Second Moments

Second Moment Calculations

The **second moment**, or **moment of inertia**, measures how an area or mass is distributed about an axis.

General Formula

The second moment of area about the (x)-axis or (y)-axis is:

$$I_x = \int y^2 \, dA, \quad I_y = \int x^2 \, dA$$

Derivation of the Moment of Inertia of a Rectangle About Its Base

We calculate the **moment of inertia** of a rectangle about its **base**, where the base is along the x-axis and the height extends from $y = 0$ to $y = h$.

1. Moment of Inertia Formula

The formula for the moment of inertia is:

$$I = \int y^2 \, dA$$

Here: - y: distance from the axis of rotation (the base of the rectangle in this case), - dA: the infinitesimal area element.

2. Setup for the Rectangle

- The rectangle has a width of b and height h.
- The height extends from $y = 0$ to $y = h$.
- The infinitesimal area element is $dA = b\,dy$.

The integral becomes:

$$I_{\text{base}} = \int_0^h y^2\,b\,dy$$

3. Solve the Integral

Factor b (constant width) outside of the integral:

$$I_{\text{base}} = b\int_0^h y^2\,dy$$

The integral of y^2 is:

$$\int y^2\,dy = \frac{y^3}{3}$$

Apply the limits $y = 0$ to $y = h$:

$$I_{\text{base}} = b\left[\frac{y^3}{3}\right]_0^h$$

Substitute the limits:

$$I_{\text{base}} = b\left(\frac{h^3}{3} - \frac{0^3}{3}\right)$$

Simplify:

$$I_{\text{base}} = b \cdot \frac{h^3}{3}$$

4. Final Result

The moment of inertia of a rectangle about its base is:

$$I_{\text{base}} = \frac{bh^3}{3}$$

Derivation of the Moment of Inertia of a Rectangle About its Centroidal Axis

We calculate the **moment of inertia** of a rectangle about its **centroidal** x-axis. The centroidal axis passes horizontally through the centroid of the rectangle, located at $y = h/2$ (mid-height of the rectangle).

1. Moment of Inertia Formula

The formula for the moment of inertia is:

$$I = \int y^2 \, dA$$

Here:

- y: distance from the axis of rotation (centroidal axis in this case),
- dA: the infinitesimal area element.

2. Setup for the Rectangle

- The rectangle is centered on the centroidal x-axis, so the height ranges from $y = -h/2$ to $y = h/2$.
- The width of the rectangle is b.
- The infinitesimal area element is $dA = b \, dy$.

The integral becomes:

$$I_{\text{centroidal}} = \int_{-h/2}^{h/2} y^2 \, b \, dy$$

3. Solve the Integral

Factor b (constant width) outside of the integral:

$$I_{\text{centroidal}} = b \int_{-h/2}^{h/2} y^2 \, dy$$

The integral of y^2 is:

$$\int y^2 \, dy = \frac{y^3}{3}$$

Apply the limits $y = -h/2$ to $y = h/2$:

$$I_{\text{centroidal}} = b \left[\frac{y^3}{3} \right]_{-h/2}^{h/2}$$

Substitute the limits:

$$I_{\text{centroidal}} = b \left(\frac{(h/2)^3}{3} - \frac{(-h/2)^3}{3} \right)$$

Since $(-h/2)^3 = -(h/2)^3$, the terms simplify:

$$I_{\text{centroidal}} = b \left(\frac{(h/2)^3}{3} + \frac{(h/2)^3}{3} \right)$$

Combine the terms:

$$I_{\text{centroidal}} = b \left(\frac{2(h/2)^3}{3} \right)$$

Simplify $(h/2)^3 = \frac{h^3}{8}$:

$$I_{\text{centroidal}} = b \cdot \frac{2}{3} \cdot \frac{h^3}{8}$$

$$I_{\text{centroidal}} = b \cdot \frac{h^3}{12}$$

4. Final Result

The moment of inertia of a rectangle about its centroidal x-axis is:

$$I_{\text{centroidal}} = \frac{bh^3}{12}$$

This is the standard formula for the moment of inertia of a rectangle about its centroidal axis.

Comparison with the Centroidal Moment of Inertia

The moment of inertia about the base is four times greater than that about the centroidal x-axis:

$$I_{\text{base}} = 4 \cdot I_{\text{centroidal}}$$

where:

$$I_{\text{centroidal}} = \frac{bh^3}{12}$$

Parallel Axis Theorem

To find the moment of inertia about an axis parallel to the centroidal axis:

$$I = I_c + Ad^2$$

Radius of Gyration

The radius of gyration of a body about an axis is a measure of the distribution of its area or mass relative to that axis. It is defined as the distance from the axis at which the total area or mass of the body could be concentrated without changing its moment of inertia.

Mathematically, the radius of gyration k is given by:

where:

- I : Moment of inertia about the axis
- A : Area of the cross-section (for area calculations)
- m : Mass of the body (for mass calculations)

Difference Between First Moment and Second Moment

The **first moment** and **second moment** are concepts from mathematics and physics, describing how quantities are distributed around a reference point (such as the mean or origin). Below is a detailed explanation:

First Moment

- **Definition**: The first moment measures the mean or center of mass relative to a reference point.
- **Formula**:
$$M_1 = \sum(x_i \cdot w_i)$$
where x_i is a value, and w_i is its weight.
- **Physical Interpretation**: In mechanics, it determines the center of mass or centroid of an object.

Second Moment

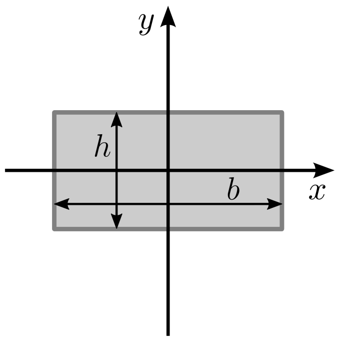
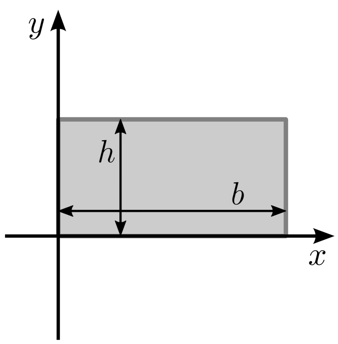
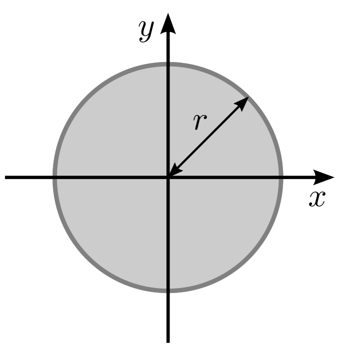
- **Definition**: The second moment measures the spread or dispersion of values around a reference point.
- **Formula**:
$$M_2 = \sum(x_i^2 \cdot w_i)$$
where x_i is a value, and w_i is its weight.
- **Physical Interpretation**: In mechanics, the second moment of area (I) is used to describe an object's resistance to bending or torsion (e.g., the moment of inertia).

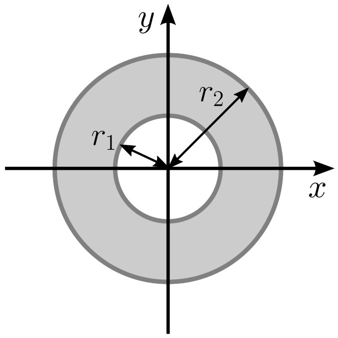
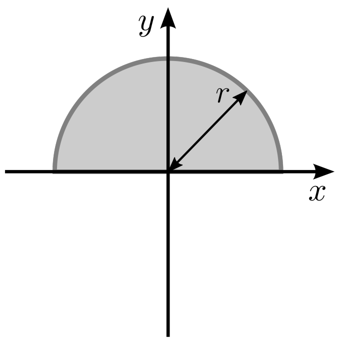
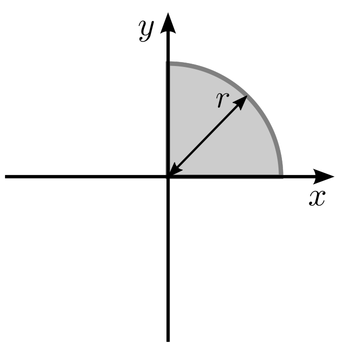
Key Differences

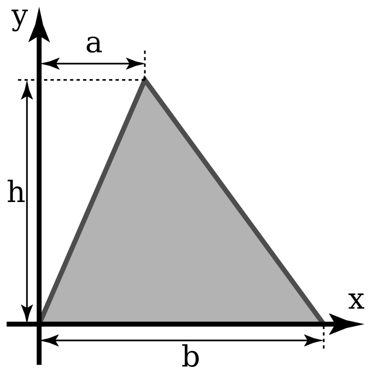
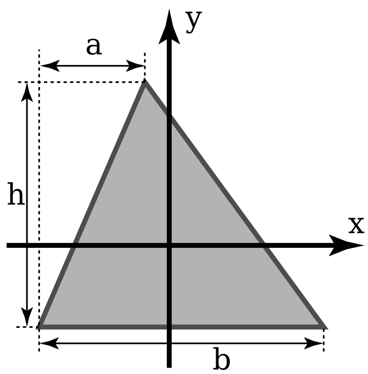
Feature	First Moment	Second Moment
Reference	Measures position relative to a point (e.g., centroid).	Measures spread relative to a point.
Physical Meaning	Indicates center or balance point.	Indicates resistance or spread.
Example (Physics)	Center of mass.	Moment of inertia.

Second Moments of Common Shapes

Table 2: Second moments

Shape	Second moment (I_x)	Second moment (I_y)
	$I_x = \frac{1}{12}bh^3$	$I_x = \frac{1}{12}b^3h$
	$I_x = \frac{1}{3}bh^3$	$I_x = \frac{1}{3}b^3h$
	$I_x = \frac{\pi}{4}r^4$	$I_y = \frac{\pi}{4}r^4$

Shape	Second moment (I_x)	Second moment (I_y)
	$I_x = \frac{\pi}{4}(r_2^4 - r_1^4)$	$I_y = \frac{\pi}{4}(r_2^4 - r_1^4)$
	$I_x = \frac{\pi}{8}r^4$	$I_y = \frac{\pi}{8}r^4$
	$I_x = \frac{\pi}{16}r^4$	$I_y = \frac{\pi}{16}r^4$

Shape	Second moment (I_x)	Second moment (I_y)
	$I_x = \frac{1}{12} bh^3$	
	$I_x = \frac{1}{36} bh^3$	