EE 628
Deep Learning
Fall 2019

Lecture 6

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Assistant Prof. Sergul Aydore

Department of Electrical and Computer Engineering

STEVENS INSTITUTE of TECHNOLOGY

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Overview

- Last lecture we covered
  - Backpropagation
  - Optimization Algorithms

- Today, we will cover
  - More on Optimization Algorithms
  - Convolutional layers

Momentum

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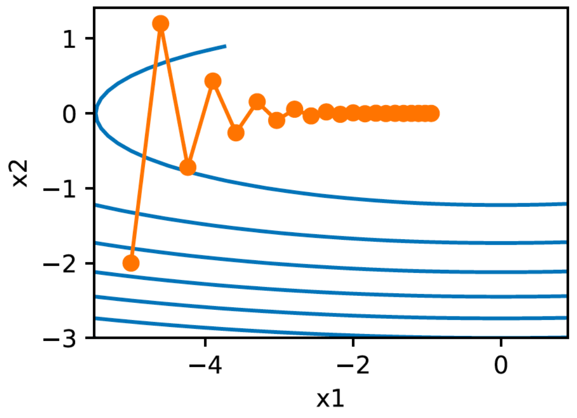
  - The slope of the objective function has a larger value in the vertical direction
  - Therefore, the variable will move more in Vertical direction



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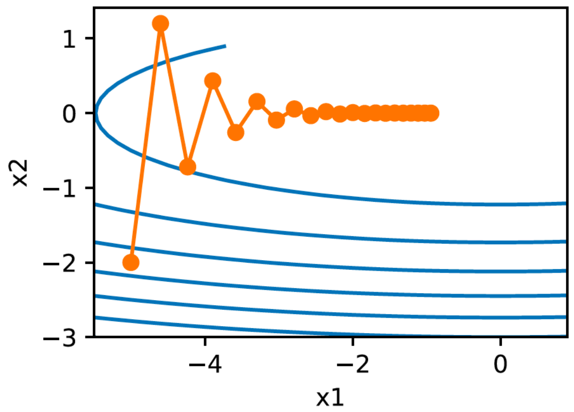

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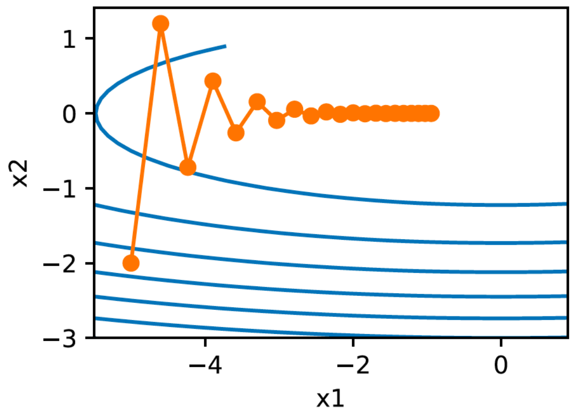

- Small learning rate will cause the variable move slower toward the optimal solution
- Large learning rate will make the variable overshoot in vertical direction

Momentum

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x1

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- At time step $t > 0$, momentum modifies the steps at each iteration as follows

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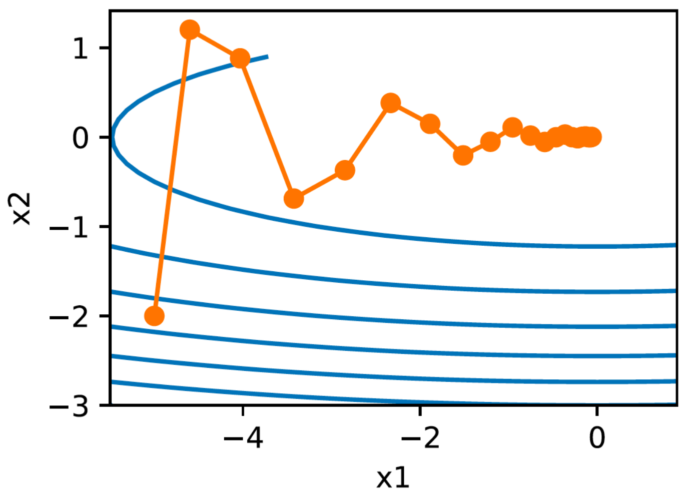

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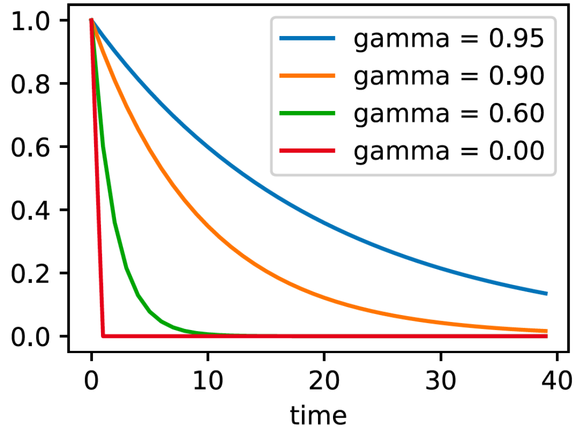

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- Now, we introduce Adagrad that adjusts the learning rate according to the gradient value of the independent variable in each direction

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- At time step t, we first sum the results of the square by element operation for the mini-batch gradient $\mathbf{g}_t$ to get the variable $\mathbf{s}_t$:

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- Next, we readjust the learning rate of each element in the independent variable of the objective function using element operations

$$\boldsymbol{x}_t \leftarrow \boldsymbol{x}_{t-1} - \frac{\eta}{\sqrt{\boldsymbol{s}_t + \epsilon}} \odot \boldsymbol{g}_t,$$

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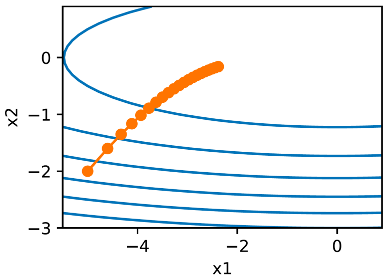
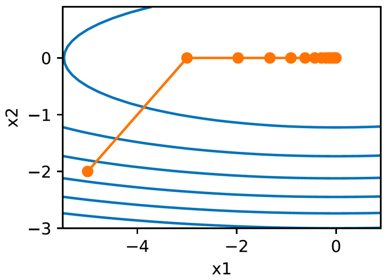
- On the contrary, if the partial derivative of such an element remains small, then its learning rate will decline more slowly.
- when the learning rate declines very fast during early iteration, yet the current solution is still not desirable, Adagrad might have difficulty finding a useful solution because the learning rate will be too small at later stages of iteration.
- If we use Adagrad for the example $f(\mathbf{x}) = 0.1\, x_1^2 + 2x_2^2$, we get





Which one uses a larger learning rate?

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- Specifically, given the hyperparameter $0 \leq \gamma < 1$, RMSProp is computed at time step $t > 0$:

$$\mathbf{s}_t \leftarrow \gamma \mathbf{s}_{t-1} + (1 - \gamma) \mathbf{g}_t \odot \mathbf{g}_t.$$

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- Like Adagrad, RMSProp readjust the learning rate of each element in the independent variable of the object function as:

$$\mathbf{x}_t \leftarrow \mathbf{x}_{t-1} - \frac{\eta}{\sqrt{\mathbf{s}_t + \epsilon}} \odot \mathbf{g}_t,$$

RMSProp

- If we expand the definition of $\mathbf{s}_t$, we see that:

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&\cdots \\
&= (1-\gamma)\left(\mathbf{g}_t \odot \mathbf{g}_t + \gamma \mathbf{g}_{t-1} \odot \mathbf{g}_{t-1} + \cdots + \gamma^{t-1} \mathbf{g}_1 \odot \mathbf{g}_1\right).
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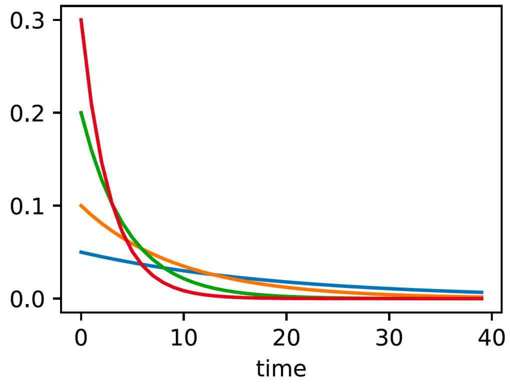
- We visualize these weights in the past 40 time steps for various γ:



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- Given the hyperparameter $0 \leq \rho < 1$, we compute $\boldsymbol{s}_t \leftarrow \rho \boldsymbol{s}_{t-1} + (1-\rho)\boldsymbol{g}_t \odot \boldsymbol{g}_t.$
- Unlike RMSprop, Adadelta maintains an additional state variable $\Delta\mathbf{x}_t$ to compute the variation of the independent variable

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- Next we update the independent variable $\boldsymbol{x}_t \leftarrow \boldsymbol{x}_{t-1} - \boldsymbol{g}_t'.$
- Finally, we use $\Delta\mathbf{x}_t$ to record EWMA on the squares of elements of $\boldsymbol{g}'$

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$$\boldsymbol{v}_t \leftarrow \beta_1 \boldsymbol{v}_{t-1} + (1 - \beta_1)\boldsymbol{g}_t. \qquad \boldsymbol{s}_t \leftarrow \beta_2 \boldsymbol{s}_{t-1} + (1 - \beta_2)\boldsymbol{g}_t \odot \boldsymbol{g}_t.$$

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- Notice that $\boldsymbol{v}_t = \underbrace{(1-\beta_1)\sum_{i=1}^{t} \beta_1^{t-i}}_{1-\beta_1^t} \boldsymbol{g}_i$

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- When t is small, the sum of the mini-batch stochastic gradient weights from each previous time step will be small. To eliminate this effect, we perform bias correction:

$$\hat{\boldsymbol{v}}_t \leftarrow \frac{\boldsymbol{v}_t}{1-\beta_1^t}, \qquad \hat{\boldsymbol{s}}_t \leftarrow \frac{\boldsymbol{s}_t}{1-\beta_2^t}.$$

Adam

- Next, the Adam algorithm will use the bias-corrected variables to re-adjust the learning rate of each element.

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- Finally, we update the independent variable as

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- Now, we introduce convolutional neural networks (CNNs), a powerful family of neural networks that were designed for precisely this purpose.
- In addition to their strong predictive performance,
  - convolutional neural networks tend to be computationally efficient,
  - both because they tend to require fewer fewer parameters than dense architectures
  - also because convolutions are easy to parallelize across GPU cores

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- A few key principles for building neural networks for computer vision:
  1. **Translation Invariance**: Our vision systems should, in some sense, respond similarly to the same object regardless of where it appears in the image
  2. **Locality**: Our visions systems should, in some sense, focus on local regions, without regard for what else is happening in the image at greater distances.

Constraining the MLP

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- We could formally express this dense layer as follows:

$$h[i,j] = u[i,j] + \sum_{k,l} W[i,j,k,l] \cdot x[k,l] = u[i,j] + \sum_{a,b} V[i,j,a,b] \cdot x[i+a, j+b]$$

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- For any given location (i, j) in the hidden layer $h[i, j]$, we compute its value by summing over pixels in x, centered around (i, j) and weighted by $V[i, j, a, b]$.

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- In other words, we have $V[i, j, a, b] = V[a, b]$ and u is a constant.
- As a result we can simplify the definition for h:

$$h[i, j] = u + \sum_{a,b} V[a, b] \cdot x[i + a, j + b]$$

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- This implies that a shift in the inputs x should simply lead to a shift in the activations h.
- This is only possible if V and u don't actually depend on (i, j).
- In other words, we have $V[i, j, a, b] = V[a, b]$ and u is a constant.
- As a result we can simplify the definition for h:

$$h[i, j] = u + \sum_{a,b} V[a, b] \cdot x[i + a, j + b]$$

- This is a convolution! We also reduced the number of parameters.

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- This, in a nutshell is the convolutional layer.

Convolutions

- In mathematics, the convolution between two functions is defined as:

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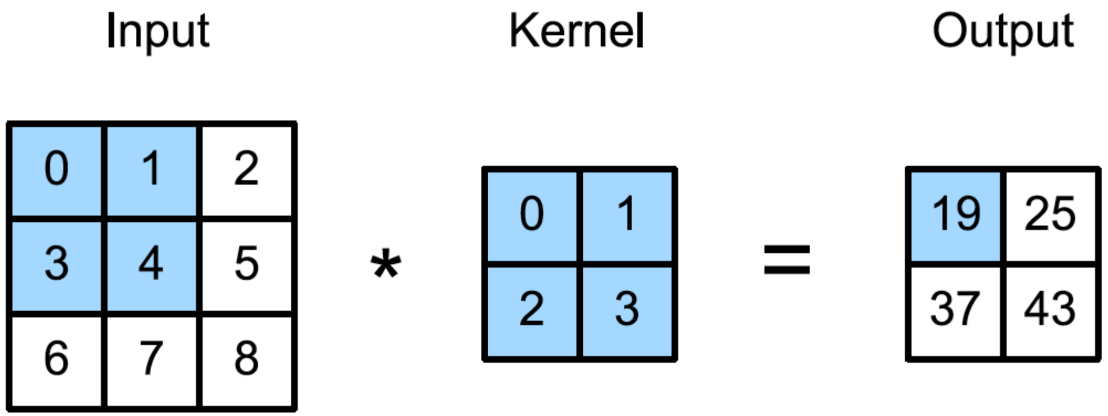
- Also note that the original definition is actually a *cross-correlation*.

Convolutions for Images

- Let’s see how convolutions work in practice.



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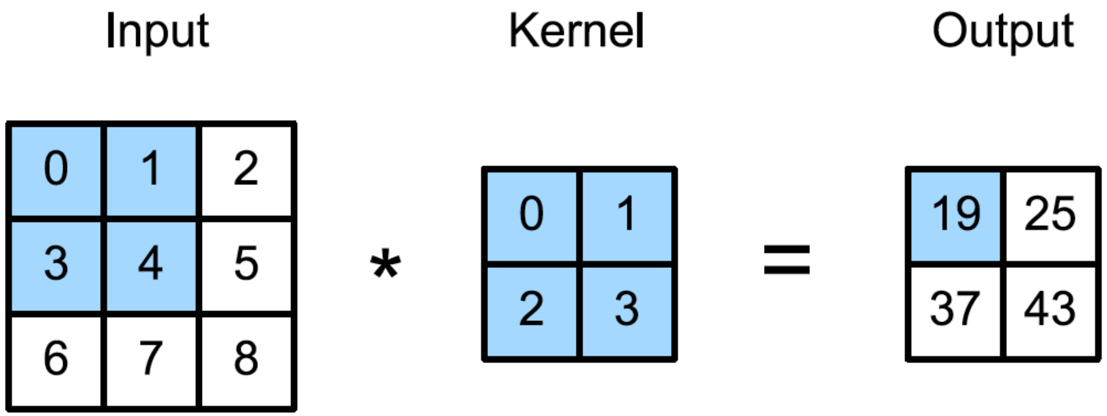

- Start the notebook IN_CLASS_convolutions

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  - Solution: **Strides**

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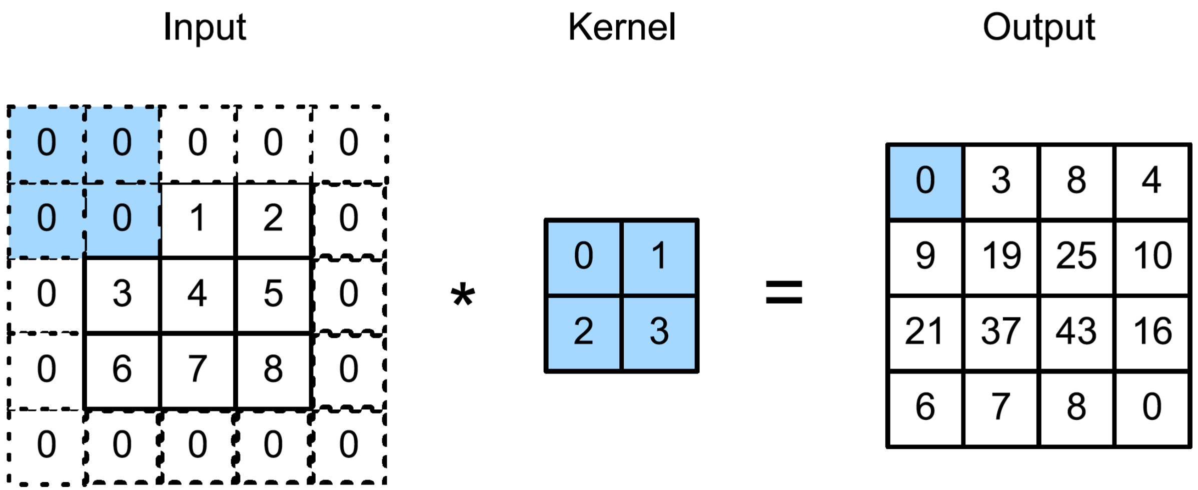
- Adding extra pixels of filler around the boundary of our input image, thus increasing the effective size of the image.



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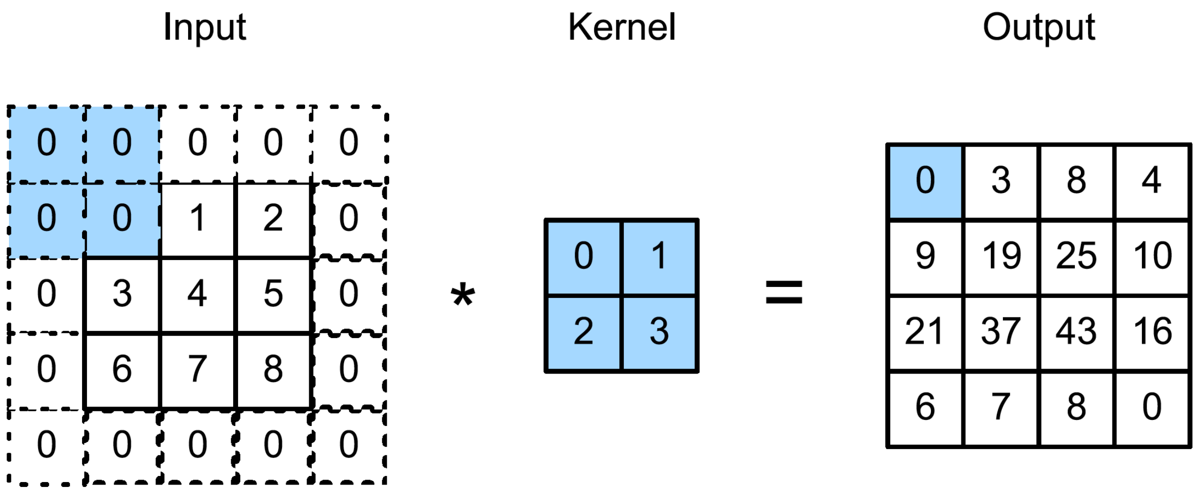

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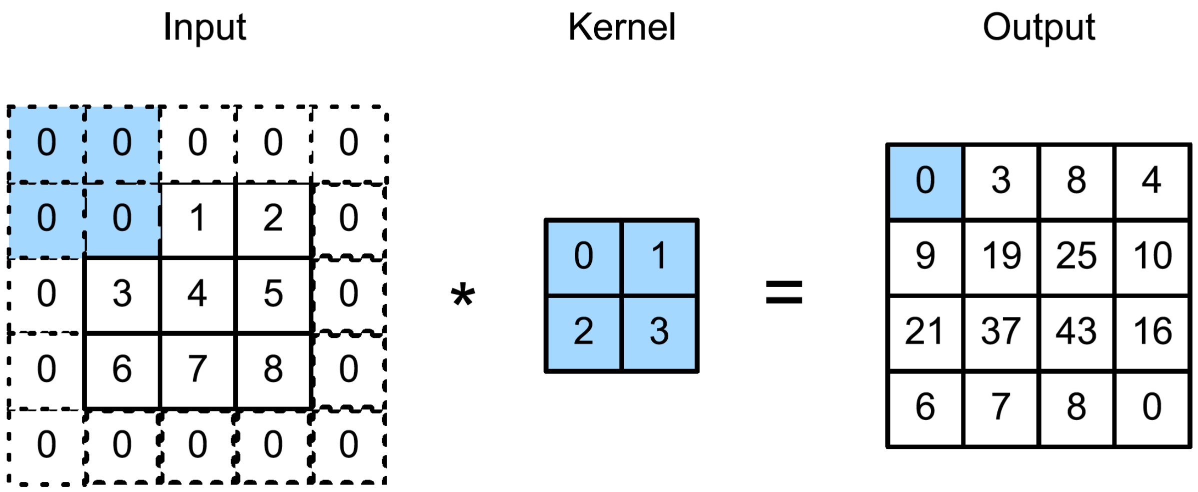

- What is the size of the output after padding?
- What do you think is a good number of padding?

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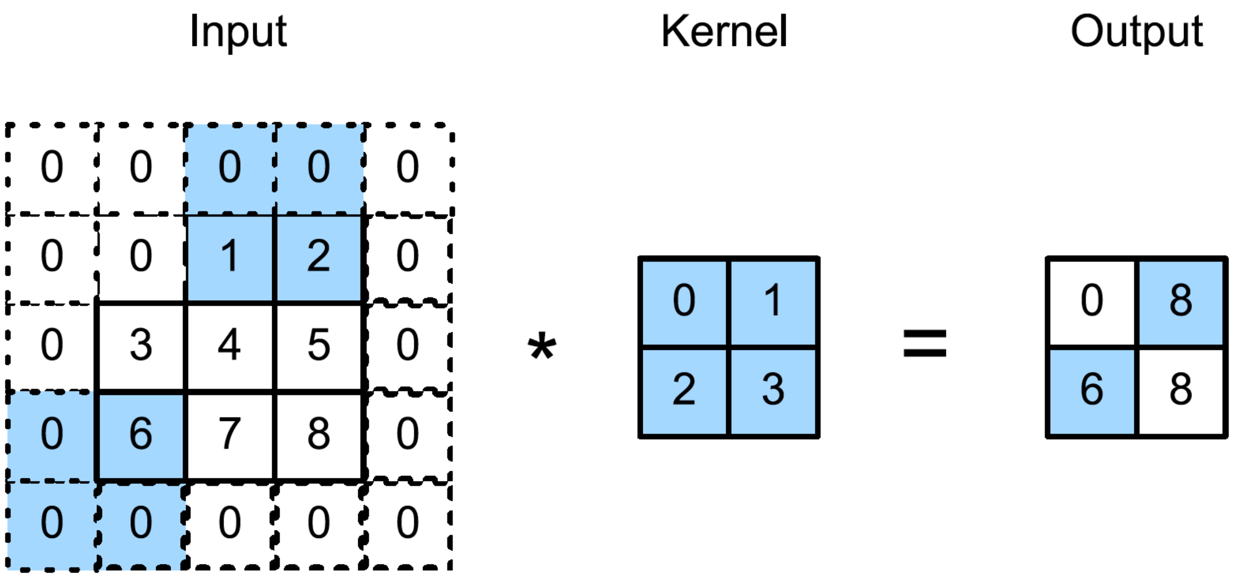
- We refer to the number of rows and columns traversed per slide as the *stride*.



Example for strides with 3 and 2 for height and width, respectively.

What is the size of the output after proper amount of padding?

Multiple Input Channels

- When we add channels into the mix, our inputs and hidden representations both become three-dimensional arrays. For example, each RGB input image has shape 3 x h x w. We refer to this axis, with a size of 3, as the channel dimension.

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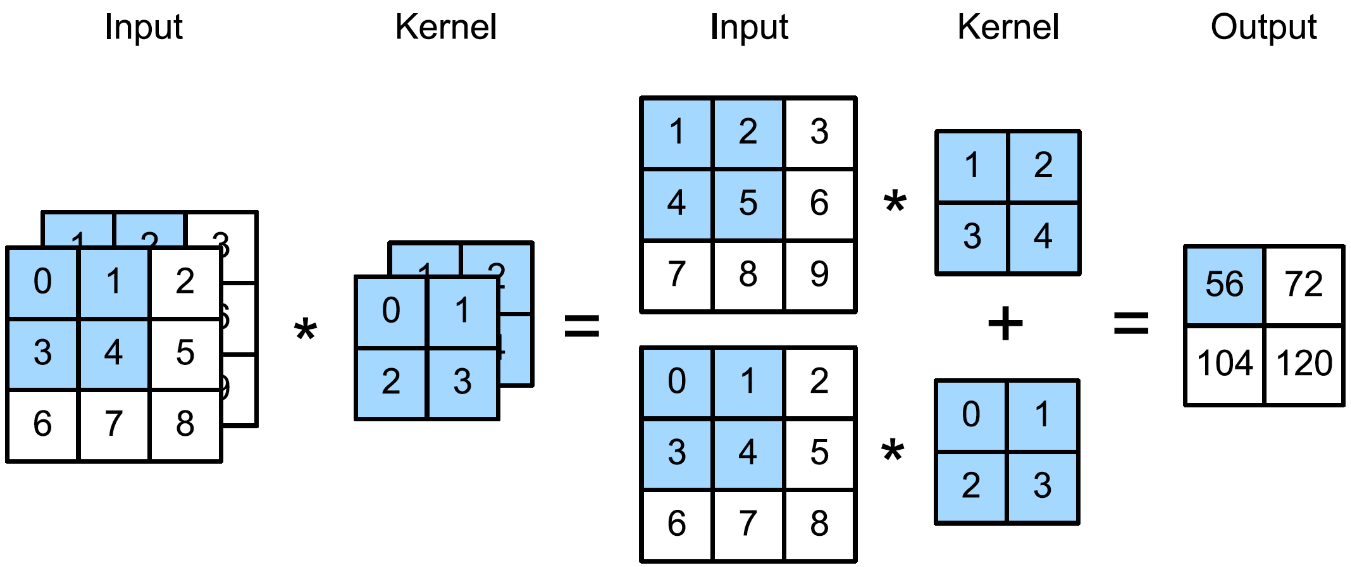
- When the input data contains multiple channels, we need to construct a convolution kernel with the same number of input channels as the input data



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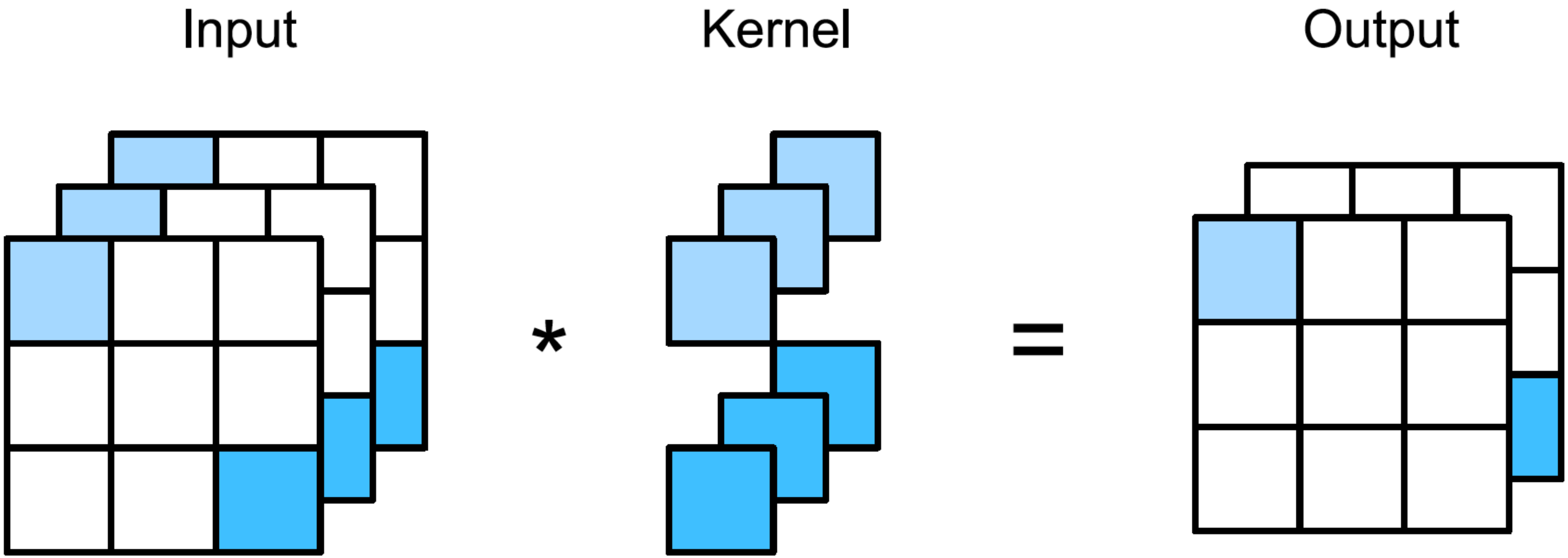
- We concatenate them on the output channel dimension, so that the shape of the convolution kernel is $c_o \times c_i \times k_h \times k_w$.

Multiple Input and Output Channels

- The figure below shows the cross-correlation computation using the 1 1 convolution kernel with 3 input channels and 2 output channels



Pooling

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- Pooling layers serve the dual purposes
  - of mitigating the sensitivity of convolutional layers to location and
  - of spatially downsampling representations

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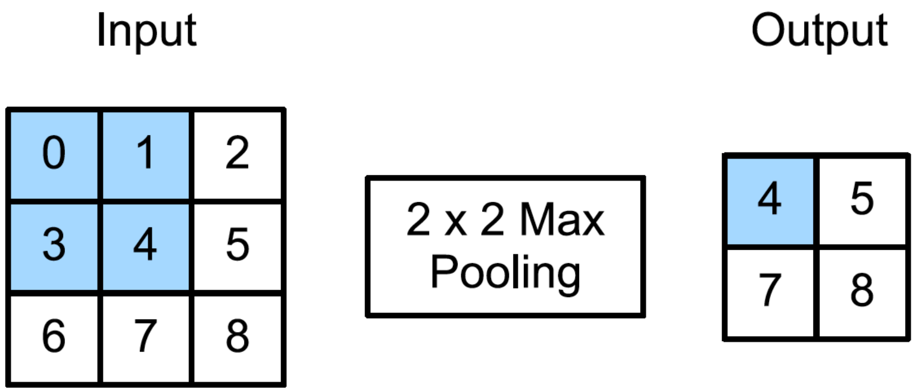
- However, the pooling layer contains no parameters.
- Instead, pooling operators are deterministic, typically calculating either the maximum or the average value of the elements in the pooling window. These operations are called *maximum pooling* (max pooling for short) and *average pooling*, respectively.



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    - This means that the number of output channels for the pooling layer is the same as the number of input channels.