Random and Stochastic Processes

$A \cup B = A + B$
$A \cap B = A \cdot B$

die: zar
dice: zarlar
outcome: sonuç
trial: deneme

Probability Space

$S\ \Omega$, a certain event and its elements "experimental outcomes" and subsets "events".

$\{\emptyset\}$, $\emptyset$, an impossible event

* outcomes of an experiment six faces of a die F_i the face number with $S = \{F_1, F_2, F_3, F_4, F_5, F_6\}$. This space has 2^6 subsets.

- Die experiment, Even, Odd

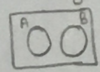
  $S = \{Even, Odd\}$ 2^2 subsets

  $\{even\}$ is a single outcome.

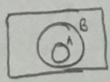
Trial: A single performence of an experiment is called a "trial". At each trial we observe a single outcome n_i.
We say that the even "A occurs" during this trial if it contains the element n_i. The certain event always occurs and the imposseble event never occurs.

- The event $A \cup B$ occurs if both A and B, or A, or B occurs.
- The event AB occurs when both A and B occurs.
- If the events A and B are mutually exclusive and event A occurs, then the event B does not occur.

S



- $A \subset B$ and occurs
  $\longrightarrow$ B occurs



- At each trial either A or $\bar{A}$ occurs.

The Axiome of Probability

We assign to each event A a number, $P(A)$ which we call "the probability of the event".

1. $P(A) \geq 0$
2. $P(S) = 1$
3. $AB = \{\emptyset\} \rightarrow P(A \cup B) = P(A) + P(B)$

Properties

$P\{\emptyset\} = 0$

$A\{\emptyset\} = \{\emptyset\}$ $A \cup \{\emptyset\} = A$

$P(A) = P(A \cup \emptyset) = P(A) + P(\emptyset)$ for any A, $P(A) = 1 - P(\bar{A}) \leq 1$

- $A \cup \bar{A} = S$ $A\bar{A} = \emptyset$
- $1 = P(S) = P(A \cup \bar{A}) = P(A) + P(\bar{A})$

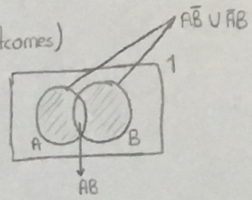
For any A and B

$P(A \cup B) = P(A) + P(B) - P(AB)$
$\leq P(A) + P(B)$

Equality of Events

Two events A and B are called equal if they consist of the same elements (outcomes)

· They are called equal with probability 1 if the set $(A \cup B)(\overline{AB}) = A\bar{B} \cup \bar{A}B$



consisting of all outcomes that are in A or B but not in AB has a zero probability.

S = certain event
Ω = sample space

Two coin tosses, $S = \{hh, ht, th, tt\}$

The class F of events

Events are subsets of S. We can present a class F of subsets

· if A and B are events → $A \cup B$ is also an event, $A.B$ is also an event.

Fields

A field, F is non-empty class of sets such that
If $A \in F$ then
If $A \in F$ and $B \in F \rightarrow A \cup B \in F$

It follows

. $A \in F$ and $B \in F \rightarrow AB \in F$ this can be shown

. $\bar{A} \cup \bar{B} \in F \rightarrow \overline{\bar{A} \cup \bar{B}} = AB \in F$

A field also contains the certain event and the impossible event $S \in F$ and $\emptyset \in F$

It follows that all sets can be written as unions and intersections of "finitely many" sets in are also in F.

This may not be the case for infinitely many sets.

BOREL FIELDS

Suppose that $A_1, A_2 \ldots, A_n \ldots$ is an infinite sequence of sets in F. If the union and the intersections of these are also in F, the F is called a Borel field.

The class of all subsets of a set S is a Borel field. Attaching to it the other subsets of S, (all subsets are necessary), we can form a field with C as its subset. It can be shown that there exists a smallest Borel field containing all elements of C.

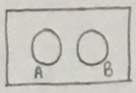
Ex: Die experiment - we are bothing even, or odd → It suffices to consider only four sets: $\{\emptyset\}$, $\{\text{even}\}$, $\{\text{odd}\}$ and S.

Ex: $S = \{a, b, c, d\}$, C consist of sets $\{a\}$ and $\{b\}$. The smallest field containing $\{a\}$ and $\{b\}$

$\emptyset$, $\{a\}$, $\{b\}$, $\{a, b\}$, $\{c, d\}$, $\{b, c, d\}$, $\{a, c, d\}$, S

Events even to are certain subsets of C forming a Borel fields.

Mutually exclusive events



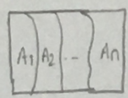
if A and B are "mutually exclusive"

$(AB = \emptyset) \rightarrow P(A \cup B) = P(A) + P(B)$

if $A_1, A_2, A_3 \ldots A_n$ are mutually exclusive (for $i \neq j$, $i, j \leq n$ $A_i A_j = \emptyset$)

$\rightarrow P(A_1 \cup A_2 \cup \ldots A_n)$

$= P(A_1) + P(A_2) + \_\_\_\_ + P(A_n)$



Axiom of Infinite Additivity

If the events $A_1, A_2 \_\_\_ A_n$ are mutually exclusive than

$P(A_1 \cup A_2 \ldots) = P(A_1) + P(A_2) + \_\_\_\_$

Axiomatic definition of an Experiment

* n_i : the single outcome

An Experiment is specified in terms of the following concepts

1. The set S of all experimental outcomes
2. The borel field of all events of S
3. The probabilities of these events

Countable Spaces

If S consists of N outcomes and $N < \infty$

$\rightarrow$ The probabilities of all event can be expressed such as

$P\{n_i\} = P_i$

of the elemantary event $\{n_i\}$.

$P_i \geq 0$ and $P_1 + P_2 + \ldots + P_N = 1$

Suppose that A is an event such as, $A = \{n_1, n_2, \ldots n_r\}$ $r < N$

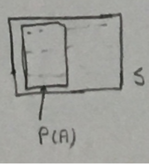
$P(A) = P_1 + P_2 + \_\_\_\_ + P_r$

Equality Likely Outcomes

If S consists of N outcomes that are equality likely, that is the probability of each outcome, p_i is the same then $p_i = \frac{1}{N}$

If A consists r elements S

the $P(A) = \frac{r}{N}$



toss a coin: yazı tura atmak

Example

a) Coin Experiment $S = \{h, t\}$

Its events $\emptyset, \{t\}, \{h\}, S$

If $P(h) = p$ and $P(t) = q$ $\rightarrow p + q = 1$

b) Toss a coin three times

$S = \{hhh, hht, hth, htt, thh, tht, tth, ttt\}$

Each have an equally likely outcome $P(n_i) = \frac{1}{8}$ $P\{\text{heads at the first two tosses}\} = \frac{2}{8}$

The Real Line $\rightarrow$ If S consists of non-countable infinity of events the its probabilities cannot be determined in the terms of $P\{n_i\}$.

Let's say S is the set of all real numbers.

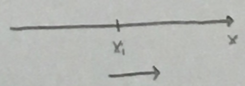
Its subsets can be considered as sets of points on the real line.

We cannot assign probabilities to all subsets of S to satisfy the axioms.

Events as intervals $x_1 \leq x < x_2$ as $x \in \mathbb{R}$.

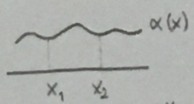
These events form a field F that can be predicted as follows:

It is the smallest Borel Field that includes all half lines $x \leq x_i$ where x_i is any number.



This field contains all open and closed intervals, all points, every set of points of the real line.

Suppose that $\alpha(x)$ is a function such that



$\int_{-\infty}^{+\infty} \alpha(x).dx = 1$

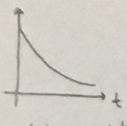
Assume

$\alpha(x) \geq 0$

We define the probability of event $\{x \leq x_i\}$ as $P\{x \leq x_i\} = \int_{-\infty}^{x} \alpha(x)\,dx$ $\boxed{P(x_i) = 0}$

Also $P\{x_1 \leq x \leq x_2\} = \int_{x_1}^{x_2} \alpha(x)\,dx$

Example-2-6 : A radioactive substance is selected $t=0$ and the time t of emission of a particle is observed

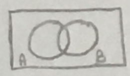


$\alpha(t) = c.e^{-c.t}$

This process defines an experiment whose outcomes are all points on the positive t axis. This experiment can be considered as a special case of the real line experiment if we assume s is the entire t axis and all events on the negative axis have zero probability.

The probability that a particle will be emitted in the time interval $(0,t_0)$ is $c\int_0^{t} e^{-c.t}.dt = 1-e^{-c.t_0}$

Probability Masses



$P(A)$ of an event A can be interpreted as the mass of corresponding figure in it venn diagram representation

We conclude $\boxed{P(A\cup B) = P(A)+P(B) - P(AB)}$* → iki Olayın Birleşme Olasılığı

Example-2-8 : A box has m white balls, n black ". Find the probability of picking a white ball by the kth draw. (w/out replacement)

$W_k = \{$a white ball is drawn by the kth draw$\}$

$W_1 = ○$

$W_2 = ● ○$

$W_3 = ● ● ○$

$X_i = \{i$ black balls followed by a white ball drawn$\}$

$$W_k = \underbrace{X_0 \cup X_1 \cup X_2 \cup \dots \cup X_{k-1}}_{\text{Mutually exclusive events}}$$

$$P(W_k) = \sum_{n=0}^{k-1} P(X_i)$$

$$P(X_0) = \frac{m}{m+n} \qquad ○$$

$$P(X_1) = \frac{n}{m+n} \cdot \frac{m}{m+n-1} \qquad ● ○$$

$$\vdots$$

$$P(X_{k-1}) = \frac{n.(n-1)\dots(n-k+1).m}{(m+n)(m+n-k+1)}$$

$$P(W_{n+1}) = 1$$

$$P(W_k) = \frac{m}{m+n} \cdot \left(1 + \frac{n}{m+n-1} + \frac{n.(n-1)}{(m+n-1).(m+n-2)} + \dots + \frac{n.(n-1)\dots(n-k+1)}{(m+n-1).(m+n-2)\dots(m+n-k+1)}\right.$$

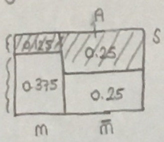
Conditional Probability

The conditional probability of an event A assuming another event M, donated by $P(A|M)$ is by definition

$$P(A|M) = \frac{P(AM)}{P(M)} \quad , P(M) \neq 0$$

$\downarrow$

$P(A \text{ given } M)$



$P(A) = 0.125 + 0.25 = 0.375$

$P(m) = 0.125 + 0.375 = 0.5$

$P(AM) = 0.125$

$P(A|m) = \frac{0.125}{0.5} = 0.25$ //

If $M \subset A$ then $P(MA) = P(M)$

$$P(A|M) = \frac{P(M)}{P(M)} = 1$$

If $A \subset M$ $\quad P(MA) = P(A)$

$$P(A|M) = \frac{P(A)}{P(M)} \geq P(A)$$

Example 2-10 In a fair die experiment determine the conditional probability of the event $\{F_2\}$ assuming the event $\{\text{even}\}$ occured

$A = \{F_2\} \qquad M = \{F_2, F_4, F_6\}$

$P(A) = 1/6 \qquad P(M) = \frac{1}{2} \qquad P(AM) = P(A) \qquad P(A/M) = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$ //

Example-2-12: 3 white balls, W_1, W_2, W_3
2 black balls, r_1, r_2

balls removed at succession

- What is the probability that the first ball is white and the 2nd is black?

First Solution

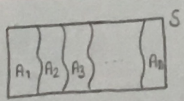
$5 \times 4 = 20$ pairs possible $\quad \{-W_1r_1 \quad -W_1r_2 \quad -W_2r_1 \quad -W_2r_2 \quad -W_3r_1 \quad -W_3r_2\}$ $\quad \frac{6}{20} = \frac{3}{10}$ //

Second Solution

$W = \{\text{white first}\} \quad P(W_1) = \frac{3}{5} \qquad B_2 = \{\text{Black second}\} = \frac{1}{2} \qquad P(B_2|W_1) = \frac{2}{4} \qquad P(W_1B_2) = P(B_2|W_1) \cdot P(W_1) =$

$\frac{2}{4} \cdot \frac{3}{5} = \frac{3}{10}$ //

Total Probability & Bayes Theorem



$U = [A_1, A_2, \dots, A_n]$

B is arbitary event

$$\boxed{P(B) = P(B|A_1)\,P(A_1) + \dots + P(B|A_n)P(A_n)}$$

it follows

"Total Probability Theorem"

Proof

$B = BS = B(A_1 \cup A_2 \cup \dots \cup A_n) = BA_1 \cup BA_2 \cup \dots BA_n$

$P(B) = P(BA_1) + \dots P(BA_n) \Rightarrow P(BA_i) = P(B|A_i)P(A_i)$

Since $P(BA_i) = P(A_i|B)P(B)$

$$P(A_i|B) = \frac{P(B|A_i)\,P(A_i)}{P(B)}$$

$$= \frac{P(B|A)}{P(B|A_1)P(A_1) + \dots + P(B/A_n)P(A_n)}$$

This is called BAYES' THEOREM.

Example 2-15 : A test for a particular cancer is known to be 95% accurate. Suppose that a person takes the test and it is positive. This person comes from a population of → those have this type of cancer. What is the probability that this person actually has this cancer?
↳ 100.000 where 2000 of →

T: {the test is positive}
H: the set of healthy patients
C: " " " sick "

$P(T/C) = 0.95$ $\quad P(\bar{T}|C) = 0.05$ $\quad P(\bar{T}/\bar{C}) = 0.95$

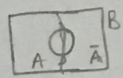
Choosing at random H: healthy C: cancer $\quad P(H): \frac{98000}{100000} = 0.98$ $\quad P(C) = 2000 = 0.02$

$$P(C/T) = \frac{P(T/C)\,P(C)}{P(T)} = \frac{0.95 \times 0.02}{P(T)} = 0.278$$

$\boxed{\begin{matrix}\bar{C} = H \\ U = [C, H] \\ C \cup H = S\end{matrix}}$

$$P(T) = P(T/C)P(C) + P(T/H)P(H) = 0.95 \times 0.02 + 0.05 \times 0.98 = \sim$$

Independence



The events A and B are called "independent" if $P(AB) = P(A)P(B)$

* We can also say • If A and B are independent then $\bar{A}$ and B, and $\bar{A}, \bar{B}$ are also independent
• The events AB and $\bar{A}B$ $(AB \cap \bar{A}B = \emptyset)$ are mutually exclusive

$B = AB \cup \bar{A}B$ $\quad P(\bar{A}) = 1 - P(A)$ $\quad P(\bar{A}B) = P(B) - P(AB) = [1 - P(A)]P(B) = P(\bar{A})P(B)$

Example-2-17 : We toss a coin twice $\quad S = \{hh, th, ht, tt\}$ $\quad P(h) = a$, $P(t) = b$ $\quad a + b = 1$, $P(hh) = a^2$, $P(ht) = P(th) = ab$
$P(tt) = b^2$

$(a+b)^2 = a^2 + 2a.b + b^2 = 1$ (It satisfies the axioms, $P(S) = 1$)

our Experiment

$H_1 = \{\text{heads at first toss}\} = \{hh, ht\}$ $\quad P(H_1) = P(hh) + P(ht) = a^2 + ab = a(a+b) = a$

$H_2 = \{\text{heads at second toss}\} = \{th, tt\}$ $\quad P(H_2) = P(tt) + P(th) = b^2 + ab = b(a+b) = b$

$P(H_1H_2) = P(hh) = a^2 = P(H_1)P(H_2)$ $\quad H_1$ and H_2 are independent!

Example-2-22 : A based coin is tossed till a head appears for the first time. What is the probability that the number of required tosses is odd?

$P(T) = q$
$P(H) = 1 - q = p$

A_i = "Head appears at the ith toss for the first time" $= \{\underbrace{T, T, T, \dots}_{i-1}, H\}$

Assuming the trials are independent $\quad P(A_i) = P(i)P(i)$ $\quad P(H) = q^{i-1} \cdot (1-q)$

$$P(\text{"Heads appear on an odd toss"}) = \sum_{i=0}^{\infty} P(A_{2i+1}) = \sum_{i=0}^{\infty} q^{2i} \cdot p = p \cdot \sum_{i=0}^{\infty} q^{2i} = \frac{p}{1-q^2} = \frac{\cancel{p}}{(1+q)\cancel{(1-q)}} = \frac{1}{1+q} = \frac{1}{2-p}$$

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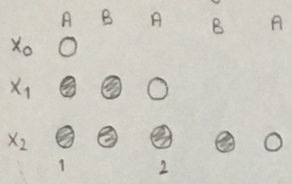
Tuesday

Ex. 2.9-) Two players A and B draw balls one at a time alternately from a box containing m white balls and n black balls. Suppose the player who picks the first white ball wins the game. What is the probability that the player who starts the game will win?

Suppose A starts the game



$X_k = \{$A and B alternately draw k black balls and draws a white ball$\}$ $k=0,1,2$

$\{A \text{ wins}\} = X_0 \cup X_1 \cup \dots$ they are mutually exclusive evented

$P_A \triangleq P\{A \text{ wins}\} = P(X_0) + P(X_1) + \dots$

$$P(X_0) = \frac{m}{m+n}$$

$$P(X_1) = \frac{n}{m+n} \cdot \frac{n-1}{m+n-1} \cdot \frac{m}{m+n-2}$$

$$P(X_2) = \frac{n}{m+n} \cdot \frac{n-1}{m+n-1} \cdot \frac{n-2}{m+n-2} \cdot \frac{n-3}{m+n-3} \cdot \frac{m}{m+n-4}$$

$$P(A) = \frac{m}{m+n}\left(1 + \frac{n(n-1)}{(m+n-1)(m+n-2)} + \dots + \right)$$

$$Q_B = P(B \text{ wins}) = \frac{m}{m+n}\left(\frac{n}{m+n-1} \cdot \frac{n(n-1)(n-2)}{(m+n-1)(m+n-2)(m+n-3)} + \dots\right)$$

$$P_A + Q_B = 1$$

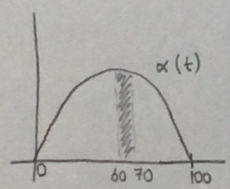
Ex 2-17-) We denote by $\underline{t}$ the age of a person who he dies.

The probability that $t \leq t_0$

$$P(t \leq t_0) = \int_a^b \alpha(t)\,dt$$

$\alpha(t) = 3 \cdot 10^{-9} \cdot t^2 (100-t)^2$ max 100

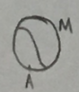
$0 \leq t \leq 100$



$$P(60 \leq t \leq 70) = \int_{60}^{70} \alpha(t)\,dt = 0.154$$

What is the probability that a person dies btwn the age of 60 and 70. provided that he is dies at 60.

$A = \{60 \leq t \leq 70\}$ $M = \{t \geq 60\}$ $AM = A$ $A \subset M$



$$P(A|M) = \frac{P(AM)}{P(M)} = \frac{P(A)}{P(M)}$$

$$= \frac{0.154}{\int_{60}^{100} \alpha(t)\,dt} = \underline{0.486}$$

Ex-2-19-) A box contains white and black balls. When two balls are drawn w/out placement suppose the probability that both are white is $1/3$.

a-) Find the smallest number of balls in the box.

b-) How small can the total number of balls be if black balls are even in number?

a white balls, b black balls

$W_k = \{$a white ball is drawn at the kth draw$\}$

$$P(W_1 \cap W_2) = \frac{1}{3}$$

$$P(W_1 \cap W_2) = P(W_2 | W_1)\, P(W_1)$$

$$= \frac{a-1}{a+b-1} \cdot \frac{a}{a+b} = \frac{1}{3} \quad , \quad \frac{a-1}{a+b-1} < \frac{a}{a+b} \quad , \quad b > 0 \quad , \quad \left(\frac{a-1}{a+b-1}\right)^2 < \frac{1}{3} < \frac{a}{a+b}$$

$$\frac{a}{a+b} \longrightarrow \left(\frac{a-1}{a+b-1}\right)^2 < \frac{1}{3} < \left(\frac{a}{a+b}\right)^2 \Rightarrow (\sqrt{3}+1)\frac{b}{2} < a < 1 + (\sqrt{3}+1)\frac{b}{2}$$

For $b=1$ $\quad 1.36 < a < 2.36 \rightarrow a = 2$

$$P(W \cap W_1) = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$$

The smallest number of balls required is $2+1=3$

Independence of three events

The events A_1, A_2 and A_3 are independent if

1-) they are independent in pairs

$$P(A_i A_j) = P(A_i)\, P(A_j)$$

2-) $P(A_1 A_2 A_3) = P(A_1) P(A_2) P(A_3)$

Generalization: The independence of n events can be defined inductively. Suppose that we have defined independence of k events for every $k < n$. We then say that the events $A_1, \dots, A_n$ are independent if any $k < n$ of them are independent and

$$P(A_1 A_2 \dots A_n) = P(A_1) \dots P(A_n)$$

Birthday Pairing

Ex-2-20 -) In a group of n people (a) what is the probability that two or more persons have the same birthday?

There are $N = 365$ days in a year.

$A = \{$two people have the same birthday$\}$

$\bar{A} = \{$no two people have the same birtday$\}$

1st person can have one of the N days as their birthday.

2nd " $N-1$

3rd " $N-2$

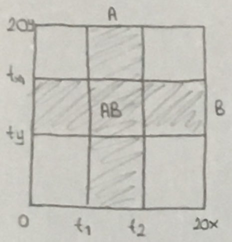
$$P(\bar{A}) = \frac{N(N-1)\dots(N-n+1)}{N^n} = \prod_{k=1}^{n-1}\left(1 - \frac{k}{N}\right)$$

$$P(A) = 1 - P(\bar{A}) = 1 - \prod_{k=1}^{n-1}\left(1 - \frac{k}{n}\right) \cong 1 - e^{-\sum_{k=1}^{n-1}\frac{k}{N}} = 1 - e^{-n(n-1)/2N}$$

$n = 23 \rightarrow P(A) = 0.5$

$n = 50 \rightarrow P(A) = 0.97$

Example 2.18 : Trains X and Y arrive at a station btwn 8.00 AM and 8.20 AM. X stops for 4 mins, and Y stops for 5 mins. Assuming that the trains arrive independently of each other.

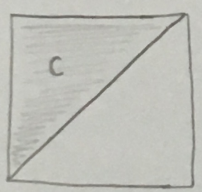


$A = \{X \text{ arrives btwn } (t_1, t_2)\} = \{t_1 \leq x \leq t_2\}$

$B = \{t_3 \leq y \leq t_4\}$

$$P(AB) = P(A)P(B) = \frac{(t_2 - t_1)(t_4 - t_3)}{20 \times 20}$$

a-) What is the probability that X arrives before Y

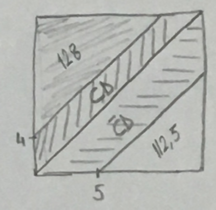


$C = \{x \leq y\}$

$$P(C) = \frac{200}{400} = \frac{1}{2}$$

b-) What is the probability that the trains meet at the station ? $\left\{\begin{matrix} x \leq y+5, \\ y \leq x+4 \end{matrix}\right\}$

$D = \{-4 \leq x - y \leq 5\}$



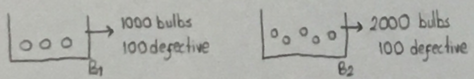
$$P(D) = \frac{400 - 128 - 112.5}{400} = \frac{159,5}{400}$$

c-) Assuming that the trains met, what is the prob. that X arrives before Y.

$C = \{x \leq y\}$ $D = \{\text{trains met}\}$

$$P(C \setminus D) = \frac{P(CD)}{P(D)} = \frac{72}{159,5}$$

Prob. 2.24 :



Two bulbs picked randomly from a randomly selected box.

a-) Find the prob. that both are defective.

$D: \{\text{two balls are defective}\}$

$$P(D \setminus B_1) = \frac{100}{1000} \times \frac{99}{999} = 0.00991$$

B_1 and B_2 are partition. $\Big\} \rightarrow \begin{matrix} B_1 \cup B_2 = S \\ B_1 B_2 = \emptyset \end{matrix}$

$$P(D \setminus B_2) = \frac{100}{2000} \times \frac{99}{1999} = 0.00248$$

$$P(D) = P(D \setminus B_1) \cdot P(B_1) + P(D \setminus B_2) P(B_2) = 0.0062$$

b-) Assuming both are defective, find the prob. that they came from B_1.

$$P(B_1 \setminus D) = \frac{P(D \setminus B_1) P(B_1)}{P(D)} = \frac{0.0091 \times 0.5}{0.0062} = \underline{0.799}$$

15.03.2016
Tuesday

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2.22 ; Shows that $2^n-(n+1)$ equations are needed to establish independence of n events.

$\underline{n=2}$ $\quad A_1, A_2 \quad P(A_1)P(A_2)=P(A_1A_2) \quad 2^2-(2+1)=1$

$\underline{n=3}$ $\quad A_1, A_2, A_3 \quad P(A_iA_j)=P(A_i)P(A_j) \quad i\neq j \quad \binom{3}{2}=3$

$P(A_1)P(A_2)P(A_3) \quad$ 1 eq $\quad 3+1=4 \quad 2^3-(3+1)=4$

$$\underline{n} \quad \binom{n}{2}+\binom{n}{3}+\dots+\binom{n}{n}=\sum_{k=2}^{n}\binom{n}{k}=2^n-(n+1)$$

2.16 : A box contains n identical balls numbered 1 through n. Supposed k balls are drawn in succession.

a-) What is the prob. that m is the largest number drawn.

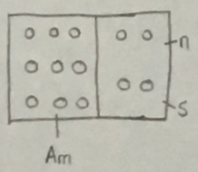
$\underbrace{1 \; 2 \; 3 \; \dots \; n}$

$\binom{n}{k}$ is the possible combinations.

$\binom{m-1}{k-1}$ possible combination that m is largest ball drawn.

$$P=\frac{\binom{m-1}{k-1}}{\binom{n}{k}}$$

2.11 : We select at random m objects from a set S of n objects and denote by A_m the set of the selected objects. Show that the prob. p of a particular element $n_0 \in S$ is in A_m equals to m/n.



How many sets A_m can be constructed ?

$\binom{n}{m}$

How many set of A_m can contain n_0

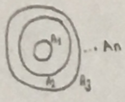
$\binom{n-1}{m-1}$

$$P=\frac{\binom{n-1}{m-1}}{\binom{n}{m}}=\frac{(n-1)!}{(n-1-m+1)(m-1)!}\cdot\frac{m!\,(n-m)!}{n!}=\frac{m}{n}$$

Theorem a-1 If $A_1, A_2, \dots,$ is an increasing sequence of events, that is, $A_1 \subset A_2 \subset A_3 \dots$ then

$$P\left(\bigcup_k A_k\right) = \lim_{n\to\infty} P(A_n)$$

Proof



$B_1 = A_1$, $B_2 = A_2\bar{A}_1 \triangleq A_2 - A_1$

$B_3 = A_3 - (B_1 \cup B_2) \dots$

$B_n = A_n - \left(\bigcup_{k=1}^{n} B_k\right)$

$A_n = \bigcup_{k=1}^{n} B_k$

$$P\left(\bigcup_k A_k\right) = P\left(\bigcup_k B_k\right) = \sum_k P(B_k) = \lim_{n\to\infty} \sum_{k=1}^{n} P(B_k) = \lim_{n\to\infty} P\left(\bigcup B_k\right) = \lim_{n\to\infty} P(A_n)$$

Theorem a-2 If $A_1, A_2 \dots$ is a decreasing sequence of events : $A_1 \supset A_2 \supset A_3 \dots$

$$P\left(\bigcap_k A_k\right) = \lim_{n\to\infty} A_n$$

Proof

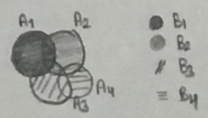
$$P\left(\bigcap_k A_k\right) = 1 - P\left(\bigcup_k \bar{A}_k\right) \qquad \text{(De Morgan's Low)}$$

$$= 1 - \lim_{n\to\infty} P(\bar{A}_n) = \lim_{n\to\infty} (1 - P(\bar{A}_n)) = \lim_{n\to\infty} P(A_n)$$

Theorem a-3 The inequality

$$P\left(\bigcup_n A_k\right) \leqslant \sum_k P(A_k)$$

holds for orbitrary events A_1, A_2, A_3



$$P\left(\bigcup_k A_k\right) = P\left(\bigcup_k B_k\right) = \sum_k P(B_k) \leqslant \sum_k P(A_k)$$

$B_1, B_2, B_3 \dots B_k \subset A_k$ $\qquad P(a_k) \leqslant P(A_k)$

Random Variables

A random variable (R.V.) is a number $X(\lambda)$ assigned to every outcome λ of a random experiment. $X \to R.V \to x$

Ex: Coin tossing, 3 coins

X: # of heads, it can take values 0,1,2,3

$P(x=0) = P(TTT) = \frac{1}{8}$ $P(x=2) = P(THH, HHT, HTH) = \frac{3}{8}$

$P(x=1) = P(TTH, THT, HTT) = \frac{3}{8}$ $P(x=3) = P(HHH) = \frac{1}{8}$

- Discreate RV → Can take on at most a countable number of possible values
- Continious RV → Can take range of real numbers

① $\{X \leq x\}$ → a subset of S containing all outcomes λ such that $X(\lambda) \leq x$

$\{X \leq x\}$ is a set experimental outcomes.

⊙ $\{x_1 \leq X \leq x_2\}$ a subset of S consisting of all outcomes $x_1 \leq X(\lambda) \leq x_2$

⊙ $\{X = x\} \to X(\lambda) = x$

⊙ $\{X \in \mathbb{R}\} \to X(\lambda)$ consisting all real numbers.

Ex: Coin tossing experiment, 3 coins

$\{X \leq 2\} = \{X=0\} \cup \{X=1\} \cup \{X=2\}$

$\{X=2\} = \{THH, HTH, HHT\}$

$\{X=5\} = \{\emptyset\}$

$\emptyset$

Distribution & Density Function

(Cumulative) Distribution Function (CDF)

$F_X(x), F(x) = P(X \leq x)$

$\{$

Ex: Coin tossing, 3 coins, X: # of heads

for $x < 0$ $F(x) = P(X < 0) = 0$

$0 \leq x < 1$ $F(x) = P(X=0) = \frac{1}{8}$

$1 \leq x < 2$ $F(x) = P(X=0) + P(X=1) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$

$2 \leq x < 3$ $F(x) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} = \frac{7}{8}$

$3 \leq x$ $F(x) = 1$

Ex (4-5): A telephone call occurs at random in the interval (0,1). In this experiment the outcomes are time distances t between 0 and 1 and the probability that t is between t_1 and t_2 is given by

$$P(t_1 \leq t \leq t_2) = t_2 - t_1$$

We define the R.V X, such that $X(t) = t \qquad 0 \leq t \leq 1$

α If $x > 1$ then $X(t) \leq x$ for every outcome

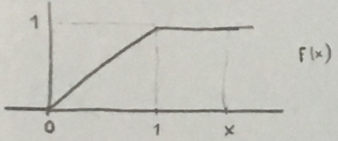
$F(x) = P(X \leq x) = P(0 \leq t \leq 1) = P(S) = 1 \qquad x > 1$

α If $0 \leq x \leq 1 \rightarrow X(t) \leq x$ for every t in the interval $(0, x)$

$F(x) = P(X \leq x) = P(0 \leq t \leq x) = x - 0 = x \qquad 0 \leq x \leq 1$

α $x < 0 \qquad F(x) = P(X \leq x) = P(\emptyset) = 0 \qquad x < 0$

$\hookrightarrow$ then $\{X \leq x\}$ is impossible because $X(t) \geq 0$



Discreate Random Variables

A random valuable can take on at most a countable number of possible values $\rightarrow$ RV is discreate

Probability mass function

$f(x) = P(X = x)$

Cumulative Distribution Function

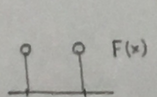
$F(x) = P(X \leq x)$

Expected value (Expectation, Mean)

$$\boxed{E(x) = M = \sum_x x\, f(x)}$$

Ex: Coin toss Ex. 1 coin $\quad$ X: number of heads

$F(0) = \frac{1}{2} \quad F(1) = \frac{1}{2}$



$E(x) = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$

Ex: We say that I is an indicator R.V. for the event A: F

$$I = \begin{cases} 1, & \text{if } A \text{ occurs} \\ 0, & \text{if } A' \text{ "} \end{cases}$$

$F_1(1) = P(A) \qquad F_2(0) = P(A') = 1 - P(A)$

$E(I) = 1P(A) + 0 \cdot [1 - P(A)] = P(A)$

Expectetion of function of a RV

X RV $g(x) \to E(g(x))$

Ex : X, RV

$F(-1) = P(X=-1) = 0.2$, $F(0) = 0.5$, $F(1) = 0.3$

$E(x^2) = ?$ $\quad y = x^2$

$F_y(1) = F_x(x^2 \cdot 1) = f_x(-1) + f_x(1) = 0.2 + 0.3 = 0.5$ $\qquad \{y=1\} = \{x=1\} \cup \{x=-1\}$

$F_y(0) = F_x(0) = 0.5$ $\qquad P(y=1) = F_y(1) = F_x(1) + F_x(-1)$

$E(y) = 1 \times 0.5 + 0 \times 0.5 = 0.5$

Proposition

$E[g(x)] = \sum_i g(x_i) . f(x_i)$

Previous Ex. : $E(x^2) = (-1)^2 \times 0.2 + 0^2 \times 0.5 + 1^2 \times 0.3 = 0.5$

Corollary : $E(ax+b) = a(x_i) + b$?

Variance : X,

$$V(x) = r^2 = E[(x-m)^2] = E(x^2 - 2xm + m^2) = E(x^2) - 2\underbrace{E(x)}_{m} m + m^2 = E(x^2) - m^2 = E(x^2) - [E(x)]^2$$

$E(x)$

Example : Calculate $V(x)$, if X is represents the outcome when a fair die is rolled.

$E(x^2) = 1^2 . \frac{1}{6} + 2^2 . \frac{1}{6} + \cdots + 6^2 . \frac{1}{6} = \frac{91}{6}$

$M = E(x) = (1+2+3+4+5+6) . \frac{1}{6} = \frac{7}{2}$

$V(x) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$

Remerk : $V(ax+b) = a^2 V(x)$ $\to$ Prove it at home

Probability Distributions

Bernoulli Trial

- Flip a coin 10 times , X = number of heads obtained
- A multiplechoice test contains 10 questions , X : the # of questions answered correctly.

1st case success as getting heads for an experiment
failure " tails

2nd case success graving correctly
failure " wrong

Bernoulli trial - is a trial with only two outcomes where probability of a success in each trial is constant.

1- The probability of success is ½ for every experiment.

2- ¼ → for any experiment

Ex : the chance that a bit is transmitted through a digital channel is received in error is 01. Transmission of each bit is independent.

X : is the number of bits in error in 4 bits transmitted.

O : outcome with no error

E : " " error

Outcome	x
OOOO	0
OOOE	1
OOEO	1
OOEE	2 ·
OEOO	1
OEOE	2 ·
OEEO	2 ·
OEEE	3
EOOO	1
EOOE	2 ·
EOEO	2 ·
EOEE	3
EEOO	2 ·
EEOE	3
EEEO	3
EEEE	4

$P(X=2) = P\{EOOE, EOEO, EEOO, OEOE, OOEE, OEEO\}$

$\binom{4}{2} = 6$

$P = (0.1)^2 \cdot (0.9)^{4-2}$

$P(X=2) = \binom{4}{2} \cdot (0.1)^2 \cdot (0.9)^2$

$P(X=x) = \binom{4}{x} P^x \cdot (1-P)^{4-x}$

Binomial Distribution

A random experiment consists of n Bernoulli trials such that

(1) The trials are independent

(2) Each trial results in only two possible outcomes "success" and "failure"

(3) The probability of a success, p remains contains

X: the number of trials that results in success, $0 \leq p \leq 1$ and $n \in \mathbb{Z}^+$

$$f(x) = \binom{n}{x} \cdot p^x \cdot (1-p)^{n-x} \qquad x = 0, 1, \dots, n$$

Ex: A company produces computers

Prob. that a computer will be defective is 0.01

The computers are sold in packages of 10 and if 1 of the computers are defective the company replaces the package. What is proportion of the packages sold must be replaced. $n = 10$, $p = 0.01$, X is binomial distribution

$$P(X=0) = \binom{10}{0} p^0 \cdot (1-p)^{10} = A$$

$$P(X=1) = \binom{10}{1} \cdot p^1 (1-p)^9 = B$$

$$1 - P(X=0) - P(X=1) = 1 - A - B = 0.004$$ packages must be replaced

Properties of Binomial R.V.

X: binomial R.V.

$$E(X^k) = \sum_{i=0}^{n} i^k \binom{n}{i} \cdot p^i (1-p)^{n-i} \qquad \text{when } i = 0$$

$$= \sum_{i=1}^{n} i^k \binom{n}{i} \cdot p^i (1-p)^{n-i}$$

$$i \binom{n}{i} = n \binom{n-1}{i-1}$$

$$E(X^k) = n.p \sum_{k=1}^{n} i^{k-1} \binom{n-1}{i-1} p^{i-1} (1-p)^{n-i} = n.p \sum_{j=0}^{n-i} (j+1)^{k-1} \binom{n-1}{j} \cdot p^j (1-p)^{n-1-j}$$

Y: Binomial RV, $n-1$, p

$$E[X^k] = n.p. E\left[(Y+1)^{k-1}\right]$$

$$k = 0 \rightarrow E(X) = n.p. \underbrace{E[(Y+1)]}_{1}$$

$$M = E[X] = n.p \qquad \text{Mean, Expected value}$$

$$E[Y+1] = E[Y] + 1$$

$$E(X^2) = np\, E[Y+1]$$
$$= np\, [(n-1)p + 1]$$

$$\sigma^2 = V(X) = E(X^2) - [E(X)]^2$$
$$= np\,[(n-1)p+1] - np^2 = \dots = np(1-p)$$

Poisson Distribution

Binomial distribution → when number of trials in a binomial experiment increases to infinity while the expected value remains the same, Poisson distribution emerges.

Ex: Let's consider transmission of n bits occurs a channel, X: number of bits in error. Error probability is p. X is a binomial RV.

$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$

what is n goes to infinity and μ remains constant $G(x) = n.p = \lambda_0$

$P(X=x) = F(x) = \binom{n}{x} p^x . (1-p)^{n-x}$

→ We can approximate B.RV when n is large and p is small enough to make np moderate.

$= \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$

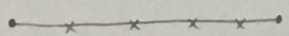
$\lim_{n\to\infty} \binom{n}{x} \left(\frac{1}{n}\right)^x \to \frac{1}{x!}$

$\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{-x} \to 1$

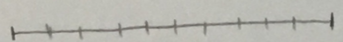
$\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{n} \to e^{-\lambda}$

$\lim_{n\to\infty} f(x) = \frac{e^{-x}.\lambda^x}{x!} \qquad x = 0,1,2\ldots$

Definition: X is a Poisson RV with a parameter X if $f(x) = \frac{e^{-x}.\lambda^x}{x!}$, $x = 0,1,2\ldots$

Ex: 

Errors occurs at random along the length of a thin copper wire. Let's X is a R.V. that counts the number of flaws in a L mm of w₁ and the average number of errors in L mm is λ.

 Portion this in n subunits. Also let's say each submit can have at most 1 error.

$E(x) = n.p = \lambda \quad p = \frac{\lambda}{n}$ when n is large enough at the same time np remains constant X can be approximated with a Poisson R.V.

Let's say the mean (expected value) of errors is $\overset{E(x)}{\lambda = 2{,}3}$ flaws/mm → for 1 mm wire. Determine the probability of having exactly 2 errors in 1 mm of wire.

$\lambda = 2{,}3$

$P(x=2) = \frac{e^{-2{,}3}.(2{,}3)^2}{2!} = 0.265$

- Probability of having at least 1 error in 10 mm wire.

$E(x) = (2{,}3)$ errors/mm $\times$ 10 mm

$= 23$ errors

$P(x \geq 1) = \sum_{x=}^{\infty} f(x) = 1 - P(x=0) \longrightarrow P(x=0) = e^{-2{,}3} \times \frac{23^0}{0!} = \ldots \rightarrow P(x \geq 1) = 1 - e^{23}$

Expected Value

$$E(x) = \sum_{x=0}^{\infty} \lambda . e^{-\lambda} . \frac{\lambda^x}{x!}$$

$$= \lambda . \sum_{x=1}^{\infty} x . e^{-x} . \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda . e^{-x} . \underbrace{\sum_{y=0}^{\infty} \frac{\lambda^y}{y!}}_{e^{\lambda}} \qquad (y = x-1)$$

$$\boxed{E(x) = \lambda}$$

Variance

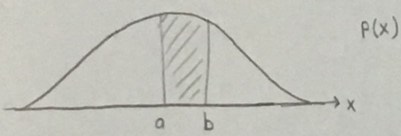
$V(x) = \sigma^2 = \lambda$

Continuous Random Variables

If a random variable takes on values from set of uncountable numbers, such R.V. is called a "continuous R.V."

Ex: X = lifetime of a transistor or X = the time that a train arrives

B is a set of $\mathbb{R}$ where $P(X \in B) = \int_B f(x)\,dx$ $\quad f(x)$ is called "probability density function" of X.



$$P(X = a) = \int_a^a f(x).dx = 0$$

Probability that a CRV will assume any fixed value is zero.

$$P(x < a) = P(x \leq a) = \int_{-\infty}^{a} f(x)dx$$

$$P(a < x < b) = P(a \leq x < b) = P(a < x \leq b) = P(a \leq x \leq b) = \int_a^b f(x).dx$$

CDF of C.R.V.

$$F(x) = P(X < x) = \int_{-\infty}^{x} F(x).dx \qquad P(x < a)$$

Ex: X : crv

$$F(x) = \begin{cases} c(4x - 2x^2 & , 0 < x < 2 \\ 0 & , \text{otherwise} \end{cases}$$

a-) $\int_{-\infty}^{\infty} F(x).dx = 0$

$$\int_0^2 c(4x - 2x^2)\,dx = 1$$

$$c.\left[\frac{4}{2}.x^2 - \frac{2}{3}.x^3\right]_0^2 = 1$$

$$c.\left(2.4 - \frac{2}{3}.8 - (0.0)\right) = 1$$

$$c = \frac{3}{8}$$

b-) $P(x > 1) = \int_1^2 \frac{3}{8}(4x - 2x^2)\,dx = \ldots = \frac{1}{2}$

* Integration by Parts

$\int$

$^*\lim_{x\to\infty} e^{-x/100} = 0$

$\int e^{\alpha . x} = \frac{1}{\alpha} . e^{\alpha x}$

Ex: The amount of time in hours that a computer functions before breaking down is a C.R.V with a PDF to given by

$$F(x) = \begin{cases} \lambda . e^{-x/100}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

What is the probability that a-) A computer will function btwn 50 and 150 hours before breaking down?
b-) it will function fewer than 100 hours?

$$\lambda \int_{-\infty}^{\infty} e^{-x/100} . dx = 1 \Rightarrow -\lambda(100) . e^{-x/100} \Big|_0^{\infty} \Rightarrow 0 - (-100\lambda) = 1 \Rightarrow \lambda = 0.01$$

a-) $P(50 < x < 150) = \int_{50}^{150} 0.01 . e^{-x/100} . dx = e^{-x/100} \Big|_{50}^{150} = e^{-1/2} - e^{-3/2} = 0.383$

b-) $P(x < 100) = \int_0^{100} F(x) . dx = \dots \cong 0.632$

Ex: The lifetime in hours of certain kind radio tube is a R.V. having a PDF given by

$$F(x) = \begin{cases} 0, & x \leq 100 \\ \frac{100}{x^2}, & x > 100 \end{cases}$$

What is the probability that exactly 2 of 5 such tubes in a radio set will have to be replaced within the first 150 hours of operation?

[Assume that E_i, $i = 1,2,3,4,5$ that ith tube will have to be replaced within this time are independent.]

$$P(x < 150) = \int_0^{150} F(x) . dx = 100 \int_{100}^{150} x^{-2} dx = \frac{100}{-1} . \left[x^{-1}\right]_{100}^{150} = -100 . \left(\frac{1}{150} - \frac{1}{100}\right) = \frac{1}{3} = P$$

X: the # of among that fails within given time frame OOOOO

y is Binomial RV $F(y) = \binom{n}{y} p^y (1-p)^{n-y}$

$P(2) = \binom{5}{2} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^3 = \frac{80}{243} = \dots$

CDF vs PDF

X: CRV F(x) is PDF

$\rightarrow F(x) = P(X \leq x) = \int_{-\infty}^{x} F(x) dx$

$\rightarrow F(x) = \frac{d}{dx} F(x)$

Expectation & Variance of a CRV

$$M_a = E(x) = \int_{-\infty}^{\infty} x . F(x) . dx$$

Ex: X: $F(x) = \begin{cases} 2x, & 0 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases}$

$$E(x) = \int_0^1 x . 2x . dx = \int_0^1 2x^2 . dx = \frac{2}{3} . x^3 \Big|_0^1 = \frac{2}{3} . x^3$$

Ex : X : PDF

$$F(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Find $E(e^x)$

$Y = e^x$ Lets find CDF of y

$$F_y(x) = P(y \leq x) = P(e^x \leq x) = P(x \leq \ln(x)) = \int_0^{\ln x} f_x(y).dy = \int_0^{\ln x} 1.dy = x \Big|_0^{\ln x} = \ln x$$

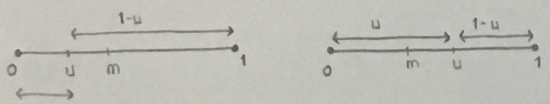
$$F_y(x) = \frac{d}{dx} F_y(x) = \frac{1}{x} \quad 1 \leq x \leq e \qquad E(e^x) = E(y) = \int_{-\infty}^{\infty} x.F_y(x) = \int_1^e x \cdot \frac{1}{x}.dx = e-1$$

Proposition : X is CRV , $g(x)$ is a function of X

$$E(g(x)) = \int_{-\infty}^{\infty} g(x).F(x).dx$$

$$E(e^x) = \int_0^1 e^x.1.dx = e-1$$

Ex : A stick of length 1 is split at point U, having density fn $F(u) = 1$, $0 < u < 1$. Determine the expected length of the piece that contains point m, $0 \leq m \leq 1$



$$L_m(u) = \begin{cases} 1-u, & u < p \\ u, & u > p \end{cases}$$

$$E(\text{lim}(u)) = \int_0^1 L_m(u).1.du = \int_0^m (1-u)du + \int_m^1 u\,du = \left(u - \frac{u^2}{2}\right)\Big|_0^m + \frac{u^2}{2}\Big|_m^1 = \left(m - \frac{m^2}{2}\right) + \left(\frac{1^2}{2} - \frac{m^2}{2}\right) = \boxed{\frac{1}{2} + m(1-m) = E(\text{lm}(u))}$$

Corollary : $E[ax+b] = a.E(x)+b$

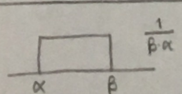
$$V(x) = E[(x-M)^2] = E[x^2 - 2xM + M^2] = E(x^2) \underbrace{-2E(x)M}_{-2M^2} + M^2 = E(x^2) - M^2$$

$\sigma = \sqrt{V(x)}$ = Standart Deviation

Also

$$V \in ax+b = a^2 V(x) = a^2.\sigma^2$$

Uniform Distribution

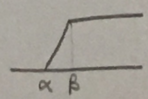


⟵ X is a uniform RV

$$F(x) = \begin{cases} \frac{1}{\beta-\alpha}, & \alpha < x < \beta \\ 0, & \text{elsewhere} \end{cases}$$

$F(x) = P(X \leq x) =$ $\quad P(X \leq \alpha) = 0 \quad F(x) = 0,\ x < \alpha$

$$\alpha < x < \beta \quad \int_\alpha^x F(x)dx = \int_\alpha^x \frac{1}{\beta-\alpha} \cdot dx = \frac{x-\alpha}{\beta-\alpha} \qquad ,\ x > \beta \quad F(x) = 1$$



$$F(x) = \begin{cases} 0, & x < \alpha \\ \frac{x-\alpha}{\beta-\alpha}, & \alpha < x < \beta \\ 1 & x > \beta \end{cases}$$

Expectation of Uniform RV

$$E(x) = \int_{\alpha}^{\beta} x \cdot \frac{1}{\beta-\alpha} \cdot dx = \frac{\beta^2-\alpha^2}{x\cdot(\beta-\alpha)} = \frac{\beta+\alpha}{2}$$

Variance

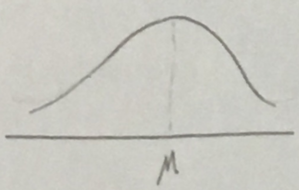
$$V(x) = E(x^2) - \mu^2 = \dots = \frac{(\beta-\alpha)^2}{12}$$

Normal (Gaussian) Distribution

X is a normal R.V. if

$$F(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{\frac{-(x-m)^2}{2\sigma^2}} \qquad -\infty < x < \infty$$

where $\sigma^{-2} = V(x)$ $\quad \mu = E(x)$



$N(\mu, \sigma^2)$ is a notation to denote normal distribution.

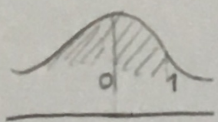
$$\int_{-\infty}^{\infty} F(x)\,dx = 1$$

Standart Normal R.V.

A normal R.V. with $\mu = 0$ and $\sigma^2 = 1$ is called a standart normal RV and is denoted Z

The CDF of Z $\quad \phi(z) = P(Z \leq z)$

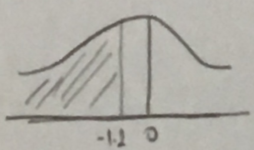
Ex: $P(Z \leq 1.5)$ Look at the table



	000...
1.5	0.933193

$P(Z < 1.5) \cong 0.9332$

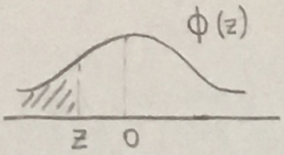
$P(Z < -1.21)$



	000	0.01
-1.2		0.1003

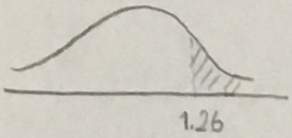
$P(Z < -1.21) = 0.1003\dots$

Example

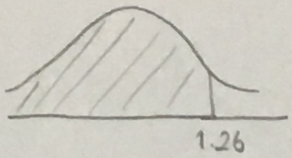


$P(Z \leq z)$, $\mu = 0$, $\sigma = 1$

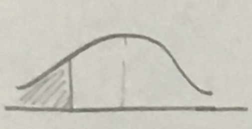
① $P(Z \geq 1.26) = 1 - P(Z \leq 1.26) = 1 - 0.88616 = 0.10384$



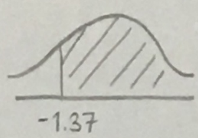
$= 1 -$



② $P(Z < -0.86) = 0.19490$



③ $P(Z > -1.37) = P(Z < 1.37) = 0.91465$



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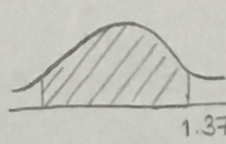
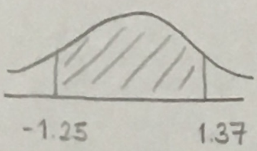


④ $P(-1.25 < Z < 1.37)$



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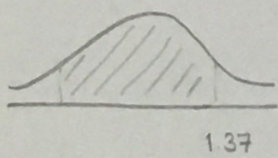

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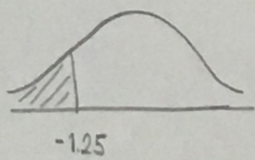


⑤ $P(Z < -4.6)$ $P(Z < -3.99) = 0.00003$ we can say $P(Z < -4.6) \approx 0$

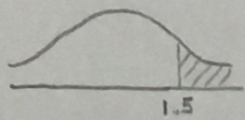
Standardizing a N.R.V *

X is a NRV with parameters μ, σ^2

The RV $Z = \dfrac{X - \mu}{\sigma}$ is standard normal R.V. w/ parameter $E(Z) = 0$
$V(Z) = 1$

Example: X is N.R.V $\mu = 10$ $\sigma^2 = 4$

$P(X > 13) = ?$



$Z = \dfrac{X - \mu}{\sqrt{4}} = \dfrac{X - 10}{2}$ $\Rightarrow P(Z > \frac{13-10}{2}) = P(Z > 1.5)$

From the table $P(Z < -1.5) = 0.06681$

Normal Approximation to the Binomial R.V.

X is a binomial R.V with parameter $n=10$, $p=0.5$ graph in the lesson's note

If X is a binomial R.V w/ parameters n, p

$$Z = \frac{X - np}{\sqrt{np(1-p)}}$$ is approximately a S.N.R.V

A better approximation is obtained by applying a continuity correction such as

$$P(X \leq x) = P(X \leq x + 0.5) = P\left(Z \leq \frac{x + 0.5 - np}{\sqrt{np(1-p)}}\right) \text{ and } P(x \leq X) = P(x - 0.5 \leq X) = P\left(\frac{x - 0.5 - np}{\sqrt{np(1-p)}} \leq Z\right)$$

This approximatis good when $np > 5\text{-}10$

$n(1-p) > 5\text{-}10$

$$P(x_1 < X < x_2) = P\left(\frac{x_1 - 0.5 - np}{\sqrt{np(1-p)}} < Z < \frac{x_2 + 0.5 - np}{\sqrt{np(1-p)}}\right)$$

$\uparrow$

$P(x_1 - 1.5 < x < x_2 + 0.5)$

Ex: X is B.N.R.V $p = 1\times10^{-7}$, $n = 16\times10^{6}$

$$P(x < 150) = \sum_{x=0}^{150} \binom{16\,000\,000}{x} 10^{-5.x} (1 - 10^{-5})$$

$np = 10^{-5} \times 16 \times 10^{16} = 160 = E(x)$

$n(1-p) = 16\times10^{6}(1 - 10^{-5}) \approx 16.10^{6}$

We can use normal approximation to the BNRV $\mu = 160$ $\sigma^2 = np(1-p) = 160 \times (1 - 10^{-5})$

$$P(x \leq 1.50) = P(x \leq 1.50{,}5)$$

$$= P\left(\frac{x - 160}{\sigma} \leq \frac{150{,}5 - 160}{\sqrt{160(1 - 10^{-5})}}\right) = P(Z \leq -0.75) = 0.227$$

Normal Approximation to Poisson Distribution

If X is a Poisson R.V. with $E(x)=X$ and $V(x)=X$, $Z=\dfrac{X-\lambda}{\sqrt{\lambda}}$ is approximately a standard N.R.V

We can apply the continuity correction used for B. distribution.

This is approximation is good when $\lambda > 10$

$P(X \le x) = P(X \le x+0.5)$

$P(x \le X) = P(x-0.5 \le X)$

Ex: X is a Poisson R.V. $\lambda = 1000$

$P(x \le 950) = ?$

$$P(x \le 950) = \sum_{x=0}^{950} e^{-1000} \cdot \frac{1000^x}{x!}$$

Difficult to calculate, instead we use Normal approximation.

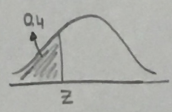
$$P(x \le 950) = P(x \le 950.5) \approx P\left(Z \le \frac{950.5-1000}{\sqrt{1000}}\right) = P(Z \le -1.57) = 0.058$$

OLD EXAMS Q = Errors on a data stream follows Poisson distribution with an average of 5 errors per minute. What is the expected amount of time in which the probability of getting at most 450 errors is 0.4 (Note: Use Normal approximation with Continuity Correction)

Y is a PRV $\lambda = 5$, X is also a PRV λ

$$P(x \le 450) = \sum_{x=0}^{450} e^{-\lambda} \cdot \frac{\lambda^x}{x!} = 0.4 \qquad E(x) = ?$$

$$P(x \le 450.50) \approx P\Big(Z \le \underbrace{\frac{450.5-\lambda}{\sqrt{\lambda}}}_{z}\Big) = 0.4$$



$P(Z \le z) = 0.4$

From the table

	-0.4	-0.05
-0.2	0.3974	0.4013

$z = -0.25$

$$\frac{450.5-\lambda}{\sqrt{\lambda}} = -0.25 \Rightarrow (450.5-\lambda)^2 = (-0.25)^2 \cdot \lambda$$

$\lambda = 455.8374$ error / t

$\lambda' = 5$ errors / m

$$\frac{455.8374}{5} = 91.1675 \text{ minutes}$$

t = 1 hours 31 minutes 10.049 seconds