LINEAR SYSTEMS

① $x_1+x_2 \rightarrow \boxed{f(-)} \rightarrow f(x_1+x_2)=f(x_1)+f(x_2)$

② $\alpha . x \rightarrow \boxed{f(.)} \rightarrow f(\alpha . x)=\alpha f(x)$

$\Rightarrow f(\alpha_1 . x_1+\alpha_2 . x_2)=\alpha_1 . f(x_1)+\alpha_2 . f(x_2)$

$\rightarrow f(x_1)=a . x_1$

1-) $f(x_1+x_2)=f(x_1)+f(x_2)$

$f(x_1+x_2)=a.(x_1+x_2)=a . x_1+a . x_2=f(x_1)+f(x_2)$

2-) $f(\alpha . x)=\alpha . f(x) \Rightarrow \alpha . a . x=\alpha . f(x)$

Matrix

row $\longleftrightarrow$ column $\updownarrow$

Homogenus Systems

$a_{11} . x_1+a_{12} . x_2+\dots+a_{1n} . x_n=0$

$\vdots$

$a_{m1} . x_m+\dots \quad a_{mn} . x_n=0$

$A . x=0$ consistent $x_1=x_2=\dots=0$

Augmented Matrix

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

Gauss Elimination Method

$x_n=\frac{\beta_n}{\alpha_{nn}}$

$$\hat{A}=\begin{bmatrix} \alpha_{11} & & & \alpha_{1n} \\ 0 & \alpha_{22} & & \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & \alpha_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}=\begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}$$

* Serbest degiskenlerin sayısı 0 olan satırların sayısı kadar.

Vectors

$$x=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \Rightarrow \text{Column Vector}$$

Inner Product

i-) $f(x,y)=f(y,x)$

ii-) $f(x+y,z)=f(x,z)+f(y,z)$

iii-) $f(\alpha . x,y)=\alpha . f(x,y)$

Matrices

Square Matrix

$$A(a_{ij})_{n \times n}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}$$

$a_{ii} \rightarrow$ diagonal elements $i=j$

* $A^{-1} . A=I$

* $C=A.B \Rightarrow A(a_{ij})_{mp}, B(b_{ij})_{pn}$

Columns of the first matrix must be equal to rows of the second matrix.

* $A=(a_{ij})_{n \times n}$ be a square matrix.

A is said to be symmetric iff $A=A^T$, that is, $a_{ij}=a_{ji}$, $\forall i,j$

Diagonal Matrix

$$\begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & d_{nn} \end{bmatrix}=D$$

Unity or Identity Matrix

$$I=\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

Inverse Matrix

$A^{-1} A=I$

If A is invertable, then A is a non-singular matrix. Otherwise, A is a singular matrix.

PROPERTIES OF MATRICES

- $(AB)^{-1}=B^{-1} . A^{-1}$
- $(AB)^T=B^T . A^T$
- $(\alpha . A)^{-1}=\alpha^{-1} . A^{-1}$
- $(A+B)^T=A^T+B^T$
- $(\alpha . A)^T=\alpha . A^T$
- $(A^T)^{-1}=(A^{-1})^T$
- $A^k=\underbrace{A.A \dots A}_{k}$
- $(\alpha . A)^k=\alpha^k . A^k$
- $(A+B)(B+A)=A^2+AB+BA+B^2$

* Matrix × Vector = Vector
* If A is invertable, Only one solution for all vector.

is non- , infinite solution

* $i=j$ and a_{ij} line is diagonal

Upper Triangular Matrices

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

Lower Triangular Matrices

$$A=\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

DETERMINANTS

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow \det(A)=|A|=a_{11} . a_{22}-a_{12} . a_{21}$$

Minors and Cofactors

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Minors: m_{ij}

Cofactors: $C_{ij}=(-1)^{i+j} . m_{ij}$

$$C=\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \Rightarrow a_{11} . c_{11}+a_{12} . c_{12}=\det(A) \quad 2\times 2$$

$$3\times 3 \Rightarrow \det(A)=a_{1j} . c_{1j}+a_{2j} . c_{2j}+\dots+a_{nj} . c_{nj}=\sum_{i=1}^{n} a_{ij} . c_{ij} \ (j, \text{column})$$

$$\det(A)=a_{i1} . c_{i1}+a_{i2} . c_{i2}+\dots+a_{in} . c_{in}=\sum_{j=1}^{n} a_{ij} . c_{ij} \ (i, \text{row})$$

* $A . C^T=\det(A) . I \Rightarrow A^{-1} . A . C^T=A^{-1} . \det(A) \Rightarrow I . C^T=A^{-1} . \det(A)$

$$\Rightarrow A^{-1}=\frac{1}{\det(A)} . C^T$$

CRAMMER'S RULE

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \vdots \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \Rightarrow$$

$A . x=b$

$x=A^{-1} . b$

$=\frac{1}{\det(A)} . C^T . b$

$\det(AB)=\det(A)\det(B)$ $\quad \det(A^T)=\det(A)$ $\quad \det(\alpha . A)=\alpha^n . \det(A)$

$\det(A^{-1})=\frac{1}{\det(A)}$ $\quad \det(A^{-1} B . A)=\det(B)$ $\quad \det(-A)=(-1)^n \det(A)$

$\det(A^k)=(\det(A))^k$

* Bir matrisin herhangi bir satır ya da sütununun bütün elemanları sıfırsa, det = 0.
* Eger herhangi iki satır ya da sütun elemanları oranı aynıysa det = 0.

Principal Minors

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \Rightarrow$$

1.order: $[a_{11}]$ $[a_{22}]$ $[a_{33}]$ $[a_{44}]$

2.order: $A_{12}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

3.order: $A_{123}=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$C(n,r)=\frac{n!}{(n-r)! . r!}$

Leading Principal minors

LINEAR ALGEBRA

Elementary Row Operations : Matris elemanları 0'a eşitleyip , x_1, x_2, x_3 değ. bulma

Gauss Elimination Method : Serbest değişken = α, β, γ , Serbest değişkenlerin sayısı = 0 olan satır sayıları kadar

Vector's Rules : INNER PRODUCT : ① $f(x,y) = f(y,x)$ DOT PRODUCT
② $f(x+y,z) = f(x,z) + f(y,z)$
③ $f(\alpha . x, y) = \alpha . f(x,y)$

→ A matrisi simetrik => $A = A^T$ ve $a_{ij} = a_{ji}$ olmalıdır.

Diagonal Matrices → $A.A^{-1} = I$

$$D = \begin{bmatrix} d_{11} & 0 & 0 & \dots & 0 \\ \vdots & d_{22} & & & \\ & & d_{33} & & \\ 0 & & & \ddots & d_{nn} \end{bmatrix}$$

→ A matrisinin tersi alınabiliyorsa tekil olmayan matris , alınamıyorsa tekil matristir.

→ $(AB)^{-1} = B^{-1}.A^{-1}$ → $(AB)^T = B^T.A^T$ → $(\alpha.A)^k = \alpha^k.A^k$ → $(A+B)^T = A^T + B^T$ → $(\alpha.A)^T = \alpha.A^T$

Upper Triangular Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

Lower Triangular Matrices

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

DETERMINANTS

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad det(A) = a_{11}.a_{22} - a_{12}.a_{21}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow det(A) = a_{11}(a_{22}.a_{33} - a_{23}.a_{32}) - a_{12}(a_{21}.a_{33} - a_{23}.a_{31}) + a_{13}(a_{21}.a_{32} - a_{22}.a_{31})$$

=> 1. Satır Eleman . Kofaktör

MINOR

$$A = \begin{bmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ \cancel{a_{21}} & a_{22} & a_{23} \\ \cancel{a_{31}} & a_{32} & a_{33} \end{bmatrix} \quad M_{11} = a_{22}.a_{33} - a_{23}.a_{32}$$

↳ det

COFACTOR

$C_{ij} = (-1)^{i+j}.M_{ij}$ => $C_{11} = (-1)^{1+1}.M_{11} = M_{11}$

→ Bir satır / sütun elemanları = 0 => $det(A) = 0$ → Herhangi iki satır / sütun elemanları eşit => $det(A) = 0$

→ Herhangi iki satır / sütun elemanları orantılı => $det(A) = 0$ → Herhangi iki satır / sütun el. yerleri değiştirilirse $det(A)$'nın işareti değişir.

→ $A_{n\times n}$ => $det(A) = det(A^T)$ → $det(A.B) = det(A).det(B)$ → $det(A)^n = det\,A^n$ → $|A^{-1}| = \frac{1}{|A|}$

→ $A = [a_{ij}]_{n\times n}$ => $|k.A| = k^n|A|$ → $det(A^{-1}) = \frac{1}{det(A)}$ → $det(\alpha.A) = \alpha^n.det(A)$

$det(-A) = (-1)^n.det(A)$

Adjoint (Ekmatris)

$$adj(A) = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T$$

Matris Tersi

$|A| \neq 0$ => Tersi ✓

$$A^{-1} = \frac{1}{|A|} . adj(A) = \frac{1}{det(A)} . C^T$$

Principal Minors

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

1. Order Principal Sub Mat. = $[a_{11}]$, $[a_{22}]$, $[a_{33}]$, $[a_{44}]$

2. Order " " " $= A_{12} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $A_{13} = \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}$, $A_{14} = \begin{bmatrix} a_{11} & a_{14} \\ a_{41} & a_{44} \end{bmatrix}$

$A_{23} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$, $A_{24} = \begin{bmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{bmatrix}$, $A_{34} = \begin{bmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{bmatrix}$

$C(n,r) = \frac{n!}{(n-r)!\,r!} = \frac{4!}{1!\,3!} = 4 \rightarrow$ 3. Order P.S.M = $A_{123} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ a_{124}, a_{234}, a_{134}

Leading Principal Minors

: $[a_{11}]$ $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ . . .

CRAMMER'S RULE

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & & a_{2n} \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{b} \Rightarrow \begin{matrix} a_{11}\cdot x_1 + a_{12}\cdot x_2 + \ldots + a_{1n}\cdot x_n = b_1 \\ a_{21}\cdot x_1 + a_{22}\cdot x_2 + \ldots + a_{2n}\cdot x_n = b_2 \\ \vdots \\ a_{n1}\cdot x_1 + a_{n2}\cdot x_2 + \ldots + a_{nn}\cdot x_n = b_n \end{matrix}$$

$$\Delta_1 = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & & \\ \vdots & \vdots & & \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} & \cdots & a_{1n} \\ a_{21} & b_2 & \vdots & & \\ \vdots & \vdots & \vdots & & \\ a_{n1} & b_n & a_{n3} & \cdots & a_{nn} \end{vmatrix}, \ldots, \Delta_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & \cdots & & b_2 \\ \vdots & & & \vdots \\ a_{n1} & \cdots & & b_n \end{vmatrix}$$

① $|A| \neq 0 \Rightarrow$ Denklemin tek çözümü var

$$x_1 = \frac{\Delta_1}{|A|} \quad x_2 = \frac{\Delta_2}{|A|}, \ldots, x_n = \frac{\Delta_n}{|A|}$$

② $|A| = 0$ ve $\Delta_1, \Delta_2, \ldots, \Delta_n$ en az biri sıfırdan farklıysa denklemin çözümü yok.

③ $|A| = 0$ ve $\Delta_1 = \Delta_2 = \ldots = \Delta_n \Rightarrow \infty$ çözüm var.

VECTOR SPACE

RANK : $A_{n\times m} \Rightarrow \max(n,m)$ if $\det(A) \neq 0$ else $\max-1 = \text{rank}$ if $\det(A) \neq 0$

Eigenvalues and Eigenvectors $\Rightarrow A.x_i = \lambda_i.x_i$, $A.v = \lambda.v$, $v \neq 0$

→ Bir matrisin öz değerleri toplamı = diagonal elemanlarının toplamı

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $\lambda_1 + \lambda_2$ (Öz deger toplamı) $= 2+2 = 4$ = diagonal el. toplamı

$\lambda_1 . \lambda_2$ (öz deger çarpımı) $= 2.2 - 1.1 = 3 = \det(A)$ $\lambda_1 = 1, \lambda_2 = 3$

→ $\det(A) = 0 \Rightarrow$ Singular Matrix , $\det(A) \neq 0 \Rightarrow$ non-singular

LINEER DEPENDENCE-INDEPENDENCE

→ $S = \{V_1.V_2, \dots, V_n\}$ ve $k_i \in \mathbb{R}$, $i = 1,2,\dots,n$

$k_1.V_1 + k_2.V_2 + \dots + k_n.V_n = 0$
- → $k_1 = k_2 = \dots = k_n = 0$ → Lineer bağımsız
- → $k_1, k_2, \dots, k_n$ = not all of them 0 → Lineer bağımlı

→ $\det(V) = 0$ → Singular Vector ⇒ $V_1.V_2,V_3$ → Lineer bağımlı

→ $A.v = \lambda.v$, $v \neq 0$
$A = (a_{ij})_{n\times n}$ > $\underbrace{(A - \lambda.I)}_{\text{eigenvalue}}.\underbrace{V}_{\text{eigenvector}} = 0$

$\det(\lambda I - A) = 0$ Characteristic Equation

$n=2$ C.E. $\Rightarrow \lambda^2 - (\overbrace{\Delta_1 + \Delta_2}^{1.\text{order}})\lambda + \Delta = 0$

$n=3$ C.E $\Rightarrow \lambda^3 - (\Delta_1 + \Delta_2 + \Delta_3)\lambda^2 + (\overbrace{\Delta_{12} + \Delta_{21} + \Delta_{13}}^{2.\text{order}})\lambda - \Delta = 0$

$n=4$ C.E. $\Rightarrow \lambda^4 - (\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4).\lambda^3 + (\Delta_{12} + \Delta_{13} + \Delta_{14} + \Delta_{23} + \Delta_{24} + \Delta_{34})\lambda^2 - (\overbrace{\Delta_{123} + \Delta_{124} + \Delta_{134} + \Delta_{234}}^{3.\text{order}})\lambda + \Delta = 0$

MATRICE NORM

$\|A\|_1 \overset{\max}{=}$ Sütun Mutlak deger toplamı $\|A\|_\infty = {}^{\max}$ Satır Mutlak deger toplamı

$\|A\|_2 = \sqrt{\lambda_{max}.A^T.A}$

12.02.2016 Cuma

LINEAR ALGEBRA

SOURCE : Elementory Linear Algebra
Howard Anton - Chris Rorres
→ John wiley · sons
1 Vize : %40   1 Final : %60

1-) Systems of Linear Equations
2-) Matrices (Matrix)
3-) Determinants
4-) Vector Space
5-) Inner Products and Norms
6-) Eigenvalues and Eigenvectors
7-) Linear Transformations

Linear Systems

R: the set of real numbers

$x \in R$, $a \in R$, $\alpha \in R$

$f(x)$?

1-)

$\xrightarrow{x} \boxed{f(-)} \rightarrow f(x)$

$\xrightarrow{x_1} \boxed{f(-)} \rightarrow f(x_1)$

$x_2 \rightarrow \boxed{f(-)} \rightarrow f(x_2)$

$x_1+x_2 \rightarrow \boxed{f(-)} \rightarrow f(x_1+x_2) = f(x_1)+f(x_2)$

2-)

$\xrightarrow{\alpha x} \boxed{f(-)} \rightarrow f(\alpha x) = \alpha . f(x)$

$\alpha_1, \alpha_2 \in R$

$x_1, x_2 \in R$

$f(\alpha_1 . x_1 + \alpha_2 . x_2) = \alpha_1 . f(x_1) + \alpha_2 . f(x_2)$

→ $f(x_1) = a . x_1$

1-) $f(x_1+x_2) = f(x_1)+f(x_2)$

$f(x_1+x_2) = a.(x_1+x_2) = a.x_1 + a.x_2 = f(x_1)+f(x_2)$

2-) $f(\alpha . x) = \alpha . f(x)$

$f(\alpha . x) = \alpha . \underline{a.x}$

$= \alpha . f(x)$

Linear function

→ $f(x) = x^2$

$f(x_1+x_2) = f(x_1)+f(x_2) = (x_1+x_2)^2 = x_1^2+x_2^2+2.x_1.x_2$

$= f(x_1)+f(x_2)+ \; ?$

→ $f(x) = x+1$

$f(x_1+x_2) = x_1 + x_2 + 1 + 1 - 1$

$= f(x_1)+f(x_2)-1$ → nonlinear function

→ $f_1(x)$ $f_2(x)$

$$\frac{d}{dx}\left(f_1(x)+f_2(x)\right) = \frac{d}{dx} f_1(x) + \frac{d}{dx} f_2(x) = \frac{d}{dx}\left(\alpha . f_1(x)\right) = \alpha . f_1(x)$$

→ $\underset{\text{constant coefficient}}{a} \cdot \underset{\text{unknown variable}}{x} = b \longrightarrow$ constant number

$x_1\ x_2\ \cdots\ x_n$ ⓜ

$a_1\ a_2\ \cdots\ a_n$

$a_1 x_1 + a_2 x_2 + \cdots a_n x_n = b_1$

$a_{1n} x_1 + a_{22} x_2 + \cdots + a_n x_n = b_1$

$a_{2n} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2$

$a_{m_1} x_1 + a_{m_2} x_2 + \cdots + a_{mn} x_n = b_m$

Matrix (Rectangular array of numbers)

→ $$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_n \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & a_{ij} & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$m \cdot 1$

row: Horizontal lines

column: Vertical lines

m rows

n columns

Dimension of a matrix is defined by $m \times n$

a_{ij} = the element of the i^{th} row and j^{th} column

$i = 1, 2, \ldots m$

$j = 1, 2, \ldots n$

→ $A = (a_{ij})_{m \times n}$

$$B = (b_{ij})_{2\times 3} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

→ $$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & & \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \end{bmatrix} \to$ row vector

$\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{n1} \end{bmatrix} \to$ column vector

Vectors

→ $x_1\ x_2\ x_3\ \cdots\ x_n$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

T → Transpose

$y^T = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}$

$$y^T z = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = y_1 z_1 + y_2 z_2 + \cdots + y_n z_n = \sum_{i=1}^{n} y_i \cdot z_i$$

→ $$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad a_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \ a_2 = \begin{bmatrix} a_{21} \\ \vdots \\ a_{2m} \end{bmatrix}, \ a_m = \begin{bmatrix} a_{m1} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1 \implies a_1^T \cdot x = b_1 \quad , a_2^T \cdot x_1 = b_2 \ , \ a_m^T x = b_m$

$a_1^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{m1} \end{bmatrix}$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$a_1^T \cdot X = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = a_{11} x_1 + a_{12} x_2 + \cdots + a_{1m} x_n$$

→ $b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} a_1^T \cdot x \\ a_2^T \cdot x \\ \vdots \\ a_m^T x \end{bmatrix} = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix} \cdot x$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \Rightarrow A \cdot x = b$$

→ $A \cdot x = b$, $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

First Equation $\overset{\alpha}{=} (a_{11} x_1 + a_{12} x_2 + \cdots + a_{1m} x_n) = b_1 . \alpha$

$a_{i1} x_1 + a_{i2} x_2 + \cdots \; a_{im} x_n = b_i$

$i = 1, 2, 3 \ldots m$

Elementary Row Operations

1-) Multiplyin any equations by a real constant number

2-) interchanging two equations

Augmented Matrix (Arttırılmış)

$$\hat{A} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

A b

$\hat{A} = [A \vdots b]$

Unity or identity

$$I_{n \times n} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & & 1 \end{bmatrix}$$

$m = n$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & \hat{b}_1 \\ 0 & 1 & & & \hat{b}_2 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 0\,1 & \hat{b}_m \end{array}\right]$$

$x_1 = \hat{b}_1$, $x_2 = \hat{b}_2$

→ $x_1 + x_2 + x_3 = 6$

$2x_1 - x_2 - 2x_3 = -6$

$x_1 + 3x_2 - x_3 = 4$

$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -2 \\ 1 & 3 & -1 \end{bmatrix}$ $b = \begin{bmatrix} 6 \\ -6 \\ 4 \end{bmatrix}$

$\hat{A} = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & -2 & -6 \\ 1 & 3 & -1 & 4 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & -4 & -18 \\ 1 & 3 & -1 & 4 \end{array}\right]$ -2 ile çarpıp 2.satır ile topla

$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & -4 & -18 \\ 0 & 2 & -2 & -2 \end{array}\right]$ -1 ile çarpıp 3.satır ile topla

$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 2 & -2 & -2 \\ 0 & -3 & -4 & -18 \end{array}\right]$ 2.satır ile 3.satır yer değiştirilir.

$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -1 & -1 \\ 0 & -3 & -4 & -18 \end{array}\right]$ 2.satır ikiye bölünür

$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -7 & -21 \end{array}\right]$ 2.satır 3 ile çarpılıp 3.satır ile toplanması

$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$ 3.satır -7'ye bölünür (Önce Alt sıfırlanır)

$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$ 3.satır 2.satıra eklenir.

$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$x_1 = 1$

$x_2 = 2$

$x_3 = 3$

3.Satırı -1 ile çarp 1'ye ilave et   2.satır -1 ile çarp 1.satıra eklenir.

Gauss Elimination Method

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Case-I - $\underline{m=n}$ → The solution is unique.

$$\hat{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & & & & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & & \cdots & a_{mn} & b_n \end{bmatrix} \Rightarrow \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} & \beta_1 \\ 0 & & & & \beta_2 \\ 0 & & & & \vdots \\ 0 & 0 & \cdots & \alpha_{nn} & \beta_n \end{bmatrix} \quad \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ 0 & \alpha_{22} & & \\ \vdots & & \alpha_{33} & \\ 0 & \cdots & 0 & \alpha_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}$$

$\alpha_{nn} \cdot x_n = \beta_n \Rightarrow x_n = \dfrac{\beta_n}{\alpha_{nn}}$ (Satır * Satır = 1 Satır ?)

Ex:

$2x_1 + 3x_2 - 2x_3 + 4x_4 = 18$

$5x_1 - 3x_2 + 4x_3 - 2x_4 = 3$

$3x_1 - 2x_3 - x_3 + 2x_4 = 4$

$-2x_1 + 4x_2 + 3x_3 - x_4 = 11$

$$\hat{A} = \begin{bmatrix} 2 & 3 & -2 & 4 & 18 \\ 5 & -3 & 4 & -2 & 3 \\ 3 & -2 & -1 & 2 & 4 \\ -2 & 4 & 3 & -1 & 11 \end{bmatrix}$$

Sol:

1. Satır -5/2 ile çarpılarak 2. Satır ile toplanması
$$= \begin{bmatrix} 2 & 3 & -2 & 4 & 18 \\ 0 & -21/2 & 9 & -12 & -42 \\ 3 & -2 & -1 & 2 & 4 \\ -2 & 4 & 3 & -1 & 11 \end{bmatrix} \Rightarrow$$

1. Satır -3/2 ile çarpılıp, 3. Satıra eklenir
$$= \begin{bmatrix} 2 & 3 & -2 & 4 & 18 \\ 0 & -21/2 & 9 & -12 & -42 \\ 0 & -13/2 & 2 & -4 & -23 \\ -2 & 4 & 3 & -1 & 11 \end{bmatrix} \Rightarrow$$

1. Satır ile 4. Satır toplanır
$$= \begin{bmatrix} 2 & 3 & -2 & 4 & 18 \\ 0 & -21/2 & 9 & -12 & -42 \\ 0 & -13/2 & 2 & -4 & -23 \\ 0 & 7 & 1 & 3 & 29 \end{bmatrix} \Rightarrow$$

2. Satır $\frac{-13}{21}$ ile çarpılıp 3. satıra ilave edilir
$$= \begin{bmatrix} 2 & 3 & -2 & 4 & 18 \\ 0 & -21/2 & 9 & -12 & -42 \\ 0 & 0 & \frac{-25}{7} & \frac{24}{7} & 3 \\ 0 & 7 & 1 & 3 & 29 \end{bmatrix} \Rightarrow$$

2. Satır 2/3 ile çarpılıp 4. Satıra eklenir
$$= \begin{bmatrix} 2 & 3 & -2 & 4 & 18 \\ 0 & -21/2 & 9 & -12 & -42 \\ 0 & 0 & \frac{-25}{7} & \frac{24}{7} & 3 \\ 0 & 0 & 7 & -5 & 1 \end{bmatrix} \Rightarrow$$

3. Satır $\frac{49}{25}$ ile çarpılıp 4. Satırla toplanır
$$= \begin{bmatrix} 2 & 3 & -2 & 4 & 18 \\ 0 & -21/2 & 9 & -12 & -42 \\ 0 & 0 & \frac{-25}{7} & \frac{24}{7} & 3 \\ 0 & 0 & 0 & \frac{43}{25} & \frac{172}{25} \end{bmatrix}$$

$\Rightarrow \frac{43}{25} \cdot x_4 = \frac{172}{25} \Rightarrow \boxed{x_4 = 4}$

$\frac{-25}{7} \cdot x_3 + \frac{24}{7} \cdot 4 = 3$

$-25 \cdot x_3 + 96 = 21 \Rightarrow$

$\boxed{x_3 = 3}$

$\frac{-21}{2} x_2 + 9 \cdot 3 + 12 \cdot 4 = -42$

$\boxed{x_2 = 2}$

$2 \cdot x_1 + 3 \cdot 2 - 2 \cdot 3 + 4 \cdot 4 = 18$

$\boxed{x_1 = 1}$

Ex:

$$x_1+2x_2+2x_3-x_4=4$$
$$-x_1+x_2+4x_3-2x_4=8$$
$$2x_1-2x_2-x_3+x_4=8$$
$$-4x_1-12x_2-2x_3+2x_4=16$$

$$\hat{A}=\begin{bmatrix}1&2&2&-1&4\\-1&1&4&-2&8\\2&-2&-1&1&8\\-4&-12&-2&2&16\end{bmatrix}$$

Sol:

1. Satır 2. Satır ile toplanır

$$=\begin{bmatrix}1&2&2&-1&4\\0&3&6&-3&12\\2&-2&-1&1&8\\-4&-12&-2&2&16\end{bmatrix}$$

$\Rightarrow$ 1. Satır -2 ile çarpılır 3. Satıra eklenir

$$=\begin{bmatrix}1&2&2&-1&4\\0&3&6&-3&12\\0&-6&-5&3&0\\-4&-12&-2&2&16\end{bmatrix}$$

$\Rightarrow$ 1 Satır 4 ile çarpılıp 4. Satıra eklenir

$$=\begin{bmatrix}1&2&2&-1&4\\0&3&6&-3&12\\0&-6&-5&3&0\\0&-4&6&-2&32\end{bmatrix}\Rightarrow$$

2. Satır 2 ile çarpılır 3. Satıra eklenir

$$=\begin{bmatrix}1&2&2&-1&4\\0&3&6&-3&12\\0&0&7&-3&24\\0&-4&6&-2&32\end{bmatrix}$$

$\Rightarrow$ 2 Satır $\frac{4}{3}$ ile çarpılır 4. Satıra eklenir

$$=\begin{bmatrix}1&2&2&-1&4\\0&3&6&-3&12\\0&0&7&-3&24\\0&0&14&-6&48\end{bmatrix}$$

$\Rightarrow$ 3 Satır -2 ile çarpılıp 4. Satıra eklenir

$$=\begin{bmatrix}1&2&2&-1&4\\0&3&6&-3&12\\0&0&7&-3&24\\0&0&0&0&0\end{bmatrix}$$

free variable $\boxed{x_4=\alpha}$

$$0.x_1+0.x_2+0.x_3+0.x_4=0$$
$$7.x_3-3x_4=24$$
$$7.x_3-3\alpha=24\Rightarrow\boxed{x_3=\frac{24+3\alpha}{7}}$$

$$3x_2+6.\frac{(24+3\alpha)}{7}-3\alpha=12$$
$$\boxed{x_2=\frac{\alpha-20}{7}}$$

$$x_1+2.\frac{(\alpha-20)}{7}+2.\frac{(24+3\alpha)}{7}-\alpha=4$$
$$\boxed{x_1=\frac{20-\alpha}{7}}$$

Serbest değişkenlerin sayısı 0 olan satırların sayıların kadardır.

Ex:

$$x_1+x_2+x_3-x_4=1$$
$$2x_1-2x_2+2x_3-2x_4=-2$$
$$2x_1+2x_2+2x_3-2x_4=2$$
$$-x_1+x_2-x_3+x_4=1$$

$$\hat{A}=\begin{bmatrix}1&1&1&-1&1\\2&-2&2&-2&-2\\2&2&2&-2&2\\-1&1&-1&1&1\end{bmatrix}$$

Sol: 1. Satır -2 ile çarpılıp 2. Satıra eklenir

$$=\begin{bmatrix}1&1&1&-1&1\\0&-4&0&0&-4\\2&2&2&-2&2\\-1&1&-1&1&1\end{bmatrix}$$

$\Rightarrow$ 1. Satır -2 ile çarpılıp 3. Satıra eklenir

$$=\begin{bmatrix}1&1&1&-1&1\\0&-4&0&0&-4\\0&0&0&0&0\\-1&1&-1&1&1\end{bmatrix}$$

$\Rightarrow$ 1. Satır 4. Satır ile toplanır

$$=\begin{bmatrix}1&1&1&-1&1\\0&-4&0&0&-4\\0&0&0&0&0\\0&2&0&0&2\end{bmatrix}\Rightarrow$$

2. Satır 2'ye böl 4 Satır ile topla

$$=\begin{bmatrix}1&1&1&-1&1\\0&-4&0&0&-4\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}\Rightarrow\boxed{x_3=\alpha}\ \boxed{x_4=\beta}$$

$$-4x_2=-4$$
$$\boxed{x_2=1}$$

$$x_1+1+\alpha-\beta=1$$
$$\boxed{x_1=\beta-\alpha}$$

Homogenuous systems

$$a_{11} \cdot x_1 + a_{12}\, x_2 + \dots + a_{1n} \cdot x_n = 0$$

$$a_{m1} \cdot x_2 + \dots + a_{mn} \cdot x_n = 0$$

$A.x = 0$

consistent

$x_1 = x_2 = x_3 = \dots = x_n = 0$

$$\left.\begin{aligned} x_1 + x_2 &= 1 \\ x_1 + x_2 &= 2 \\ -x_1 - x_2 &= -2 \end{aligned}\right\} 0 = -1$$

Ex:

$$x_1 + 2x_2 + 2x_3 - x_4 = 8$$
$$-x_1 + x_2 + 4x_3 - 2x_4 = 8$$
$$2x_1 - 2x_2 - x_3 + x_4 = 12$$
$$-4x_1 - 12x_2 - 2x_3 + 2x_4 - 16$$

$$A = \begin{bmatrix} 1 & 2 & 2 & -1 & 8 \\ -1 & 1 & 4 & -2 & 8 \\ 2 & -2 & -1 & 1 & 12 \\ -4 & -12 & -2 & 2 & 16 \end{bmatrix}$$

$$-4x_1 + 4x_2 + 2x_3 - 2x_4 = -24$$
$$-8x_1 - 8x_2 + 0 + 0 = -8$$

$x_1 + x_2 = 1$

$$-2x_1 - 4x_2 - 4x_3 + 2x_4 = -16$$
$$-3x_1 - 3x_2 + 0 + 0 = -8$$

$x_1 + x_2 = \frac{-8}{-3}$

Vectors

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

$\alpha \in \mathbb{R}, \beta \in \mathbb{R}$

Column Vector

① $x + y = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix}$

② $x+y = y+x$ (Commutativity)

③ $x + (y+z) = (x+y) + z$ (Associativity)

④ $\alpha \cdot x = \begin{bmatrix} \alpha \cdot x_1 \\ \alpha \cdot x_2 \\ \vdots \\ \alpha \cdot x_n \end{bmatrix}$

⑤ $(\alpha + \beta).x = \alpha.x + \beta.x$

⑥ $\alpha(x+y) = \alpha.x + \alpha.y$

⑦ <u>Inner Product</u> $x, y \quad f(x,y)$

i-) $f(x,y) = f(y,x)$

ii-) $f(x+y, z) = f(x,z) + f(y,z)$

iii-) $f(\alpha x, y) = \alpha . f(x,y)$

<u>Dot Product</u>

✓ i-) $f(x,y) = x^T . y = [x_1, x_2, \dots, x_n] . \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^{n} x_i . y_i = \boxed{x_1 . y_1 + \dots + x_n y_n}$

$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \; y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \Rightarrow f(y,x) = y^T . x = [y_1 \dots y_n] . \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \boxed{y_1 x_1 \dots y_n x_n}$

✓ ii-) $f(x+y, z) = (x+y)^T . z$

$$= \begin{bmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{bmatrix}^T . \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = [x_1+y_1, \dots, x_n+y_n] . \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \Rightarrow$$

$$= \sum_{i=1}^{n} (x_i + y_i) . z_i \Rightarrow$$

$$= \sum_{i=1}^{n} x_i . z_i + \sum_{i=1}^{n} y_i . z_i = x^T . z + y^T . z = f(x,z) + f(y,z)$$

✓ iii-) $f(\alpha x, y) = ((\alpha . x)^T, y) = \alpha . x^T . y + f(x,y) = \alpha . f(x,y)$

Matrices

$$A_{(a_{ij})_{n\times n}} = \begin{bmatrix} a_{11} & & \cdots & a_{1n} \\ & a_{22} & & \vdots \\ \vdots & & \ddots & \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}$$

Square Matrix

$i = j$

$a_{ii} \rightarrow$ diagonal elements

$a_{11} \; a_{22} \cdots a_{nn}$

a_{ij}, $i \neq j \rightarrow$ off-diagonal elements

$A = (a_{ij})_{n\times n}$

$B = (b_{ij})_{n\times n}$

$C = A + B$

$(c_{ij})_{n\times n} = (a_{ij})_{n\times n} + (b_{ij})_{n\times n}$

$c_{ij} = a_{ij} + b_{ij} \quad , i,j = 1,2,\ldots,n$

$C = A - B \Rightarrow c_{ij} = a_{ij} - b_{ij} \quad , i,j = 1,2,\ldots,n$

$C = \alpha . A \Rightarrow c_{ij} = \alpha . a_{ij} \quad i,j = 1,2,\ldots,n$

$$C = A.B = \overset{A}{\begin{bmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mp} \end{bmatrix}} \overset{B}{\begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{p1} & \cdots & b_{pn} \end{bmatrix}}$$

m rows, p columns

p rows, n columns

$a_i = \begin{bmatrix} a_{i1} \\ \vdots \\ a_{ip} \end{bmatrix} \quad i = 1,2,\ldots,n$

$b_j = \begin{bmatrix} b_{1j} \\ \vdots \\ b_{pj} \end{bmatrix} \quad j = 1,2,\ldots,n$

Columns of the first matrix must be equal to Rows of the second matrix.

$$c_{ij} = a_i^T . b_j = \begin{bmatrix} a_{i1} \cdots a_{ip} \end{bmatrix} . \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{pj} \end{bmatrix}$$

$$c_{11} = \begin{bmatrix} a_{11} & \cdots & a_{1p} \end{bmatrix} \begin{bmatrix} b_{11} \\ \vdots \\ b_{p1} \end{bmatrix} = a_{11}.b_{11} + \ldots + a_{1p}.b_{p1} = c_{11}$$

* Let $A = (a_{ij})_{n\times n}$ be a square matrix

A is said to be symmetric iff $A = A^T$, that is, $a_{ij} = a_{ji}$, $\forall i,j$

Diagonal Matrices

$$D = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & & \vdots \\ 0 & & \ddots & 0 \\ 0 & \cdots & 0 & d_{nn} \end{bmatrix}$$

Unity or Identity Matrix

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 1 \end{bmatrix}$$

* **Inverse Matrix**

$A^{-1}.A = I$

If A is invertable, then A is a non-singular matrix.

Otherwise

A is a singular matrix.

PROPERTIES

$A+B=B+A$	$(\alpha+\beta)A=\alpha A+\beta A$	$(AB)^{-1}=B^{-1}A^{-1}$	$(A^T)^{-1}=(A^{-1})^T$
$A+(B+C)=(A+B)+C$	$\alpha(A+B)=\alpha.A+\alpha.B$	$(AB)^T=B^TA^T$	$A^k=\underbrace{A.A.\_\_A}_{k}$
$A(BC)=(AB)C$	$\alpha(BA)=(\alpha\beta)A$	$(\alpha A)^{-1}=\alpha^{-1}A^{-1}$	
$A(B+C)=AB+AC$	$\alpha(AB)=(\alpha A)B$	$(A+B)^T=A^T+B^T$	$(\alpha A)^k=\alpha^kA^k$ (if A is invertable)
$(B+C)A=BA+CA$		$(\alpha A)^T=\alpha.A^T$	$(A+B)(A+B)=A^2+AB+BA+B^2$

Ex:

$$A=\begin{bmatrix}1&2\\3&4\end{bmatrix}\quad B=\begin{bmatrix}1&1\\1&2\end{bmatrix}\Rightarrow A.B=\begin{bmatrix}3&5\\7&11\end{bmatrix}$$

$A=(a_{ij})_{n\times n}$

$x=[x_1 \dots x_n]$ $\qquad Ax=b$

$b=[b_1 \dots b_n]^T$

$\underline{A^{-1}.A}.x=A^{-1}.b$

$I.x=A^{-1}.b$

$x=A^{-1}.b$

(Tek Çözüm)

- Matrix x Vector = Vector
- If A is invertable, Only one solution for all vector
  is non- , infinite solution
- $i=j$ and a_{ij} line is diagonal

Upper Triangular Matrices

$$A=\begin{bmatrix}a_{11}&a_{12}&a_{13}&a_{14}\\0&a_{22}&a_{23}&a_{24}\\0&0&a_{33}&a_{34}\\0&0&0&a_{44}\end{bmatrix}$$

Lower Triangular Matrices

$$A=\begin{bmatrix}a_{11}&0&0&0\\a_{21}&a_{22}&0&0\\a_{31}&a_{32}&a_{33}&0\\a_{41}&a_{42}&a_{43}&a_{44}\end{bmatrix}$$

Ex:

$$A=\begin{bmatrix}1&2&2&-1\\-1&1&4&-2\\2&-2&-1&1\\-4&-8&-2&2\end{bmatrix}\qquad\begin{bmatrix}1&2&2&-1\\0&3&6&-3\\0&0&7&-3\\0&0&0&\frac{4}{7}\end{bmatrix}\ 12\qquad\begin{bmatrix}2&0&0&0\\-3&-3&0&0\\5&4&-1&0\\-4&-8&-2&2\end{bmatrix}\ 12$$

DETERMINANTS

$$A=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}\Rightarrow\begin{bmatrix}a_{11}&a_{12}\\0&a_{22}-a_{12}\cdot\frac{a_{21}}{a_{11}}\end{bmatrix}\qquad \det(A)=a_{11}\left(a_{22}-\frac{a_{12}a_{21}}{a_{11}}\right)$$

$$=a_{11}a_{22}-a_{12}a_{21}$$

$$A=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}\Rightarrow\begin{bmatrix}a_{11}&a_{12}&a_{13}\\0&\frac{a_{11}a_{22}-a_{12}a_{21}}{a_{11}}&\frac{a_{11}a_{13}-a_{13}a_{21}}{a_{11}}\\a_{31}&a_{32}&a_{33}\end{bmatrix}\Rightarrow\begin{bmatrix}a_{11}&a_{12}&a_{13}\\0&\frac{a_{11}a_{22}-a_{12}a_{21}}{a_{11}}&\frac{a_{11}a_{23}-a_{13}a_{21}}{a_{11}}\\0&\frac{a_{11}a_{32}-a_{12}a_{31}}{a_{11}}&\frac{a_{11}a_{33}-a_{13}a_{31}}{a_{11}}\end{bmatrix}$$

$\frac{-a_{21}}{a_{11}}\qquad\frac{-a_{31}}{a_{11}}\qquad\Rightarrow 0$

$$=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\0&\frac{a_{11}a_{22}-a_{12}a_{21}}{a_{11}}&\frac{a_{11}a_{23}-a_{13}a_{21}}{a_{11}}\\0&0&\frac{a_{11}a_{33}-a_{13}a_{31}}{a_{11}}+\frac{a_{11}a_{23}-a_{13}a_{21}}{a_{11}}\cdot\frac{a_{12}a_{31}-a_{11}a_{32}}{a_{11}a_{22}-a_{12}a_{21}}\end{bmatrix}\Rightarrow \det(A)=a_{11}a_{22}a_{33}-a_{11}a_{23}a_{32}-a_{22}a_{13}a_{31}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}$$

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$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Minors and Cofactors

m_{ij} $\quad c_{ij} = (-1)^{i+j} m_{ij}$

$$* \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \det(A)$$

$m_{11} = a_{22} \quad m_{21} = a_{12} \Rightarrow c_{11} = a_{22} \quad c_{21} = -a_{12}$

$m_{12} = a_{21} \quad m_{22} = a_{11} \qquad c_{12} = -a_{21} \quad c_{22} = a_{11}$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}$$

$a_{11} \cdot c_{11} + a_{12} \cdot c_{12} = a_{11}a_{22} - a_{12}a_{21} = \det(A)$ First Row

$a_{11}a_{22} - a_{12}a_{21}$ First Column

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$A = [a_n]$

Sub-matrix

$m_{11} \quad m_{12} \quad a_{13} \qquad c_{11} \quad c_{12} \quad c_{13}$

$$m_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22} \cdot a_{33} - a_{32}a_{23} \qquad a_{11} \cdot c_{11} = a_{22} \cdot a_{33} - a_{32}a_{23}$$

$$m_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{31}a_{23} \qquad a_{12} \cdot c_{12} = a_{31}a_{23} - a_{21}a_{33}$$

$$m_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22} \qquad a_{13} \cdot c_{13} = a_{21}a_{32} - a_{31}a_{22}$$

$$\det(A) = a_{11} \cdot a_{22} \cdot a_{33} - a_{11} \cdot a_{32} \cdot a_{23} + a_{12}a_{31}a_{23} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} \Rightarrow \text{First Row}$$

$$\det(A) = a_{1j}c_{1j} + a_{2j}c_{2j} + \dots + a_{nj}c_{nj} = \sum_{i=1}^{n} a_{ij} \cdot c_{ij} \quad (j.\text{column})$$

$$\det(A) = a_{i1}c_{i1} + a_{i2}c_{i2} + \dots + a_{in}c_{in} = \sum_{j=1}^{n} a_{ij}c_{ij} \quad (i.\text{ row})$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$m_{11} = a_{22} \qquad c_{11} = a_{22}$

$m_{12} = -a_{21} \qquad c_{12} = -a_{21}$

$m_{21} = -a_{12} \qquad c_{21} = -a_{12}$

$m_{22} = a_{11} \qquad c_{22} = a_{11}$

$$C = \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}$$

$a_{11}(-a_{12}) + a_{12} \cdot a_{11} = 0$ (1. First Row 2. Second Row)

$a_{12} \cdot a_{22} - a_{22} \cdot a_{12} = 0$

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \quad C^T = \begin{bmatrix} c_{11} & c_{21} & \cdots & c_{n1} \\ c_{12} & & & \vdots \\ \vdots & & & \\ c_{1n} & \cdots & & c_{nn} \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} \end{bmatrix}$$

$$a_i = \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix}, \quad c_j = \begin{bmatrix} c_{j1} \\ c_{j2} \\ \vdots \\ c_{jn} \end{bmatrix}, \quad A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{bmatrix}, \quad C^T = [c_1 c_2 \dots c_n], \quad A.C^T = \begin{bmatrix} a_1^T c_1 & a_1^T c_2 & \dots & a_1^T c_n \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ a_n^T c_n & \dots & & a_n^T c_n \end{bmatrix} = \begin{bmatrix} \det(A) & 0 & 0 & \dots & 0 \\ 0 & \det(A) & & & 0 \\ 0 & & & & \vdots \\ \vdots & & & & 0 \\ 0 & \cdots & & & \det(A) \end{bmatrix}$$

$i = 1, 2, \dots, n \qquad j = 1, 2, \dots, n$

$a_i^T \cdot c_j = 0, \; i \neq j$

$C^T = adj(A)$ $\quad$ adj(A) = adjoint of A

$$A.C^T = det(A).I \Rightarrow \underbrace{A^{-1}.A}_{I}.C^T = det(A).A^{-1} \Rightarrow C^T.I = det(A).A^{-1} \Rightarrow \boxed{A^{-1} = \frac{1}{det(A)}.C^T}$$

→ $A.x = b$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$A.x = b$
$X = A^{-1}.b$
$= \frac{1}{det(A)}.C^T.b = \begin{bmatrix} b_1c_{11} + b_2c_{21} + \ldots + b_nc_{n1} \\ b_1c_{12} + b_2c_{22} + \ldots + b_nc_{n2} \\ \vdots \\ b_1c_{1n} + b_2c_{2n} \cdots + b_nc_{nn} \end{bmatrix}$ = CRAMER'S RULE

Ex: $x_1 + x_2 = 3$
$x_1 - x_2 = 1$

$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $\quad b = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ $\quad det(A) = -2$ $\quad$ *Δ = delta

$x_1 = 2, x_2 = 1$

$\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow -4 \quad \frac{-4}{-2} = x_1$ $\qquad \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \Rightarrow \frac{-2}{-2} = x_2$

$$x_j = \frac{1}{det(A)} . \underbrace{\boxed{\sum_{i=1}^{n} b.c_{ij}}}_{\Delta_j(A)}, \; j = 1,2 \ldots, n \qquad x_j = \frac{\Delta_j(A)}{\Delta(A)}$$

$det(AB) = det(A)det(B)$

$det(A^{-1}) = \frac{1}{det(A)} \Rightarrow det(AA^{-1}) = 1 \Rightarrow det(A)\,det(A^{-1}) = 1$

$det(A^T) = det(A)$ $\qquad det(A^k) = (det(A))^k \Rightarrow det(A.A.A)$

$det(A^{-1}BA) = det(B)$

$det(\alpha A) = \alpha^n.det(A)$

$det(-A) = (-1)^n.det(A)$

* Bir matrisin herhangi bir satır ya da sütununun bütün elemanları sıfırsa determinant 0'dır ✓
* Eğer herhangi iki satır ya da sütun elemanları oranı aynıysa determinant 0'dır. ✓

Exam Q: $\begin{bmatrix} \alpha & \beta & \gamma \\ \alpha & \beta & \gamma \\ \alpha & \beta & \gamma \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $\quad$ $a = b = c$ Inconsistent No solution

Principal Minors

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \Rightarrow \begin{bmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{bmatrix}$$

1.Order principal sub matrices

$[a_{11}]$ $[a_{22}]$ $[a_{33}]$ $[a_{44}]$

134 124 123 234

2.Order

$$A_{12} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \underbrace{\begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}}_{= A_{13}}, \underbrace{\begin{bmatrix} a_{11} & a_{14} \\ a_{41} & a_{44} \end{bmatrix}}_{= A_{14}}, A_{23} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}, A_{24} = \begin{bmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{bmatrix}, A_{34} = \begin{bmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{bmatrix}$$

$$C(n,r) = \frac{n!}{(n-r)!.r!} \Rightarrow \frac{4!}{1!.3!} = 4$$

$$A_{123} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

A_{124}

A_{234}

A_{134}

Leading Principal Minors

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \quad [a_{11}] \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \dots$$

Vector Space

F: Real or Complex Number

$x_1, x_2, \dots x_n \in F^n$, $\alpha_1, \alpha_2, \dots \alpha_n \in F^n$

$$y = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n = \sum_{i=1}^{n} \alpha_i x_i$$

$S = [x_1, x_2, \dots, x_n]$

$Y = \text{span}(S)$

$S = \{x_1, x_2, \dots, x_n\}$ Linearly Independent

$\alpha_1 x_1 + \dots + \alpha_n x_n = 0$

$\alpha_1 = \alpha_2 = \alpha_3 = \dots = \alpha_n = 0$

Ex: $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $x_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\alpha_1 x_1 + \alpha_2 x_2 = 0$

$$\alpha_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\alpha_1 + \alpha_2 = 0$ $\quad \alpha_1 = 0$

$\alpha_1 + 2\alpha_2 = 0$ $\quad \alpha_2 = 0$

$S = \{a_1, a_2, \dots, a_n\}$

$\alpha_1, \alpha_2, \dots, \alpha_n$

$$a_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix}, i = 1, 2, \dots, n$$

$$\alpha_1 . a_1 + \alpha_2 a_2 + \dots + \alpha_n a_n = \alpha_1 . \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + \dots + \alpha_n . \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$A.\alpha = 0$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{m1} & \dots & & a_{mn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

(Değişken)
Bilinmeyen > Denklem

m,n

1-) $n > m$ $\downarrow$ Linearly Dependent

$x_1 + x_2 + x_3 = 1$ Keyfi değişken

$x_1 + x_2 - x_3 = 4$

$n = 3, m = 2$

r: 0'dan farklı satır sayısı

2-) $n = m$ $\quad B = 0$ Çözüm Tek Olursa $\alpha = 0$ $\quad n = r$ ve $m = n$'dir

3-) $n < m$ $\quad B >$ Elem. Row Operation $m - n$ kadar 0 olacak

Linearly Independent , Örnek bul ve çöz.

Tersi varsa $Det(A) \neq 0$