Задачи к экзамену по ТЕОРИИ ПОЛЯ, ФАКИ

1. $\operatorname{rot}\left(\frac{[\vec{a},\vec{r}]}{r}\right)=\left(\operatorname{rot}\left(\frac{[\vec{a},\vec{r}]}{r}\right)\right)_\alpha=e_{\alpha\beta\gamma}\frac{\partial}{\partial x_\beta}\frac{e_{\gamma\mu\nu}a_\mu x_\nu}{r}=e_{\alpha\beta\gamma}e_{\gamma\mu\nu}a_\mu\left(\frac{\delta_{\beta\nu}}{r}-\frac{x_\nu x_\beta}{r^3}\right)=$
$=a_\mu(\delta_{\alpha\mu}\delta_{\beta\nu}-\delta_{\alpha\nu}\delta_{\beta\mu})\left(\frac{\delta_{\beta\nu}}{r}-\frac{x_\nu x_\beta}{r^3}\right)=a_\mu\left(\frac{3\delta_{\alpha\mu}}{r}-\frac{\delta_{\alpha\mu}}{r}-\frac{\delta_{\alpha\mu}x_\beta^2}{r^3}+\frac{x_\alpha x_\mu}{r^3}\right)=a_\mu\left(\frac{3\delta_{\alpha\mu}}{r}-\frac{\delta_{\alpha\mu}}{r}-\right.$
$\left.-\frac{\delta_{\alpha\mu}}{r}+\frac{x_\alpha x_\mu}{r^3}\right)=\frac{a_\alpha}{r}+x_\alpha\frac{a_\mu x_\mu}{r^3}=\left(\frac{\vec{a}}{r}+\frac{(\vec{a},\vec{r})\,\vec{r}}{r^3}\right)_\alpha=\frac{\vec{a}}{r}+\frac{(\vec{a},\vec{r})\,\vec{r}}{r^3}$

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2. $\operatorname{grad}\left(\vec{a}\,\vec{r}\,e^{i\vec{k}\vec{r}}\right)=\left(\operatorname{grad}\left(\vec{a}\,\vec{r}\,e^{i\vec{k}\vec{r}}\right)\right)_\alpha=\frac{\partial}{\partial x_\alpha}a_\beta x_\beta e^{ik_\gamma x_\gamma}=a_\beta\frac{\partial x_\beta}{\partial x_\alpha}e^{ik_\gamma x_\gamma}+a_\beta x_\beta\frac{\partial}{\partial x_\alpha}e^{ik_\gamma x_\gamma}=$
$=a_\beta\delta_{\alpha\beta}e^{ik_\gamma x_\gamma}+a_\beta x_\beta ik_\gamma\frac{\partial x_\gamma}{\partial x_\alpha}e^{ik_\gamma x_\gamma}=a_\beta\delta_{\alpha\beta}e^{ik_\gamma x_\gamma}+a_\beta x_\beta ik_\gamma\delta_{\alpha\gamma}e^{ik_\gamma x_\gamma}=(a_\alpha+ik_\alpha a_\beta x_\beta)e^{ik_\gamma x_\gamma}=$
$=\left(\left(\vec{a}+i(\vec{a},\vec{r})\vec{k}\right)e^{i\vec{k}\vec{r}}\right)_\alpha=\left(\vec{a}+i(\vec{a},\vec{r})\vec{k}\right)e^{i\vec{k}\vec{r}}$

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3. $\operatorname{rot}\left(\frac{[\vec{a},\vec{r}]}{r^3}\right)=\left(\operatorname{rot}\left(\frac{[\vec{a},\vec{r}]}{r^3}\right)\right)_\alpha=e_{\alpha\beta\gamma}\frac{\partial}{\partial x_\beta}\frac{e_{\gamma\mu\nu}a_\mu x_\nu}{r^3}=e_{\alpha\beta\gamma}e_{\gamma\mu\nu}a_\mu\left(\frac{\delta_{\beta\nu}}{r^3}-3\frac{x_\nu x_\beta}{r^5}\right)=$
$=a_\mu(\delta_{\alpha\mu}\delta_{\beta\nu}-\delta_{\alpha\nu}\delta_{\beta\mu})\left(\frac{\delta_{\beta\nu}}{r^3}-3\frac{x_\nu x_\beta}{r^5}\right)=a_\mu\left(\frac{3\delta_{\alpha\mu}}{r^3}-\frac{\delta_{\alpha\mu}}{r^3}-3\frac{\delta_{\alpha\mu}x_\beta^2}{r^5}+3\frac{x_\alpha x_\mu}{r^5}\right)=$
$=a_\mu\left(\frac{3\delta_{\alpha\mu}}{r^3}-\frac{\delta_{\alpha\mu}}{r^3}-3\frac{\delta_{\alpha\mu}}{r^3}+3\frac{x_\alpha x_\mu}{r^5}\right)=-\frac{a_\alpha}{r}+3x_\alpha\frac{a_\mu x_\mu}{r^5}=\left(-\frac{\vec{a}}{r^3}+3\frac{(\vec{a},\vec{r})\,\vec{r}}{r^5}\right)_\alpha=-\frac{\vec{a}}{r^3}+3\frac{(\vec{a},\vec{r})\,\vec{r}}{r^5}$

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4. $\operatorname{grad}\left(\frac{\vec{a}\vec{d}(t-r/c)}{r}\right)=\left(\operatorname{grad}\left(\frac{\vec{a}\vec{d}(t-r/c)}{r}\right)\right)_\alpha=\frac{\partial}{\partial x_\alpha}\frac{a_\beta d_\beta}{r}=-x_\alpha\frac{a_\beta d_\beta}{r^3}-\frac{a_\beta}{r}\frac{x_\alpha}{rc}\dot{d}_\beta=$
$=\left(-\frac{\vec{a}\vec{d}(t-r/c)}{r^3}\vec{r}-\frac{\vec{a}\dot{\vec{d}}(t-r/c)}{cr^2}\vec{r}\right)_\alpha=-\frac{\vec{a}\vec{d}(t-r/c)}{r^3}\vec{r}-\frac{\vec{a}\dot{\vec{d}}(t-r/c)}{cr^2}\vec{r}$

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5. $\operatorname{div}\frac{[\vec{a},\vec{r}]}{r^2}=\frac{\partial}{\partial x_\alpha}\frac{e_{\alpha\beta\gamma}a_\beta x_\gamma}{r^2}=e_{\alpha\beta\gamma}a_\beta\frac{\partial}{\partial x_\alpha}\frac{x_\gamma}{r^2}=e_{\alpha\beta\gamma}a_\beta\left(\frac{\delta_{\alpha\gamma}}{r^2}-2\frac{x_\alpha x_\gamma}{r^4}\right)=e_{\alpha\beta\gamma}\delta_{\alpha\gamma}\frac{a_\beta}{r^2}-2e_{\alpha\beta\gamma}\frac{x_\alpha x_\gamma}{r^4}a_\beta=$
$=0$

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6. $\operatorname{div}\frac{[\vec{a},\vec{r}]}{r^3}=\frac{\partial}{\partial x_\alpha}\frac{e_{\alpha\beta\gamma}a_\beta x_\gamma}{r^3}=e_{\alpha\beta\gamma}a_\beta\frac{\partial}{\partial x_\alpha}\frac{x_\gamma}{r^3}=e_{\alpha\beta\gamma}a_\beta\left(\frac{\delta_{\alpha\gamma}}{r^3}-3\frac{x_\alpha x_\gamma}{r^5}\right)=e_{\alpha\beta\gamma}\delta_{\alpha\gamma}\frac{a_\beta}{r^3}-3e_{\alpha\beta\gamma}\frac{x_\alpha x_\gamma}{r^5}a_\beta=$
$=0$

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7. $\operatorname{div}\left[\vec{r},\vec{d}(t-r/c)\right]=\frac{\partial}{\partial x_\alpha}e_{\alpha\beta\gamma}x_\beta d_\gamma=e_{\alpha\beta\gamma}\delta_{\alpha\beta}d_\gamma-e_{\alpha\beta\gamma}\frac{x_\beta x_\alpha}{rc}\dot{d}_\gamma=0$

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8. $\operatorname{rot}\operatorname{rot}\frac{\vec{a}}{r^3}=\left(\operatorname{rot}\operatorname{rot}\frac{\vec{a}}{r^3}\right)_\alpha=e_{\alpha\beta\gamma}\frac{\partial}{\partial x_\beta}e_{\gamma\mu\nu}\frac{\partial}{\partial x_\mu}\frac{a_\nu}{r^3}=a_\nu e_{\alpha\beta\gamma}e_{\gamma\mu\nu}\frac{\partial^2}{\partial x_\beta\partial x_\mu}\frac{1}{r^3}=a_\beta(\delta_{\alpha\mu}\delta_{\beta\nu}-\delta_{\alpha\nu}\delta_{\beta\mu})\cdot$

$$\cdot\left(-3\frac{\delta_{\beta\mu}}{r^5}+15\frac{x_\mu x_\beta}{r^7}\right)=-3a_\nu\left(\frac{\delta_{\alpha\nu}}{r^5}+5\frac{\delta_{\alpha\nu}x_\beta^2}{r^7}-3\frac{\delta_{\alpha\nu}}{r^5}-5\frac{x_\alpha x_\nu}{r^7}\right)=-3a_\nu\left(3\frac{\delta_{\alpha\nu}}{r^5}-5\frac{x_\alpha x_\nu}{r^7}\right)=$$
$$=-9\frac{a_\alpha}{r^5}+15x_\alpha\frac{a_\nu x_\nu}{r^7}=\left(-9\frac{\vec a}{r^5}+15\vec r\frac{(\vec a,\vec r)}{r^7}\right)_\alpha=-9\frac{\vec a}{r^5}+15\frac{(\vec a,\vec r)\vec r}{r^7}$$

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9. $$\operatorname{rot}\operatorname{rot}\frac{\vec a}{r}=\left(\operatorname{rot}\operatorname{rot}\frac{\vec a}{r}\right)_\alpha=e_{\alpha\beta\gamma}\frac{\partial}{\partial x_\beta}e_{\gamma\mu\nu}\frac{\partial}{\partial x_\mu}\frac{a_\nu}{r}=a_\nu e_{\alpha\beta\gamma}e_{\gamma\mu\nu}\frac{\partial^2}{\partial x_\beta\partial x_\mu}\frac{1}{r}=a_\nu(\delta_{\alpha\mu}\delta_{\beta\nu}-\delta_{\alpha\nu}\delta_{\beta\mu})\cdot$$
$$\cdot\left(-\frac{\delta_{\beta\mu}}{r^3}+3\frac{x_\mu x_\beta}{r^5}\right)=-a_\nu\left(\frac{\delta_{\alpha\nu}}{r^3}+3\frac{\delta_{\alpha\nu}x_\beta^2}{r^5}-3\frac{\delta_{\alpha\nu}}{r^3}-3\frac{x_\alpha x_\nu}{r^5}\right)=3a_\nu\frac{x_\alpha x_\nu}{r^5}-\frac{a_\alpha}{r^3}=\left(3\frac{(\vec a,\vec r)}{r^5}\vec r-\frac{\vec a}{r^3}\right)_\alpha=$$
$$=3\frac{(\vec a,\vec r)}{r^5}\vec r-\frac{\vec a}{r^3}$$

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10. $$\operatorname{grad}\left((\vec a,\vec r)^2e^{i\vec k\vec r}\right)=\left(\operatorname{grad}\left((\vec a,\vec r)^2e^{i\vec k\vec r}\right)\right)_\alpha=\frac{\partial}{\partial x_\alpha}a_\beta x_\beta a_\gamma x_\gamma e^{ik_\mu x_\mu}=a_\beta\delta_{\alpha\beta}a_\gamma x_\gamma e^{ik_\mu x_\mu}+$$
$$+a_\beta x_\beta a_\gamma\delta_{\alpha\gamma}e^{ik_\mu x_\mu}+a_\beta x_\beta a_\gamma x_\gamma ik_\mu\delta_{\alpha\mu}e^{ik_\mu x_\mu}=(a_\alpha a_\gamma x_\gamma+a_\alpha a_\beta x_\beta+ik_\alpha a_\beta x_\beta a_\gamma x_\gamma)e^{ik_\mu x_\mu}=$$
$$=\left(\left(2(\vec a,\vec r)\vec a+i(\vec a,\vec r)^2\vec k\right)e^{i\vec k\vec r}\right)_\alpha=\left(2(\vec a,\vec r)\vec a+i(\vec a,\vec r)^2\vec k\right)e^{i\vec k\vec r}$$

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11. $$\operatorname{div}\left([\vec a,\vec r]e^{i\vec k\vec r}\right)=\frac{\partial}{\partial x_\alpha}e_{\alpha\beta\gamma}a_\beta x_\gamma e^{ik_\mu x_\mu}=e_{\alpha\beta\gamma}a_\beta\delta_{\alpha\gamma}e^{ik_\mu x_\mu}+e_{\alpha\beta\gamma}a_\beta x_\gamma ik_\mu\delta_{\alpha\mu}e^{ik_\mu x_\mu}=e_{\alpha\beta\gamma}a_\beta x_\gamma ik_\mu\delta_{\alpha\mu}e^{ik_\mu x_\mu}$$
$$=ik_\alpha e_{\alpha\beta\gamma}a_\beta x_\gamma e^{ik_\mu x_\mu}=i(\vec k,\vec a,\vec r)e^{i\vec k\vec r}$$

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12. $$\operatorname{rot}\left(\left[\vec r,\vec d(t-r/c)\right]\right)=\left(\operatorname{rot}\left(\left[\vec r,\vec d(t-r/c)\right]\right)\right)_\alpha=e_{\alpha\beta\gamma}\frac{\partial}{\partial x_\beta}e_{\gamma\mu\nu}x_\mu d_\nu=e_{\alpha\beta\gamma}e_{\gamma\mu\nu}\frac{\partial}{\partial x_\beta}x_\mu d_\nu=$$
$$=(\delta_{\alpha\mu}\delta_{\beta\nu}-\delta_{\alpha\nu}\delta_{\beta\mu})\left(d_\nu\delta_{\beta\mu}-\frac{x_\beta x_\mu}{rc}\frac{\partial d_\nu}{\partial t}\right)=\left(\delta_{\alpha\nu}d_\nu-3\delta_{\alpha\nu}d_\nu-\frac{x_\nu\dot d_\nu}{rc}x_\alpha+\frac{x_\beta^2}{rc}\dot d_\alpha\right)=\left(-2\vec d(t-\right.$$
$$\left.-r/c)-\frac{\left(\dot{\vec d},\vec r\right)\vec r}{rc}+\frac{r}{c}\dot{\vec d}\right)_\alpha=-2\vec d(t-r/c)-\frac{\left(\dot{\vec d},\vec r\right)\vec r}{rc}+\frac{r}{c}\dot{\vec d}$$

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13. $$\operatorname{grad}\left(\frac{f(t-r/c)}{r}\right)=\left(\operatorname{grad}\left(\frac{f(t-r/c)}{r}\right)\right)_\alpha=\frac{1}{r}\frac{\partial}{\partial x_\alpha}f(t-r/c)+f(t-r/c)\frac{\partial r}{\partial x_\alpha}=-\frac{x_\alpha}{cr^2}\frac{\partial}{\partial t}f(t-$$
$$-r/c)-f(t-r/c)\frac{x_\alpha}{r^3}=\left(-\frac{\dot f(t-r/c)}{cr^2}\vec r-\frac{f(t-r/c)}{r^3}\vec r\right)_\alpha=-\frac{\dot f(t-r/c)}{cr^2}\vec r-\frac{f(t-r/c)}{r^3}\vec r$$

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14. $$\operatorname{rot}\left[\vec a,\operatorname{grad}\left(e^{i\vec k\vec r}\right)\right]=\left(\operatorname{rot}\left[\vec a,\operatorname{grad}\left(e^{i\vec k\vec r}\right)\right]\right)_\alpha=e_{\alpha\beta\gamma}\frac{\partial}{\partial x_\beta}e_{\gamma\mu\nu}a_\mu\frac{\partial}{\partial x_\nu}e^{ik_\eta x_\eta}=e_{\alpha\beta\gamma}e_{\gamma\mu\nu}a_\mu\frac{\partial}{\partial x_\beta}\frac{\partial}{\partial x_\nu}e^{ik_\eta x_\eta}$$
$$=-a_\mu(\delta_{\alpha\mu}\delta_{\beta\nu}-\delta_{\alpha\nu}\delta_{\beta\mu})k_\eta\delta_{\nu\eta}k_\beta\delta_{\beta\eta}e^{ik_\eta x_\eta}=-a_\mu(\delta_{\alpha\mu}\delta_{\beta\nu}-\delta_{\alpha\nu}\delta_{\beta\mu})k_\nu k_\beta e^{ik_\eta x_\eta}=-\left(a_\alpha k_\beta^2-k_\alpha a_\mu k_\mu\right)e^{ik_\eta x_\eta}$$
$$=\left(\left((\vec a,\vec k)\vec k-\left|\vec k\right|^2\vec a\right)e^{i\vec k\vec r}\right)_\alpha=\left((\vec a,\vec k)\vec k-\left|\vec k\right|^2\vec a\right)e^{i\vec k\vec r}$$

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15. $\operatorname{div}\left(\vec{r}e^{i\vec{k}\vec{r}}\right) = \dfrac{\partial}{\partial x_\alpha} x_\alpha e^{ik_\beta x_\beta} = 3e^{ik_\beta x_\beta} + ik_\beta \delta_{\alpha\beta} x_\alpha e^{ik_\beta x_\beta} = (3 + ik_\alpha x_\alpha) e^{ik_\beta x_\beta} = \left(3 + i\vec{k}\vec{r}\right) e^{i\vec{k}\vec{r}}$

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16. $\operatorname{grad}\left(\dfrac{\exp\left(i\vec{k}\vec{r}\right)}{r^3}\right) = \left(\operatorname{grad}\left(\dfrac{\exp\left(i\vec{k}\vec{r}\right)}{r^3}\right)\right)_\alpha = \dfrac{\partial}{\partial x_\alpha} \dfrac{e^{ik_\beta x_\beta}}{r^3} = i\delta_{\alpha\beta} \dfrac{k_\beta}{r^3} e^{ik_\beta x_\beta} - 3\dfrac{x_\alpha}{r^5} e^{ik_\beta x_\beta} =$
$= \left(\left(i\dfrac{\vec{k}}{r^3} - 3\dfrac{\vec{r}}{r^5}\right) e^{i\vec{k}\vec{r}}\right)_\alpha = \left(i\dfrac{\vec{k}}{r^3} - 3\dfrac{\vec{r}}{r^5}\right) e^{i\vec{k}\vec{r}}$

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17. $\operatorname{grad}\left(r^2 e^{i\vec{k}\vec{r}}\right) = \left(\operatorname{grad}\left(r^2 e^{i\vec{k}\vec{r}}\right)\right)_\alpha = \dfrac{\partial}{\partial x_\alpha} x_\beta^2 e^{ik_\gamma x_\gamma} = 2\delta_{\alpha\beta} x_\beta e^{ik_\gamma x_\gamma} + x_\beta^2 ik_\gamma \delta_{\alpha\gamma} e^{ik_\gamma x_\gamma} = \Big(2x_\alpha +$
$ix_\beta^2 k_\alpha\Big) e^{ik_\gamma x_\gamma} = \left(\left(2\vec{r} + ir^2\vec{k}\right) e^{i\vec{k}\vec{r}}\right)_\alpha = \left(2\vec{r} + ir^2\vec{k}\right) e^{i\vec{k}\vec{r}}$