MATHEMATICAL LOGIC

→**Proposition** (önerme): A proposition is a statement that has an exact truth value. (It is either true or false)

Ex: 1-) $2+2=5$ (A proposition)
2-) The earth is round. (A proposition)
3-) Only CS students are in this class (A proposition)

∝ Questions are not propositions such as What is your name? How are you? } Not propositions

∝ Come here
Go away
Do this } Not propositions

∝ $x+2=5$ → Not a proposition ∝ $x+y=z$ → Not a proposition } Predicates

Notation: We'll denote (ifade edilmek) propositions with lower-case letters. p, q, r, s, t..
$p: 2+2=5$
q: The earth is round.

The truth value of a proposition

A proposition p will have truth value T or 1 if it is true,
" " " F or 0 if it is false.

→ Propositions are classified structrally as (1) simple propositions → Simple statement
(2) compound propositions are made of least two simple propositions, using some special connectives.

Operations on Propositions

(1) **Negation** (Olumsuzlama): The negation of a proposition p will be denoted by p'.
p' is obtained from p by adding "not" in many cases.

Truth table

p	p'
1	0
0	1

$p: 2+2=5$
$p': 2+2\neq 5$

$p: 2^5 < 5^2$
$p': 2^5 \not< 5^2$ or $2^5 > 5^2$

p: It's raining today
p': It is not raining today

(2) Disjunction ("and", $\wedge$) (Ayırma)

The disjunction of two propositions p, q is obtained by the connection "and" $p\wedge q = p$ and q

p: It's raining today
q: It's cold today

$p \wedge q$ = It's cold and raining today.

The truth value of $p\wedge q$: For $p\wedge q$ to be true, both p and q must be true

p	q	$p\wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

③ Conjuction ("or", $\vee$) (Bağlama)

The conjuction of two propositions p, q is done by using the "or" connective.

$p \vee q \rightarrow p$ or q

p : You have to take Discrete Mathematics.

q : You have to take Calculus.

$p \vee q$: You have to take Disc. Math. and Calc.

Truth Table

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

For $p \vee q$ to be true it's enough for one of p and q to be true (In other words $p \vee q$ is false only when with p and q are false.)

③,5 ("xor", "exclusive or", $\oplus$)

$p + q$ is true if exactly one of p and q is true.

Truth Table

p	q	$p \oplus q$
1	1	0
1	0	1
0	1	1
0	0	0

Simple addition Modulo 2.

④ Implication, Conditional, "$\Rightarrow$") (Koşullu, Şartlı)

$p \Rightarrow q \rightarrow$ If p then q whenever p, then q

q when p

p implies q

$p \Rightarrow q$

p: the condition, the promise

q: the conclusion

Truth Value = $p \Rightarrow q$ is false only when p is true and q is false

Ex: If I have breakfast then I don't have lunch.

Such a person will have lied to us only if he has had breakfast and had lunch.

Truth Table

p	q	$p \Rightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1

⑤ Biconditional ($\Leftrightarrow$, "if and only if") (Ancak ve Ancak)

p if and only if q is true when p and q have the same truth value.

p	q	$\Leftrightarrow$
1	1	1
1	0	0
0	1	0
0	0	1

(Notice that $p \Leftrightarrow q$ and $(p \Rightarrow q) \wedge (q \Rightarrow p)$ have the same truth value.)

You take Math 110 $\Leftrightarrow$ you are a CS student.

You take the flight $\Leftrightarrow$ you buy a ticket.

There are three basic "logical gates"

not → $'$ → negative

and → $\wedge$

or → $\vee$

Remark: Compound propositions can be obtained by bringing together many propositions with connections and operations using parantheses in suitable places.

Ex: $((p \wedge q') \Rightarrow q)' \vee r$

$[(p \Rightarrow q') \vee r] \Rightarrow (q \wedge p')$

Finding the thruth value of compound propositions on truth table

Write down the truth table for Ex: $(p \wedge q') \Rightarrow (p \vee q)$

p	q	q'	$p \wedge q'$	$p \vee q$	$(p \wedge q') \Rightarrow (p \vee q)$
1	1	0	0	1	1
1	0	1	1	1	1
0	1	0	0	1	1
0	0	1	0	0	1

Ex: $((p \wedge q') \Rightarrow q)' \vee r$

p	q	r	q'	$p \wedge q'$	$(p \wedge q') \Rightarrow q$	$((p \wedge q') \Rightarrow q)'$	$((p \wedge q') \Rightarrow q)' \vee r$
1	1	1	0	0	1	0	1
1	1	0	0	0	1	0	0
1	0	1	1	1	0	1	1
1	0	0	1	1	0	1	1
0	1	1	0	0	1	0	1
0	1	0	0	0	1	0	0
0	0	1	1	0	1	0	1
0	0	0	1	0	1	0	0

$\Leftarrow$ or

Equvalence of Propositions

Two propositions p and q are said to be "equilavent", "$p \equiv q$", if p and q have two same truth value.

Two compound propositions are said to be equilavent if they have some truth value for all possible values of their components.

Examples

① $p \Rightarrow q \equiv p' \vee q$

we'll establish the equilavence by using truth table.

p	q	p'	$p \Rightarrow q$	$p' \vee q$
1	1	0	1	1
1	0	0	0	0
0	1	1	1	1
0	0	1	1	1

Identical

$p \Rightarrow q$ is equilavent to $p' \vee q$.

② $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$p \Leftrightarrow q$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$
1	1	1	1	1	1
1	0	0	1	0	0
0	1	1	0	0	0
0	0	1	1	1	1

Identical

Ex: $p \Rightarrow q \equiv q' \Rightarrow p'$ (Rule of Contrapositive)

p	q	p'	q'	$p \Rightarrow q$	$q' \Rightarrow p'$
1	1	0	0	1	1
1	0	0	1	0	0
0	1	1	0	1	1
0	0	1	1	1	1

The Algebra of Operations

① $(p')' = p$

② Commutativity of $\vee$ and $\wedge$

$p \wedge q = q \wedge p \quad , \quad p \vee q = q \vee p$

③ Associativity of $\vee$ and $\wedge$

$p \wedge (q \wedge r) = (p \wedge q) \wedge r \quad , \quad p \vee (q \vee r) = (p \vee q) \vee r$

④ Distributivity Laws

$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r) \quad , \quad p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

⑤ De Morgan Rules

$(p \wedge q)' = p' \vee q' \quad , \quad (p \vee q)' = p' \wedge q' \qquad \left(\begin{array}{l} \wedge' = \vee \\ \vee' = \wedge \end{array} \right)$

Applications

① $p \Rightarrow q = q' \Rightarrow p'$

$q' \Rightarrow p' = (q')' \vee p' \underset{①}{=} q \vee p' \underset{②}{=} p' \vee q = p \Rightarrow q$

Some well-known identities

$p \wedge 1 = p$	$p \wedge 0 = 0$	$p \vee p = p$	$p \vee p' = 1$
$p \vee 0 = p$	$p \vee 1 = 1$	$p \wedge p = p$	$p \wedge p' = 0$

A past midterm problem

Simplify the following propositions w/out using truth tables

a) $(p \Rightarrow q) \vee (p' \Rightarrow q) = (p' \vee q) \vee (p \vee q) = \underbrace{(p \vee p')}_{1} \vee \underbrace{(q \vee q)}_{q} = 1 \vee q = \underline{\underline{1}}$

b) $(p \Rightarrow q) \wedge (p' \Rightarrow q) = (p' \vee q) \wedge (p \vee q) = \underbrace{(p' \wedge p)}_{0} \vee q = 0 \vee q = \underline{\underline{q}}$ (Distributive Property)

Definition of Tautology and Contradiction

A compound proposition that is true for all values of the components is said to be a "tautology" $\rightarrow p \vee p'$

" " " " " false " " " " " " " " " " " "contradiction" $\rightarrow p \wedge p'$

Ex (Past Exam Problem)

Show that $(p \vee q)' \vee (p' \wedge q)$ is logically equivivalent to p' by

a-) using truth tables

b-) using identities

a-)

p	q	p'	$p \vee q$	$(p \vee q)'$	$p' \wedge q$	$(p \vee q)' \vee (p' \wedge q)$
1	1	0	1	0	0	0
1	0	0	1	0	0	0
0	1	1	1	0	1	1
0	0	1	0	1	0	1

$\vee$

Identical

b-) $(p \vee q)' \vee (p' \wedge q) \underset{\text{(De Morgan)}}{=} (p' \wedge q') \vee (p' \wedge q) \underset{\text{(Distributive)}}{=}$

$\Rightarrow p' \wedge \underbrace{(q' \vee q)}_{1} = p' \wedge 1 = \underline{p'}$

Ex = Prove that $[p \wedge (p \Rightarrow q)] \Rightarrow q$ is a tautology

a-) using truth tables

b-) using identities

b-) $[p \wedge (p \Rightarrow q)] \Rightarrow q = [p \wedge (p \Rightarrow q)]' \vee q \underset{\text{Distributivity}}{=} [\underbrace{(p \wedge p')}_{0} \vee (p \wedge q)]' \vee q = [0 \vee (p \wedge q)]' \vee q = (p \wedge q)' \vee q =$

$(p' \vee q') \vee p = p' \vee (q' \vee q) = p' \vee 1 = 1$

Logic on Electrical Switchboards

Gates on a switchboard

no current goes through

current goes through

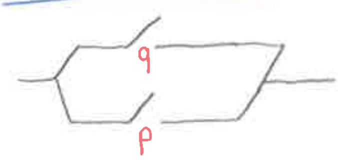
A gate can be shown by a proposition p.

Two main ways to connect gates

① Serial connection

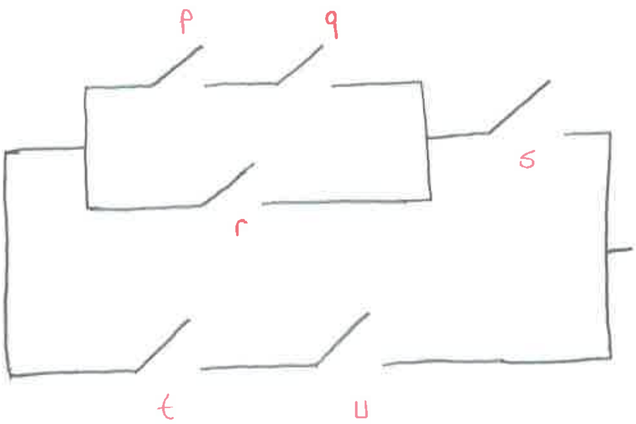
$\rightarrow p \wedge q$

② Parallel connection



$\rightarrow p \vee q$

Ex :



$[[(p \wedge q) \vee r] \wedge s] \vee (t \wedge u)$

Predicates, Variables, Quantifiers

→ $P(x) : \underbrace{x}_{\text{Variable}} < 3$ → a predicate (or open sentence)
→ not a proposition

By giving certain values to x, $P(x)$ can be turned into a proposition.

$P(2) : 2 < 3$ → TRUE (Proposition) $P(4) : 4 < 3$ → FALSE (Proposition)

→ $P(\underbrace{x,y}_{\text{Variables}}) : x = y + 1$

$P(2,1)$ is TRUE , $P(1,1)$ is FALSE , $P(x, x-1)$ is TRUE for any value of x.

Quantifiers : There are two quantifiers in Math.

1. The universal quantifier → "for all" → $\forall$
2. The existantial quantifier → "for some" → $\exists$

1-) $\forall x, p(x)$ This reads "for all x, $p(x)$ holds"

Ex= $\forall x, x < 3$ A proposition

When is it True : The proposition is true only when $p(x)$ holds for all values of x.

When is it False : The proposition is false if $p(x)$ does not hold for at least one value of x.

Ex : Everybody in this class owns an Iphone.
If one in the class doesn't own an Iphone, the above statement will be false
The statement will be true only if all people in the class own an Iphone

Ex : $\forall x \in R, x^2 \geq x$ $\forall x \geq 1, x^2 \geq x$ TRUE

$x = \frac{1}{2} \Rightarrow \left(\frac{1}{2}\right)^2 = \frac{1}{4} < \frac{1}{2}$, FALSE $x \geq 1$

2-) $\exists x, p(x)$ → When is it True ? is true if $P(x)$ holds for at least one value of x
$\exists x, x < 3$ is false if $P(x)$ fails for all values of x.

Ex : Some people in this class own an Iphone.
TRUE if one person has an Iphone.
FALSE if nobody in this class owns an Iphone.

NEGATING Statements with Quantifiers

$[\forall x \; P(x)]' = \exists x \; P(x)'$

$[\exists x \; P(x)]' = \forall x \; P(x)'$

Some special number sets we'll see frequently

$\mathbb{N} \to$ Set of natural numbers $= \{1,2,3,\ldots\}$

$\mathbb{Z} \to$ Set of integers $= \{\ldots,-2,-1,0,1,2,\ldots\}$

$\mathbb{Q} \to$ Set of rational numbers $= \{a/b \mid a,b \in \mathbb{Z}, b \neq 0\}$

$\mathbb{R} \to$ Set of real numbers

$\mathbb{R} \setminus \mathbb{Q} \to$ Set of irrational numbers $(\sqrt{2}, e, \pi, \sqrt[3]{2}, \ldots)$

$\mathbb{C} \to$ Set of complex numbers $(a+bi \mid a,b \in \mathbb{R} \quad i = \sqrt{-1})$

$[\forall x \quad P(x) \wedge Q(x)]' = \exists x \quad P(x)' \vee Q(x)'$, $[\exists x \quad P(x) \vee Q(x)]' = \forall x \quad P(x)' \wedge Q(x)'$

Ex: Find the truth of the following proposition, and negate it.

$\forall x \in \mathbb{Z} \, , \; 2^x \in \mathbb{Z}$ if $x = -1 \to 2^{-1} = \frac{1}{2} \notin \mathbb{Z}$ (FALSE)

NEGATION: $\exists x \in \mathbb{Z} \, , \; 2^x \notin \mathbb{Z}$

Ex: $\forall x \in \mathbb{R} \, , \; x^2 > 0$ if $x = 0 \Rightarrow 0^2 = 0 \not> 0$ FALSE

NEGATION $\exists x \in \mathbb{R} \, , \; x^2 \not> 0 \quad (x^2 \leq 0)$

$\forall x \in \mathbb{R} \, , x^2 \geq 0$

TRUE

NESTED QUANTIFIERS (More Variables)

The two variable case :

$\ldots x \quad \ldots y \qquad P(x,y)$

Four possibilities of use of quantifiers.

1. $\forall x \quad \forall y \quad P(x,y) \to$ For all x and for all y $P(x,y)$
2. $\exists x \quad \exists y \quad P(x,y) \to$ For some x and some y $P(x,y)$
3. $\forall x \quad \exists y \quad P(x,y) \to$ For all x, there exists some y such that $P(x,y)$ (The Most Common Case)
4. $\exists x \quad \forall y \quad P(x,y) \to$ There exists an value of x such that for all y, $P(x,y)$ (Rare)

When are they true? When are they false?

① $\forall x \ \forall y \ P(x,y)$ TRUE if $P(x,y)$ holds for all possible values of x and y

FALSE if we can find one value of x and one value of y for which $P(x,y)$ fails.

$(\forall x \ \forall y \ P(x,y))' = \exists x \ \exists y \ , P(x,y)'$

② $\exists x \ \exists y \ P(x,y)$ TRUE if we can find one value of x and one value of y for which $P(x,y)$ holds.

$[\exists x \ \exists y \ P(x,y)]'$ FALSE if $P(x,y)$ fails for all possible values of x and y.

$= \forall x \ \forall y \ P(x,y)'$

Determine the truth value, and negate

Ex: $\forall m \in \mathbb{N} \ , \forall n \in \mathbb{N} \quad (m+n)^{m-n} \in \mathbb{Z}$

$m=1 \ , n=2 \Rightarrow (m+n)^{m-n} = 3^{-1} = \frac{1}{3} \notin \mathbb{Z}$ → FALSE

$m=2 \ , n=1 \Rightarrow 3^1 \in \mathbb{Z}$ → TRUE

Negation: $\exists m \in \mathbb{N} \ , \exists n \in \mathbb{N} \quad (m+n)^{m-n} \notin \mathbb{Z}$

Ex: $\forall m \in \mathbb{N} \ , \forall n \in \mathbb{N} \ , m^m + n^n > m^n + n^m$

$m=n=1 \qquad 1^1+1^1 = 1^1+1^1$, FALSE

Negation: $\exists m \in \mathbb{N} \ , \exists n \in \mathbb{N} \quad m^m + n^n \leqslant m^n + n^m$ ($\cancel{\ }$)

Ex: $\forall m \in \mathbb{N} \ , \forall n \in \mathbb{N} \quad m^n + n^m \geqslant m^m + n^n$

$m=3, n=2 \qquad 3^2+2^3 = 17 \ , \ 3^3+2^2 = 31 \qquad 3^2+2^3 < 3^3+2^2$ FALSE

Negation: $\exists m \in \mathbb{N} \ , \exists n \in \mathbb{N} \ , m^n + n^m < m^m + n^n$

Ex: $\exists x \in \mathbb{Z} \ , \exists y \in \mathbb{Z} \quad 2x+y=5$ and $x-3y=-8$

For the statement to be true, the equation system $\begin{matrix} 3/2x+y=5 \\ x-3y=-8 \end{matrix}$ must have integer solutions.

for $x=1$ and $y=3$, $2x+y=5$ and $x-3y=-8$ The statement = TRUE

Negation: $\forall x \in \mathbb{Z} \ , \forall y \in \mathbb{Z} \quad 2x+y \neq 5$ or $x-3y \neq -8$

③ $\forall x \quad \exists y \quad P(x,y)$

x: the independent variable

y: the dependent variable (usually y will depend on x)

The statement is true if for any given X, we can find some y (depending on x) such that $P(x,y)$ holds.

The statement is false if we can final one value for x such that $P(x,y)$ fails no matter what y is.

Negation: $[\forall x \quad \exists y \quad P(x,y)]' = \exists x \quad \forall y \quad P(x,y)'$

Ex: $\forall x \in R \;, \exists y \in R \quad x \cdot y = 0$

Given any $x \in R$, let $y=0$, then $x \cdot 0 = 0$ (TRUE)

Negation: $\exists x \in R \;, \forall y \in R \quad x \cdot y \neq 0$

Ex: $\forall x \in R \;, \exists y \in R \;, y^2 = x^2 + 2x - 1$

if $x=0 \Rightarrow y^2 = -1$, no solutions in $\mathbb{R}$ (FALSE)

Negation: $\exists x \in R \;, \forall y \in R \;, y^2 \neq x^2 + 2x - 1$

Ex: $\forall x \in R \;, \exists y \in \mathbb{R} \;, x^2 + xy + y^2 = 1$

Suppose x is given $y^2 + yx + 1 - x^2 = 0$

if $\Delta = x^2 - 4(x^2 - 1) < 0 \qquad 4 - 3x^2 < 0 \qquad x = 2$

Suppose $x = 2$, $y^2 + 2y + 3 = 0 \qquad \Delta = 4 - 12 = -8 < 0$ No solutions in $\mathbb{R}$ (FALSE)

Negation: $\exists x \in \mathbb{R} \;, \forall y \in \mathbb{R} \quad x^2 + xy + y^2 \neq 1$

Ex: $\forall x \in N \;, \exists y \in \mathbb{Z} \;, \dfrac{x + 2y + 1}{x + y + 1} \in \mathbb{N}$

Given any will $x \in \mathbb{N}$, choose $y = 0 \qquad \dfrac{x+1}{x+1} = 1 \in \mathbb{N}$ (TRUE)

Negation: $\exists x \in \mathbb{N} \;, \forall y \in \mathbb{Z} \;, \dfrac{x + 2y + 1}{x + y + 1} \notin \mathbb{N}$

Ex : $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \quad xy = 1$

if $x = 0$, $x.y = 0$ for all $y \in \mathbb{R}$ (FALSE)

Negation : $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x.y \neq 1$

Ex : $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \quad x+y \in \mathbb{R}/\mathbb{Q}$ or $xy \in \mathbb{R}\setminus\mathbb{Q}$

Given any $x \in \mathbb{R}$, choose $y = \sqrt{2} - x \Rightarrow x+y = \sqrt{2} \in \mathbb{R}/\mathbb{Q}$ (TRUE)

Ex : $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \quad x+y \in \mathbb{R}/\mathbb{Q}$ and $xy \in \mathbb{R}/\mathbb{Q}$

let $x = 0$, for all $y \in \mathbb{R}$, $xy = 0 \notin \mathbb{R}\setminus\mathbb{Q}$ (FALSE)

Negation : $\exists x \in \mathbb{R}, \forall y \in \mathbb{R} \quad x+y \in \mathbb{Q}$ or $xy \in \mathbb{Q}$

Ex : $\forall x \in \mathbb{R}\setminus\mathbb{Q}, \exists y \in \mathbb{R}, x^y \in \mathbb{Q}$

Given any $x \in \mathbb{R}\setminus\mathbb{Q}$, let $y = 0$ then $x^y = x^0 = 1 \in \mathbb{Q}$ (since $x \neq 0$) (TRUE)

Negation : $\exists x \in \mathbb{R}\setminus\mathbb{Q}, \forall y \in \mathbb{R}, x^y \notin \mathbb{Q}$

Ex : $\forall x \in \mathbb{Q}, \forall y \in \mathbb{Q}, x^y \in \mathbb{R}$

$x = -1$, $y = \frac{1}{2}$ $\quad x^y = (-1)^{1/2} = \sqrt{-1} \notin \mathbb{R}$ (FALSE)

Negation : $\exists x \in \mathbb{Q}, \exists y \in \mathbb{Q}, x^y \notin \mathbb{R}$

Ex : $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}\setminus\mathbb{Q}, x+y \in \mathbb{R}\setminus\mathbb{Q}$ or $xy \in \mathbb{Q}$

Suppose $x \in \mathbb{Q}$, then choose $y = \sqrt{2} \Rightarrow \sqrt{2} + x \in \mathbb{R}\setminus\mathbb{Q}$ (Irrational + Rational = Irrational) (TRUE)

Suppose $x \in \mathbb{R}\setminus\mathbb{Q}$, choose $y = \frac{1}{x} \in \mathbb{R}\setminus\mathbb{Q}$ and $xy = 1 \in \mathbb{Q}$ (TRUE)

Negation : $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}\setminus\mathbb{Q}, x+y \in \mathbb{Q}$ and $xy \in \mathbb{R}\setminus\mathbb{Q}$

Ex : $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}\setminus\mathbb{Q}, x+y \in \mathbb{R}\setminus\mathbb{Q}$ or $x.y \in \mathbb{Q}$

$x = \sqrt[3]{2}$	$x+y = 0 \in \mathbb{Q}$	$x = 1+\sqrt{2}$	$x+y = 1 \in \mathbb{Q}$	(FALSE)
$y = -\sqrt[3]{2}$	$x.y = -\sqrt[3]{4} \notin \mathbb{Q}$	$y = -\sqrt{2}$	$x.y = -\sqrt{2}-2 \notin \mathbb{Q}$	

Ex : $\forall m \in \mathbb{Z}, \exists n \in \mathbb{Z}, m+n=4$

given $m \in \mathbb{Z}$, let $n = 4 - m \in \mathbb{Z}$ then $m+n=4$ (TRUE)

$\exists m \in \mathbb{Z}, \exists n \in \mathbb{Z}, m+n=4$

$m=2, n=2 \quad 2+2=4$ (TRUE)

Ex : $\forall x \in \mathbb{Z} \quad x^3 - x \equiv 0 \pmod 6$

$$x^3 - x = x(x-1)(x+1)$$

$$= \underbrace{(x-1).x}_{\text{divisible by } 2}.(x+1) \longrightarrow \text{divisible by } 3$$

So it is TRUE

PROOF METHOD

① Direct Proof

$p \Rightarrow q \rightarrow p$ implies q

Many mathematical statements and theorems are of their form.

(hypothesis) $\Rightarrow$ (assertion)

if x is even then x^2 is divisible by 4.

if x is rational then $\frac{1}{x}$ is rational.

In mathematics many theorems consist of a chain of implications. "implication" is transitive.

if $p \Rightarrow q$ and $q \Rightarrow r$ then $p \Rightarrow r \qquad p \Rightarrow q \Rightarrow r$

- If you study hard, you can get good grades.

If you get good grades, you will be able to find a good job.

Ex : If $x \in \mathbb{Z}$ is even, then x^2 is divisible by 4

Proof : if x is even $x = 2k$ for some $k \in \mathbb{Z}$ then $x^2 = (2k)^2 = 4k^2$ then x^2 is divisible by 4.

Ex : If $x \in \mathbb{R}$, then $x^2 - 2x + 2 \geq 1$

Proof : if $x \in \mathbb{R}$, $x^2 - 2x + 2 = x^2 + 2x + 1 - 1 = (x-1)^2 + 1 \geq 0 + 1 \geq 1$

Ex : If $x \in \mathbb{R}$, then $x^2 + x + 1 > 0$

$$x^2 + x + 1 = x^2 + 2.x.\frac{1}{2} + \overbrace{\frac{1}{4} + \frac{3}{4}}^{1} = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} \geq 0 + \frac{3}{4} \geq \frac{3}{4} > 0$$

② Contrapositive (Proof By Contra Positive)

This proof method is based on the following logical equivalance.

$p \Rightarrow q \equiv q' \Rightarrow p'$

Instead of proving "p implies q", we prove "not q implies not p".

Ex: if x^2 is odd then x is odd.

if x is even x^2 is even

if x is even, then $x = 2k$ for some $k \in \mathbb{Z}$ $x^2 = (2k)^2 = 4k^2$ so x^2 is even.

Ex: if $x^2 + x < 6$ then $x < 2$

by contrapositive, this is equivalent to if $x \geq 2$, then $x^2 + x \geq 6$ indeed,

if $x \geq 2$ then $x^2 \geq 4$, now $\left.\begin{array}{l} x \geq 2 \\ x^2 \geq 4 \end{array}\right\} x^2 + x \geq 2 + 4 = 6$

③ Proof By Contradiction

Assume that we want to show a proposition p is true.

if $p' \Rightarrow 0$, then p must be 1. So if by assuming that p is false, we arrive at a contradiction (sth. that is completely false) then this shows that p is true.

Examples

① If $m > 1$ is a positive integer and $k \in \mathbb{Z}$ is any integer, then $m \nmid km+1$ (m does not divide $km+1$)

Proof: Assume $m \mid km+1 \Rightarrow k.m+1 = m.n$ for some $n \in \mathbb{Z}$

$\Rightarrow 1 = \underbrace{m}_{>1}.\underbrace{(n-k)}_{\in \mathbb{Z}} \rightarrow$ Contradiction   Thus $m \nmid km+1$

② $\sqrt{2}$ is irrational.

Assume $\sqrt{2} \in \mathbb{Q}$ $\sqrt{2} = \frac{a}{b}$, $a,b \in \mathbb{Z}_+$ $\frac{1}{2} = \frac{3}{6} = \frac{17}{34} = \frac{-75}{-150} \cdots$

$2 = \frac{a^2}{b^2} \Rightarrow a^2 = 2.b^2$ $\frac{a}{b}$ can not be cancelled further. (i.e., a and b have no common divisor)

a^2 is even $\Rightarrow a$ is even $\Rightarrow a = 2a_1$ for some $a_1 \in \mathbb{Z}_+$

$\Rightarrow 4a_1^2 = 2b^2 \Rightarrow b^2 = 2a_1^2 \Rightarrow b^2$ is even

$\Rightarrow b$ is even, so $b = 2b_1$ for some $b_1 \in \mathbb{Z}_+$

$\left.\begin{matrix} a = 2a_1 \\ b = 2b_1 \end{matrix}\right\}$ Contradiction

→ $p > 1$ is prime if a/p implies $a = 1$ or p

2, 3, 5, 7, 11, 13, 17, 19

Every positive integer > 1, can be written as a product of primes.

Theorem (Euclid): There are infinitely many primes.

Proof: Assume there are finitely many primes.

Assume that all primes are given by $\{P_1, P_2, \dots, P_n\}$

$N = P_1.P_2 \dots P_n + 1$

$N > P_1, P_2, \dots, P_n$

Two cases:

(1) if N is prime ✓

(2) if N is not prime, then those exists a prime number q such that q/N

by Example(1), $P_1 \nmid N$, $P_2 \nmid N$, ..., $P_n \nmid N$, So $q \neq P_1, P_2 \dots, P_n$ ✓

Example There exists 13 consecutive integers, none of which is prime.

i.e., there exists $a \in \mathbb{Z}_+$
s.t., $\underbrace{a+1, a+2, \ldots, a+13}$

↳ None of these is prime

$\underbrace{14!+2}_{\substack{2 \\ \text{even}}}, \; 14!+\underbrace{3}_{\substack{\text{divisible} \\ \text{by } 3}}, \ldots, \underbrace{14!+14}_{\substack{\text{divisible} \\ \text{by } 14}}$ $\qquad 14! = 1 . 2 . 3 . 4 \ldots 14$

None generally for any $n \geqslant 2$, $\underbrace{(n+1)!+2, (n+1)!+3, \ldots (n+1)!+(n+1)}_{n \text{ Consecutive integers, none is prime.}}$

Example $\exists a \in \overset{(\text{irrational})}{\mathbb{R} \setminus \mathbb{Q}}, \; \exists b \in \mathbb{R} \setminus \mathbb{Q}, \; a^b \in \mathbb{Q}$ → TRUE

- if $\sqrt{2}^{\sqrt{2}} \in \mathbb{Q}$ then
- if $\sqrt{2}^{\sqrt{2}} \in \mathbb{R}$, then $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = (\sqrt{2})^{\sqrt{2} . \sqrt{2}} = (\sqrt{2})^2 = 2 \in \mathbb{Q}$

SETS

A set is a collection of "objects", listed without repetition.

There are 3 basic ways of describing a set.
- Verbally
- Through Venn-diagrams
- Listing

Sets will be denoted by capital letters. A, B, C ...

The objects in the sets will be called "element", this relationship will be expressed as "$X \in A$" → X is an element of A.

A= All students in MATH 110 class

=

$= \{A, E \ldots, \ldots\}$

$\{x \in \mathbb{Z} \mid x \text{ is even}\} = \{\ldots, -4, -2, 0, 2, 4 \ldots\}$ = All even integers

$\{x \in \mathbb{Z}_+ \mid x \text{ is prime}\} = \{2, 3, 5, 7 \ldots\}$

$\{x \in \mathbb{R} \mid 2 \leqslant x < 3\} = [2,3)$

$\{x \in \mathbb{R} \mid 2 < x < 3\} = (2,3)$

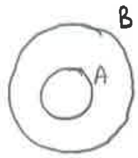
$\{x \in \mathbb{R} \mid 2 \leqslant x \leqslant 3\} = [2,3]$

2 Special Sets

- The empty set is the set that has no elements. $\emptyset$ or $\{\}$
- The universal set (it depends on the context) is the set that contains all the sets within a given context. Denoted by U.

Subset: We say $A \subseteq B$ (A is a subset of B)

if every element of A is also in B or $x \in A \Rightarrow x \in B$



$\emptyset \subseteq A$, $A \subseteq U$, $A \subseteq A$

$(2,3] \subseteq [2,3]$

$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$

if $A \subseteq B$ and there exists $x \in B$ such that we say A is a proper subset of B.

Equality of sets

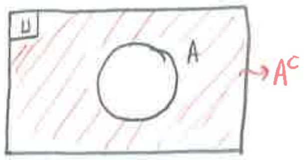
We say $A = B$ if they consists of the exact same elements.

Proposition: $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$ in other words $A = B \Leftrightarrow x \in A \Rightarrow x \in B$ and $x \in B \Rightarrow x \in A$

Operations of sets

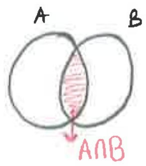
① **Complement:** Let A be a set, by the complement of A, we'll mean the set of elements that are not in A.

$A^c = \{x \in U \mid x \notin A\}$



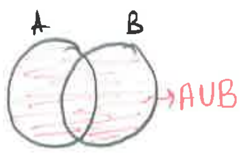
② **Intersection:** The intersection of A and B is the set of all elements that are in both A and B

$A \cap B = \{x \mid \underbrace{x \in A}_{p} \text{ and } \underbrace{x \in B}_{q}\}$ $\quad p \wedge q$



③ **Union:** The union of A and B is the set of all elements that are in A or in B.

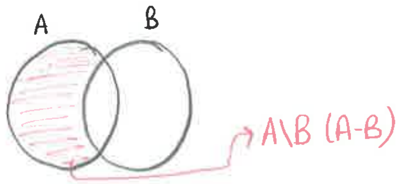
$A \cup B = \{x \mid \underbrace{x \in A}_{p} \text{ or } \underbrace{x \in B}_{q}\}$ $\quad p \vee q$

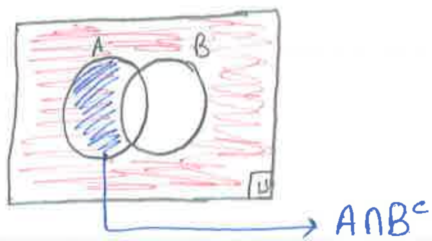


④ **Difference:** $A \setminus B$ (or $A - B$) $= \{x \mid x \in A \text{ and } x \in B\}$

$= \{x \mid x \in A \text{ and } x \in B^c\}$

$= A \cap B^c$



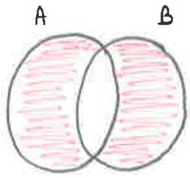


13.03.2015
Thursday

⑤ Symmetric Difference

$A \Delta B = \{x \mid x$ belongs to exactly one of A or $B\}$

$= \{x \mid x \in A \text{ xor } x \in B\}$



$A \Delta B = (A \setminus B) \cup (B \setminus A)$
$= (A \cap B^c) \cup (B \cap A^c)$

*

$U = \mathbb{R}$

$A = \{0\}$, $A^c =$ All nonzero real numbers $= \mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$

Properties of operations on Sets

① Commutativity

$A \cup B = B \cup A$, $A \cap B = B \cap A$

② Associativity

$A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$

③ Distributivity

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

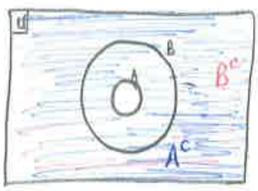
④ De Morgan Rules

$(A \cap B)^c = A^c \cup B^c$, $(A \cup B)^c = A^c \cap B^c$

⑤ $A \subseteq B \iff B^c \subseteq A^c$

$\underbrace{x \in A}_{p} \Rightarrow \underbrace{x \in B}_{q}$ $\qquad x \notin B \Rightarrow x \notin A$

$p \Rightarrow q \equiv q' \Rightarrow p'$ (Rule of Contrapositive)



$B^c \subseteq A^c$

Some Further Properties (Observations)

① $A \cap B \subseteq A$, $A \cap B \subseteq B$

$x \in A \cap B \Rightarrow x \in A$ and $x \in B \Rightarrow A \cap B \subseteq A$, $A \cap B \subseteq B$

1.5 $A \cap B = A \iff A \subseteq B$

Proof: ($\Rightarrow$) Hypothesis : $A \cap B = A$
we want to show : $A \subseteq B$

From (1) $A \cap B \subseteq B$
$A \subseteq B$

($\Leftarrow$) Hypothesis : $A \subseteq B$
Conclusion : $A \cap B = A$

$A \cap B = A \iff A \subseteq A \cap B$ and $A \cap B \subseteq A$

From (1) $A \cap B \subseteq A$

To show $A \subseteq A \cap B$, assume $x \in A$.

Since $A \subseteq B$, $x \in A \Rightarrow x \in B$

So since $x \in A$ and $x \in B \Rightarrow x \in A \cap B$

$\Rightarrow A \subseteq A \cap B$

1.75 If $C \subseteq A$ and $C \subseteq B$ then $C \subseteq A \cap B$

$C \subseteq A$: $x \in C \to x \in A$
$C \subseteq B$: $x \in C \to x \in B$

$x \in C \Rightarrow x \in A$ and $x \in B$
$x \in C \Rightarrow x \in A \cap B$

2 $A \subseteq A \cup B$
$B \subseteq A \cup B$

2.5 $A \cup B = B \iff A \subseteq B$

2.75 If $A \subseteq C$ and $B \subseteq C$ then $A \cup B \subseteq C$

3 $A \cap A = A$

$A \cap \emptyset = \emptyset$

$A \cap \mathcal{U} = A$

$A \cap A^c = \emptyset$ → Def: If $A \cap B = \emptyset$, A and B are called "disjoint sets"

4 $A \cup A = A$

$A \cup \emptyset = A$

$A \cup \mathcal{U} = \mathcal{U}$

$A \cup A^c = \mathcal{U}$

5 $\emptyset^c = \mathcal{U}$

$\mathcal{U}^c = \emptyset$

$(A^c)^c = A$

Example: $\mathcal{U} = \mathbb{R}$

$A = [-2,3) = \{x \in \mathbb{R} \mid -2 \leq x < 3\}$

$B = [2,4] = \{x \in \mathbb{R} \mid 2 \leq x \leq 4$

$A \cup B = [-2,4] = \{x \in \mathbb{R} \mid -2 \leq x \leq 4\}$

$A \cap B = [2,3) = \{x \in \mathbb{R} \mid 2 \leq x < 3\}$

$A^c = (-\infty,-2) \cup [3,\infty)$

$B^c = (-\infty,2) \cup (4,\infty)$

Ex: Show that for any set A, B and C

$(A\setminus B)\setminus C = A\setminus(B\cup C)$

$(A\setminus B)\setminus C = (A\cap B^c)\cap C^c \underset{\text{Associavity}}{=} A\cap(B^c\cap C^c) \underset{\text{De Morgan}}{=} A\cap(B\cup C)^c = A\setminus(B\cup C)$

Ex (Past Midterm Problem)

$A=\{2n+1 \mid n\in\mathbb{Z}\}$, $B=\{3n-1\mid n\in\mathbb{Z}\}$, $C=\{4n-1\mid n\in\mathbb{Z}\}$, $D=\{6n\mid n\in\mathbb{Z}\}$ Determine whether the following are True or False

a-) $B\subseteq A$ → FALSE, Because $2\in B$, but $2\notin A$
b-) $D\subseteq A^c$
c-) $B\cap D=\emptyset$
d-) $A\cup D\cup B=\mathbb{Z}$
e-) $A\setminus(B\setminus C)=D^c$
f-)

Sol: $A=\{\ldots,-5,-3,-1,1,3,5,\ldots\}$ → All odd integers

$B=\{\ldots,-7,-4,-1,2,5,8,\ldots\}$

$C=\{\ldots,-9,-5,-1,3,7,\ldots\}$

$D=\{\ldots-12,-6,0,6,12,\ldots\}$

b-) $x\in D \Rightarrow x=6.k=2.3k \Rightarrow x$ is even $\Rightarrow x\in A^c$, TRUE

c-) TRUE, Because every integer in D is divisible by 3, but integers in B are not.

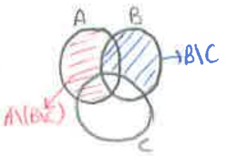
d-) $4\notin A$, $4\notin D$, $4\notin B$ because $4=3n-1$ has no solutions in integers
$4\notin A\cup D\cup B$, FALSE

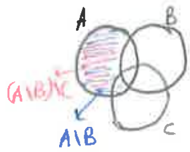
e-) $2\in D^c$, but $A\setminus(B\setminus C)\subseteq A$ → Odd integers
$2\notin A\setminus(B\setminus C)$, FALSE

Ex: For arbitrary sets A, B, C. Determine whether the following are true or false

a-) $A\cup B=A \Rightarrow A=B$, $A=\{1\}$, $B=\emptyset$, $A\cup B=A$ but $A\neq B$ (FALSE)

b-) $A\setminus(B\setminus C)=(A\setminus B)\setminus C$,

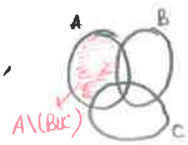


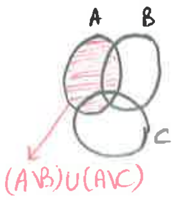


, $A=\{1,2\}$, $B=\emptyset$, $C=\{1\}$, $A\setminus(B\setminus C)=\{1,2\}$
$(A\setminus B)\setminus C=\{2\}$, FALSE

c-) $(A\cup B)\cap C\subseteq A\cup C$, $(A\cup B)\cap C\subseteq C\subseteq A\cup C$ TRUE

d-) $A\setminus(B\cup C)=(A\setminus B)\cup(A\setminus C)$,





, $A=\{1,2\}$, $B=\emptyset$, $C=\{1\}$, $A\setminus(B\cup C)=\{2\}$
$(A\setminus B)\cup(A\setminus C)=\{1,2\}\cup\{2\}$
$=\{1,2\}$
FALSE

e-) $A\cap B=A\cap B^c \Rightarrow A=\emptyset$

$(A\cap B)\cap(A\cap B^c)=(A\cap A)\cap(B\cap B^c)=\emptyset$ since $A\cap B=A\cap B^c$

$A\cap B=A\cap B^c=(A\cap B)\cap(A\cap B^c)=\emptyset$

Arbitrary unions and Intersections

$A \cap B \cap C = \{x \mid x \in A \wedge x \in B \wedge x \in C\}$

$A \cup B \cup C = \{x \mid x \in A \vee x \in B \vee x \in C\}$

$A_1 \cap A_2 \cap A_3 \cap \dots A_n = \{x \in A_1 \wedge x \in A_2 \wedge x \in A_3 \dots \wedge x \in A_n\} = \{x \mid x$ belongs to all $A_i\}$

$A_1 \cup A_2 \cup A_3 \cup \dots A_n = \{x \mid x \in A_1 \vee x \in A_2 \vee x \in A_3 \dots \vee x \in A_n\} = \{x \mid x$ belongs to some $A_i\}$

Infinite intersection and union

Suppose $\{A_1, A_2 \dots\}$ is an infinite collection of sets.

Then $\underbrace{\bigcup_{n=1}^{\infty} A_n}_{A_1 \cup A_2 \cup A_3 \cup \dots} = \{x \mid \exists\, i \in \mathbb{N},\ x \in A_i\}$, $\underbrace{\bigcap_{n=1}^{\infty} A_n}_{A_1 \cap A_2 \cap A_3 \cap \dots} = \{x \mid \forall i \in \mathbb{N},\ x \in A_i\}$

PROPERTIES

① $\forall i \in \mathbb{N}$, $\bigcap_{n=1}^{\infty} A_n \subseteq A$

$\forall i \in \mathbb{N}$, $A_i \subseteq \bigcup_{n=1}^{\infty} A$

② If $\forall i \in \mathbb{N}$, $A_i \subseteq B$, then $\bigcup_{n=1}^{\infty} A_n \subseteq B$

If $\forall i \in \mathbb{N}$, $B \subseteq A_i$, then $B \subseteq \bigcap_{n=1}^{\infty} A_n$

③ De Morgan Rules

$\left(\bigcap_{n=1}^{\infty} A_n\right)^c = \bigcup_{n=1}^{\infty} A_n^c$ $\quad \left(\left(\forall i \in \mathbb{N}, x \in A_i\right)' = \exists\, i \in \mathbb{N}, x \in A_i, (x \in A_i^c)\right)$

$\left(\bigcup_{n=1}^{\infty} A_n\right)^c = \bigcap_{n=1}^{\infty} A_n^c$ $\quad \left(\left(\exists\, i \in \mathbb{N}, x \in A_i\right)' = \forall i \in \mathbb{N}, x \notin A_i, (x \in A_i^c)\right)$

④ Distributive Property

$A \cap \left(\bigcup_{n=1}^{\infty} B_n\right) = \bigcup_{n=1}^{\infty} (A \cap B_n)$ $\quad \left[\begin{array}{l} A \cap (B_1 \cup B_2) \\ = (A \cap B_1) \cup (A \cap B_2) \end{array}\right]$, $A \cup \left(\bigcap_{n=1}^{\infty} B_n\right) = \bigcap_{n=1}^{\infty} (A \cup B_n) = \left[\begin{array}{l} A \cup (B_1 \cap B_2) \\ = (A \cup B_1) \cap (A \cup B_2) \end{array}\right]$

Nested Sets

Increasing nested sets: $A_1 \subseteq A_2 \subseteq A_3 \subseteq A_4 \subseteq A_5 \dots$

Decreasing nested sets: $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \supseteq A_5 \dots$

Remark: If $A_1 \subseteq A_2 \subseteq A_3 \dots$ then $\bigcap_{n=1}^{\infty} A_n = A_1$, union requires extra work.

If $A_1 \supseteq A_2 \supseteq A_3 \dots$ then $\bigcup_{n=1}^{\infty} A_n = A_1$, intersection requires extra work.

Ex: Find $\bigcup_{n=1}^{\infty}\left(2-\frac{1}{n}, 2+\frac{1}{n}\right)$

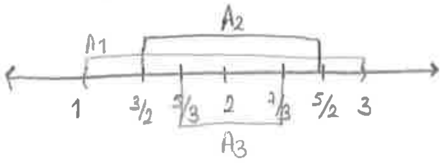
$\bigcap_{n=1}^{\infty}\left(2-\frac{1}{n}, 2+\frac{1}{n}\right)$

Sol: $A_n = \left(2-\frac{1}{n}, 2+\frac{1}{n}\right)$

$A_1 = (1,3)$

$A_2 = (3/2 + 5/2)$

$A_3 = (5/3 + 7/3)$



$A_1 \supseteq A_2 \supseteq A_3 \dots$

$\bigcup_{n=1}^{\infty}\left(2-\frac{1}{n}, 2+\frac{1}{n}\right) = (1,3)$

$\bigcap_{n=1}^{\infty}\left(2-\frac{1}{n}, 2+\frac{1}{n}\right) = \{2\}$ (This requires proof)

$\lim_{x\to\infty}\left(2-\frac{1}{n}\right) = \lim_{n\to\infty}\left(2+\frac{1}{n}\right) = 2$

Ex: $A_n = \left(0, 2-\frac{1}{n}\right)$ Find $\bigcup_{n=1}^{\infty} A_n$ and $\bigcap_{n=1}^{\infty} A_n$

Sol: $A_1 = (0,1)$

$A_2 = (0, 3/2)$

$A_3 = (0, 5/3)$

Since $\lim_{n\to\infty}\left(2-\frac{1}{n}\right) = 2$

and since $2-\frac{1}{n} < 2$ for all n, $\Rightarrow$

$A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$

$\bigcap_{n=1}^{\infty} A_n = A_1 = (0,1)$

$\bigcup_{n=1}^{\infty} A_n = (0,2)$

Ex: $A_n = \left(-\infty, 1+\frac{1}{n}\right]$ Find $\bigcap_{n=1}^{\infty} A_n$, $\bigcup_{n=1}^{\infty} A_n$

Sol: $A_1 = (-\infty, 2]$

$A_2 = (-\infty, 3/2]$

$A_3 = (-\infty, 4/3]$

$A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots A_n$

$\bigcup_{n=1}^{\infty} A_n = A_1 = (-\infty, 2]$

Since $\lim_{x\to\infty}\left(1+\frac{1}{n}\right) = 1$

and $1 \in A_n$ for all n

$\bigcap_{n=1}^{\infty} A_n = (-\infty, 1]$

Ex: $A_n = \left(-\infty, 1-\frac{1}{n}\right]$ Find $\bigcup_{n=1}^{\infty} A_n$ and $\bigcap_{n=1}^{\infty} A_n$

$A_1 = (-\infty, 0]$

$A_2 = (-\infty, 1/2]$

$A_3 = (-\infty, 2/3]$

$A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots \subseteq A_n$

$\bigcap_{n=1}^{\infty} A_n = A_1 = (-\infty, 0]$

Since $\lim_{n\to\infty}\left(1-\frac{1}{n}\right) = 1$

and $1-\frac{1}{n} < 1$ for all n,

which means $1 \notin A_n$, for all n

$\bigcup_{n=1}^{\infty} A_n = (-\infty, 1)$

Power set

Let A be any set, by the power set of A, we mean the set of all subsets of A.

Mathematically speaking, $\mathbb{P}(A) = \{x \mid x \subseteq A\}$

Remark. The power set is never empty.

For any set A, $\emptyset \in \mathbb{P}(A)$ in particular if $A \neq O$, the $P(A) = \{\emptyset\}$ $\qquad A \in \mathbb{P}(A)$

$\{\text{EVEN INTEGERS}\} \in \mathbb{P}(\mathbb{Z})$

$\{\text{PRIME NUMBERS}\} \in \mathbb{P}(\mathbb{Z})$

Ex: if $A = \{1,2,3\}$

$\mathbb{P}(A) = \{\emptyset, \{1\}, \{1,2\}, \{1,2,3\}, \{2\}, \{2,3\}, \{3\}, \{1,3\}\}$

The number of subsets of a finite set

Suppose A has n elements, $|A| = n$. Then $|\mathbb{P}(A)| = 2^n$

Suppose $A = \{1,2,3,\dots,n\}$, Any subset of A is formed by taking some elements from A, and by omitting the rest.

If we pick an element we'll denote that by a "1" omitting an element will be denoted by a "0"

The binomial expansion theorem

$$(x+y)^n = \binom{n}{n} \cdot x^n \cdot y^0 + \binom{n}{n-1} \cdot x^{n-1} \cdot y^1 + \dots + \binom{n}{k} x^k \cdot y^{n-k} + \dots + \binom{n}{0} \cdot x^0 \cdot y^n = \sum_{k=0}^{n} \binom{n}{k} x^k \cdot y^{n-k}$$, for all x and y

Put $x = y = 1$

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot 1^k \cdot 1^{n-k} = \underbrace{\binom{n}{0}}_{\#\text{ of 0-element subsets of } A} + \overbrace{\binom{n}{1}}^{\#\text{ of 1-element subsets}} + \dots \binom{n}{n}$$

The number of subsets

Properties

(1) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Proof : $B \cap C \subseteq B \Rightarrow A \times (B \cap C) \subseteq A \times B$

$B \cap C \subseteq C \Rightarrow A \times (B \cap C) \subseteq A \times C$

$A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$

Suppose $(a, x) \in (A \times B) \cap (A \times C)$

$\Rightarrow a \in A,\ x \in B,\ x \in C$

$\Rightarrow a \in A,\ x \in B \cap C \Rightarrow (a, x) \in A \times (B \cap C)$

$(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$

(2) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(3) $(A \times C) \cap (B \times D) = (A \cap B) \times (C \cap D)$

Indeed : $A \cap B \subseteq A, B$

$C \cap D \subseteq C, D$

$(A \cap B) \times (C \cap D) \subseteq A \times C$

$(A \cap B) \times (C \cap D) \subseteq B \times C \quad \Rightarrow (A \cap B) \times (C \cap D) \subseteq (A \times C) \cap (B \times D)$

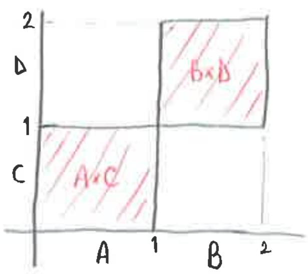
Suppose $(x, y) \in (A \times C) \cap (B \times D)$

$\Rightarrow x \in A,\ x \in B \Rightarrow x \in A \cap B$

$y \in C,\ y \in D \Rightarrow y \in C \cap D \quad \Rightarrow (x, y) \in (A \cap B) \times (C \cap D)$

(4) In general $(A \times C) \cup (B \times D) \neq (A \cup B) \times (C \cup D)$ ($\subseteq$ always True)

Example :



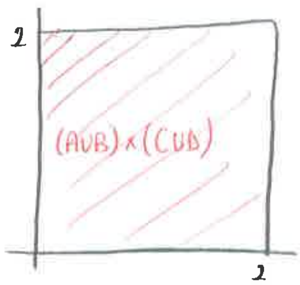
$(A \times C) \cup (B \times D)$

$A = [0,1]$

$B = [1,2]$

$C = [0,1]$

$D = [1,2]$



$A \cup B = [0,2]$

$C \cup D = [0,2]$