Integration By Parts

$$\int u.dv = u.v - \int v.du \quad \text{(LAPTE)}$$

Reduction Formula

$$\int \cos^n x.dx = \frac{\cos^{n-1}.\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x.dx$$

$$\int \sin^m x.\cos^n x.dx =$$

I-) m is even $u = \sin x$, $du = \cos x.dx$ } Substitution

II-) n is even $u = \cos x$, $du = -\sin x.dx$

III) If both are given $\cos^2 x = \frac{(\cos 2x+1)}{2}$

$\sin^2 x = \frac{(1-\cos 2x)}{2}$

$$\int \sqrt{1 \mp \sin 2x}\,.dx = \int \sqrt{(\sin x - \cos x)^2}\,dx$$

$\sqrt{x^2+a^2}$, $x = a.\tan\theta$

$\sqrt{a^2-x^2}$, $x = a.\sin\theta$

$\sqrt{x^2-a^2}$, $x = a.\sec\theta$

Integral

$f'(x) = \frac{1}{\sqrt{x}}$ | $f'(x) = \frac{1}{e^x}$ | $f'(x) = \sec^2 x = 1+\tan^2 x = \frac{1}{\cos^2 x}$ $\Big\}\rightarrow$ $f(x)$ $\tan x + c$ | $f'(x) = \sec x . \tan x$

$f(x) = 2\sqrt{x} + c$ | $f(x) = -e^{-x}$ | $f(x) = \sec x + c$

$f(x) = \operatorname{cosec} x + c$

$f'(x) = \operatorname{cosec} x . \cot x$

$$\text{Average of } f(x) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

→ Köklü ifadelerin integralinde Alan, yarım çemberden bulunur.

$\cos(2x) =>$

Extra

→ $1+2+3+ \dots +n = \sum_{k=1}^{n} k = \frac{n.(n+1)}{2}$ → $2+4+6+ \dots + 2n = \sum_{k=1}^{n} 2k = n(n+1)$

→ $1+3+5+ \dots + (2n-1) = \sum_{k=1}^{n} (2k-1) = n^2$ → $1^2+2^2+3^2+ \dots n^2 = \sum_{k=1}^{n} k^2 = \frac{n.(n+1)(2n+1)}{6}$

→ $1^3+2^3+3^3+ \dots + n^3 = \sum_{k=1}^{n} k^3 = \left[\frac{n.(n+1)}{2}\right]^2$

→ $\sum_{k=p}^{n} a_k = \sum_{k=p}^{n} f(k) = \sum_{k=p+r}^{n+r} a_{k-r} = \sum_{k=p+r}^{n+r} f(k-r)$

Volume

1-) Disc Method

$$\pi . \int_a^b f^2(x) . dx = V$$

2-) Shell Method

y-axis

$$V = \int_a^b 2.\pi . x . f(x) dx = \int_a^b 2\pi . (\text{Kabuk Yarıçapı})(\text{Kabuk Yüksekliği})\,dx$$

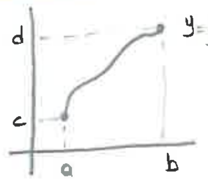
Definite Integral

→ $\int_a^a f(x)dx = 0$ → $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Average of $f(x)$;

$$\text{Avr}(f) = \frac{1}{b-a}\int_a^b f(x)dx$$

ARC LENGTH



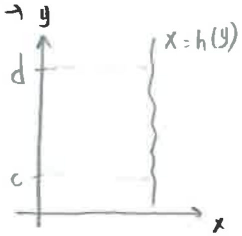
$$\text{Arc Length } (L) = \int_a^b \sqrt{1+f'(x)^2}\,dx$$

Length of Curve

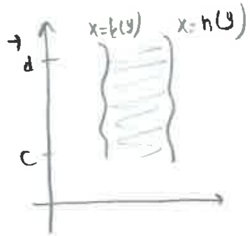
$$L = \int_c^d \sqrt{1+g'(y)^2}\,.dy$$

→ $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$

$\cos^2 x = \frac{1}{2}.(1+\cos 2x)$, $\sin^2 x = \frac{1}{2}(1-\cos 2x)$

→ 

$A = \int_c^d h(y)$

→ 

$A = \int_c^d [h(y) - k(y)] . dy$

SURFACE AREA

$$S = \int_a^b 2.\pi . f(x) \sqrt{1+f'(x)^2}\,.dx$$

$$S = \int_c^d 2.\pi . g(y) . \sqrt{1+g'(y)^2}\,.dy$$

CENTER OF MASS

$$M = \int_a^b \ell . f(x) . dx$$

$$\tilde{x} = \frac{1}{M} . \int_a^b \ell . x . [f(x) - g(x)] . dx$$

$$\tilde{y} = \frac{1}{M} . \int_a^b \frac{\ell}{2} . [f^2(x) - g^2(x)] . dx$$

13.06.2015
Monday

Integration By Parts

$$d(u(x).V(x)) = d(u(x)).V(x) + d(V(x)).u(x)$$

$$u.V = \int V.du + \int u.dV$$

$$\boxed{\int u.dV = u.V - \int V.du}$$

LAPTÜ, LAPTE

Ex: $\int x.e^x.dx = \begin{bmatrix} u = x & du = dx \\ dV = e^x.dx & V = e^x \end{bmatrix} = x.e^x - \int e^x.dx = x.e^x - e^x + c$

Ex: $\int \ln x.dx = \begin{bmatrix} u = \ln x & du = \frac{1}{x}.dx \\ dV = dx & V = x \end{bmatrix} = x.\ln x - \int x.\frac{1}{\cancel{x}}.dx = x.\ln x - x + c$

Ex: $\int \arctan x.dx = \begin{bmatrix} u = \arctan x & du = \frac{1}{1+x^2}.dx \\ dV = dx & V = x \end{bmatrix} = x.\arctan x - \frac{1}{2}\int \frac{2.x}{1+x^2}.dx = x.\arctan x - \frac{1}{2}\ln|1+x^2| + c$

Ex: $\int x.\sin x.dx = \begin{bmatrix} u = x & du = dx \\ dV = \sin x.dx & V = -\cos x \end{bmatrix} = -x.\cos x - \int -\cos x.dx = -x.\cos x + \sin x + c$

Ex: $\int (x^2+1).e^{2x}.dx = \begin{bmatrix} u = x^2+1 & du = 2x.dx \\ dV = e^{2x}.dx & V = \frac{e^{2x}}{2} \end{bmatrix} = (x^2+1).\frac{e^{2x}}{2} - \int \frac{e^{2x}}{\cancel{2}}.\cancel{2}x.dx = \frac{e^{2x}(x^2+1)}{2} - \int e^{2x}.x.dx = \begin{bmatrix} u = x & du = dx \\ dV = e^{2x}.dx & V = \frac{e^{2x}}{2} \end{bmatrix} \Rightarrow$

$$= \frac{(x^2+1).e^{2x}}{2} - \left[\frac{x.e^{2x}}{2} - \int \frac{e^{2x}}{2}.dx\right] = e^{2x}\left[\frac{x^2+1}{2} - \frac{x}{2} + \frac{1}{4}\right] + c$$

Ex: $\int x^3.e^x.dx = \begin{bmatrix} \text{Differentiate} & \text{Int} \\ & e^x \\ x^3 & e^x \\ 3x^2 & e^x \\ 6x & e^x \\ 6 & e^x \\ 0 & e^x \end{bmatrix} = x^3.e^x - 3x^2e^x + 6x.e^x - 6.e^x + c$

Ex: $\int (x^3+2x).\sin x.dx = \begin{bmatrix} \text{Diff.} & \text{Int} \\ x^3+2x & \sin x \\ 3x^2+2 & -\cos x \\ 6x & -\sin x \\ 6 & \cos x \\ 0 & \sin x \end{bmatrix} = -(x^3+2x).\cos x + (3x^2+2).\sin x + (6x).\cos x - 6.\sin x + c$

Ex: $\int (x^2+1).e^{2x}.dx = \begin{bmatrix} x^2+1 & e^{2x} \\ 2x & e^{2x}/2 \\ 2 & e^{2x}/4 \\ 0 & e^{2x}/8 \end{bmatrix} = e^{2x}\left[\frac{x^2+1}{2} - \frac{x}{2} + \frac{1}{4}\right] + c$

// One of the [illegible] polinomed

Ex: $I = \int e^x . \sin x . dx = \begin{bmatrix} u=\sin x & dv = e^x . dx \\ du=\cos x dx & v=e^x \end{bmatrix} = e^x . \sin x - \int e^x . \cos x . dx = \begin{bmatrix} u=\cos x & dv = e^x . dx \\ du=-\sin x . dx & V=e^x \end{bmatrix} = e^x . \sin x - \left[e^x . \cos x - \underbrace{\int e^x . \sin x . dx}_{I} \right] + c$

$$\int e^x . \sin x . dx = e^x . \sin x - e^x . \cos x - \int e^x . \sin x . dx \Rightarrow 2 . \int e^x . \sin x . dx = e^x (\sin x - \cos x) \Rightarrow \int e^x . \sin x . dx = \frac{e^x}{2} (\sin x - \cos x) + c$$

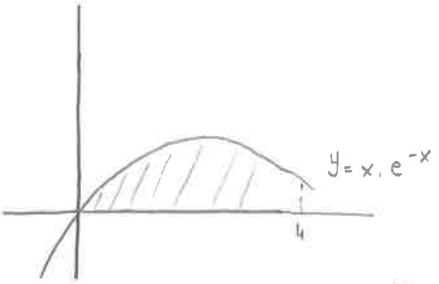
HW: $\int e^x . \cos x . dx =$ $\qquad$ $\int \arcsin x . dx =$

$\int x^2 . 2^x . dx =$

$\rightarrow \int \frac{x}{\sin^2 x} . dx = ?$

Ex: $\int x^2 . \ln x . dx = \begin{bmatrix} u=\ln x & dV = x^2 . dx \\ du = \frac{1}{x} . dx & V = \frac{x^3}{3} \end{bmatrix} = \frac{x^3}{3} . \ln x - \int \frac{x^3}{3} . \frac{1}{x} . dx = \frac{x^3 . \ln x}{3} - \int \frac{x^2}{3} . dx = \frac{x^3 . \ln x}{3} - \frac{x^3}{9} + c$

Ex:



Find the area of shaded region.

$$A = \int_0^4 x . e^{-x} . dx = \begin{bmatrix} x & & e^{-x} \\ & + & \\ 1 & & -e^{-x} \\ & - & \\ 0 & & e^{-x} \end{bmatrix} = -x . e^{-x} - e^{-x} \Big|_0^4 = -e^{-x}(x+1) \Big|_0^4 = -5 . e^{-4} - (-1) = \underline{1 - 5e^{-4}}$$

Reduction Formula

$$\int \cos^n x . dx = \frac{\cos^{n-1} x . \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x . dx$$

Ex: $\int \cos^3 x . dx = \left[\frac{\cos^2 x . \sin x}{3} + \frac{2}{3} \int \cos x . dx \right] = \underline{\frac{\cos^2 x . \sin x}{3} + \frac{2}{3} . \sin x + c}$

Ex: $\int \cos^n x . dx = \int \cos^{(n-1)} x . \cos x . dx = \begin{bmatrix} U = \cos^{n-1} x & dV = \cos x . dx \\ du = -(n-1) . \cos^{(n-2)} x . \sin x . dx & V = \sin x \end{bmatrix} = \cos^{n-1} x . \sin x + \int (n-1) . \cos^{(n-2)} x . \sin^2 x . dx$

$= \cos^{(n-1)} x . \sin x + (n-1) \int \cos^{(n-2)} x (1 - \cos^2 x) . dx$

$= \quad " \quad + (n-1) \left[\int \cos^{(n-2)} x \, dx - \int \cos^n x . dx \right]$

$\Rightarrow \cancel{n} . \int \cos^n x . dx = \frac{\cos^{(n-1)} x . \sin x}{n} + \frac{(n-1)}{n} \int \cos^{(n-2)} x . dx$

$$\int \sin^m x \cdot \cos^n x \cdot dx =$$

(I) m is even $u = \sin x$, $du = \cos x \cdot dx$ $\Big\}$ => Substitution Method

(II) n is even $u = \cos x$

(III) If both are even

$$\cos^2 x = \frac{1}{2}(\cos 2x + 1)$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

Ex : $\int \cos^4 x \cdot \underbrace{\sin^3 x}_{(\sin^2 x(\sin x))} \cdot dx = \int \cos^4 x \cdot (1-\cos^2 x) \cdot \sin x \cdot dx = \int u^4 (1-u^2) \cdot du = \int (u^4 - u^6) du = \frac{u^5}{5} - \frac{u^7}{7} + c = \frac{\cos^5 x}{5} - \frac{\cos^7 x}{7} + c$

$\left[\begin{array}{l} \cos x = u \\ -\sin x = du \end{array}\right]$

Ex : $\int \sin^6 x \cdot \underbrace{\cos^3 x}_{\underbrace{\cos^2 x \cdot \cos x}_{(1-\sin^2 x)}} \ dx = \int \sin^6 x \cdot (1-\sin^2 x) \cdot \cos x \cdot dx = \int u^6 (1-u) \, du$

$u = \sin x$

Ex : $\int \sin^3 x \cdot dx = \int \sin^2 x (\sin x) \, dx = \int (1-\cos^2 x) \cdot \sin x \cdot dx = -\int (1-u^2) \cdot du =$

$\cos x = u$

Ex : $\int \cos^5 x \cdot dx = \int [\cos^2 x]^2 \cdot \cos x \cdot dx = \int (1-\sin^2 x)^2 \cdot \cos x \cdot dx = \int (1-u)^2 \cdot du =$

$\sin x = u$

Ex : $\int \sin^2 x \cdot dx = \frac{1}{2} \cdot \int (1 - \cos 2x) \, dx = \frac{1}{2} \cdot \left[x - \frac{\sin 2x}{2} \right] + c$

Ex : $\int \cos^4 x \cdot dx = \int (\cos^2 x)^2 \, dx = \int \left(\frac{1}{2} \cdot (\cos 2x + 1) \right)^2 \cdot dx = \frac{1}{4} \cdot \int (\cos^2 2x + 2 \cdot \cos 2x + 1) dx = \frac{1}{4} \int \left[\frac{1}{2} \cdot (\cos 4x + 1) + 2 \cdot (\cos 2x) + 1 \right] dx$

$$= \frac{1}{4} \cdot \left[\frac{1}{2} \left(\frac{\sin 4x}{4} + x \right) + \sin 2x + x \right] + c$$

$\sin(a+b) = \sin a \cos b + \sin b . \cos a$

$\sin(a-b) = $

$+ \; 2 . \sin a . \cos b = \sin(a+b) + \sin(a-b)$

$\cos(a+b) = \cos a . \cos b - \sin a \sin b$

$+ \; \cos(a-b) = $

$-2 \sin a \cos b = \cos(a+b) - \cos(a-b)$

$2 \cos . \cos b = \cos(a+b) + \cos(a-b)$ HW

$\rightarrow$ Ex: $\int \sqrt{1 \mp \sin 2x} \, . dx = \int \sqrt{(\sin x - \cos x)^2} \, . dx$

Ex: $\int \sqrt{1 \pm \cos 2x} \, . dx =$

Ex: $\int \tan^3 x . dx = \int (\tan^3 x + \tan x - \tan x) dx =$

Ex: $\int \tan^4 x . dx = \int (\tan^4 x + \tan^2 x - \tan^2 x) dx =$

Ex: $\int \sec^3 x . dx = \int \sec x . \sec^2 x . dx = \begin{bmatrix} u = \sec x & dv = \sec^2 x \, dx \\ du = \sec x . \tan x & v = \tan x \end{bmatrix} = \sec x . \tan x - \int \underbrace{\sec x . \tan^2 x . dx}_{\substack{\sec x (\sec^2 x - 1) \\ \sec x^3 (-\sec x) \\ \sec^3 x + \sec x}} \Longrightarrow$ $\boxed{\sec^2 x = \tan^2 x + 1}$

$\int \sec^3 x . dx = \sec x . \tan x - \int \sec^3 x . dx + \int \sec x . dx \Rightarrow 2 \int \sec^3 x . dx = \sec x . \tan x + \ln |\sec x + \tan x| + c$

Ex: $\int \cos 5x . \sin 3x . dx = \frac{1}{2} \int \left[\sin(3x+5x) + \sin(3x - 5x) \right] . dx = \frac{1}{2} \int \left[\sin 8x - \sin 2x \right] . dx = \frac{1}{2} \left[- \frac{\cos 8x}{8} + \frac{\cos 2x}{2} \right] + c$

Ex: $\int \sin 5x . \sin 3x . dx = -\frac{1}{2} \int \left[\cos(8x) - \cos(2x) \right] dx = -\frac{1}{2} \left[\frac{\sin 8x}{8} - \frac{\sin(2x)}{2} \right] + c$

HW: $\int \cos 3x . \cos 5x . dx$

$\rightarrow \sqrt{x^2 + a^2}$, $x = a . \tan\theta$

$\sqrt{a^2 - x^2}$, $x = a . \sin\theta$

$\sqrt{x^2 - a^2}$, $x = a . \sec\theta$

Ex: $\int \frac{dx}{\sqrt{1 - x^2}} = \int \frac{\cancel{\cos\theta} . d\theta}{\cancel{|\cos\theta|}} = \theta + c = \arcsin x + c$

$\begin{pmatrix} x = \sin\theta \\ dx = \cos\theta \, d\theta \end{pmatrix}$

Ex: $\int \frac{dx}{\sqrt{9 - x^2}} = \int \frac{3 . \cos\theta . d\theta}{\sqrt{9 - 9\sin^2\theta}} = \int \frac{3 . \cancel{\cos\theta}}{3 \cancel{\cos\theta}} . d\theta = \theta + c = \arcsin\left(\frac{x}{3}\right) + c$

$\begin{pmatrix} x = 3\sin\theta \\ dx = 3 . \cos\theta . d\theta \end{pmatrix}$

$\sin\theta = \frac{x}{3}$

15.04.2015
Wednesday

Ex: $\int \sqrt{4-x^2}\,.dx = \int \sqrt{4-4\sin^2\theta}\,.2\cos\theta.d\theta = 4.\int \cos^2\theta.d\theta = 4.\int \frac{1+\cos 2\theta}{2}.d\theta = 2\left[\theta + \frac{\sin 2\theta}{2}\right]+c$

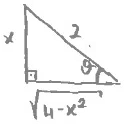
$= 2\left[\arcsin\left(\frac{x}{2}\right) + \frac{x.\sqrt{4-x^2}}{4}\right]+c$

$x = 2.\sin\theta$

$dx = 2.\cos\theta.d\theta$

$\sin\theta = \frac{x}{2}$, $\sin 2\theta = 2.\sin\theta.\cos\theta$

$= \cancel{2}.\frac{x}{\cancel{2}}.\frac{\sqrt{4-x^2}}{2}$



Ex: $\int \sqrt{9+x^2}\,.dx = \int \sqrt{9+9\tan^2\theta}\,.3\sec^2\theta.d\theta = 9.\int \sec^2\theta.d\theta = \frac{9}{2}.\left[\sec\theta.\tan\theta + \ln|\sec\theta+\tan\theta|\right]+c$

$= \frac{9}{2}\left[\frac{\sqrt{x^2+9}}{3}.\frac{x}{3} + \ln\left|\frac{\sqrt{x^2+9}}{3}+\frac{x}{3}\right|\right]+c$

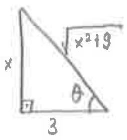
$x = 3.\tan\theta$

$dx = 3.\sec^2\theta.d\theta$

$\tan\theta = \frac{x}{3}$

$\cos\theta = \frac{3}{\sqrt{x^2+9}}$

$\sec\theta = \frac{\sqrt{x^2+9}}{3}$

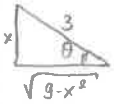


Ex: $\int \frac{\sqrt{9-x^2}}{x^2}.dx = \int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta}.3\cos\theta.d\theta = \int \frac{9.\cos^2\theta}{9.\sin^2\theta}.d\theta = \int \left[\cot^2\theta+1-1\right]d\theta = -\cot\theta - \theta + c$

$= -\frac{\sqrt{9-x^2}}{x} - \arcsin\left(\frac{x}{3}\right)+c$

$x = 3.\sin\theta$

$\sin\theta = \frac{x}{3}$



Ex(HW): ($x=\tan\theta$) $\int \frac{dx}{x^2.\sqrt{x^2+1}}$

Ex(HW) ($x=\sec\theta$) $\int \frac{dx}{\sqrt{x^2-1}}$

Ex: $\int \sqrt{5+4x-x^2}\,.dx = \int \sqrt{9-(x-2)^2}\,.dx$

$5+4x+4-4-x^2$

$9-(x^2-4x+4)$

$9-(x-2)^2$

$x-2 = 3.\sin\theta$

Ex : $\int_0^{\pi}\sqrt{1-\sin^2 x}\,.dx = \int_0^{\pi}|\cos x|\,dx = \int_0^{\pi/2}\cos x\,.dx + \int_{\pi/2}^{\pi} -\cos x\,.dx$

I.way

Ex : $\int_0^{2\pi}\sqrt{\frac{1-\cos x}{2}}\,.dx = \int_0^{2\pi}\sqrt{\frac{1-1+2.\sin^2\left(\frac{x}{2}\right)}{2}}\,.dx = \int_0^{2\pi}\left|\sin\left(\frac{x}{2}\right)\right|dx = \int_0^{2\pi}\sin\left(\frac{x}{2}\right).dx$

$\cos 2x = 1-2\sin^2 x$

$\cos a = 1-2\sin^2\left(\frac{a}{2}\right)$

II.way

$= \frac{1}{\sqrt{2}}\int_0^{2\pi}\frac{\sqrt{1-\cos^2 x}}{\sqrt{1+\cos x}}\,.dx = \frac{1}{\sqrt{2}}\int_0^{2\pi}\frac{|\sin x|.dx}{\sqrt{1+\cos x}} =$ $(1+\cos x = u)$

Ex : $\int\frac{\sin^2 x}{\sqrt{1-\cos x}}\,.dx = \int\frac{\sqrt{1+\cos x}\,.\sin^2 x}{\cancel{\sqrt{1-\cos^2 x}}}\,.dx = \int\sqrt{1+\cos x}\,.\sin x\,.dx$, $1+\cos x = u^2$

$\sqrt{1-\cos^2 x}$

$\sqrt{\sin^2 x}$

$\sin x$

Ex . $\int\frac{dx}{x^2+2x+2} = \int\frac{dx}{(x+1)^2+1} = \left[\begin{matrix}x+1=u\\ dx=du\end{matrix}\right] = \int\frac{du}{u^2+1} = \arctan u + c = \arctan(x+1)+c$

Ex : $\int\frac{dx}{x^2+6x+13} = \int\frac{dx}{(x+3)^2+4} = \frac{1}{2}.\arctan\left(\frac{x+3}{2}\right)+c \Rightarrow \frac{1}{4}.\int\frac{dx}{\left(\frac{x+3}{2}\right)^2+1} = \left[u=\frac{x+3}{2}\right]$

$\hat{9}\,4$

Ex : $\int\frac{dx}{\sqrt{2x-x^2}} = \int\frac{dx}{\sqrt{1-(x-1)^2}} = \left[x-1=u\right] = \arcsin(x-1)+c$

$\forall x$,

$\rightarrow \frac{2x+1}{x^2-4} = \frac{A}{(x-2)} + \frac{B}{(x+2)} \Rightarrow \frac{2x+1}{x^2-4} = \frac{A.(x+2)+B.(x-2)}{x^2-4} \Rightarrow 2x+1 = A(x+2)+B(x-2) \Rightarrow \frac{2x+1}{x^2-4} = \frac{1}{4}\left[\frac{5}{x-2}+\frac{3}{x+2}\right]$

I.way Polinomial Equality

$> 2x+1 = x(A+B)+2A-2B$ $\quad A+B=2 \quad 2A-2B=1$

II.way

$> x=2 \Rightarrow 5=4A \Rightarrow A=\frac{5}{4} \qquad x=-2 \Rightarrow -3=-4B \Rightarrow B=\frac{3}{4}$

Ex : $\int\frac{2x+1}{x^2-4}\,.dx = \frac{1}{4}\int\left(5.\frac{1}{x-2}+3.\frac{1}{x+2}\right).dx = \frac{1}{4}.\left[5.\ln|x-2|+3.\ln|x+2|\right]+c$

$$\rightarrow \int \frac{Q(x)}{P(x)} \cdot dx \quad , \ \deg\left[Q(x)\right] < \deg\left[P(x)\right]$$

$$\frac{\cdots}{(ax+b)^n} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} \qquad + \frac{A_n}{(ax+b)^n}$$

Ex: $$\frac{1}{x^3-8} = \frac{1}{(x-2)(x^2+2x+4)} = \frac{A}{(x-2)} + \frac{Bx+C}{x^2+2x+4}$$

Ex: $$\frac{1}{x^2.(x-1)(x^2+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{Dx+E}{x^2+4}$$

Ex: $$\int \frac{5x-3}{x^2-2x-3} \cdot dx = \int \left(3 \cdot \frac{1}{(x-3)} + 2 \cdot \frac{1}{(x+1)}\right) . dx = 3.\ln|x-3| + 2.|x+1| + c$$

$$\left[\begin{array}{l} \dfrac{5x-3}{x^2-2x-3} = \dfrac{A}{(x-3)} + \dfrac{B}{(x+1)} \\ 5x-3 = A.(x+1) + B(x-3) \\ x=-1 \Rightarrow B=2 \\ x=3 \Rightarrow A=3 \end{array}\right]$$

Ex: $$\int \frac{x^2+4x+1}{(x^2-1)(x+3)} \cdot dx = \left[\frac{x^2+4x+1}{(x^2-1)(x+3)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+3)} \Rightarrow x^2+4x+1 = A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1) \Rightarrow \begin{array}{l} x=1,\ A=\frac{3}{4} \\ x=-1,\ B=\frac{1}{2}=\frac{2}{4} \\ x=-3,\ C=\frac{-1}{4} \end{array}\right]$$

$$= \frac{1}{4} \int \left[\frac{3}{x-1} + \frac{2}{x+1} - \frac{1}{x+3}\right] . dx = \frac{1}{4} \cdot \left[3.\ln|x-1| + 2.\ln|x+1| - \ln|x+3|\right] + c$$

Ex: $$\int \frac{6x+7}{(x+2)^2} \cdot dx = \frac{A}{x+2} + \frac{B}{(x+2)^2} \qquad \begin{array}{l} x=-2 \\ x=0 \end{array} \quad \begin{array}{l} B=-5 \\ A=6 \end{array} \qquad 6\ln|x+2| + \frac{5}{x+2} + c$$

Ex: $$\int x^n . \cos x \, dx \stackrel{?}{=} x^n . \sin x - n\int x^{n-1} . \sin x . dx$$ Show that

$$\left[\begin{array}{ll} u = x^n & dV = \cos x . dx \\ du = n.(x)^{(n-1)} . dx & V = \sin x \end{array}\right] =$$

Ex: $$\int x^n . e^x . dx = x^n . e^x - n\int x^{n-1} . e^x . dx \Rightarrow \left[\begin{array}{ll} u = x^n & dV = e^x . dx \\ du = n . x^{(n-1)} . dx & V = e^x \end{array}\right]$$

20.04.2015
Monday

Ex: $\displaystyle\int_0^{\pi/2} x^3 \cdot \cos 2x \cdot dx = \left[\begin{matrix} u = x^3 & dV = \cos 2x \cdot dx \\ du = 3x^2 \cdot dx & V = \frac{\sin 2x}{2} \end{matrix}\right] = x^3 \cdot \frac{\sin 2x}{2} - \frac{3}{2}\int x^2 \cdot \sin 2x \cdot dx =$

$$\left[\begin{matrix} x^3 & \cos 2x \\ 3x^2 & \sin 2x/2 \\ 6x & -\cos 2x/4 \\ 6 & -\sin 2x/8 \\ 0 & \cos 2x/16 \end{matrix}\right] = x^3 \cdot \frac{\sin 2x}{2} + 3x^2 \cdot \frac{\cos 2x}{4} - 6x \cdot \frac{\sin 2x}{8} - \frac{6}{16} \cdot \cos 2x \Big|_0^{\pi/2} = \left[0 - \frac{3\pi^2}{16} - 0 + \frac{3}{8}\right] - \frac{3}{8} = \frac{-3\pi^2}{16}$$

Ex: $\displaystyle\int_{-\pi}^{\pi} (1-\cos^2 t)^{3/2} dt = 2\int_0^{\pi} \sqrt{(1-\cos^2 x)^3} \cdot dx = 2 \cdot \int_0^{\pi} |\sin x|^3 \cdot dx = 2 \cdot \int_0^{\pi} \sin^3 x \cdot dx = 2 \int_2^{\pi} (1-\cos^2 x) \cdot \sin x \cdot dx = \left[\begin{matrix} \cos x = u \\ -\sin x \cdot dx = du \end{matrix}\right] = 2 \cdot \int_1^{-1} (1-u^2)(-du) \Rightarrow$

$[f(-x) = f(x) \Rightarrow f(x)$ is even$]$
func $\displaystyle\int_{-a}^{a} f(x)\,dx = 2 \cdot \int_0^a f(x) \cdot dx$

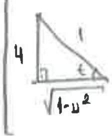
$f(x) = \left(\sqrt{1-\cos^2 x}\right)^3$
$f(-x) = \left(\sqrt{1-\cos^2 x}\right)^3$ $\Rightarrow f(x)$ is even function

$$= 2 \cdot \int_{-1}^{1} (1-u^2)\,du = 2 \cdot \left[u - \frac{u^3}{3}\right]_{-1}^{1} = 2 \cdot \left[\left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right)\right] = \frac{8}{3}$$

Ex: $\displaystyle\int \sqrt{x} \cdot \sqrt{1-x} \cdot dx = \left[\begin{matrix} x = u^2 \\ dx = 2u\,du \end{matrix}\right] = \int 2u^2 \cdot \sqrt{1-u^2} \cdot du = \left[\begin{matrix} u = \sin t \\ du = \cos t\,dt \end{matrix}\right] = \int 2 \cdot \sin^2 t \cdot \sqrt{1-\sin^2 t} \cdot \cos t \cdot dt = 2 \cdot \int \sin^2 t \cdot \cos^2 t \cdot dx = \frac{1}{2}\int (\sin 2t)^2 \cdot dt$

$$= \frac{1}{2} \cdot \int \frac{(1-\cos 4t)}{2} \cdot dt = \frac{1}{4} \cdot \left(t - \frac{\sin 4t}{4}\right) + c =$$

$\sin 4t = 2 \cdot \sin 2t \cdot \cos 2t$
$= 4 \cdot \sin t \cdot \cos t \cdot (1 - 2\sin^2 t)$



$\sin t = u$

$$= \frac{1}{4}\left(\arcsin u - \frac{4u\sqrt{1-u^2}\,(1-2u^2)}{4}\right)$$

$$= \frac{1}{4} \cdot \left[\arcsin\sqrt{x} - \sqrt{x} \cdot \sqrt{1-x}\,(1-2x)\right] + c$$

Ex: $\displaystyle\int \frac{5x-3}{(x+1)(x-3)} \cdot dx = \int \left(\frac{A}{x+1} + \frac{B}{(x-3)}\right) dx = \left[\begin{matrix} 5x-3 = A(x-3) + B(x+1) \\ x=3 \Rightarrow \\ 12 = 4B,\ B=3 \\ x=-1 \Rightarrow -8 = -4A \\ A=2 \end{matrix}\right] = \int \left(\frac{2}{x+1} + \frac{3}{x-3}\right) dx = 2 \cdot \ln|x+1| + 3 \cdot \ln|x-3| + c$

Ex: $\displaystyle\int \frac{6x+7}{(x+2)^2} \cdot dx = \int \left[\frac{A}{x+2} + \frac{B}{(x+2)^2}\right] dx = \left[\begin{matrix} 6x+7 = A \cdot (x+2)^2 + B \\ x=-2 \Rightarrow B=-5 \\ x=0 \Rightarrow A=6 \end{matrix}\right] = \int \left[\frac{6}{x+2} - \frac{5}{(x+2)^2}\right] dx = 6 \cdot \ln|x+2| + \frac{5}{x+2} + c$ $\displaystyle = \left[\int \frac{1}{(x+2)^2} \cdot dx = \int (x+2)^{-2}\,dx \quad (x+2) = u\right]$

Ex: $\displaystyle\int \frac{dx}{x(x^2+1)^2} = \int \left[\frac{\overset{1}{A}}{x} + \frac{\overset{1}{B}x + \overset{0}{C}}{x^2+1} + \frac{\overset{-1}{D}x + \overset{0}{E}}{(x^2+1)^2}\right] dx = \left[\begin{matrix} 1 = (x^2+1)^2 A + (x^2+1) \cdot x(Bx+C) + x \cdot (Dx+E) \\ x=0 \Rightarrow 1=A \\ x=1 \Rightarrow,\ x=-1,\ x=2,\ x-2 \end{matrix}\right] = \int \left[\frac{1}{x} + \frac{-x}{x^2+1} - \frac{x}{(x^2+1)^2}\right] dx = \ln|x| + \frac{1}{2}\ln|x^2+1| + \frac{1}{2(x^2+1)}$

Ex: $\displaystyle\int \frac{x+4}{x^3+3x^2-10x} \cdot dx = \int \left(\frac{A}{x} + \frac{B}{x+5} + \frac{C}{x-2}\right) dx$

$x(x^2+3x-10)$

Ex: $\displaystyle\int \frac{x^2+1}{(x-1)(x-2)(x-3)} = \int \left(\frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}\right) dx$

$$\text{Ex}: \int \frac{9x^3-3x+1}{x^3-x^2}\cdot dx = \left[\begin{array}{l|l} 9x^3-3x+1 & x^3-x^2 \\ \underline{9x^3-9x^2} & 9 \\ 9x^2-3x+1 & \end{array}\right] = \int\left(9+\frac{9x^2-3x+1}{x^3-x^2}\right)\cdot dx = 9\int dx + \int \frac{9x^2-3x+1}{x^2(x-1)}\cdot dx = 9x + \int\left(\frac{A}{x}+\frac{B}{x^2}+\frac{C}{(x-1)}\right)dx =$$

$$\left[\begin{array}{l} 9x^2-3x+1 = A.(x)(x-1)+B(x-1)+C(x^2) \\ x=0 \Rightarrow B=-1 \quad , \; x=1 \Rightarrow C=7 \quad , \; x=-1 \Rightarrow A=2 \end{array}\right] = 9x + \int\left(\frac{2}{x}-\frac{1}{x^2}+\frac{7}{x-1}\right)dx = 9x+2\ln|x|+\frac{1}{x}+7\ln|x-1|+c = 9x+\frac{1}{x}+\ln|x^2.(x-1)^7|+c$$

$$\text{Ex}: \frac{x^3}{x^2+2x+1}\cdot dx = \left[\begin{array}{l|l} x^3 & x^2+2x+1 \\ \underline{x^3+2x^2+x} & x-2 \\ -2x^2-x & \\ \underline{-2x^2-4x-2} & \\ 3x+2 & \end{array}\right] = \int\left[x-2+\frac{3x+2}{(x+1)^2}\right]dx = \int (x-2).dx + \int\left(\frac{3x+3}{(x+1)^2}-\frac{1}{(x+1)^2}\right)dx = \frac{x^2}{2}-2x+3\ln|x+1|+\frac{1}{(x+1)}+c$$

(1) $$\int \frac{e^t.dt}{e^{2t}+3e^t+2} = \left[\begin{array}{l} e^t=x \\ e^t.dt=dx \end{array}\right] = \int \frac{dx}{x^2+3x+2} = \int\left(\frac{A}{x+2}+\frac{B}{x+1}\right)dx$$

(2) $$\int \frac{\cos y.dy}{\sin^2 y+\sin y-6} = \left[\begin{array}{l} x=\sin y \\ dx=\cos y.dy \end{array}\right] =$$

(3) $$\int \frac{dx}{\sqrt{x^3}-\sqrt{x}} = 2.\int \frac{1}{2.\sqrt{x}.(x-1)}dx = 2.\int \frac{1}{t^2-1}dt = 2.\int\left(\frac{A}{t-1}+\frac{B}{t+1}\right)dt =$$

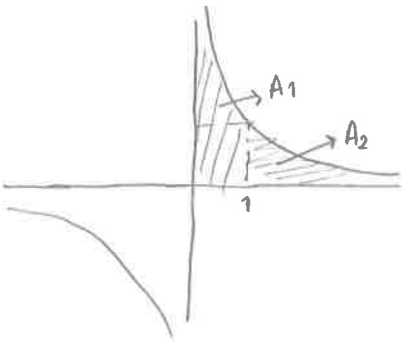
(4) $$\int \frac{\sqrt{x+1}}{x} = \int \frac{u}{u^2-1}\cdot 2u.du = \int \frac{2u^2-2+2}{u^2-1}\cdot du = \int\left(2+\frac{2}{u^2-1}\right).du = \int\left(2+\frac{A}{u-1}+\frac{B}{u+1}\right)du$$

$$\left[\begin{array}{l} \sqrt{x+1}=u \\ x+1=u^2 \\ x=u^2-1 \end{array}\right]$$

(5) $$\int \frac{1}{x\sqrt{x+9}}\cdot dx = \left[\begin{array}{l} \sqrt{x+9}=u \\ x+9=u^2 \\ x=u^2-9 \\ dx=2u.du \end{array}\right] = \int \frac{2u.du}{(u^2-9)u} = \int\left(\frac{A}{u-3}+\frac{B}{u+3}+\frac{C}{u}\right)du$$

Repeat it!

IMPROPER INTEGRAL

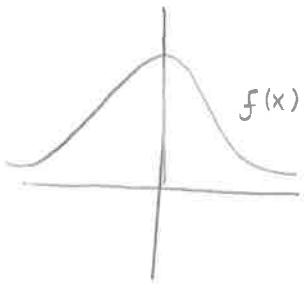


$$A_1 = \int_0^1 \frac{1}{x} \cdot dx = \lim_{R \to 0^+} \left[\int_R^1 \frac{1}{x} \cdot dx \right]$$

$$= \lim_{R \to 0^+} \ln x \Big|_R^1 = \lim_{R \to 0^+} (\ln 1 - \ln R) = \infty$$

$$A_2 = \int_1^\infty \frac{1}{x} \cdot dx = \lim_{R \to \infty} \int_1^R \frac{1}{x} dx = \lim_{R \to \infty} [\ln R - \ln 1] = \infty$$

Ex :



$f(x) = \frac{1}{x^2+1}$

$$A = \int_{-\infty}^{\infty} \frac{1}{x^2+1} dx = \lim_{R \to -\infty} \int_R^0 \frac{1}{x^2+1} dx + \lim_{R \to \infty} \int_0^R \frac{1}{x^2+1} \cdot dx$$

$$= 2 . \lim_{R \to \infty} \int_0^R \frac{1}{x^2+1} dx = 2 . \lim_{R \to \infty} [\arctan R - \arctan 0] = 2\left[\frac{\pi}{2} - 0\right] = \pi$$

TYPE - I

i-) If $f(x)$ is continuous on $[a, \infty)$ then , $\int_a^\infty f(x) dx = \lim_{R \to \infty} \int_a^R f(x) . dx$

ii-) $\int_{-\infty}^a f(x) dx = \lim_{R \to -\infty} \int_R^a f(x) . dx$, $f(x)$ is continuous on $(-\infty, a]$

iii-) $f(x)$ is continuous on $(-\infty, \infty)$, $\int_{-\infty}^{\infty} f(x) . dx = \lim_{R \to -\infty} \int_R^c f(x) dx + \int_c^R f(x) dx$

NOTE: In each case if the limit values converges to a real number (value) then we can say that the improper integral convergent. Otherwise, divergent.

Ex: $\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{R\to\infty} \int_1^R x^{-p}\,dx = \lim_{R\to\infty} \frac{x^{1-p}}{1-p}\Bigg|_1^R = \lim_{R\to\infty} \frac{1}{1-p}\left[R^{1-p}-1\right] = \begin{cases} \frac{1}{p-1}, & p>1 \\ \infty, & p<1 \end{cases}$ $\begin{matrix} \to \text{conv.}, p>1 \\ \to \text{div.}, p<1 \end{matrix}$

$$= \int_1^{\infty} \frac{1}{x} = \lim_{R\to\infty} \ln x \Big|_1^R = \infty$$

Ex: $\int_2^{\infty} \frac{1}{\sqrt{x}}\,dx$ show that is divergent

Ex: $\int_1^{\infty} \frac{1}{x^3}\,dx$ show that is convergent

TYPE - II

i-) if $f(x)$ is cont. on $[a,b)$ and discont. $x=b$ then, $\int_a f(x)\,dx = \lim_{R\to b^-} \int_a^R f(x)\,dx$

ii-) If $f(x)$ is cont. on $(a,b]$ but not cont. at $x=a$ then, $\int_a^b f(x)\,dx = \lim_{R\to a^+} \int_R^b f(x)\,dx$

iii-) $f(x)$ is cont. on $[a,c)\cup(c,b]$ but not cont. at $x=c$ then, $\int_a^b f(x)\,dx = \lim_{R\to c^-} \int_a^R f(x)\,dx + \lim_{R\to c^+} \int_R^b f(x)\,dx$

Ex: $\int_1^{e+1} \frac{1}{x-1}\,dx$, $x=1$ disc. $\Rightarrow \lim_{R\to 1^+} \int_R^{e+1} \frac{1}{x-1}\,dx = \lim_{R\to 1^+} \Big[\ln|x-1|\Big]_R^{e+1} = \lim_{R\to 1^+} \Big[\ln e - \ln|R-1|\Big] = 1+\infty = \infty$

Divergent

Ex: $\int_{-2}^{1} \frac{1}{x^3}\,dx$, $x=0$ disc. type III $= \int_{-2}^{0} + \int_{0}^{1} = \lim_{R\to 0^-} \int_{-2}^{R} (x^{-3})\,dx + \lim_{R\to 0^+} \int_R^1 (x^{-3})\,dx$

$$= \lim_{R\to 0^-} \left[\frac{-1}{2x^2}\right]_{-2}^{R} + \lim_{R\to 0^+} \left[\frac{-1}{2x^2}\right]_R^1$$

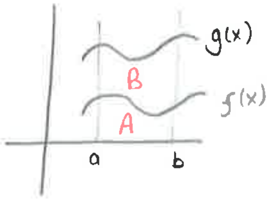
$$= \left(-\infty + \frac{1}{8}\right) + \left(\frac{-1}{2} + \infty\right)$$

Divergent

One of them is divergent

Ex : $\int_1^\infty e^{-x^2}\,dx =$ div. or conv. ?

if $0 \leq f(x) \leq g(x)$ on $[a,b]$ then $\underbrace{\int_a^b f(x)\,dx}_{A} \leq \underbrace{\int_b^a g(x)\,dx}_{A+B}$



$$0 \leq \int_1^\infty e^{-x^2}\,dx \leq \int_1^\infty e^{-x}\,dx = \frac{1}{e}$$

Theorem Comparison Test

Let $f(x)$ and $g(x)$ be a cont. func. on $[a,\infty)$ and $0 \leq f(x) \leq g(x)$, $\forall x \geq a$ then,

i-) If $\int_a^b g(x)\,dx$ is convergent then $\int_a^b f(x)\,dx$ is also conv.

ii-) If $\int_a^b f(x)\,dx$ is divergent then $\int_a^b g(x)\,dx$ is also div.

Ex : $\int_1^\infty \frac{1}{\sqrt{x^2-0,5}}\,dx$ test for convergency , $[1,\infty)$: $\sqrt{x^2-0,5} \leq \sqrt{x^2} = x$

$\frac{1}{\sqrt{x^2-0,5}} \geq \frac{1}{x}$ $\Rightarrow$ $\int_1^\infty \frac{1}{x}\,dx$ is div. (Small One)

div.

Ex : $\int_1^\infty \frac{\cos^2 x}{x^2}\,dx =$

$-1 \leq \cos x \leq 1$

$0 \leq \cos^2 x \leq 1$

$\frac{\cos^2 x}{x^2} \leq \frac{1}{x^2}$ $\int_1^\infty \frac{1}{x^2}\,dx$ is conv. (Big One)

conv

27.04.2015
Monday

Theorem: Limit Comperision Test

If the positive functions $f(x)$ and $g(x)$ are cont. on $[a,\infty)$, and if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$, $0<L<\infty$

then $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ both converge or both diverge.

Ex: $\int_1^\infty \frac{1}{1+x^2}dx$ test for conv. $\frac{1}{1+x^2} \approx \frac{1}{x^2}$, $\int_1^\infty \frac{1}{1+x^2}dx$ is $\boxed{\text{conv.}}$

conv.

$$\lim_{x\to\infty} \frac{\frac{1}{x^2}}{\frac{1}{x^2+1}} = \lim_{x\to\infty} \frac{x^2+1}{x^2} = \boxed{1}$$

Ex: $\int_2^\infty \frac{dx}{\sqrt{x-1}}$ test for convergency , $\boxed{\text{I. WAY}}$ $\frac{1}{\sqrt{x-1}} \approx \frac{1}{\sqrt{x}}$

$\int_2^\infty \frac{1}{\sqrt{x}}dx$ is $\boxed{\text{divergent}}$, $\lim_{x\to\infty} \frac{\frac{1}{\sqrt{x}}}{\frac{1}{\sqrt{x-1}}} = \boxed{1}$

div

$\boxed{\text{II. WAY}}$

$\sqrt{x-1} < \sqrt{x}$

$\frac{1}{\sqrt{x-1}} > \frac{1}{\sqrt{x}}$ by comp. test $\int_2^\infty \frac{dx}{\sqrt{x}}$ is div. $\Rightarrow \int_2^\infty \frac{dx}{\sqrt{x-1}}$ is also div.

$\boxed{\text{III. WAY}}$ (Evaluate Integral)

$$2.\int_2^\infty \frac{dx}{2.\sqrt{x-1}} = 2.\lim_{R\to\infty} \sqrt{x-1}\Big|_2^R = 2\left[\infty - 1\right] = \infty \text{ (Not Real) } \underline{\text{Div.}}$$

Ex : $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ evoluate the int.

$$= \lim_{R\to 0^+} 2\int \frac{1}{2\sqrt{x}}\,dx = \lim_{R\to 0^+} \sqrt{x}\,\Big|_0^1 = \sqrt{1}-0 = 1 \text{ Conv.}$$

(Real Value)

Ex : $\int_2^{\infty} \frac{2}{x^2-x}\,dx$ evaluate the int.

$$= \lim_{R\to\infty}\int_2^R \frac{2}{x(x+1)} = \left[\frac{2}{x(x+1)} = \frac{A}{x}+\frac{B}{(x+1)} \Rightarrow 2 = A(x+1)+B.x \Rightarrow \begin{matrix} B=2 \\ A=-2 \end{matrix}\right] = \lim_{R\to\infty}\int_2^R \left(\frac{-2}{x}+\frac{2}{x-1}\right)dx = 2.\lim_{R\to\infty}\left[\ln\left|\frac{x-1}{x}\right|\right]_2^R$$

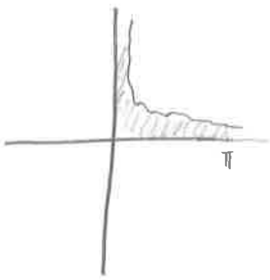
$$= 2\left[0-\ln\frac{1}{2}\right] = 2\ln 2 = \boxed{\ln 4}$$

conv.

Ex : $\int_0^{\ln 2} x^{-2}.\,e^{-1/x}\,dx$ test for conv. $\left[\lim_{x\to 0^+} \frac{-1}{x} = -\infty,\ u = -\frac{1}{\ln 2} = \frac{-\ln e}{\ln 2} = -\log_2 e\right]$

$$\left[\begin{matrix} u = \frac{-1}{x} \\ du = \frac{-1}{x^2}\,dx \end{matrix}\right] \qquad = \int_{-\infty}^{-\log_2 e} e^u.du = \lim_{R\to-\infty} e^u\Big|_R^{-\log_2 e} = e^{-\log_2 e} - 0 \quad (\text{Conv.})$$

Ex : $\int_0^{\pi} \frac{dx}{\sqrt{x}+\sin x}$ test for conv.



$$\frac{1}{\sqrt{x}+\sin x} > 0 \text{ at } [0,\pi]$$

$$\sqrt{x}+\sin x \geq \sqrt{x} \Rightarrow \frac{1}{\sqrt{x}+\sin x} \leq \frac{1}{\sqrt{x}} \rightarrow 2\sqrt{\pi}-0 = 2\sqrt{\pi} \quad (\text{Conv.})$$

Ex : $\int_0^1 \frac{1}{x^p}\,dx = \lim_{R\to 0^+} \frac{x^{-p+1}}{1-p}\Big|_R^1 = \begin{cases} \frac{1}{1-p}-\infty, & p>1 \\ \frac{1}{1-p}-0, & p<1 \\ \infty & p=1 \end{cases}$

$$\int_0^1 \frac{1}{x^p} = \begin{cases} \text{div} & P \geq 1 \\ \text{conv.} & P<1 \end{cases}$$

Ex: $\displaystyle\int_1^\infty \frac{1}{\sqrt{e^x - x}}\,dx$ test for conv.

I. WAY COMP.

$0 \leqslant \sqrt{e^x - x} \leqslant \sqrt{e^x} = e^{\frac{x}{2}}$, $\dfrac{1}{\sqrt{e^x - x}} \geqslant e^{\frac{-x}{2}}$, $\displaystyle\int_1^\infty e^{\frac{-x}{2}}\,dx = \left.\frac{e^{-\frac{x}{2}}}{\frac{-1}{2}}\right|_1^\infty = -2e^{\frac{-x}{2}}\Big|_1^\infty = -2\left(0 - \frac{1}{e}\right) = 2e^{-1}$

FAILS

CONV.

II. WAY Limit Comp Test

$f(x) = \dfrac{1}{\sqrt{e^x - x}}$

$\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \sqrt{\frac{e^x}{e^x - x}} = 1$

Both conv. or div.

Both Conv.

$g(x) = \dfrac{1}{\sqrt{e^x}} = e^{\frac{-x}{2}}$

Ex: (HW) $\displaystyle\int_1^\infty \frac{e^x}{x}\,dx$

$f(x) = \dfrac{e^x}{x}$

$\displaystyle\lim_{x\to\infty} \frac{\frac{e^x}{x}}{e^x} = 0$, For $g(x) = e^x$ limit comp. test does not work.

$g(x) = e^x$

$\dfrac{e^x}{x} \leqslant e^x$, $[1,\infty)$

29.04.2015
Wednesday

① $$\int \frac{\sin x\,.dx}{\cos^2 x - \cos x - 2} = \begin{bmatrix} \cos x = u \\ -\sin x.dx = du \end{bmatrix} = \int \frac{-du}{u^2 - u - 2} = \int \frac{A}{(u-2)} + \frac{B}{(u+1)}$$

② $$\int \frac{e^x.dx}{e^{2x} - e^x - 2} = \begin{bmatrix} e^x = u \\ e^x.dx = du \end{bmatrix} = \int \frac{du}{u^2 - u - 2} = \int \frac{A}{(u-2)} + \frac{B}{(u+1)}$$

③ $$\int \ln(x+1)dx = \begin{bmatrix} u = \ln(x+1) & dv = dx \\ du = \frac{1}{x+1}.dx & v = x \end{bmatrix} = x.\ln(x+1) - \int \frac{x+1-1}{x+1}.dx = x - \ln|x+1| + c$$

④ $$\int \frac{dx}{e^x - 1} = \begin{bmatrix} e^x = u \\ e^x.dx = du \Rightarrow dx = \frac{du}{u} \end{bmatrix} = \int \frac{du}{u.(u-1)} = \int \left(\frac{A}{u} + \frac{B}{u-1}\right)du$$

⑤ $$\int \frac{y.dy}{\sqrt{16 - y^2}} = \int \frac{-x.dx}{x} = -x + c = -\sqrt{16 - y^2} + c$$

$$\begin{bmatrix} 16 - y^2 = x^2 \\ -2y\,dy = 2x\,dx \end{bmatrix}$$

⑥ $$\int_1^\infty \frac{\ln x}{x} dx = \int u.du = \frac{u^2}{2}\Bigg| = \frac{(\ln x)^2}{2}\Bigg|_1^\infty = \infty - 0 = \infty \quad \text{div}$$

***
⑦ $$\int \frac{x.dx}{\sqrt{2-x}} = \begin{bmatrix} 2 - x = t^2 \\ x = 2 - t^2 \\ dx = -2t.dt \end{bmatrix} = \int \frac{(2-t^2)(-2t)\,dt}{t} = \int (2t^2 - 4)dt =$$

⑧ $$\int x^3.e^{x^2}.dx = \int x^2.e^{x^2}.x.dx = \frac{1}{2}.\int u.e^u.du = \left[\begin{array}{ll} x^2 = u & \\ 2x.dx = du & \\ x.dx = du/2 & \\ \hline u & e^u \\ 1 & e^u \\ 0 & e^u \end{array}\right] = \frac{1}{2}.\left[u.e^u - e^u\right] + c = \frac{e^{x^2}(x^2 - 1)}{2} + c$$

⑨ $$\int \frac{x^3+2}{4-x^2}.dx = \left[\begin{array}{l|l} x^3+2 & 4-x^2 \\ -x^3+4x & -x \\ \hline 4x+2 & \end{array}\right] = \int \left(-x + \frac{4x+2}{4-x^2}\right)dx = \frac{-x^2}{2} + \int \left(\frac{A}{2-x} + \frac{B}{2+x}\right)dx$$

⑩ $$\int \frac{dx}{\sqrt{-2x - x^2}} = \int \frac{dx}{\sqrt{1 - 1 - 2x - x^2}} = \int \frac{dx}{\sqrt{1 - (x+1)^2}} = \int \frac{dt}{\sqrt{1 - t^2}} = \arcsin(x+1) + c$$

⑪ $$\int \frac{1 - \cos x}{\underset{(1-\cos x)}{1 + \cos x}} dx = \int \frac{1 - 2\cos x + \cos^2 x}{\sin^2 x} dx = \int \csc^2 x.dx - 2\int \frac{\cos x.dx}{\sin^2 x} + \int \cot^2 x.dx = -\cot x - 2\int \frac{du}{u^2} + \int (\cot^2 x + 1 - 1)dx = -\cot x + \frac{2}{u} - \cot x -$$
+c

(12) $\int_0^\infty x^2 \cdot e^{-x} \cdot dx = e^{-x}\left[-x^2-2x-2\right]_0^\infty = \lim_{R\to\infty} -\left[\frac{R^2+2R-2}{e^R} - \frac{2}{e^0}\right] = -(0-2) = 2$

$$\begin{bmatrix} x^2 & e^{-x} \\ 2x & -e^{-x} \\ 2 & e^{-x} \\ 0 & -e^{-x} \end{bmatrix}$$

$\int_0^1 \frac{1}{x^2} \cdot dx$ is div

$\int_1^\infty \frac{1}{x^2} \cdot dx$ is conv.

(13) $\int_6^\infty \frac{dx}{\sqrt{x^2+1}}$ test for conv.

I. WAY $\frac{1}{\sqrt{x^2+1}} \approx$, $\lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{\sqrt{x^2+1}}} = 1$

$\int_6^\infty \frac{1}{x} dx$ is div.

$\int_6^\infty \frac{dx}{\sqrt{x^2+1}}$ is also div.

II. WAY COMPARISON TEST

~~NO~~ $\sqrt{x^2+1} \geq \sqrt{x^2} = x$ $\qquad ? \leq \frac{1}{\sqrt{x^2+1}} \leq \frac{1}{x}$

$\int_6^\infty \frac{1}{x} dx$ div. $\not\Rightarrow \int_6^\infty \frac{1}{\sqrt{x^2+1}} dx$ is div.

(14) $\int_0^\infty e^{-x} \cdot \cos x \, dx$ test for conv.

I. WAY Evaluate by int. by parts

II. WAY Comp. Test

$\frac{\cos x}{e^x} \leq \frac{1}{e^x} \qquad [0,\infty)$

(15) $\int_1^\infty \frac{e^{-t}}{\sqrt{t}} dt$ test for conv.

$\int_0^\infty e^{-x} dx = -e^{-x}\Big|_0^\infty = -(0-1) = 1$ conv.

$\frac{1}{e^t\sqrt{t}} \leq \frac{1}{e^t} \qquad [1,\infty)$

$\int_1^\infty \frac{1}{e^t} dt$ is conv. $\xRightarrow{\text{by C.T.}} \int_1^\infty \frac{1}{e^t\sqrt{t}} dt$ is also conv.

(17) $\int e^{\ln\sqrt{x}} . dx = \int \sqrt{x} . dx$

(18) $\int \sqrt{x} . \sqrt{1+\sqrt{x}}\, dx = \begin{bmatrix} 1+\sqrt{x} = t^2 \\ \sqrt{x} = t^2 - 1 \\ \frac{1}{2\sqrt{x}} \cdot dx = 2t . dt \\ dt = 4t(t^2-1)dt \end{bmatrix} = \int (t^2-1) . t . 4t(t^2-1) dt = 4\int t^2 . (t^2-1)^2 dt =$

(19) $\int \frac{x^3}{1+x^2} dx = \int \left(\frac{x . (x^2+1)}{x^2+1} - \frac{x}{x^2+1} \right) dx = \int \left(x - \frac{1}{2} \cdot \frac{2x}{x^2+1} \right) dx = \frac{x^2}{2} - \frac{1}{2} \ln |x^2+1| + c$

(20) $\int \frac{dx}{1+\sqrt{x}} = \begin{bmatrix} x = t^2 \\ dx = 2t . dt \end{bmatrix} = \int \frac{2t . dt}{1+t} = \int \left(\frac{2t+2}{t+1} - \frac{2}{t+1} \right) dt = \int \left(2 - \frac{2}{t+1} \right) dt = 2t - 2\ln(t+1) + c = 2\sqrt{x} - 2\ln(\sqrt{x}+1) + c$

(21) $\int \frac{dx}{\sqrt{e^{2x}-1}} = \begin{bmatrix} e^x = t \\ e^x . dx = dt \\ dx = dt/t \end{bmatrix} = \int \frac{dt}{t\sqrt{t^2-1}} = \begin{bmatrix} t = sec \\ dt = secv . tanv . dv \end{bmatrix} = \int \frac{secv . tanv . dv}{secv . \sqrt{tan^2 v}} = \begin{bmatrix} \frac{1}{cosv} = t \\ cosv = 1/t \\ V = arccos\, 1/t \end{bmatrix} = \int dV = V + C$

$= arccos\left(\frac{1}{t}\right) + c$

$= arccos(e^{-x}) + c$

04.05.2015
Monday

SEQUENCES

$a_n : N^+ \to R$

Domain : N^+ natural numbers

$\{a_n\} = \{a_1, a_2, a_3, \dots, a_n\} = \{a_n\}_1^\infty$

Ex: $a_n = \sqrt{n}$

$\{a_n\} = \{\sqrt{n}\} = \{0, 1, \sqrt{2}, \sqrt{3}, 2 \dots\}$

Ex: $b_n = (-1)^n$

$\{b_n\} = \{(-1)^n\} = \{-1, 1, -1, 1, \dots\}$

Ex: $c_n = \cos(n\pi)$

$\{c_n\} = \{\cos(n\pi)\} = \{-1, 1, -1, 1, \dots\}$

Two sequences are equal.

Ex: $d_n = \dfrac{n-1}{n} \Rightarrow \{d_n\} = \left\{\dfrac{n-1}{n}\right\} = \left\{0, \dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4} \dots\right\}$

Limit of a sequence,

Ex: $a_n = \sqrt{n}$ (div)

$\lim a_n = \lim\limits_{n\to\infty} \sqrt{n} = \infty$

Ex: $\lim (-1)^n = \begin{cases} 1 & , n \text{ is even} \\ -1 & , n \text{ is odd} \end{cases} =$ the limit does not exist (div)

Ex: $\lim \left(\dfrac{n-1}{n}\right) = 1$ (conv.)

Ex: $\lim\limits_{n\to\infty} \dfrac{1-2^n}{1+2^n} = -1$ (conv.)

Ex: $a_n = \dfrac{1}{n} \Rightarrow \lim a_n = 0$ (conv.)

Ex: $\lim\limits_{n\to\infty} \dfrac{\ln n}{n} = \dfrac{\infty}{\infty} = \dfrac{1}{n.1} = \infty$

Ex: $\lim\limits_{n\to\infty} \dfrac{\cos n}{n} = 0$

Ex: $\lim\limits_{n\to\infty} \sqrt{\dfrac{n}{2^n}} = \sqrt{\lim\limits_{n\to\infty} \dfrac{n}{2^n}} = \sqrt{\lim \dfrac{1}{2^n . \ln 2}} = 0$

Ex: $\lim\limits_{n\to\infty} \left(\dfrac{n-1}{1+n}\right)^n = L \Rightarrow \lim\limits_{n\to\infty} \left[n . \ln\left(\dfrac{n-1}{1+n}\right)\right] = \ln L \Rightarrow \infty . 0 \Rightarrow \lim\limits_{n\to\infty} \dfrac{\ln\left(\dfrac{n-1}{n+1}\right)}{\dfrac{1}{n}} = \dfrac{0}{0} \Rightarrow \lim\limits_{n\to\infty} \dfrac{\dfrac{\dfrac{(\cancel{n}+1)-(\cancel{n}-1)}{(n-1)^2}}{\dfrac{n-1}{n+1}}}{-\dfrac{1}{n^2}} \Rightarrow$

$\lim \dfrac{2.(n+1).n^2}{(n+1)^2(n-1)} = \lim \dfrac{-2n^2}{n^2-1} = -2 = \ln L \quad \boxed{L = e^{-2}}$

Theorem :

① $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

② $\lim_{n\to\infty} x^{\frac{1}{n}} = 1 \ , \ x>0$

③ $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$

④ $\lim_{n\to\infty} \sqrt[n]{n} = 1$

⑤ $\lim_{n\to\infty} x^n = 0 \ , \ |x|<1$

⑥ $\lim_{n\to\infty} \frac{x^n}{n!} = 0$

Ex : $\lim \frac{\ln(n^2)}{n} = \lim 2\left(\frac{\ln n}{n}\right)^{\to 0} = 0$

Ex : $\lim \sqrt[n]{n^2} = \lim \Big(\underbrace{\sqrt[n]{n}}_{=1}\Big)^2 = 1^2 = 1$

Ex : $\lim \sqrt[n]{5n} = \lim \underbrace{5^{\frac{1}{n}}}_{1} \cdot \underbrace{\sqrt[n]{n}}_{1} = 1.1 = 1$

Ex : $\lim \left(\frac{n-2}{n}\right)^n = \lim\left(1+\frac{(-2)}{n}\right)^n = e^{-2}$

Exercises

① $a_n = \frac{1-5n^4}{n^4+8n^3} = -5$ (Conv.)

② $a_n = (-1)^n \Big(\underbrace{1-\frac{1}{n}}_{1}\Big)$ (Div.)

③ $\sqrt{\frac{2n}{n+1}} = \sqrt{2}$

④ $\lim n - \lim(n+1) = \lim(n-(n+1)) = \lim(-1) = -1$

⑤ $\lim \frac{\ln n}{n^{1/n}} = \infty$ (div)

⑥ $\lim \frac{3^n \cdot 6^n}{2^{-n} \cdot n!} = \frac{(6^n)^2}{n!} = 0$

⑦ $a_n = n - \sqrt{n^2-1} = \left(\frac{n+\sqrt{n^2-1}}{n+\sqrt{n^2-1}}\right) = \lim a_n = \ln \frac{n^2-(n^2-1)}{n+\sqrt{n^2-1}} = \ln \frac{1}{n+\sqrt{n^2+1}} = 0$
$(\infty-\infty)$

⑧ $a_n = \frac{1}{n}\int_1^n \frac{1}{x}\,dx = \frac{1}{n}\cdot \ln x\Big|_1^n = \frac{\ln n - 0}{n} = 0$

(9) $\sqrt{n}\cdot\sin\left(\frac{1}{\sqrt{n}}\right) = \lim \dfrac{\sin\left(\frac{1}{\sqrt{n}}\right)}{\frac{1}{\sqrt{n}}} \to \dfrac{0}{0} = \dfrac{\cancel{\frac{-1}{2}\cdot n^{-3/2}}\cdot\cos\left(\frac{1}{\sqrt{n}}\right)}{\cancel{-\frac{1}{2}\cdot n^{-\frac{3}{2}}}} = \lim\cos\left(\frac{1}{\sqrt{n}}\right) = 1$

$\infty\cdot 0$

(10) $\dfrac{n!}{n^n} = \dfrac{\cancel{n}\cdot(n-1)\cdots 1}{\cancel{n}\cdot n\cdots n} < \dfrac{1}{n} \to 0 \Rightarrow \lim_{n\to\infty}\dfrac{n!}{n^n} = 0$

$\leq 1 \leq 1$

(11) $\dfrac{n!}{2^n\cdot 3^n} = \dfrac{n!}{6^n} = \infty$

(12) $\ln n - \ln(n+1) = \ln\left|\dfrac{n}{n+1}\right| \to \ln 1 = 0$

Ex $\boxed{\text{Limit of } a_n \text{ is converges}}$ and $\boxed{a_{n+1} = \dfrac{72}{1+a_n}}$, $a_1 = 2 \Rightarrow \lim a_n = ?$

at $\infty = L$
at $\infty + 1 = L$

$\lim_{n\to\infty} a_n = L$

$\lim_{n\to\infty} a_{n+1} = L$

$\lim \underbrace{a_{n+1}}_{L} = \lim \dfrac{72}{1+\underbrace{a_n}_{L}} \Rightarrow L = \dfrac{72}{1+L} \Rightarrow L\cdot(L+1) = 72$ $L=8$ or $\cancel{L=-9}$

$a_n > 0 \Rightarrow \boxed{\lim a_n = 8}$

a is the general form of a sequence. Then the sqn.

$\{a_n\} = [a_n]_{n=1}^{\infty} = \{a_1, a_2, a_3 \dots, a_k, \dots a_\infty\}$

The Series : $a_1 + a_2 + \dots + a_k + \dots = \sum_{n=1}^{\infty} a_n$

S_n = The sum of first n.th term

$S_1 = a_1$, $S_2 = a_1 + a_2$, $S_3 = a_1 + a_2 + a_3$ ..... , $S_n = a_1 + a_2 + \dots + a_n$

Geometric Series : $a, a.r, a.r^2, a.r^3, \dots, a.r^{n-1}$

These are first n.th elements of the geometric series.

$S_n = a + \cancel{a.r} + \cancel{a.r^2} + \dots + \cancel{a.r^{n-1}}$

$-\quad r.S_n = \cancel{a.r} + \cancel{a.r^2} + \dots + a.r^n$

$S_n - r.S_n = a - a.r^n$

$S_n = a\cdot\dfrac{1-r^n}{1-r}$

06.05.2015
Wednesday

HW: By using the series sum show that $0,\bar{1} = \frac{1}{9}$

HW: $\sum_{n=p}^{\infty} a.r^n \overset{?}{=} a.r^p \cdot \frac{1}{1-r}$, $|r|<1$ If $|r| \geq 1$ => The series diverge

Prove

$-1<r<1$

Ex: $\sum_{k=0}^{\infty} \left(\frac{-2}{3}\right)^k = ?$ $\left(\frac{-2}{3}\right)^0 \cdot \frac{1}{1-\left(\frac{-2}{3}\right)}$ $\Leftarrow \left|\frac{-2}{3}\right| < 1$ $= \frac{1}{1+\frac{2}{3}} = \frac{3}{5}$

Ex: $\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots = ?$

$\frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \cdots \Rightarrow \sum_{n=2}^{\infty} \left(\frac{1}{3}\right)^n = \left(\frac{1}{3}\right)^2 \cdot \frac{1}{1-\frac{1}{3}} = \frac{1}{6}$

HW: Prove that $0,\bar{1} = \frac{1}{9}$
(Ans)

$0,\bar{1} = 0,111\ldots = 0,1+0,01+0,001+\ldots = \frac{1}{10} + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3 + \cdots = \sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n = \frac{1}{10} \cdot \frac{1}{1-\frac{1}{10}} = \frac{1}{9}$

Ex: $\sum_{n=1}^{\infty} \frac{2^n - 3^n}{6^n} = \sum_{n=1}^{\infty} \left(\frac{2}{6}\right)^n - \sum_{n=1}^{\infty} \left(\frac{3}{6}\right)^n = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n - \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 3 \cdot \frac{1}{1-\frac{1}{3}} - 2 \cdot \frac{1}{1-\frac{1}{2}} = \frac{1}{2}$

Ex: $\sum_{n=2}^{\infty} \frac{4}{2^n} = 4 \cdot \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n$

Ex: $\sum_{n=2} (-1)^{n+1} \cdot \left(\frac{2}{3}\right)^n = -\sum (-1)^n \left(\frac{2}{3}\right)^n = -\sum_{n=2} \left(\frac{-2}{3}\right)^n = -\left(\frac{-2}{3}\right)^2 \cdot \frac{1}{1-\left(-\frac{2}{3}\right)} = \frac{-4}{15}$

Ex: $\sum_{n=1}^{\infty} \frac{2^{n-1}+3^n}{6^{n-1}} = \sum \frac{2^{-1} \cdot 2^n}{6^{-1} \cdot 6^n} + \sum \frac{3^n}{6^n \cdot 6^{-1}} = 3 \cdot \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n + 6 \cdot \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 3 \cdot \frac{1}{3} \cdot \frac{1}{1-\frac{1}{3}} + 6 \cdot \frac{1}{2} \cdot \frac{1}{1-\frac{1}{2}}$

$\sum_{k=1}^{n} a.r^{k-1} = a \cdot \frac{1-r^{n} \to 0}{1-r} = \infty$

Ex: $\sum_{n=1}^{\infty} (\sqrt{2})^n$ = divergent (∞)

$|\sqrt{2}| \geq 1$

Ex: $2{,}2\overline{3} = ?$

$$= 2{,}2 + 0{,}0333\ldots$$

$$= 2{,}2 + 3 \cdot \left[0{,}01 + 0{,}001 + \ldots \right]$$

$$= \frac{22}{10} + 3 \sum_{n=2}^{\infty} \left(\frac{1}{10} \right) = \ldots$$

Ex: $\frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \ldots = ? \quad \left(\frac{1}{n \cdot (n+1)} \right)$

$$= \sum_{n=1}^{\infty} \frac{1}{n \cdot (n+1)} = \sum \left(\frac{A}{n} + \frac{B}{n+1} \right) = \left[\frac{1}{n \cdot (n+1)} = \frac{A}{n} + \frac{B}{n+1} \Rightarrow 1 = (n+1)A + B \cdot n \Rightarrow A = 1, B = -1 \right]$$

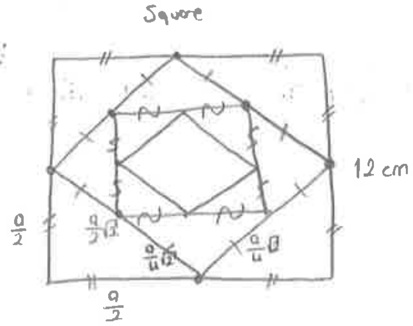
$$= \lim_{k \to \infty} \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \lim_{k \to \infty} \left[\frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} \,/\, + \cancel{\frac{1}{k}} - \frac{1}{k+1} \right] = \lim_{k \to \infty} \left(1 - \frac{1}{k+1} \right) = 1$$

Ex: $\sum_{k=1}^{\infty} \left(\ln\sqrt{k} - \ln\sqrt{k+1} \right) = \lim_{n \to \infty} \sum_{k=1}^{n} \left(\ln\sqrt{k} - \ln\sqrt{k+1} \right) = \lim_{n \to \infty} \left(\ln 1 - \cancel{\ln\sqrt{2}} + \cancel{\ln\sqrt{2}} - \cancel{\ln\sqrt{3}} + \cancel{\ldots} + \cancel{\ln\sqrt{n}} - \ln\sqrt{n+1} \right) = \lim_{n \to \infty} \left(-\ln\sqrt{n+1} \right) = -\infty$

(div.)

Ex: $\sum_{k=2}^{\infty} \left(\sqrt{k+4} - \sqrt{k+5} \right) = \ldots = \lim_{n \to \infty} \left(\sqrt{6} - \sqrt{n+5} \right) = -\infty$

Ex:



Find sum of area of all squares

	I	II	III		k
Sides $\rightarrow$	a	$a \cdot \frac{\sqrt{2}}{2}$	$a \cdot \left(\frac{\sqrt{2}}{2}\right)^2$		$a \cdot \left(\frac{\sqrt{2}}{2}\right)^{k-1}$
	$\downarrow$	$\downarrow$	$\downarrow$		
Area $\rightarrow$	a^2	$a^2 \cdot \frac{1}{2}$	$a^2 \left(\frac{1}{2}\right)^2$		

$$\text{Area} = a^2 \left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \ldots \right) = a^2 \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = a^2 \cdot \frac{1}{1 - \frac{1}{2}} = 2 \cdot a^2 \underset{a=12}{\Rightarrow} 2 \cdot 12^2 = 288 \text{ cm}^2$$

Theorem : (n-term test)

(I) $\sum_{n=1}^{\infty} a_n$ is convergent $\Rightarrow \lim a_n = 0$

(II) $\left.\begin{array}{c} \lim a_n \neq 0 \\ \text{OR} \\ \text{the limit does not exist} \end{array}\right\} \Rightarrow \sum a_n$ divergent

Ex : Test for convergency

1- $\sum_{n=1}^{\infty} (\sqrt{2})^n$, $a_n = (\sqrt{2})^n \to \infty \neq 0 \Rightarrow$ div.

2- $\sum_{n=0}^{\infty} (-1)^{n+1}$, $\lim (-1)^{n+1}$ not exist $\Rightarrow$ div.

3- $\sum_{n=0}^{\infty} \cos n$, $\lim (\cos n)$ not exist $\Rightarrow$ div.

4- $\sum_{n=2}^{\infty} \frac{n}{2n-1}$, $\lim \left(\frac{n}{2n-1}\right) = \frac{1}{2} \neq 0 \Rightarrow$ div.

5- $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{2^n} \Rightarrow \lim_{n\to\infty} \frac{(-1)^{n+1}}{2^n} = 0 \Rightarrow$ n term test does not work

$$\Rightarrow \sum_{n=1}^{\infty} -1 \cdot \left(\frac{-1}{2}\right)^n = (-1) \cdot \left(\frac{-1}{2}\right) \cdot \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{1}{3} \Rightarrow \text{conv.}$$

6- $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right) \Rightarrow \lim \cos\left(\frac{1}{n}\right) = 1 \neq 0 \Rightarrow$ Div.

7- $\sum_{n=1}^{\infty} \ln\left(\frac{1}{n}\right) = \lim \ln\left(\frac{1}{n}\right) = -\infty \neq 0 \Rightarrow$ Div.

INTEGRAL TEST

Let $\{a_n\}$ be a sequence of positive terms

Suppose that $a_n = f(n)$, where f is cont., positive, decreasing function, f x, $\forall x \geq N$ (N a positive integer)

Then $\sum\limits_{n=N}^{\infty} a_n$ and $\int\limits_{n}^{\infty} f(x)\,dx$ both conv. or div.

$\left\{ a_n = f(n),\ \text{cont}\ (+),\ \searrow \right\} \Rightarrow \sum, \int$ both conv. or div.

Ex: $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ test for conv.

(n-th term test does not work)

Lets try to use Integral test

$a_n = f(n) = \frac{1}{n}$ $\searrow$, cont, $(+)$ $\Rightarrow \int\limits_{1}^{\infty} \frac{1}{x}\,dx = \lim\limits_{R\to\infty} \ln R - \ln 1 = \infty$

$\int\limits_{1}^{\infty} \frac{1}{x}\,dx$ is diver. $\Rightarrow \sum\limits_{n=1}^{\infty} \frac{1}{n}$ is also div.

Ex= Test for conv.

$\sum\limits_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow \left\{ a_n = f(n) = \frac{1}{n^2},\ (+),\ \text{cont},\ \searrow \right\} = \int\limits_{1}^{\infty} \frac{1}{x^2}\,dx = \lim\limits_{R\to\infty} \left(\frac{-1}{x}\right)\Bigg|_{1}^{\infty} = \lim\limits_{R\to\infty} 0-(-1) = 1$ conv. $\Rightarrow \sum \frac{1}{n^2}$ is also Conv.

Ex= $\sum\limits_{n=2}^{\infty} \frac{1}{n^2+1} = \left\{ f(x) = \frac{1}{x^2+1},\ [2,\infty) \right\}$

$\int\limits_{2}^{\infty} \frac{dx}{x^2+1} = \lim\limits_{R\to\infty} \left[\arctan R - \arctan 2\right] = \frac{\pi}{2} - \arctan 2 \Rightarrow$ conv. Then, $\sum \frac{1}{n^2+1}$ is conv.

11.05.2015
Monday

Comparison Test :

Let $\sum a_n$, $\sum c_n$ and $\sum d_n$ be series with nonnegative terms. Suppose that for some integer N.

$d_n \leq a_n \leq c_n \quad \forall n \quad n \geq N$

i-) if $\sum c_n$ convergent, then $\sum a_n$ also conv.

ii-) if $\sum d_n$ divergent, then $\sum a_n$ also div.

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{2}{5n-1}$ test for conv.

$$\frac{2}{5} \sum_{n=1}^{\infty} \frac{1}{n-\frac{1}{5}} \xleftarrow{\text{comp.}} \left\{ \frac{1}{n-\frac{1}{5}} > \frac{1}{n} \quad \sum \frac{1}{n} \text{ is div.} \right.$$

$\int_1^{\infty} \frac{1}{x}\,dx$ (div) (p-test)

$\Downarrow$

$\Rightarrow \int_1^{\infty} , \sum \frac{1}{n} \Leftarrow$ Int. Test $\frac{1}{n}$, cont, (+), $\searrow$

Limit Comp. Test

Suppose than, $a_n > 0$, $b_n > 0$, $\forall n$, $n \geq N$ (N an integer)

i-) If $\lim \frac{a_n}{b_n} = c > 0$, then both $\sum a_n$ and $\sum b_n$ conv. or div.

ii-) If $\lim \frac{a_n}{b_n} = 0$ and $\sum b_n$ conv. then $\sum a_n$ is also conv.

iii-) If $\lim \frac{a_n}{b_n} = \infty$ and $\sum b_n$ div then $\sum a_n$ is also div.

Ex: Test for conv.

$$\frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} + \cdots + \frac{2n-1}{n^2} + \cdots$$

$$= \sum_{n=2}^{\infty} \frac{2n-1}{n^2} \quad \left(\frac{2n-1}{n^2} \approx \frac{1}{n} \right) \longrightarrow \lim_{n\to\infty} \frac{\frac{2n-1}{n^2}}{\frac{1}{n}} = 2 \quad (\text{Div})$$

$\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ is div

$\Uparrow$

(p-test + int-test)

Ex : Test for conv.

$$\frac{1}{3}+\frac{1}{5}+\frac{1}{9}+\frac{1}{17}+\frac{1}{33}+\cdots+\frac{1}{2^n+1}$$

$$=\sum_{n=1}^{\infty}\frac{1}{2^n+1} \Rightarrow \lim \frac{\frac{1}{2^n+1}}{\frac{1}{2^n}}=1=c>0 \quad \Rightarrow \underline{\text{CONV.}}$$

$$\rightarrow \sum_{n=1}^{\infty}\frac{1}{2^n}=\frac{1}{2}\cdot\frac{1}{1-\frac{1}{2}}=\underline{1} \quad (\text{Conv.})$$

Ex : $\displaystyle\sum_{n=1}^{\infty}\frac{1+n.\ln n}{n^2+4}$

I. WAY

$$\frac{n.\ln n}{n^2}=\frac{\ln n}{n}\approx\frac{1+n.\ln n}{n^2+4}$$

For the integral test $\left(\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n}\right)$

$\frac{\ln n}{n}$ + , ↘ , cont

$$\int_1^{\infty}\frac{\ln x}{x}dx=\int u.du=\frac{u^2}{2}=\frac{(\ln x)^2}{2}\Bigg|_1^{\infty}=\infty \quad \text{div.} \Rightarrow \sum_{n=1}^{\infty}\frac{\ln n}{n} \quad \text{div.}$$

$$\Rightarrow \lim\frac{\frac{1+n.\ln}{n^2+4}}{\frac{\ln n}{n}}=\lim\frac{n+n^2.\ln n}{(n^2+4)\ln n}=\lim\frac{\frac{1}{n}+\ln n}{\left(1+\frac{4}{n^2}\right)\ln n}=1\geq 0$$

$\displaystyle\sum\frac{1+n.\ln n}{n^2+4}$ is also div.

II. WAY

$$\frac{1+n.\ln n}{n^2+4}\approx\frac{\ln n}{n}\approx\frac{1}{n}, \quad \sum_{n=1}^{\infty}\frac{1}{n} \text{ div.} \quad \left(\begin{matrix}\text{int - test}\\ \text{p - test}\end{matrix}\right)$$

$$\lim_{n\to\infty}\frac{\frac{1+n.\ln n}{n^2+4}}{\frac{1}{n}}=\lim\frac{n+n^2.\ln n}{n^2+4}=\infty \rightarrow \sum\frac{1+n.\ln n}{n^2+4} \text{ is div.}$$

[1] $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}=\begin{cases}p>1 & \text{conv.}\\ p\leq 1 & \text{div.}\end{cases}$

p-test for series

[2] Comparison test

$\sum a_n , \sum b_n , \sum c_n$

$a_n , b_n , c_n > 0$

$a_n \leq b_n \leq c_n$

$\sum a_n$ div. $\Rightarrow \sum b_n$ div.

$\sum c_n$ conv. $\Rightarrow \sum b_n$ conv.

[3] integral test

$\sum a_n$, a_n , + , ↘ , cont

$a_n = f(n)$

$$\sum_{n=1}^{\infty}a_n \approx \int_1^{\infty}f(x)dx \equiv \left(\begin{matrix}\text{both}\\ \text{conv. or}\\ \text{div.}\end{matrix}\right)$$

Exercises

→ Test for conv.

(1) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ => (I) limit comp $\left(\frac{1}{n}\right)$

(II) Int test

(III) Comp. test $\left(\frac{1}{n^2}\right)$

(2) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ (1) Limit comp $\left(\frac{1}{\sqrt{n}}\right)$

(2) Comp test $\left(\frac{1}{\sqrt{n}}\right)$

(3) $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}}$ (1) Comp. test

(4) $\sum_{n=1}^{\infty} \frac{1}{n.3^n}$ $\frac{1}{n.3^n} < \frac{1}{3^n}$

$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{3} \cdot \frac{1}{1-\frac{3}{2}}$ (Conv.)

$\sum \frac{1}{n.3^n}$ (Conv.) (by comp. test)

(5) $\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n^2+2}} \approx \sqrt{\frac{1}{n}} = \frac{1}{\sqrt{n}}$ $\sum \frac{1}{\sqrt{n}}$ (Div.)

(6) $\sum_{n=1}^{\infty} \frac{2^n}{3+4^n} \approx \frac{2^n}{4^n} = \frac{1}{2^n}$ $\sum \frac{1}{2^n}$ (Conv.)

(7) $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+4}\right)^n$ (n-th teorem)

(8) $\sum_{n=1}^{\infty} \frac{1}{n.(1+\ln^2 n)}$ (int. test)

$a_n = \frac{1}{n.(1+\ln^2 n)} = f(n)$ ↘, (+), conv.

$\int_1^{\infty} \frac{1}{x+\ln^2 x} \cdot dx$ $\ln x = u$, $\frac{1}{x} \cdot dx = du$ => $\int \frac{du}{u^2} = -\frac{1}{u}\Big|_1^{\infty} = -\frac{1}{\ln x}\Big|_1^{\infty} = -(0-\infty) = \infty$ (Div.)

4) n-term test

$\lim a_n \neq 0$ => $\sum a_n$ div.

9 $\sum_{n=2}^{\infty} \frac{n-4}{n^2-2n+1}$

1. Lim. Comp. Test
2. Int.

10 $\sum_{n=2}^{\infty} \frac{\ln(n^2)}{n}$

1. Limit Comp.
2. Int.

11 $\sum_{n=1}^{\infty} n \cdot \sin\left(\frac{1}{n}\right)$

$$\lim n \cdot \sin\left(\frac{1}{n}\right) = \infty \cdot 0$$
$$= \lim \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \frac{0}{0}$$
$$= \lim \frac{\left(\frac{-1}{n^2}\right) \cdot \cos\left(\frac{1}{n}\right)}{\left(\frac{-1}{n^2}\right)} = 1 \neq 0 \quad \text{(Div.)}$$

Root Test

$$\lim \sqrt[n]{a_n}$$

$$= \lim \sqrt[n]{\frac{2^n+1}{3^n}} = \lim \frac{\sqrt[n]{2^n+1}}{3} = \frac{2}{3} < 1 \Rightarrow \text{Convergent.}$$

Ex : $\sum_{n=1}^{\infty} \frac{(2n)!}{n! \cdot n!}$

$$\Rightarrow \lim \frac{a_{n+1}}{a_n} = \lim \frac{(2n+2)!}{(n+1)!(n+1)} \cdot \frac{n! \cdot n!}{(2n)!}$$

$$= \lim \frac{(2n+2) \cdot (2n+1) \cdot (2n)! \cdot n! \cdot n!}{(n+1)(n!)(n+1)\, n!\, (2n)!} = \lim \frac{4n+2}{n+1} = 4 > 1 \Rightarrow \text{Divergent.}$$

Ex : $\sum_{n=1}^{\infty} \frac{4^n \cdot n! \cdot n!}{(2n)!}$

$$= \lim \frac{a_{n+1}}{a_n} = \frac{4^{n+1} \cdot (n+1) \cdot n! \cdot (n+1) \cdot n! \cdot (2n)!}{(2n+2)(2n+1)(2n)! \cdot 4^n \cdot n! \cdot n!} = \lim \frac{4n+4}{4n+2} = 1 \quad \text{test fails.}$$

Ex : $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

$$\lim \sqrt[n]{\frac{n^2}{2^n}} = \lim \frac{\left(\sqrt[n]{n}\right)^2}{2} = \frac{1^2}{2} < 1 \quad \text{(Conv.)}$$

Ex : $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^3}$

$$= \lim \sqrt[n]{\frac{2^n}{n^3}} = \lim \frac{2}{(\sqrt[n]{n})^3} = 2 > 1 \quad (\text{Div.})$$

Ex : $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$

$$= \lim \sqrt[n]{\left(\frac{1}{1+n}\right)^n} = \lim \frac{1}{1+n} = 0 < 1 \quad (\text{Conv.})$$

Ex : $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$

* $a_n = \dfrac{1}{n}$

i-) $\lim a_n = 0$

ii-) $\dfrac{1}{n} > 0$

iii-) $\dfrac{1}{n} > \dfrac{1}{n+1}$

The series is alternating $\Rightarrow$ Conv.

Ex : $\displaystyle\sum (-1) \cdot \frac{1}{2^n}$

$a_n = \dfrac{1}{2^n} \Rightarrow \lim \dfrac{1}{2^n} = 0$, $\dfrac{1}{2^n} > 0 \Rightarrow \dfrac{1}{2^n} > \dfrac{1}{2^{n+1}}$

The series is alternating $\Rightarrow$ Conv.

Ex : $\displaystyle\sum (-1)^{n+1} \cdot \frac{1}{n^2}$

$$= -\left[\sum (-1)^n \cdot \frac{1}{n^2}\right] \qquad \frac{1}{n^2} \begin{cases} \to 0 \\ \to \text{decreasing} \\ \to \text{Alternating} \end{cases} \Rightarrow \text{Conv.}$$

$\downarrow\downarrow\downarrow$

$$\sum \left|(-1)^n \cdot \frac{1}{n}\right| = \sum \frac{1}{n} \quad (\text{Div.}) \quad (\text{p-test})$$

$$\sum \left|(-1)^n \cdot \frac{1}{2^n}\right| = \sum_{n=1} \frac{1}{2^n} = \frac{1}{2} \cdot \frac{1}{1-\frac{1}{2}} = 1 \Rightarrow \text{Conv.}$$

$$\sum \left|(-1)^{n+1} \cdot \frac{1}{n^2}\right| = \sum \frac{1}{n^2} \quad \text{Conv. (p-test)}$$

① $\sum_{n=1}^{\infty} (-5)^{-n} = \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{5}\right)^n$

I. Way : Alternating $\Rightarrow$ Conv.

II. Way : $\sum \left| (-1)^n \cdot \left(\frac{1}{5}\right)^n \right| = \sum_{n=1} \left(\frac{1}{5}\right)^n = \frac{1}{5} \cdot \frac{1}{1-\frac{1}{5}} = 4$ (Conv.)

② $\sum_{n=1} \frac{\ln n}{n^3}$ (Limit-Comparison Test) $\left(\frac{1}{n^2}\right)$

③ $\sum_{n=1} \frac{n!}{10^n}$ (ratio)

④ $\sum_{n=1}^{\infty} \frac{(-100)^n}{n!} \Rightarrow \sum | \quad | = \sum_{n=1}^{\infty} \frac{100^n}{n!}$

$= \lim \frac{a_n+1}{a_n} = \lim \frac{100 \cdot 100^n}{(n+1) \cdot n!} \cdot \frac{n!}{100^n} = \lim \frac{100}{n+1} = 0 < 1 \Rightarrow$ Conv.

$\sum \frac{(-100)^n}{n!}$ is absolutely Conv.

✱ $a_n = \frac{\sin n}{n^2}$ we can not use

$$\left.\begin{array}{ll} \text{comp. test} & \\ \text{limit.} & '' \\ \text{int.} & '' \\ \text{root} & '' \\ \text{ratio} & '' \end{array}\right\} a_n > 0$$

Let's take the absolute value : $\sum \left| \frac{\sin n}{n^2} \right|$ conv. comp. test $\left| \frac{\sin n}{n^2} \right| \leq \frac{1}{n^2}$, $\sum \frac{1}{n^2}$ is conv. (p-test)

Ex : $\sum_{n=1}^{\infty} \left(\frac{x}{2}\right)^n$ find the interval of convergence and radius of conv.

Root Test

$\lim \sqrt[n]{\left|\frac{x}{2}\right|^n} = \left|\frac{x}{2}\right| < 1 \Rightarrow$ Conv.

$|x| < 2$, $-2 < x < 2$

✱ $x=2 \Rightarrow \sum_{n=1}^{\infty} \left(\frac{-2}{2}\right)^n = \sum_{n=1}^{\infty} (-1)^n \Rightarrow$ Div.

n-term test $\lim ((-1)^n) \neq 0$

$x=-2 \Rightarrow \sum_{n=1}^{\infty} \left(\frac{2}{2}\right)^n = \sum_{n=1}^{\infty} 1^n = 0$ (Div)

Interval of conv. = $(-2,2)$

Radius " " $= 2$

Ex: $\sum_{n=1} (-1)^{n-1} \cdot \frac{x^n}{n}$ find the interval of convergency

Ratio Test

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left| \frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}} \right| = \lim \left| \frac{x \cdot x^n \cdot n}{(n+1) \cdot x^n} \right| = \lim_{n\to\infty} |x| \cdot \frac{n}{n+1} = |x| < 1$$

$-1 < x < 1$

$x = -1 \Rightarrow \sum \frac{(-1)^{n-1} \cdot (-1)^n}{n} = \sum \frac{(-1)^{2n-1}}{n} = -\sum \frac{1}{n}$ Div. (p-test)

$x = 1 \Rightarrow \sum (-1)^{n-1} \cdot \frac{1^n}{n} = -\sum (-1)^n \cdot \frac{1}{n}$ Alternating series that (Conv.)

Interval of conv. $= (-1, 1]$

Radius of conv. $= 1$

Ex: $\sum_{n=0}^{\infty} \frac{n \cdot (x+3)^n}{5^n}$ find the interval of conv.

Root Test

$$\lim \sqrt[n]{\left| \frac{n \cdot (x+3)^n}{5^n} \right|} = \lim \sqrt[n]{n} \left| \frac{x+3}{5} \right| = \left| \frac{x+3}{5} \right| < 1$$

$|x+3| < 5$, $-5 < x+3 < 5$

$-8 < x < 2$

$x = -8$ $\quad \sum \frac{n \cdot (-5)^n}{5^n} = \sum_{n=0}^{\infty} (-1)^n \cdot n$

n-term test

$\lim ((-1)^n \cdot n)$ not exists $\Rightarrow$ divergent

$x = 2 \Rightarrow \sum \frac{n \cdot 5^n}{5^n} = \sum n = \infty \Rightarrow$ div.

int. of conv. $= (-8, 2)$

radius of conv. $= 5$

18.05.2015
Monday

Defn : A power series about $x=0$ is a series of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n + \dots$

A power series about $x=a$ is the series $\sum_{n=0}^{\infty} c_n (x-a)^n$ in which the center a and the coefficients $c_0, c_1, \dots, c_n \dots$ are constants.

For Example

The power series extensions of $\frac{1}{1-x}$ about $x=0$ is $\sum_{n=0}^{\infty} x^n$, $|x|<1$

Equivalently, $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + \dots + x^n + \dots$, $|x|<1$

*

$$\boxed{\int \frac{1}{1-x}\,dx = \int \sum_{n=0}^{\infty} x^n\,dx} = \int (1+x+\dots)\,dx$$

$1-x=u$

$-dx=du$

$\int \frac{-du}{u}$

$-\ln u = \sum \frac{x^n}{n}$ $\qquad x + \frac{x^2}{2} + \frac{x^3}{3}$ $\qquad \Rightarrow -\ln|1-x| = \sum_{n=1}^{\infty} \frac{x^n}{n}$, $|x|<1$

Ex : Find the power series expansion of $\frac{1}{1+x}$ about $x=0$

We know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $|x|<1$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n , \quad |-x|<1$$

Ex : Find the power series of $\ln|1+x|$ about $x=0$

$$\ln|1+x| = \int \frac{1}{1+x}\,dx = \int \sum_{n=0}^{\infty} (-x)^n\,dx = \int [1-x+x^2-\dots]\,dx$$

$$\boxed{\ln|1+x| = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{n+1}}{n+1}} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

Ex : $\sin x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n-1}}{(2n-1)!}$, $x=0$ (about) , Find power series of $\cos x$ about $x=0$

$$\frac{d}{dx} \sin x = \frac{d}{dx}\left[\sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n-1}}{(2n-1)!}\right]$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(2n-1)\cdot x^{2n-2}}{(2n-1)(2n-2)!} \Rightarrow \cos x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n-2}}{(2n-2)!} = \sum_{n=-1}^{\infty} (-1)^{n+1} \cdot \frac{x^{2n}}{(2n)!}$$

Taylor Series

If a function $f(x)$ can be differentiable (about $x=x_0$) infinitely many times. then we can write it in the Taylor Series form about $x=x_0$, $\boxed{\sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!}(x-x_0)^n = f(x)}$

If it is about $x=0$ we called this series as Maclaurian Series, $f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(x-0)^n = \boxed{\sum_{n=0}^{\infty} \frac{f^n(0)}{n!} \cdot x^n}$

Ex: Find the Taylor series generated by $\frac{1}{x}$ about $x=1$

$f(x) = \frac{1}{x}$, $f(x) = \sum_{n=0}^{\infty} \frac{f^n(1)}{n!} \cdot (x-1)^n = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \cdots + \frac{f^n(1)}{n!}(x-1)^n$

$f'(x) = -x^{-2}$

$f''(x) = (-1)(-2) \cdot x^{-3}$

$f'''(x) = (-1) \cdot n \cdot x^{-(n+1)}$

$= 1 - (x-1) + (x-1)^2 + (x-1)^3 + \cdots (-1)^n (x-1)^n$

$$\boxed{\frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n \cdot (x-1)^n}$$

Ex: Find the Maclaurian series of e^x

$e^x = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} \cdot x^n = f(0) + f'(0) \cdot x' + \cdots + \frac{f^n(0)}{n!} \cdot x^n + \cdots$

$f(x) = e^x \Rightarrow f(0) = 1$

$f'(x) = e^x \Rightarrow f'(1) = e$

$f^{(n)}(x) = e^x \Rightarrow f^n(0) = 1$

$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \longrightarrow x = 1 \Rightarrow e = 1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} + \cdots$

2.5

2.6

Ex: Find the Maclaurian series of

1. $\sin x$
2. $\cos x$
3. $\frac{1}{1-x}$
4. $\frac{1}{1+x}$

2. $f(x) = \cos x$ — $f(0) = 1$

$f'(x) = -\sin x$ — $f'(0) = 0$

$f''(x) = -\cos x$ — $f''(0) = -1$

$f'''(x) = \sin x$ — $f'''(0) = 0$

$\cos x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot (x)^n$

$= f(0) + f'(0) \cdot x + f''(0) \cdot x^2 + \cdots$

$= 1 + 0 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} - \cdots + (-1)^n \cdot \frac{x^{2n}}{(2n)!}$

$$\boxed{\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{(2n)!}}$$

Ex: Find the radius of convergency of $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^3 \cdot 3^n}$

Root Test

$$\lim \sqrt[n]{\left|\frac{(x-1)^n}{n^3 \cdot 3^n}\right|} < 1 \qquad x = -2$$

$$\lim \frac{|x-1|}{(\sqrt[n]{n})^3 \cdot 3} < 1$$

$$\frac{|x-1|}{3} < 1$$

$$|x-1| < 3$$

$$-3 < x-1 < 3$$

$$-2 < x < 4$$

$x = -2$

$$\sum_{n=1}^{\infty} \frac{(-1)^n \cdot \cancel{3^n}}{n^3 \cdot \cancel{3^n}} = \sum (-1)^n \cdot \frac{1}{n^3}$$

$a_n = \frac{1}{n^3}$ $\longrightarrow \searrow$
$\longrightarrow +$
$\longrightarrow$ dec.

Alternating.

$x = 4$ $\quad \sum \frac{3^n}{n^3 \cdot 3^n}$

$$= \sum_{n=0}^{\infty} \frac{1}{n^3}$$

p-test
conv.

Int. of conv. $= [-2, 4]$

Radius of conv. $= 3$

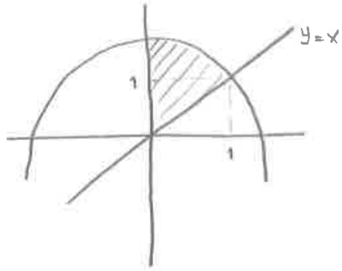
FINAL EXAM EXERCISE

20.05.2015
Wednesday

① a-) $\int \frac{\cos x}{1+\sin^2 x}\,dx = \left[\sin x = u\right] = \arctan u + c$

b-) $\int_0^1 \frac{4}{x^2-4}\,dx = \int_0^1 \left(\frac{A}{x-2} + \frac{B}{x+2}\right) dx = \left[A.\ln|x-2| + B.\ln|x+2|\right]_0^1 =$

② $y = 2 - x^2$
$y = x$



$2 - x^2 = x$
$x^2 + x - 2 = 0$
$x = -2 \ ,\ x = 1$

b-) $A = \int_0^1 \left[(2-x^2) - x\right] dx$

c-) $V = \pi \int_0^L \left[(2-x^2)^2 - (x)^2\right] dx$

③ a-) $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+1}{n^3+1} = \left[\begin{array}{l} a_n \to 0 \\ a_n \downarrow \end{array} \Rightarrow \text{Alt.} \to \text{Conv}\right] \Rightarrow \lim_{n\to\infty} \frac{n^2+1}{n^3+1} = 0$

b-) Ex
$\sum \ln\left(\frac{n+1}{n+2}\right) = \sum \left[\ln(n+1) - \ln(n+2)\right] = \infty$ (Div) | $\lim \left[\ln\left(\frac{n+1}{n+2}\right)\right] = \ln\left(\frac{1}{2}\right) \neq 0 \Rightarrow$ Div.

c-) $\overbrace{\frac{1}{n^2}}^{\text{conv}} \leq \frac{2+\sin n}{n^2} \leq \overbrace{\frac{3}{n^2}}^{\text{conv.}}$

④ $\lim \sqrt[n]{\left|\frac{x^n}{2^n(n+1)}\right|} < 1 \Rightarrow \lim \frac{|x|}{2\sqrt[n]{n+1}} < 1 \Rightarrow \frac{|x|}{2} < 1 \Rightarrow \boxed{-2 < x < 2}$

I.C $= [-2, 2)$

R.C $= \frac{4}{2} = 2$

$x = -2$
$\sum \frac{(-2)^n}{2^n(n+1)} = \sum (-1)^n \cdot \frac{1}{n+1}$ Alt. (Conv.)

$x = 2$
$\sum \frac{2^n}{2^n(n+1)} = \sum \frac{1}{n+1}$ (Div.) Lim K.T. $\left(\frac{1}{n}\right)$

---

② $\int \frac{dx}{(x+2)^2 + 1^2} = \arctan(x+2) + c$

③ Ex
$\int \log(\quad)dx$
$\int \text{arc}(\quad)\,dx$
$\Rightarrow$ Use Int. by Parts method

$\int \ln(x+1)dx = \left[\begin{array}{ll} u = \ln(x+1) & dV = dx \\ du = \frac{1}{x+1} & V = x \end{array}\right] = \int \ln(x+1)dx = x.\ln(x+1) - \int \frac{x+1-1}{x+1}\,dx - \int\left(1 - \frac{1}{x+1}\right)dx$

$= x.\ln(x+1) - x + \ln(x+1) + c$

(4) $\sum_{n=3}^{\infty} (-1)^{n-1} \cdot \frac{5}{2^n} = \frac{5}{8} - \frac{5}{16} + \frac{5}{32} - \frac{5}{64} + \cdots$

$\sum -5 \cdot \frac{(-1)^n}{2^n} = -5 \cdot \sum_{n=3}^{\infty} \left(\frac{-1}{2}\right)^3 \cdot \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{5}{12}$

(5) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}} = \left[\begin{matrix} \to + \\ \to \text{cont.} \\ \to \text{decr.} \end{matrix} \right] = \int_{n=2}^{\infty} \frac{dx}{x \cdot \sqrt{\ln x}} = 2 \cdot \int \frac{du}{2\sqrt{u}} = 2\sqrt{u} \Big| = 2 \cdot \sqrt{\ln x} \Big|_2^{\infty} = \infty$ (Div.) $= \sum$ is also div.

$\ln x = u$

(6) $\sum_{n=2}^{\infty} \left(\frac{3n+4}{4n-5}\right)^n = \lim \sqrt[n]{\left|\frac{3n+4}{4n-5}\right|^n} < 1 \Rightarrow$ Conv. $\quad \lim_{n\to\infty} \frac{3n+4}{4n-5} = \frac{3}{4} < 1 \Rightarrow$ Conv.

(7) $\sum_{n=2}^{\infty} \left(1-\frac{1}{n}\right)^n \neq 0 \Rightarrow$ Div. (n-term test) $= e^{-1}$ $\quad (\times)$

(8) $\sum_{n=1}^{\infty} \frac{x^n}{3^n(n+2)} = \lim \sqrt[n]{\frac{x^n}{3^n(n+2)}} < 1 \Rightarrow \frac{|x|}{3} < 1 \Rightarrow -3 < x < 3 \Rightarrow$ I.C. $= [-3,3)$, R.C $= \frac{6}{2} = 3$

$x=-3 \quad \sum \frac{(-3)^n}{3^n(n+1)} = \sum (-1)^n \frac{1}{(n+1)}$ conv. $\qquad x=3 \quad \sum \frac{3^n}{3^n(n+2)} = \sum \frac{1}{n+2}$ div.

(9) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow \sin x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \Rightarrow \cos x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!} \Rightarrow \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

Ex: $\sum \frac{f(x_0)}{n!} (x-x_0)^n =$

$e^x = \sum \frac{x^n}{n!} \Rightarrow e^{2x} = \sum \frac{(2x)^n}{n!}$

$\sin(3x)$

$\sin(x)$

$\sin(x) = \sum (-1) \cdot \frac{x^{2n+1}}{(2n+1)!} \Rightarrow \sin(3x) = \sum (-1) \cdot \frac{(3x)^{2n+1}}{(2n+1)!}$

$\ln(x+1)$

$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n \quad \Rightarrow \ln(x+1) = \int \frac{1}{x+1} \cdot dx = \int \sum (-1)^n \cdot x^n = \sum (-1)^n \cdot \frac{x^{n+1}}{n+1}$