CALCULUS - II

16.02.2015
Monday

Antiderivative ;

$F(x) = x^2 + 2 \quad F'(x) = 2x$

$F(x) = x^2 + e^{-2} \quad F'(x) = 2x$

$F(x) = x^2 + \ln\left(\frac{1}{\pi}\right) \quad F'(x) = 2x$

Def.: If the derivative of $F(x)$ is $f(x)$, then we called that $F(x)$ is the antiderivative of $f(x)$

$f(x) = 2x \Rightarrow F(x) = x^2 + c$, c is arbitrary constant

Note that Most common antiderivative is $F(x) + c$.

$F'(x) = f(x)$	Antiderivative $F(x)$
$2x$	$x^2 + c$
$\cos x$	$\sin x + c$
$\sin x$	$-\cos x + c$
e^x	$e^x + c$
a^x	$\frac{a^x}{\ln a} + c$, $a > 0$
$\cos(2x)$	$\frac{\sin 2x}{2} + c$
$\frac{1}{\sqrt{x}}$	$2\sqrt{x} + c$
$\cot x$	$\ln\lvert\sin x\rvert + c$
$\frac{1}{e^x} + x^2$	$-e^{-x} + \frac{x^3}{3} + c$
$\cos\frac{x}{2}$	$2 \sin\frac{x}{2}$
$\sec^2 x = 1 + \tan^2 x = \frac{1}{\cos^2 x}$	$\tan x + c$
$\sec x \cdot \tan x$	$\sec x + c$

Def: Let $F(x)$ is the antiderivative of $f(x)$. Then the most general form of $F(x)$ is indefinite integral.

$$\int f(x).dx \qquad F(x) = \int f(x).dx$$

Ex: $F(x) = \int \sec^2 x \, dx = \tan x + c$

Ex

① $\int \tan^2 x \, dx = \int (\tan^2 x + 1 - 1) . dx = \tan x - x + c$

② $\int (x^4 - 4x^3 + 2) \, dx = \frac{x^5}{5} - x^4 + 2x + c$

③ $\int \frac{x^3 - 2x^2 + 1}{x} . dx = \frac{x^3}{3} - x^2 + \ln\lvert x\rvert + c$

④ $\int \frac{\sqrt{x} + 1}{\sqrt{x}} = x + 2.\sqrt{x} + c$

23.02.2015
Monday

Ex: Solve the initial value problem

$$\left.\begin{aligned} y' &= x^2+2x+5 \\ y(0) &= 5 \end{aligned}\right\} \quad y(x) = ?$$

Sol: $y(x) = \int (x^2+2x+5).dx = \frac{x^3}{3} + x^2 + 5x + c$

$y(0) = 5 = c$

$$y(x) = \frac{x^3}{3} + x^2 + 5x + 5$$

→ Some properties (In the page)

$\int c.f(x).dx = c.\int f(x).dx \quad , c \in R$, $\int (f(x) \mp g(x)).dx = \int f(x).dx \mp \int g(x).dx$

Ex: → $\int (2x+1).dy = (2x+1).y + c$ → $\int 3.dy = 3y + c$

Ex: $\int 2^x.dx = \frac{2^x}{\ln 2} + c$ Ex: $\int e^{2x}.dx = \frac{e^{2x}}{2} + c$ Ex: $\int 4x.e^{x^2}.dx = 2.e^{x^2} + c$

Ex: $\int \tan x.dx = \int \frac{\sin x}{\cos x}.dx = -\ln|\cos x| + c$ Ex: $\int \frac{x^2-3x+2}{x-1}.dx = \int \frac{(x-2)\cancel{(x-1)}}{\cancel{(x-1)}}.dx = \frac{x^2}{2} - 2x + c$

Ex: $\int \sec x.\tan x \, dx = \sec x + c$

Ex: 1-) $\int (3e^x + 5.\cos x - 10\sec^2 x) dx = 3e^x + 5\sin x - 10\tan x + c$

2-) $\int \left(\frac{23}{y^2+1} + 6\operatorname{cosec} y.\cot y + \frac{9}{y}\right) dy = 23.\tan^{-1} y - 6.\operatorname{cosec} y + 9.\ln|y| + c$

3-) $\int \frac{7-6.\sin^2\theta}{\sin^2\theta}.d\theta = \int (7.\operatorname{cosec}^2\theta - 6).d\theta = -7.\cot\theta - 6\theta + c$

4-) $\int 2.\sin\left(\frac{t}{2}\right).\cos\left(\frac{t}{2}\right).dt = \int \sin t.dt = -\cos t + c$

5-) $f''(x) = 15\sqrt{x} + 5x^3 + 6 \quad , f(1) = -\frac{5}{4} \quad , f(4) = 404$

$f'(x) = \int 15.x^{1/2} + 5x^3 + 6 = 10.x^{3/2} + \frac{5}{4}.x^4 + 6x + c_1$

$f(x) = \int 10.x^{3/2} + \frac{5.x^4}{4} + 6x + c_1 = 4.x^{5/2} + \frac{x^5}{4} + 3x^2 + c_1.x + c_2$

$f(1) = 4 + \frac{1}{4} + 3 + c_1 + c_2 = -\frac{5}{4}$ $f(4) = 4.\sqrt{4^5} + \frac{4^5}{4} + 3.4^2 + 4c_1 + c_2 = 404$

$c_1 = -\frac{13}{2}$

$c_2 = -2$

$c_1 + c_2 = -\frac{34}{4}$

$$f(x) = 4\sqrt{x^5} + \frac{x^5}{4} + 3x^2 - \frac{13}{2}x - 2$$

Extra

→ $a_0+a_1+a_2+\dots+a_n=\sum_{k=0}^{n} a_k$

→ $2+4+6+\dots+2n=\sum_{k=1}^{n}2k=n.(n+1)$

→ $1+2+3+\dots+n=\sum_{k=1}^{n} k=\frac{n.(n+1)}{2}$

→ $1^2+2^2+3^2+\dots+n^2=\sum_{k=1}^{k} k^2=\frac{n.(n+1)(2n+1)}{6}$

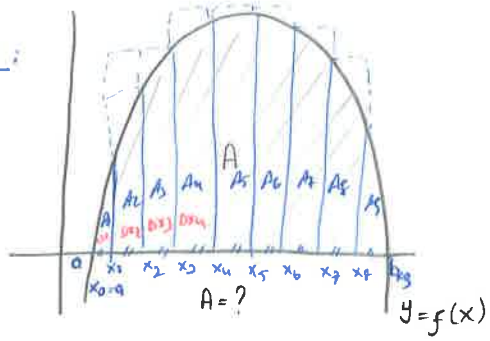
→ $1+3+5+\dots+(2n-1)=\sum_{k=1}^{n}(2k-1)=n^2$

→ $1^3+2^3+3^3+\dots+n^3=\sum_{k=1}^{n} k^3=\left[\frac{n.(n+1)}{2}\right]^2$

$$\sum_{k=p}^{n} a_k=\sum_{k=p}^{n} f(k) \Rightarrow \sum_{k=p+r}^{n+r} a_{k-r}=\sum_{k=p+r}^{n+r} f(k-r)$$

Ex: $\sum_{k=-2}^{10}\frac{\sin k+k^2}{2k+1}=\sum_{k=0}^{12}\frac{\sin(k-2)+(k-2)^2}{2.(k-2)+1}$

Ex:



$A\cong A_1+A_2+\dots+A_9=\sum_{k=1}^{k=9}A_k$

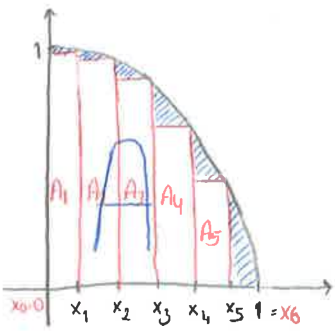
$A_1=|x_1-x_0|.f(x_1)$

$A_2=|x_2-x_1|.f(x_2)$

$A_3=|x_3-x_2|.f(x_3)$

$|x_k-x_{k-1}|=\Delta x_k$

Ex: Find the area btwn $f(x)=1-x^2$, $[0,1]$, x-axis and y-axis



$A_1=|x_1-x_0|.f(x_1)$

$A_2=|x_2-x_1|.f(x_2)$

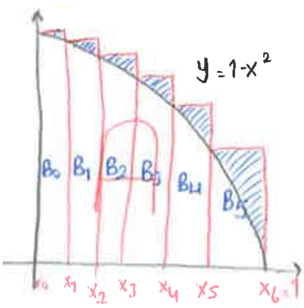
$\vdots$

$A_5=|x_5-x_4|.f(x_5)$

$\sum_{k=1}^{5}A_k=\sum_{k=1}^{5}f(x_k).\Delta x_k$, $\Delta x_k=|x_k-x_{k-1}|$

$A>\sum_{k=1}^{5}A_k=A_1+A_2+A_3+A_4+A_5$ → Lower sum

Upper Sum



$B_0=f(x_0).|x_1-x_0|$

$B_1=f(x_1).|x_2-x_1|$

$\vdots$

$B_5=f(x_5).|x_6-x_5|$

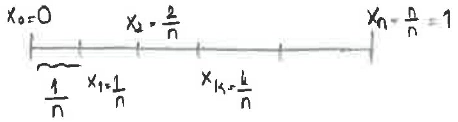
$\Delta x_k=|x_k-x_{k-1}|$

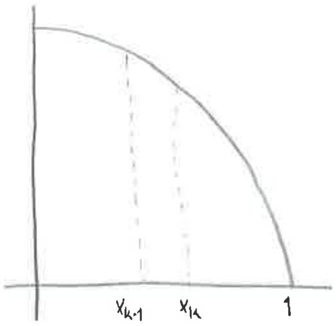
$A<B_0+B_1+\dots+B_5=\sum_{k=0}^{5}f(x_k).\Delta x_k$

25.02.2015
Wednesday

Claim

$$\Delta X_k \to 0 \quad \Rightarrow \quad \sum_{k=1}^{\infty} A_k = A = \sum_{k=0}^{\infty} B_k$$





Lower sum $= \displaystyle\sum_{k=1}^{n} f(x_k)\cdot \Delta X_k \quad , f(x) = 1-x^2$

$$= \sum_{k=1}^{n} (1-x_k^2)\cdot \Delta X_k$$

$$= \sum_{k=1}^{n} \left(1-\frac{k^2}{n^2}\right)\cdot \frac{1}{n}$$

$$= \sum_{k=1}^{n} \frac{1}{n} - \sum_{k=1}^{n} \frac{k^2}{n^3}$$

$$\rightarrow \frac{1}{n}\cdot\sum_{k=1}^{n} 1 - \frac{1}{n^3}\cdot\sum_{k=1}^{n} k^2 =$$

$$\frac{1}{n}\cdot n - \frac{1}{n^3}\cdot\frac{n\cdot(n+1)(2n+1)}{6} =$$

$$\boxed{1 - \frac{2n^3-3n^2+n}{6n^3}}$$

$$\Delta X_k = |X_k - X_{k-1}| = \frac{1-0}{n} \Rightarrow \Delta X_k = \frac{1}{n}$$

$\Downarrow$

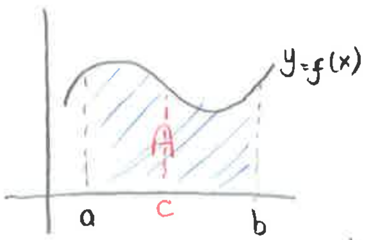
$$\boxed{A = \int_0^1 (1-x^2)\cdot dx = \left(x-\frac{x^3}{3}\right)\Bigg|_0^1 = \left(1-\frac{1}{3}\right)-0 = \frac{2}{3}}$$

$$(\Delta X_k \to 0) = (n\to\infty)$$

$$\Delta X_k = \frac{1}{n} \xrightarrow{n\to\infty} 0$$

$$A = \lim_{n\to\infty}\left(1-\frac{2n^3+3n^2+n}{6n^3}\right) = \left(1-\frac{2}{6}\right) = \underline{\underline{\frac{2}{3}}}$$

Definite Integral



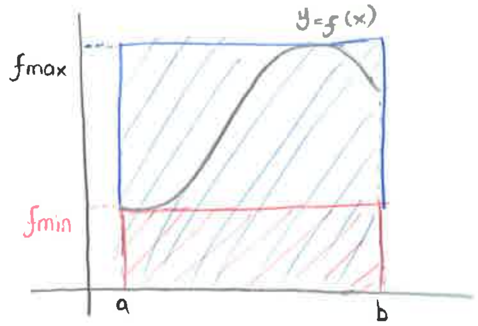
$0 \leq f(x)$ and piece-wice continuous

$$A = \lim_{x\to\infty} \sum_{k=1}^{n} f(x_k)\cdot\Delta X_k = \int_a^b f(x)\cdot dx$$

$\rightarrow \displaystyle\int_a^a f(x)\cdot dx = 0$

$\rightarrow \displaystyle\int_a^b f(x)\cdot dx = \int_a^c f(x)\cdot dx + \int_c^b f(x)\cdot dx$

$\rightarrow \displaystyle\int_a^b f(x)\, dx = -\int_b^a f(x)\cdot dx$



$$\min_{a \leq x \leq b} f \cdot (b-a) \leq \int_a^b f(x).dx \quad \max_{a \leq x \leq b} f \cdot (b-a)$$

Ex : Show that $\int_0^1 \sqrt{1+\cos x} \leq \sqrt{2}$

$f(x) = \sqrt{1+\cos x}$

$f_{max} = \sqrt{2}$

$$\int_0^1 \sqrt{1+\cos x}\,.dx \leq \max_{0 \leq x \leq 1} f \cdot (1-0)$$

$$\leq \sqrt{2}\,.\,1$$

Ex : $\int_{-2}^{4} f(x).dx = 10$, $\int_{2}^{4} f(x).dx = 2$, $\int_{-2}^{2} h(x).dx = -5$

a-) $\int_{2}^{-2} f(x).dx = ? \rightarrow -8$

$$\Downarrow$$

$$-\int_{-2}^{2} f(x).dx$$

$$\int_{-2}^{4} f(x).dx = \int_{-2}^{2} f(x).dx + \int_{2}^{4} f(x).dx = 10$$

$$\int_{-2}^{2} f(x).dx = 8$$

b-) $\int_{-2}^{2} \left[3.f(x) - 2.h(x)\right].dx = ?$

$$3\int_{-2}^{2} f(x).dx - 2\int_{-2}^{2} h(x).dx$$

$$= 3.8 - 2.(-5) = \underline{\underline{34}}$$

* Average of $f(x) = Avr(f) = \frac{1}{b-a} \int_a^b f(x)\,dx$

Ex : Find the average of $f(x) = \sqrt{4-x^2}$ on $[-2,2]$

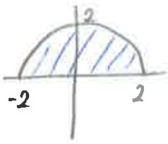
$$\text{Average } f(x) = \frac{1}{2-(-2)} \int_{-2}^{2} \sqrt{4-x^2}\,.dx$$

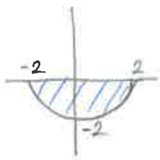
$$= \frac{1}{\cancel{4}} \cdot \frac{\pi.\cancel{2^2}}{2} = \underline{\underline{\frac{\pi}{2}}}$$

$x^2+y^2=4$

$y^2 = 4-x^2$

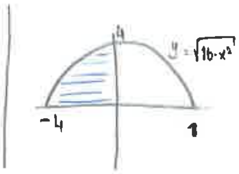
$y = \sqrt{4-x^2}$ $\quad$ $y = -\sqrt{4-x^2}$





$\sqrt{f(x)} > 0$

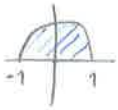
① $\int_{-4}^{0}\sqrt{16-x^2}\,.dx$



$= \frac{\pi . 4^{\cancel{2}}}{\cancel{4}} = 4.\pi$

② $\int_{-1}^{1}\left(1+\sqrt{1-x^2}\right).dx$

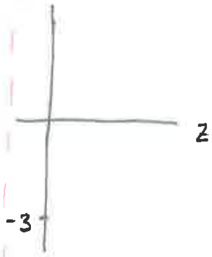
$= \int_{-1}^{1} dx + \int_{-1}^{1}\sqrt{1-x^2}\,.dx$



$= x\Big|_{-1}^{1} + \frac{\pi . 1^2}{2}$

$= 1-(-1) + \frac{\pi}{2} = 2+\frac{\pi}{2}$

③ $\int_{3}^{0}(2z-3)\,dz$



④ $\int_{1}^{2} 3u^2 .du = u^3\Big|_{1}^{2}$

$= 8-1 = 7$

⑤ $\int_{-2}^{1}|x|.dx$ $|x| = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$

$\int_{-2}^{1}|x|.dx = \int_{-2}^{0}-x.dx + \int_{0}^{1}x.dx = \frac{-x^2}{2}\Big|_{-2}^{0} + \frac{x^2}{2}\Big|_{0}^{1} = -(0-2)+\frac{1}{2}-0 = \frac{5}{2}$

Ex: $\int_{-2}^{4}(2-|x|).dx = \int_{-2}^{4}2.dx - \int_{-2}^{4}|x|.dx = \int_{-2}^{4}2.dx + \int_{-2}^{0}x.dx - \int_{0}^{4}x.dx = 2x\Big|_{-2}^{4} + \frac{x^2}{2}\Big|_{-2}^{0} - \frac{x^2}{2}\Big|_{0}^{4} = 8+4+(0-2)-(8-0) = 2$

Ex: $\int_{0}^{3}|1-x|.dx$ $|1-x| = \begin{cases} x-1 & , x \geq 1 \\ 1-x & , x < 1 \end{cases}$

$= \int_{0}^{1}(1-x).dx + \int_{1}^{3}(x-1)\,dx = (0-1)+(2-0) = 1$

SUBSTITUTION METHOD FOR INTEGRATION

Up to this stage, we could do simple integration using formulas and simple rules. For more complicated ones, like $\int x.e^{x^2}dx$, we have t
use some techniques.

Theorem: On an interval I, if $g(x)$ is differentiable and $f(x)$ is continuous, then $\int f(g(x)).g'(x)\,dx = \int f(u).du$ where $u=g(x)$

Examples

a) $\int (x^2+1)^4\, 2x\,dx = \begin{bmatrix} x^2+1=u \\ 2x.dx=du \end{bmatrix} = \int u^4.du = \frac{u^5}{5}+c = \frac{(x^2+1)^5}{5}+c$

b) $\int 2x.e^{x^2}dx = \begin{bmatrix} x^2=u \\ 2x=du \end{bmatrix} = \int e^u.du = e^u+c = e^{x^2}+c$

02.03.2015
Monday
Mother's Birthday :))

c-) $\int e^x . \sin(e^x).dx = \begin{bmatrix} e^x = u \\ e^x.dx = du \end{bmatrix} = \int \sin u . du = -\cos u + c = -\cos(e^x) + c$

d-) $\int \cos(2x+1).dx = \begin{bmatrix} 2x+1 = u \\ 2 = du \\ dx = \frac{du}{2} \end{bmatrix} = \int \cos u . \frac{du}{2} = \frac{\sin u}{2} = \frac{\sin(2x+1)}{2} + c$

e-) $\int \frac{dx}{e^x + e^{-x}} = \int \frac{dx}{e^x + \frac{1}{e^x}} = \int \frac{e^x.dx}{e^{2x}+1} = \begin{bmatrix} e^x = u \\ e^x dx = du \end{bmatrix} = \int \frac{du}{u^2+1} = \arctan u + c = \arctan(e^x) + c$

f-) $\int \frac{x.dx}{\sqrt[3]{x^2+1}} = \begin{bmatrix} x^2+1 = u \\ 2x.dx = du \end{bmatrix} = \frac{1}{2} \int \frac{du}{u^{1/3}} = \frac{1}{\cancel{2}} . \frac{\cancel{2}}{3} . u^{2/3} = \frac{1}{3} . (x^2+1)^{2/3} + c$

g-) $\int \sqrt{x+4} . dx = \begin{bmatrix} x+4 = u \\ dx = du \end{bmatrix} = \int u^{1/2} du = \frac{3}{2} . u^{3/2} + c = \frac{3}{2} . (x+4)^{3/2} + c$

h-) $\int x . \sqrt{2x+1} . dx = \begin{bmatrix} 2x+1 = u \\ 2.dx = du \\ dx = du/2 \\ x = \frac{u-1}{2} \end{bmatrix} = \int \frac{u-1}{2} . u^{1/2} . \frac{du}{2} = \frac{1}{4} \int u^{3/2} . du - \frac{1}{4} \int u^{1/2} du = \frac{1}{4} . \frac{5}{2} u^{5/2} - \frac{1}{4} . \frac{3}{2} u^{3/2} + c$

$= \frac{5}{8} (2x+1)^{5/2} - \frac{3}{8} . (2x+1)^{3/2} + c$

i-) $\int x . \sin(2.x^2) . dx = \begin{bmatrix} 2x^2 = u \\ 4x.dx = du \\ x.dx = du/4 \end{bmatrix} = \int \sin u . \frac{du}{4} = \frac{-\cos u}{4} + c = -\frac{1}{4} . \cos(2x^2) + c$

j-) $\int \frac{9.r^2}{\sqrt{1-r^3}} dr = \begin{bmatrix} 1-r^3 = u \\ -3r^2.dr = du \\ 9r^2 dr = -3du \end{bmatrix} = -3 \int \frac{du}{u^{1/2}} = -3 . \frac{1}{2} . u^{1/2} + c = \frac{-3}{2} . (1-r^3)^{1/2} + c$

k-) $\int \sqrt{x} . \sin(x^{3/2} - 1) dx = \begin{bmatrix} x^{3/2} - 1 = u \\ \frac{3}{2} . x^{1/2} dx = du \\ x^{1/2} . dx = \frac{2}{3} . du \end{bmatrix} = \frac{2}{3} \int \sin u . du = \frac{-2}{3} . \cos u + c = \frac{-2}{3} . \cos(x^{3/2} - 1) + c$

l-) $\int \frac{1}{x^2} . \cos\left(\frac{1}{x}\right) dx = \begin{bmatrix} \frac{1}{x} = u \\ \frac{-1}{x^2} . dx = du \\ \frac{1}{x^2} . dx = -du \end{bmatrix} = -\int \cos u . du = -\sin u + c = -\sin\left(\frac{1}{x}\right) + c$

m-) $\int \frac{r.dr}{(r^2-4)^2} = \begin{bmatrix} r^2 - 4 = u \\ 2r.dr = du \\ r.dr = du/2 \end{bmatrix} = \frac{1}{2} \int \frac{du}{u^2} = \frac{-1}{2} . \frac{1}{u} + c = \frac{-1}{2} . \frac{1}{r^2-4} + c$

n-) $\int \frac{dx}{x.\ln x} = \left[\begin{array}{l} \ln|x| = u \\ \frac{1}{x}\cdot dx = du \end{array}\right] = \int \frac{du}{u} = \ln|u| + c = \underline{\ln|\ln|x|| + c}$

o-) $\int \tan x . dx = \int \frac{\sin x}{\cos x} \cdot dx = \left[\begin{array}{l} \cos x = u \\ -\sin x = du \end{array}\right] = \int \frac{-du}{u} = -\ln|u| + c = \underline{-\ln|\cos x| + c}$

p-) $\int \tan^3 x \, dx = \int \tan x . \tan^2 x . dx = \int \tan x \, (\tan^2 x + 1 - 1)\, dx = \underbrace{\int \tan x \, (\tan^2 x + 1)\, dx}_{I_1} - \underbrace{\int \tan x}_{I_2}$

Remark: $(\arctan x)' = \frac{1}{1+x^2}$, $(\text{arccot} x)' = \frac{-1}{1+x^2}$

$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$, $(\arccos x)' = \frac{-1}{\sqrt{1-x^2}}$

$I_1 = \left[\begin{array}{l} \tan x = u \\ (\tan^2 x + 1) dx = du \end{array}\right] = \int u . du = \frac{u^2}{2} + c = \frac{\tan^2 x}{2} + c$

Then, from (o), we have that $\int \tan^3 x \, dx = \underline{\frac{\tan^2 x}{2} - \ln|\ln(x)| + c}$

r-) $\int \frac{\arctan x . dx}{1 + x^2} = \left[\begin{array}{l} \arctan x = u \\ \frac{dx}{1+x^2} = du \end{array}\right] = \int u . du = \frac{u^2}{2} + c = \underline{\frac{(\arctan x)^2}{2} + c}$

s-) $\int \frac{\sin x . dx}{1 + \cos^2 x} = \left[\begin{array}{l} \cos x = u \\ -\sin x dx = du \\ \sin x dx = -du \end{array}\right] = -\int \frac{du}{1+u^2} = \text{arccot} u + c = \underline{\text{arccot}(\cos x) + c}$

t-) $\int \frac{1}{x^2} \cdot \sin \frac{1}{x} \, dx = \left[\begin{array}{l} \frac{1}{x} = u \\ \frac{-1}{x^2} \cdot dx = du \end{array}\right] = -\int \sin u . du = \cos u + c = \underline{\cos \frac{1}{x} + c}$

u-) $\int \frac{x . dx}{\sqrt{x+1}} = \left[\begin{array}{l} x+1 = u \\ dx = du \end{array} \Rightarrow x = u - 1\right] = \int \frac{u-1}{u^{1/2}} \cdot du = \int u^{1/2} du - \int \frac{1}{u^{1/2}} du = \frac{3}{2} \cdot u^{3/2} - \frac{1}{2} \cdot u^{1/2} + c = \underline{\frac{3}{2}(x+1)^{3/2} - \frac{1}{2}(x+1)^{1/2} + c}$

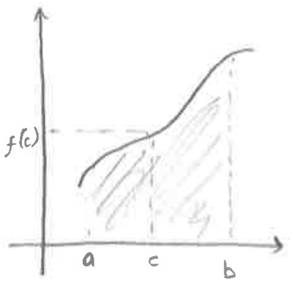
v-) $\int 3x^5 . \sqrt{x^3+1} \, . dx = \left[\begin{array}{l} x^3+1 = u \\ 3x^2 . dx = du \end{array} \Rightarrow x^3 = u - 1\right] = \int 3x^2 . x^3 . \sqrt{x^3+1} \, . dx = \int (u-1)\, u^{1/2} du = \int u^{3/2} du - \int u^{1/2} du = \frac{5}{2} . u^{5/2} - \frac{3}{2} . u^{3/2} + c$

$= \underline{\frac{5}{2}(x^3+1)^{5/2} - \frac{3}{2}(x^3+1)^{3/2} + c}$

Theorem : Mean-Value

$f(x)$ is cont. on $[a,b]$ then at some $c \in [a,b]$

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx$$



Thm : Fundemental thm of Calculus part I

$$\frac{d}{dx} \int_a^x f(t)\,dt = f(x)$$

Thm : fundemental thm of Calculus part II

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Proof : let $F(x)$ is the anti-derivative of $f(x)$

i.e. $\dfrac{dF(x)}{dx} = f(x) \iff \int f(x)\,dx$

Derivative of constant is 0

$$\int_a^x F(t)\,dt = F(x) - F(a)$$

$$\frac{d}{dx} \int_a^x f(t)\,dt = \frac{d}{dx}\Big[F(x) - F(a)\Big] = f(x) - 0$$

Ex : $\dfrac{d}{dx}\displaystyle\int_a^x \cos t\,dt = \cos x$

I. Way (Fund. thm)

II. Way $\dfrac{d}{dx}\displaystyle\int_a^x \cos t\,dt = \dfrac{d}{dx}\Big[\sin t \Big|_a^x\Big] = \dfrac{d}{dx}\Big[\sin x - \sin a\Big]$

$= \cos x - 0$

Ex : $\displaystyle\int_{-\pi/4}^{0} \sec x \cdot \tan x\,dx = \sec x \Big|_{-\pi/4}^{0} = \sec 0 - \sec\left(-\frac{\pi}{4}\right) = 1 - \frac{2}{\sqrt{2}}$

Leipnitz-Rule [use (*)]

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt = f(v(x)).v'(x) - f(u(x)).u'(x)$$

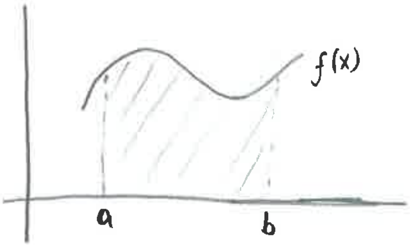
Proof: $\int_{u(x)}^{V(x)} f(t)dt = F(v(x)) - F(u(x))$ (from (*))

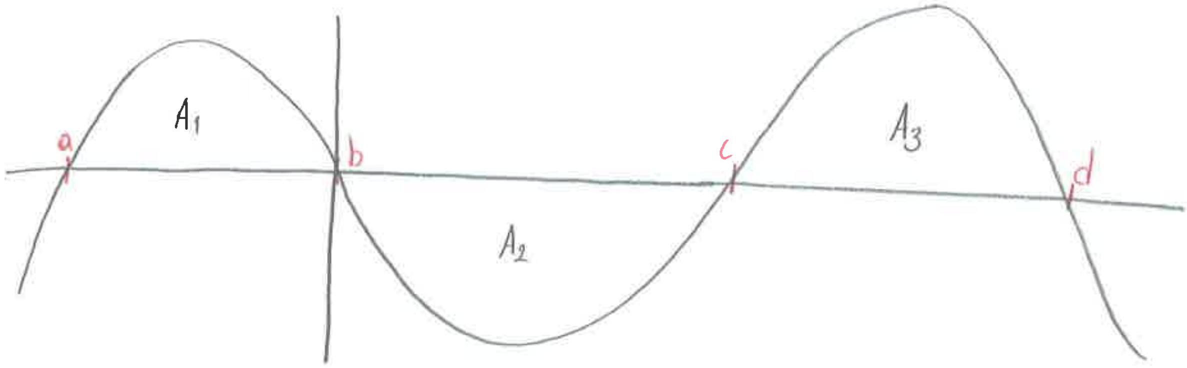
$$\frac{d}{dx}\int_{u(x)}^{V(x)} f(t)\,dt = \frac{d}{dx}\left[F(V(x)) - F(u(x))\right] = f(V(x)).V'(x) - f(u(x)).u'(x)$$

Ex: $y(x) = \int_0^{\sqrt{x}} \sin(t^2)dt$ $\quad \frac{dy}{dx} = ? = \frac{d}{dx}\int_0^{\sqrt{x}} \sin(t^2)dt = \sin(\sqrt{x})^2 \cdot \frac{1}{2\sqrt{x}} - \sin(0^2).0 = \frac{\sin x}{2\sqrt{x}}$

Ex: $\frac{d}{dx}\int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}} = ? = \frac{1}{\cancel{\sqrt{1-\sin^2 x}}} \cdot \cancel{\cos x} - \frac{1}{\sqrt{1-0^2}} \cdot 0 = 1$

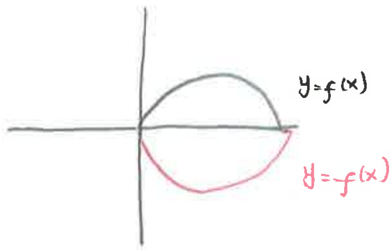
$f(x) \geqslant 0$ on $[a,b]$ $\Rightarrow$ $\int_a^b f(x)dx$ = area btwn the curve, $a \leqslant x \leqslant b$ and x-axis





→ $\int_a^d f(x)dx = A_1 - A_2 + A_3 = \int_a^b f(x).dx + \int_b^c f(x).dx + \int_c^d f(x)dx$

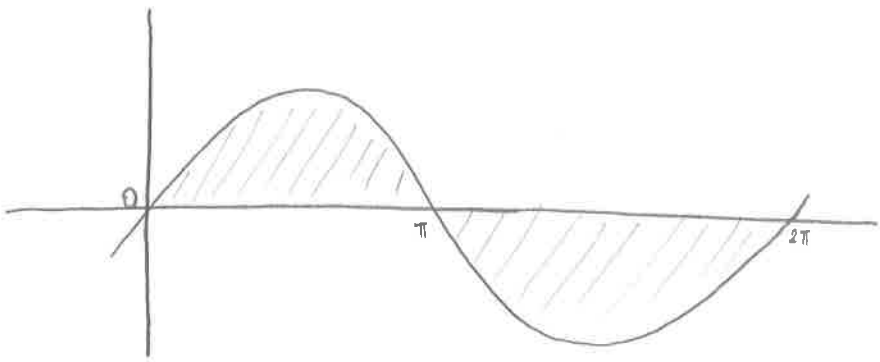
$y = -f(x)$ and $y = f(x)$ are symmetric w.r.t x-axis.



→ $\int_a^d |f(x)|dx = \int_a^b |f(x)|\,dx + \int_b^c |f(x)|dx + \int_c^d |f(x)|dx$

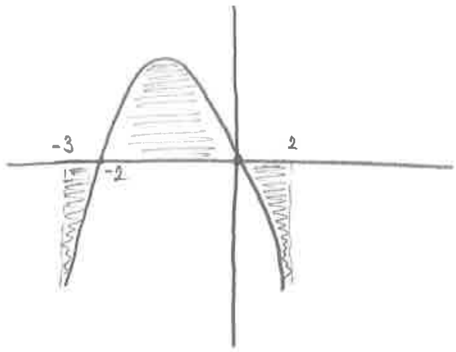
$= \int_a^b f(x)dx + \int_b^c -f(x)dx + \int_c^d f(x)dx = A_1 + A_2 + A_3$

Ex : Find the area bounded by $f(x) = \sin x$, $[0, 2\pi]$ and x-axis



$$\int_0^{2\pi} |\sin x| dx = \int_0^{\pi} |\sin x| dx + \int_{\pi}^{2\pi} |\sin x| dx = \int_0^{\pi} \sin x dx + \int_{\pi}^{2\pi} -\sin x = -\cos x \Big|_0^{\pi} + \cos x \Big|_{\pi}^{2\pi} = -(-1-1) + (1-(-1)) = \underline{\underline{4}}$$

Ex : find the area bold by $\overbrace{y = -x^2 - 2x}^{y = -x(x-2)}$, $-3 \leq x \leq 2$ and x-axis



Area : $$\int_{-3}^{2} |-x^2 - 2x| dx = \int_{-3}^{-2} \underbrace{|-x^2-2x|}_{-} dx + \int_{-2}^{0} \underbrace{|-x^2-2x|}_{+} dx + \int_{0}^{2} \underbrace{|-x^2-2x|}_{-} dx$$

$$= \int_{-3}^{-2} (x^2+2x) dx - \int_{-2}^{0} (x^2+2x) dx + \int_{0}^{2} (x^2+2x) dx$$

$$= \frac{x^3}{3} + x^2 \Big|_{-3}^{-2} - \left(\frac{x^3}{3} + x^2\right)\Big|_{-2}^{0} + \left(\frac{x^3}{3} + x^2\right)\Big|_{0}^{2}$$

$$= \left(\frac{-8}{3} + 4\right) - (-9+9) - \left[0 - \left(\frac{-8}{3} + 4\right)\right] + \left[\frac{8}{3} + 4 - 0\right] = 12 - \frac{8}{3} = \underline{\underline{\frac{26}{3}}}$$

Hw : $\int \sec x \, dx = ?$

Exercises

① $$\int_0^{\pi/4} \tan^2 x \, dx = \int_0^{\pi/4} (\tan^2 x + 1 - 1) dx = \tan x - x \Big|_0^{\pi/4} = \left(1 - \frac{\pi}{4}\right) - 0 = 1 - \frac{\pi}{4} //$$

② $$\int_{\pi/2}^{\pi} \frac{\sin 2x}{2.\sin x} dx = \int_{\pi/2}^{\pi} \frac{2.\cancel{\sin x}.\cos x}{2.\cancel{\sin x}} dx = \sin x \Big|_{\pi/2}^{\pi} = 0 - 1 = -1 //$$

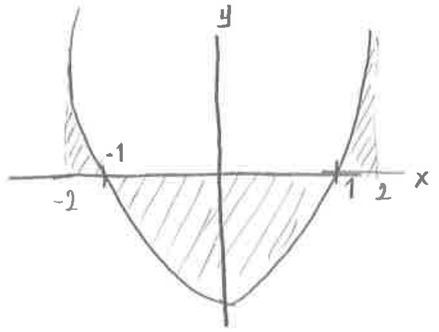
③ $$\int_0^{\pi} \frac{1}{2} \cdot (\cos x + |\cos x|) dx = \int_0^{\pi/2} \frac{1}{2} \cdot (\cos x + \cos x) dx + \int_{\pi/2}^{\pi} \frac{1}{2} (\cancel{\cos x} - \cancel{\cos x}) dx = \int_0^{\pi/2} \cos x . dx = \sin x \Big|_0^{\pi/2} = 1 //$$

④ $$\frac{d}{dx} \int_{\tan x}^{0} \frac{dt}{1+t^2} = \frac{1}{1+0^2} . 0 - \frac{1}{1+\cancel{\tan^2 x}} . \cancel{\sec x} = 0 - 1 = \underline{\underline{-1}}$$

04.03.2015
Wednesday

(5) Find the total area btwn the region $\underbrace{y=3x^2-3}$, $-2\leq x\leq 2$ and x-axis

$y=3(x-1)(x+1)$



Area : $\displaystyle\int_{-2}^{2}|3x^2-3|\,dx = \int_{-2}^{-1}(3x^2-3)dx - \int_{-1}^{1}(3x^2-3)dx + \int_{1}^{2}(3x^2-3)\,dx$

(6) Solve the initial value problem $\underbrace{\frac{dy}{dx}=\frac{1}{x}}$, $y(\pi)=-3$

$dy=\frac{1}{x}\cdot dx$

$y(x)=\int\frac{1}{x}\cdot dx = \ln|x|+c \Rightarrow -3=\ln|\pi|+c \qquad c=-3-\ln|\pi| \qquad \underline{y(x)=\ln|x|-\ln\pi-3}$

(7) $\displaystyle\int\tan^3x\,dx = \int(\tan^3x+\tan x-\tan x)\,dx = \int\underbrace{\tan x}_{u}\,\underbrace{(\tan^2x+1)dx}_{du} + \int\frac{\overset{dv}{-\sin x}}{\underset{v}{\cos x}} = \int u\,du - \int\frac{dv}{v} = \frac{u^2}{2}-\ln|u|+c =$

$\displaystyle = \underline{\frac{\tan^2x}{2}-\ln|\cos x|+c}$

(8) $\displaystyle\int\cos^2x\,dx = \int\frac{\cos 2x+1}{2}\,dx = \frac{1}{2}\int(\cos^2x+1)\,dx = \underline{\frac{1}{2}\left(\frac{\sin 2x}{2}+x\right)+c}$

(9) $\displaystyle\int\cos^3x\,dx = \int\cos^2x\cdot\cos x\,dx = \int(1-\sin^2x)\cdot\cos x\,dx = \left|\begin{matrix}\sin x=u\\ \cos x=du\end{matrix}\right| = \int(1-u^2)du = u-\frac{u^3}{3}+c = \underline{\sin x-\frac{\sin^3x}{3}+c}$

(10) $\displaystyle\int\frac{e^{1/x}}{x^2}\cdot dx = \left|\begin{matrix}u=\frac{1}{x}\\ du=\frac{-1}{x^2}dx\end{matrix}\right| = -\int e^u\cdot du = -e^u+c = \underline{-e^{1/x}+c}$

(11) $\displaystyle\int\frac{dx}{\cos^2x\sqrt{1-\tan^2x}} = \left|\begin{matrix}\tan x=u\\ \sec^2x\cdot dx=du\end{matrix}\right| = \int\frac{du}{\sqrt{1-u^2}} = \arcsin u+c = \underline{\arcsin(\tan x+c)}$

(12) $\displaystyle\int\frac{2\cdot dx}{x\cdot\cos^2(\ln x)} = \left|\begin{matrix}u=\ln x\\ du=\frac{1}{x}\cdot dx\end{matrix}\right| = \int\frac{2\cdot du}{\cos^2u} = 2\int\sec^2u\,du = 2\cdot\tan u+c = \underline{2\tan(\ln x)+c}$

(13) $\int \frac{\cos x \,.\, dx}{\sqrt{1+\sin x}} = \left| \begin{matrix} u = \sin x + 1 \\ du = \cos x \,.\, dx \end{matrix} \right| = 2 . \int \frac{du}{2\sqrt{x}} = 2\sqrt{u} + c = \underline{2\sqrt{1+\sin x} + c}$

Ex: $\int \sin x . \cos x \, dx$

I. Way

$\sin x . \cos x = \frac{1}{2} . \sin 2x$

$\frac{1}{2} \int \sin(2x) dx =$

$= \frac{1}{2} \left(- \frac{\cos 2x}{2} \right) + c$

$= \frac{-\cos 2x}{4} + c$

II. Way

$u = \sin x$

$du = \cos x . dx$

$\int u . du = \frac{u^2}{2} + c$

$= \frac{\sin^2 x}{2} + c$

III. Way

$u = \cos x$

$du = -\sin x . dx$

$= - \int u . du$

$= \frac{-u^2}{2} + c = - \frac{\cos^2 x}{2} + c$

Ex: $\int x . (x-10)^{10} dx = \left| \begin{matrix} u = x - 10 \\ du = dx \\ u + 10 = x \end{matrix} \right| = \int (u+10)^{10} . du = \int (u^{11} + 10u^{10}) du = \frac{u^{12}}{12} + \frac{10u^{11}}{11} + c = \underline{\frac{(x-10)^{12}}{12} + \frac{10(x-10)^{11}}{11} + c}$

Ex: $\int x^3 . \sqrt{x^2+1} \, dx =$

I. Way

$\left| \begin{matrix} x^2 + 1 = u \Rightarrow x^2 = u - 1 \\ 2x . dx = du \\ x . dx = \frac{du}{2} \end{matrix} \right| = \int (u-1) . \sqrt{u} . \frac{du}{2} = \ldots$

II. Way

$\left. \begin{matrix} x^2 + 1 = t^2 \\ \cancel{2} x dx = \cancel{2} t dt \\ x . dx = t . dt \end{matrix} \right\} = \int \underbrace{x^2}_{t^2-1} . \underbrace{\sqrt{x^2+1}}_{t} . \underbrace{x dx}_{t . dt} = \int (t^2 - 1) . t . t \, dt = \int (t^4 - t^2) dt = \frac{t^5}{5} - \frac{t^3}{3} + c =$

$= \underline{\frac{\sqrt{(x^2+1)^5}}{5} - \frac{\sqrt{(x^2+1)^3}}{3} + c}$

Ex: $\int_{-1}^{1} \underbrace{3x^2}_{du} . \sqrt{\underbrace{x^3+1}_{u}} . dx = \int_0^2 \sqrt{u} . du = \int u^{1/2} . du = \left. \frac{u^{3/2}}{\frac{3}{2}} \right|_0^2 = \left. \frac{2}{3} . \sqrt{u^3} \right|_0^2 = \frac{2\sqrt{8}}{3} - 0 = \underline{\frac{4\sqrt{2}}{3}}$

$u = x^3 + 1$

$du = 3x^2 . dx$

$x = -1 \Rightarrow u = (-1)^3 + 1 = 0$

$x = 1 \Rightarrow 2 = u$

09.03.2015
Monday

Ex: $\int_{\pi/4}^{\pi/2} \underbrace{\cot x}_{u} \cdot \underbrace{\csc^2 x \cdot dx}_{-du} = -\int_1^0 u\,du = \frac{u^2}{2}\Big|_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$

$u = \cot x$

$x = \frac{\pi}{4} \Rightarrow u = 1$

$x = \frac{\pi}{2} \Rightarrow u = 0$

ODD FUNCTION $= f(-x) = -f(x)$

EVEN FUNCTION $= f(-x) = f(x)$

If $f(x)$ is even function $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

If $f(x)$ is odd function $\int_{-a}^{a} f(x)dx = 0$

Ex: $\int_{-1}^{1} x^3 \cdot \sqrt{x^2+1}\, dx = 0$ (ODD)

Ex: $\int_{-\pi/2}^{\pi/2} \sin x\, dx = 0$ (ODD)

Ex: $\int_{-\pi/4}^{\pi/4} \tan x\, dx = 0$ (ODD)

Ex: $\int_{-\pi/4}^{\pi/4} \cos x \cdot dx = 2 \cdot \int_{-\pi/4}^{\pi/4} \cos x \cdot dx = 2 \cdot \sin x \Big|_0^{\pi/4} = \frac{2\cdot\sqrt{3}}{2} - 0 = \sqrt{2}$ (EVEN)

Ex: $I = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\, dx \Rightarrow$

I. Way

$I = 0$, $f(x) = \frac{4x}{\sqrt{x^2+1}}$, $f(-x) = \frac{-4x}{\sqrt{x^2+1}} \Rightarrow f(-x) = -f(x)$ (ODD)

II. Way

$x^2+1 = u$

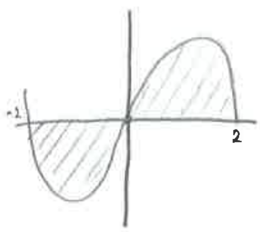
$2x \cdot dx = du$

$x = -\sqrt{3} \Rightarrow u = 4$

$x = \sqrt{3} \Rightarrow 4 = u$

$I = \int_4^4 \frac{2 \cdot du}{u} = 0$

Ex: $y = x \cdot \sqrt{4-x^2}$



$A = \int_{-2}^{2} |f(x)|\,dx = \int_{-2}^{0} -f(x)dx + \int_0^2 f(x)dx \Rightarrow \int_{-2}^{0} -x\cdot\sqrt{4-x^2}\,dx + \int_0^2 x\cdot\sqrt{4-x^2}\,dx$.

$4 - x^2 = u$

$-2x\, dx = du$

$x \cdot dx = -\frac{du}{2}$

$= \int_0^4 \sqrt{u}\cdot\frac{du}{2} - \int_4^0 \sqrt{u}\cdot\frac{du}{2} = \frac{1}{2}\cdot\int_0^4 \sqrt{u}\,du + \frac{1}{2}\cdot\int_0^4 \sqrt{u}\,du$

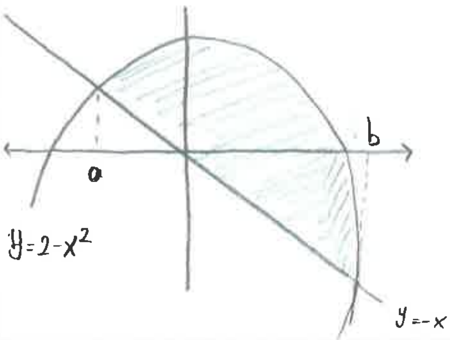
$= 2\cdot\frac{1}{2}\int_0^4 \sqrt{u}\,du = \frac{u^{3/2}}{\frac{3}{2}}\Big|_0^4 = \frac{2\cdot 8}{3} - 0 = \frac{16}{3}$

Ex: Find the area enclosed by the parabola $f(x) = 2 - x^2$ and the line $y = -x$

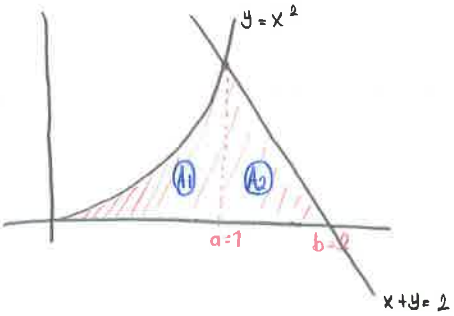
$A = \int_{a=-1}^{b=2} \left[(2-x^2) - (-x)\right] dx$

$2 - x^2 = -x$

$x^2 - x - 2 = 0$, $(x-2)(x-1) = 0$, $x = 2$, $x = -1$



Ex :

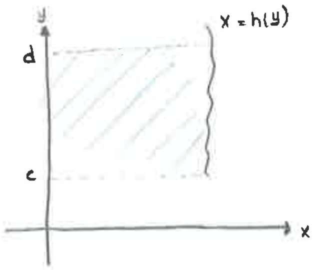


$$A_1+A_2=\int_0^{a=1} x^2\,dx+\int_{a=1}^{b=2}(2-x)\,dx$$

$x^2=2-x$
$x^2+x-2=0$
$(x+2)(x-1)=0$
$x=-2,\ x=1$

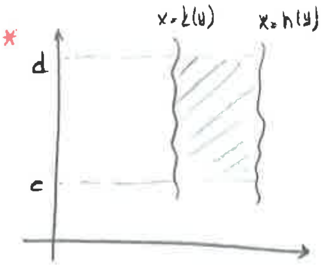
$= \dots$

*



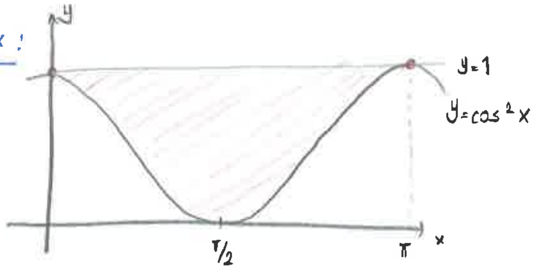
$$A=\int_c^d h(y)$$

*



$$A=\int_c^d \left[h(y)-l(y)\right]dy$$

Ex :

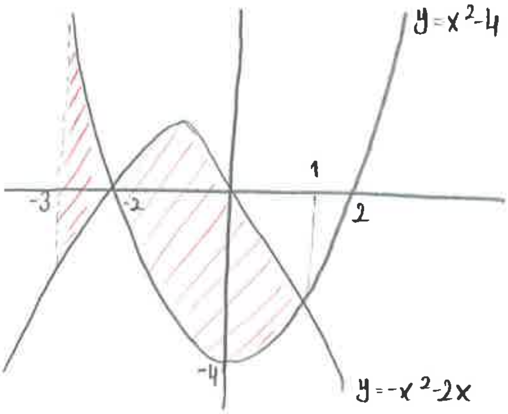


$$A=\int_0^{\pi}(1-\cos^2 x)\,dx = \dots$$

Ex : $y=-x^2-2x=-x(x+2)$

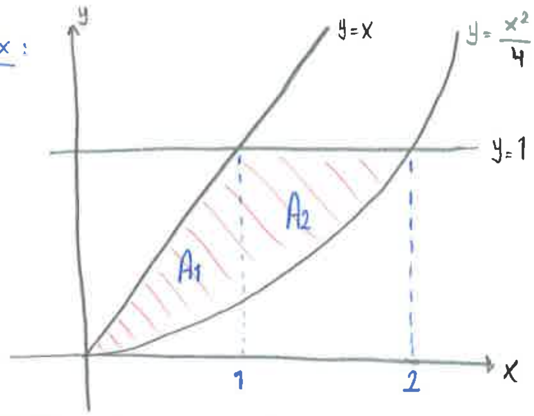
$y=x^2-4$

$-3\leqslant x\leqslant 1$



$$A=\int_{-3}^{-2}\left[(x^2-4)-(-x^2-2x)\right]dx+\int_{-2}^{1}\left[(-x^2-2x)-(x^2-4)\right]dx = \dots$$

Ex :



Short Way

$$\int_0^1 (2\sqrt{y}-y)\,dy =$$

$x^2=4y$
$x=2\sqrt{y}$
$x=y$

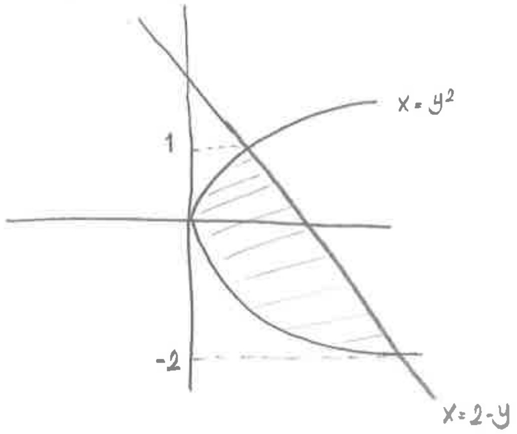
Long Way

$$A=\int_0^1\left(x-\frac{x^2}{4}\right)dx+\int_1^2\left(1-\frac{x^2}{4}\right)dx$$

$x = y^2 = 2-y$

$y^2+y-2=0 \Rightarrow y=-2\ ,\ y=1$

Ex: $x = y^2$ and $x+y=2$



$$A = \int_{-2}^{1}\left[(2-y)-y^2\right]dy = \ldots$$

Ex:

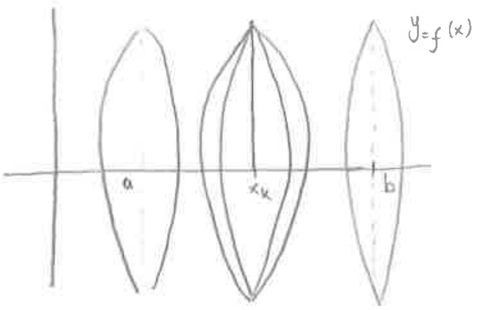
$$\int_{-\pi}^{\pi}\frac{\cos x}{\sqrt{4+3.\sin x}}\,.dx = \left|\begin{array}{l} 4+3\sin x = y \\ 3\cos x\,.dx = dy \\ \cos x\,dx = \dfrac{dy}{3}\end{array}\right| , \left|\begin{array}{l} x=-\pi \Rightarrow y=4 \\ x=\pi \Rightarrow y=4\end{array}\right| = \int_{4}^{4}\frac{\frac{dy}{3}}{\sqrt{y}} = \underline{\underline{0}}$$

Ex:

$$\int_{1}^{4}\frac{dy}{2\sqrt{y}\,(1+\sqrt{y})^2} = \left|\begin{array}{l} 1+\sqrt{y} = x \\ \dfrac{1}{2\sqrt{y}}\,.dy = dx\end{array}\right| = \int_{2}^{3}\frac{dx}{x^2} = \ldots$$

VOLUME

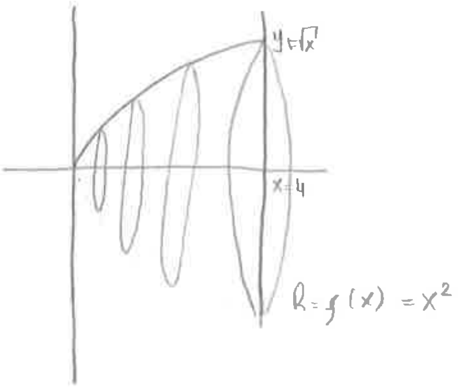
1-) Disc Method



$$V_k = \pi.R^2.\Delta x = \pi.f(x_k)^2.\Delta x$$

$$V = \sum_{k=1}^{n}\pi.f^2(x_k).\Delta x \Rightarrow V_{\lim_{n\to\infty}}\sum_{k=1}^{n}\pi.f^2.(x_k).\Delta x = \pi\int_{a}^{b}f^2(x).dx = V$$

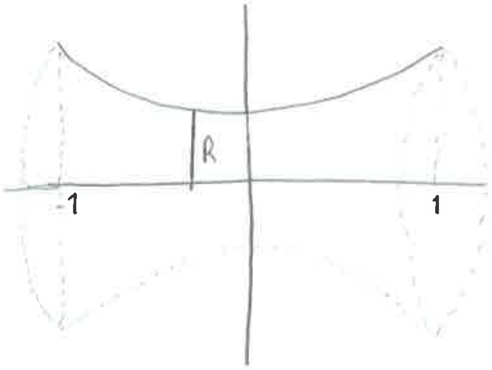
Ex: $f(x)=\sqrt{x}$, $x=4$ and x-axis rotated the x-axis



$$V = \pi\int_{a}^{b}A(x).dx = \pi\int_{0}^{4}(\sqrt{x})^2.dx$$

Ex: $y=x^2+1$, $-1\leq x\leq 1$

That graph rotated around the x-axis. Find the volume of the solid region.

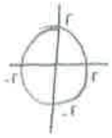


$R=f(x)=x^2+1$

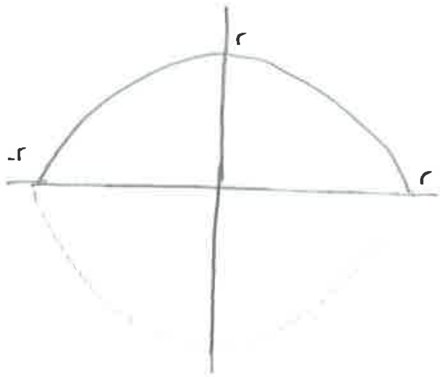
$$V=\pi\int_{-1}^{1}(x^2+1)^2\,dx$$

Ex: Show that the volume of sphere is $\frac{4.\pi.r^3}{3}$

$x^2+y^2=r^2$



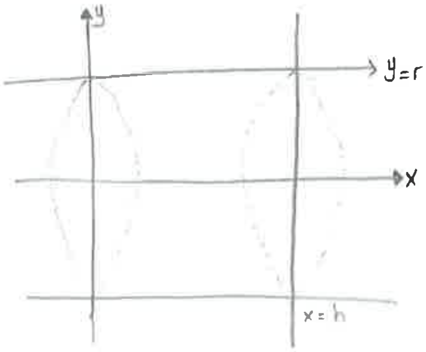
$y=\sqrt{r^2-x^2}$



$$V=\pi.\int_{-r}^{r}\left[\sqrt{r^2-x^2}\right]^2 dx$$

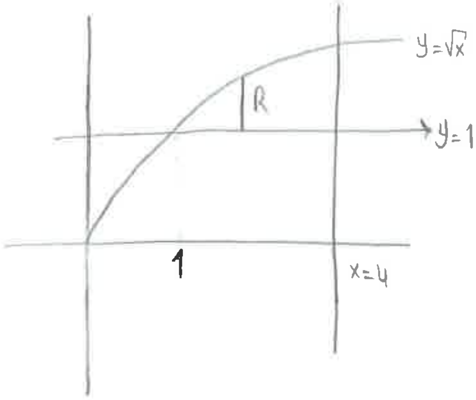
$$=\frac{4.\pi.r^3}{3}$$

Ex: Volume of S



$$V=\pi.\int_0^h (r)^2.dx=\pi.\int_0^h r^2.dx=\pi.r^2.h$$

Ex: $f(x)=\sqrt{x}$, $x=4$ and $y=1$ rotated around the line $y=1$



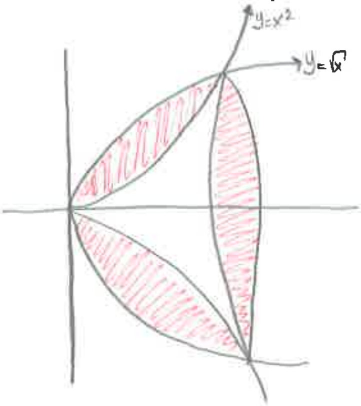
$R=\sqrt{x}-1$

$$V=\pi.\int R^2.dx$$

$$V=\pi.\int_1^4(\sqrt{x}-1)^2.dx$$

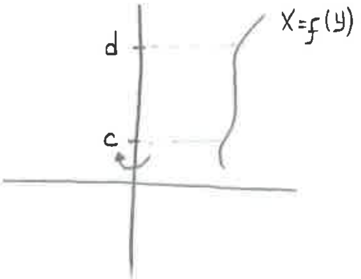
Ex: The region bounded by $y=\sqrt{x}$ and $y=x^2$

Find the volume of solid region rotated around x-axis.



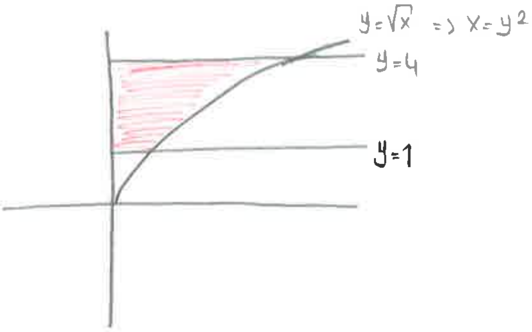
$$V=\pi.\int_0^1 (\sqrt{x})^2.dx-\pi.\int_0^1 (x^2)^2.dx$$

$$=\pi.\int_0^1 \left[(\sqrt{x})^2-(x^2)^2\right].dx$$



$$\boxed{V=\pi.\int_c^d f^2(y).dy}$$

Ex: $f(x)=\sqrt{x}$, $f(x)=1$, $f(x)=4$ and y-axis rotated around y-axis.

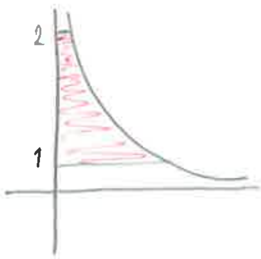


$$V=\int_1^4 (y^2)^2.dy$$

Ex: $x.y=1$, $y=1$, $y=2$ and y-axis rotated around y-axis.

$x=1/y$

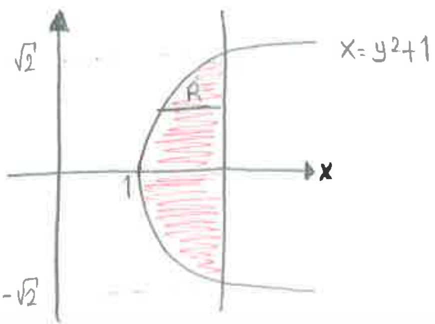
$x=f(y)=\frac{1}{y}$



$$V=\pi.\int_1^2 \left(\frac{1}{y}\right)^2.dy$$

Ex: Bdd by $x=y^2+1$ and $x=3$ and revolved around $x=3$

$x=y^2+1$



$y^2+1=3$

$y^2=2$

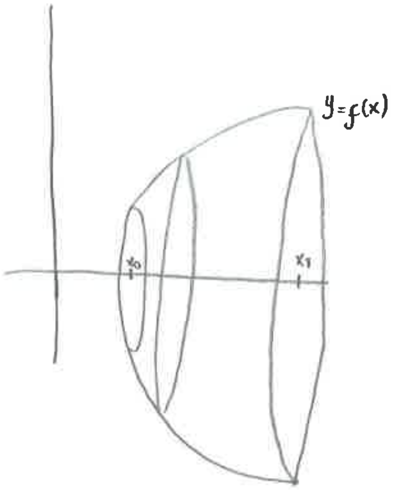
$y=\mp\sqrt{2}$

$R=3-(y^2+1)$

$V=\pi.\int (R)^2.dy$

$$V=\pi.\int_{-\sqrt{2}}^{\sqrt{2}} (R)^2.dy=\pi.\int_{-\sqrt{2}}^{\sqrt{2}} (2-y^2)^2.dy$$

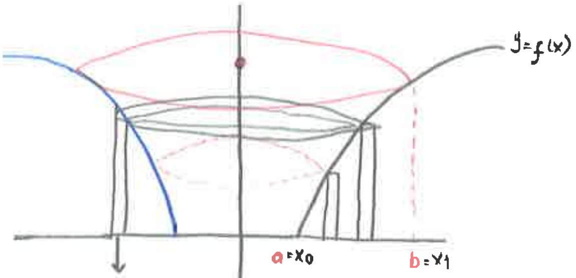
Disc



$$\lim_{n\to\infty}\sum_{k=0}^{n}\pi\cdot f(x)^2\cdot\Delta x_i = \pi\cdot\int_{x_0}^{x_1} f^2(x)\cdot dx = V$$

$$n\to\infty = \Delta x\to 0$$

SHELL METHOD



$\pi\cdot r^2 h$

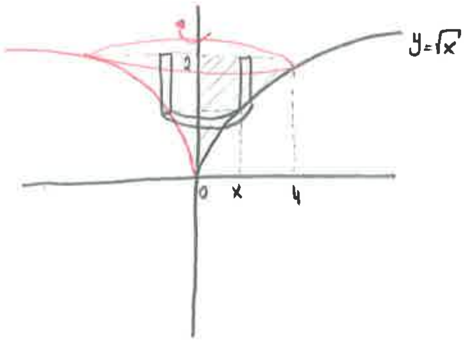
$V=a.b.c$
$b=\Delta x_i$
$a=f(x_i)$
$c=2.\pi.x$

Volume $=V_i = 2.\pi.x_i.f(x_i).\Delta x_i$

$$\sum V_i = \sum_{i=0}^{n} 2.\pi.x_i.f(x_i).\Delta x_i \xrightarrow{n\to\infty} \int_a^b 2.\pi.x.f(x).dx = V$$

Ex: $f(x)=\sqrt{x}$, $0\leq x\leq 4$ Find the volume of solid region rotated around the y-axis.

The region bold by



I. Way (Shell): $h=2-\sqrt{x}$

$l=2.\pi.x$

Δx

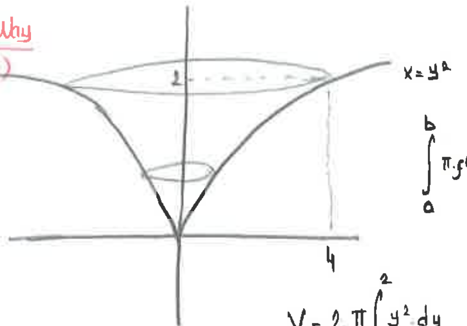
$$V=\int_0^4 2.\pi.x.(2-\sqrt{x}).dx$$

$$=2.\pi.\int_0^4\left(2x-x^{3/2}\right)dx$$

$$=2.\pi.\left(x^2-\frac{x^{5/2}}{5/2}\right)\Bigg|_0^4$$

$$=2.\pi.\left[16-\frac{2}{5}.32\right]=\frac{32\pi}{5}$$

II. Way (Disc)

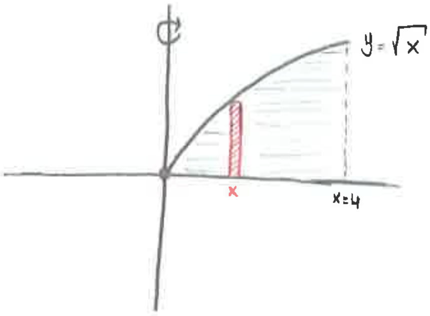


$$\int_a^b \pi.f(y)^2\,dy$$

$$V=2.\pi\int_0^2 y^2\,dy$$

$$V=\pi.\int_0^2 (y^2)^2.dy=\frac{\pi.y^5}{5}\Bigg|_0^2=\frac{32\pi}{5}$$

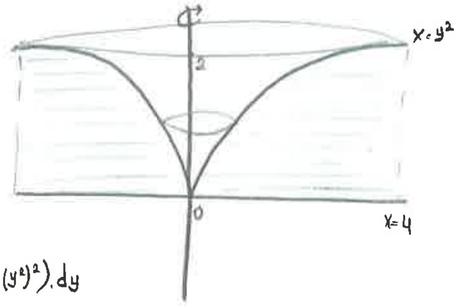
Ex :



I. Way (Shell)

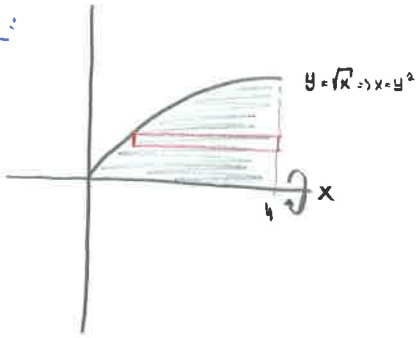
$$V=\int_0^4 2.\pi.x.\sqrt{x}\,.dx$$

II. Way (Disc)



$$V=\pi.\int_0^2 (4^2-(y^2)^2).dy$$

Ex :



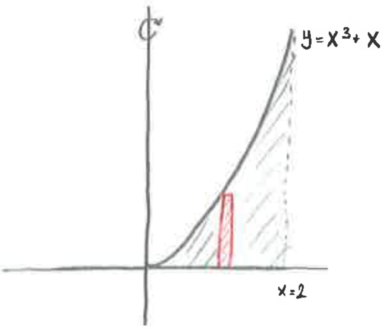
Shell

$$V=\int_0^2 2.\pi.y.y^2.dy$$

Disc

$$V=\pi.\int_0^4 (\sqrt{x})^2.dx$$

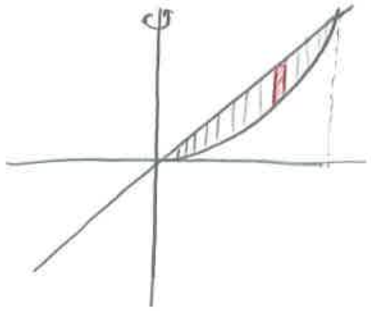
Ex : $f(x)=x^3+x$, $x=2$ and x-axis rotate the region about y-axis.



Shell

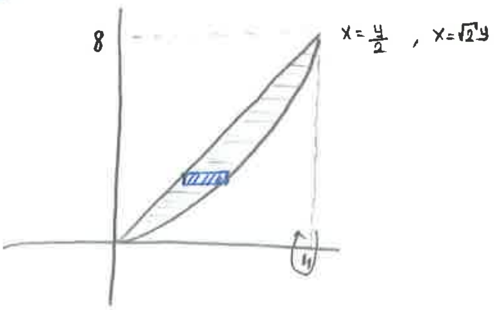
$$V=\int_0^2 2.\pi.x.(x^3+x).dx$$

Ex : Bold by $y=2x$, $y=\frac{x^2}{2}$ Rotated around y-axis. $\left\{ 2x=\frac{x^2}{2} \Rightarrow 4x=x^2 \Rightarrow x=0, x=4 \right\}$



$$V=\int_0^4 2.\pi\,x\left(2x-\frac{x^2}{2}\right)dx$$

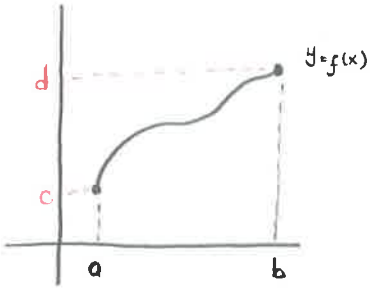
Ex : Previous



$$V=\int_0^8 2.\pi.y\left(\sqrt{2y}-\frac{y}{2}\right).dy$$

$f(y) = x \Leftrightarrow y = f^{-1}(x)$

ARC LENGTH



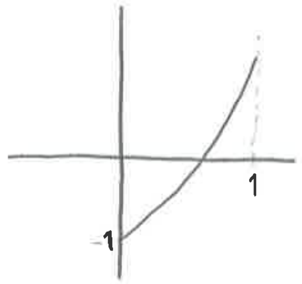
arc length $= \boxed{\int_a^b \sqrt{1+(f'(x))^2}\, dx}$

If $y=f(x)$ fails to derivative then $x = f^{-1}(y) = g(y)$

$x = g(y)$

$$\boxed{\int_c^d \sqrt{1+g'(y)^2}\, dy}$$

Ex: $f(x) = \frac{4\sqrt{2}}{3} \cdot x^{3/2} - 1$ $\quad 0 \leqslant x \leqslant 1$ $\quad$ Find the length of curve.



$$l = \int_0^1 \sqrt{1+(2\sqrt{2}\cdot\sqrt{x})^2}\, dx$$

Ex: Find the length of the curve $y = \left(\frac{x}{2}\right)^{\frac{2}{3}}$ from $x=0$ to $x=2$

$y' = \frac{2}{3} \cdot \left(\frac{x}{2}\right)^{-1/3}$, at $x=0$ not cont.

Let

$\frac{x}{2} = y^{3/2} \Rightarrow x = 2 . y^{3/2}$ $\qquad g(y) = 2 . y^{3/2}$

$$l = \int_0^1 \sqrt{1+(3\sqrt{y})^2}\, dy$$

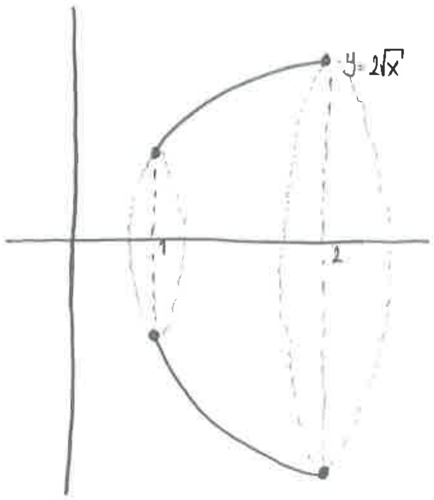
SURFACE AREA

Defn. = If $f(x) \geqslant 0$ ($g(y) \geqslant 0$) is continuously differentiable on $[a,b]$ ($[c,d]$), the area of the surface generated by the graph of $y = f(x)$ ($x = g(y)$) about the

x-axis is $$S = \int_a^b 2 \cdot \pi \cdot f(x) \cdot \sqrt{1 + (f'(x))^2} \cdot dx$$

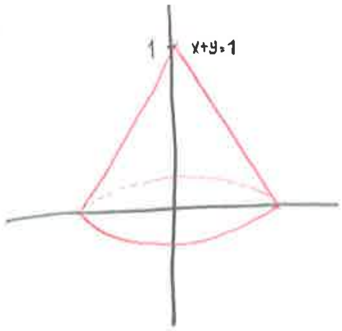
y-axis $$S = \int_c^d 2\pi \cdot g(y) \cdot \sqrt{1 + g'(y)^2} \cdot dy$$

Ex: $y = 2\sqrt{x}$, $1 \leqslant x \leqslant 2$ about x-axis



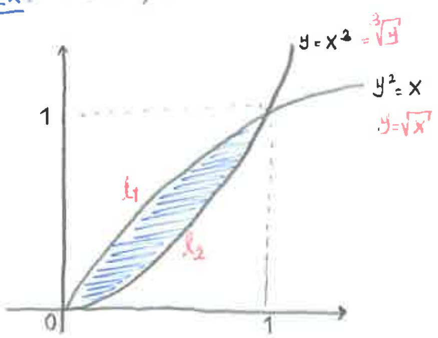
$$S = \int_1^2 2 \cdot \pi \cdot 2\sqrt{x} \cdot \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2} \cdot dx$$

Ex: $x + y = 1$, $0 \leqslant y \leqslant 1$ is revolved about y-axis to generate the cone. Find the surface area.



$$S = \int_0^1 2 \cdot \pi \cdot (1-y) \sqrt{1 + (-1)^2} \cdot dy$$

Ex: $y^2 = x$, $y = x^3$



$$\left.\begin{matrix} y = x^3 \\ y^2 = x \end{matrix}\right\} \Rightarrow \begin{matrix} x = 0 \\ x = 1 \end{matrix}$$

a.) Find the area of shaded region

$$A = \int_0^1 (\sqrt{x} - x^3)\, dx$$

$$A = \int_0^1 (\sqrt[3]{y} - y^2)\, dy$$

b.) Find the perimeter of the shaded region (Arc length)

$l_1 + l_2$ = perimeter

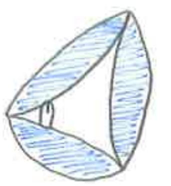
$$l_1 = \int_0^1 \sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2} \cdot dx = \int_0^1 \sqrt{1 + (2y)^2} \cdot dy$$

$$l_2 = \int_0^1 \sqrt{1 + (3x^2)^2} \cdot dx = \int_0^1 \sqrt{1 + \left(\frac{1}{3} y^{-2/3}\right)^2}\, dy$$

18.03.2015
Wednesday

c-) Rotate the shaded region around x-axis.

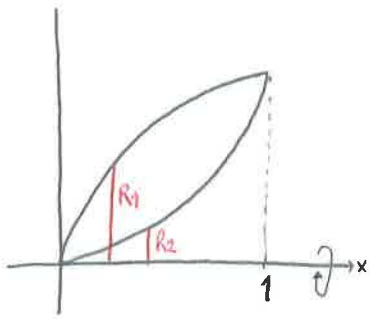
i-) Find surface area $(S_{out} + S_{in})$



$$S_{out} = \int_0^1 2.\pi.\sqrt{x}.\sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2}\,.dx$$

$$S_{in} = \int_0^1 2.\pi.x^3\sqrt{1+(3x^2)^2}\,.dx$$

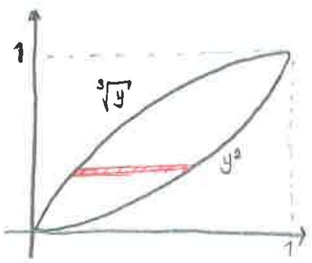
ii-) Volume



Disc Method

$$V = \pi.\int_0^1 \Big[\underbrace{(\sqrt{x})^2}_{\substack{Out\\R_1}} - \underbrace{(x^3)^2}_{\substack{In\\R_2}}\Big]\,.dx$$

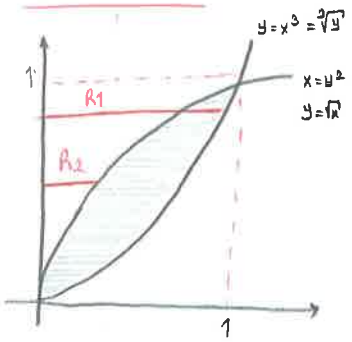
Shell Method



$$V = \int_0^1 2.\pi.y.\left(y^2 - \sqrt[3]{y}\right).dy$$

d-) Rotate around y-axis

Disc Method

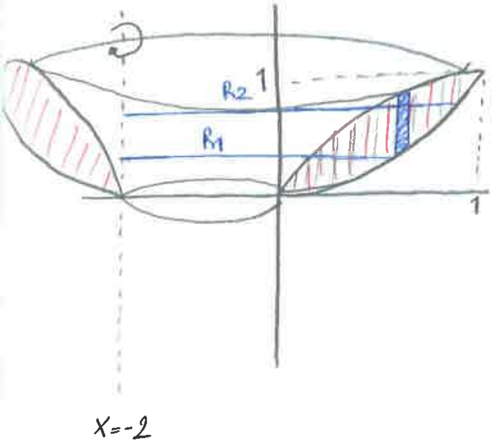


$$V = \pi.\int\left[(\sqrt[3]{y})^2 - (y^2)^2\right]dy$$

Shell Method

$$V = \int_0^1 2.\pi.x.(\sqrt{x} - x^3).dx$$

e-) Rotate around $x=-2$



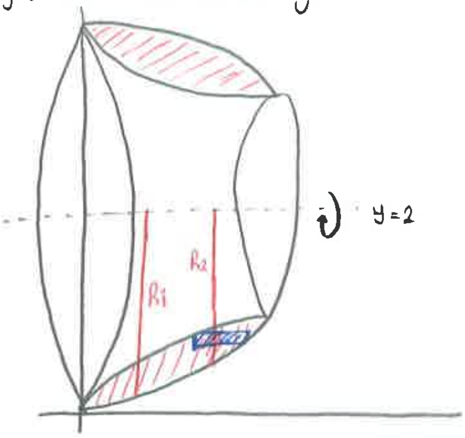
Disc Method

$$V = \pi\int\Big[\underbrace{\left(\sqrt[3]{y}-(-2)\right)^2}_{R_1} - \underbrace{\left(y^2-(-2)\right)^2}_{R_2}\Big]dy$$

Shell Method

$$V = \int_0^1 2\pi(x-(-2))(\sqrt{x} - x^3).dx$$

f·) Rotate the shaded region around $y=2$



$R_1 = 2 - x^3$

$R_2 = 2 - \sqrt{x}$

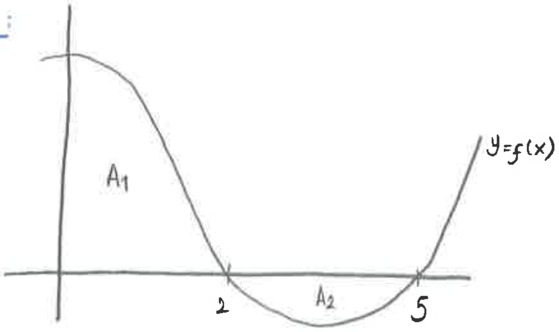
Disc

$$V = \pi \int_0^1 \left[(2-x^3)^2 - (2-\sqrt{x})^2\right] dx$$

Shell

$$V = \int_0^1 2\pi \underbrace{(2-y)}_{\text{Radius}} \cdot \underbrace{(\sqrt[3]{y} - y^2)}_{\text{Length of shell}} . dy$$

Ex :



$A_1 = 12$, $A_2 = 10$

$$\int_0^2 f(x).dx = A_1 = 12$$

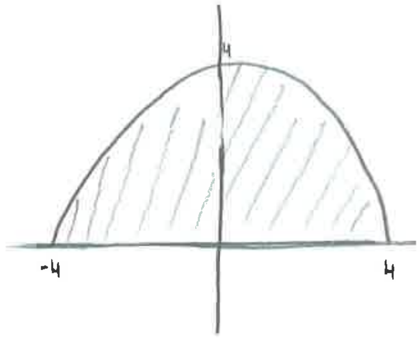
$$\int_2^5 f(x).dx = -A_2 = -10$$

$$\int_5^2 f(x).dx = -\int_2^5 f(x).dx = 10$$

$$\int_0^5 f(x)\, dx = \int_0^2 f(x).dx + \int_2^5 f(x).dx = 12 - 10 = 2$$

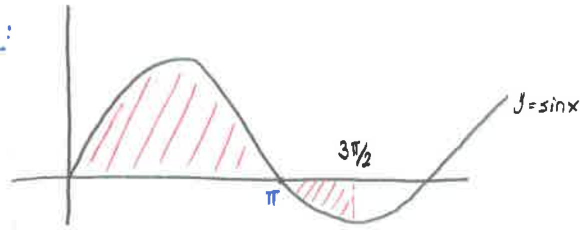
$$\int_0^5 |f(x)| . dx = A_1 + A_2 = 12 + 10 = 22$$

Ex: $y=\sqrt{16-x^2}$ Find the btwn the curve and x-axis



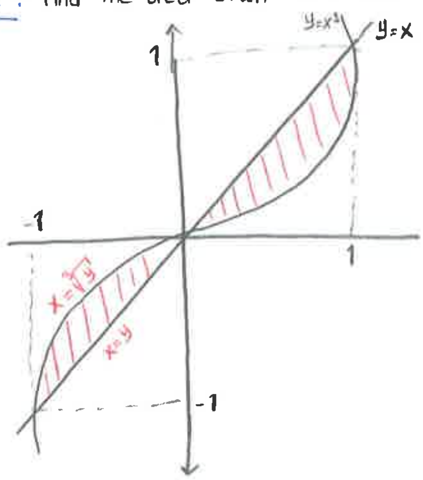
$$A=\int_{-4}^{4}\sqrt{16-x^2}\,dx = \frac{\pi \cdot 4^2}{2}$$

Ex:



$$\int_0^{3\pi/2}|\sin x|\,dx = \int_0^{\pi}\sin x\,dx + \int_{\pi}^{3\pi/2}-\sin x\,dx = -\cos x\Big|_0^{\pi} + \cos x\Big|_{\pi}^{3\pi/2} = 3$$

Ex: Find the area btwn $y=x$ and $y=x^3$



$x^3=x$

$x^3-x=0$

$x(x^2-1)=0$

$x=0$

$x=\mp 1$

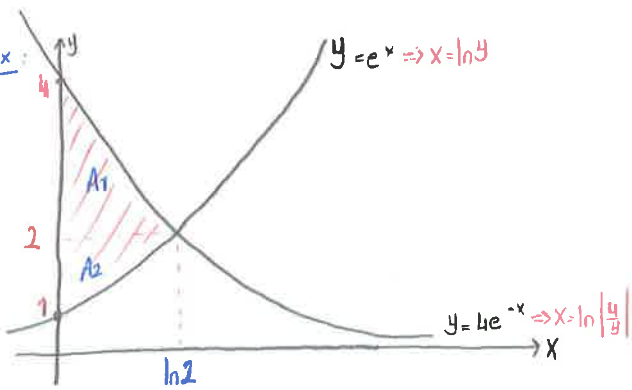
I. Way

$$\int_{-1}^{0}(x^3-x)dx + \int_0^1 (x-x^3)\,dx =$$

II. Way

$$\int_{-1}^{0}(y-\sqrt[3]{y})\,dy + \int_0^1(\sqrt[3]{y}-y)\,dy =$$

Ex:



$$A=\int_0^{\ln 2}(4e^{-x}-e^{x})dx$$

$$A=\underbrace{\int_1^2 \ln y\,dy}_{A_2} + \underbrace{\int_2^4 \ln\left|\frac{4}{y}\right|\,dy}_{A_1}$$

$e^x=4e^{-x}$

$e^{2x}=4$

$e^x=2$

$x=\ln 2$

$y=e^x$

$x=\ln y$

$y=4.e^{-x}$

$e^x=\frac{y}{4}$

$e^x=\frac{4}{y}$

$x=\ln\left|\frac{4}{y}\right| = \ln 4-\ln|y|$

Ex: $\int e^{x^2+\ln x}\cdot dx = ?$

$$= \int e^{x^2}\cdot e^{\ln x}\cdot dx = \int \underbrace{x\cdot e^{x^2}\cdot dx}_{\frac{du}{2}} = \frac{1}{2}\cdot\int e^u\cdot du = \frac{1}{2}\cdot e^u + c = \frac{1}{2}\cdot e^{x^2} + c$$

Ex: $\int 2^{3x}\cdot dx = \int 8^x\cdot dx = \frac{8^x}{\ln 8} + c$

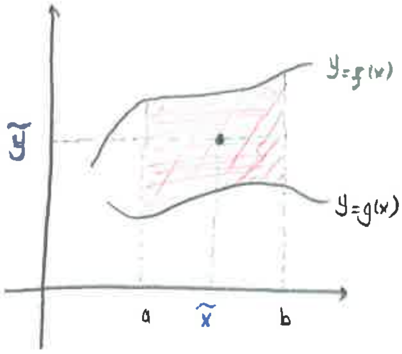
CENTER OF MASS

$$\tilde{x} = \frac{1}{M}\cdot\int_a^b \ell\cdot x\,[f(x)-g(x)]\cdot dx$$

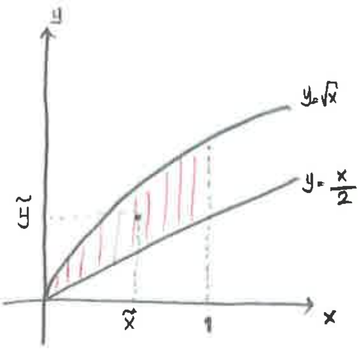
$$\tilde{y} = \frac{1}{M}\cdot\int_a^b \frac{\ell}{2}\cdot[f^2(x)-g^2(x)]\cdot dx$$

ℓ: density

$$M = \int_a^b dm = \int_a^b \ell\cdot f(x)\cdot dx =$$



Ex: Find the center of mass for the thin plate bold by $g(x)=\frac{x}{2}$, $f(x)=\sqrt{x}$ and $0 \leq x \leq 1$ with the density function $\ell(x)=x^2$



$$m = \int_0^1 x^2\cdot(\sqrt{x}-\frac{x}{2})\cdot dx = \frac{9}{56}$$

$$\tilde{x} = \frac{56}{9}\int_0^1 x^2\cdot x\,(\sqrt{x}-\frac{x}{2})\cdot dx = \frac{308}{405}$$

$$\tilde{y} = \frac{56}{9}\int_0^1 \frac{x^2}{2}\left[(\sqrt{x})^2-\left(\frac{x}{2}\right)^2\right]dx = \frac{252}{405}$$

Mid-Term Questions

23.03.2015
Monday

① $\int \tan^3 x \cdot dx = \int (\tan^3 x + \tan x - \tan x) \cdot dx = \int \underbrace{\tan x}_{u} (\underbrace{\tan^2 x + 1}_{du}) dx + \int \frac{\overbrace{-\sin x}^{dt}}{\underbrace{\cos x}_{t}} \cdot dx = \int u \cdot du + \int \frac{dt}{t} = \frac{u^2}{u} + \ln t + c$

② (Half Angle) $\int \left[\cos^4\left(\frac{x}{2}\right) - \sin^4\left(\frac{x}{2}\right)\right] \cdot dx = \int \overbrace{\left[\cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)\right]}^{\cos x} \cdot \overbrace{\left[\cos^2\left(\frac{x}{2}\right) + \sin^2\left(\frac{x}{2}\right)\right]}^{=1} \cdot dx = \int \cos x \cdot dx = \sin x + c$

③ $\int (1+x^2) \cdot d(\arctan x) = \int \cancel{(1+x^2)} \cdot \frac{1}{\cancel{1+x^2}} \cdot dx = \int dx = x + c$

④ $f'(x) = x+1$, $f(2) = -1$ $\Rightarrow$ $f(x) = \int (x+1) \cdot dx = \frac{x^2}{2} + x + c$, $f(2) = -1 = \frac{2^2}{2} + 2 + c \Rightarrow c = -5$ $\Big\}$ $f(0) = -5$

$f(0) = ?$

⑤ $\int \frac{x^2+2}{x^2+1} \cdot dx = \int \left(\frac{x^2+1}{x^2+1} + \frac{1}{x^2+1}\right) \cdot dx = \int \left(1 + \frac{1}{x^2+1}\right) \cdot dx = x + \arctan x + c$

⑥ $\int \frac{\cos^2 x}{1+\sin x} \cdot dx = \int \frac{1-\sin^2 x}{1+\sin x} \cdot dx = \int (1-\sin x) \cdot dx = x + \cos x + c$

⑦ $\int \frac{1}{\cos x} \cdot dx = \int \frac{\cos^2 x + \sin^2 x}{\cos x} \cdot dx = \int \left(\cos x + \frac{\sin^2 x}{\cos x}\right) \cdot dx = \ldots$ $\Big\|$ $\int \sec x \cdot \frac{(\sec x + \tan x)}{\sec x + \tan x} \cdot dx$

⑧ $\int (x^2+1)^3 \cdot x \cdot dx = \left| \begin{array}{l} x^2+1 = u \\ 2x = du \\ x = du/2 \end{array} \right. = \int \frac{u^3 \cdot du}{2} =$

⑨ $\int e^{\overbrace{\sin x}^{u}} \cdot \underbrace{\cos x}_{du} \, dx =$

⑩ $\int \frac{\overbrace{dx}^{}}{\underbrace{x}_{du} \underbrace{\ln x}_{u}} = \int \frac{du}{u} = \ln|u| + c = \ln|\ln x| + c$

⑪ $\int \frac{\overbrace{\arctan x}^{u}}{\underbrace{1+x^2}_{du}} \, dx = \int u \cdot du$

⑫ $\int \frac{\overbrace{\arcsin x}^{u}}{\underbrace{\sqrt{1-x^2}}_{du}} \, dx = \int u \cdot du$

⑬ $\int \frac{\overbrace{2x-1}^{du}}{\underbrace{x^2-x-2}_{u}} \cdot dx = \int \frac{du}{u} = \ln|u| + c$

⑭ $\int \frac{\overbrace{e^x - e^{-x}}^{du}}{\underbrace{e^x + e^{-x}}_{u}} \cdot dx =$

⑮ $\int \frac{\overbrace{\sin 2x}^{du}}{\underbrace{1+\sin^2 x}_{u}} \cdot dx$

⑯ $\int \frac{\overbrace{e^x}^{u}}{\sqrt{1-e^{2x}}} \cdot dx = \int \frac{du}{\sqrt{1-u^2}} = \arcsin u + c$

⑰ $\int \frac{\overbrace{dx}^{du}}{x \cdot \cos^2(\ln x)} = \Big| \ln x = u = \int \frac{du}{\cos^2 u} = \tan u + c = \tan(\ln x) + c$

⑱ $\int \cos(\underbrace{2x+1}_{u}) \cdot dx =$

18) $\int \cos(\underbrace{2x+1}_{u})\,dx$

19) $\int \cos(\underbrace{\cos^2 x}_{u}) . \underbrace{\sin 2x}_{du}\,dx$

20) $\int \sin^2 x . \underbrace{\cos x . dx}_{du}$ $\left| \sin x = u \right.$

21) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} . dx = \left| \begin{array}{l} \sqrt{x} = u \\ \frac{1}{2\sqrt{x}} . dx = du \\ \frac{dx}{\sqrt{x}} = 2 . du \end{array} \right. = 2 . \int e^u . du$

22) $\int \frac{\cot x}{\ln(\sin x)} . dx = \int \frac{du}{u} \Rightarrow \left| \begin{array}{l} \ln(\sin x) = u \\ \frac{\cos x}{\sin x} = du \end{array} \right|$

23) $\int \frac{\tan x}{\cos^2 x} . dx = \left| \tan x = u \right|$

24) $\int \frac{d.(x^2+1)}{x^2} = \int \frac{2x}{x^2} . dx = \ln|x^2| + c$

25) $\int_4^9 \frac{1+\sqrt{x}}{1-\sqrt{x}} . dx$ apply $u = \sqrt{x}$ $\Rightarrow \left| \begin{array}{ll} u = \sqrt{x} & x = 4 \Rightarrow u = 2 \\ du = \frac{1}{2\sqrt{x}} . dx & x = 9 \Rightarrow u = 3 \\ 2u . du = dx & \end{array} \right| \Rightarrow I = \int_2^3 \frac{1+u}{1-u} . 2u . du$

** 26) $\int_1^2 \frac{e^{2x} - e^x}{e^x + 1}$, $x = \ln t$, $\left| \begin{array}{ll} x = \ln t \Rightarrow t = e^x & x = 1 \quad t = e \\ dt = e^x . dx & x = 2 \quad t = e^2 \\ dt / t = dx & \end{array} \right| \Rightarrow I = \int_e^{e^2} \frac{t^2 - t}{t+1} . \frac{dt}{t}$

27) $\int \frac{x^2}{1+x^6} . dx = \left| \begin{array}{l} x^3 = u \\ 3x^2 . dx = du \\ x^2 . dx = du/3 \end{array} \right| = \frac{1}{3} . \int \frac{du}{1+u^2} = \frac{1}{3} . \arctan u + c = \frac{1}{3} . \arctan(x^3) + c$

28) $\int x . (x+1)^5 . dx = \left| \begin{array}{l} x+1 = u \Rightarrow x = u-1 \\ dx = du \end{array} \right| = \int (u-1) u^5 . du = \ldots$

29) $\int \frac{\sin^3 x}{\cos^5 x} . dx = \int \tan^3 x . \underbrace{\sec^2 x . dx}_{du} = \left| \tan x = u \right| = \int u^3 . du =$

30) $\int (2 - 4 . \sin^2 x) . dx = 2 . \int (1 - 2\sin^2 x) . dx = 2 . \int \cos 2x . dx = \sin 2x + c$

31) $\int (\tan^5 x + \tan^3 x) . dx = \int \tan^3 x \underbrace{(\tan^2 x + 1) . dx}_{du}$
$\tan x = u$

32) $\int \frac{1}{\cos^4 x} . dx = \int \frac{\sin^2 x + \cos^2 x}{\cos^4 x} . dx = \int \left(\underbrace{\frac{\sin^2 x}{\cos^4 x}}_{\tan^2 x . \sec^2 x} + \frac{\cos^2 x}{\cos^4 x} \right) = \int (\tan^2 x . \sec^2 x + \sec^2 x) . dx = \int (\tan^2 x + 1) . \underbrace{\sec^2 x . dx}_{du}$ $\tan x = u$

33) $\int \frac{dx}{e^x + e^{-x}} =$

34) $\int e^{(e^x + x)} dx$

35) $\int_0^{\pi/3} \sqrt{1 - \cos^2 x} = \sqrt{2} . \int_0^{\pi/3} |\sin x| \, dx = \sqrt{2} . \int_0^{\pi/3} \sin x \, dx$

$\sqrt[2n]{x^{2n}} = |x| \quad n \in \mathbb{Z}^+$

*

$$\sqrt{1-\cos 2x} = \sqrt{\cancel{1}-(\cancel{1}-2.\sin^2 x)} = \sqrt{2.\sin^2 x} = \sqrt{2}|\sin x|$$

$$\sqrt{1+\cos 2x} = \sqrt{1+2\cos^2 x-1} = \sqrt{2}.|\cos x|$$

$$\sqrt{1-\sin 2x} = \sqrt{\sin^2 x+\cos^2 x-2.\sin x.\cos x} = \sqrt{(\sin x-\cos x)^2} = |\sin x-\cos x|$$

$$\sqrt{1+\sin 2x} = \quad '' \quad = \sqrt{(\sin x+\cos x)^2} = |\sin x+\cos x|$$

25.03.2015
Wednesday

Midterm

1-) $\int_1^e \frac{\ln x}{x} \cdot dx = \left| \begin{matrix} \ln x = t \\ \frac{1}{x} \cdot dx = dt \end{matrix} \right| \Rightarrow I = \int_0^1 t \cdot dt = \frac{t^2}{2} \Big|_0^1 = \frac{1}{2}$

$x = 1 \Rightarrow t = \ln 1 = 0$
$x = e \Rightarrow t = \ln e = 1$

2-) $\int \frac{1-\cos 2x}{1+\cos 2x} \cdot dx \Rightarrow$

I. Way

$\cos 2x = 1 - 2\sin^2 x$
$= 2\cos^2 x - 1$

$\Rightarrow \int \frac{1-(1-2\sin^2 x)}{1+2\cos^2 x - 1} \cdot dx = \int \frac{\sin^2 x}{\cos^2 x} = \int \tan^2 x \cdot dx \overset{+1-1}{=} \tan x - x + c$

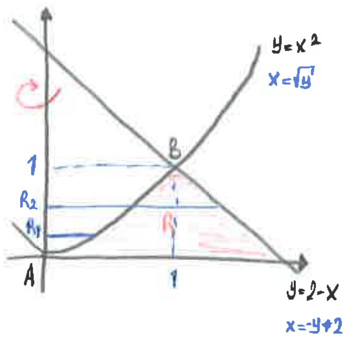
II. Way

$\int \frac{1-\cos 2x}{1+\cos 2x} \cdot \frac{(1-\cos 2x)}{(1-\cos 2x)} = \int \frac{(1-\cos^2 x)^2}{1-\cos^2(2x)} \cdot dx = \int \frac{1-2\cdot\cos(2x)+\cos^2(2x)}{\sin^2(2x)} \cdot dx = \int \underbrace{\csc^2(2x)\cdot dx}_{=} - 2\int \frac{\cos 2x}{\sin^2(2x)} \cdot dx + \int \cot^2(2x) \cdot dx \Rightarrow$

$= \frac{-\cot(2x)}{2} - \int \frac{du}{u^2} - \cot x - x + c$

$\to \frac{1}{\sin 2x}$

3-)



a-) $y = x^2$, $y = 2 - x$ x-axis Area?

b-) $y = x^2$, $y = 2 - x$ y-axis Area? Rotate it around y-axis.

c-) Rotate around x-axis

I. WAY

a-) $x^2 = 2 - x$
$x^2 + x - 2 = 0$
$(x+2)(x-1) = 0$

$A = \int_0^1 x^2 \cdot dx + \int_1^2 (2-x) \cdot dx$

II. WAY

a-) $A = \int_0^1 \left[(2-y) - \sqrt{y}\right] dy$

DISC METHOD

b-) $V = \pi \cdot \int_0^1 \left[(2-y)^2 - (\sqrt{y})^2\right] \cdot dy$

SHELL METHOD

b-) $V = \int_0^1 2\cdot\pi\cdot x \cdot x^2 \cdot dx + \int_1^2 2\cdot\pi\cdot x \cdot (2-x) \cdot dx$

SHELL METHOD

c-) $V = \int_0^1 2\cdot\pi\cdot y\,(2 - y - \sqrt{y})\,dy$

c-) DISC METHOD

$V = \pi \int_0^1 (x^2)^2 \cdot dx + \pi \cdot \int_1^2 (2-x)^2 \cdot dx$

ARC LENGTH

$\ell = \int_0^1 \sqrt{1+(2x)^2} \cdot dx + \int_1^2 \sqrt{1+(-1)^2} \cdot dx + 2$

OR

$\ell = \int_0^1 \sqrt{1+\left(\frac{1}{2\sqrt{y}}\right)^2} \cdot dy + \int_0^1 \sqrt{1+(-1)^2} \cdot dy + 2$

Ne olacak ki dine sonra kaybedip, pişman diye, [illegible]

4-) a) $\int \frac{x}{x+1}\cdot dx = \left| \begin{array}{l} x+1=u \\ x=u-1 \\ dx=du \end{array} \right| \Rightarrow I = \int \frac{u-1}{u}\cdot du = \int \left(1-\frac{1}{u}\right)\cdot du = u - \ln u + c \Rightarrow x+1-\ln(x+1)+c$

b-) $\int \frac{\ln^3 x}{3x}\cdot dx = \left| \begin{array}{l} u=\ln x \\ du=\frac{1}{x}\cdot dx \end{array} \right| \Rightarrow I = \int \frac{u^3}{u}\cdot du = \frac{u^4}{12}+c = \frac{\ln^4 x}{12}+c$

c-) $\int \frac{x\cdot e^{(x^2)}}{\sqrt{e^{(x^2)}+2}}\cdot dx = \left| \begin{array}{l} e^{(x^2)}+2=t^2 \\ 2x\cdot e^{(x^2)}dx = 2\cdot t\cdot dt \end{array} \right| \Rightarrow I = \int \frac{t\cdot dt}{t} = t+c = \sqrt{x^2+2}+c$

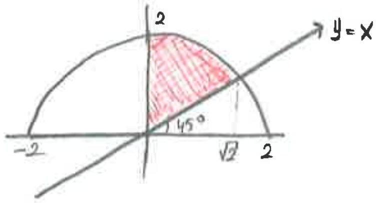
d-) $g(x) \int_1^{\tan x} \sqrt{1+t^3}\cdot dt \Rightarrow \frac{d}{dx}\int_{v(x)=1}^{u(x)=\tan x} f(t)\cdot dt = f(v)\cdot v' - f(u)\cdot u' \Rightarrow g(x) = \sqrt{1+\tan^3 x}\cdot \sec^2 x - \sqrt{1+1^3}\cdot 0$

5-) a) $1-x^2=t^2$
$-2x\cdot dx = 2t$

$\int_{-1}^{1} x\cdot\sqrt{1-x^2}\cdot dx \Rightarrow I = \int_0^0 \sqrt{t^2}\cdot t\cdot dt = 0$

güzel it

b-) $\int_0^{\sqrt{2}} \left(\sqrt{4-x^2}-x\right)\cdot dx = \left| \begin{array}{l} 4-x^2=y^2 \Rightarrow y=\sqrt{4-x^2} \\ x^2+y^2=4 \end{array} \right|$



$I = \frac{\pi\cdot 2^2}{8} = \frac{\pi}{2}$

6-) a-) $\int \frac{\sqrt[3]{x+2}+1}{\sqrt{x+2}-1}\cdot dx = \left| \begin{array}{l} x+2=t^6 \\ dx=6t^5\cdot dt \end{array} \right| = \int \frac{\sqrt[3]{t^6}+1}{\sqrt{t^6}-1} = \int \frac{t^2+1}{t^3}\cdot 6t^5\cdot dt = 6\cdot\int (t^4-t^2)\cdot dt =$

b-) $^* \int \sin^2 x\cdot dx = \int \frac{1-\cos 2x}{2}\cdot dx$

$\sin x = u$
(odd)

$\int \sin^3 x\cdot dx = \int \sin^2 x\cdot \sin x\cdot dx$

$\left| \begin{array}{l} \cos x = u \\ -\sin x\, dx = du \end{array} \right| = \int (1-\cos^2 x)\cdot \sin x\cdot dx = -\int (1-u^2)\cdot du$

Ex $\int \cos^5 x\cdot dx = \int (\cos^2 x)^2\cdot \cos x\cdot dx$ (even)

$= \int (1-\cos^2 x)\cdot \sin x\cdot dx = -\int (1-u^2)\cdot du$

c-) $\int x\sqrt{x+2}\cdot dx = \left| \begin{array}{l} x+2=t^2 \Rightarrow x=t^2-2 \\ dx = 2t\cdot dt \end{array} \right| = \int (t^2-2)\sqrt{t^2}\cdot 2t\cdot dt = \int (t^2-2)2t^2\cdot dt = \int (2t^4-4t^2)dt$

d-) $\int \frac{2\cdot dx}{x\cdot\cos^2(\ln x)} = \left| \begin{array}{l} \ln x = t \\ \frac{1}{x}\cdot dx = dt \end{array} \right| \Rightarrow I = \int \frac{2\cdot dt}{\cos^2 t} = 2\cdot\tan t + c = 2\cdot\tan(\ln x)+c$

7-) a) $\int_0^{\pi/8} \cos^2 x\cdot\sin^2 x\cdot dx = \int (\cos x\cdot\sin x)^2\cdot dx = \frac{1}{4}\cdot\int (2\cdot\cos x\cdot\sin x)^2\cdot dx = \frac{1}{4}\cdot\int \sin^2(2x)\cdot dx = \frac{1}{4}\int \frac{1-\cos(4x)}{2}\cdot dx = \frac{1}{8}\cdot\int (1-\cos(4x))dx \Rightarrow$

$= \frac{1}{8}\left[x - \frac{\sin 4x}{4}\Big|_0^{\pi/8}\right]$

$= \frac{1}{8}\cdot\left[\frac{\pi}{8}-\frac{1}{4}-0\right]$

b-) $\int \sin^2 x\cdot\sin x\cdot dx = \int (1-\cos^2 x)\cdot\sin x\cdot dx$

c-) $\int \sin^4 x\cdot\cos^3 x\cdot dx = \int \sin^4 x\cdot(1-\sin^2 x)\cdot\underbrace{\cos x\cdot dx}_{du}$ ($\sin x = u$) $= \int u^4\cdot(1-u^2)\cdot du$

Sequences

$\lim_{n\to\infty} \frac{\ln n}{n} = 0$, $\lim_{n\to\infty} x^{\frac{1}{n}} = 1$, $x>0$

$\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$, $\lim_{n\to\infty} \sqrt[n]{n} = 1$

$\lim_{n\to\infty} x^n = 0$, $\lim_{n\to\infty} \frac{x^n}{n!} = 0$

The Series

$a_1+a_2+\dots+a_k+\dots = \sum_{n=1}^{\infty} a_n$

Geometric Series

$S_n = a.\frac{1-r^n}{1-r} \Rightarrow \sum_{n=p}^{\infty} a.r^n = a.r^p.\frac{1}{1-r}$

Theorem : (n-term test) n terimin toplamın n'den büyük

(I) $\sum_{n=1}^{\infty} a_n$ is convergent $\Rightarrow \lim a_n = 0$ (0'a yaklaşıyorsa)

(II) $\lim a_n \neq 0$ or the limit does not exist $\Big\} \Rightarrow \sum a_n$ is divergent

Theorem : (Integral-Test)

$a_n = f(n)$ is Continious , Positive , Decreasing function

$\sum_{n=N}^{\infty} a_n$ and $\int_n^{\infty} f(x)dx$ Both Conv. or Div.

$\propto \arctan(\infty) = \frac{\pi}{2}$

Theorem : (Comparison Test)

$\sum a_n , \sum c_n , \sum d_n$ Nonnegative terms $a_n, b_n, c_n > 0$

if $\sum c_n$ div. then $\sum a_n$ also div.

if $\sum d_n$ conv. then $\sum a_n$ also conv.

Theorem : (Limit-Comparison Test)

$a_n > 0$, $b_n > 0$, $n \geq N$

i-) If $\lim \frac{a_n}{b_n} = c > 0$ then both $\sum a_n$ and $\sum b_n$ conv. or div

ii-) If $\lim \frac{a_n}{b_n} = 0$ and both are Conv.

iii-) If $\lim \frac{a_n}{b_n} = \infty$ and both are Div.

Root Test

$\sum a_n$ $n \geq N$ $a_n \geq 0$

$\lim_{n\to\infty} \sqrt[n]{a_n} = p$

$p<1$ Conv.
$p>1$ Div.
$p=1$ This test does not use

Ratio-Test

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = p$

$p<1$ Conv.
$p>1$ Div.
$p=1$ This test does not use.

Integration By Parts

$$\int u \, dv = u \cdot v - \int v \, du$$

(LAPTÜ)

Reduction Formula

$$\int \cos^n x = \frac{\cos^{n-1} x \cdot \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \cdot dx$$

$$\int \sin^m x \cdot \cos^n x \cdot dx \Rightarrow$$

(I) m is even, $u = \sin x$, $du = \cos x$ $\Big\}$ → Substitution

(II) n is even, $u = \cos x$

(III) If both are even, $\cos^2 x = \frac{1}{2} \cdot (\cos 2x + 1)$

$\sin^2 x = \frac{1}{2} \cdot (1 - \cos 2x)$

$\sqrt{1 \mp \sin 2x}\, dx = \sqrt{(\sin x - \cos x)^2} \cdot dx$	$\sec^2 x = \tan^2 x + 1$
$U = \sec x$ $dv = \sec^2 x \cdot dx$ $du = \sec x \cdot \tan x$ $v = \tan x$	$\cos 2x = \cos^2 x - \sin^2 x$ $= 1 - 2\sin^2 x$ $= 2\cos^2 x - 1$
$2 \cdot \sin a \cdot \cos b = \sin(a+b) + \sin(a-b)$ $2 \cdot \cos a \cdot \cos b = \cos(a+b) + \cos(a-b)$ $-2 \cdot \sin a \cdot \sin b = \cos(a+b) - \cos(a-b)$	$\sqrt{x^2+a^2} \Rightarrow x = a \cdot \tan\theta$ $\sqrt{a^2-x^2} \Rightarrow x = a \cdot \sin\theta$ $\sqrt{x^2-a^2} \Rightarrow x = a \sec\theta$

$$\int \frac{Q(x)}{P(x)} \cdot dx \Rightarrow \deg[Q(x)] < \deg[P(x)]$$

$$\frac{\quad}{(ax+b)^n} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} \cdots \frac{A_n}{(ax+b)^n}$$

$\cos 2x = \cos^2 x - \sin^2 x$ $= 1 - 2\sin^2 x$ $= 2\cos^2 x - 1$	If function is even; $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)\,dx$
$\int_0^1 \frac{1}{x^2} \cdot dx$ is div. $\int_1^\infty \frac{1}{x^2} \cdot dx$ is conv.	

IMPROPER INTEGRAL

TYPE-I

a-) $f(x)$ is cont. $[a, \infty)$ then, $\int_a^\infty f(x)dx = \lim_{R\to\infty} \int_a^R f(x) \cdot dx$

b-) $f(x)$ is cont. $(-\infty, a)$ then, $\int_{-\infty}^a f(x)dx = \lim_{R\to\infty} \int_R^a f(x)dx$

c-) $f(x)$ is cont. $(-\infty, \infty)$ then, $\int_{-\infty}^\infty f(x)dx = \lim_{R\to\infty} \int_R^c f(x)dx + \int_c^R f(x)\,dx$

→ Limit value is a real number = convergent, Otherwise = divergent.

TYPE-II

a-) $f(x)$ is cont. $[a, b)$ and disc. $x = b$ then,

$$\int_a^b f(x)\,dx = \lim_{R\to b^-} \int_a^R f(x)dx$$

b-) $f(x)$ is cont. $(a, b]$ and disc. $x = a$ then,

$$\int_a^b f(x)\,dx = \lim_{R\to a^+} \int_R^b f(x)dx$$

c-) $f(x)$ is cont. $[a, c) \cup (c, b]$ and disc. $x = c$ then,

$$\int_a^b f(x)dx = \lim_{R\to c^-} \int_a^R f(x)dx + \lim_{R\to c^+} \int_R^b f(x)dv$$

Theorem - Comparison Test

$f(x)$ and $g(x)$ cont. func. $[a, \infty)$ and $0 < f(x) < g(x)$, $\forall x \geq a$ then,

a-) If $\int_a^b g(x)dx$ is convergent then $\int_a^b f(x)\,dx$ is also conv. (Big)

b-) If $\int_a^b f(x)dx$ is divergent then $\int_a^b g(x)\,dx$ is also div. (Small)

$$* \int_1^\infty \frac{1}{x^P} \cdot dx = \begin{cases} P > 1, & \text{Conv.} \\ P < 1, & \text{Div} \end{cases}$$

Theorem - Limit Comparison Test

If the pos. func. $f(x)$ and $g(x)$ cont $[a, \infty)$, and if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$,

$0 < L < \infty$ then $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ both conv. or div.