

- In the general case the direction of stress is inclined to the area and we usually resolve it into two components:
  - a *normal stress* perpendicular to the area
  - a *shearing stress* acting in the plane of the area

Elasticity

If we take a very small cubic element at a point *O,* with sides parallel to the coordinate axes, the components of stress acting on the sides of this element and the directions taken as positive are as indicated in the figure



Components of stress at a point:

$\sigma_x, \sigma_y, \sigma_z, \tau_{xy} = \tau_{yx}, \tau_{xz} = \tau_{zx}, \tau_{yz} = \tau_{zy}$

σ – normal stress

τ – shearing stress

Elasticity

- The small displacements of particles of a deformed body will usually be resolved into components *u, v, w* parallel to the coordinate axes *x, y, z*, respectively
- It will be assumed that these components are very small quantities varying continuously over the volume of the body

Elasticity

$$\epsilon_x = \frac{\partial u}{\partial x}, \qquad \epsilon_y = \frac{\partial v}{\partial y}, \qquad \epsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

ϵ - unit elongation,

γ - unit shearing strain

Elasticity

For a thin plate, the stress components σ_z, τ_{xz}, τ_{yz} are zero on both faces of the plate, and it may be assumed that they are zero also within the plate

The state of the stress is specified by σ_x, σ_y si τ_{xy}

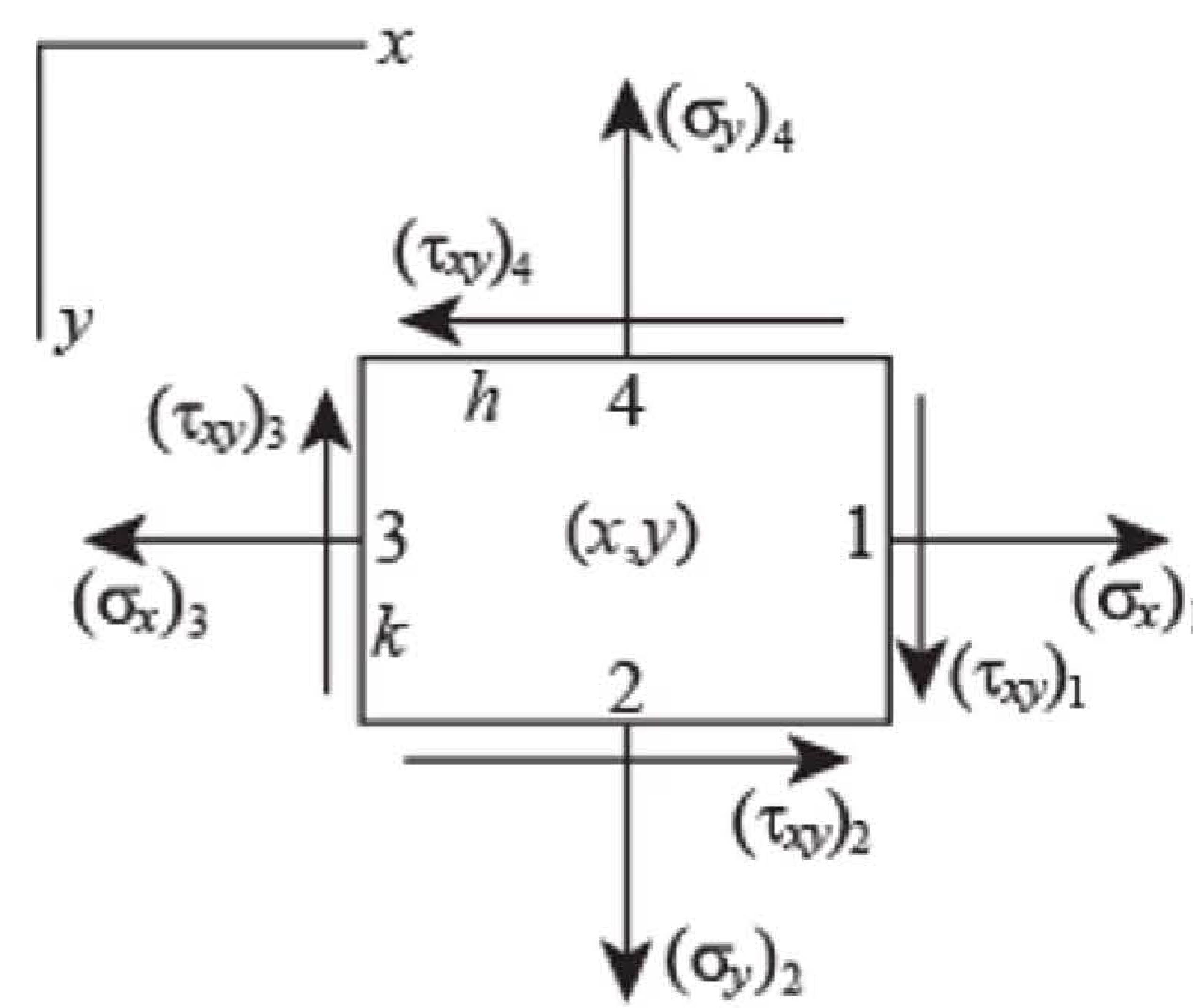
and is called ***plain stress***

Equations of equilibrium for an interior point (x, y),

$$(\sigma_x)_1 k - (\sigma_x)_3 k + (\tau_{xy})_2 h - (\tau_{xy})_4 h + Xhk = 0$$

$$\frac{(\sigma_x)_1 - (\sigma_x)_3}{h} + \frac{(\tau_{xy})_2 - (\tau_{xy})_4}{k} + X = 0$$

where *X*, *Y* components of body force per unit volume



Elasticity

For an infinitesimal bloc $(h \to 0, k \to 0)$:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0$$
$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + Y = 0$$

In practical applications the weight of the body is usually the only body force. Taking the *y*-axis downward and denoting by ρ the mass per unit volume of the body,

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$
$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \rho g = 0$$

Remarks:

-The equations must be satisfied at all points throughout the volume of the body.

-At the boundary they must be such as to be in equilibrium with the external forces on the boundary of the plate, so that external forces may be regarded as a continuation of the internal stress distribution

Elasticity

Denoting by $\bar{X}$ and $\bar{Y}$ the components of the surface forces per unit area, at a point (x, y) on the boundary:

$$\bar{X} = l\sigma_x + m\tau_{xy}$$

$$\bar{Y} = m\sigma_y + l\tau_{xy}$$

where *l* and *m* are the direction cosines of the normal *N* to the boundary

In case of a rectangular plate the coordinate axes are taken parallel to the sides of the plate. For a side of the plate parallel to the x-axis

$$\bar{X} = \pm\tau_{xy}\,, \bar{Y} = \pm\sigma_y$$

Elasticity

- The problem of the theory of elasticity usually is to determine the state of stress in a body submitted to the action of given forces
- In the case of a 2D problem it is necessary to solve the differential equations of equilibrium and the solution must be such as to satisfy the boundary conditions

Elasticity

The mathematical formulation of the condition for compatibility of stress distribution with the existence of continuous functions *u, v, w* defining the deformation:

- Strain components considered in the case of 2D problems:

$$\epsilon_x = \frac{\partial u}{\partial x}, \epsilon_y = \frac{\partial v}{\partial y}, \tau_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

- *Condition of compatibility* that must be satisfied by the strain components:

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

- By using Hookes's law the condition can be transformed into a relation between the components of stress

Elasticity

If the **weight of the body is the only body force**, the compatibility equation in terms of stress components becomes

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0$$

The equations of equilibrium, the boundary conditions and the compatibility equations give **a system of equations which is usually sufficient for the complete determination of the stress distribution in a 2D problem**

Elasticity

- In the case when the weight of the body is the only body force, the equations to be satisfied are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \rho g = 0 \qquad \text{(a)}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0 \qquad \text{(b)}$$

To these equations the boundary conditions should be added

The usual method of solving these equations is by introducing a new function, called **Airy *stress function***

Elasticity

Eqs. (a) are satisfied by taking any function ϕ of x and y and putting the following expressions for the stress components

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} - \rho g y \, , \sigma_y = \frac{\partial^2 \phi}{\partial x^2} - \rho g y \, , \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

A variety of solutions of the equations of equilibrium (*a*) are obtained. The true solution of the problem is that which satisfies also the compatibility equation (*b*)

The stress function ϕ must satisfy the equation

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

The solution of a 2D problem, when the weight is the only body force, reduces to finding a solution of the above equation which satisfies the boundary condition of the problem

Airy *stress function*

Solution by Polynomials

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

By taking polynomials of various degrees, and suitably adjusting their coefficients, a number of practically important problems can be solved

Stress function in the form of a polynomial of second and third degree

$$\phi_2 = \frac{a_2}{2}x^2 + b_2xy + \frac{c_2}{2}y^2$$

$$\phi_3 = \frac{a_3}{3 \cdot 2}x^3 + \frac{b_3}{2}x^2y + \frac{c_3}{2}xy^2 + \frac{d_3}{3 \cdot 2}y^3$$

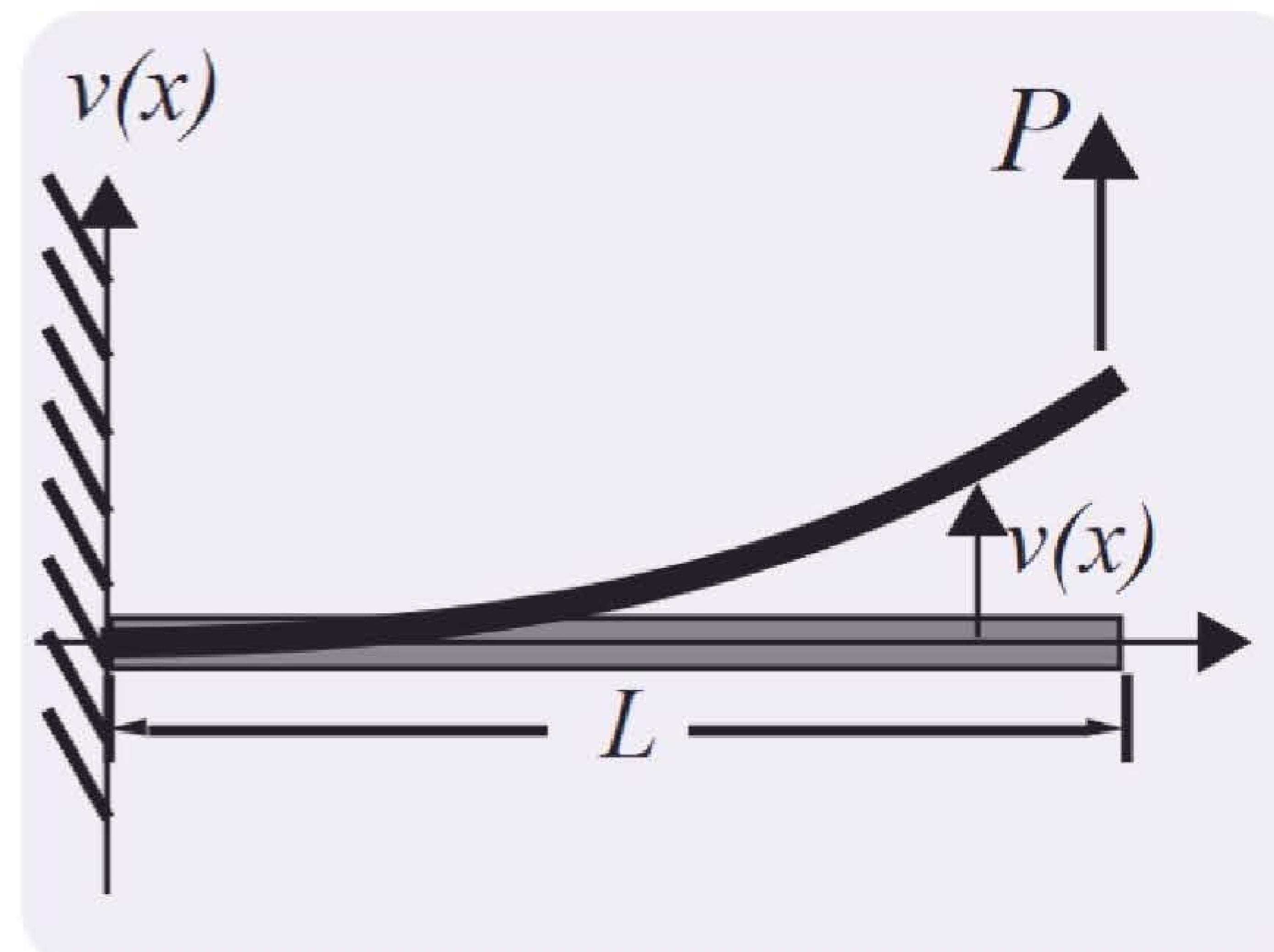
$$\Phi_4 = \frac{a_4}{4 \cdot 3}x^4 + \frac{b_4}{3 \cdot 2}x^3y + \frac{c_4}{2}x^2y^2 + \frac{d_4}{3 \cdot 2}xy^3 + \frac{e_4}{4 \cdot 3}y^4$$

Example

<u>Bending of a Cantilever Loaded at the End</u>

The main steps toward the construction of a model:

- Choose a degree for the stress function polynomial
- Impose boundary conditions for forces, stresses and shear
- Identify the model coefficients



in order to obtain the expression of the displacement

Identification of the analytical model

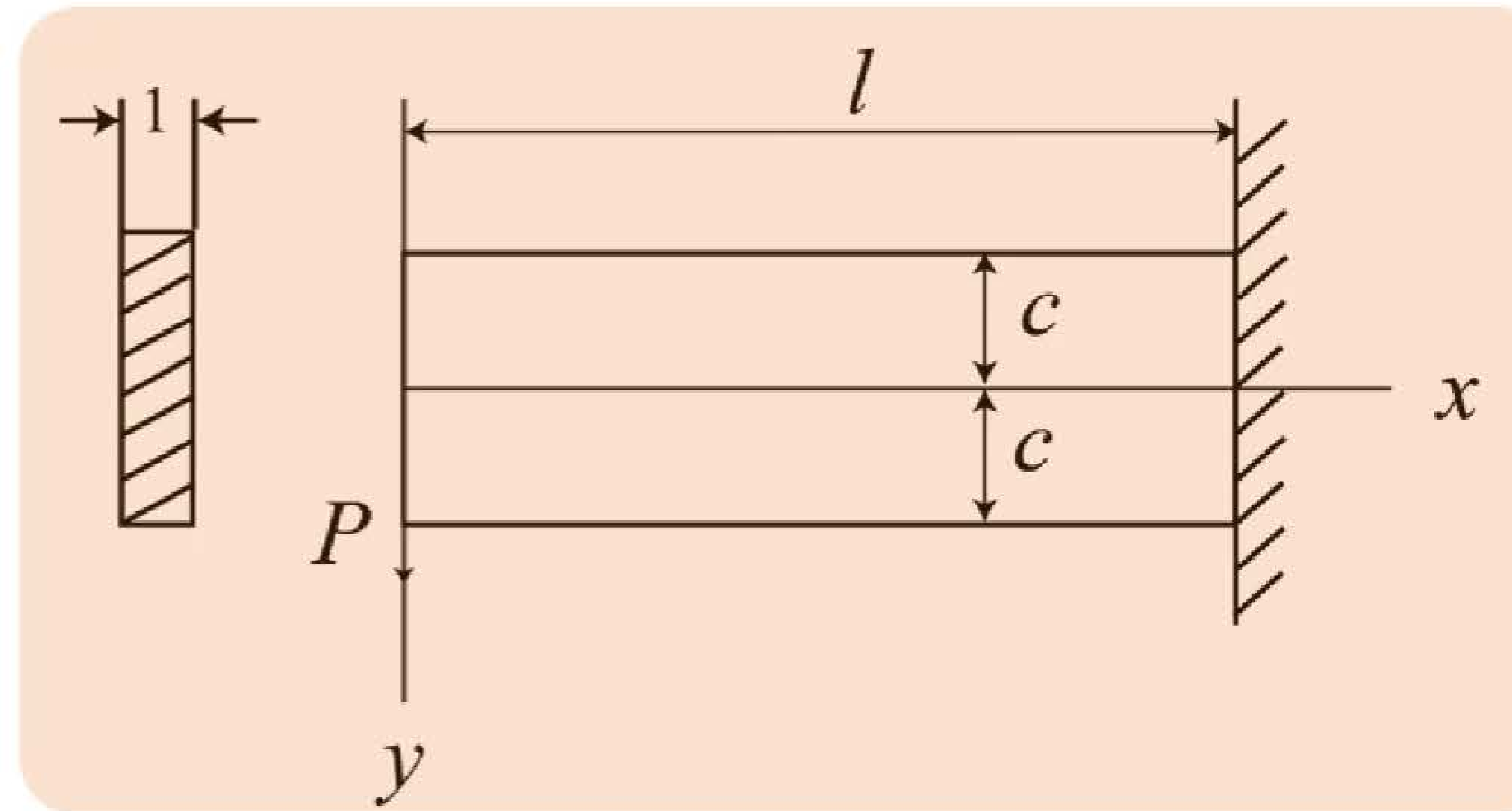
Cantilever Loaded at the End

$$\sigma_x = d_4 xy\,,\quad \sigma_y = 0$$

$$\tau_{xy} = -b_2 - \frac{d_4}{2} y^2$$

$$\left(\tau_{xy}\right)_{y=\pm c} = -b_2 - \frac{d_4}{2} c^2 = 0$$

$$d_4 = -\frac{2b_2}{c^2}$$



Cantilever having a narrow rectangular cross section of unit width

The longitudinal sides (y=c and y=-c) must be free from forces

Identification of the analytical model

$$-\int_{-c}^{c} \tau_{xy} d\,y = \int_{-c}^{c} \left(b_2 - \frac{b_2}{c^2} y^2\right) d\,y = P$$

$$b_2 = \frac{3}{4}\frac{P}{c}$$

$$\sigma_x = -\frac{3}{2}\frac{P}{c^3} xy\,, \quad \sigma_y = 0$$

$$\tau_{xy} = -\frac{3P}{4c}\left(1 - \frac{y^2}{c^2}\right)$$

On the loaded end, the sum of the shearing forces must be equal with P

Identification of the analytical model

$$I = \frac{2}{3}c^3$$

The moment of inertia I of the cross section of the cantilever

$$\sigma_x = -\frac{Pxy}{I}\ , \quad \sigma_y = 0$$

$$\tau_{xy} = -\frac{P}{I}\frac{1}{2}(c^2 - y^2)$$

$$\epsilon_x = \frac{\partial\, u}{\partial\, x} = \frac{\sigma_x}{E} = -\frac{Pxy}{EI}\ , \epsilon_y = \frac{\partial\, v}{\partial\, y} = -\frac{\nu\sigma_x}{E} = \frac{\nu Pxy}{EI}$$

$$\gamma_{xy} = \frac{\partial\, u}{\partial\, y} + \frac{\partial\, v}{\partial\, x} = \frac{\tau_{xy}}{G} = -\frac{P}{2IG}(c^2 - y^2)$$

$$u = -\frac{Px^2y}{2EI} + f(y)\ , v = \frac{\nu Pxy^2}{2EI} + f_1(x)$$

$$-\frac{Px^2}{2EI} + \frac{d\, f(y)}{d\, y} + \frac{\nu Py^2}{2EI} + \frac{d\, f_1(x)}{d\, x} = -\frac{P}{2IG}(c^2 - y^2)$$

$$F(x) = -\frac{Px^2}{2EI} + \frac{d\,f_1(x)}{d\,x},$$

$$G(y) = \frac{d\,f(y)}{d\,y} + \frac{\nu P y^2}{2EI} - \frac{Py^2}{2IG}$$

$$K = -\frac{Pc^2}{2IG}$$

The equality must be valid even if x alone or y alone would vary: a situation that is valid only if both F(x) and G(y) are constant

$$F(x) + G(y) = K$$

$$e + d = -\frac{Pc^2}{2IG}$$

$$\frac{d\, f_1(x)}{d\, x} = \frac{Px^2}{2EI} + d\ ,$$

$$\frac{d\, f(y)}{d\, y} = -\frac{Py^2}{2EI} + \frac{Py^2}{2IG} + e$$

$$f(x) = -\frac{\nu P y^3}{6EI} + \frac{Py^3}{6IG} + ey + g$$

$$f_1(x) = \frac{Px^3}{6EI} + d\, x + h$$

$$u = -\frac{Px^2y}{2EI} - \frac{\nu Py^3}{6EI} + \frac{Py^3}{6IG} + ey + g$$

$$v = \frac{\nu Pxy^2}{2EI} + \frac{Px^3}{6EI} + d\,x + h$$

$$g = 0\,, h = -\frac{Pl^3}{6EI} - dl$$

$$(v)_{y=0} = \frac{Px^3}{6EI} - \frac{Pl^3}{6EI} - d(l - x)$$

$$\left(\frac{\partial v}{\partial x}\right)_{x=l,y=0} = 0$$

$$\left(\frac{\partial u}{\partial y}\right)_{x=l,y=0} = 0$$

$$d = -\frac{Pl^2}{2EI}$$

$$e = \frac{Pl^2}{2EI} - \frac{Pc^2}{2IG}$$

$$u = -\frac{Px^2y}{2EI} - \frac{\nu Py^3}{6EI} + \frac{Py^3}{6IG} + \left(\frac{Pl^2}{2EI} - \frac{Pc^2}{2IG}\right)y$$

$$v = \frac{\nu Pxy^2}{2EI} + \frac{Px^3}{6EI} - \frac{Pl^2x}{2EI} + \frac{Pl^3}{3EI}$$

$$(v)_{y=0} = \frac{Px^3}{6EI} - \frac{Pl^2x}{2EI} + \frac{Pl^3}{3EI}$$

Bibliography

1. Meleshko , V. "Selected topics in the history of the two-dimensional biharmonic problem“. ASME. Appl. Mech. Rev. January 2003; 56(1): 33–85. https://doi.org/10.1115/1.1521166
2. S. Biswas, B. C. Lovell . *Bezier and Splines in Image Processing and Machine Vision*. Springer 2008. ISBN: 978-1-84628-956-9
3. H. Prautzsch, W. Boehm, M. Paluszny. *Bezier and B-Spline Techniques*. Springer 2002. ISBN: 978-3-642-07842-2
4. A.P. S. Selvadurai. *Partial Differential Equations in Mechanics 2. The Biharmonic Equation, Poisson's Equation*. Springer 2000. ISBN: 978-3-642-08667-0
5. S.P. Timoshenko, J.N. Goodier. *Theory of Elasticity*. McGraw Hill 1970, 3rd edition