Let $A_1, A_2, \ldots, A_n$ be n sets. The cartesian product of $A_1, A_2, \ldots A_n$ x the set of all n-tuples

$$A_1 \times A_2 \times \ldots \times A_n = \{a_1, a_2, \ldots, a_n\} \;\Big|\; \begin{array}{l} a_1 \in A_1 \\ a_2 \in A_2 \\ \vdots \\ a_n \in A_n \end{array}$$

In particular if $A_1 = A_2 = \ldots = A_n$

$$A \times A \times \ldots \times A = A^n = \{(a_1, a_2, \ldots, a_n) \mid a_1, a_2, \ldots, a_n \in A\}$$

RELATIONS

Def: Let A and B be two non-empty sets.

A "relation" from A to B.

is just a non-empty subset of $A \times B$

Mathematically, a relation R is $R \subseteq A \times B$

R consits of order pairs.

If $(a,b) \in R$, we say a is related to (via R) and we denote this by "aRb" $\to$ a is related to b or $(a,b) \in R$

Ex: $A = \{1,2,3\}$

$B = \{a,b,c,d\}$ there are some examples of relations from A to B.

$R_1 := \{(1,c)\}$ $\to 1R_1c$, $1 \not R d$

$R_2 := \{(1,a),(1,b),(1,c),(1,d)\}$ $\to 1R_2a$, $1R_2b$, $1R_2c$, $1R_2d$

$R_3 := \{(1,b),(2,a),(3,c),(3,d)\}$ $\to 1R_3b$ ...

$R_4 := A \times B$ (the trivial relation) $\to 2R_4b$, $2R_4c$ ...

Ex: When we say R is a relation on A, we mean, R is a relation from A to A, i.e., $R \subseteq A \times A$

$A = \{1,2,3\}$

$B = \{(1,1),(2,2),(3,3),(1,2),(2,3),(3,1)\}$ $\to 1R_11$, $2R2$, $3R3$, ...

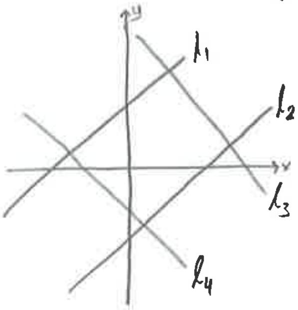
Ex: Let $A = \mathbb{Z}$, Define R on $\mathbb{Z}$

$aRb \iff a \equiv b \pmod 3 \iff 3 \mid (a-b)$ or $a-b$ is multiple of 3.

$1R4$, $2R5$, $-2R1$, $-11R1$

This can be generalized to any $m \geq 2$, so for such an m, $aRb \iff a \equiv b \pmod m$

(4) Suppose a is the set of all lines in the real plane.



Define a relation on A:

- $l_1 \parallel l_2 \iff l_1 R_1 l_2$ [$m_1 = m_2$, i.e, they have the same slope]
- $l_1 R_2 l_2 \iff l_1 \perp l_2$ [$m_1 \cdot m_2 = -1$]

(5) For some $n \geq 1$,

let $A = \mathbb{P}[\{1,2,\dots,n\}] = \{S \mid S \subseteq \{1,2,\dots,n\}\} = \{\emptyset, \{1\}, \{2\}, \dots, \{1,2,\dots,n\}\}$

Define :

- $S R_1 T \iff S = T$, $\{1\} R_1 \{1\}$, $\{1,2\} R_1 \{1,2\}$, $\{1\} \not R_1 \{2\}$
- $S R_2 T \iff S \subseteq T$, $\{1\} R_2 \{1\}$, $\{1\} R_2 \{1,2\}$, $\{1\} \not R_2 \{2\}$
- $S R_3 T \iff S \cap T = \emptyset$, $\{1\} R_3 \{2\}$, $\{1\} \not R_3 \{1\}$, $\{1\} \not R_3 \{1,2\}$
- $S R_3 T \iff |S \setminus T| = 1$, $\{1\} R_4 \{2\}$, $\{2\} \not R_4 \{2\}$, $\{2\} \not R_4 \{1,2\}$, $\{1,2\} R_4 \{2\}$

(6) Let $A = \mathbb{R}$ → The set of real numbers

$R_1 : a R_1 b \iff a \leq b$ $\quad$ $1 R_1 1$, $2 R_1 3$, $3 \not R_1 2$

$R_2 : a R_2 b \iff a < b$ $\quad$ $1 \not R_2 1$ $\quad$ $2 R_2 3$, $3 \not R_2 2$

(7) Let A → All people in the world.

R_1: $a R_1 b \iff$ a and b have the same eye color.

R_2: $a R_2 b \iff$ a is married to b.

R_3: $a R_3 b \iff$ a is an ancestor of b.

(8) $A = \mathbb{N} = \{1,2,3,\dots\}$

Define R and A, $\quad a R b \iff a / b$ (a divides b , b is divisible by a , b is a multiple of a)

$1 R n$ for all $n \in \mathbb{N}$

$2 R 4$, $3 R 15$, n/n for all $n \in \mathbb{N}$,

$4 \not R 2$, $5 \not R 12$ ...

Types of Relations

1) Reflexive Relations

Let R be a relation on A. R is said to be "reflexive" if aRa for all ($\forall$) $a \in A$

R is reflexive $\iff \forall a \in A$, $(a,a) \in R$

Examples

1) $A = \{1,2,3,4\}$

$R_1 : \{(1,1),(2,2),(3,3),(4,1),(1,3)\}$ → Not REFLEXIVE $(4,4) \notin R_1$

$R_2 : \{(1,1),(2,2)\}$ → Not REFLEXIVE $(3,3) \notin R_2$ $(4,4) \notin R_2$

$R_3 : \{(1,1),(2,2),(3,3),(4,4),(4,1)\}$ → REFLEXIVE

$R_4 : \{(1,2),(2,3),(3,4),(4,1)\}$ → Not REFLEXIVE

2) $A = \mathbb{Z}$, $aRb \iff a \equiv b \pmod 3$ Is R reflexive? Is aRa ?

$a \equiv a \pmod 3$

REFLEXIVE

3) A → All lines in the plane

$R_1 : l_1 R_1 l_2 \iff l_1$ is parallel to l_2 → Reflexive $\quad l R_1 l$ because they both have the same slope.

$R_2 : l_1 R_2 l_2 \iff l_1$ is perpendicular to l_2

R_2 is not reflexive, $l R_2 l \iff m \cdot m = -1 \Rightarrow m^2 = -1$ no solutions in real numbers.

4) $A = P(\{1,2,\dots n\})$

$S R_1 T \iff S = T$ → Reflexive, since $S = S$

$S R_2 T \iff S \subseteq T$ → Reflexive, since $S \subseteq S$ for all S

$S R_3 T \iff S \cap T = \emptyset$ → Not Reflexive $\quad S R_3 S \iff S \cap S = \emptyset \iff S = \emptyset$ so $\emptyset R_3 \emptyset$, $\{1\} R_3 \{1\}$

5) $A = \mathbb{R}$

$R_1 : aR_1 b \iff a \leq b$ → Reflexive, $a \leq a$ for all a.

$R_2 : aR_2 b \iff a < b$ → Not Reflexive, $1 \not< 1$

6) $A = \mathbb{R}^2 = \{(x,y) \mid x,y \in \mathbb{R}\}$

Define $\mathbb{R}$ on A $\quad (x_1,y_1) R (x_2,y_2) \iff x_1^2 + y_1^2 = x_2^2 + y_2^2$ $\quad$ [In other words, two points in $\mathbb{R}^2$ are related $\iff$ their distance to the origin is the same.]

Is this reflexive? $\quad (x,y) R (x,y)$? (same)

$\Updownarrow$

$x^2+y^2 = x^2+y^2$ ✓ (for all $x,y \in \mathbb{R}$)

Thus, R is reflexive.

(7) $A \to$ People in the world.

aR_1b if and only if a and b have the same eye color. REFLEXIVE

aR_2b if a is maried to b. NOT REFLEXIVE

aR_3b if a is taller then b NOT REFLEXIVE

(8) $A = \mathbb{N}$

$aRb \iff a/b$

Since $\forall a \in \mathbb{N}$, a/a , R is REFLEXIVE.

2 Symmetric Relations

Let R be a relation.

We say that R is "symmetric" if $(a,b) \in R \Rightarrow (b,a) \in R$ or $aRb \Rightarrow bRa$ $p \Rightarrow q$

R is not symmetric if $\exists a \neq b$ such that aRb but $b \not R a$.

Examples

(1) $A = \{1,2,3,4\}$

$R_1 = \{(1,2)\}$, Not SYMMETRIC because $1R_12$ but $2 \not R_1 1$

$R_2 = \{(1,2), (2,1)\} \to$ Symmetric

$R_3 = \{(1,1), (2,2), (3,3), (4,4)\} \to$ Symmetric and Reflexive

$R_4 = \{(1,3), (2,3), (3,2), (4,4)\} \to$ Not SYMMETRIC because $1R_43$ but $3 \not R_4 1$

(2) $A = \mathbb{Z}$,

$R: xRy \iff x \equiv y \pmod 3$ SYMMETRIC

$$\Updownarrow$$

$$3 \mid (x-y)$$

$$\Downarrow$$

$$x-y = 3.m \;, m \in \mathbb{Z}$$

$$y-x = 3.(-m) \;, -m \in \mathbb{Z}$$

$$\Downarrow$$

$$y \equiv x \pmod 3$$

(3) $A = \mathbb{P}[\{1,2,\dots,n\}]$

$R_1: SR_1T \iff S \subseteq T$ $\{1\} \subseteq \{1,2\}$ but $\{1,2\} \not\subseteq \{1\}$ NOT SYMMETRIC

$R_2: SR_2T \iff S \cap T \neq \emptyset$ if $S \cap T = \emptyset \Rightarrow T \cap S = \emptyset \Rightarrow$ SYMMETRIC

$R_3: SR_3T \iff |S \setminus T| = 1$, $S = \{1,2\}$ then $|S \setminus T| = 1$, SR_3T

$T = \{1\}$ but $T \setminus S = \emptyset$, $|T \setminus S| = 0$ $T \not R_3 S$ NOT SYMMETRIC

(4) $A = \mathbb{R}^2 = \{(x,y) \mid x,y \in \mathbb{R}\}$

$(x_1,y_1)\,R\,(x_2,y_2) \iff x_1^2+y_1^2 = x_2^2+y_2^2$

$x_1^2+y_1^2 = x_2^2+y_2^2 \Rightarrow x_2^2+y_2^2 = x_1^2+y_1^2$ $\Rightarrow$ if $(x_1,y_1)R(x_2,y_2)$, then $(x_2,y_2)R(x_1,y_1)$ SYMMETRIC

(5) $A \to$ lines in the plane

$l_1 R_1 l_2 \iff l_1 \parallel l_2$

$l_1 R_2 l_2 \iff l_1 \perp l_2$

$\Big\}$ SYMMETRIC

(6) $A \to$ People in the world.

R_1: Having the same eye color. $\to$ SYMMETRIC

R_2: $aR_2b \iff a$ is married to b $\to$ SYMMETRIC

R_3: $aR_3b \iff a$ is taller than b $\to$ NOT SYMMETRIC

(7) $A = \mathbb{N}$

$aRb \iff a/b$

$1/3$ but $3 \nmid 1$ or $2/4$ but $4 \nmid 2$ NOT SYMMETRIC

3 ANTISYMMETRIC RELATIONS

Let R be a relation

R is said to be antisymmetric

if "aRb and $bRa \Rightarrow a=b$"

R is not antisymmetric if $\exists\, a,b$ s.t. aRb and bRa but $a \neq b$

Examples

(1) $A = \mathbb{R}$, $aRb \iff a \leq b$

$a \leq b$ and $b \leq a$ then $a=b$ Thus R is Anti-Symmetric

(2) $A = \mathcal{P}[\{1,2,\dots,n\}]$

$SRT \iff S \subseteq T$ | $S \subseteq T$ and $T \subseteq S \Rightarrow S=T$ so R is Anti-Symmetric

(3) $A = \mathbb{N}$

$aRb \iff a \backslash b$

$\underbrace{a \backslash b}$ and $b \backslash a$ implies $a=b$ Anti-Symmetric

$\Downarrow$

$b = a.k$ $a = b.m$

$b = bk.m \Rightarrow k.m = 1 \iff k = m = 1$

Remark: The concept of symmetric and anti-symmetric are not mutually exclusive.

Example

$A = \{1,2,3,4\}$

$R_1 = \{(1,1),(2,2)\}$ → Symmetric and Antisymmetric

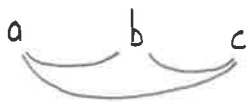
$R_2 = \{(1,2),(2,1)\}$ → Symmetric, NOT Anti-symmetric $1R_2 2$, $2R_2 1$ but $1 \neq 2$

$R_3 = \{(1,2),(2,1),(2,3)\}$ → NOT symmetric, NOT Antisymmetric

$R_4 = \{(1,2)\}$ → NOT Symmetric, Anti-symmetric

(4) Transitive Relations

Let R b l a relation R is said to be "transitive", if $\underbrace{aRb}_{p}$ and $\underbrace{bRc}_{q}$ implies $\underbrace{aRC}_{r} \rightarrow \underbrace{(p \wedge q)}_{1} \Rightarrow \underbrace{r}_{0}$



When is a relation not transitive

if $\exists$ a,b,c such that aRb and bRC but $a \not R c$

Examples

(1) $A = \{1,2,3,4\}$

$R_1 = \{(1,1),(2,2),(3,3)\}$ → Transitive

$R_1 = \{(1,1),(2,2),(3,3),(4,4)\}$
Reflexive
Symmetric
Anti-Symmetric
Transitive

$R_2 = \{(1,2),(2,1)\}$ → $1R2$, $2R1$ but $1R_1$ → Not Transitive

$R_3 = \{(1,2),(2,1),(1,1)\}$ → $2R1$, $1R2$ but $1 \not R 2$ → Not Transitive

$R_4 = \{(1,3),(2,3)\}$ → Transitive

$R_5 = \{(1,2),(2,3),(1,3)\}$ → Transitive

(2) $A = \mathbb{Z}$,

$aRb \iff a \equiv b \pmod 7$

$aRb \Rightarrow a-b = 7k$

$bRc \Rightarrow b-c = 7m$

$a-c = 7(k+m)$

So aRc → Transitive

(3) $A = \mathbb{R}$, "$\leqslant$"

if $a \leqslant b$ and $b \leqslant c \Rightarrow a \leqslant c$ Transitive

(4) $A = P[\{1,2,\dots,n\}]$

$R_1 = S R_1 T \iff S \subseteq T$ → Transitive

$R_2 = S R_2 T \iff S \cap T = \emptyset$

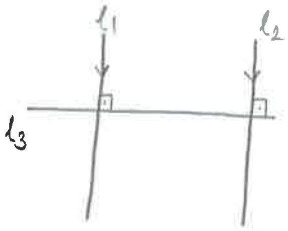
$\left.\begin{array}{l}\{1\} \cap \{3\} = \emptyset \\ \{3\} \cap \{1,2\} = \emptyset\end{array}\right)$ but $\{1\} \cap \{1,3\} = \{1\} \neq \emptyset$ → Transitive

$\{1\} R_2 \{3\}$, $\{3\} R_2 \{1,2\}$ but $\{1\} \not R_2 \{1,2\}$

(5) A = the set of all lines in the plane

R_1 = being parallel → Transitive

R_2 = " perpendicular



but $l_1 \not R_2\, l_3$ → NOT Transitive

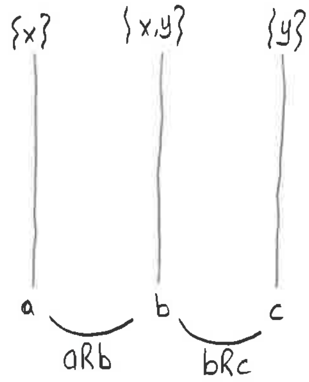
$l_1 R_2 l_2$, $l_2 R_2 l_3$

(6) A = The people in the world

R_1 = Having the same eye color → Transitive

$R_2 = a R_2 b$ if a is a descendent of b → Transitive (speak the same language)

$R_3 = a R_3 b$ if a and b have a common friend → NOT Transitive



$x \neq y$

but $a \not R c$

(7) $A = \mathbb{R}^2 = \{(x,y) \mid x, y \in \mathbb{R}\}$

$(x_1, y_1) R (x_2, y_2) \iff x_1^2 + y_1^2 = x_2^2 + y_2^2$

if $(x_1, y_1) R (x_2, y_2) \Rightarrow x_1^2 + y_1^2 = x_2^2 + y_2^2$ then $(x_2, y_2) R (x_3, y_3) \Rightarrow x_2^2 + y_2^2 = x_3^2 + y_3^2$ → Transitive

(8) $A = \mathbb{N}$

$a R b \iff a / b$

if a/b and b/c

$\Downarrow \qquad \Downarrow$

$b = a.k \qquad c = b.m \qquad \Rightarrow \quad c = a.k.m \Rightarrow a/c$

Transitive

Question-1

let $A = \{1,2,3,4,5\}$

Give an example to a relation b on A that is both symmetric and anti-symmetric.

$\{(1,1)\}$ is enough ✓

Any such relation has to be a subject of $\{(1,1),(2,2),(3,3),(4,4),(5,5)\}$

Question-2 (Exam)

$A\mathbb{R}$, for the following relations, determine whether it's reflexive, symmetric or transitive

(a) $xRy \iff x = y-1$ or $x = y + \frac{1}{2}$ NOT Reflexive

(b) $xRy \iff x-y \in \mathbb{Z}$

(c) $xRy \iff x = 2y$

(d) $xRy \iff x \cdot y > 0$

(a) $1R2$ and $2R3$ but $1 \not R 3$ → Not Transitive

(b) $\forall x \in \mathbb{R}$, $x - x = 0 \in \mathbb{Z}$, xRx → Reflexive

if $x-y \in \mathbb{Z}$ then $(-1)(x-y) \in \mathbb{Z} \Rightarrow y-x \in \mathbb{Z}$ → Symmetric

if $x-y \in \mathbb{Z}$ and $y-z \in \mathbb{Z}$, then $x-z = \underbrace{(x-y)}_{\in \mathbb{Z}} + \underbrace{(y-z)}_{\in \mathbb{Z}} \in \mathbb{Z}$

(c) $1 \not R 1$ → not Reflexive

$2R1$ but $1 \not R 2$ → Not Symmetric

$(2 = 2.1)$ $(1 \neq 2.2)$

$4R2$ and $2R1$ but $4 \not R 1$ → Not Transitive $(4 \neq 2.1)$

(d) Not Reflexive since $0 \not R 0$

$x.y = y.x$ so if $x.y > 0$, then $y.x > 0$ → Symmetric

$$\left.\begin{array}{l} \text{if } x.y > 0 \text{ and } y.z > 0 \text{ then } (x)(y)(y.z) > 0 \\ \Rightarrow (xz)y^2 > 0 \Rightarrow x.z > 0 \end{array}\right\} \text{Transitive}$$

Ex: $A = \mathbb{Z}$ for the following relations, determine the types.

(a) $xRy \iff x \neq y$

(b) $xRy \iff x \cdot y \geq 1$ (in $\mathbb{Z}$, $x.y \geq 1$), $1 \iff x.y > 0$

(c) $xRy \iff x \equiv y \pmod{2}$

(d) $xRy \iff x$ is a multiple of y

(e) $xRy \iff x$ and y are both negative or both non-negative $\rightarrow x.y \geq 0$

(a) Not Reflexive

if $x \neq y$ then $y \neq x$ → SYMMETRIC

$1 \neq 2$ and $2 \neq 1$ but $1 = 1$ → NOT TRANSITIVE

(b) $0.0 \not\geq 1$ → NOT REFLEXIVE

$x \cdot y = y.x$ → SYMMETRIC

$x.y > 0$ and $y.z > 0 \Rightarrow x.z > 0$ → TRANSITIVE

(c) REFLEXIVE, SYMMETRIC, TRANSITIVE

(d) if $x \in \mathbb{Z}$, $x = x, 1$, xRx → Reflexive

$2R_1$ but $1 \not R 2$ → NOT SYMMETRIC

$2R(-2)$ and $(2)R2$ but $-2 \neq 2$ → NOT ANTI-SYMMETRIC

if $x = y.k$ and $y = z.m$ then $x = z.m.k$ → Transitive

(e) For any integer x, $x^2 \geq 0$ → Reflexive

$x.y = y.x$ → Symmetric

if $x.y \geq 0$ and $y.z \geq 0$

then $(x.y)(y.z) \geq 0 \Rightarrow (x.z)y^2 \geq 0$ → Transitive

$= (x.z) \geq 0$

29.04.2015
Wednesday

Equivalance Relations

Def: Let R be a relation defined on a set ARi, sort to be on "equivalence relation"

If R is
- reflexive
- symmetric and
- transitive

Ex:

(1) Let $A = \mathbb{Z}$

$R = xRy \iff x \equiv y \pmod{2}$

R is an equivalence relation on $\mathbb{Z}$

(2) Let A → all lines in the plane

R → being an equivalence relation

(3) Let $A = \mathbb{R}$

$(x_1, y_1) R (x_2, y_2) \iff x_1^2 + y_1^2 = x_2^2 + y_2^2$

Reflexive $\iff (x,y) R (x,y)$? $x^2 + y^2 = x^2 + y^2$ ✓

if $(x_1, y_1) R (x_2, y_2)$ then $(x_2, y_2) R (x_1, y_1)$

$x_1^2 + y_1^2 = x_2^2 + y_2^2$ ✓ SYMMETRIC

TRANSITIVE

If $(x_1, y_1) R (x_2, y_2)$ and $(x_2, y_2) R (x_3, y_3)$ then $(x_1, y_1) R (x_3, y_3)$

$$\left.\begin{array}{l} x_1^2 + y_1^2 = x_2^2 + y_2^2 \quad \text{and} \\ x_2^2 + y_2^2 = x_3^2 + y_3^2 \end{array}\right\} x_1^2 + y_1^2 = x_3^2 + y_3^2$$

TRANSITIVE ✓

(4) A → People in the world

R → Having the same eye color

R is an equivalence relation

Def : (Equivalence Class)

Let R be an equivalence relation on B. By the equivalence of an element $a \in A$.

We mean the set of all elements in A, that are related to a.

In other words, $R(a) = \{x \in A \mid xRa\}$

Observe : For any $a \in A$,

$R(a)$ contains at least one element.

$\forall a \in A$, $a \in R(a)$ because R is refluxive and hence aRa for all $a \in A$.

Examples

$A = \mathbb{Z}$

R: $xRy \iff x \equiv y \pmod 2$

Consider equivalence classes

$R(0) = \{x \in \mathbb{Z} \mid xR0\}$

$= \{x \in \mathbb{Z} \mid x \equiv 0 \pmod 2\}$ = All even integers $= \{\dots, -4, -2, 0, 2, 4, \dots\}$

$R(1) = \{x \in \mathbb{Z} \mid xR_1\}$

$= \{x \in \mathbb{Z} \mid x \equiv 1 \pmod 2\}$ = All odd numbers $= \{\dots, -5, -3, -1, 1, 3, 5, \dots\}$

$R(4) = R(0)$ $\quad R(4) = \{x \in \mathbb{Z} \mid x \equiv 4 \pmod 2 \iff x \equiv 0 \pmod 2\}$

Theorem : Let R be an equivalence relation on A.

1. If xRy, then $R(x) = R(y)$
2. If $x \not R y$, then $R(x) \cap R(y) = \emptyset$

Proof : 1. To proof $R(x) = R(y)$, we'll show $R(x) \subseteq R(y)$ and $R(y) \subseteq R(x)$

Assume that $a \in R(x)$, So, aRx. But xRy. Since R is transitive,

and aRx xRy $\Rightarrow aRy \Rightarrow a \in R(y)$ $\quad$, $R(x) \subseteq R(y)$

Suppose $a \in R(y) \Rightarrow aRy$. Since R is symmetric, $xRy \Rightarrow yRx$

Now aRy and yRa $\Rightarrow aRx$ $\quad a \in R(x)$ $\quad R(y) \subseteq R(x)$

R is transitive

(2) Assume that $R(x) \cap R(y) \neq \emptyset$, so $\exists a \in R(x) \cap R(y)$

$\Rightarrow a \in R(x)$ and $a \in R(y)$

$\Rightarrow aRx$ and aRy

$\Rightarrow xRa$ and aRy $\Rightarrow xRy$ (Contradiction)

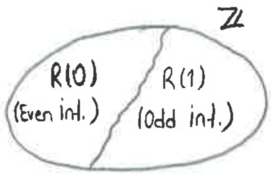
R is symmetric R is transitive

Corollary

(1) Two equivalence classes are either identical or disjoint.

(2) The set A is partitioned into a union of equivalence classes.

For example, in $\mathbb{Z}$, with R : equivalence (mod 2)



Examples

Let $A = \mathbb{Z}$, Define R on $\mathbb{Z}$ by $xRy \Leftrightarrow x^2+x = y^2+y$

(1) Prove that R is an equivalence relation

(2) Find $R(0)$, $R(3)$. In general, Find $R(a)$ for any at $\mathbb{Z}$

Solution

(1) $x^2+x = x^2+x$ ✓ (Reflexive)

if $x^2+x = y^2+y$, then $y^2+y = x^2+x$ (Symmetric).

if $x^2+x = y^2+y$, and $y^2+y = z^2+z$ (Transitive) R is equivalence relation

(2) $R(0) = \{x \in \mathbb{Z} \mid xR0\} = \{x \in \mathbb{Z} \mid x^2+x = 0\}$

$x^2+x = 0 \Leftrightarrow x \cdot (x+1) = 0 \Leftrightarrow x = 0$ or $x = -1 \in \mathbb{Z}$ So, $R(0) = \{0, -1\}$

- $R(3) = \{x \in \mathbb{Z} \mid xR3\} = \{x \in \mathbb{Z} \mid x^2+x = 3^2+3\}$

$x^2+x = 12 \Leftrightarrow x^2+x-12 = (x-3)(x+4) = 0 \Leftrightarrow x = 3$ or $x = -4$

$R(3) = \{3, -4\}$

- $R(a) = \{x \in \mathbb{Z} \mid xRa\} = \{x \in \mathbb{Z} \mid x^2+x = a^2+a\}$

$x^2+x = a^2+a \Leftrightarrow (x^2-a^2) + (x-a) = 0$

$\Leftrightarrow (x-a)(x+a)+(x-a) = 0$

$\Leftrightarrow (x-a)(x+a+1) = 0$

$\Leftrightarrow x = a$, $x = -1-a$ So for any $a \in \mathbb{Z}$, $R(a) = \{a, -1-a\}$

Ex: Let $A = \mathbb{R}$ → The set of real numbers

Define R on $\mathbb{R}$ as:

$$xRy \iff x^3+x^2+x = y^3+y^2+y$$

① Show that R is an equivalence relation

② Find $R(0)$. Find $R(1)$. In general Find $R(a) = ?$

② $R(0) = \{x \in \mathbb{R} \mid xR0\} = \{x \in \mathbb{R} \mid x^3+x^2+x = 0\}$

$$x^3+x^2+x = 0 \iff x(x^2+x+1) = 0 \Rightarrow \underbrace{x=0}_{\checkmark} \text{ or } \underbrace{x^2+x+1=0}_{\times}$$

$\Delta = 1-4 = -3 < 0$ No Real Solutions

$R(0) = \{0\}$

$R(1) = \{x \in \mathbb{R} \mid xR1\} = \{x \in \mathbb{R} \mid x^3+x^2+x = 1^3+1^2+1\}$

$$(x^3-1^3)+(x^2-1^2)+(x-1) = 0$$

$$\Rightarrow (x-1)(x^2+x+1)+(x-1)(x+1)+(x-1) = 0 \Rightarrow (x-1)\left[x^2+x+1+x+1+1\right] = 0 \Rightarrow (x-1)(x^2+2x+3) \Rightarrow \underset{\checkmark}{x=1} \text{ or } x^2+2x+3 = 0$$

$\Delta = -8 < 0$ $\times$

$R(1) = \{1\}$

$R(a) = \{x \in \mathbb{R} \mid xRa\}$

$= \{x \in \mathbb{R} \mid x^3+x^2+x = a^3+a^2+a\}$

$$(x^3-a^3)+(x^2-a^2)+(x-a) = 0 \Rightarrow (x-a)\left[x^2+ax+x+a+1\right] = 0 \Rightarrow x=a \text{ or } x^2+(a-1)x+(a^2+a+1) = 0$$

$R(a) = \{a\}$ for all $a \in \mathbb{R}$

$$\Delta = (a+1)^2 - 4(a^2+a+1) = -3a^2-2a-3$$

$$= -a^2-2a-1-2a^2-2$$

$$= \underbrace{-(a+1)^2}_{\leq 0} \underbrace{-2a^2-2}_{\leq 0} < 0 \text{ for all } a \in \mathbb{R}$$

Ex : $A = \mathbb{R}_{\geq 0}$

Define R on A : $xRy \iff x\sqrt{x} - \sqrt{x} = y\sqrt{y} - \sqrt{y}$

① Show that R is an equivalence relation

② Find $R(0)$. Find $R(2)$. For which a, does $R(a)$ contain more than one element?

—

② $R(0) = \{x \in \mathbb{R}_{\geq 0} \mid xR0\}$

$= \{x \in \mathbb{R}_{\geq 0} \mid x\sqrt{x} - \sqrt{x} = 0\}$

$x\sqrt{x} - \sqrt{x} = 0 \iff \sqrt{x}(x-1) = 0$

$\iff x = 0, x = 1$ $\qquad R(0) = \{0,1\}$

—

$R(2) = \{x \in \mathbb{R}_{\geq 0} \mid xR2\}$

$= \{x \in \mathbb{R}_{\geq 0} \mid x\sqrt{x} - \sqrt{x} = 2\sqrt{2} - \sqrt{2}\}$

$\iff (x\sqrt{x} - 2\sqrt{2}) - (\sqrt{x} - \sqrt{2}) = 0$

$\iff [(\sqrt{x})^3 - (\sqrt{2})^3] - [\sqrt{x} - \sqrt{2}] = 0 \Rightarrow (\sqrt{x} - \sqrt{2}) \cdot [x + \sqrt{x}.\sqrt{2} + 2] - (\sqrt{x} - \sqrt{2}) = 0 \Rightarrow (\sqrt{x} - \sqrt{2})[x + \sqrt{x}.\sqrt{2} + 1] = 0$

$\Rightarrow x = 2$ or $x + \sqrt{x}.\sqrt{2} + 1 = 0$

$R(2) = \{2\}$

$\sqrt{x} = t$, $t^2 + \sqrt{2}t + 1 = 0$, $\Delta = (\sqrt{2})^2 - 4 = 2 - 4 = -2 < 0$ } → No Solutions

—

$R(a) = \{x \in \mathbb{R}_{\geq 0} \mid xRa\} = \{x \in \mathbb{R}_{\geq 0} \mid x\sqrt{x} - \sqrt{x} = a\sqrt{a} - \sqrt{a}\}$

$x\sqrt{x} - \sqrt{x} = a\sqrt{a} - \sqrt{a} \Rightarrow (x\sqrt{x} - a\sqrt{a}) - (\sqrt{x} - \sqrt{a}) = 0$

$((\sqrt{x})^3 - (\sqrt{a})^3) - (\sqrt{x} - \sqrt{a}) = 0 \Rightarrow (\sqrt{x} - \sqrt{a})(x + \sqrt{x}.\sqrt{a} + a) - (\sqrt{x} - \sqrt{a}) = 0 \Rightarrow \overbrace{(\sqrt{x} - \sqrt{a})}^{x=a}[x + \sqrt{x}\sqrt{a} + a - 1] = 0$

$|R(a)| > 1 \iff x + \sqrt{x}.\sqrt{a} + a - 1 = 0$ has solutions in $\mathbb{R}$.

substuting $\sqrt{x} = t$, $t^2 + \sqrt{a}.t + a - 1 = 0$

$|R(a)| > 1 \iff \Delta \geq 0 \iff (\sqrt{a})^2 - 4.(a-1) \geq 0$

$\iff a - 4(a-1) \geq 0$

$\iff -3a + 4 \geq 0$

$\iff a \leq \frac{4}{3}$

As a result; if $0 \leq a \leq \frac{4}{3}$, $|R(a)| > 1$

if $a > \frac{4}{3}$, $R(a) = \{a\}$

FUNCTIONS

Def: Let f be a relation from A to B (So, $f \subseteq A \times B$)

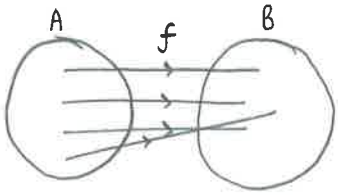
If for every $a \in A$, there exists a unique $b \in B$ such that $(a,b) \in f$ (or afb), then we say f is a function from A to B.

In such a case, we say $f : A \to B$ is a function.

Two remarks

(1) Every element in A must be related to some element in B.

(2) The element B in that is related to $a \in A$ must be unique.



From each element in A, there is one and only one outgoing arrow.

Notation: If f is a function from A to B, and $(a,b) \in f$ or afb we'll denote this by $f(a) = b$ (a: input, b: output)

$A = \{a, b\}$

$B = \{x, y\}$

(a) $f_1 = \{(a,x), (b,y)\}$ → Function

(b) $f_2 = \{(a,x), (b,x)\}$ → Function

(c) $f_3 = \{(a,y)\}$ → Not Function

(d) $f_4 = \{(a,x), (a,y), (b,x)\}$ → Not Function

Examples

(1) R is defined on $\mathbb{Z}$, $xRy \iff x \equiv y \pmod{7}$, Is R a function?

$1R1$, $1R8$, $1R15$, ... Not a function, since 1 is related to more than one element in $\mathbb{Z}$.

More generally:

Proposition: Let R be an equivalence relation on A.
Then R defines a function from A to A
if and only if R is the equality, i.e., $R = \{(x,x) \mid x \in A\}$

Prove: Suppose R is an equivalence relation that is a function. Since, R is reflexive, $\{(x,x) \mid x \in A\} \subseteq R$

Suppose for $x \neq y$; $(x,y) \in R$

xRx, xRy, $x \neq y$ → x has two distinct images. Contradiction.

So $R = \{(x,x) \mid x \in A\}$

(2) $A = \mathbb{P}[\{1,2,3,\dots,n\}]$

$R_1: SR_1T \iff S \subseteq T \rightarrow \{1\}R_1\{1\}, \{1\}R_1\{1,2\}$ => Not a Function

$R_2: SR_2T \iff S \cap T \neq 0 \rightarrow \{1\} \cap \{2\} = \emptyset, \{1\} \cap \{3\} = \emptyset$ => Not a Function

$R_3: SR_3T \iff S \cap T \neq 0$ and $S \cup T = \{1,2,\dots,n\} \rightarrow$ If $S = \emptyset$, $T = \{1,2,\dots,n\}$
If $S = \{1\}$, $T = \{2,3,\dots,n\}$

Which of these relations define a function on A?

In fact, I claim $SR_3T \iff T = S^c$
So, R_3 is defines a function
$f: \mathbb{P}[(1,2,\dots,n)] \rightarrow \mathbb{P}[\{1,2,\dots,n\}]$
$S \subseteq \{1,2,\dots,n\}: f(S) = S^c$

(3) $A \rightarrow$ People in the world.

$aRb \iff b$ is a's father

a function

(4) $f: \mathbb{R} \rightarrow \mathbb{R}$

$f(x) = x^2$ is a function

$f: \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = \begin{cases} 2x+3, & x < 3 \\ x+5, & x \geq 3 \end{cases}$$

Is f a function?

f is not function. $f(3) \begin{matrix} \nearrow 9 \\ \searrow 8 \end{matrix}$

Domain and Range Functions

$f: A \rightarrow B$

$A \rightarrow$ Domain of f (the set on which f is defined)
(the largest set on which f is defined)

Range $(f) \neq B$ in general

Range $(f) = f(A) = \{f(a) \mid a \in A\}$

—

$f: \mathbb{N} \rightarrow \mathbb{N}$
$f(n) = n + 7$

Domain $(f) = \mathbb{N}$, Range $(f) = \{8,9,10,\dots\}$

—

$f(x) = \sqrt{x}$, Domain $= [0,\infty)$
Range $= [0,\infty)$

—

$f(x) = e^x$, Domain $= \mathbb{R}$
Range $= (0,\infty)$

$f(x) = \ln x$
Domain $= (0,\infty)$
Range $= \mathbb{R}$

Ex: $f(x) = \frac{2x+3}{x-1}$, Domain $= \mathbb{R}\setminus\{1\}$, Range $= \mathbb{R}\setminus\{2\}$ $\quad f: \mathbb{R}\setminus\{1\} \to \mathbb{R}\setminus\{2\}$ $\quad 3 = -2$ (Impossible)

Ex: $f(x) = \ln(\cos x)$ $\quad \cos x > 0$

Domain $= \dots \cup \left(\frac{-5\pi}{2}, \frac{-3}{2}\pi\right) \cup \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$

Range $= (-\infty, 0]$

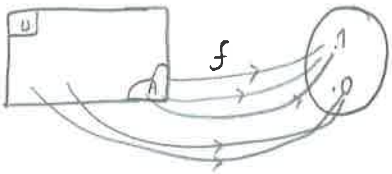
Some special functions

(1) The characteristic function

Let $\mathcal{U}$ be a universal set, and $A \subseteq \mathcal{U}$. The "characteristic function" of A is denoted by $\mathcal{X}_A$,

$\mathcal{X}_A : \mathcal{U} \to \{0,1\}$

$$\mathcal{X}_A(x) = \begin{cases} 1 & , \text{if } x \in A \\ 0 & , \text{if } x \notin A \end{cases}$$



Examples: $\mathcal{X}_\emptyset(x) = 0$ for all $x \in \mathcal{U}$, so $\mathcal{X}_\emptyset$ is the constant zero function.

$\mathcal{X}_{\mathcal{U}}(x) = 1$ for every $x \in \mathcal{U}$

$\mathcal{U} = \{1,2,3,4,5\}$

$A = \{2,3\}$ $\quad \mathcal{X}_A(1) = 0$, $\mathcal{X}_A(2) = \mathcal{X}_A(3) = 1$, $\mathcal{X}_A(4) = \mathcal{X}_A(5) = 0$

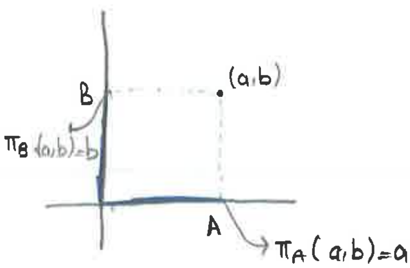
(2) Projection

Let A and B be two sets.

We can define to projections from $A\times B$ to A and B.

$\pi_A : A\times B \longrightarrow A$, $\pi_A(a,b) = a$ for all $(a,b) \in A\times B$

$\pi_B : A\times B \longrightarrow B$, $\pi_B(a,b) = b$ for all $(a,b) \in A\times B$



(3) The identity function

$f: A \to A$

$f(a) = a$, for all $a \in A$

—

$f: [0,\infty) \to [0,\infty)$

$f(x) = \sqrt{x^2}$ → Identity function

$f: \mathbb{R} \to \mathbb{R}$

$f(x) = \sqrt{x^2}$ → Not Identity f. $\sqrt{x^2} = |x|$

(4) Equal Functions

Two functions f and g are said to be equal $(f = g)$

if (i) $Dom(f) = Dom(g)$

(ii) $Range(f) = Range(g)$

(iii) $f(x) = g(x)$ for all x in the domain

Ex: $f, g: \mathbb{N} \to \mathbb{N}$

$f(n) = n^2 - 2n - 1$

$g(n) = (n-1)^2$

$f = g$

$f, g: \mathbb{R} \to \mathbb{R}$

$f(x) = \max\{x, -x\}$

$g(x) = x$ then

$f = g$

$f, g: \mathbb{Z} \to \mathbb{Z}$

$f(n) = \cos(\pi n)$

$g(n) = \begin{cases} 1, & \text{if } n \text{ is even} \\ -1, & \text{if } n \text{ is odd} \end{cases} = (-1)^n$

$f(0) = 1$ $f(-1) = -1$

$f(1) = -1$ $f(3) = -1$

Increasing - Decreasing Functions

Suppose f is defined on a subset of $\mathbb{R}$.

if $x < y \Rightarrow f(x) < f(y)$ → f is increasing

if $x < y \Rightarrow f(x) > f(y)$ → f is decreasing

Example: Give examples (if possible) to functions

$f: \mathbb{N} \to \mathbb{N}$ such that

(1) f is increasing → $f(n) = n$, $f(n) = 2n$

(2) f is decreasing → There is no decreasing function , $f: \mathbb{N} \to \mathbb{N}$ Why? Suppose $f: \mathbb{N} \to \mathbb{N}$ and is decreasing , $f(1) = M$ $\Rightarrow$

(3) $x < y \Rightarrow f(x) = f(y)$ → $f(n) = 2$, for all $n \in \mathbb{N}$ (Constant Function)

(4) f Is increasing on even integers x, y odd and $x < y \Rightarrow f(x) = f(y)$ → $f(n) = \begin{cases} n, & \text{if } n \text{ is even} \\ 2, & \text{if } n \text{ is odd} \end{cases}$

$\Rightarrow$ Since f is decreasing $f(2) < f(1) = M \Rightarrow f(2) \leq M-1$

$f(3) < f(2) \leq M-1 \Rightarrow f(3) < M-1 \Rightarrow f(3) \leq M-2$

$\overbrace{f(M)}^{\in \mathbb{N}} \leq 1 \Rightarrow f(M) = 1 \Rightarrow f(M+1) < f(M) = 1 \Rightarrow$ Contradiction

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Ex: $f: \mathbb{Z} \to \mathbb{Z}$ find example of

① Decreasing functions on $\mathbb{Z}$ $\longrightarrow f(n) = -n$

② f is decreasing on even integers, increasing on odd integers $\longrightarrow f(n) = \begin{cases} -n & \text{, if } n \text{ is even} \\ n & \text{, if } n \text{ is odd} \end{cases}$

One-to-One and onto Functions

① One-to-one Functions

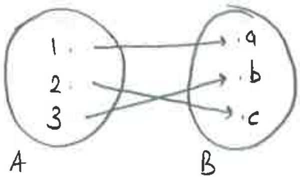
Let $f: A \to B$ be a function.

We say f is one-to-one if $\overset{p}{a_1 \neq a_2} \Rightarrow \overset{q}{f(a_1) \neq f(a_2)}$ (In other words, different elements in A have different images)

$\equiv$

$f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

$(p \Rightarrow q) \equiv (q' \Rightarrow p')$ Contrapositive



Ex: $f: \mathbb{Z} \to \mathbb{Z}$

$f(x) = 2x+3$ Is f one-to-one ?

$f(x) = f(y) \Rightarrow 2x+3 = 2y+3 \Rightarrow 2x = 2y \Rightarrow x = y$ ✓ f is one-to-one

Ex: $f: \mathbb{Q} \setminus \{1/3\} \to \mathbb{Q} \setminus \{2/3\}$

$f(x) = \dfrac{2x+1}{3x-1}$ Is f one-to-one ?

$f(x) = f(y) \Rightarrow \dfrac{2x+1}{3x-1} = \dfrac{2y+1}{3y-1} \Rightarrow (2x+1)(3y-1) = (3x-1)(2y+1) \Rightarrow 6xy - 2x + 3y - 1 = 6xy + 3x - 2y - 1 \Rightarrow -2x + 3y = -2y + 3x \Rightarrow$
$5x = 5y \Rightarrow x = y$

f is one-to-one

Ex: $f: \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$ Is f one-to-one ?

$f(1) = f(-1) = 1$ Not one-to-one

Ex: $f: [0, \infty) \to [0, \infty)$

$f(x) = x^2$

$f(x) = f(y) \Rightarrow x^2 = y^2 \Rightarrow \sqrt{x^2} = \sqrt{y^2} \Rightarrow |x| = |y| \Rightarrow x = y \quad (x, y \geq 0)$

Ex : $f: \mathbb{R} \to \mathbb{R}$

$f(x) = x^3$

$f(x) = f(y) \Rightarrow x^3 = y^3 \Rightarrow x^3 - y^3 = 0 \Rightarrow (x-y)\underbrace{(x^2+xy+y^2)}_{} = 0 \Rightarrow x = y$

$\hookrightarrow$ an equation in x. $\Delta = y^2 - 4y^2 = -3y^2 < 0$ for all $y \neq 0$

if $y \neq 0$, $x^2 + xy + y^2 = 0$ has no solutions
if $y = 0$, $x^2 = 0 \Rightarrow x = 0$ ✓

$\Big\}$ f is one-to-one.

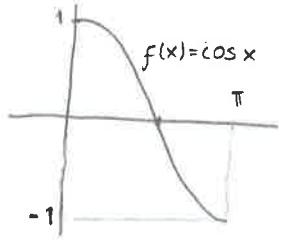
Ex : $f: [0, 2\pi] \to [-1, 1]$

$f(x) = \cos x$ Is f one-to-one ?

$\cos\left(\frac{\pi}{2}\right) = \cos\left(\frac{3\pi}{2}\right) = 0$ Not one-to-one

Ex : $f: [0, \pi] \to [-1, 1]$

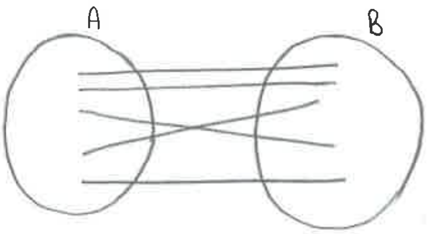
$f(x) = \cos x$ Is f one-to-one ?



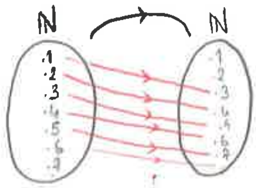
$\cos x$ is decreasing in $[0, \pi]$
So, it is one-to-one.

ONTO FUNCTION

Let $f: A \to B$ b/a Function. If $\forall b \in B$, $\exists a \in A$ such that $f(a) = b$
In other words, no isolated elements in B.



Examples



$f: \mathbb{N} \to \mathbb{N}$, $f(x) = x + 2$, Is f onto ?

f is onto $\Leftrightarrow$ for any $y \in \mathbb{N}$, $f(x) = y$ has solutions in $\mathbb{N}$

Take $y = 1$ then $f(x) = x + 2 = 1 \Leftrightarrow x = -1 \notin \mathbb{N}$

So, there is no solutions to $f(x) = 1$, f is not onto.

Ex : $f: \mathbb{Z} \to \mathbb{Z}$

$f(x) = x + 2$

given any $y \in \mathbb{Z}$, $f(x) = x + 2 = y \Leftrightarrow x = y - 2 \in \mathbb{Z}$ So f is onto.

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Question: For each function below, determine whether its one-to-one or onto

(a) $f: \mathbb{R}\setminus\{0\} \to \mathbb{R}\setminus\{0\}$, $f(x) = \frac{1}{x}$

(b) $g: \mathbb{R}\setminus\{1\} \to \mathbb{R}\setminus\{1\}$, $g(x) = \frac{x}{x-1}$

(c) $h: \mathbb{R} \to \mathbb{R}$, $h(x) = x^2 - x + 1$

(a) $f(x) = f(y)$

$\Rightarrow \frac{1}{x} = \frac{1}{y} \quad \Rightarrow x = y$ ✓ $\quad f$ is one-to-one

If $y \neq 0$ is any real number then $f(x) = y \iff \frac{1}{x} = y \iff x = \frac{1}{y} \in \mathbb{R}\setminus\{0\}$ So f is into.

(b) $g(x) = g(y)$

$\Rightarrow \frac{x}{x-1} = \frac{y}{y-1} \quad \Rightarrow x.(y-1) = y.(x-1) \quad \Rightarrow \cancel{xy} - x = \cancel{xy} - y \quad \Rightarrow -x = -y \quad \Rightarrow x = y \quad g$ is one-to-one

Let $y \neq 1$ be any real number. $g(x) = y \iff \frac{x}{x-1} = y \iff x = yx - y \Rightarrow y = x.(y-1) \iff x = \frac{y}{y-1} \in \mathbb{R}$ g is onto

(c) $h: \mathbb{R} \to \mathbb{R}$

$h(x) = x^2 - x + 1 \longrightarrow h(x) = h(y) \iff x^2 - x + 1 = y^2 - y + 1 \iff x^2 - y^2 - (x-y) = 0 \iff (x-y)(x+y-1) = 0 \iff x = y$ or $x + y = 1$

$x = 0$, $y = 1 \Rightarrow h(0) = 1$, $h(1) = 1$ $\rangle$ $h(0) = h(1)$
h is not one-to-one

h is onto $\iff$ for every $y \in \mathbb{R}$

$x^2 - x + 1 = y$ has solutions in $\mathbb{R}$ $\quad$ for $y = 0$ $\quad x^2 - x + 1 = 0$ has no solutions in $\mathbb{R}$, $\Delta = 1 - 4 = -3 < 0$ Not onto

Example : Find example of $f : \mathbb{N} \to \mathbb{N}$ that is

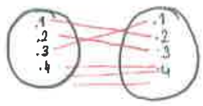
(a) one-to-one and onto

(b) neither one-to-one and nor onto (Constant Function)

(c) one-to-one but not onto

(d) onto but not one-to-one

(a) $f(x) = x$ one-to-one and onto

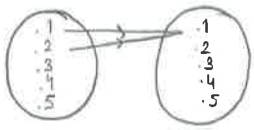


$$f(x) = \begin{cases} 1 & \text{if } x = 2 \\ 3 & \text{if } x = 1 \\ 2 & \text{if } x = 3 \\ x & \text{if } x \geq 4 \end{cases}$$

(b) $f(x) = 1$ or $f(x) = 2$

(c) $f(x) = 2x$

(d)



$$f(x) = \begin{cases} 1 & \text{if } x = 1 \\ x-1 & \text{if } x \geq 2 \end{cases}$$

onto, but not one-to-one (1-1)

Example : Give examples to $f : \mathbb{N} \to \mathbb{Z}$ such that f is

(a) Neither one-to-one nor onto

(b) One-to-one but not onto

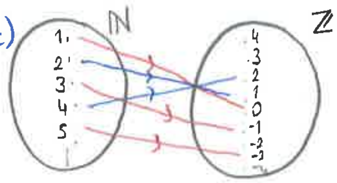
(c) One-to-one and onto

(d) Onto but not one-to-one

(a) $f(x) = -1$ or $f(x) = 0$

(b) $f(x) = x$ One-to-one, but not onto

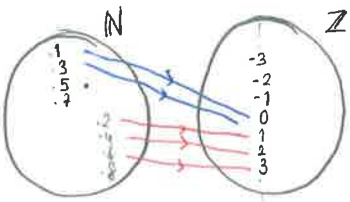
(c)



$$f(x) = \begin{cases} \dfrac{x}{2} & , \text{if } x \text{ is even} \\[2ex] \dfrac{1-x}{2} & , \text{if } x \text{ is odd} \end{cases}$$

One-to-one and onto

(d)



$$f(x) = \begin{cases} \dfrac{3-x}{2} & , \text{if } x \geq 3, x \text{ is odd} \\[2ex] 0 & , \text{if } x \text{ is } 1 \\[1ex] \dfrac{x}{2} & , \text{if } x \text{ is even} \end{cases}$$

Ex: $A = \{1, 3, 5, \dots\}$ → the set of odd natural numbers ($2n+1$)

$B = \{2, 6, 10, \dots\}$ → set of all natural numbers of the form $4n+2$

Find examples of functions $f: A \to B$ so that

(1) f is not one-to-one, not onto

(2) f is one-to-one and onto

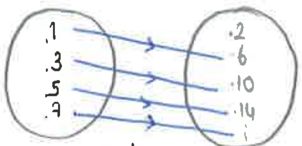
(3) f is one-to-one but not onto

(4) f is onto but not one-to-one

Solution: (1) $f(x) = 2$ or $f(x) = 10$

(2) $f(x) = 2x$

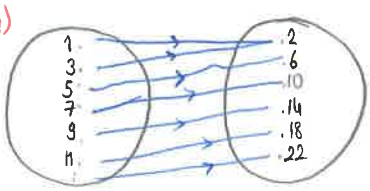
(3)



$f(x) = 2x+4$

One-to-one
Not onto

(4)



$f(x) = \begin{cases} 1, & \text{if } x = 1 \\ 2x-4, & \text{if } x > 1 \end{cases}$ $\quad f(1) = f(3) = 2$ $\quad$ Onto, but not one-to-one

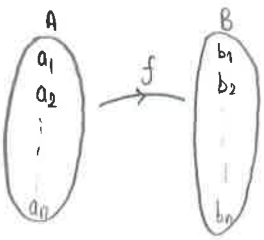
FINITE SETS

Theorem: Let A and B two finite sets.

(1) If $\exists f: A \to B$ that is one-to-one, then $|A| \leq |B|$

(2) If $\exists f: A \to B$ that is onto, then $|A| \geq |B|$

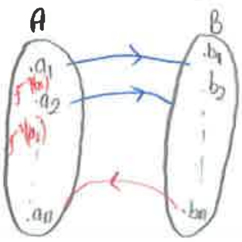
Proof: (1)



$\underbrace{\{f(a_1), f(a_2), \dots, f(a_n)\}}_{\text{All distinct}} \subseteq B$

Since B contains a subset of m elements,
$|B| \geq M = |A|$ ✓

(2)



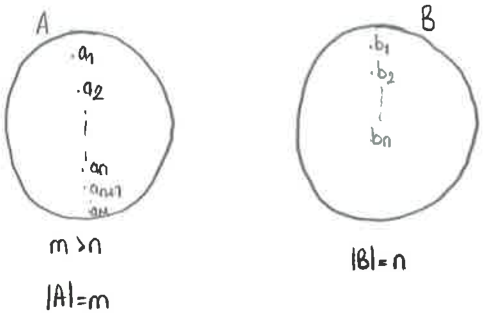
$\underbrace{\{f^{-1}(b_1), f^{-1}(b_2), \dots, f^{-1}(b_n)\}}_{\text{All distinct}} \subseteq A$

Since A contains n distinct elements,
$|A| \geq n = |B|$

Corollary: Let A and B finite sets. Then,

(1) if $|A|>|B|$, then there cannot be a one-to-one function from A to B.

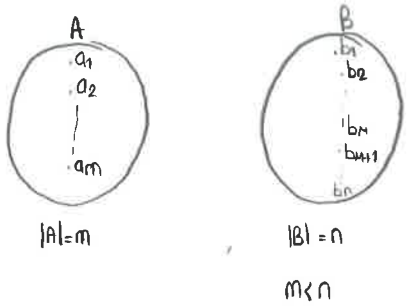
(2) If $|A|<|B|$, then there cannot be an onto function from A to B.



if f is one-to-one

$f(a_i)\neq f(a_.)$ if it's $f(a_{n+1})$ will have to the equal to $f(a_i)$

for some $i=1,2,\ldots,n$ $\quad f$ cannot be one-to-one



The set of images of elements under $f=\{f(a_1),f(a_2),\ldots f(a_m)\}$

So, at must m elements in B are covered by f.

But B has $n>m$ elements. Thus f cannot be onto.

Theorem: Suppose A and B are finite sets with $|A|=|B|$

(1) If $f:A\to B$ is one-to-one then it has to be onto.

(2) If $f:A\to B$ is onto, then it has to be one-to-one.

(1) and (2) can be combined.

$f:A\to B$ is one-to-one $\Longleftrightarrow f$ is onto.

Proof: (1) $f:A\to B$ $\qquad A=\{a_1,a_2,\ldots,a_n\}$

$B=\{b_1,b_2,\ldots,b_m\}$

Suppose f is one-to-one

This means $\underbrace{\{\underline{f(a_1),f(a_2),\ldots f(a_m)}_{\text{are all distinct}}\}}_{\text{contains } m \text{ elements}}\subseteq B$ $\qquad$ Since $|B|=m$, $\{f(a_1),f(a_2),\ldots,f(a_m)\}=B$

$\Longrightarrow f$ is onto

(2) $f:A\to B$ is onto.

Since f is onto, $\underbrace{\{f(a_1),f(a_2),\ldots f(a_n)\}}=B$ $\rightarrow$ contains m elements

$\Downarrow$

must contain m elements

$\Downarrow$

$f(a_i)$'s are all distinct

$\Downarrow$

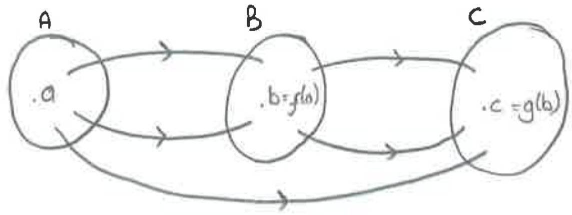
f is one-to-one

Composition of Functions

Let $f : A \to B$ and $g : B \to C$ be two functions

Then the composition of f and g.

$g \circ f : A \to C$ is a function that acts on A as: $(g \circ f)(a) = g(f(a))$



Note : The order in compositions is important, in general $g \circ f \neq f \circ g$. In fact in many cases one is defined while the other is not.

A special case

Let $f : A \to A$

$f \circ f : A \to A$, $(f \circ f)(a) = f(f(a))$

$f \circ f \circ f : A \to A$, $(f \circ f \circ f)(a) = f(f(f(a)))$

Example

$f, g, h : \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$

$g(x) = 2x+1$ $\qquad (f \circ f)(x) = f(f(x)) = f(x^2) = x^4$, $(f \circ f \circ f)(x) = x^8$, $(\underbrace{f \circ f \circ \cdots f}_{n \text{ times}})(x) = x^{2^n}$

$h(x) = x-2$

$(f \circ g)(x) = f(g(x)) = f(2x+1) = (2x+1)^2 = 4x^2+4x+1$

$\neq$

$(g \circ f)(x) = g(f(x)) = g(x^2) = 2x^2+1$

$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x-2)) = f(2(x-2)+1) = f(2x-3) = (2x-3)^2$

Theorem-1

$f : A \to B$, $g : B \to C$ are functions.

(i) If both f and g are one-to-one then $(g \circ f)$ is one-to-one

(ii) If both f and g are onto, then $(g \circ f)$ is onto.

Proof: (i) $(g \circ f)(x) = (g \circ f)(y) \overset{\text{composition}}{\Longrightarrow} g(f(x)) = g(f(y))$

$\underset{g \text{ is one-to-one}}{\Longrightarrow} f(x) = f(y) \underset{f \text{ is one-to-one}}{\Longrightarrow} x = y$

$g \circ f$ is one-to-one.

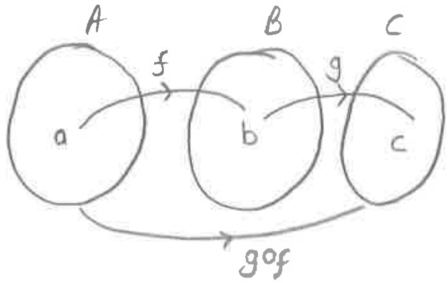
(ii) $g \circ f : A \to C$

We would like to show $g \circ f$ is onto.

Let $c \in C$. Then since $g : B \to C$ is onto, $\exists b \in B$ such that $g(b) = c$

Since $f : A \to B$ is onto, $\exists a \in A$ such that $f(a) = b$

But then $(g \circ f)(a) = c$. Thus $g \circ f$ is onto.



Theorem-2

$f : A \to B$, $g : B \to C$ are functions.

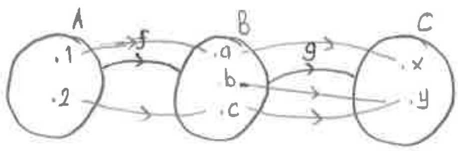
(i) If $g \circ f$ is one-to-one, then f must be one-to-one. (No information about g. g doesn't have to be one-to-one)

(ii) If $g \circ f$ is onto, then g must be onto. (No info. about f, ie., f does not have to be onto)

Proof

(i) Let $f(x) = f(y)$ $\Rightarrow g(f(x)) = g(f(y))$ $\Rightarrow (g \circ f)(x) = (g \circ f)(y)$ $\Big\rangle$ f is one-to-one $\Rightarrow x = y$

An example of f, g such that $g \circ f$ is one-to-one, but g is not.



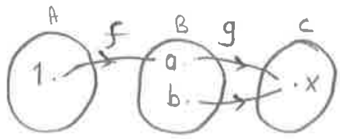
$g \circ f : A \to C$, $(g \circ f)(1) = x$
$(g \circ f)(2) = y$ #

$g \circ f$ is one-to-one. Since, $g(b) = g(c) = y$, g is not one-to-one

(ii) We will show $g : B \to C$ is onto if $(g \circ f) : A \to C$ is onto.

Pick $c \in C$. Since $g \circ f$ is onto, $\exists a \in A$ such that $(g \circ f)(a) = c$. Now let $b = f(a) \in B$

But then $g(b) = g(f(a)) = (g \circ f)(a) = c$ g is onto.



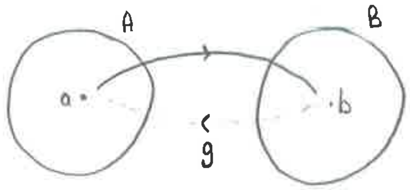
Give an example of $f : A \to B$, $g : B \to C$ such that $g \circ f$ is onto, but f is not.

$g \circ f : A \to C$

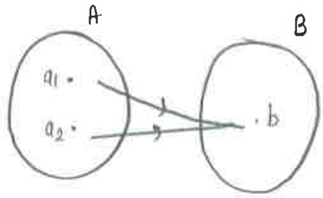
$(g \circ f)(1) = x$, $g \circ f$ is onto. But f is not onto.

The Inverse of a Function

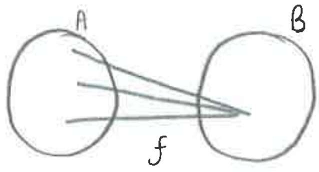
Let $f: A \to B$ be a function. If there exists a function $g: B \to A$ such that $(g \circ f)(a) = a$, for all $a \in A$ and $(f \circ g)(b) = b$, for all $b \in B$, then the function g is said to be the inverse of f and is denoted by f^{-1}.



if f is not one-to-one
f cannot have an inverse,



if f is not onto,
f cannot have an inverse



Theorem

Let $f: A \to B$ be a function.

f has an inverse (f^{-1} exists). If and only if f is one-to-one and onto.

Proof: We already know that if f^{-1} exists, f has to be one-to-one and onto.

Now suppose f is one-to-one and onto.
We'll show f^{-1} exists.

Since is onto, for any $b \in B$, $\exists a \in A$, $f(a) = b$

Define $f^{-1}: B \to A$ such that $f^{-1}(b) = a$ (if $f(a) = b$)

If $f^{-1}(b) = a_1$ and $f^{-1}(b) = a_2$ this means $f(a_1) = f(a_2) = b$

This is impossible since f is one-to-one.

→ How to find the inverse ?

$f(x) = y \iff f^{-1}(y) = x$

Examples

① Let $f: \mathbb{Q} \to \mathbb{Q}$, $f(x) = 3x - 2$

- Show that f has an inverse (f is one-to-one and onto)
- Find $f^{-1}(x) = ?$

(1)

$f(x)=f(y)$

$\Rightarrow 3x-2=3y-2$

$3x=3y$

$x=y$

f is one-to-one

Let $y\in\mathbb{Q}$ be any element

$f(x)=y \Rightarrow 3x-2=y \Rightarrow x=\frac{y+2}{3}\in\mathbb{Q}$

For any $y\in\mathbb{Q}$, $f\left(\frac{y+2}{3}\right)=y$, f is onto.

f has an inverse

$f(x)=3x-2 \Rightarrow f^{-1}(\underbrace{3x-2}_{y})=x$

$\Rightarrow f^{-1}(y)=\frac{y+2}{3}$ ✓

(2) $f:\mathbb{R}\setminus\{\frac{-3}{2}\}\to\mathbb{R}\setminus\{\frac{1}{2}\}$

$f(x)=\frac{x-1}{2x-3}$

- Show that f has an inverse
- Find $f^{-1}(x)=?$

(2)

$f(x)=f(y)$

$\Rightarrow \frac{x-1}{2x+3}=\frac{y-1}{2y+3}$

$2xy+3x-2y-3=2xy+3y-2x-3$

$x=y$

Let $y\neq\frac{1}{2}$ be any real number.

$f(x)=y \Rightarrow \frac{x-1}{2x+3}=y \Rightarrow$

$x-1=2xy+3y$

$x=\frac{1+3y}{1-2y}\in\mathbb{R}$

$f\left(\frac{1+3y}{1-2y}\right)=y$ f is onto

$f\left(\frac{1+3y}{1-2y}\right)=y \Rightarrow f^{-1}(y)=\frac{1+3y}{1-2y}$ ✓

$y\neq\frac{1}{2}$

(3) $f:[2,\infty)\to[3,\infty)$

$f(x)=x^2-4x+7$

- Show that f has an inverse
- Find $f^{-1}(x)=?$

(3)

$f(x)=f(y)$

$x^2-4x+7=y^2-4y+7$

$x^2-y^2-4x+4y=0$

$(x-y)\underbrace{(x+y-4)}_{>0}=0$

$x=y$

$x,y\in[2,\infty)$

$x+y>4$

except when $x=y=2$

Let $y\geqslant 3$ be a real number.

$f(x)=y \Rightarrow x^2-4x+7=y$

$\Rightarrow x^2-2.2.x+2^2+3=y$

$(x-2)^2+3=y$

$(x-2)^2=y-3\geqslant 0$

$\Rightarrow\sqrt{(x-2)^2}=\sqrt{(y-3)} \Rightarrow$

$|x-2|=\sqrt{y-3}$

$x\geqslant 2 \to x-2=\sqrt{y-3}$

$x=2+\sqrt{y-3}$

$\Rightarrow f(2+\sqrt{y-3})=y$, f is onto.

$f^{-1}(y)=2+\sqrt{y-3}$ ✓

⑫ $\forall x \in R, \exists y \in R, x.y = 1$

if $x=0$, $x.y=0$ for all $y \in R$ FALSE

Neg: $\exists x \in R, \forall y \in R, x.y \neq 1$

⑬ $\forall x \in R, \exists y \in R$, $x+y \in R \setminus Q$ or $x.y \in R \setminus Q$

choose $y = \sqrt{2} - x \Rightarrow x+y = \sqrt{2} \in R \setminus Q$ TRUE

Neg: $\exists x \in R, \forall y \in R$, $x+y \in R \setminus Q$ and $x.y \in R \setminus Q$

⑭ $\forall x \in R, \exists y \in R$, $x+y \in R \setminus Q$ and $x.y \in R \setminus Q$

Let $x=0$, for all $y \in R$ $x.y = 0 \notin R \setminus Q$ FALSE

⑮ $\forall x \in R \setminus Q, \exists y \in R$ $x^y \in Q$

Let $y=0$ $x^y = x^0 = 1 \in Q$ (since $x \neq 0$) TRUE

⑯ $\forall x \in Q, \forall y \in Q$, $x^y \in R$

$x=-1$, $y=\frac{1}{2}$ $x^y = (-1)^{1/2} = \sqrt{-1} \notin R$ FALSE

⑰ $\forall x \in R, \exists y \in Q$ $x+y \in R \setminus Q$ or $xy \in Q$

$y = \sqrt{2}$ $\sqrt{2}+x \in R \setminus Q$, $y = \frac{1}{x} \in R \setminus Q$ $x.y=1$ TRUE

⑱ $\forall x \in R, \forall y \in R \setminus Q$, $x+y \in R \setminus Q$ or $xy \in Q$

$x = \sqrt[3]{2}$ $x+y = 0 \in Q$ | $x = 1+\sqrt{2}$ $x+y = 1 \in Q$

$y = -\sqrt[3]{2}$ $x.y = -\sqrt[3]{4} \notin Q$ | $y = -\sqrt{2}$ $x.y = -\sqrt{2}-2 \notin Q$ FALSE

⑲ $\forall m \in Z, \exists n \in Z$, $m+n=4$

$n = 4-m$ (TRUE)

⑳ $\forall x \in Z$, $x^3 - x \equiv 0 \pmod 6$

$x^3 - x = \overbrace{x.\underbrace{(x-1)}_{\text{divisible by 2}}(x+1)}$ divisible by 3 } TRUE

SETS

Complement: A^c

Intersection: $A \cap B$

Union: $A \cup B$

Difference: $A \setminus B$

Symmetric Difference

$A \Delta B = (A \setminus B) \cup (B \setminus A)$

$\rightarrow \emptyset^c = U$, $U^c = \emptyset$, $(A^c)^c = A$

Ex: $(A \setminus B) \setminus C = A \setminus (B \cup C)$

$(A \setminus B) \setminus C = (A \cap B^c) \cap C^c = A \cap (B^c \cap C^c) = A \setminus (B \cup C)$

Infinite Intersection and Union

Increasing nested sets. $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots A_n$

Decreasing nested sets: $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots A_n$

If sets is increasing nested sets then $\bigcap_{n=1}^{\infty} A_n = A_1$, Union find with limit

If sets in decreasing nested sets then $\bigcup_{n=1}^{\infty} A_n = A_1$, Intersection find with limit.

RELATIONS

① Reflexive (Yansıyan)

$A = \{1,2,3\}$ $R = \{(1,1),(2,2),(3,3)\}$ aRa for all $a \in A$ (x,x)

② Symmetric

$A = \{1,2,3\}$ $R_1: \{(1,2),(2,1)\}$, $R_2: \{2,2\}$, $(x,y) = (y,x)$

③ Anti Symmetric

if "aRb and $bRa \Rightarrow a=b$"

$A = \{1,2,3\}$ $R_1: \{(1,1),(2,2)\}$, $R_2: \{(1,2)\}$, $R_3: \{(1,2)(2,1)\}$ Not Anti-Symmetric

④ Transitive

$A = \{1,2,3\}$ $R_1: \{(1,2),(2,1),(1,1)\}$, $R_2: \{(1,3),(2,3)\}$, $R_3: \{1,1\}$

$(x,y)\ (y,z) = (x,z)$

Equivalence: - Reflexive, Symmetric and Transitive

FUNCTIONS

① One-to-one

Domain: A, Range: B, Diff. elements in A have diff. images in B.

$x = y$

② Onto

No isolated elements in B. $x=y \Leftrightarrow y=x$ x'i yalnız bırak

$\rightarrow$ One-to-one and Onto $\Rightarrow y=x \Rightarrow f(x)=x$

Neither one-to-one nor Onto $\Rightarrow f(x)=2$ (Constant)

$\rightarrow$ Inverse: One-to-one and Onto

DISCRETE MATH FINAL NOTES

MATHEMATICAL LOGIC

¬Proposition : Exact truth value
Question (-)
Come here, Go away (-)
$x+2=5$ (-) , $x+y=z$ (-)

Disjunction ("and", $\wedge$) : Only $1\text{-}1=1$

Conjuction ("or", $\vee$) : Only $0\text{-}0=0$

("xor", "exclusive or", $\oplus$) : $1\text{-}0=1$, $0\text{-}1=1$, Others are 0.

Implication ("=>") : Only $1\text{-}0=0$, $(p' \vee q)$

Biconditional ("if and only if", <=>) : $1\text{-}1=1$, $0\text{-}0=1$

$(p \Leftrightarrow q) = (p \Rightarrow q) \wedge (q \Rightarrow p)$

* => does not convert with Distributivity Laws

Tautology : True for all values , Contradiction : False for all values

$\forall \to$ "for all" , $\exists \to$ "for some" ... Ex ⟶

Negating

$[\forall x\, P(x)]' = \exists x\, p(x)'$, $[\exists x\, P(x)]' = \forall x\, p(x)'$

$R \setminus Q$: Irrational Numbers ($\sqrt{2}, e, \pi$)

PROOF METHOD

① Direct Proof

Ex : If $x \in \mathbb{Z}$ is even, then x^2 is divisible by 4.

$x=2k$ $4k^2$ then x^2 is divisible by 4

Ex: If $x \in R$ then $x^2+x+1>0$

$x^2+x+1 = x^2+2.x.\frac{1}{2}+\frac{1}{4}+\frac{3}{4} = (x+\frac{1}{2})^2+\frac{3}{4} \Rightarrow \frac{3}{4}>0$

Ex: There exists 13 consecutive integers, none of which is prime

s.t : $(a+1), (a+2), \ldots, (a+13)$

$\underset{\text{even}\ (2)}{14!+2}, \underset{\text{odd}}{14!+3}, \ldots\ 14!+14$

Ex: $\exists a\, R\setminus Q, \exists b\, R\setminus Q, a^b \in Q$

$a=(\sqrt{2})^{\sqrt{2}}$ $b=\sqrt{2}$ $(\sqrt{2})^{\sqrt{2}.\sqrt{2}}=2 \in Q$ (TRUE)

Examples

① $\forall x \in \mathbb{R}$, $x^2 \geq x$

$x=\frac{1}{2} \Rightarrow \frac{1}{4} < \frac{1}{2}$ FALSE

② $\forall x \in \mathbb{Z}$, $2^x \in \mathbb{Z}$

if $x=-1$ $\frac{1}{2} \notin \mathbb{Z}$ FALSE, Neg: $\exists x \in \mathbb{Z}, 2^x \notin \mathbb{Z}$

③ $\forall x \in \mathbb{R}$, $x^2 > 0$

if $x=0$, $0=0$ FALSE, Neg : $\exists x \in \mathbb{R}, x^2 \leq 0$

④ $\forall m \in \mathbb{N}, \forall n \in \mathbb{N}$ $(m+n)^{m-n} \in \mathbb{Z}$

$m=1, n=2$ $(1+2)^{1-2} \notin \mathbb{Z}$ FALSE Neg: $\exists m \in \mathbb{N}, \exists n \in \mathbb{N}$ $(m+n)^{m-n} \notin \mathbb{Z}$

⑤ $\forall m \in \mathbb{N}, \forall n \in \mathbb{N}$, $m^m+n^n > m^n+n^m$

$m=n=1$ $1^1+1^1=1^1+1^1$ FALSE Neg : $m^m+n^n \leq m^n+n^m$

⑥ $\forall m \in \mathbb{N}, \forall n \in \mathbb{N}$, $m^n+n^m \geq m^m+n^2$

$m=3, n=2$ $3^2+2^3=17$, $3^3+2^2=31$ FALSE

⑦ $\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}$, $2x+y=5$ and $x-3y=-8$

$3/\ 2x+y=5$, $x-3y=-8$ $x=1$, $y=3$ TRUE Neg : $\forall x \in \mathbb{Z}$, $\forall y \in \mathbb{Z}$ $2x+y=5$ or $x-3y=-8$

⑧ $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$, $x.y=0$

Given any $x \in \mathbb{R}$, let $y=0$ then $x.0=0$ TRUE

Neg : $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x.y \neq 0$

⑨ $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$, $y^2=x^2+2x-1$

if $x=0$, $y^2=-1$, no sol. in $\mathbb{R}$ FALSE Neg. : $\exists x \in \mathbb{R}$ $\forall y \in \mathbb{R}$ $y^2 \neq x^2+2x-1$

⑩ $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$, $x^2+xy+y^2=1 \Rightarrow$

if $\Delta = x^2-4(x^2-1)<0$ $y^2+yx+1-x^2=0$, Suppose $x=2$
$4-3x^2<0$ $x=2$ $\Delta=-8<0$ No sol. in $\mathbb{R}$ FALSE

Neg : $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}$, $x^2+xy+y^2 \neq 1$

⑪ $\forall x \in \mathbb{N}, \exists y \in \mathbb{Z}$, $\frac{x+2y+1}{x+y+1} \in \mathbb{N}$

Given any will $x \in \mathbb{N}$, choose $y=0$ $\frac{x+1}{x+1}=1 \in \mathbb{N}$ TRUE

Neg : $\exists x \in \mathbb{N}, \forall y \in \mathbb{Z}$, $\frac{x+2y+1}{x+y+1} \notin \mathbb{N}$

⟶