CLASSIFICATION OF QUOTIENT BUNDLES ON THE FARGUES-FONTAINE CURVE

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ABSTRACT. We completely classify all quotient bundles of a given vector bundle on the Fargues-Fontaine curve. As consequences, we have two additional classification results: a complete classification of all vector bundles that are generated by a fixed number of global sections, and a nearly complete classification of subsheaves of a given vector bundle. For the proof, we combine the dimension counting argument for moduli of bundle maps developed in [BFH+17] with a series of reduction arguments based on some reinterpretation of the classifying conditions.

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1. Introduction

In [FF18], Fargues and Fontaine constructed a remarkable scheme, now commonly referred to as the *Fargues-Fontaine curve*, which serves as the "fundamental curve" for p-adic Hodge theory and the local Langlands program. In fact, many constructions in these fields have geometric interpretations in terms of vector bundles on the Fargues-Fontaine curve. Most notably, Fargues-Scholze [FS21] builds upon the idea of Fargues [Far16] to construct the local Langlands correspondence in terms of certain sheaves on the stack of vector bundles on the Fargues-Fontaine curve.

In this paper we obtain several classification results regarding vector bundles on the Fargues-Fontaine curve. Our main result is a complete classification of all quotient bundles of a given vector bundle. As a special case, we obtain a complete classification of all vector bundles that are generated by a fixed number of global sections. In addition, a dual statement of our main result gives a nearly complete classification of subsheaves of a given vector bundle.

1.1. **Statement of results.**

For a precise statement of our results, we briefly recall the classification of vector bundles on the Fargues-Fontaine curve.

Theorem 1.1.1 (Fargues-Fontaine [FF18, Théorème 8.2.10], Kedlaya [Ked05, Theorem 4.5.7])**.** *Fix a prime number p. Let E be a finite extension of $\mathbb{Q}_p$, and let F be an algebraically closed perfectoid field of characteristic p. Denote by $X = X_{E,F}$ the Fargues-Fontaine curve associated to the pair (E, F).*

> *(1) The scheme X is complete in the sense that the divisor of an arbitrary nonzero rational function on X has degree zero. As a consequence, there is a well-defined notion of the slope of a vector bundle on X.*
>
> *(2) For every rational number λ, there is a unique stable bundle of slope λ on X, denoted by $\mathcal{O}(\lambda)$.*
>
> *(3) Every semistable bundle of slope λ is of the form $\mathcal{O}(\lambda)^{\oplus m}$.*
>
> *(4) Every vector bundle $\mathcal{V}$ on X admits a canonical Harder-Narasimhan filtration which splits into a direct decomposition*
> $$\mathcal{V} \simeq \bigoplus_i \mathcal{O}(\lambda_i)^{\oplus m_i}$$
> *where λ_i's run over the Harder-Narasimhan slopes of $\mathcal{V}$; in other words, the isomorphism class of $\mathcal{V}$ is determined by the Harder-Narasimhan polygon $\mathrm{HN}(\mathcal{V})$ of $\mathcal{V}$ (as defined in Definition 2.2.8).*

For a vector bundle $\mathcal{V}$ with a direct sum decomposition as in (4) of Theorem 1.1.1, we set

$$\mathcal{V}^{\leq \mu} := \bigoplus_{\lambda_i \leq \mu} \mathcal{O}(\lambda_i)^{\oplus m_i} \qquad \text{and} \qquad \mathcal{V}^{\geq \mu} := \bigoplus_{\lambda_i \geq \mu} \mathcal{O}(\lambda_i)^{\oplus m_i} \qquad \text{for every } \mu \in \mathbb{Q}.$$

Now we can state our main result as follows:

Theorem 1.1.2. *Let $\mathcal{E}$ be a vector bundle on X. Then a vector bundle $\mathcal{F}$ on X is a quotient bundle of $\mathcal{E}$ if and only if the following equivalent conditions are satisfied:*

> *(i) For every $\mu \in \mathbb{Q}$, we have $\mathrm{rank}(\mathcal{E}^{\leq \mu}) \geq \mathrm{rank}(\mathcal{F}^{\leq \mu})$ with equality if and only if $\mathcal{E}^{\leq \mu}$ and $\mathcal{F}^{\leq \mu}$ are isomorphic.*

(ii) *If we align the Harder-Narasimhan polygons* $\mathrm{HN}(\mathcal{E})$ *and* $\mathrm{HN}(\mathcal{F})$ *so that their right endpoints lie at the origin, then for each* $i = 1, \cdots, \mathrm{rank}(\mathcal{F})$, *the slope of* $\mathrm{HN}(\mathcal{F})$ *on* $[-i, -i+1]$ *is greater than or equal to the slope of* $\mathrm{HN}(\mathcal{E})$ *on* $[-i-1, -i]$ *unless* $\mathrm{HN}(\mathcal{E})$ *and* $\mathrm{HN}(\mathcal{F})$ *agree on* $[-i, 0]$.
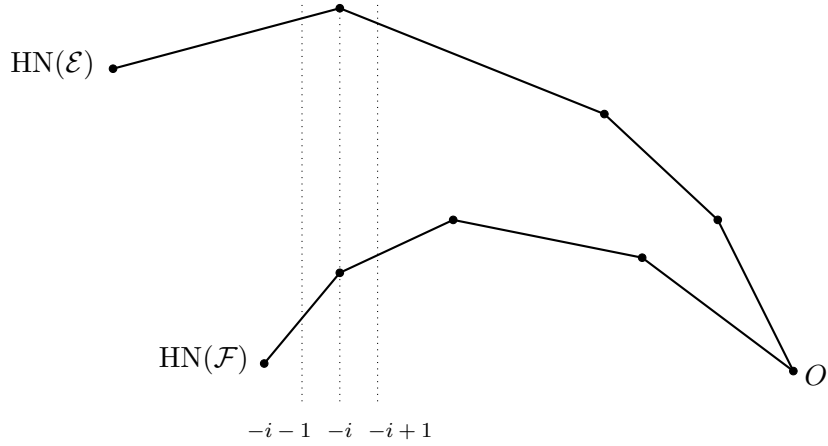

FIGURE 1. Illustration of the condition (ii) in Theorem 1.1.2.

The necessity part of Theorem 1.1.2 is a consequence of the slope formalism for vector bundles on the Fargues-Fontaine curve. Hence Theorem 1.1.2 asserts that the slope formalism is the only obstruction for the existence of a surjective bundle map between two given vector bundles on X. We emphasize that this is a very unique feature for the slope category of vector bundles on the Fargues-Fontaine curve. The main reason for this feature is that for any given vector bundles $\mathcal{V}, \mathcal{W}$ on X the space $\mathrm{Hom}(\mathcal{V}, \mathcal{W})$ is either empty or huge, where the nonemptiness is determined by the slope formalism.

If we take $\mathcal{E} = \mathcal{O}_X^{\oplus n}$ for some positive integer n in Theorem 1.1.2, we obtain the following classification of finitely globally generated vector bundles on X.

Corollary 1.1.3. *A vector bundle* $\mathcal{F}$ *on* X *is generated by* n *global sections if and only if the following conditions are satisfied:*

(i) *All Harder-Narasimhan slopes of* $\mathcal{F}$ *are nonnegative.*
(ii) $\mathrm{rank}(\mathcal{F}) \leq n$ *with equality if and only if* $\mathcal{F} \simeq \mathcal{O}_X^{\oplus n}$.

In addition, dualizing the statement of Theorem 1.1.2 yields a classification of a majority of subsheaves of a given vector bundle on X.

Corollary 1.1.4. *Let* $\mathcal{E}$ *be a vector bundle on* X. *Then a vector bundle* $\mathcal{D}$ *on* X *is a subsheaf of* $\mathcal{E}$ *if the following equivalent conditions are satisfied:*

(i) *For every* $\mu \in \mathbb{Q}$, *we have* $\mathrm{rank}(\mathcal{E}^{\geq \mu}) \geq \mathrm{rank}(\mathcal{D}^{\geq \mu})$ *with equality if and only if* $\mathcal{E}^{\geq \mu}$ *and* $\mathcal{D}^{\geq \mu}$ *are isomorphic.*
(ii) *If we align the Harder-Narasimhan polygons* $\mathrm{HN}(\mathcal{D})$ *and* $\mathrm{HN}(\mathcal{E})$ *so that their left endpoints lie at the origin, then for each* $i = 1, \cdots, \mathrm{rank}(\mathcal{D})$, *the slope of* $\mathrm{HN}(\mathcal{D})$ *on* $[i-1, i]$ *is less than or equal to the slope of* $\mathrm{HN}(\mathcal{E})$ *on* $[i, i+1]$ *unless* $\mathrm{HN}(\mathcal{D})$ *and* $\mathrm{HN}(\mathcal{E})$ *agree on* $[0, i]$.
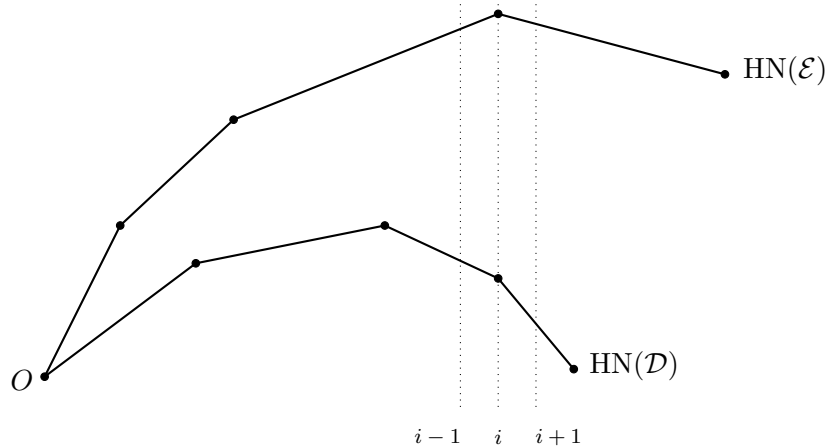
FIGURE 2. Illustration of the condition (ii) in Corollary 1.1.4.

We remark that Corollary 1.1.4 does not give a complete classification of all subsheaves. The main issue is that the cokernel of an injective bundle map may have a torsion while the kernel of a surjective bundle map is always torsion-free. In fact, Corollary 1.1.4 gives a complete classification of all subsheaves with torsion-free cokernel.

It is also worthwhile to note that our main result holds verbatim for the projective line $\mathbb{P}^1$ over an arbitrary field. While the statement for $\mathbb{P}^1$ can be proved by some elementary linear algebra, it can also be seen by the same proof for Theorem 1.1.2 using a dimension formula for spaces of bundle maps between two vector bundles on $\mathbb{P}^1$. We refer the readers to the appendix of this article for details.

1.2. **Applications of the main result.**

Our main result and its proof have applications in some problems that naturally arise in p-adic geometry.

First, Theorem 1.1.2 has an application towards the problem of classifying all vector bundles $\mathcal{E}$ on X that arise as an extension of two fixed vector bundles $\mathcal{D}$ and $\mathcal{F}$ on X. In fact, this is the main motivating problem for our work, as it naturally arises in the study of geometric objects such as the stack of vector bundles on X and the flag varieties. When both $\mathcal{D}$ and $\mathcal{F}$ are semistable, we have a complete answer by the work of the author and his collaborators in [BFH$^+$17], which in turn leads to an explicit description of the connected components of the stack of vector bundles on X by Hansen [Han17]. In the subsequent paper [Hon20], the author applies Theorem 1.1.2 to extend the main result of [BFH$^+$17] as follows:

Theorem 1.2.1 ([Hon20, Theorem 3.2.1]). *Let $\mathcal{D}, \mathcal{E}$, and $\mathcal{F}$ be vector bundles on X such that one of $\mathcal{D}, \mathcal{E}$, and $\mathcal{F}$ is semistable. Assume that the maximum slope of $\mathcal{D}$ is less than the minimum slope of $\mathcal{F}$. Then there exists a short exact sequence of the form*

$$0 \longrightarrow \mathcal{D} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0$$

if and only if the following conditions are satisfied:

(i) *For every $\mu \in \mathbb{Q}$, we have $\mathrm{rank}(\mathcal{E}^{\leq \mu}) \geq \mathrm{rank}(\mathcal{F}^{\leq \mu})$ with equality if and only if $\mathcal{E}^{\leq \mu}$ and $\mathcal{F}^{\leq \mu}$ are isomorphic.*

(ii) *For every $\mu \in \mathbb{Q}$, we have $\mathrm{rank}(\mathcal{E}^{\geq \mu}) \geq \mathrm{rank}(\mathcal{D}^{\geq \mu})$ with equality if and only if $\mathcal{E}^{\geq \mu}$ and $\mathcal{D}^{\geq \mu}$ are isomorphic.*

(iii) $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F}) \geq \mathrm{HN}(\mathcal{E})$, *which means that* $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$ *lies above* $\mathrm{HN}(\mathcal{E})$ *with the same endpoints.*

The conditions (i) and (ii) can be stated purely in terms of HN polygons as in Theorem 1.1.2. Note that these conditions are clearly necessary; indeed, they are equivalent to existence of surjective bundle maps $\mathcal{E} \twoheadrightarrow \mathcal{F}$ and $\mathcal{E}^{\vee} \twoheadrightarrow \mathcal{D}^{\vee}$ by Theorem 1.1.2, where $\mathcal{E}^{\vee}$ and $\mathcal{D}^{\vee}$ denote the duals of $\mathcal{E}$ and $\mathcal{D}$. In addition, the necessity of the condition (iii) is a consequence of the slope formalism.

We expect that Theorem 1.2.1 holds without the semistability assumption on one of $\mathcal{D}, \mathcal{E}$, or $\mathcal{F}$. With this generalization of Theorem 1.2.1, we should be able to obtain an explicit description on the geometry of the p-adic flag variety in terms of two natural stratifications, namely the Harder-Narasimhan stratification and the Newton stratification, in line with the work of many authors including Caraiani-Scholze [CS17], Chen-Fargues-Shen [CFS17], Shen [She19], Chen [Che20], Viehmann [Vie21], and Nguyen-Viehmann [NV21].

As another application, the author in the sequel paper [Hon21] adapts our argument and several constructions from this paper to establish a complete classification for subsheaves as follows:

Theorem 1.2.2 ([Hon21, Theorem 3.1.1])**.** *Let* $\mathcal{E}$ *be a vector bundle on* X*. Then a vector bundle* $\mathcal{D}$ *is a subsheaf of* $\mathcal{E}$ *if and only if it satisfies the following equivalent conditions:*

(i) *For every* $\mu \in \mathbb{Q}$*, we have* $\mathrm{rank}(\mathcal{E}^{\geq \mu}) \geq \mathrm{rank}(\mathcal{D}^{\geq \mu})$*.*

(ii) *If we align the Harder-Narasimhan polygons* $\mathrm{HN}(\mathcal{D})$ *and* $\mathrm{HN}(\mathcal{E})$ *so that their left endpoints lie at the origin, then for each* $i = 1, \cdots, \mathrm{rank}(\mathcal{D})$*, the slope of* $\mathrm{HN}(\mathcal{D})$ *on* $[i-1, i]$ *is less than or equal to the slope of* $\mathrm{HN}(\mathcal{E})$ *on* $[i-1, i]$*.*

Theorem 1.2.2 in particular gives a classification of all subsheaves $\mathcal{D}$ of $\mathcal{E}$ such that $\mathcal{E}/\mathcal{D}$ is a torsion sheaf. Such subsheaves are of particular interest as they arise from *modifications of vector bundles*, which play a pivotal role in the study of various geometric objects such as the B_{dR}^{+}-affine Grassmannians and the moduli of local shtukas.

1.3. **Outline of the strategy.**

It is relatively easy to see that the condition (i) in Theorem 1.1.2 is indeed necessary and that it is equivalent to the condition (ii). Therefore the main part of our proof will concern the sufficiency of the condition (i) in Theorem 1.1.2.

Our argument will be based on the dimension counting method for certain moduli spaces of bundle maps as developed in [BFH^{+}17]. We define the moduli functors

- $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})$ which parametrizes bundle maps $\mathcal{E} \to \mathcal{F}$, and
- $\mathcal{S}\mathrm{urj}(\mathcal{E}, \mathcal{F})$ which parametrizes surjective bundle maps $\mathcal{E} \twoheadrightarrow \mathcal{F}$.

These functors are represented by diamonds in the sense of Scholze [Sch18]. The goal is to show that the diamond $\mathcal{S}\mathrm{urj}(\mathcal{E}, \mathcal{F})$ is not empty if the condition (i) in Theorem 1.1.2 is satisfied. To this end, we consider auxiliary spaces $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})_{\mathcal{Q}}$ which (roughly) parametrizes bundle maps $\mathcal{E} \to \mathcal{F}$ with image isomorphic to a specified subsheaf $\mathcal{Q}$ of $\mathcal{F}$. Then showing nonemptiness of $\mathcal{S}\mathrm{urj}(\mathcal{E}, \mathcal{F})$ boils down to establishing the following inequality on dimensions of the topological spaces:

$$\dim |\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})_{\mathcal{Q}}| < \dim |\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})| \qquad \text{if } \mathcal{Q} \neq \mathcal{F}. \tag{1.1}$$

The dimension theory for diamonds allows us to rewrite this inequality in terms of degrees of certain vector bundles related to $\mathcal{E}, \mathcal{F}$ and $\mathcal{Q}$.

However, the details of our arguments are completely different from those in [BFH$^+$17]. The main reason is that, unlike the quantities considered in [BFH$^+$17], the quantities we need to study in this paper do not generally have good interpretations in terms of areas of polygons related to the Harder-Narasimhan slopes. In fact, our proof of the inequality (1.1) will consist of a series of reduction steps as follows:

> Step 1. We reduce the proof of (1.1) to the case when all slopes of $\mathcal{E}, \mathcal{F}$ and $\mathcal{Q}$ are integers.
> Step 2. We further reduce the proof of (1.1) to the case $\mathrm{rank}(\mathcal{Q}) = \mathrm{rank}(\mathcal{F})$.
> Step 3. When $\mathrm{rank}(\mathcal{Q}) = \mathrm{rank}(\mathcal{F})$, we complete the proof of (1.1) by gradually "reducing" the slopes of $\mathcal{F}$ to the slopes of $\mathcal{Q}$.

As a key ingredient of our reduction argument, we introduce and study the notion of *slopewise dominance* for vector bundles on the Fargues-Fontaine curve. This notion provides a combinatorial interpretation of the inequality in the condition (i) of Theorem 1.1.2 in terms of Harder-Narasimhan polygons, and allows us to use the equivalence between the conditions (i) and (ii) of Theorem 1.1.2 to its full capacity. In particular, this notion yields several implications of the condition (ii) which are difficult to directly deduce from the condition (i), and plays a pivotal role in our process of "reducing" the slopes of $\mathcal{F}$ to the slopes of $\mathcal{Q}$ in Step 3.. This notion is also crucial for applications of Theorem 1.1.2, such as Theorem 1.2.2 and Theorem 1.2.1 which are discussed in the sequel papers [Hon20] and [Hon21].

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2. Preliminaries on the Fargues-Fontaine curve

2.1. The construction.

Throughout this paper, we fix the following data:

- p is a prime number;
- E is a finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$;
- F is an algebraically closed perfectoid field of characteristic p.

The Fargues-Fontaine curve can be constructed in two different flavors, namely as a scheme and as an adic space. We first present the construction as an adic space since it is simpler to describe than the construction as a scheme is.

Definition 2.1.1. Denote by E° and F° the rings of integers of E and F, respectively. Let π be a uniformizer of E, and let ϖ be a pseudouniformizer of F. We write $W_{E^\circ}(F^\circ) := W(F^\circ) \otimes_{W(\mathbb{F}_q)} E^\circ$ for the ring of ramified Witt vectors of F° with coefficients in E°, and $[\varpi]$ for the Teichmuller lift of ϖ. Define

$$\mathcal{Y}_{E,F} := \mathrm{Spa}(W_{E^\circ}(F^\circ)) \setminus \{|p[\varpi]| = 0\},$$

and let $\phi : \mathcal{Y}_{E,F} \to \mathcal{Y}_{E,F}$ be the Frobenius automorphism of $\mathcal{Y}_{E,F}$ induced by the q-Frobenius φ_q on $W_{E^\circ}(F^\circ)$. The (mixed-characteristic) *adic Fargues-Fontaine curve* associated to the pair (E, F) is

$$\mathcal{X}_{E,F} := \mathcal{Y}_{E,F}/\phi^{\mathbb{Z}}.$$

Remark. This definition makes sense since the action of ϕ on $\mathcal{Y}_{E,F}$ turns out to be properly discontinuous.

Proposition 2.1.2 ([Ked16, Theorem 4.10]). *$\mathcal{X}_{E,F}$ is a Noetherian adic space over $\mathrm{Spa}(E)$.*

Remark. When E is replaced by a finite extension of $\mathbb{F}_p((t))$, there is a related construction by Hartl-Pink [HP04] which we may regard as the equal-characteristic Fargues-Fontaine curve. Our main results are equally valid for vector bundles on the equal-characteristic Fargues-Fontaine curve with the same proof.

Our next goal is to relate the above construction of $\mathcal{X}_{E,F}$ to the schematic construction of the Fargues-Fontaine curve. To this end, we first define some vector bundles on $\mathcal{X}_{E,F}$. By descent, giving a vector bundle $\mathcal{V}$ on $\mathcal{X}_{E,F}$ amounts to giving a ϕ-equivariant vector bundle $\mathring{\mathcal{V}}$ on $\mathcal{Y}_{E,F}$, that is, a vector bundle $\mathring{\mathcal{V}}$ on $\mathcal{Y}_{E,F}$ together with an isomorphism $\phi^*\mathring{\mathcal{V}} \xrightarrow{\sim} \mathring{\mathcal{V}}$.

Definition 2.1.3. Let $\lambda = r/s$ be a rational number written in lowest terms with $s > 0$. Let $v_1, v_2, \cdots, v_s$ be a trivializing basis of $\mathcal{O}_{\mathcal{Y}_{E,F}}^{\oplus s}$. Define an isomorphism $\phi^*\mathcal{O}_{\mathcal{Y}_{E,F}}^{\oplus s} \xrightarrow{\sim} \mathcal{O}_{\mathcal{Y}_{E,F}}^{\oplus s}$ by

$$v_1 \mapsto v_2, \quad v_2 \mapsto v_3, \quad \cdots, \quad v_{s-1} \mapsto v_s, \quad v_s \mapsto \pi^{-r}v_1,$$

where we abuse notation to view $v_1, v_2, \cdots, v_s$ as a trivializing basis for $\phi^*\mathcal{O}_{\mathcal{Y}_{E,F}}^{\oplus s}$ as well. We write $\mathcal{O}(\lambda)$ for the vector bundle on $\mathcal{X}_{E,F}$ corresponding to the vector bundle $\mathcal{O}_{\mathcal{Y}_{E,F}}^{\oplus s}$ with the isomorphism $\phi^*\mathcal{O}_{\mathcal{Y}_{E,F}}^{\oplus s} \xrightarrow{\sim} \mathcal{O}_{\mathcal{Y}_{E,F}}^{\oplus s}$ as defined above.

The following fact suggests that we can regard $\mathcal{O}(1)$ as an "ample" line bundle on $\mathcal{X}_{E,F}$.

Proposition 2.1.4 ([KL15, Lemma 8.8.4 and Proposition 8.8.6]). *Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{X}_{E,F}$. Then for all sufficiently large $n \in \mathbb{Z}$, the twisted sheaf $\mathcal{F}(n) := \mathcal{F} \otimes \mathcal{O}(1)^{\otimes n}$ satisfies the following properties:*

   (i) $H^1(\mathcal{X}_{E,F}, \mathcal{F}(n)) = 0$.
   (ii) The sheaf $\mathcal{F}(n)$ is generated by finitely many global sections.

We now recover the schematic construction of the Fargues-Fontaine curve as follows:

Definition 2.1.5. We define the *schematic Fargues-Fontaine curve* associated to the pair (E, F) by

$$X_{E,F} := \mathrm{Proj}\left(\bigoplus_{n \geq 0} H^0(\mathcal{X}_{E,F}, \mathcal{O}(n))\right).$$

Remark. The original construction of the schematic Fargues-Fontaine curve in [FF18, §6] was given in terms of the period ring B_{cris}^+ in p-adic Hodge theory (see also [FF14], §4.1):

$$X_{E,F} = \mathrm{Proj}\left(\bigoplus_{n \geq 0} (B_{\mathrm{cris}}^+)^{\varphi_q = \pi^n}\right).$$

This definition agrees with Definition 2.1.5 via the identification $H^0(\mathcal{X}_{E,F}, \mathcal{O}(n)) \simeq (B_{\mathrm{cris}}^+)^{\varphi_q = \pi^n}$.

Proposition 2.1.6 ([FF18, Théorème 6.5.2]). *The scheme $X_{E,F}$ is noetherian, connected, and regular of Krull dimension one.*

Remark. The scheme $X_{E,F}$ admits a natural structure morphsim $X_{E,F} \to \mathrm{Spec}(E)$ induced by a canonical isomorphism $(B_{\mathrm{cris}}^+)^{\varphi_q = 1} \cong E$. However, it is not a curve in the usual sense; in fact, one can show that the residue field at a closed point on $X_{E,F}$ is a complete algebraically closed extension of E, which in particular implies that $X_{E,F}$ is not of finite type over E.

For our purpose, the two constructions of the Fargues-Fontaine curve are essentially equivalent, as we have a version of GAGA for the Fargues-Fontaine curve.

Theorem 2.1.7 ([KL15, Theorems 6.3.12 and 8.7.7]). *There is a natural map*

$$\mathcal{X}_{E,F} \to X_{E,F}$$

which induces by pullback an equivalence of the categories of vector bundles.

Following Kedlaya-Liu [KL15, §8.7], we can extend the construction of the adic Fargues-Fontaine curve to relative settings.

Definition 2.1.8. Let $S = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space over $\mathrm{Spa}(F)$, and let ϖ_R be a pseudouniformizer of R. Denote by E° the ring of integers of E, and by R° the ring of power bounded elements of R. We take the ring of ramified Witt vectors $W_{E^\circ}(R^+) := W(R^+) \otimes_{W(\mathbb{F}_q)} E^\circ$ and write $[\varpi_R]$ for the Teichmuller lift of ϖ_R. Define

$$\mathcal{Y}_{E,S} := \mathrm{Spa}(W_{E^\circ}(R^+), W_{E^\circ}(R^+)) \setminus \{|p[\varpi_R]| = 0\},$$

and let $\phi : \mathcal{Y}_{E,S} \to \mathcal{Y}_{E,S}$ be the Frobenius automorphism of $\mathcal{Y}_{E,S}$ induced by the q-Frobenius φ_q on $W_{E^\circ}(R^+)$. The *relative adic Fargues-Fontaine curve* associated to the pair (E, S) is

$$\mathcal{X}_{E,S} := \mathcal{Y}_{E,S}/\phi^{\mathbb{Z}}.$$

More generally, for an arbitrary perfectoid space S over $\mathrm{Spa}(F)$, we choose an affinoid cover $S = \bigcup S_i = \bigcup \mathrm{Spa}(R_i, R_i^+)$ and define the relative adic Fargues-Fontaine curve $\mathcal{X}_{E,S}$ by gluing the adic spaces $\mathcal{X}_{E,S_i}$.

Remark. By construction, the relative curve $\mathcal{X}_{E,S}$ comes with a natural map $\mathcal{X}_{E,S} \to \mathcal{X}_{E,F}$. However, the relative curve $\mathcal{X}_{E,S}$ cannot be obtained from $\mathcal{X}_{E,F}$ by base change; indeed, neither $\mathcal{X}_{E,S}$ nor $\mathcal{X}_{E,F}$ is defined over $\mathrm{Spa}(F)$.

2.2. **Classification of vector bundles and Harder-Narasimhan polygons.**

For the rest of this paper, we will simply write $\mathcal{X} := \mathcal{X}_{E,F}$ and $X := X_{E,F}$. Moreover, we will speak interchangeably about vector bundles on $\mathcal{X}$ and X in light of Theorem 2.1.7.

In this subsection we review the main classification theorem for vector bundles on the Fargues-Fontaine curve and discuss some of its immediate consequences.

Proposition 2.2.1 ([FF18, §8.2.1]). *There exists a natural isomorphism from the Picard group* $\mathrm{Pic}(X)$ *to* $\mathbb{Z}$, *which maps* $\mathcal{O}(d)$ *to* d *for any* $d \in \mathbb{Z}$.

Definition 2.2.2. Let $\mathcal{V}$ be a nonzero vector bundle on X.

(1) We write $\mathrm{rk}(\mathcal{V})$ for the rank of $\mathcal{V}$, and $\mathcal{V}^\vee$ for the dual bundle of $\mathcal{V}$.
(2) We define the *degree* of $\mathcal{V}$, denoted by $\deg(\mathcal{V})$, to be the image of the determinant line bundle $\wedge^{\mathrm{rk}(\mathcal{V})}(\mathcal{V})$ under the natural isomorphism $\mathrm{Pic}(X) \cong \mathbb{Z}$ in Proposition 2.2.1.
(3) We define the *slope* of $\mathcal{V}$ by

$$\mu(\mathcal{V}) := \frac{\deg(\mathcal{V})}{\mathrm{rk}(\mathcal{V})}.$$

Let us now recall the usual notions of stability and semistability.

Definition 2.2.3. Let $\mathcal{V}$ be a nonzero vector bundle on $\mathcal{X}$.

(1) We say that $\mathcal{V}$ is *stable* if $\mu(\mathcal{W}) < \mu(\mathcal{V})$ for all nonzero proper subbundles $\mathcal{W} \subset \mathcal{V}$.
(2) We say that $\mathcal{V}$ is *semistable* if $\mu(\mathcal{W}) \leq \mu(\mathcal{V})$ for all nonzero proper subbundles $\mathcal{W} \subset \mathcal{V}$.

We collect some fundamental facts about semistable vector bundles on $\mathcal{X}$.

Proposition 2.2.4 ([FF18, Théorème 8.2.10]). *Let λ be a rational number.*

(1) *The vector bundle $\mathcal{O}(\lambda)$ represents the unique isomorphism class of stable vector bundles on $\mathcal{X}$ of slope λ.*

(2) *Every semistable vector bundle of slope λ is isomorphic to $\mathcal{O}(\lambda)^{\oplus n}$ for some n.*

Lemma 2.2.5. *Let r and s be relatively prime integers with $s > 0$.*

(1) *The bundle $\mathcal{O}(r/s)$ has rank s, degree r, and slope r/s.*

(2) *For any relatively prime integers r' and s' with $s' > 0$, we have*

$$\mathcal{O}\left(\frac{r}{s}\right) \otimes \mathcal{O}\left(\frac{r'}{s'}\right) \simeq \mathcal{O}\left(\frac{r}{s} + \frac{r'}{s'}\right)^{\oplus \gcd(ss', rs' + r's)}.$$

In particular, the bundle $\mathcal{O}(r/s) \otimes \mathcal{O}(r'/s')$ has rank ss', degree $rs' + r's$, and slope $r/s + r'/s'$.

(3) *$\mathcal{O}(r/s)^{\vee} \simeq \mathcal{O}(-r/s)$.*

Proof. All statements are straightforward to check using Definition 2.1.3. $\qquad\square$

Theorem 2.2.6 ([FF18, Proposition 8.2.3],[Ked05, Proposition 4.1.3]). *We have the following cohomological computations:*

(1) *$H^0(\mathcal{X}, \mathcal{O}(\lambda)) = 0$ if and only if $\lambda < 0$.*

(2) *$H^1(\mathcal{X}, \mathcal{O}(\lambda)) = 0$ if and only if $\lambda \geq 0$.*

It turns out that every vector bundle on $\mathcal{X}$ admits a direct sum decomposition into stable bundles, as stated in the following theorem:

Theorem 2.2.7 ([FF18, Théorème 8.2.10]). *Every vector bundle $\mathcal{V}$ on $\mathcal{X}$ admits a unique filtration*

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_l = \mathcal{V} \tag{2.1}$$

such that the successive quotients $\mathcal{V}_i/\mathcal{V}_{i-1}$ are semistable vector bundles with

$$\mu(\mathcal{V}_1/\mathcal{V}_0) > \mu(\mathcal{V}_2/\mathcal{V}_1) > \cdots > \mu(\mathcal{V}_l/\mathcal{V}_{l-1}).$$

Moreover, the filtration (2.1) splits into a direct sum decomposition

$$\mathcal{V} \simeq \bigoplus_{i=1}^{l} \mathcal{O}(\lambda_i)^{\oplus m_i} \tag{2.2}$$

where $\lambda_i = \mu(\mathcal{V}_i/\mathcal{V}_{i-1})$.

Remark. The existence and uniqueness of the filtration (2.1) is a formal consequence of Propositions 2.1.6 and 2.2.1, as noted by Kedlaya [Ked17, §3.4]. The existence of the direct sum decomposition 2.2 then follows by Proposition 2.2.4 and Theorem 2.2.6. We note that the recent work of Fargues-Scholze [FS21] gives a new proof of Proposition 2.2.4 (and thus Theorem 2.2.7) which conceptualize the original proof of Fargues-Fontaine [FF18].

Definition 2.2.8. Let $\mathcal{V}$ be a vector bundle on $\mathcal{X}$.

(1) We refer to the filtration (2.1) in Theorem 2.2.7 as the *Harder-Narasimhan (HN) filtration* of $\mathcal{V}$.

(2) We refer to the decomposition (2.2) in Theorem 2.2.7 as a *Harder-Narasimhan (HN) decomposition* of $\mathcal{V}$.

(3) We define the *Harder-Narasimhan (HN) polygon* of $\mathcal{V}$ as the upper convex hull of the points $(\mathrm{rk}(\mathcal{V}_i), \deg(\mathcal{V}_i))$ where $\mathcal{V}_i$'s are subbundles in the HN filtration of $\mathcal{V}$.

(4) We refer to the slopes of $\mathrm{HN}(\mathcal{V})$ as the *Harder-Narasimhan (HN) slopes* of $\mathcal{V}$, or simply the *slopes* of $\mathcal{V}$. These are precisely the numbers $\lambda_i = \mu(\mathcal{V}_i/\mathcal{V}_{i-1})$ in Theorem 2.2.7.

We can restate Theorem 2.2.7 in terms of HN polygons as follows:

Corollary 2.2.9. *Every vector bundle $\mathcal{V}$ on $\mathcal{X}$ is determined up to isomorphism by its HN polygon* $\mathrm{HN}(\mathcal{V})$.

Let us now introduce some notations that we will frequently use.

Definition 2.2.10. Let $\mathcal{V}$ be a vector bundle on $\mathcal{X}$ with Harder-Narasimhan filtration

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \ldots \subset \mathcal{V}_m = \mathcal{V}.$$

(1) We write $\mu_{\max}(\mathcal{V})$ (resp. $\mu_{\min}(\mathcal{V})$) for the maximum (resp. minimum) slope of $\mathcal{V}$.

(2) For every $\mu \in \mathbb{Q}$, we define $\mathcal{V}^{\geq \mu}$ (resp. $\mathcal{V}^{>\mu}$) to be the subbundle of $\mathcal{V}$ given by $\mathcal{V}_i$ for the largest value of i such that $\mu(\mathcal{V}_i/\mathcal{V}_{i-1}) \geq \mu$ (resp. such that $\mu(\mathcal{V}_i/\mathcal{V}_{i-1}) > \mu$). We also define $\mathcal{V}^{<\mu} := \mathcal{V}/\mathcal{V}^{\geq \mu}$ and $\mathcal{V}^{\leq \mu} := \mathcal{V}/\mathcal{V}^{>\mu}$.

Lemma 2.2.11. *Let $\mathcal{V}$ be a vector bundle on $\mathcal{X}$ with Harder-Narasimhan decomposition*

$$\mathcal{V} \simeq \bigoplus_{i=1}^{l} \mathcal{O}(\lambda_i)^{\oplus m_i}.$$

Then we have the following identifications:

$$\mathcal{V}^{\geq \mu} \simeq \bigoplus_{\lambda_i \geq \mu} \mathcal{O}(\lambda_i)^{\oplus m_i} \qquad and \qquad \mathcal{V}^{>\mu} \simeq \bigoplus_{\lambda_i > \mu} \mathcal{O}(\lambda_i)^{\oplus m_i},$$

$$\mathcal{V}^{\leq \mu} \simeq \bigoplus_{\lambda_i \leq \mu} \mathcal{O}(\lambda_i)^{\oplus m_i} \qquad and \qquad \mathcal{V}^{<\mu} \simeq \bigoplus_{\lambda_i < \mu} \mathcal{O}(\lambda_i)^{\oplus m_i}.$$

Proof. This is an immediate consequence of Definition 2.2.10. $\qquad\square$

Lemma 2.2.12. *Given a vector bundle $\mathcal{V}$ on $\mathcal{X}$, we have identities*

$$\mathrm{rk}(\mathcal{V}) = \mathrm{rk}(\mathcal{V}^{\vee}) \quad and \quad \deg(\mathcal{V}) = -\deg(\mathcal{V}^{\vee}).$$

More generally, for every $\mu \in \mathbb{Q}$ we have identities

$$\mathrm{rk}(\mathcal{V}^{\geq \mu}) = \mathrm{rk}((\mathcal{V}^{\vee})^{\leq -\mu}) \quad and \quad \deg(\mathcal{V}^{\geq \mu}) = -\deg((\mathcal{V}^{\vee})^{\leq -\mu}).$$

Proof. When $\mathcal{V}$ is stable, the first statement is an immediate consequence of Lemma 2.2.5. From this, we deduce the first statement for the general case using HN decomposition of $\mathcal{V}$. The second statement then follows from the first statement since we have $(\mathcal{V}^{\geq \mu})^{\vee} \simeq (\mathcal{V}^{\vee})^{\leq -\mu}$ by Lemma 2.2.5 and Lemma 2.2.11. $\qquad\square$

Lemma 2.2.13. *Given two vector bundles $\mathcal{V}$ and $\mathcal{W}$ on $\mathcal{X}$, we have*

$$\mathrm{Hom}(\mathcal{V}, \mathcal{W}) = 0 \qquad if\ and\ only\ if \qquad \mu_{min}(\mathcal{V}) > \mu_{max}(\mathcal{W}).$$

Proof. It suffices to consider the case when both $\mathcal{V}$ and $\mathcal{W}$ are stable; indeed, the general case will follow from this special case using the HN decompositions of $\mathcal{V}$ and $\mathcal{W}$. Let us now write $\mathcal{V} = \mathcal{O}(\lambda)$ and $\mathcal{W} = \mathcal{O}(\mu)$ for some $\lambda, \mu \in \mathbb{Q}$. Then using Lemma 2.2.5 we find

$$\mathrm{Hom}(\mathcal{V}, \mathcal{W}) \simeq H^0(\mathcal{X}, \mathcal{V}^{\vee} \otimes \mathcal{W}) \simeq H^0(\mathcal{X}, \mathcal{O}(\lambda)^{\vee} \otimes \mathcal{O}(\mu)) \simeq H^0(\mathcal{X}, \mathcal{O}(\mu - \lambda)^{\oplus n})$$

for some n. Since the condition $\mu_{\min}(\mathcal{V}) > \mu_{\max}(\mathcal{W})$ is equivalent to $\lambda > \mu$, the assertion follows from Theorem 2.2.6. $\qquad\square$

3. Moduli of bundle maps

3.1. Definitions and key properties.

In this section we define certain moduli spaces of bundle maps and collect some of their key properties. We refer the readers to [BFH+17, §3.3] for details.

Definition 3.1.1. Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on $\mathcal{X}$. We denote by $\mathrm{Perf}_{/\mathrm{Spa}(F)}$ the category of perfectoid spaces over $\mathrm{Spa}(F)$.

(1) For each $S \in \mathrm{Perf}_{/\mathrm{Spa}(F)}$, we write $\mathcal{E}_S$ and $\mathcal{F}_S$ for the pullbacks of $\mathcal{E}$ and $\mathcal{F}$ along the natural map $\mathcal{X}_S \to \mathcal{X}$.

(2) $\mathcal{H}^0(\mathcal{E})$ is the functor associating to $S \in \mathrm{Perf}_{/\mathrm{Spa}(F)}$ the set $H^0(\mathcal{X}_S, \mathcal{E}_S)$.

(3) $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})$ is the functor associating to $S \in \mathrm{Perf}_{/\mathrm{Spa}(F)}$ the set of $\mathcal{O}_{\mathcal{X}_S}$-module maps $\mathcal{E}_S \to \mathcal{F}_S$.

(4) $\mathcal{S}\mathrm{urj}(\mathcal{E}, \mathcal{F})$ is the subfunctor of $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})$ whose S-points parametrize surjective $\mathcal{O}_{\mathcal{X}_S}$-module maps $\mathcal{E}_S \twoheadrightarrow \mathcal{F}_S$.

(5) $\mathcal{I}\mathrm{nj}(\mathcal{E}, \mathcal{F})$ is the subfunctor of $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})$ whose S-points parametrize "fiberwise-injective" $\mathcal{O}_{\mathcal{X}_S}$-module maps. Precisely, this functor parametrizes $\mathcal{O}_{\mathcal{X}_S}$-module maps $\mathcal{E}_S \to \mathcal{F}_S$ whose pullback along the map $\mathcal{X}_{\overline{x}} \to \mathcal{X}_S$ for any geometric point $\overline{x} \to S$ gives an injective $\mathcal{O}_{\mathcal{X}_{\overline{x}}}$-module map.

Remark. The condition defining $\mathcal{I}\mathrm{nj}(\mathcal{E}, \mathcal{F})$ is much stronger than the condition that $\mathcal{E}_S \to \mathcal{F}_S$ is injective. We impose this enhanced condition to ensure that $\mathcal{I}\mathrm{nj}(\mathcal{E}, \mathcal{F})$ is a sheaf; indeed, the functor associating $S \in \mathrm{Perf}_{/\mathrm{Spa}(F)}$ to the set of injective $\mathcal{O}_{\mathcal{X}_S}$-module maps $\mathcal{E}_S \in \mathcal{F}_S$ is not even a presheaf. The fiberwise injectivity guarantees that a short exact sequence $0 \to \mathcal{E}_S \to \mathcal{F}_S \to \mathcal{F}_S/\mathcal{E}_S \to 0$ remains exact after tensoring with $\mathcal{O}_{\mathcal{X}_T}$ for any perfectoid space T over S as $\mathrm{Tor}_1(\mathcal{F}_S/\mathcal{E}_S)$ vanishes for all $S \in \mathrm{Perf}_{/\mathrm{Spa}(F)}$; however, we don't have this property without fiberwise injectivity.

Scholze's theory of diamonds in [Sch18] provides a framework for making sense of these functors as moduli spaces, thereby allowing us to study their geometric properties.

Proposition 3.1.2 ([BFH+17, Proposition 3.3.2, Proposition 3.3.5 and Proposition 3.3.6]). *Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on $\mathcal{X}$. The functors $\mathcal{H}^0(\mathcal{E})$, $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})$, $\mathcal{S}\mathrm{urj}(\mathcal{E}, \mathcal{F})$ and $\mathcal{I}\mathrm{nj}(\mathcal{E}, \mathcal{F})$ are all locally spatial and partially proper diamonds in the sense of Scholze [Sch18]. Moreover, their dimensions are given as follows:*

(1) The diamond $\mathcal{H}^0(\mathcal{E})$ is equidimensional of dimension $\deg(\mathcal{E})^{\geq 0}$.

(2) The diamond $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})$ is equidimensional of dimension $\deg(\mathcal{E}^\vee \otimes \mathcal{F})^{\geq 0}$.

(3) The diamonds $\mathcal{S}\mathrm{urj}(\mathcal{E}, \mathcal{F})$ and $\mathcal{I}\mathrm{nj}(\mathcal{E}, \mathcal{F})$ are either empty or equidimensional of dimension $\deg(\mathcal{E}^\vee \otimes \mathcal{F})^{\geq 0}$.

Remark. For a diamond Y, its dimension refers to the Krull dimension of its topological space $|Y|$. For locally spatial diamonds, this notion of dimension turns out to behave pleasantly well as their topological spaces are locally spectral.

We also note that, by the work of Le Bras [LB18], the functors $\mathcal{H}^0(\mathcal{E})$ and $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})$ are also Banach-Colmez spaces as defined by Colmez [Col02]. Moreover, their dimension as a diamond is equal to their *principal dimension* as a Banach-Colmez space.

Proposition 3.1.3 ([BFH+17, Theorem 3.3.11]). *Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on $\mathcal{X}$ satisfying the following properties:*

(i) There exists a nonzero bundle map $\mathcal{E} \to \mathcal{F}$.

(ii) For any $\mathcal{Q} \subsetneq \mathcal{F}$ which also occurs as a quotient of $\mathcal{E}$ we have an inequality

$$\deg(\mathcal{E}^\vee \otimes \mathcal{Q})^{\geq 0} + \deg(\mathcal{Q}^\vee \otimes \mathcal{F})^{\geq 0} < \deg(\mathcal{E}^\vee \otimes \mathcal{F})^{\geq 0} + \deg(\mathcal{Q}^\vee \otimes \mathcal{Q})^{\geq 0}.$$

Then there exists a surjective bundle map $\mathcal{E} \twoheadrightarrow \mathcal{F}$.

Remark. Let us provide an interpretation of the properties (i) and (ii) in line with Proposition 3.1.2. Let S be the set of isomorphism classes of subsheaves $\mathcal{Q} \subsetneq \mathcal{F}$ which also occur as a quotient of $\mathcal{E}$. For each $\mathcal{Q} \in S$, composition of bundle maps induces a natural map of diamonds

$$\mathcal{S}\mathrm{urj}(\mathcal{E}, \mathcal{Q}) \times_{\mathrm{Spd}\, F} \mathcal{I}\mathrm{nj}(\mathcal{Q}, \mathcal{F}) \to \mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F}).$$

Let us define $|\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})_\mathcal{Q}| \subset |\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})|$ to be the image of the induced map on topological spaces; indeed, $|\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})_\mathcal{Q}|$ is the underlying topological space of a subdiamond $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})_\mathcal{Q}$ of $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})$, which essentially parametrizes bundle maps $\mathcal{E} \to \mathcal{F}$ with image isomorphic to $\mathcal{Q}$ at all geometric points. Then $|\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})_\mathcal{Q}|$ is either empty or satisfies

$$\dim |\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})_\mathcal{Q}| = \deg(\mathcal{E}^\vee \otimes \mathcal{Q})^{\geq 0} + \deg(\mathcal{Q}^\vee \otimes \mathcal{F})^{\geq 0} - \deg(\mathcal{Q}^\vee \otimes \mathcal{Q})^{\geq 0},$$

as shown in [BFH+17, Lemma 3.3.10]. Hence the properties (i) and (ii) together imply that $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})$ admits an F-point and satisfies

$$\dim |\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})_\mathcal{Q}| < \dim |\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})| \quad \text{for } \mathcal{Q} \in S.$$

3.2. Dimension counting by Harder-Narasimhan polygons.

Our discussion in §3.1 suggests that we will have to understand quantities of the form $\deg(\mathcal{V}^\vee \otimes \mathcal{W})^{\geq 0}$ for fairly arbitrary vector bundles $\mathcal{V}$ and $\mathcal{W}$ on $\mathcal{X}$. In this subsection, we prove some useful lemmas for this purpose following the strategy developed in [BFH+17, §2.3].

Definition 3.2.1. Let v and w be arbitrary vectors in $\mathbb{R}^2$.

(1) We denote by v_x (resp. v_y) the x-coordinate (resp. y-coordinate) of v.

(2) If $v_x \neq 0$, we write $\mu(v) := v_y/v_x$ for the slope of v.

(3) If both v and w have nonzero x-coordinates, we will often write $v \prec w$ (resp. $v \preceq w$) in lieu of $\mu(v) < \mu(w)$ (resp. $\mu(v) \leq \mu(w)$).

(4) We write $v \times w$ for the (two-dimensional) cross product of v and w, regarded as a scalar by the formula $v \times w = v_x w_y - v_y w_x$.

In this paper, we are exclusively interested in vectors with a positive x-coordinate. For such vectors, we can characterize the relations $\preceq$ and $\prec$ in terms of the two dimensional cross product as follows:

Lemma 3.2.2. *Let v and w be vectors in $\mathbb{R}^2$ with $v_x, w_x > 0$. Then we have $v \prec w$ (resp. $v \preceq w$) if and only if $v \times w > 0$ (resp. $v \times w \geq 0$).*

Proof. This is straightforward to check using Definition 3.2.1. $\qquad\qquad\square$

We will make use of Lemma 3.2.2 by expressing HN polygons in terms of vectors.

Definition 3.2.3. Let $\mathcal{V}$ be a vector bundle on $\mathcal{X}$ with Harder-Narasimhan decomposition

$$\mathcal{V} = \bigoplus_{i=1}^{l} \mathcal{O}(\lambda_i)^{m_i}$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_l$. We define the *HN vectors of $\mathcal{V}$* by

$$\overrightarrow{\mathrm{HN}}(\mathcal{V}) := (v_i)_{1 \leq i \leq l}$$

where v_i is the vector representing the i-th line segment in $\mathrm{HN}(\mathcal{V})$; more precisely, writing $\lambda_i = r_i/s_i$ in lowest terms with $s_i > 0$, we set $v_i := (m_i s_i, m_i r_i)$.

The following simple lemma is pivotal to our discussion in this section.

Lemma 3.2.4. *Let $\mathcal{V}$ and $\mathcal{W}$ be vector bundles on $\mathcal{X}$ with $\overrightarrow{\mathrm{HN}}(\mathcal{V}) = (v_i)$ and $\overrightarrow{\mathrm{HN}}(\mathcal{W}) = (w_j)$. Then we have*

$$\deg(\mathcal{V}^\vee \otimes \mathcal{W}) = \sum_{i,j} v_i \times w_j \qquad \text{and} \qquad \deg(\mathcal{V}^\vee \otimes \mathcal{W})^{\geq 0} = \sum_{v_i \preceq w_j} v_i \times w_j$$

Proof. When $\mathcal{V}$ and $\mathcal{W}$ are both semistable, we quickly verify both identities in (3.2.4) using Lemma 2.2.5 and Lemma 3.2.2. Then we deduce the general case using the HN decompositions of $\mathcal{V}$ and $\mathcal{W}$. $\qquad\square$

Corollary 3.2.5. *For arbitrary vector bundles $\mathcal{V}$ and $\mathcal{W}$ on $\mathcal{X}$, we have*

$$\dim \mathcal{H}om(\mathcal{V}, \mathcal{W}) = 0 \qquad \text{if and only if} \qquad \mu_{min}(\mathcal{V}) \geq \mu_{max}(\mathcal{W}).$$

Proof. This is an immediate consequence of Proposition 3.1.2 and Lemma 3.2.4 $\qquad\square$

Remark. This is not a consequence of Lemma 2.2.13; in fact, When $\mu_{\min}(\mathcal{V}) = \mu_{\max}(\mathcal{W})$, Lemma 2.2.13 and Corollary 3.2.5 respectively yield $\mathrm{Hom}(\mathcal{V}, \mathcal{W}) \neq 0$ and $\dim \mathcal{H}om(\mathcal{V}, \mathcal{W}) = 0$.

Definition 3.2.6. Given a vector bundle $\mathcal{V}$ on $\mathcal{X}$, we write $\mathcal{V}(\lambda) := \mathcal{V} \otimes \mathcal{O}(\lambda)$ for any $\lambda \in \mathbb{Q}$.

Lemma 3.2.7. *Let $\mathcal{V}$ and $\mathcal{W}$ be vector bundles on $\mathcal{X}$. For any $\lambda \in \mathbb{Q}$ we have*

$$\deg(\mathcal{V}(\lambda)^\vee \otimes \mathcal{W}(\lambda))^{\geq 0} = \mathrm{rk}(\mathcal{O}(\lambda))^2 \cdot \deg(\mathcal{V}^\vee \otimes \mathcal{W})^{\geq 0}.$$

Proof. By Lemma 2.2.5 we find

$$\mathcal{V}(\lambda)^\vee \otimes \mathcal{W}(\lambda) \simeq \mathcal{V}^\vee \otimes \mathcal{W} \otimes \mathcal{O}(\lambda) \otimes \mathcal{O}(\lambda)^\vee \simeq \mathcal{V}^\vee \otimes \mathcal{W} \otimes \mathcal{O}^{\mathrm{rk}(\mathcal{O}(\lambda))^2}$$

and consequently obtain the desired assertion. $\qquad\square$

Lemma 3.2.8. *Let $\mathcal{V}$ and $\mathcal{W}$ be vector bundles on $\mathcal{X}$. Take $\tilde{\mathcal{V}}$ and $\tilde{\mathcal{W}}$ to be vector bundles on $\mathcal{X}$ whose HN polygons are obtained by vertically stretching $\mathrm{HN}(\mathcal{V})$ and $\mathrm{HN}(\mathcal{W})$ by a positive integer factor C. Then we have*

$$\deg(\tilde{\mathcal{V}}^\vee \otimes \tilde{\mathcal{W}})^{\geq 0} = C \cdot \deg(\mathcal{V}^\vee \otimes \mathcal{W})^{\geq 0}.$$

Proof. Let us consider the HN vectors

$$\overrightarrow{\mathrm{HN}}(\mathcal{V}) = (v_i), \qquad \overrightarrow{\mathrm{HN}}(\mathcal{W}) = (w_j), \qquad \overrightarrow{\mathrm{HN}}(\tilde{\mathcal{V}}) = (\tilde{v}_i), \qquad \overrightarrow{\mathrm{HN}}(\tilde{\mathcal{W}}) = (\tilde{w}_j).$$

By construction, we have the following relations between these HN vectors.

$$\tilde{v}_{i,x} = v_{i,x}, \qquad \tilde{w}_{j,x} = w_{j,x}, \qquad \tilde{v}_{i,y} = Cv_{i,y}, \qquad \tilde{w}_{j,y} = Cw_{j,y}.$$

Now for each i and j we have

$$\tilde{v}_i \times \tilde{w}_j = \tilde{v}_{i,x}\tilde{w}_{j,y} - \tilde{v}_{i,y}\tilde{w}_{j,x} = C(v_{i,x}w_{j,y} - v_{i,y}w_{j,x}) = C \cdot (v_i \times w_j) \qquad (3.1)$$

We thus use Lemma 3.2.2, Lemma 3.2.4 and (3.1) to find

$$\deg(\tilde{\mathcal{V}}^\vee \otimes \tilde{\mathcal{W}})^{\geq 0} = \sum_{\tilde{v}_i \preceq \tilde{w}_j} \tilde{v}_i \times \tilde{w}_j = \sum_{v_i \preceq w_j} C(v_i \times w_j)$$

$$= C \sum_{v_i \preceq w_j} v_i \times w_j = C \cdot \deg(\mathcal{V}^\vee \otimes \mathcal{W})^{\geq 0},$$

completing the proof. $\qquad\square$

Remark. The equation (3.1) represents the fact that the vertical stretch by a factor C scales the area of an arbitrary parallelogram by the same factor.

4. Classification of quotient bundles

4.1. **The main statement and its consequences.**

Let us state our main theorem, which gives a complete classification of all quotient bundles of a given vector bundle on $\mathcal{X}$.

Theorem 4.1.1. *Let $\mathcal{E}$ be a vector bundle on $\mathcal{X}$. Then a vector bundle $\mathcal{F}$ on $\mathcal{X}$ is a quotient bundle of $\mathcal{E}$ if and only if the following equivalent conditions are satisfied:*

(i) For every $\mu \in \mathbb{Q}$, we have $\mathrm{rk}(\mathcal{E}^{\leq \mu}) \geq \mathrm{rk}(\mathcal{F}^{\leq \mu})$ with equality if and only if $\mathcal{E}^{\leq \mu}$ and $\mathcal{F}^{\leq \mu}$ are isomorphic.

(ii) If we align $\mathrm{HN}(\mathcal{E})$ and $\mathrm{HN}(\mathcal{F})$ so that their right endpoints lie at the origin, then for each $i = 1, \cdots, \mathrm{rk}(\mathcal{F})$, the slope of $\mathrm{HN}(\mathcal{F})$ on $[-i, -i+1]$ is greater than or equal to the slope of $\mathrm{HN}(\mathcal{E})$ on $[-i-1, -i]$ unless $\mathrm{HN}(\mathcal{E})$ and $\mathrm{HN}(\mathcal{F})$ agree on $[-i, 0]$.
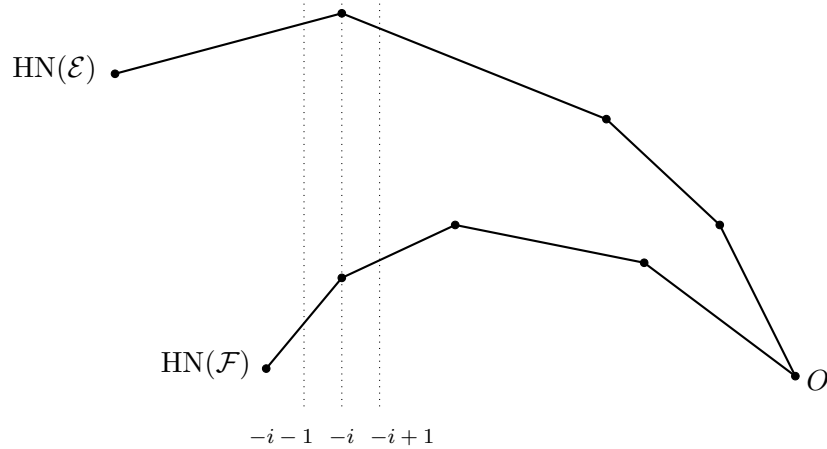

FIGURE 3. Illustration of the condition (ii) in Theorem 4.1.1.

We will discuss our proof of Theorem 4.1.1 in the subsequent subsections. In this subsection we explain some classification results as consequences of Theorem 4.1.1.

Our first corollary of Theorem 4.1.1 dualizes the statement of Theorem 4.1.1 to classify *almost* all subsheaves of a given vector bundle on $\mathcal{X}$.

Corollary 4.1.2. *Let $\mathcal{E}$ be a vector bundle on $\mathcal{X}$. Then a vector bundle $\mathcal{D}$ on $\mathcal{X}$ is a subsheaf of $\mathcal{E}$ if the following equivalent conditions are satisfied:*

(i) For every $\mu \in \mathbb{Q}$, we have $\mathrm{rk}(\mathcal{E}^{\geq \mu}) \geq \mathrm{rk}(\mathcal{D}^{\geq \mu})$ with equality if and only if $\mathcal{E}^{\geq \mu}$ and $\mathcal{D}^{\geq \mu}$ are isomorphic.

(ii) If we align $\mathrm{HN}(\mathcal{D})$ and $\mathrm{HN}(\mathcal{E})$ so that their left endpoints lie at the origin, then for each $i = 1, \cdots, \mathrm{rk}(\mathcal{D})$, the slope of $\mathrm{HN}(\mathcal{D})$ on $[i-1, i]$ is less than or equal to the slope of $\mathrm{HN}(\mathcal{E})$ on $[i, i+1]$ unless $\mathrm{HN}(\mathcal{D})$ and $\mathrm{HN}(\mathcal{E})$ agree on $[0, i]$.
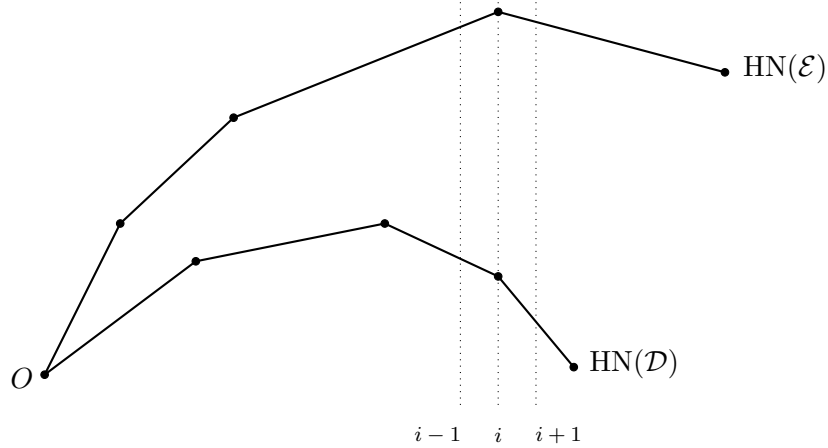
FIGURE 4. Illustration of the condition (ii) in Corollary 4.1.2.

Proof. Let us first to show that there exists an injective bundle map $\mathcal{D} \hookrightarrow \mathcal{E}$ if $\mathcal{D}$ satisfies the condition (i). By means of dualizing, it suffices to show that there exists a surjective bundle map $\mathcal{E}^{\vee} \twoheadrightarrow \mathcal{D}^{\vee}$, or equivalently that $\mathcal{D}^{\vee}$ is a quotient bundle of $\mathcal{E}^{\vee}$. This follows from Theorem 4.1.1 since by Lemma 2.2.12 we can rewrite the condition (i) as follows:

(i)$^{\vee}$ For every $\mu \in \mathbb{Q}$, we haev $\mathrm{rk}((\mathcal{E}^{\vee})^{\leq -\mu}) \geq \mathrm{rk}((\mathcal{D}^{\vee})^{\leq -\mu})$ with equality if and only if $(\mathcal{E}^{\vee})^{\leq -\mu}$ and $(\mathcal{D}^{\vee})^{\leq -\mu}$ are isomorphic.

It remains to verify that the conditions (i) and (ii) are equivalent. By reflecting the HN polygons $\mathrm{HN}(\mathcal{D})$ and $\mathrm{HN}(\mathcal{E})$ about the y-axis, we obtain the HN polygons $\mathrm{HN}(\mathcal{D}^{\vee})$ and $\mathrm{HN}(\mathcal{E}^{\vee})$ with their right endpoints at the origin. We thus find that the condition (ii) is equivalent to the following condition:

(ii)$^{\vee}$ For each $i = 1, \cdots, \mathrm{rk}(\mathcal{D}^{\vee})$, the slope of $\mathrm{HN}(\mathcal{D}^{\vee})$ on the interval $[-i, -i+1]$ is greater than or equal to the slope of $\mathrm{HN}(\mathcal{E}^{\vee})$ on $[-i-1, -i]$ unless $\mathrm{HN}(\mathcal{D}^{\vee})$ and $\mathrm{HN}(\mathcal{E}^{\vee})$ agree on $[-i, 0]$.

Moreover, the conditions (i)$^{\vee}$ and (ii)$^{\vee}$ are equivalent by Theorem 4.1.1. Hence we obtain the equivalence between the conditions (i) and (ii) as desired, thereby completing the proof. $\square$

Remark. Corollary 4.1.2 does not give a complete classification of subsheaves since the equality part of the condition (i) is not necessary. The main underlying issue is that the cokernel of an injective bundle map is not necessarily a vector bundle. It turns out that the inequality in the condition (i) alone gives a complete classification of subsheaves of $\mathcal{E}$, as proved by the author in the sequel paper [Hon21].

As another consequence of Theorem 4.1.1, we have a complete classification of finitely globally generated vector bundles on $\mathcal{X}$.

Corollary 4.1.3. *A vector bundle $\mathcal{E}$ on $\mathcal{X}$ is generated by n global sections if and only if the following conditions are satisfied:*

(i) $\mathrm{HN}(\mathcal{E})$ *has only nonnegative slopes, i.e., $\mathcal{E}^{<0} = 0$.*
(ii) *We have $\mathrm{rk}(\mathcal{E}) \leq n$ with equality if and only if $\mathcal{E}$ is trivial (i.e., isomorphic to $\mathcal{O}^{\oplus n}$).*

Proof. A vector bundle $\mathcal{E}$ on $\mathcal{X}$ is generated by n global sections if and only if there is a surjective bundle map $\mathcal{O}^{\oplus n} \twoheadrightarrow \mathcal{E}$, which amounts to saying that $\mathcal{E}$ is a quotient bundle of $\mathcal{O}^{\oplus n}$. By Theorem 4.1.1, this precisely means that we have $\mathrm{rk}((\mathcal{O}^{\oplus n})^{\leq \mu}) \geq \mathrm{rk}(\mathcal{E}^{\leq \mu})$ for every

$\mu \in \mathbb{Q}$ with equality if and only if $(\mathcal{O}^{\oplus n})^{\leq \mu}$ and $\mathcal{E}^{\leq \mu}$ are isomorphic. Since $(\mathcal{O}^{\oplus n})^{\leq \mu}$ is equal to $\mathcal{O}^{\oplus n}$ for $\mu \geq 0$ and zero for $\mu < 0$, we find that $\mathcal{E}$ on $\mathcal{X}$ is generated by n global sections if and only if the following conditions are satisfied:

(i)' For every $\mu < 0$, we have $\mathcal{E}^{\leq \mu} = 0$.

(ii)' For every $\mu \geq 0$, we have $n \geq \mathrm{rk}(\mathcal{E}^{\leq \mu})$ with equality if and only if $\mathcal{E}^{\leq \mu}$ is trivial.

The conditions (i) and (i)' are evidently equivalent. In addition, the condition (ii)' implies the condition (ii) as we have $\mathcal{E}^{\leq \mu_{\max}(\mathcal{E})} = \mathcal{E}$. Hence it remains to prove that the condition (ii) implies the condition (ii)'.

The inequality in the condition (ii) implies the inequality in the condition (ii)' as we have $\mathrm{rk}(\mathcal{E}^{\leq \mu}) \leq \mathrm{rk}(\mathcal{E})$ for every $\mu \in \mathbb{Q}$. Therefore we only need to consider the equality part in the condition (ii)'. Let us now suppose that we have $\mathrm{rk}(\mathcal{E}^{\leq \mu}) = n$ for some $\mu \in \mathbb{Q}$. The condition (ii) yields $\mathrm{rk}(\mathcal{E}^{\leq \mu}) \leq \mathrm{rk}(\mathcal{E}) \leq n$, which in turn implies $\mathrm{rk}(\mathcal{E}^{\leq \mu}) = \mathrm{rk}(\mathcal{E}) = n$. The first equality means that $\mathcal{E}^{\leq \mu}$ is isomorphic to $\mathcal{E}$, while the second equality implies that $\mathcal{E}$ is trivial by the condition (ii). We thus deduce that $\mathcal{E}^{\leq \mu}$ is trivial as desired. $\qquad \square$

4.2. Slopewise dominance of vector bundles.

The rest of this paper will be devoted to proving Theorem 4.1.1. In this section, we introduce and study the notion of *slopewise dominance* which will be crucial for our proof.

Definition 4.2.1. Let $\mathcal{V}$ and $\mathcal{W}$ be vector bundles on $\mathcal{X}$. Assume that their HN polygons $\mathrm{HN}(\mathcal{V})$ and $\mathrm{HN}(\mathcal{W})$ are aligned as usual so that their **left endpoints** lie at the origin. We say that $\mathcal{V}$ *slopewise dominates* $\mathcal{W}$ if for $i = 1, \cdots, \mathrm{rk}(\mathcal{W})$, the slope of $\mathrm{HN}(\mathcal{W})$ on the interval $[i-1, i]$ is less than or equal to the slope of $\mathrm{HN}(\mathcal{V})$ on this interval.
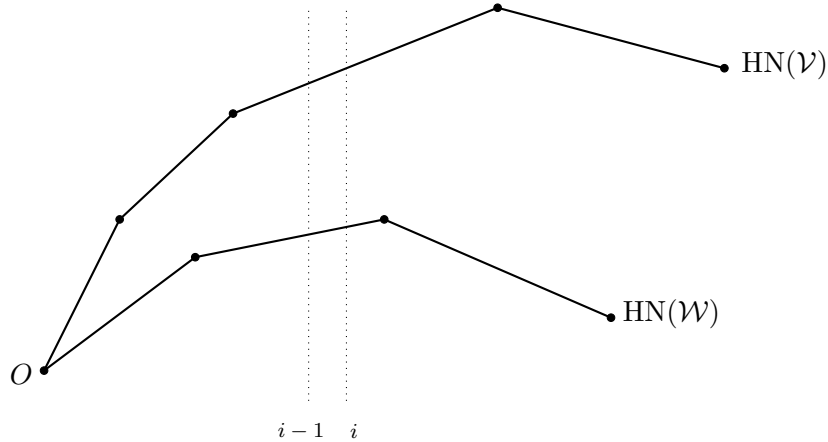

FIGURE 5. Illustration of Definition 4.2.1

Remark. Slopewise dominance of $\mathcal{V}$ on $\mathcal{W}$ implies that $\mathrm{rk}(\mathcal{V}) \geq \mathrm{rk}(\mathcal{W})$.

The notion of slopewise dominance gives us a characterization of the condition (i) in Theorem 4.1.1 (and the condition (i) in Corollary 4.1.2).

Lemma 4.2.2 (cf. [BFH$^+$17] Corollary 4.2.2). *Let $\mathcal{V}$ and $\mathcal{W}$ be vector bundles on $\mathcal{X}$.*

(1) $\mathcal{V}$ slopewise dominates $\mathcal{W}$ if and only if $\mathrm{rk}(\mathcal{V}^{\geq \mu}) \geq \mathrm{rk}(\mathcal{W}^{\geq \mu})$ for every $\mu \in \mathbb{Q}$.

(2) $\mathcal{V}^{\vee}$ slopewise dominates $\mathcal{W}^{\vee}$ if and only if $\mathrm{rk}(\mathcal{V}^{\leq \mu}) \geq \mathrm{rk}(\mathcal{W}^{\leq \mu})$ for every $\mu \in \mathbb{Q}$.

Proof. We first note that the statement (2) follows from the statement (1) as a dual statement. In fact, by Lemma 2.2.12 we can rewrite the inequality $\mathrm{rk}(\mathcal{V}^{\leq \mu}) \geq \mathrm{rk}(\mathcal{W}^{\leq \mu})$ as $\mathrm{rk}((\mathcal{V}^{\vee})^{\geq -\mu}) \geq \mathrm{rk}((\mathcal{W}^{\vee})^{\geq -\mu})$. Hence we only need to prove the statement (1).

We now assume the inequality $\mathrm{rk}(\mathcal{V}^{\geq \mu}) \geq \mathrm{rk}(\mathcal{W}^{\geq \mu})$ for every $\mu \in \mathbb{Q}$ and assert that $\mathcal{V}$ slopewise dominates $\mathcal{W}$. For each $i = 1, \cdots, \mathrm{rk}(\mathcal{W})$, we let μ_i be the slope of $\mathrm{HN}(\mathcal{W})$ on the interval $[i-1, i]$. If some μ_i is greater than the slope of $\mathrm{HN}(\mathcal{V})$ on $[i-1, i]$, convexity of HN polygons yields $\mathrm{rk}(\mathcal{V}^{\geq \mu_i}) < i \leq \mathrm{rk}(\mathcal{W}^{\geq \mu_i})$ which contradicts the inequality we assumed. We thus deduce that $\mathcal{V}$ slopewise dominates $\mathcal{W}$ as desired.

Conversely, we claim the inequality $\mathrm{rk}(\mathcal{V}^{\geq \mu}) \geq \mathrm{rk}(\mathcal{W}^{\geq \mu})$ for every $\mu \in \mathbb{Q}$ assuming that $\mathcal{V}$ slopewise dominates $\mathcal{W}$. Suppose for contradiction that $\mathrm{rk}(\mathcal{V}^{\geq \mu}) < \mathrm{rk}(\mathcal{W}^{\geq \mu})$ for some μ. Then for $i = \mathrm{rk}(\mathcal{W}^{\geq \mu})$, the slope of $\mathrm{HN}(\mathcal{W})$ on the interval $[i-1, i]$ is at least μ whereas the slope of $\mathrm{HN}(\mathcal{V})$ on this interval is less than μ. In particular, the slope of $\mathrm{HN}(\mathcal{W})$ on $[i-1, i]$ is greater than the slope of $\mathrm{HN}(\mathcal{V})$ on this interval, yielding a desired contradiction. □

The notion of slopewise dominance also yields an interesting inequality on degrees which will be useful to us.

Lemma 4.2.3. *Let $\mathcal{V}$ and $\mathcal{W}$ be vector bundles on $\mathcal{X}$ such that $\mathcal{V}$ slopewise dominates $\mathcal{W}$. We have an inequality*

$$\deg(\mathcal{V})^{\geq 0} \geq \deg(\mathcal{W})^{\geq 0}.$$

Proof. We align $\mathrm{HN}(\mathcal{V})$ and $\mathrm{HN}(\mathcal{W})$ as in Definition 4.2.1 so that their left endpoints lie at the origin.
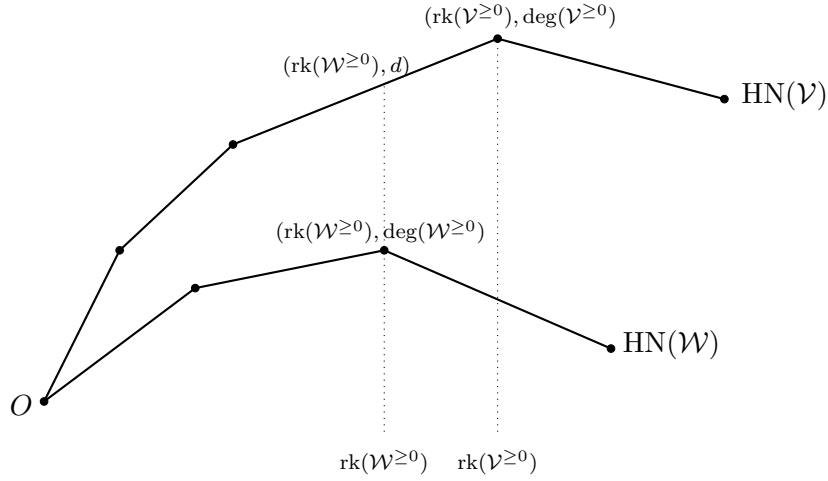

FIGURE 6. Comparison of nonnegative parts using slopewise dominance

We denote by d the y-value of $\mathrm{HN}(\mathcal{V})$ at $\mathrm{rk}(\mathcal{W}^{\geq 0})$. Since $\mathrm{HN}(\mathcal{V})$ lies above $\mathrm{HN}(\mathcal{W})$ by slopewise dominance, we compare the y-values of $\mathrm{HN}(\mathcal{V})$ and $\mathrm{HN}(\mathcal{W})$ at $\mathrm{rk}(\mathcal{W}^{\geq 0})$ and obtain

$$\deg(\mathcal{W}^{\geq 0}) \leq d.$$

Moreover, we observe that the y-value of $\mathrm{HN}(\mathcal{V})$ increases on the interval $[0, \mathrm{rk}(\mathcal{V}^{\geq 0})]$. Since $\mathrm{rk}(\mathcal{W}^{\geq 0}) \leq \mathrm{rk}(\mathcal{V}^{\geq 0})$ by Lemma 4.2.2, we compare the y-values of $\mathrm{HN}(\mathcal{V})$ at $\mathrm{rk}(\mathcal{W}^{\geq 0})$ and $\mathrm{rk}(\mathcal{V}^{\geq 0})$ to find

$$d \leq \deg(\mathcal{V}^{\geq 0}).$$

We thus combine the two inequalities to obtain the desired inequality. □

Remark. By the same argument we can prove the inequality $\deg(\mathcal{V})^{\geq\mu} \geq \deg(\mathcal{W})^{\geq\mu}$ for all $\mu > 0$. However, this inequality does not necessarily hold for $\mu < 0$. In fact, when μ is sufficiently small the inequality is equivalent to $\deg(\mathcal{V}) \geq \deg(\mathcal{W})$, which doesn't necessarily hold as shown by $\mathcal{V} = \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(-2)$ and $\mathcal{W} = \mathcal{O}(1/2)$.

A number of our reduction arguments will use the following decomposition lemma regarding slopewise dominance.

Lemma 4.2.4. *Let $\mathcal{V}$ and $\mathcal{W}$ be vector bundles on $\mathcal{X}$ such that $\mathcal{V}$ slopewise dominates $\mathcal{W}$. Then we have decompositions*

$$\mathcal{V} \simeq \mathcal{U} \oplus \mathcal{V}' \qquad and \qquad \mathcal{W} \simeq \mathcal{U} \oplus \mathcal{W}' \tag{4.1}$$

satisfying the following properties:

(i) $\mathcal{V}'$ *slopewise dominates* $\mathcal{W}'$.
(ii) *If* $\mathcal{W}' \neq 0$, *we have* $\mu_{max}(\mathcal{V}') > \mu_{max}(\mathcal{W}')$.
(iii) *If* $\mathcal{U} \neq 0$ *and* $\mathcal{W}' \neq 0$, *we have* $\mu_{min}(\mathcal{U}) \geq \mu_{max}(\mathcal{V}') > \mu_{max}(\mathcal{W}')$.

Proof. We assume that $\mathrm{HN}(\mathcal{V})$ and $\mathrm{HN}(\mathcal{W})$ are aligned as in Definition 4.2.1. For each $x \in [0, \mathrm{rk}(\mathcal{W})]$, we define $d(x)$ to be the vertical distance between $\mathrm{HN}(\mathcal{V})$ and $\mathrm{HN}(\mathcal{W})$ at x. Note that $d(x)$ is nonnegative and increasing by slopewise dominance of $\mathcal{V}$ on $\mathcal{W}$.

Let us take the maximum r with $d(r) = 0$. The interval $[0, r]$ corresponds to the common part of $\mathrm{HN}(\mathcal{V})$ and $\mathrm{HN}(\mathcal{W})$. Moreover, unless we have $r = \mathrm{rk}(\mathcal{W})$ the polygon $\mathrm{HN}(\mathcal{W})$ changes its slope at r so that $d(x)$ becomes positive after this point. Hence r must be an integer.

We take $\mathcal{U}$ to be the vector bundle on $\mathcal{X}$ whose HN polygon is given by the common part of $\mathrm{HN}(\mathcal{V})$ and $\mathrm{HN}(\mathcal{W})$, as illustrated by the red polygon in the figure below. We also take $\mathcal{V}'$ and $\mathcal{W}'$ to be vector bundles on $\mathcal{X}$ whose HN polygons are given by the complement subpolygons of $\mathrm{HN}(\mathcal{V})$ and $\mathrm{HN}(\mathcal{W})$, as illustrated by the blue and green polygons in the figure below. Note that these definitions are valid since r is an integer.

FIGURE 7. Illustration of the decompositions (4.1) in terms of HN polygons.

It remains to check the desired properties for $\mathcal{U}$, $\mathcal{V}'$ and $\mathcal{W}'$. By construction we have decompositions

$$\mathcal{V} \simeq \mathcal{U} \oplus \mathcal{V}' \qquad and \qquad \mathcal{W} \simeq \mathcal{U} \oplus \mathcal{W}'.$$

Moreover, we obtain slopewise dominance of $\mathcal{V}'$ on $\mathcal{W}'$ from slopewise dominance of $\mathcal{V}$ on $\mathcal{W}$. If $\mathcal{W}' \neq 0$, we have $r < \mathrm{rk}(\mathcal{W})$ and therefore deduce the strict inequality $\mu_{\max}(\mathcal{V}') >$

$\mu_{\max}(\mathcal{W}')$ from the fact that $d(x)$ becomes positive after r. If $\mathcal{U} \neq 0$ and $\mathcal{W}' \neq 0$, we also have $\mu_{\min}(\mathcal{U}) \geq \mu_{\max}(\mathcal{V}')$ by convexity of $\mathrm{HN}(\mathcal{V})$, thereby obtaining a combined inequality $\mu_{\min}(\mathcal{U}) \geq \mu_{\max}(\mathcal{V}') > \mu_{\max}(\mathcal{W}')$. $\qquad\square$

We will also need the following duality of slopewise dominance for vector bundles of equal ranks.

Lemma 4.2.5. *Let $\mathcal{V}$ and $\mathcal{W}$ be vector bundles on $\mathcal{X}$ with $\mathrm{rk}(\mathcal{V}) = \mathrm{rk}(\mathcal{W})$. Then $\mathcal{V}$ slopewise dominates $\mathcal{W}$ if and only if $\mathcal{W}^{\vee}$ slopewise dominates $\mathcal{V}^{\vee}$.*

Proof. We align the polygons $\mathrm{HN}(\mathcal{V})$ and $\mathrm{HN}(\mathcal{W})$ so that their left points lie at the origin. By reflecting $\mathrm{HN}(\mathcal{V})$ and $\mathrm{HN}(\mathcal{W})$ about the y-axis, we obtain the polygons $\mathrm{HN}(\mathcal{V}^{\vee})$ and $\mathrm{HN}(\mathcal{W}^{\vee})$ with their right points at the origin.
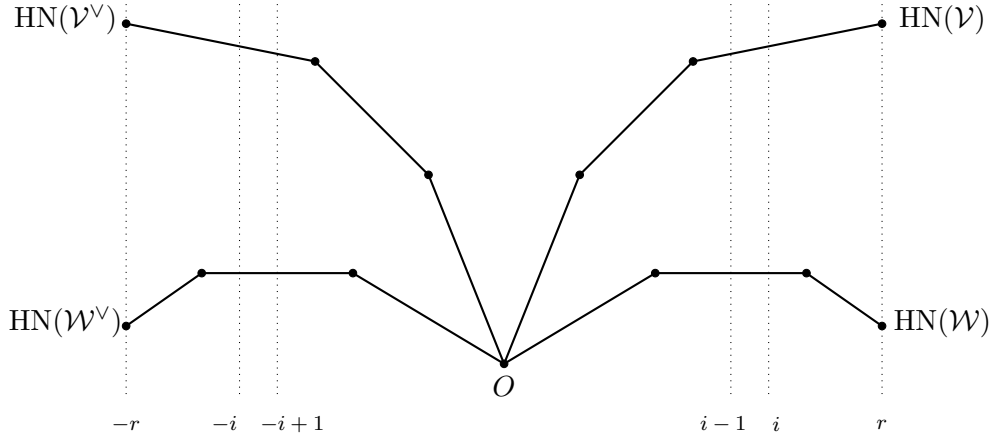

FIGURE 8. Duality of slopewise dominance for vector bundles of equal ranks

Note that $\mathcal{V}, \mathcal{W}, \mathcal{V}^{\vee}$ and $\mathcal{W}^{\vee}$ all have equal rank by our assumption $\mathrm{rk}(\mathcal{V}) = \mathrm{rk}(\mathcal{W})$. We let r denote this common rank of $\mathcal{V}, \mathcal{W}, \mathcal{V}^{\vee}$ and $\mathcal{W}^{\vee}$.

With this setup, we can establish our assertion by proving equivalence of the following statements:

(a) $\mathcal{W}^{\vee}$ slopewise dominates $\mathcal{V}^{\vee}$.
(b) For each $i = 1, 2, \cdots, r$, the slope of $\mathrm{HN}(\mathcal{V}^{\vee})$ on the interval $[-i, -i+1]$ is less than or equal to the slope of $\mathrm{HN}(\mathcal{W}^{\vee})$ on this interval.
(c) For each $i = 1, 2, \cdots, r$, the slope of $\mathrm{HN}(\mathcal{W})$ on the interval $[i-1, i]$ is less than or equal to the slope of $\mathrm{HN}(\mathcal{V})$ on this interval.
(d) $\mathcal{V}$ slopewise dominates $\mathcal{W}$.

Equivalence between (a) and (b) is a consequence of the fact that the left points of $\mathrm{HN}(\mathcal{V}^{\vee})$ and $\mathrm{HN}(\mathcal{W}^{\vee})$ have the same x-values of $-r$ in our alignment; in fact, to compare the slopes as per Definition 4.2.1 we only have to align the left points at the same x-values. Equivalence between (b) and (c) is immediate since the slope of $\mathrm{HN}(\mathcal{V}^{\vee})$ (resp. $\mathrm{HN}(\mathcal{W}^{\vee})$) on $[-i, -i+1]$ is the negative of the slope of $\mathrm{HN}(\mathcal{V})$ (resp. $(\mathrm{HN}(\mathcal{W}))$ on $[i-1, i]$. Equivalence between (c) and (d) is evident by Definition 4.2.1. $\qquad\square$

4.3. **Formulation of the key inequality.**

Our primary goal in this subsection is to reduce the statement of Theorem 4.1.1 to a quantitative statement which we can prove using the results from §3.2.

We begin by establishing the equivalence of the two characterizations of quotient bundles in the statement of Theorem 4.1.1.

Proposition 4.3.1. *For arbitrary vector bundles $\mathcal{E}$ and $\mathcal{F}$ on $\mathcal{X}$, the conditions (i) and (ii) in Theorem 4.1.1 are equivalent.*

Proof. As in the statement of Theorem 4.1.1, we align $\mathrm{HN}(\mathcal{E})$ and $\mathrm{HN}(\mathcal{F})$ so that their right endpoints lie at the origin. By reflecting $\mathrm{HN}(\mathcal{E})$ and $\mathrm{HN}(\mathcal{F})$ about the y-axis, we obtain the HN polygons $\mathrm{HN}(\mathcal{E}^{\vee})$ and $\mathrm{HN}(\mathcal{F}^{\vee})$ with their left endpoints at the origin.

Let us now assert that the condition (i) implies the condition (ii). Suppose that for some positive integer $i \leq \mathrm{rk}(\mathcal{F})$ the slope of $\mathrm{HN}(\mathcal{F})$ on $[-i, -i+1]$ is less than the slope of $\mathrm{HN}(\mathcal{E})$ on $[-i-1, -i]$. By Lemma 4.2.2 the inequality in the condition (i) is equivalent to slopewise dominance of $\mathcal{E}^{\vee}$ on $\mathcal{F}^{\vee}$. Then our observation in the preceding paragraph shows that the slope of $\mathrm{HN}(\mathcal{F})$ on $[-i, -i+1]$ must be greater than or equal to the slope of $\mathrm{HN}(\mathcal{E})$ on $[-i, -i+1]$. Therefore our assumption implies that $\mathrm{HN}(\mathcal{E})$ has a breakpoint at $-i$. Taking μ to be the slope of $\mathrm{HN}(\mathcal{F})$ on $[-i, -i+1]$, we find

$$\mathrm{rk}(\mathcal{E}^{\leq \mu}) \leq i = \mathrm{rk}(\mathcal{F}^{\leq \mu}).$$

Now the condition (i) yields $\mathrm{rk}(\mathcal{E}^{\leq \mu}) = \mathrm{rk}(\mathcal{F}^{\leq \mu}) = j$ and $\mathcal{E}^{\leq \mu} \simeq \mathcal{F}^{\leq \mu}$, thereby implying that $\mathrm{HN}(\mathcal{E})$ and $\mathrm{HN}(\mathcal{F})$ must agree on $[-i, 0]$ as the condition (ii) states.

It remains to prove that the condition (ii) implies the condition (i). By our observation in the first paragraph and convexity of HN polygons, the condition (ii) implies that $\mathcal{E}^{\vee}$ slopewise dominates $\mathcal{F}^{\vee}$, and consequently yields the inequality in the condition (i) by Lemma 4.2.2. Let us now suppose that we have $\mathrm{rk}(\mathcal{E}^{\leq \mu}) = \mathrm{rk}(\mathcal{F}^{\leq \mu})$ for some $\mu \in \mathbb{Q}$. Taking $i = \mathrm{rk}(\mathcal{E}^{\leq \mu}) = \mathrm{rk}(\mathcal{F}^{\leq \mu})$ we make the following observations:

(a) Both $\mathrm{HN}(\mathcal{E})$ and $\mathrm{HN}(\mathcal{F})$ have vertices at $-i$.
(b) The slope of $\mathrm{HN}(\mathcal{F})$ on $[-i, -i+1]$ is less than or equal to μ.
(c) The slope of $\mathrm{HN}(\mathcal{E})$ on $[-i-1, -i]$ is greater than μ unless $i = \mathrm{rk}(\mathcal{E}) = \mathrm{rk}(\mathcal{F})$.

Hence the condition (ii) implies that $\mathrm{HN}(\mathcal{E})$ and $\mathrm{HN}(\mathcal{F})$ must agree on $[-i, 0]$, and consequently yields an isomorphism $\mathcal{E}^{\leq \mu} \simeq \mathcal{F}^{\leq \mu}$ as desired. $\qquad\square$

As our next step, we verify the necessity part of Theorem 4.1.1.

Proposition 4.3.2. *Given a vector bundle $\mathcal{E}$ on $\mathcal{X}$, every quotient bundle $\mathcal{F}$ of $\mathcal{E}$ satisfies the condition (i) of Theorem 4.1.1.*

Proof. Let μ be an arbitrary rational number, and consider the decomposition $\mathcal{E} \simeq \mathcal{E}^{\leq \mu} \oplus \mathcal{E}^{>\mu}$. Since any bundle map from $\mathcal{E}^{>\mu}$ to $\mathcal{F}^{\leq \mu}$ must be zero by Lemma 2.2.13, the composite surjective map $\mathcal{E} \twoheadrightarrow \mathcal{F} \twoheadrightarrow \mathcal{F}^{\leq \mu}$ should factor through $\mathcal{E}^{\leq \mu}$. We thus find $\mathrm{rk}(\mathcal{E}^{\leq \mu}) \geq \mathrm{rk}(\mathcal{F}^{\leq \mu})$.

Let us now assume that $\mathrm{rk}(\mathcal{E}^{\leq \mu}) = \mathrm{rk}(\mathcal{F}^{\leq \mu})$ for some $\mu \in \mathbb{Q}$. Then the kernel of the surjective map $\mathcal{E}^{\leq \mu} \twoheadrightarrow \mathcal{F}^{\leq \mu}$ must be zero since it is a subbundle of $\mathcal{E}^{\leq \mu}$ whose rank is equal to $\mathrm{rk}(\mathcal{E}^{\leq \mu}) - \mathrm{rk}(\mathcal{F}^{\leq \mu}) = 0$. Hence we obtain an isomorphism $\mathcal{E}^{\leq \mu} \simeq \mathcal{F}^{\leq \mu}$, thereby verifying the condition (i). $\qquad\square$

We also note that Proposition 4.3.2 has the following dual statement:

Proposition 4.3.3. *Given a vector bundle $\mathcal{E}$ on $\mathcal{X}$, every subsheaf $\mathcal{D}$ of $\mathcal{E}$ satisfies the inequality $\mathrm{rk}(\mathcal{E}^{\geq\mu}) \leq \mathrm{rk}(\mathcal{D}^{\geq\mu})$ for every $\mu \in \mathbb{Q}$.*

Proof. Let μ be an arbitrary rational number, and consider the decomposition $\mathcal{E} = \mathcal{E}^{<\mu} \oplus \mathcal{E}^{\geq\mu}$. Since any bundle map from $\mathcal{D}^{\geq\mu}$ to $\mathcal{E}^{<\mu}$ must be zero by Lemma 2.2.13, the composite injective map $\mathcal{D}^{\geq\mu} \hookrightarrow \mathcal{D} \hookrightarrow \mathcal{E}$ should factor through $\mathcal{E}^{\geq\mu}$. We thus find $\mathrm{rk}(\mathcal{E}^{\geq\mu}) \leq \mathrm{rk}(\mathcal{D}^{\geq\mu})$, which is equivalent to the desired inequality by Lemma 2.2.12. $\square$

By Proposition 4.3.1 and Proposition 4.3.2, it remains to prove the sufficiency of the condition (i) in Theorem 4.1.1. For this, we have the following elementary and important reduction:

Lemma 4.3.4. *We may prove the sufficiency of the condition (i) in Theorem 4.1.1 under the assumption that $\mathcal{E}$ and $\mathcal{F}$ have no common slopes.*

Proof. Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on $\mathcal{X}$ satisfying the condition (i) in Theorem 4.1.1. By their HN decompositions, we may write

$$\mathcal{E} \simeq \mathcal{G} \oplus \acute{\mathcal{E}} \qquad \text{and} \qquad \mathcal{F} \simeq \mathcal{G} \oplus \acute{\mathcal{F}} \tag{4.2}$$

for some vector bundles $\acute{\mathcal{E}}$ and $\acute{\mathcal{F}}$ with no common slopes.

Let μ be an arbitrary rational number. We assert that $\mathrm{rk}(\acute{\mathcal{E}}^{\leq\mu}) \geq \mathrm{rk}(\acute{\mathcal{F}}^{\leq\mu})$ with equality if only if $\acute{\mathcal{E}}^{\leq\mu}$ is isomorphic to $\acute{\mathcal{F}}^{\leq\mu}$. By the decompositions in (4.2) we have

$$\mathcal{E}^{\leq\mu} \simeq \mathcal{G}^{\leq\mu} \oplus \acute{\mathcal{E}}^{\leq\mu} \qquad \text{and} \qquad \mathcal{F}^{\leq\mu} \simeq \mathcal{G}^{\leq\mu} \oplus \acute{\mathcal{F}}^{\leq\mu}, \tag{4.3}$$

which consequently yield

$$\mathrm{rk}(\mathcal{E}^{\leq\mu}) = \mathrm{rk}(\mathcal{G}^{\leq\mu}) + \mathrm{rk}(\acute{\mathcal{E}}^{\leq\mu}) \qquad \text{and} \qquad \mathrm{rk}(\mathcal{F}^{\leq\mu}) = \mathrm{rk}(\mathcal{G}^{\leq\mu}) + \mathrm{rk}(\acute{\mathcal{F}}^{\leq\mu}). \tag{4.4}$$

We thus deduce the desired inequality $\mathrm{rk}(\acute{\mathcal{E}}^{\leq\mu}) \geq \mathrm{rk}(\acute{\mathcal{F}}^{\leq\mu})$ from the corresponding inequality $\mathrm{rk}(\mathcal{E}^{\leq\mu}) \geq \mathrm{rk}(\mathcal{F}^{\leq\mu})$ for $\mathcal{E}$ and $\mathcal{F}$. Moreover, if we have $\mathrm{rk}(\acute{\mathcal{E}}^{\leq\mu}) = \mathrm{rk}(\acute{\mathcal{F}}^{\leq\mu})$, then by (4.4) we find $\mathrm{rk}(\mathcal{E}^{\leq\mu}) = \mathrm{rk}(\mathcal{F}^{\leq\mu})$, which in turn yields an isomorphism $\mathcal{E}^{\leq\mu} \simeq \mathcal{F}^{\leq\mu}$ and consequently implies by (4.3) that $\acute{\mathcal{E}}^{\leq\mu}$ is isomorphic to $\acute{\mathcal{F}}^{\leq\mu}$.

Now we observe from (4.2) that a surjective bundle map $\acute{\mathcal{E}} \twoheadrightarrow \acute{\mathcal{F}}$ gives rise to a surjective bundle map $\mathcal{E} \twoheadrightarrow \mathcal{F}$ by direct summing with the identity map for $\mathcal{G}$. Hence our discussion in the preceding paragraph implies that we can prove the sufficiency of the condition (i) in Theorem 4.1.1 after replacing $\mathcal{E}$ and $\mathcal{F}$ by $\acute{\mathcal{E}}$ and $\acute{\mathcal{F}}$, yielding our desired reduction as $\acute{\mathcal{E}}$ and $\acute{\mathcal{F}}$ have no common slopes by construction. $\square$

Remark. After our reduction in Lemma 4.3.4, the rank inequality of the condition (i) in Theorem 4.1.1 becomes essentially strict. In fact, if $\mathcal{E}$ and $\mathcal{F}$ have no common slopes, $\mathcal{E}^{\leq\mu}$ is isomorphic to $\mathcal{F}^{\leq\mu}$ only holds for $\mu \in \mathbb{Q}$ with $\mathcal{E}^{\leq\mu} = \mathcal{F}^{\leq\mu} = 0$.

We now state our key inequality for proving the sufficiency of the condition (i) in Theorem 4.1.1.

Proposition 4.3.5. *Let $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{Q}$ be vector bundles on $\mathcal{X}$ with the following properties:*

(i) $\mathrm{rk}(\mathcal{E}^{\leq\mu}) \geq \mathrm{rk}(\mathcal{F}^{\leq\mu})$ for every $\mu \in \mathbb{Q}$ with equality only when $\mathcal{E}^{\leq\mu} \simeq \mathcal{F}^{\leq\mu}$.
(ii) $\mathrm{rk}(\mathcal{E}^{\leq\mu}) \geq \mathrm{rk}(\mathcal{Q}^{\leq\mu})$ for every $\mu \in \mathbb{Q}$ with equality only when $\mathcal{E}^{\leq\mu} \simeq \mathcal{Q}^{\leq\mu}$.
(iii) $\mathrm{rk}(\mathcal{F}^{\geq\mu}) \geq \mathrm{rk}(\mathcal{Q}^{\geq\mu})$ for every $\mu \in \mathbb{Q}$.
(iv) $\mathcal{E}$ and $\mathcal{F}$ have no common slopes.

Then we have an inequality

$$\deg(\mathcal{E}^{\vee} \otimes \mathcal{Q})^{\geq 0} + \deg(\mathcal{Q}^{\vee} \otimes \mathcal{F})^{\geq 0} \leq \deg(\mathcal{E}^{\vee} \otimes \mathcal{F})^{\geq 0} + \deg(\mathcal{Q}^{\vee} \otimes \mathcal{Q})^{\geq 0} \tag{4.5}$$

with equality if and only if $\mathcal{Q} = \mathcal{F}$.

Example 4.3.6. We present an example which shows that our reduction in Lemma 4.3.4 is crucial for the formulation of Proposition 4.3.5. Take $\mathcal{E} = \mathcal{O}^{\oplus 3}, \mathcal{F} = \mathcal{O}^{\oplus 2}$ and $\mathcal{Q} = \mathcal{O}$. Note that our choice does not satisfy the property (iv). However, we check the other properties (i), (ii) and (iii) by Proposition 4.3.2 and Proposition 4.3.3 after observing that $\mathcal{F}$ and $\mathcal{Q}$ are quotient bundles of $\mathcal{E}$ while $\mathcal{Q}$ is a subbundle of $\mathcal{F}$. We now observe that all terms in (4.5) are zero, thereby obtaining an equality even though $\mathcal{Q} \neq \mathcal{F}$. Hence the equality condition in Proposition 4.3.5 can be broken when $\mathcal{E}$ and $\mathcal{F}$ have common slopes.

We will prove Proposition 4.3.5 in §4.4. Here we explain why establishing Proposition 4.3.5 finishes the proof of Theorem 4.1.1.

Proposition 4.3.7. *Proposition 4.3.5 implies the sufficiency of the condition (i) in Theorem 4.1.1.*

Proof. Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on $\mathcal{X}$ satisfying the condition (i) in Theorem 4.1.1. We further assume that $\mathcal{E}$ and $\mathcal{F}$ have no common slopes in light of Lemma 4.3.4. We wish to prove existence of a surjective bundle map $\mathcal{E} \twoheadrightarrow \mathcal{F}$ assuming Proposition 4.3.5. For this, it suffices to check that $\mathcal{E}$ and $\mathcal{F}$ satisfy the assumptions (i) and (ii) in Proposition 3.1.3.

We first check the assumption (i) in Proposition 3.1.3 for $\mathcal{E}$ and $\mathcal{F}$. By Lemma 2.2.13, it is enough to prove $\mu_{\min}(\mathcal{E}) \leq \mu_{\max}(\mathcal{F})$. We verify this by observing

$$\mathrm{rk}(\mathcal{E}^{\leq \mu_{\max}(\mathcal{F})}) \geq \mathrm{rk}(\mathcal{F}^{\leq \mu_{\max}(\mathcal{F})}) = \mathrm{rk}(\mathcal{F}) > 0.$$

It remains to check the assumption (ii) in Proposition 3.1.3 for $\mathcal{E}$ and $\mathcal{F}$. Let $\mathcal{Q}$ be an arbitrary proper subsheaf of $\mathcal{F}$ which also occurs as a quotient of $\mathcal{E}$. Then $\mathcal{E}, \mathcal{F}$ and $\mathcal{Q}$ satisfy the assumptions of Proposition 4.3.5 by Proposition 4.3.2, Proposition 4.3.3 and our assumption on $\mathcal{E}$ and $\mathcal{F}$. Since we have $\mathcal{Q} \neq \mathcal{F}$, Proposition 4.3.5 now yields a strict inequality

$$\deg(\mathcal{E}^{\vee} \otimes \mathcal{Q})^{\geq 0} + \deg(\mathcal{Q}^{\vee} \otimes \mathcal{F})^{\geq 0} < \deg(\mathcal{E}^{\vee} \otimes \mathcal{F})^{\geq 0} + \deg(\mathcal{Q}^{\vee} \otimes \mathcal{Q})^{\geq 0}$$

as required by the assumption (ii) in Proposition 3.1.3. $\square$

4.4. **Proof of the key inequality.**

We now aim to establish Proposition 4.3.5.

Definition 4.4.1. For arbitrary vector bundles $\mathcal{E}, \mathcal{F}$ and $\mathcal{Q}$ on $\mathcal{X}$, we define

$$c_{\mathcal{E},\mathcal{F}}(\mathcal{Q}) := \deg(\mathcal{E}^{\vee} \otimes \mathcal{F})^{\geq 0} + \deg(\mathcal{Q}^{\vee} \otimes \mathcal{Q})^{\geq 0} - \deg(\mathcal{E}^{\vee} \otimes \mathcal{Q})^{\geq 0} - \deg(\mathcal{Q}^{\vee} \otimes \mathcal{F})^{\geq 0}.$$

Remark. The inequality (4.5) in Proposition 4.3.5 can be stated as $c_{\mathcal{E},\mathcal{F}}(\mathcal{Q}) \geq 0$.

Our proof of Proposition 4.3.5 will consist of a series of reduction steps as follows:

Step 1. We reduce the proof to the case where all slopes of $\mathcal{E}, \mathcal{F}$ and $\mathcal{Q}$ are integers.
Step 2. We further reduce the proof to the case $\mathrm{rk}(\mathcal{Q}) = \mathrm{rk}(\mathcal{F})$.
Step 3. After these reductions, we complete the proof by gradually "reducing" the slopes of $\mathcal{F}$ to the slopes of $\mathcal{Q}$.

Throughout these reduction steps, we will establish the following key facts:

(1) The quantity $c_{\mathcal{E},\mathcal{F}}(\mathcal{Q})$ monotone decreases to 0 as we reduce $\mathrm{rk}(\mathcal{F})$ to $\mathrm{rk}(\mathcal{Q})$ and the slopes of $\mathcal{F}$ to the slopes of $\mathcal{Q}$.
(2) When $\mathrm{rk}(\mathcal{Q}) < \mathrm{rk}(\mathcal{F})$, the equality $c_{\mathcal{E},\mathcal{F}}(\mathcal{Q}) = 0$ never holds.
(3) When $\mathrm{rk}(\mathcal{Q}) = \mathrm{rk}(\mathcal{F})$, the equality $c_{\mathcal{E},\mathcal{F}}(\mathcal{Q}) = 0$ holds only when $\mathcal{Q} = \mathcal{F}$.

We will then obtain the desired inequality $c_{\mathcal{E},\mathcal{F}}(\mathcal{Q}) \geq 0$ from the first fact and the equality condition $\mathcal{Q} = \mathcal{F}$ from the second and the third facts.

Remark. For curious readers, we briefly describe how each assumption in Proposition 4.3.5 will be used to establish the key facts (1), (2) and (3) above.

The fact (1) relies on the rank inequalities from the assumptions (i), (ii) and (iii) of Proposition 4.3.5. As we will see in Lemma 4.4.2, these rank inequalities can be interpreted as slopewise dominance relations between the vector bundles $\mathcal{E}, \mathcal{F}$ and $\mathcal{Q}$. We will use those relations to "gradually reduce" $\mathrm{HN}(\mathcal{F})$ to $\mathrm{HN}(\mathcal{Q})$ in a way that $c_{\mathcal{E},\mathcal{F}}(\mathcal{Q})$ always decreases.

The fact (2) is essentially a consequence of the assumption (iv) of Proposition 4.3.5. In Proposition 4.4.6, we will use this assumption to prove that $c_{\mathcal{E},\mathcal{F}}(\mathcal{Q})$ strictly decreases while we reduce the rank of $\mathcal{F}$ by cutting down $\mathrm{HN}(\mathcal{F})$ from the right (cf. Example 4.3.6).

The fact (3) comes from the assumption (iv) along with the equality conditions in the assumptions (i) and (ii) of Proposition 4.3.5. As we will see in Lemma 4.4.10, these assumptions ensure that $c_{\mathcal{E},\mathcal{F}}(\mathcal{Q})$ strictly decreases during the first reduction cycle in Step 3.

Before proceeding to our reduction steps, let us make some useful observations about the assumptions of Proposition 4.3.5.

Lemma 4.4.2. *Let $\mathcal{E}, \mathcal{F}$ and $\mathcal{Q}$ be as in the statement of Proposition 4.3.5. Then we have the following slopewise dominance relations:*

(1) $\mathcal{E}^\vee$ *slopewise dominates* $\mathcal{F}^\vee$.
(2) $\mathcal{E}^\vee$ *slopewise dominates* $\mathcal{Q}^\vee$.
(3) $\mathcal{F}$ *slopewise dominates* $\mathcal{Q}$.

Proof. By Lemma 4.2.2, these slopewise dominance relations follow from the rank inequalities in the assumptions (i), (ii) and (iii) of Proposition 4.3.5. $\qquad\square$

Remark. Based on Lemma 4.4.2, we can give an intuitive explanation on how our reduction steps work. Let us rewrite $c_{\mathcal{E},\mathcal{F}}(\mathcal{Q})$ as

$$c_{\mathcal{E},\mathcal{F}}(\mathcal{Q}) := \deg((\mathcal{F}^\vee)^\vee \otimes \mathcal{E}^\vee)^{\geq 0} + \deg(\mathcal{Q}^\vee \otimes \mathcal{Q})^{\geq 0} - \deg((\mathcal{Q}^\vee)^\vee \otimes \mathcal{E}^\vee)^{\geq 0} - \deg(\mathcal{Q}^\vee \otimes \mathcal{F})^{\geq 0}.$$

Then by Lemma 4.4.2 every term on the right side is of the form $\deg(\mathcal{V}^\vee \otimes \mathcal{W})^{\geq 0}$ where $\mathcal{W}$ slopewise dominates $\mathcal{V}$. During our reduction steps, we will utilize the slopewise dominance relations as stated in Lemma 4.4.2 so that all terms in $c_{\mathcal{E},\mathcal{F}}(\mathcal{Q})$ behave as we want.

Lemma 4.4.3. *Let $\mathcal{E}, \mathcal{F}$ and $\mathcal{Q}$ be as in the statement of Proposition 4.3.5. Choose a positive integer C, and let $\tilde{\mathcal{E}}, \tilde{\mathcal{F}}$ and $\tilde{\mathcal{Q}}$ be vector bundles on $\mathcal{X}$ whose HN polygons are obtained by vertically stretching $\mathrm{HN}(\mathcal{E}), \mathrm{HN}(\mathcal{F})$ and $\mathrm{HN}(\mathcal{Q})$ by a factor C. Then we have the following properties of $\tilde{\mathcal{E}}, \tilde{\mathcal{F}}$ and $\tilde{\mathcal{Q}}$.*

(i) $\mathrm{rk}(\tilde{\mathcal{E}}^{\leq \mu}) \geq \mathrm{rk}(\tilde{\mathcal{F}}^{\leq \mu})$ *for every $\mu \in \mathbb{Q}$ with equality only when $\tilde{\mathcal{E}}^{\leq \mu} \simeq \tilde{\mathcal{F}}^{\leq \mu}$.*
(ii) $\mathrm{rk}(\tilde{\mathcal{E}}^{\leq \mu}) \geq \mathrm{rk}(\tilde{\mathcal{Q}}^{\leq \mu})$ *for every $\mu \in \mathbb{Q}$ with equality only when $\tilde{\mathcal{E}}^{\leq \mu} \simeq \tilde{\mathcal{Q}}^{\leq \mu}$.*
(iii) $\mathrm{rk}(\tilde{\mathcal{F}}^{\geq \mu}) \geq \mathrm{rk}(\tilde{\mathcal{Q}}^{\geq \mu})$ *for every $\mu \in \mathbb{Q}$.*
(iv) $\tilde{\mathcal{E}}$ *and $\tilde{\mathcal{F}}$ have no common slopes.*

Proof. By construction, we have the following facts:

(a) For $\mathcal{V} = \mathcal{E}, \mathcal{F}$ and $\mathcal{Q}$, we have $\mathrm{rk}(\tilde{\mathcal{V}}^{\leq \mu}) = \mathrm{rk}(\mathcal{V}^{\leq \mu/C})$ and $\mathrm{rk}(\tilde{\mathcal{V}}^{\geq \mu}) = \mathrm{rk}(\mathcal{V}^{\geq \mu/C})$ for every $\mu \in \mathbb{Q}$.
(b) For $\mathcal{W} = \mathcal{F}$ and $\mathcal{Q}$, we have $\tilde{\mathcal{E}}^{\leq \mu} \simeq \tilde{\mathcal{W}}^{\leq \mu}$ if $\mathcal{E}^{\leq \mu/C} \simeq \mathcal{W}^{\leq \mu/C}$.
(c) The slopes of $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{F}}$ are given by multiplying the slopes of $\mathcal{E}$ and $\mathcal{F}$ by C.

Hence we deduce the properties (i) - (iv) from the corresponding properties of $\mathcal{E}, \mathcal{F}$ and $\mathcal{Q}$. $\quad\square$

Lemma 4.4.4. *Let $\mathcal{E}, \mathcal{F}$ and $\mathcal{Q}$ be as in the statement of Proposition 4.3.5. For any integer λ, the vector bundles $\mathcal{E}(\lambda), \mathcal{F}(\lambda)$ and $\mathcal{Q}(\lambda)$ satisfy the following properties:*

(i) $\mathrm{rk}(\mathcal{E}(\lambda)^{\leq\mu}) \geq \mathrm{rk}(\mathcal{F}(\lambda)^{\leq\mu})$ *for every $\mu \in \mathbb{Q}$ with equality only when $\mathcal{E}(\lambda)^{\leq\mu} \simeq \mathcal{F}(\lambda)^{\leq\mu}$.*

(ii) $\mathrm{rk}(\mathcal{E}(\lambda)^{\leq\mu}) \geq \mathrm{rk}(\mathcal{Q}(\lambda)^{\leq\mu})$ *for every $\mu \in \mathbb{Q}$ with equality only when $\mathcal{E}(\lambda)^{\leq\mu} \simeq \mathcal{Q}(\lambda)^{\leq\mu}$.*

(iii) $\mathrm{rk}(\mathcal{F}(\lambda)^{\geq\mu}) \geq \mathrm{rk}(\mathcal{Q}(\lambda)^{\geq\mu})$ *for every $\mu \in \mathbb{Q}$.*

(iv) $\mathcal{E}(\lambda)$ *and* $\mathcal{F}(\lambda)$ *have no common slopes*

Proof. Since the vector bundle $\mathcal{O}(\lambda)$ has rank 1 and degree λ, tensoring a vector bundle with $\mathcal{O}(\lambda)$ is the same as adding λ to all slopes. Therefore we have the following observations:

(a) For $\mathcal{V} = \mathcal{E}, \mathcal{F}$ or $\mathcal{Q}$, we have $\mathrm{rk}(\mathcal{V}(\lambda)^{\leq\mu}) = \mathrm{rk}(\mathcal{V}^{\leq\mu-\lambda})$ and $\mathrm{rk}(\mathcal{V}(\lambda)^{\geq\mu}) = \mathrm{rk}(\mathcal{V}^{\geq\mu-\lambda})$ for every $\mu \in \mathbb{Q}$.

(b) For $\mathcal{W} = \mathcal{F}$ and $\mathcal{Q}$, we have $\mathcal{E}(\lambda)^{\leq\mu} \simeq \mathcal{W}(\lambda)^{\leq\mu}$ if $\mathcal{E}^{\leq\mu-\lambda} \simeq \mathcal{W}^{\leq\mu-\lambda}$.

(c) The slopes of $\mathcal{E}(\lambda)$ and $\mathcal{F}(\lambda)$ are given by adding λ to the slopes of $\mathcal{E}$ and $\mathcal{F}$.

We thus deduce the properties (i) - (iv) from the corresponding properties of $\mathcal{E}, \mathcal{F}$ and $\mathcal{Q}$. □

We are now ready to carry out Step 1 and Step 2.

Proposition 4.4.5. *To prove Proposition 4.3.5, we may assume that all slopes of $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{Q}$ are integers.*

Proof. Let $\mathcal{E}, \mathcal{F}$ and $\mathcal{Q}$ be as in the statement of Proposition 4.3.5. Take C to be the least common multiple of all denominators of the slopes of $\mathcal{E}, \mathcal{F}$ and $\mathcal{Q}$, and let $\tilde{\mathcal{E}}, \tilde{\mathcal{F}}$ and $\tilde{\mathcal{Q}}$ be vector bundles on $\mathcal{X}$ whose HN polygons are obtained by vertically stretching $\mathrm{HN}(\mathcal{E}), \mathrm{HN}(\mathcal{F})$ and $\mathrm{HN}(\mathcal{Q})$ by a factor C. Note that all slopes of $\tilde{\mathcal{E}}, \tilde{\mathcal{F}}$ and $\tilde{\mathcal{Q}}$ are integers by construction. We now use Lemma 3.2.8 to obtain an identity

$$c_{\tilde{\mathcal{E}},\tilde{\mathcal{F}}}(\tilde{\mathcal{Q}}) = C \cdot c_{\mathcal{E},\mathcal{F}}(\mathcal{Q})$$

which implies that the inequality (4.5) for $\mathcal{E}, \mathcal{F}$ and $\mathcal{Q}$ follows from the corresponding inequality for $\tilde{\mathcal{E}}, \tilde{\mathcal{F}}$ and $\tilde{\mathcal{Q}}$. In addition, our construction translates the equality condition $\mathcal{Q} = \mathcal{F}$ for the former inequality to the equality condition $\tilde{\mathcal{Q}} = \tilde{\mathcal{F}}$ for the latter inequality. Now Lemma 4.4.3 implies that we may prove Proposition 4.3.5 after replacing $\mathcal{E}, \mathcal{F}$ and $\mathcal{Q}$ by $\tilde{\mathcal{E}}, \tilde{\mathcal{F}}$ and $\tilde{\mathcal{Q}}$, thereby yielding our desired reduction. □

Proposition 4.4.6. *It suffices to prove Proposition 4.3.5 under the additional assumptions that $\mathrm{rk}(\mathcal{Q}) = \mathrm{rk}(\mathcal{F})$ and that all slopes of $\mathcal{E}, \mathcal{F}$ and $\mathcal{Q}$ are integers.*

Proof. Suppose that Proposition 4.3.5 holds in the special case where the additional assumptions are satisfied. We assert that the general case of Proposition 4.3.5 follows from this special case by induction on $\mathrm{rk}(\mathcal{F}) - \mathrm{rk}(\mathcal{Q})$. We assume that all slopes of $\mathcal{E}, \mathcal{F}$ and $\mathcal{Q}$ are integers in light of Proposition 4.4.5.

We first reduce our induction step to the case $\mu_{\min}(\mathcal{F}) = 0$. For this, we take $\lambda = -\mu_{\min}(\mathcal{F})$ so that we have $\mu_{\min}(\mathcal{F}(\lambda)) = \mu_{\min}(\mathcal{F}) + \lambda = 0$. Our assumption implies that λ is an integer, and consequently that all slopes of $\mathcal{E}(\lambda), \mathcal{F}(\lambda)$ and $\mathcal{Q}(\lambda)$ are integers as well. We now apply Lemma 3.2.7 to get an identity

$$c_{\mathcal{E}(\lambda),\mathcal{F}(\lambda)}(\mathcal{Q}(\lambda)) = c_{\mathcal{E},\mathcal{F}}(\mathcal{Q}),$$

which implies that the inequality (4.5) for $\mathcal{E}, \mathcal{F}$ and $\mathcal{Q}$ is equivalent to the corresponding inequality for $\mathcal{E}(\lambda), \mathcal{F}(\lambda)$ and $\mathcal{Q}(\lambda)$. In addition, we translate the equality condition $\mathcal{Q} = \mathcal{F}$ for the former inequality to the equality condition $\mathcal{Q}(\lambda) = \mathcal{F}(\lambda)$ for the latter inequality. We also have $\mathrm{rk}(\mathcal{F}) - \mathrm{rk}(\mathcal{Q}) = \mathrm{rk}(\mathcal{F}(\lambda)) - \mathrm{rk}(\mathcal{Q}(\lambda))$ as tensoring with $\mathcal{O}(\lambda)$ does not change ranks. Now Lemma 4.4.4 implies that we may proceed to the induction step after replacing $\mathcal{E}, \mathcal{F}$ and $\mathcal{Q}$ by $\mathcal{E}(\lambda), \mathcal{F}(\lambda)$ and $\mathcal{Q}(\lambda)$, thereby yielding our desired reduction.

Let us now assume that $\mu_{\min}(\mathcal{F}) = 0$. For our induction step we have $\mathrm{rk}(\mathcal{F}) - \mathrm{rk}(\mathcal{Q}) > 0$, or equivalently $\mathrm{rk}(\mathcal{F}) > \mathrm{rk}(\mathcal{Q})$. Hence we can write

$$\mathcal{F} = \breve{\mathcal{F}} \oplus \mathcal{O}$$

for some vector bundle $\breve{\mathcal{F}}$ with $\mu_{\min}(\breve{\mathcal{F}}) \geq 0$ and $\mathrm{rk}(\breve{\mathcal{F}}) \geq \mathrm{rk}(\mathcal{Q})$.
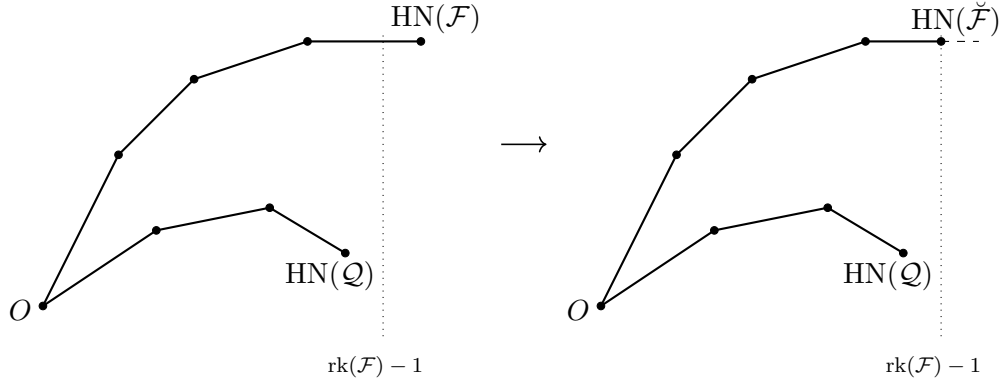

FIGURE 9. Illustration of the induction step

We assert that our assumptions on $\mathcal{E}, \mathcal{F}$ and $\mathcal{Q}$ yield the corresponding conditions on $\mathcal{E}, \breve{\mathcal{F}}$ and $\mathcal{Q}$. In other words, we claim that $\mathcal{E}, \breve{\mathcal{F}}$ and $\mathcal{Q}$ satisfy the following properties:

   (i) $\mathrm{rk}(\mathcal{E}^{\leq \mu}) \geq \mathrm{rk}(\breve{\mathcal{F}}^{\leq \mu})$ for every $\mu \in \mathbb{Q}$ with equality only when $\mathcal{E}^{\leq \mu} \simeq \breve{\mathcal{F}}^{\leq \mu}$.
   (ii) $\mathrm{rk}(\mathcal{E}^{\leq \mu}) \geq \mathrm{rk}(\mathcal{Q}^{\leq \mu})$ for every $\mu \in \mathbb{Q}$ with equality only when $\mathcal{E}^{\leq \mu} \simeq \mathcal{Q}^{\leq \mu}$.
   (iii) $\mathrm{rk}(\breve{\mathcal{F}}^{\geq \mu}) \geq \mathrm{rk}(\mathcal{Q}^{\geq \mu})$ for every $\mu \in \mathbb{Q}$.
   (iv) $\mathcal{E}$ and $\breve{\mathcal{F}}$ have no common slopes.
   (v) the slopes of $\mathcal{E}, \breve{\mathcal{F}}$ and $\mathcal{Q}$ are integers.

By construction, the properties (ii), (iv) and (v) immediately follow from the corresponding assumptions on $\mathcal{E}, \mathcal{F}$ and $\mathcal{Q}$. The inequality $\mathrm{rk}(\mathcal{E}^{\leq \mu}) \geq \mathrm{rk}(\breve{\mathcal{F}}^{\leq \mu})$ in (i) follows from the corresponding inequality $\mathrm{rk}(\mathcal{E}^{\leq \mu}) \geq \mathrm{rk}(\mathcal{F}^{\leq \mu})$ after observing

$$\mathrm{rk}(\breve{\mathcal{F}}^{\leq \mu}) = \begin{cases} \mathrm{rk}(\mathcal{F}^{\leq \mu}) - 1 & \text{if } \mu \geq 0, \\ \mathrm{rk}(\mathcal{F}^{\leq \mu}) & \text{if } \mu < 0. \end{cases}$$

This observation further shows that equality in $\mathrm{rk}(\mathcal{E}^{\leq \mu}) \geq \mathrm{rk}(\breve{\mathcal{F}}^{\leq \mu})$ never holds for $\mu \geq 0$. Moreover, for $\mu < 0$ we have $\breve{\mathcal{F}}^{\leq \mu} = 0$ by the fact $\mu_{\min}(\breve{\mathcal{F}}) \geq 0$. Hence we deduce that equality in $\mathrm{rk}(\mathcal{E}^{\leq \mu}) \geq \mathrm{rk}(\breve{\mathcal{F}}^{\leq \mu})$ can hold only if $\mathcal{E}^{\leq \mu} = \breve{\mathcal{F}}^{\leq \mu} = 0$, thereby verifying the property (i). The remaining property (iii) is equivalent to slopewise dominance of $\breve{\mathcal{F}}$ on $\mathcal{Q}$ by Lemma 4.2.2, and thus follows from the following observations:

   (a) $\mathcal{F}$ slopewise dominates $\mathcal{Q}$ by Lemma 4.4.2.

(b) $\mathrm{HN}(\check{\mathcal{F}})$ is obtained from $\mathrm{HN}(\mathcal{F})$ by removing the line segment over the interval $(\mathrm{rk}(\mathcal{F}) - 1, \mathrm{rk}(\mathcal{F})]$ (and thereby having the same left endpoint).

(c) Since $\mathrm{rk}(\mathcal{F}) > \mathrm{rk}(\mathcal{Q})$, the removal process in (b) does not affect slopewise dominance.

Now since $\mathrm{rk}(\check{\mathcal{F}}) - \mathrm{rk}(\mathcal{Q}) < \mathrm{rk}(\mathcal{F}) - \mathrm{rk}(\mathcal{Q})$, our induction hypothesis yields

$$c_{\mathcal{E},\check{\mathcal{F}}}(\mathcal{Q}) \geq 0 \tag{4.6}$$

with equality if and only if $\mathcal{Q} \simeq \check{\mathcal{F}}$. For the desired inequality $c_{\mathcal{E},\mathcal{F}}(\mathcal{Q}) \geq 0$ we compute

$$\begin{aligned}
\deg(\mathcal{E}^\vee \otimes \mathcal{F})^{\geq 0} &= \deg(\mathcal{E}^\vee \otimes (\check{\mathcal{F}} \oplus \mathcal{O}))^{\geq 0} \\
&= \deg(\mathcal{E}^\vee \otimes \check{\mathcal{F}})^{\geq 0} + \deg(\mathcal{E}^\vee \otimes \mathcal{O})^{\geq 0} \\
&= \deg(\mathcal{E}^\vee \otimes \check{\mathcal{F}})^{\geq 0} + \deg(\mathcal{E}^\vee)^{\geq 0}, \\
\deg(\mathcal{Q}^\vee \otimes \mathcal{F})^{\geq 0} &= \deg(\mathcal{Q}^\vee \otimes (\check{\mathcal{F}} \oplus \mathcal{O}))^{\geq 0} \\
&= \deg(\mathcal{Q}^\vee \otimes \check{\mathcal{F}})^{\geq 0} + \deg(\mathcal{Q}^\vee \otimes \mathcal{O})^{\geq 0} \\
&= \deg(\mathcal{Q}^\vee \otimes \check{\mathcal{F}})^{\geq 0} + \deg(\mathcal{Q}^\vee)^{\geq 0}.
\end{aligned}$$

Then we have

$$c_{\mathcal{E},\mathcal{F}}(\mathcal{Q}) = c_{\mathcal{E},\check{\mathcal{F}}}(\mathcal{Q}) - \deg(\mathcal{E}^\vee)^{\geq 0} + \deg(\mathcal{Q}^\vee)^{\geq 0}.$$

Since $\mathcal{E}^\vee$ slopewise dominates $\mathcal{Q}^\vee$ as noted in Lemma 4.4.2, we use Lemma 4.2.3 to find

$$c_{\mathcal{E},\mathcal{F}}(\mathcal{Q}) \geq c_{\mathcal{E},\check{\mathcal{F}}}(\mathcal{Q}) \tag{4.7}$$

with equality if and only if $\deg(\mathcal{E}^\vee)^{\geq 0} = \deg(\mathcal{Q}^\vee)^{\geq 0}$ or equivalently $\deg(\mathcal{E})^{\leq 0} = \deg(\mathcal{Q})^{\leq 0}$. Combining (4.6) and (4.7) we obtain the desired inequality

$$c_{\mathcal{E},\mathcal{F}}(\mathcal{Q}) \geq 0. \tag{4.8}$$

It remains to check the equality condition for (4.8). Note that the equality condition $\mathcal{Q} = \mathcal{F}$ as asserted in Proposition 4.3.5 never holds in our induction step as we have $\mathrm{rk}(\mathcal{F}) > \mathrm{rk}(\mathcal{Q})$. We thus have to prove that the inequality (4.8) is strict. Suppose for contradiction that we have equality in (4.8). From the equality conditions for (4.6) and (4.7) we get $\mathcal{Q} \simeq \check{\mathcal{F}}$ and $\deg(\mathcal{E})^{\leq 0} = \deg(\mathcal{Q})^{\leq 0}$. Since $\mu_{\min}(\check{\mathcal{F}}) \geq 0$ by construction, the condition $\mathcal{Q} \simeq \check{\mathcal{F}}$ implies $\deg(\mathcal{Q})^{\leq 0} = 0$. Hence we must have $\deg(\mathcal{E})^{\leq 0} = 0$, which implies $\mu_{\min}(\mathcal{E}) \geq 0$. Furthermore, since we assume $\mu_{\min}(\mathcal{F}) = 0$, the assumption (iv) of Proposition 4.3.5 yields

$$\mu_{\min}(\mathcal{E}) > 0 = \mu_{\min}(\mathcal{F}).$$

We thus find

$$\mathrm{rk}(\mathcal{E}^{\leq \mu_{\min}(\mathcal{F})}) = 0 < \mathrm{rk}(\mathcal{F}^{\leq \mu_{\min}(\mathcal{F})}),$$

yielding a contradiction to the assumption (i) of Proposition 4.3.5 as desired. $\qquad\square$

We now proceed to the final reduction step. Here we aim to reduce the slopes of $\mathcal{F}$ to the slopes of $\mathcal{Q}$ in a certain way that the quantity $c_{\mathcal{E},\mathcal{F}}(\mathcal{Q})$ can only decrease throughout the procedure. For a precise description of our procedure, we introduce the following construction:

Definition 4.4.7. Let $\mathcal{V}$ and $\mathcal{W}$ be nonzero vector bundles on $\mathcal{X}$ with integer slopes such that $\mathcal{V}$ slopewise dominates $\mathcal{W}$. Let $\overline{\mathcal{V}}$ be the vector bundle on $\mathcal{X}$ obtained from $\mathcal{V}$ by reducing all slopes of $\mathcal{V}^{>\mu_{\max}(\mathcal{W})}$ to $\mu_{\max}(\mathcal{W})$. More precisely, we set

$$\overline{\mathcal{V}} := \mathcal{O}(\mu_{\max}(\mathcal{W}))^{\oplus \mathrm{rk}(\mathcal{V}^{>\mu_{\max}(\mathcal{W})})} \oplus \mathcal{V}^{\leq \mu_{\max}(\mathcal{W})}.$$

We say that $\overline{\mathcal{V}}$ is the *maximal slope reduction* of $\mathcal{V}$ to $\mathcal{W}$.
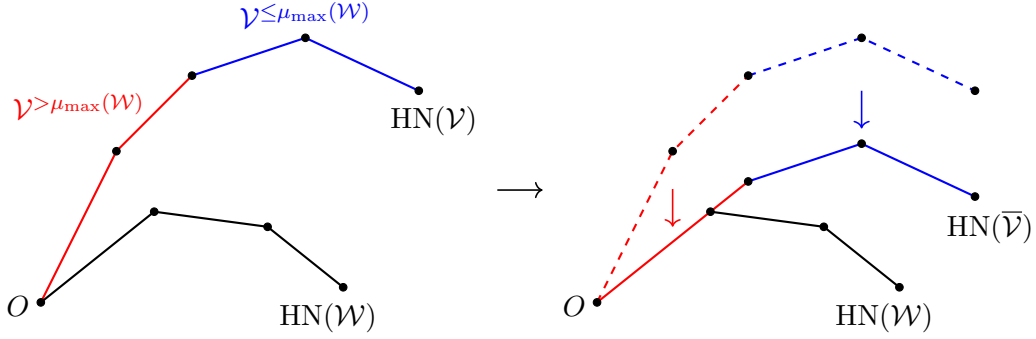
FIGURE 10. Illustration of the maximal slope reduction

Remark. The assumption that $\mathcal{V}$ and $\mathcal{W}$ have integer slopes is crucial in Definition 4.4.7. In fact, if $\mu_{\max}(\mathcal{W})$ is not an integer, reducing all slopes of $\mathcal{V}^{>\mu_{\max}(\mathcal{W})}$ to $\mu_{\max}(\mathcal{W})$ may not make sense. For example, if we consider $\mathcal{V} = \mathcal{O}(1)^{\oplus 3}$ and $\mathcal{W} = \mathcal{O}\left(\frac{1}{2}\right)$, reducing all slopes of $\mathcal{V}$ to $\mu_{\max}(\mathcal{W}) = \frac{1}{2}$ should yield a semistable vector bundle of slope $\frac{1}{2}$ and rank 3, which does not exist.

On the other hand, slopewise dominance of $\mathcal{V}$ on $\mathcal{W}$ is not essential for the definition to make sense. However, there are a couple of reasons that we don't consider the case when $\mathcal{V}$ does not slopewise dominates $\mathcal{W}$. First, our terminology doesn't quite make sense in this case as $\mathcal{V}$ may have no slopes to reduce down to $\mathcal{W}$, for example when $\mu_{\max}(\mathcal{V}) < \mu_{\min}(\mathcal{W})$. Second, we won't need this case for our purpose; indeed, we will only apply the notion of maximal slope reduction to (some direct summands of) $\mathcal{F}$ and $\mathcal{Q}$ for which we have a slopewise dominance relation given by Lemma 4.4.2.

We note some basic properties of the maximal slope reduction.

Lemma 4.4.8. *Let $\mathcal{V}$ and $\mathcal{W}$ be nonzero vector bundles on $\mathcal{X}$ with integer slopes such that $\mathcal{V}$ slopewise dominates $\mathcal{W}$. Let $\overline{\mathcal{V}}$ denote the maximal slope reduction of $\mathcal{V}$ to $\mathcal{W}$. Then $\overline{\mathcal{V}}$ satisfies the following properties:*

(i) $\mu_{max}(\overline{\mathcal{V}}) = \mu_{max}(\mathcal{W})$.
(ii) $\mathrm{rk}(\overline{\mathcal{V}}) = \mathrm{rk}(\mathcal{V})$.
(iii) $\mathcal{V} = \overline{\mathcal{V}}$ if and only if $\mu_{max}(\mathcal{V}) = \mu_{max}(\mathcal{W})$.
(iv) $\overline{\mathcal{V}}$ slopewise dominates $\mathcal{W}$.
(v) all slopes of $\overline{\mathcal{V}}$ are integers.

Proof. All properties are immediate consequences of Definition 4.4.7 $\qquad\square$

We can now recursively define our procedure for reducing the slopes of $\mathcal{F}$ to the slopes of $\mathcal{Q}$ as follows:

(I) Since $\mathcal{F}$ slopewise dominates $\mathcal{Q}$ as noted in Lemma 4.4.2, we use Lemma 4.2.4 to obtain decompositions

$$\mathcal{F} \simeq \mathcal{U} \oplus \mathcal{F}' \qquad \text{and} \qquad \mathcal{Q} \simeq \mathcal{U} \oplus \mathcal{Q}'$$

satisfying the following properties:
(i) $\mathcal{F}'$ slopewise dominates $\mathcal{Q}'$.
(ii) If $\mathcal{Q}' \neq 0$, we have $\mu_{\max}(\mathcal{F}') > \mu_{\max}(\mathcal{Q}')$.

(iii) If $\mathcal{U} \neq 0$ and $\mathcal{Q}' \neq 0$, we have $\mu_{\min}(\mathcal{U}) \geq \mu_{\max}(\mathcal{F}') > \mu_{\max}(\mathcal{Q}')$.

(II) If $\mathcal{Q}' = 0$, we terminate the process. Otherwise, we go back to (I) after replacing $\mathcal{F}$ by $\mathcal{U} \oplus \overline{\mathcal{F}}'$, where $\overline{\mathcal{F}}'$ denotes the maximal slope reduction of $\mathcal{F}'$ to $\mathcal{Q}'$.
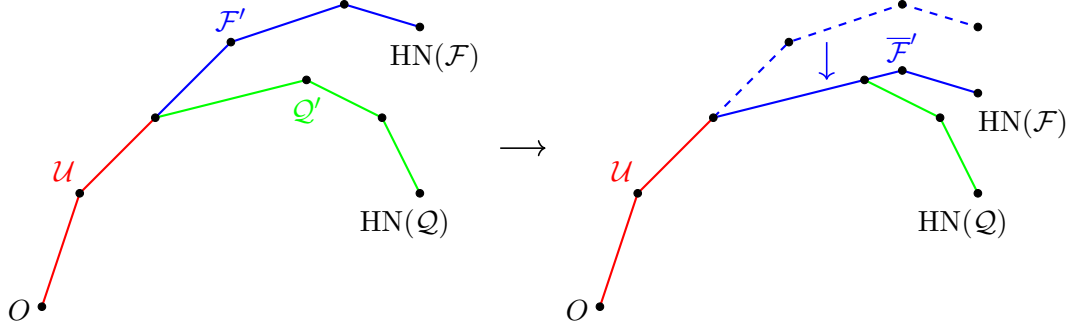

FIGURE 11. Illustration of the slope reduction process

With this procedure, we will obtain the desired inequality $c_{\mathcal{E},\mathcal{F}}(\mathcal{Q}) \geq 0$ by establishing the following facts:

(A) The quantity $c_{\mathcal{E},\mathcal{F}}(\mathcal{Q})$ never increases throughout the process.
(B) The process eventually terminates with the condition $\mathcal{Q} = \mathcal{F}$ and $c_{\mathcal{E},\mathcal{F}}(\mathcal{Q}) = 0$.

For the equality condition, we will further show that

(C) $c_{\mathcal{E},\mathcal{F}}(\mathcal{Q})$ strictly decreases after the first cycle of the process,

thereby deducing that the equality $c_{\mathcal{E},\mathcal{F}}(\mathcal{Q}) = 0$ can be only achieved by starting with the terminal state $\mathcal{Q} = \mathcal{F}$.

The main subtlety for our procedure arises from the fact that some of our assumptions on $\mathcal{E}, \mathcal{F}$ and $\mathcal{Q}$ may be lost during our process. In the following lemma, we give a list of all assumptions that are maintained throughout the entire process.

Lemma 4.4.9. *Let $\mathcal{E}, \mathcal{F}$ and $\mathcal{Q}$ be nonzero vector bundles on $\mathcal{X}$ with the following properties:*

(i) $\mathrm{rk}(\mathcal{E}^{\leq \mu}) \geq \mathrm{rk}(\mathcal{F}^{\leq \mu})$ for every $\mu \in \mathbb{Q}$

(ii) $\mathrm{rk}(\mathcal{E}^{\leq \mu}) \geq \mathrm{rk}(\mathcal{Q}^{\leq \mu})$ for every $\mu \in \mathbb{Q}$ with equality only when $\mathcal{E}^{\leq \mu} \simeq \mathcal{Q}^{\leq \mu}$.

(iii) $\mathrm{rk}(\mathcal{F}^{\geq \mu}) \geq \mathrm{rk}(\mathcal{Q}^{\geq \mu})$ for every $\mu \in \mathbb{Q}$.

(iv) all slopes of $\mathcal{E}, \mathcal{F}$ and $\mathcal{Q}$ are integers.

(v) $\mathrm{rk}(\mathcal{Q}) = \mathrm{rk}(\mathcal{F})$.

Then all properties (i) - (v) are invariant under replacing $\mathcal{F}$ by $\overline{\mathcal{F}}$, the maximal slope reduction of $\mathcal{F}$ to $\mathcal{Q}$.

Proof. Let us first remark that the maximal slope reduction of $\mathcal{F}$ to $\mathcal{Q}$ makes sense. Indeed, the property (iii) implies slopewise dominance of $\mathcal{F}$ on $\mathcal{Q}$ by Lemma 4.4.2 while the property (iv) says that $\mathcal{F}$ and $\mathcal{Q}$ have integer slopes.

We now assert that the property (i) is a formal consequence of the other properties. Note that the property (iii) is equivalent to slopewise dominance of $\mathcal{F}$ on $\mathcal{Q}$ by Lemma 4.2.2. Combining this with the property (v), we obtain slopewise dominance of $\mathcal{Q}^{\vee}$ on $\mathcal{F}^{\vee}$ by Lemma 4.2.5. Hence Lemma 4.2.2 now yields an inequality

$$\mathrm{rk}(\mathcal{Q}^{\leq \mu}) \geq \mathrm{rk}(\mathcal{F}^{\leq \mu}) \qquad \text{for every } \mu \in \mathbb{Q}.$$

We then deduce the desired inequality $\text{rk}(\mathcal{E}^{\leq \mu}) \geq \text{rk}(\mathcal{F}^{\leq \mu})$ for every $\mu \in \mathbb{Q}$ by combining the above inequality with the inequality in the property (ii).

Hence we only need to check the invariance of the other properties (ii) - (v). The invariance of the property (ii) is obvious since $\mathcal{E}$ and $\mathcal{Q}$ remain unchanged. The property (iii) is equivalent to slopewise dominance of $\mathcal{F}$ on $\mathcal{Q}$ by Lemma 4.2.2, so its invariance under replacing $\mathcal{F}$ by $\overline{\mathcal{F}}$ follows from Lemma 4.4.8. The invariance of the properties (iv) and (v) also follow immediately from Lemma 4.4.8. $\qquad\square$

Lemma 4.4.9 suggests that during our process we may lose the following assumptions:

- $\mathcal{E}$ and $\mathcal{F}$ have no common slopes.
- the equality in $\text{rk}(\mathcal{E}^{\leq \mu}) \geq \text{rk}(\mathcal{F}^{\leq \mu})$ holds only when $\mathcal{E}^{\leq \mu} \simeq \mathcal{F}^{\leq \mu}$.

Fortunately, losing either of these assumptions during our procedure will do no harm to our proof. In fact, these assumptions will be only necessary for establishing the fact that $c_{\mathcal{E},\mathcal{F}}(\mathcal{Q})$ strictly decreases after the first cycle of our procedure. In other words, our proof will be valid as long as we begin our procedure with all assumptions in Proposition 4.3.5.

Let us now prove the key inequality for Step 3.

Proposition 4.4.10. *Let $\mathcal{E}, \mathcal{F}$ and $\mathcal{Q}$ be nonzero vector bundles on $\mathcal{X}$ with the following properties:*

(i) $\text{rk}(\mathcal{E}^{\leq \mu}) \geq \text{rk}(\mathcal{F}^{\leq \mu})$ for every $\mu \in \mathbb{Q}$

(ii) $\text{rk}(\mathcal{E}^{\leq \mu}) \geq \text{rk}(\mathcal{Q}^{\leq \mu})$ for every $\mu \in \mathbb{Q}$ with equality only when $\mathcal{E}^{\leq \mu} \simeq \mathcal{Q}^{\leq \mu}$.

(iii) $\text{rk}(\mathcal{F}^{\geq \mu}) \geq \text{rk}(\mathcal{Q}^{\geq \mu})$ for every $\mu \in \mathbb{Q}$.

(iv) all slopes of $\mathcal{E}, \mathcal{F}$ and $\mathcal{Q}$ are integers.

(v) $\text{rk}(\mathcal{Q}) = \text{rk}(\mathcal{F})$.

Let $\overline{\mathcal{F}}$ be the maximal slope reduction of $\mathcal{F}$ to $\mathcal{Q}$. Then we have an inequality

$$c_{\mathcal{E},\mathcal{F}}(\mathcal{Q}) \geq c_{\mathcal{E},\overline{\mathcal{F}}}(\mathcal{Q}). \tag{4.9}$$

Moreover, equality in (4.9) never holds if $\mathcal{E}, \mathcal{F}$ and $\mathcal{Q}$ satisfy the following additional properties:

(vi) $\mathcal{E}$ and $\mathcal{F}$ have no common slopes.

(vii) For every $\mu \in \mathbb{Q}$, an equality $\text{rk}(\mathcal{E}^{\leq \mu}) = \text{rk}(\mathcal{F}^{\leq \mu})$ holds only when $\mathcal{E}^{\leq \mu} \simeq \mathcal{F}^{\leq \mu}$.

(viii) $\mu_{max}(\mathcal{F}) > \mu_{max}(\mathcal{Q})$

Proof. Set $\lambda = \mu_{\max}(\mathcal{Q})$ and $r = \text{rk}(\mathcal{F}^{> \lambda})$. By definition, we may write

$$\mathcal{F} = \mathcal{F}^{> \lambda} \oplus \mathcal{F}^{\leq \lambda} \qquad \text{and} \qquad \overline{\mathcal{F}} = \mathcal{O}(\lambda)^{\oplus r} \oplus \mathcal{F}^{\leq \lambda}. \tag{4.10}$$

Then we have

$$
\begin{aligned}
\deg(\mathcal{E}^{\vee} \otimes \mathcal{F})^{\geq 0} &= \deg(\mathcal{E}^{\vee} \otimes (\mathcal{F}^{>\lambda} \oplus \mathcal{F}^{\leq\lambda}))^{\geq 0} \\
&= \deg(\mathcal{E}^{\vee} \otimes \mathcal{F}^{>\lambda})^{\geq 0} + \deg(\mathcal{E}^{\vee} \otimes \mathcal{F}^{\leq\lambda})^{\geq 0}, \\
\deg(\mathcal{E}^{\vee} \otimes \overline{\mathcal{F}})^{\geq 0} &= \deg(\mathcal{E}^{\vee} \otimes (\mathcal{O}(\lambda)^{\oplus r} \oplus \mathcal{F}^{\leq\lambda}))^{\geq 0} \\
&= \deg(\mathcal{E}^{\vee} \otimes \mathcal{O}(\lambda)^{\oplus r})^{\geq 0} + \deg(\mathcal{E}^{\vee} \otimes \mathcal{F}^{\leq\lambda})^{\geq 0}, \\
\deg(\mathcal{Q}^{\vee} \otimes \mathcal{F})^{\geq 0} &= \deg(\mathcal{Q}^{\vee} \otimes (\mathcal{F}^{>\lambda} \oplus \mathcal{F}^{\leq\lambda}))^{\geq 0} \\
&= \deg(\mathcal{Q}^{\vee} \otimes \mathcal{F}^{>\lambda})^{\geq 0} + \deg(\mathcal{Q}^{\vee} \otimes \mathcal{F}^{\leq\lambda})^{\geq 0}, \\
\deg(\mathcal{Q}^{\vee} \otimes \overline{\mathcal{F}})^{\geq 0} &= \deg(\mathcal{Q}^{\vee} \otimes (\mathcal{O}(\lambda)^{\oplus r} \oplus \mathcal{F}^{\leq\lambda}))^{\geq 0} \\
&= \deg(\mathcal{Q}^{\vee} \otimes \mathcal{O}(\lambda)^{\oplus r})^{\geq 0} + \deg(\mathcal{Q}^{\vee} \otimes \mathcal{F}^{\leq\lambda})^{\geq 0}.
\end{aligned}
$$

Thus we obtain

$$
\deg(\mathcal{E}^{\vee} \otimes \mathcal{F})^{\geq 0} - \deg(\mathcal{E}^{\vee} \otimes \overline{\mathcal{F}})^{\geq 0} = \deg(\mathcal{E}^{\vee} \otimes \mathcal{F}^{>\lambda})^{\geq 0} - \deg(\mathcal{E}^{\vee} \otimes \mathcal{O}(\lambda)^{\oplus r})^{\geq 0}, \tag{4.11}
$$

$$
\deg(\mathcal{Q}^{\vee} \otimes \mathcal{F})^{\geq 0} - \deg(\mathcal{Q}^{\vee} \otimes \overline{\mathcal{F}})^{\geq 0} = \deg(\mathcal{Q}^{\vee} \otimes \mathcal{F}^{>\lambda})^{\geq 0} - \deg(\mathcal{Q}^{\vee} \otimes \mathcal{O}(\lambda)^{\oplus r})^{\geq 0}. \tag{4.12}
$$

Let us write $\overrightarrow{\mathrm{HN}}(\mathcal{E}) = (e_i), \overrightarrow{\mathrm{HN}}(\mathcal{F}^{>\lambda}) = (f_j), \overrightarrow{\mathrm{HN}}(\mathcal{Q}) = (q_k), \overrightarrow{\mathrm{HN}}(\mathcal{O}(\lambda)^{\oplus r}) = (\overline{f})$, and set $f := \sum f_j$. Note that we can write $\overline{f} = \sum \overline{f}_j$ where $\overline{f}_j$ denotes the vector obtained by reducing the slope of f_j to λ. By construction, we have the following observations:

(1) $f_x = \mathrm{rk}(\mathcal{F}^{>\lambda}) = r = \overline{f}_x$
(2) $f_y \geq \overline{f}_y$ with equality if and only if $f = \overline{f} = 0$.

We now use Lemma 3.2.4 to write the right side of (4.11) as

$$
\deg(\mathcal{E}^{\vee} \otimes \mathcal{F}^{>\lambda})^{\geq 0} - \deg(\mathcal{E}^{\vee} \otimes \mathcal{O}(\lambda)^{\oplus r})^{\geq 0} = \sum_{e_i \preceq f_j} e_i \times f_j - \sum_{\mu(e_i) \leq \lambda} e_i \times \overline{f}. \tag{4.13}
$$

Note that each e_i with $\mu(e_i) \leq \lambda$ satisfies $e_i \preceq f_j$ for all j since by construction we have $\mu(f_j) > \lambda$ for all j. We then find

$$
\sum_{\mu(e_i) \leq \lambda} e_i \times f_j \leq \sum_{e_i \preceq f_j} e_i \times f_j \tag{4.14}
$$

as each term on the right hand side is nonnegative. Now (4.13) yields

$$
\begin{aligned}
\deg(\mathcal{E}^{\vee} \otimes \mathcal{F}^{>\lambda})^{\geq 0} - \deg(\mathcal{E}^{\vee} \otimes \mathcal{O}(\lambda)^{\oplus r})^{\geq 0} &\geq \sum_{\mu(e_i) \leq \lambda} e_i \times f_j - \sum_{\mu(e_i) \leq \lambda} e_i \times \overline{f} \\
&= \sum_{\mu(e_i) \leq \lambda} e_i \times \sum f_j - \sum_{\mu(e_i) \leq \lambda} e_i \times \overline{f} \\
&= \sum_{\mu(e_i) \leq \lambda} e_i \times (f - \overline{f}).
\end{aligned}
$$

Since $(f - \overline{f})_x = 0$ as noted in (1), we have

$$
\sum_{\mu(e_i) \leq \lambda} e_i \times (f - \overline{f}) = \left(\sum_{\mu(e_i) \leq \lambda} e_i \right)_x \cdot (f - \overline{f})_y = \mathrm{rk}(\mathcal{E}^{\leq\lambda}) \cdot (f - \overline{f})_y.
$$

We thus obtain an inequality

$$
\deg(\mathcal{E}^{\vee} \otimes \mathcal{F}^{>\lambda})^{\geq 0} - \deg(\mathcal{E}^{\vee} \otimes \mathcal{O}(\lambda)^{\oplus r})^{\geq 0} \geq \mathrm{rk}(\mathcal{E}^{\leq\lambda}) \cdot (f - \overline{f})_y
$$

which is equivalent by (4.11) to an inequality

$$\deg(\mathcal{E}^\vee \otimes \mathcal{F})^{\geq 0} - \deg(\mathcal{E}^\vee \otimes \overline{\mathcal{F}})^{\geq 0} \geq \mathrm{rk}(\mathcal{E}^{\leq\lambda}) \cdot (f - \overline{f})_y. \tag{4.15}$$

Let us now use Lemma 3.2.4 to write the right side of (4.12) as

$$\deg(\mathcal{Q}^\vee \otimes \mathcal{F}^{>\lambda})^{\geq 0} - \deg(\mathcal{Q}^\vee \otimes \mathcal{O}(\lambda)^{\oplus r})^{\geq 0} = \sum_{q_k \preceq f_j} q_k \times f_j - \sum_{\mu(q_k) \leq \lambda} q_k \times \overline{f}.$$

Note that the conditions $q_k \preceq f_j$ and $\mu(q_k) \leq \lambda$ hold for all j and k; indeed, by construction we have $\mu(q_k) \leq \mu_{\max}(\mathcal{Q}) = \lambda < \mu(f_j)$ for all j and k. Hence we can simplify the above equation as

$$\begin{aligned} \deg(\mathcal{Q}^\vee \otimes \mathcal{F}^{>\lambda})^{\geq 0} - \deg(\mathcal{Q}^\vee \otimes \mathcal{O}(\lambda)^{\oplus r})^{\geq 0} &= \sum q_k \times f_j - \sum q_k \times \overline{f} \\ &= \sum q_k \times \sum f_j - \sum q_k \times \overline{f} \\ &= \sum q_k \times (f - \overline{f}). \end{aligned}$$

Now, as in the previous paragraph, we use the fact $(f_j - f_j')_x = 0$ from (1) to write

$$\sum q_k \times (f - \overline{f}) = \left(\sum q_k\right)_x - (f - \overline{f})_y = \mathrm{rk}(\mathcal{Q}) \cdot (f - \overline{f})_y$$

and consequently obtain an equation

$$\deg(\mathcal{Q}^\vee \otimes \mathcal{F}^{>\lambda})^{\geq 0} - \deg(\mathcal{Q}^\vee \otimes \mathcal{O}(\lambda)^{\oplus r})^{\geq 0} = \mathrm{rk}(\mathcal{Q}) \cdot (f - \overline{f})_y.$$

By (4.12), this equation is equivalent to

$$\deg(\mathcal{Q}^\vee \otimes \mathcal{F})^{\geq 0} - \deg(\mathcal{Q}^\vee \otimes \overline{\mathcal{F}})^{\geq 0} = \mathrm{rk}(\mathcal{Q}) \cdot (f - \overline{f})_y. \tag{4.16}$$

Note that we have

$$c_{\mathcal{E},\mathcal{F}}(\mathcal{Q}) - c_{\mathcal{E},\overline{\mathcal{F}}}(\mathcal{Q}) = \left(\deg(\mathcal{E}^\vee \otimes \mathcal{F})^{\geq 0} - \deg(\mathcal{E}^\vee \otimes \overline{\mathcal{F}})^{\geq 0}\right) - \left(\deg(\mathcal{Q}^\vee \otimes \mathcal{F})^{\geq 0} - \deg(\mathcal{Q}^\vee \otimes \overline{\mathcal{F}})^{\geq 0}\right).$$

Hence (4.15) and (4.16) together yields an inequality

$$c_{\mathcal{E},\mathcal{F}}(\mathcal{Q}) - c_{\mathcal{E},\overline{\mathcal{F}}}(\mathcal{Q}) \geq (\mathrm{rk}(\mathcal{E}^{\leq\lambda}) - \mathrm{rk}(\mathcal{Q})) \cdot (f - \overline{f})_y. \tag{4.17}$$

Now we observe $\mathcal{Q} = \mathcal{Q}^{\leq\mu_{\max}(\mathcal{Q})} = \mathcal{Q}^{\leq\lambda}$ and find

$$\mathrm{rk}(\mathcal{E}^{\leq\lambda}) - \mathrm{rk}(\mathcal{Q}) = \mathrm{rk}(\mathcal{E}^{\leq\lambda}) - \mathrm{rk}(\mathcal{Q}^{\leq\lambda}) \geq 0$$

where the inequality follows from the assumption (ii). Since we also have $(f - \overline{f})_y \geq 0$ as noted in (2), we obtain

$$(\mathrm{rk}(\mathcal{E}^{\leq\lambda}) - \mathrm{rk}(\mathcal{Q})) \cdot (f - \overline{f})_y \geq 0 \tag{4.18}$$

We thus deduce the desired inequality (4.9) from (4.17) and (4.18).

It remains to prove the last statement of Proposition 4.4.10. For the rest of the proof, we therefore assume that $\mathcal{E}, \mathcal{F}$ and $\mathcal{Q}$ satisfy the properties (vi), (vii) and (viii). We also suppose for contradiction that equality in (4.9) holds. Then we must have equality in both (4.17) and (4.18). The equality in (4.18) gives us two cases to consider, namely

(a) when $\mathrm{rk}(\mathcal{E}^{\leq\lambda}) = \mathrm{rk}(\mathcal{Q})$,
(b) when $(f - \overline{f})_y = 0$.

We first investigate the case (b). The defining condition $(f - \overline{f})_y = 0$ yields $f = 0$ by (2), which implies $\mathrm{rk}(\mathcal{F}^{>\lambda}) = r = 0$ by (1). The decompositions (4.10) then yield $\mathcal{F} = \overline{\mathcal{F}}$, implying $\mu_{\max}(\mathcal{F}) = \mu_{\max}(\mathcal{Q})$ by Lemma 4.4.8. We thus have a contradiction to the property (viii).

Let us now consider the case (a). We may assume that $\mathcal{F}^{>\lambda} \neq 0$, since otherwise we can argue as in the preceding paragraph to obtain a contradiction. The equality in (4.17) implies equality in (4.15), and consequently equality in (4.14). Then every term on the right side of (4.14) appears on the left side of (4.14), since every term in (4.14) is positive by the property (vi). Therefore every e_i that satisfies $e_i \preceq f_j$ for some j also satisfies $\mu(e_i) \leq \lambda$. In particular, we obtain $\mu(e_i) \leq \lambda$ for all e_i with $e_i \preceq f_1$. Since $\mu(f_1) = \mu_{\max}(\mathcal{F}^{>\lambda}) = \mu_{\max}(\mathcal{F})$, we deduce

$$\mathcal{E}^{\leq \mu_{\max}(\mathcal{F})} = \mathcal{E}^{\leq \lambda}. \tag{4.19}$$

We then use the defining condition $\mathrm{rk}(\mathcal{E}^{\leq \lambda}) = \mathrm{rk}(\mathcal{Q})$ and the assumption (v) to find

$$\mathrm{rk}(\mathcal{E}^{\leq \mu_{\max}(\mathcal{F})}) = \mathrm{rk}(\mathcal{E}^{\leq \lambda}) = \mathrm{rk}(\mathcal{Q}) = \mathrm{rk}(\mathcal{F}) = \mathrm{rk}(\mathcal{F}^{\leq \mu_{\max}(\mathcal{F})}),$$

which yields an isomorphism

$$\mathcal{E}^{\leq \mu_{\max}(\mathcal{F})} \simeq \mathcal{F}^{\leq \mu_{\max}(\mathcal{F})} = \mathcal{F}. \tag{4.20}$$

by the property (vii). Moreover, we observe $\mathcal{Q} = \mathcal{Q}^{\leq \mu_{\max}(\mathcal{Q})} = \mathcal{Q}^{\leq \lambda}$ to rewrite the defining condition $\mathrm{rk}(\mathcal{E}^{\leq \lambda}) = \mathrm{rk}(\mathcal{Q})$ as $\mathrm{rk}(\mathcal{E}^{\leq \lambda}) = \mathrm{rk}(\mathcal{Q}^{\leq \lambda})$, thereby obtaining another isomorphism

$$\mathcal{E}^{\leq \lambda} \simeq \mathcal{Q}^{\leq \lambda} = \mathcal{Q} \tag{4.21}$$

by the assumption (ii). Now (4.19), (4.20) and (4.21) together imply that $\mathcal{F} \simeq \mathcal{Q}$, thereby yielding a contradiction to the property (viii). $\qquad\square$

The following proposition translates the results of Lemma 4.4.9 and Proposition 4.4.10 in the setting of our slope reduction procedure.

Proposition 4.4.11. *Let $\mathcal{E}, \mathcal{F}$ and $\mathcal{Q}$ be nonzero vector bundles on $\mathcal{X}$ with the following properties:*

(i) $\mathrm{rk}(\mathcal{E}^{\leq \mu}) \geq \mathrm{rk}(\mathcal{F}^{\leq \mu})$ for every $\mu \in \mathbb{Q}$

(ii) $\mathrm{rk}(\mathcal{E}^{\leq \mu}) \geq \mathrm{rk}(\mathcal{Q}^{\leq \mu})$ for every $\mu \in \mathbb{Q}$ with equality only when $\mathcal{E}^{\leq \mu} \simeq \mathcal{Q}^{\leq \mu}$.

(iii) $\mathrm{rk}(\mathcal{F}^{\geq \mu}) \geq \mathrm{rk}(\mathcal{Q}^{\geq \mu})$ for every $\mu \in \mathbb{Q}$.

(iv) all slopes of $\mathcal{E}, \mathcal{F}$ and $\mathcal{Q}$ are integers.

(v) $\mathrm{rk}(\mathcal{Q}) = \mathrm{rk}(\mathcal{F})$.

Consider the decompositions

$$\mathcal{F} \simeq \mathcal{U} \oplus \mathcal{F}' \qquad \text{and} \qquad \mathcal{Q} \simeq \mathcal{U} \oplus \mathcal{Q}' \tag{4.22}$$

given by Lemma 4.2.4. Assume that $\mathcal{Q}' \neq 0$, and let $\overline{\mathcal{F}}'$ denote the maximal slope reduction of $\mathcal{F}'$ to $\mathcal{Q}'$.

(1) The properties (i) - (v) are invariant under replacing $\mathcal{F}$ by $\tilde{\mathcal{F}} := \mathcal{U} \oplus \overline{\mathcal{F}}'$.

(2) We have an inequality

$$c_{\mathcal{E}, \mathcal{F}}(\mathcal{Q}) \geq c_{\mathcal{E}, \tilde{\mathcal{F}}}(\mathcal{Q}). \tag{4.23}$$

(3) The inequality (4.23) becomes strict if $\mathcal{E}$ and $\mathcal{F}$ satisfy the following additional properties:

(vi) $\mathcal{E}$ and $\mathcal{F}$ have no common slopes.

(vii) For every $\mu \in \mathbb{Q}$, an equality $\mathrm{rk}(\mathcal{E}^{\leq \mu}) = \mathrm{rk}(\mathcal{F}^{\leq \mu})$ holds only when $\mathcal{E}^{\leq \mu} \simeq \mathcal{F}^{\leq \mu}$.
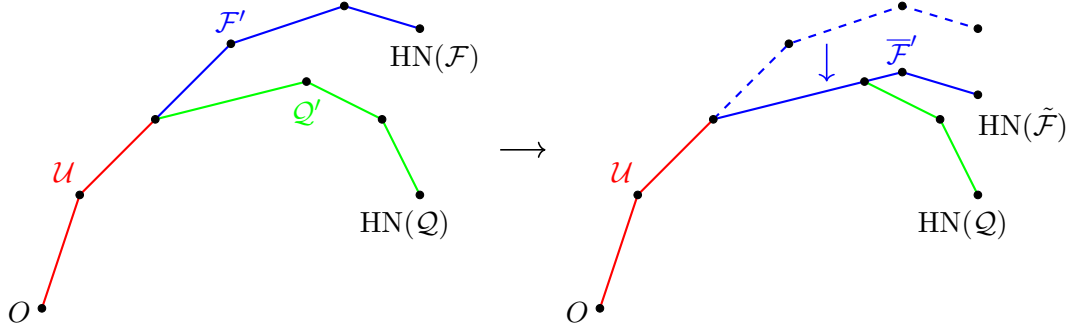
FIGURE 12. Illustration of the constructions in Proposition 4.4.11

Proof. Let us first verify that all constructions in the statement are valid. The validity of the decompositions (4.22) relies on slopewise dominance of $\mathcal{F}$ on $\mathcal{Q}$, which follows from the property (iii) by Lemma 4.2.2. For the validity of the maximal slope reduction of $\mathcal{F}'$ to $\mathcal{Q}'$, we verify slopewise dominance of $\mathcal{F}'$ on $\mathcal{Q}'$ by Lemma 4.2.4 and integer slopes of $\mathcal{F}'$ and $\mathcal{Q}'$ by the property (iv).

We assert that the properties (i) - (v) yield the corresponding properties for $\mathcal{E}, \mathcal{F}'$ and $\mathcal{Q}'$ as follows:

(i)' $\mathrm{rk}(\mathcal{E}^{\leq \mu}) \geq \mathrm{rk}(\mathcal{F}'^{\leq \mu})$ for every $\mu \in \mathbb{Q}$.

(ii)' $\mathrm{rk}(\mathcal{E}^{\leq \mu}) \geq \mathrm{rk}(\mathcal{Q}'^{\leq \mu})$ for every $\mu \in \mathbb{Q}$ with equality only when $\mathcal{E}^{\leq \mu} \simeq \mathcal{Q}'^{\leq \mu}$.

(iii)' $\mathrm{rk}(\mathcal{F}'^{\geq \mu}) \geq \mathrm{rk}(\mathcal{Q}'^{\geq \mu})$ for every $\mu \in \mathbb{Q}$.

(iv)' all slopes of $\mathcal{E}, \mathcal{F}'$ and $\mathcal{Q}'$ are integers.

(v)' $\mathrm{rk}(\mathcal{Q}') = \mathrm{rk}(\mathcal{F}')$.

We only need to check the properties (ii)' - (v)' since the property (i)' will then follow as a formal consequence of these properties as in the proof of Lemma 4.4.9. The property (ii)' follows from the property (ii) since $\mathcal{Q}' = \mathcal{Q}^{\leq \lambda}$ with $\lambda = \mu_{\max}(\mathcal{Q}')$. The property (iii)' is equivalent to slopewise dominance of $\mathcal{F}'$ on $\mathcal{Q}'$ which follows from Lemma 4.2.4. The properties (iv)' and (v)' follow immediately from the corresponding properties (iv) and (v) by construction.

With the properties (i)' - (v)' established, Lemma 4.4.9 and Proposition 4.4.10 now yield the following facts:

(1)' The properties (i)' - (v)' are invariant under replacing $\mathcal{F}'$ by $\overline{\mathcal{F}}'$.

(2)' We have an inequality
$$c_{\mathcal{E}, \mathcal{F}'}(\mathcal{Q}') \geq c_{\mathcal{E}, \overline{\mathcal{F}}'}(\mathcal{Q}'). \tag{4.24}$$

(3)' The inequality (4.24) becomes strict if $\mathcal{E}, \mathcal{F}'$ and $\mathcal{Q}'$ satisfy the following additional properties:

(vi)' $\mathcal{E}$ and $\mathcal{F}'$ have no common slopes.

(vii)' For every $\mu \in \mathbb{Q}$, an equality $\mathrm{rk}(\mathcal{E}^{\leq \mu}) = \mathrm{rk}(\mathcal{F}'^{\leq \mu})$ holds only when $\mathcal{E}^{\leq \mu} \simeq \mathcal{F}^{\leq \mu}$.

(viii)' $\mu_{\max}(\mathcal{F}') > \mu_{\max}(\mathcal{Q}')$.

We wish to deduce the statements (1), (2) and (3) respectively from the above facts (1)', (2)' and (3)'.

Let us now prove the statement (1). Note that, as in the proof of Lemma 4.4.9, we only need to show the invariance of the properties (ii) - (v). The invariance of the property (ii) is

evident since $\mathcal{E}$ and $\mathcal{Q}$ remain unchanged. For the invariance of the remaining properties (iii), (iv) and (v), we have to show that $\mathcal{E}, \tilde{\mathcal{F}}$ and $\mathcal{Q}$ satisfy the following properties:

(iiĩ) $\mathrm{rk}(\tilde{\mathcal{F}}^{\geq \mu}) \geq \mathrm{rk}(\mathcal{Q}^{\geq \mu})$ for every $\mu \in \mathbb{Q}$.

(iṽ) all slopes of $\mathcal{E}, \tilde{\mathcal{F}}$ and $\mathcal{Q}$ are integers.

(ṽ) $\mathrm{rk}(\mathcal{Q}) = \mathrm{rk}(\tilde{\mathcal{F}})$.

After writing $\tilde{\mathcal{F}} = \mathcal{U} \oplus \overline{\mathcal{F}}'$ by definition and also $\mathcal{Q} = \mathcal{U} \oplus \mathcal{Q}'$ as in (4.22), we deduce all of these properties from the invariance of the properties (iii)', (iv)' and (v)' noted in (1)'

We move on to the statement (2). Since $\mathcal{Q}' \neq 0$ by our assumption, Lemma 4.2.4 yields

$$\mu_{\min}(\mathcal{U}) \geq \mu_{\max}(\mathcal{F}') > \mu_{\max}(\mathcal{Q}') = \mu_{\max}(\overline{\mathcal{F}}') \qquad \text{if } \mathcal{U} \neq 0. \tag{4.25}$$

Then by Corollary 3.2.5 we obtain

$$\deg(\mathcal{U}^\vee \otimes \mathcal{F}')^{\geq 0} = \deg(\mathcal{U}^\vee \otimes \overline{\mathcal{F}}')^{\geq 0} = 0. \tag{4.26}$$

Note that (4.26) does not require the condition $\mathcal{U} \neq 0$ from (4.25) since it evidently holds when $\mathcal{U} = 0$. Now we use (4.26) and the decompositions in (4.22) to find

$$\begin{aligned}
\deg(\mathcal{E}^\vee \otimes \mathcal{F})^{\geq 0} &= \deg(\mathcal{E}^\vee \otimes (\mathcal{U} \oplus \mathcal{F}'))^{\geq 0} \\
&= \deg(\mathcal{E}^\vee \otimes \mathcal{U})^{\geq 0} + \deg(\mathcal{E}^\vee \otimes \mathcal{F}')^{\geq 0}, \\
\deg(\mathcal{E}^\vee \otimes \tilde{\mathcal{F}})^{\geq 0} &= \deg(\mathcal{E}^\vee \otimes (\mathcal{U} \oplus \overline{\mathcal{F}}'))^{\geq 0} \\
&= \deg(\mathcal{E}^\vee \otimes \mathcal{U})^{\geq 0} + \deg(\mathcal{E}^\vee \oplus \overline{\mathcal{F}}')^{\geq 0}, \\
\deg(\mathcal{Q}^\vee \otimes \mathcal{F})^{\geq 0} &= \deg((\mathcal{U} \oplus \mathcal{Q}')^\vee \otimes (\mathcal{U} \oplus \mathcal{F}'))^{\geq 0} \\
&= \deg(\mathcal{U}^\vee \otimes \mathcal{U})^{\geq 0} + \deg(\mathcal{Q}'^\vee \otimes \mathcal{U})^{\geq 0} + \deg(\mathcal{U}^\vee \otimes \mathcal{F}')^{\geq 0} + \deg(\mathcal{Q}'^\vee \otimes \mathcal{F}')^{\geq 0} \\
&= \deg(\mathcal{U}^\vee \otimes \mathcal{U})^{\geq 0} + \deg(\mathcal{Q}'^\vee \otimes \mathcal{U})^{\geq 0} + \deg(\mathcal{Q}'^\vee \otimes \mathcal{F}')^{\geq 0}, \\
\deg(\mathcal{Q}^\vee \otimes \tilde{\mathcal{F}})^{\geq 0} &= \deg((\mathcal{U} \oplus \mathcal{Q}')^\vee \otimes (\mathcal{U} \oplus \overline{\mathcal{F}}'))^{\geq 0} \\
&= \deg(\mathcal{U}^\vee \otimes \mathcal{U})^{\geq 0} + \deg(\mathcal{Q}'^\vee \otimes \mathcal{U})^{\geq 0} + \deg(\mathcal{U}^\vee \otimes \overline{\mathcal{F}}')^{\geq 0} + \deg(\mathcal{Q}'^\vee \otimes \overline{\mathcal{F}}')^{\geq 0} \\
&= \deg(\mathcal{U}^\vee \otimes \mathcal{U})^{\geq 0} + \deg(\mathcal{Q}'^\vee \otimes \mathcal{U})^{\geq 0} + \deg(\mathcal{Q}'^\vee \otimes \overline{\mathcal{F}}')^{\geq 0}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
c_{\mathcal{E}, \mathcal{F}}(\mathcal{Q}) - c_{\mathcal{E}, \tilde{\mathcal{F}}}(\mathcal{Q}) &= \big(\deg(\mathcal{E}^\vee \otimes \mathcal{F})^{\geq 0} - \deg(\mathcal{E}^\vee \otimes \tilde{\mathcal{F}})^{\geq 0} \big) - \big(\deg(\mathcal{Q}^\vee \otimes \mathcal{F})^{\geq 0} - \deg(\mathcal{Q}^\vee \otimes \tilde{\mathcal{F}})^{\geq 0} \big) \\
&= \big(\deg(\mathcal{E}^\vee \otimes \mathcal{F}')^{\geq 0} - \deg(\mathcal{E}^\vee \otimes \overline{\mathcal{F}}')^{\geq 0} \big) - \big(\deg(\mathcal{Q}'^\vee \otimes \mathcal{F}')^{\geq 0} - \deg(\mathcal{Q}'^\vee \otimes \overline{\mathcal{F}}')^{\geq 0} \big) \\
&= c_{\mathcal{E}, \mathcal{F}'}(\mathcal{Q}') - c_{\mathcal{E}, \overline{\mathcal{F}}'}(\mathcal{Q}'). \tag{4.27}
\end{aligned}$$

Hence the statement (2) now follows directly from the fact (2)'.

We now consider the final statement (3). In accordance with the statement, we assume that $\mathcal{E}$ and $\mathcal{F}$ satisfy the properties (vi) and (vii). By 4.27, the inequality (4.23) becomes strict if and only if the inequality (4.24) is strict. Therefore we can prove the statement (3) by verifying that $\mathcal{E}, \mathcal{F}'$ and $\mathcal{Q}'$ satisfy the properties (vi)', (vii)' and (viii)' as stated in the fact (3)'. The property (vi)' follows from the property (vi) since $\mathcal{F}'$ is a direct summand of $\mathcal{F}$ by construction. Moreover, by Lemma 4.2.4 the property (viii)' follows from our assumption $\mathcal{Q}' \neq 0$. Hence it remains to verify the property (vii)'. Suppose that we have $\mathrm{rk}(\mathcal{E}^{\leq \mu}) = \mathrm{rk}(\mathcal{F}'^{\leq \mu})$ for some μ. We wish to prove that $\mathcal{E}^{\leq \mu} \simeq \mathcal{F}'^{\leq \mu}$. Since $\mathcal{F}'$ is a direct summand of $\mathcal{F}$, we find

$$\mathrm{rk}(\mathcal{F}'^{\leq \mu}) \leq \mathrm{rk}(\mathcal{F}^{\leq \mu}) \qquad \text{for every } \mu \in \mathbb{Q} \tag{4.28}$$

with equality if and only if $\mathcal{F}'^{\leq\mu} = \mathcal{F}^{\leq\mu}$. We thus obtain a series of inequalities

$$\mathrm{rk}(\mathcal{F}'^{\leq\mu}) \leq \mathrm{rk}(\mathcal{F}^{\leq\mu}) \leq \mathrm{rk}(\mathcal{E}^{\leq\mu}) \qquad \text{for every } \mu \in \mathbb{Q} \tag{4.29}$$

combining (4.28) and the property (i). Now the equality $\mathrm{rk}(\mathcal{E}^{\leq\mu}) = \mathrm{rk}(\mathcal{F}'^{\leq\mu})$ implies that both equalities should hold in (4.29). Hence the equality condition for (4.28) and the property (vii) together yield

$$\mathcal{F}'^{\leq\mu} = \mathcal{F}^{\leq\mu} \simeq \mathcal{E}^{\leq\mu}.$$

We thus complete the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We are finally ready to complete Step 3.

Proposition 4.4.12. *Proposition 4.3.5 holds under the additional assumptions that* $\mathrm{rk}(\mathcal{Q}) = \mathrm{rk}(\mathcal{F})$ *and that all slopes of* $\mathcal{E}, \mathcal{F}$ *and* $\mathcal{Q}$ *are integers.*

Proof. Let $\mathcal{E}, \mathcal{F}$ and $\mathcal{Q}$ be vector bundles on $\mathcal{X}$ satisfying the following properties:

   (i) $\mathrm{rk}(\mathcal{E}^{\leq\mu}) \geq \mathrm{rk}(\mathcal{F}^{\leq\mu})$ for every $\mu \in \mathbb{Q}$ with equality only when $\mathcal{E}^{\leq\mu} \simeq \mathcal{F}^{\leq\mu}$.
   (ii) $\mathrm{rk}(\mathcal{E}^{\leq\mu}) \geq \mathrm{rk}(\mathcal{Q}^{\leq\mu})$ for every $\mu \in \mathbb{Q}$ with equality only when $\mathcal{E}^{\leq\mu} \simeq \mathcal{Q}^{\leq\mu}$.
   (iii) $\mathrm{rk}(\mathcal{F}^{\geq\mu}) \geq \mathrm{rk}(\mathcal{Q}^{\geq\mu})$ for every $\mu \in \mathbb{Q}$.
   (iv) all slopes of $\mathcal{E}, \mathcal{F}$ and $\mathcal{Q}$ are integers.
   (v) $\mathrm{rk}(\mathcal{Q}) = \mathrm{rk}(\mathcal{F})$.
   (vi) $\mathcal{E}$ and $\mathcal{F}$ have no common slopes.

We wish to prove that the inequality (4.5) holds with equality if and only if $\mathcal{Q} = \mathcal{F}$.

Let us define a sequence $(\mathcal{F}_n)$ of vector bundles on $\mathcal{X}$ as follows:

   (I) Set $\mathcal{F}_0 := \mathcal{F}$.
   (II) For each $n \geq 0$, consider the decompositions

$$\mathcal{F}_n \simeq \mathcal{U}_n \oplus \mathcal{F}'_n \qquad \text{and} \qquad \mathcal{Q} \simeq \mathcal{U}_n \oplus \mathcal{Q}'_n \tag{4.30}$$

   given by Lemma 4.2.4. If $\mathcal{Q}'_n = 0$, we make $\mathcal{F}_n$ the final term of the sequence. Otherwise, we set

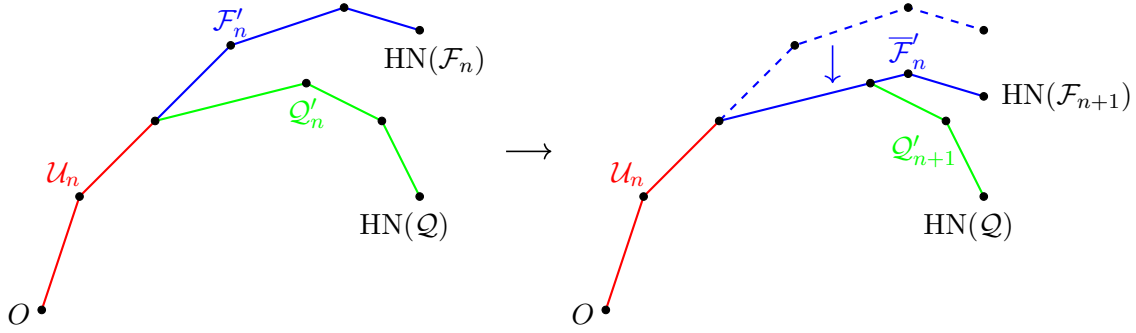$$\mathcal{F}_{n+1} := \mathcal{U}_n \oplus \overline{\mathcal{F}}'_n$$

   where $\overline{\mathcal{F}}'_n$ denotes the maximal slope reduction of $\mathcal{F}'_n$ to $\mathcal{Q}'_n$.

An induction argument using Proposition 4.4.11 yields the following facts:

   (a) The sequence $(\mathcal{F}_n)$ is well-defined with the following properties:
      (i_n) $\mathrm{rk}(\mathcal{E}^{\leq\mu}) \geq \mathrm{rk}(\mathcal{F}_n^{\leq\mu})$ for every $\mu \in \mathbb{Q}$
      (ii_n) $\mathrm{rk}(\mathcal{E}^{\leq\mu}) \geq \mathrm{rk}(\mathcal{Q}^{\leq\mu})$ for every $\mu \in \mathbb{Q}$ with equality only when $\mathcal{E}^{\leq\mu} \simeq \mathcal{Q}^{\leq\mu}$.
      (iii_n) $\mathrm{rk}(\mathcal{F}_n^{\geq\mu}) \geq \mathrm{rk}(\mathcal{Q}^{\geq\mu})$ for every $\mu \in \mathbb{Q}$.
      (iv_n) all slopes of $\mathcal{E}, \mathcal{F}_n$ and $\mathcal{Q}$ are integers.
      (v_n) $\mathrm{rk}(\mathcal{Q}) = \mathrm{rk}(\mathcal{F}_n)$.
   (b) We have an inequality

$$c_{\mathcal{E},\mathcal{F}_n}(\mathcal{Q}) \geq c_{\mathcal{E},\mathcal{F}_{n+1}}(\mathcal{Q}). \tag{4.31}$$

   (c) The inequality (4.31) is strict if $\mathcal{E}$ and $\mathcal{F}_n$ satisfy the following additional properties:
      (vi_n) $\mathcal{E}$ and $\mathcal{F}_n$ have no common slopes.
      (vii_n) For every $\mu \in \mathbb{Q}$, an equality $\mathrm{rk}(\mathcal{E}^{\leq\mu}) = \mathrm{rk}(\mathcal{F}_n^{\leq\mu})$ holds only when $\mathcal{E}^{\leq\mu} \simeq \mathcal{F}_n^{\leq\mu}$.

FIGURE 13. Construction of the sequence $(\mathcal{F}_n)$

We assert that the sequence $(\mathcal{F}_n)$ is finite. It suffices to prove

$$\mathrm{rk}(\mathcal{U}_n) < \mathrm{rk}(\mathcal{U}_{n+1}) \tag{4.32}$$

since we have $\mathrm{rk}(\mathcal{U}_n) \leq \mathrm{rk}(\mathcal{Q})$ by (4.30). To this end, we align the polygons $\mathrm{HN}(\mathcal{F}_n)$ and $\mathrm{HN}(\mathcal{Q})$ so that their left endpoints lie at the origin. The proof of Lemma 4.2.4 shows that $\mathcal{U}_n$ represents the common part of $\mathrm{HN}(\mathcal{F}_n)$ and $\mathrm{HN}(\mathcal{Q})$. Moreover, since $\overline{\mathcal{F}}'_n$ is the maximal slope reduction of $\mathcal{F}'_n$ to $\mathcal{Q}'_n$, the polygons $\mathrm{HN}(\overline{\mathcal{F}}'_n)$ and $\mathrm{HN}(\mathcal{Q}'_n)$ with their left endpoints aligned have some nontrivial common part which we represent by a nonzero vector bundle $\mathcal{T}_n$. Let us now consider the decompositions

$$\mathcal{F}_{n+1} = \mathcal{U}_n \oplus \overline{\mathcal{F}}'_n \qquad \text{and} \qquad \mathcal{Q} = \mathcal{U}_n \oplus \mathcal{Q}'_n \tag{4.33}$$

given by the definition of $\mathcal{F}_{n+1}$ and (4.30). The definition of $\mathcal{F}'_{n+1}$ assumes that $\mathcal{Q}'_n \neq 0$, so Lemma 4.2.4 yields

$$\mu_{\min}(\mathcal{U}_n) > \mu_{\max}(\mathcal{Q}'_n) = \mu_{\max}(\overline{\mathcal{F}}'_n) \qquad \text{if } \mathcal{U}_n \neq 0.$$

Hence the decompositions (4.33) imply that the common part of $\mathrm{HN}(\mathcal{F}_{n+1})$ and $\mathrm{HN}(\mathcal{Q})$ (with their left endpoints at the origin) is given by $\mathcal{U}_n \oplus \mathcal{T}_n$. Now we find $\mathcal{U}_{n+1} \simeq \mathcal{U}_n \oplus \mathcal{T}_n$ by the proof of Lemma 4.2.4, and consequently obtain the inequality (4.32) as $\mathcal{T}$ is nonzero.

Let us now denote by r the index of the final term in the sequence $(\mathcal{F}_n)$. Since $\mathcal{Q}'_r = 0$ by (II), the decompositions in (4.30) yield

$$\mathcal{F}_r \simeq \mathcal{U}_r \oplus \mathcal{F}'_r \qquad \text{and} \qquad \mathcal{Q} \simeq \mathcal{U}_r.$$

Hence the property (v_n) for $n = r$ implies

$$\mathcal{F}_r \simeq \mathcal{Q}, \tag{4.34}$$

which in turn yields $c_{\mathcal{E},\mathcal{F}_r}(\mathcal{Q}) = 0$ by Definition 4.4.1. Now by the fact (b) we find

$$c_{\mathcal{E},\mathcal{F}}(\mathcal{Q}) = c_{\mathcal{E},\mathcal{F}_0}(\mathcal{Q}) \geq c_{\mathcal{E},\mathcal{F}_1}(\mathcal{Q}) \geq \cdots \geq c_{\mathcal{E},\mathcal{F}_r}(\mathcal{Q}) = 0, \tag{4.35}$$

thereby establishing the desired inequality (4.5).

Our final task is to show that equality in (4.5) holds if and only if $\mathcal{Q} = \mathcal{F}$. Since equality in (4.5) evidently holds if $\mathcal{Q} = \mathcal{F}$, we only need to show that equality in (4.5) implies $\mathcal{Q} = \mathcal{F}$. The properties (i) and (vi) together imply that whenever $r \geq 1$ we have a strict inequality

$$c_{\mathcal{E},\mathcal{F}_0}(\mathcal{Q}) > c_{\mathcal{E},\mathcal{F}_1}(\mathcal{Q})$$

by the fact (c). Hence we deduce from (4.35) that equality in (4.5) holds only if $r = 0$, which implies $\mathcal{F} = \mathcal{F}_0 \simeq \mathcal{Q}$ by (4.34). □

We thus complete the proof of Proposition 4.3.5, and therefore the proof of Theorem 4.1.1.

APPENDIX A. CLASSIFICATION OF QUOTIENT BUNDLES ON $\mathbb{P}^1$

by SERIN HONG AND HANNAH LARSON

A.1. Main statement.

In this appendix, we establish an analogue of Theorem 4.1.1 for vector bundles on the projective line $\mathbb{P}^1$ over an arbitrary field k. For each integer d, we denote by $\mathcal{O}(d)$ the d-th Serre twist of the trivial line bundle on $\mathbb{P}^1$. It is a classical theorem (often attributed to Grothendieck) that every vector bundle $\mathcal{V}$ on $\mathbb{P}^1$ admits a direct sum decomposition

$$\mathcal{V} \simeq \bigoplus_{i=1}^{r} \mathcal{O}(d_i) \qquad \text{with } d_i \in \mathbb{Z}. \tag{A.1}$$

Now we can state the main statement of this appendix as follows:

Theorem A.1.1. *Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on $\mathbb{P}^1$ with direct sum decompositions*

$$\mathcal{E} \simeq \bigoplus_{i=1}^{r} \mathcal{O}(a_i) \qquad and \qquad \mathcal{F} \simeq \bigoplus_{j=1}^{s} \mathcal{O}(b_j) \tag{A.2}$$

for some integers $a_1 \leq \cdots \leq a_r$ and $b_1 \leq \cdots \leq b_s$. Then $\mathcal{F}$ arises as a quotient of $\mathcal{E}$ if and only if for each $j = 1, \cdots, s$, we have either $b_j \geq a_{j+1}$ or $b_i = a_i$ for all $i = 1, \cdots, j$.

This theorem is indeed an analogue of Theorem 4.1.1 for vector bundles on $\mathbb{P}^1$. For every vector bundle $\mathcal{V}$ on $\mathbb{P}^1$, we can use a direct sum decomposition as in (A.1) to define its Harder-Narasimhan polygon $\mathrm{HN}(\mathcal{V})$ and vector bundles $\mathcal{V}^{\leq \mu}$ for every $\mu \in \mathbb{Q}$. Then we can state Theorem A.1.1 in exact accordance with Theorem 4.1.1.

In the subsequent sections, we present two proofs of Theorem A.1.1. The first proof is based on some elementary linear algebra, and is largely inspired by the argument of Eisenbud-Harris [EH16, Proposition 6.30]. The second proof is based on dimension analysis on moduli spaces of bundle maps, and is essentially identical to our proof of Theorem 4.1.1.

A.2. First proof: elementary linear algebra.

For the rest of this appendix, we take $\mathcal{E}$ and $\mathcal{F}$ to be vector bundles on $\mathbb{P}^1$ with direct sum decompositions as in (A.2).

Lemma A.2.1. *The existence of a surjective bundle map $\mathcal{E} \twoheadrightarrow \mathcal{F}$ amounts to the existence of an $s \times r$ matrix M over the polynomial ring $k[x, y]$ with the following properties:*

(i) The (p, q)-th entry of M is either zero or homogeneous of degree $b_p - a_q$.

(ii) The $s \times s$ minors of M have no common zeros.

Proof. For each integer $d \geq 0$, we can canonically identify $H^0(\mathbb{P}^1, \mathcal{O}(d))$ as the space of degree d homogeneous polynomials in $k[x, y]$. In addition, we have a natural identification $\mathrm{Hom}_{\mathbb{P}^1}(\mathcal{O}(a_q), \mathcal{O}(b_p)) \cong H^0(\mathbb{P}^1, \mathcal{O}(b_p - a_q))$ for each $p = 1, \cdots, s$ and $q = 1, \cdots, r$. Therefore every bundle map $\mathcal{E} \to \mathcal{F}$ can be represented by an $s \times r$ matrix M over $k[x, y]$ with the property (i). Moreover, the map $\mathcal{E} \to \mathcal{F}$ is surjective if and only if M has rank s at all points on $\mathbb{P}^1$, which amounts to having the property (ii). $\square$

Remark. In the proof, when we identify $H^0(\mathbb{P}^1, \mathcal{O}(d))$ as the space of degree d homogeneous polynomials over k, we take the convention that the zero polynomial is homogeneous of all nonnegative degrees.

Proposition A.2.2. *If $\mathcal{F}$ arises as a quotient of $\mathcal{E}$, then for each $j = 1, \cdots, s$, we have either $b_j \geq a_{j+1}$ or $b_i = a_i$ for all $i = 1, \cdots, j$.*

Proof. Suppose that we have $b_j < a_{j+1}$ for some $j = 1, \cdots, s$. We wish to show $b_i = a_i$ for $i = 1, \cdots, j$. Since $\mathcal{F}$ arises as a quotient of $\mathcal{E}$, we can take an $s \times r$ matrix M over $k[x, y]$ as in Lemma A.2.1. Then for each $p = 1, \cdots, j$ and $q = j+1, \cdots, r$, we find $b_p \leq b_j < a_{j+1} \leq a_q$ and consequently deduce that the (p, q)-th entry of M is zero by the property (i) in Lemma A.2.1. In other words, M has a block decomposition

$$M = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$$

where A is a $j \times j$ matrix over $k[x, y]$. Hence any nonzero $s \times s$ minor of M should be divisible by the determinant of A. Now the property (ii) in Lemma A.2.1 implies that the determinant of A should be a constant nonzero polynomial; otherwise it has a nontrivial zero at which all $s \times s$ minors of M vanish.

We then observe that for each $i = 1, \cdots, j$ we must have $b_i \geq a_i$; otherwise, we find $b_p \leq b_i < a_i \leq a_q$ for each $p = 1, \cdots, i$ and $q = i+1, \cdots, r$, and consequently deduce by the property (i) in Lemma A.2.1 that A has a block decomposition

$$A = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}$$

with A_{11} being an $i \times (i-1)$ matrix and thus has a zero determinant. Similarly, for each $i = 1, \cdots, j$ we must have $b_i \leq a_i$; otherwise, we find $b_p \geq b_i > a_i \geq a_q$ for each $p = i, \cdots, s$ and $q = 1, \cdots, i$, and consequently deduce by the property (i) in Lemma A.2.1 that A has a block decomposition

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where A_{21} is an $(s-i+1) \times i$ matrix with entries of positive degree, and thus has a determinant which is either zero or of positive degree. Therefore we conclude that a_i and b_i are equal for each $i = 1, \cdots, j$ as desired. $\qquad\square$

Proposition A.2.3. *Assume that for each $j = 1, \cdots, s$, we have either $b_j \geq a_{j+1}$ or $b_i = a_i$ for all $i = 1, \cdots, j$. Then $\mathcal{F}$ arises as a quotient of $\mathcal{E}$.*

Proof. Let l be the largest integer with the property $a_l = b_l$. Take M to be the $s \times r$ matrix whose nonzero entries are given as follows:

$$M = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ \hline & & & x^{b_{l+1}-a_{l+1}} & y^{b_{l+1}-a_{l+2}} & & \\ & & & & \ddots & & \ddots \\ & & & & & x^{b_s-a_s} & y^{b_s-a_{s+1}} \end{pmatrix}$$

It suffices to show that M satisfies the properties (i) and (ii) in Lemma A.2.1. The property (i) is evident by construction. The property (ii) follows from the fact that the $s \times s$ minor with columns $1, \cdots, s$ is a power of x while the $s \times s$ minor with columns $1, \cdots, l, l+2, \cdots, s+1$ is a power of y. $\qquad\square$

We now deduce Theorem A.1.1 from Proposition A.2.2 and Proposition A.2.3.

A.3. **Second proof: dimension analysis on moduli spaces of bundle maps.**

Let us denote by $\mathrm{Sch}_{/k}$ the category of k-schemes. For every vector bundle $\mathcal{V}$ on $\mathbb{P}^1$, we will write $\chi(\mathcal{V}) := \mathrm{rk}(\mathcal{V}) + \deg(\mathcal{V})$. In addition, for arbitrary vector bundles $\mathcal{V}$ and $\mathcal{W}$ on $\mathbb{P}^1$, we can define the moduli functors $\mathcal{H}\mathrm{om}_{\mathbb{P}^1}(\mathcal{V}, \mathcal{W})$, $\mathcal{S}\mathrm{urj}_{\mathbb{P}^1}(\mathcal{V}, \mathcal{W})$, and $\mathcal{I}\mathrm{nj}_{\mathbb{P}^1}(\mathcal{V}, \mathcal{W})$ on $\mathrm{Sch}_{/k}$ as in Definition 3.1.1. Then we have the following analogue of Proposition 3.1.2:

Proposition A.3.1. *Let $\mathcal{V}$ and $\mathcal{W}$ be vector bundles on $\mathbb{P}^1$.*

(1) *$\mathcal{H}\mathrm{om}_{\mathbb{P}^1}(\mathcal{V}, \mathcal{W})$ is represented by the affine scheme $\mathbb{A}_k^{\chi(\mathcal{V}^\vee \otimes \mathcal{W})^{\geq 0}}$.*
(2) *$\mathcal{S}\mathrm{urj}_{\mathbb{P}^1}(\mathcal{V}, \mathcal{W})$ and $\mathcal{I}\mathrm{nj}_{\mathbb{P}^1}(\mathcal{V}, \mathcal{W})$ are represented by an open subscheme of $\mathcal{H}\mathrm{om}_{\mathbb{P}^1}(\mathcal{V}, \mathcal{W})$, and thus are either empty or of dimension $\chi(\mathcal{V}^\vee \otimes \mathcal{W})^{\geq 0}$.*

Proof. The functor $\mathcal{H}\mathrm{om}_{\mathbb{P}^1}(\mathcal{V}, \mathcal{W})$ is indeed represented by $\mathrm{Spec}\,(\mathrm{Sym}_k\, H^0(\mathbb{P}^1, \mathcal{V}^\vee \otimes \mathcal{W})^\vee)$, whose Krull dimension is given by $\dim_k H^0(\mathbb{P}^1, \mathcal{V}^\vee \otimes \mathcal{W}) = \chi(\mathcal{V}^\vee \otimes \mathcal{W})^{\geq 0}$. Now we can argue exactly as in [BFH$^+$17, §3.3] to establish the second statement. $\qquad\square$

Moreover, we have an analogue of Proposition 3.1.3 as follows:

Proposition A.3.2. *$\mathcal{F}$ arises as a quotient of $\mathcal{E}$ if the following conditions are satisfied:*

(i) *There exists a nonzero bundle map $\mathcal{E} \to \mathcal{F}$.*
(ii) *For any $\mathcal{Q} \subsetneq \mathcal{F}$ which also occurs as a quotient of $\mathcal{E}$ we have an inequality*

$$\chi(\mathcal{E}^\vee \otimes \mathcal{Q})^{\geq 0} + \chi(\mathcal{Q}^\vee \otimes \mathcal{F})^{\geq 0} < \chi(\mathcal{E}^\vee \otimes \mathcal{F})^{\geq 0} + \chi(\mathcal{Q}^\vee \otimes \mathcal{Q})^{\geq 0}.$$

Proof. Let S be the set of isomorphism classes of subsheaves $\mathcal{Q} \subseteq \mathcal{F}$ which also occur as a quotient of $\mathcal{E}$. We assert that S is finite. The Harder-Narasimhan theory for vector bundles on $\mathbb{P}^1$ is almost identical to the Harder-Narasimhan theory for vector bundles on $\mathcal{X}$, as we only need an additional requirement that all Harder-Narasimhan slopes are integers. In particular, for vector bundles on $\mathbb{P}^1$ we can define the notion of slopewise dominance and obtain analogues of Propositions 4.3.2 and 4.3.3. It follows that the slopes of every $\mathcal{Q} \in S$ are bounded by $\mu_{\min}(\mathcal{E})$ and $\mu_{\max}(\mathcal{F})$. Since vector bundles on $\mathbb{P}^1$ have integer slopes in their HN polygons, we deduce that S is finite.

Let us now assume for contradiction that $\mathcal{F}$ does not arise as a quotient of $\mathcal{E}$. Then S does not contain the isomorphism class of $\mathcal{F}$. For each $\mathcal{Q} \in S$, we define $\mathcal{H}\mathrm{om}_{\mathbb{P}^1}(\mathcal{E}, \mathcal{F})_{\mathcal{Q}}$ to be the image of the natural map

$$\mathcal{S}\mathrm{urj}_{\mathbb{P}^1}(\mathcal{E}, \mathcal{Q}) \times_{\mathrm{Spec}\,(k)} \mathcal{I}\mathrm{nj}_{\mathbb{P}^1}(\mathcal{Q}, \mathcal{F}) \to \mathcal{H}\mathrm{om}_{\mathbb{P}^1}(\mathcal{E}, \mathcal{F})$$

induced by composition of bundle maps, and write $\overline{\mathcal{H}\mathrm{om}_{\mathbb{P}^1}(\mathcal{E}, \mathcal{F})_{\mathcal{Q}}}$ for its closure in $\mathcal{H}\mathrm{om}_{\mathbb{P}^1}(\mathcal{E}, \mathcal{F})$. Since $\mathcal{H}\mathrm{om}_{\mathbb{P}^1}(\mathcal{E}, \mathcal{F})$ is represented by $\mathbb{A}_k^{\chi(\mathcal{V}^\vee \otimes \mathcal{W})^{\geq 0}}$ as noted in Proposition A.3.1, for each locally closed subscheme $\dim \mathcal{H}\mathrm{om}_{\mathbb{P}^1}(\mathcal{E}, \mathcal{F})_{\mathcal{Q}}$ with $\mathcal{Q} \in S$ we find

$$\dim \mathcal{H}\mathrm{om}_{\mathbb{P}^1}(\mathcal{E}, \mathcal{F})_{\mathcal{Q}} = \dim \overline{\mathcal{H}\mathrm{om}_{\mathbb{P}^1}(\mathcal{E}, \mathcal{F})_{\mathcal{Q}}}.$$

By construction, $\mathcal{H}\mathrm{om}_{\mathbb{P}^1}(\mathcal{E}, \mathcal{F})$ is covered by the subschemes $\overline{\mathcal{H}\mathrm{om}_{\mathbb{P}^1}(\mathcal{E}, \mathcal{F})_{\mathcal{Q}}}$ with $\mathcal{Q} \in S$. As the set S is finite, we find

$$\dim \mathcal{H}\mathrm{om}_{\mathbb{P}^1}(\mathcal{E}, \mathcal{F}) = \sup_{\mathcal{Q} \in S} \dim \overline{\mathcal{H}\mathrm{om}_{\mathbb{P}^1}(\mathcal{E}, \mathcal{F})_{\mathcal{Q}}} = \sup_{\mathcal{Q} \in S} \dim \mathcal{H}\mathrm{om}_{\mathbb{P}^1}(\mathcal{E}, \mathcal{F})_{\mathcal{Q}}. \qquad (\mathrm{A}.3)$$

In addition, we can argue exactly as in [BFH$^+$17, Lemma 3.3.10] to show that $\mathcal{H}\mathrm{om}_{\mathbb{P}^1}(\mathcal{E}, \mathcal{F})_{\mathcal{Q}}$ for each $\mathcal{Q} \in S$ is either empty or satisfies

$$\dim \mathcal{H}\mathrm{om}_{\mathbb{P}^1}(\mathcal{E}, \mathcal{F})_{\mathcal{Q}} = \chi(\mathcal{E}^\vee \otimes \mathcal{Q})^{\geq 0} + \chi(\mathcal{Q}^\vee \otimes \mathcal{F})^{\geq 0} - \chi(\mathcal{Q}^\vee \otimes \mathcal{Q})^{\geq 0}.$$

Now the assumption (ii) and Proposition A.3.1 together imply

$$\dim \mathcal{H}\mathrm{om}_{\mathbb{P}^1}(\mathcal{E}, \mathcal{F})_{\mathcal{Q}} < \dim \mathcal{H}\mathrm{om}_{\mathbb{P}^1}(\mathcal{E}, \mathcal{F}) \quad \text{for every } \mathcal{Q} \in S,$$

thereby yielding a contradiction by (A.3) as desired. $\square$

Remark. The proof of Proposition A.3.2 is slightly different from the original proof of [BFH$^+$17, Theorem 3.3.11]. Here we established (A.3) by using the fact that $\mathcal{H}\mathrm{om}_{\mathbb{P}^1}(\mathcal{E}, \mathcal{F})$ is represented by an affine algebraic space. For [BFH$^+$17, Theorem 3.3.11], we get an analogous identity from the fact that the topological space $|\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})_{\mathcal{Q}}|$ is stable under generalization and specialization inside $|\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})|$.

Let us also record some basic properties of the function χ.

Lemma A.3.3. *Let $\mathcal{V}$ and $\mathcal{W}$ be vector bundles on $\mathbb{P}^1$.*

(1) *We have $\chi(\mathcal{V} \oplus \mathcal{W}) = \chi(\mathcal{V}) + \chi(\mathcal{W})$.*

(2) *We have $\chi(\mathcal{V}^\vee \otimes \mathcal{W})^{\geq 0} = 0$ if and only if $\mu_{min}(\mathcal{V})$ is greater than $\mu_{max}(\mathcal{W})$.*

(3) *For any $d \in \mathbb{Z}$, we have $\chi(\mathcal{V}(d)^\vee \otimes \mathcal{W}(d))^{\geq 0} = \chi(\mathcal{V}^\vee \otimes \mathcal{W})^{\geq 0}$.*

(4) *If $\mathcal{V}$ slopewise dominates $\mathcal{W}$, then we have $\chi(\mathcal{V})^{\geq 0} \geq \chi(\mathcal{W})^{\geq 0}$.*

(5) *For any $d \in \mathbb{Z}$, we have $\chi(\mathcal{V}^\vee \otimes \mathcal{W}^{>d})^{\geq 0} \geq \chi(\mathcal{V}^\vee \otimes \mathcal{O}(d)^{\mathrm{rk}(\mathcal{W}^{>d})})^{\geq 0}$.*

Proof. The statement (1) is evident by the additivity of rank and degree for vector bundles. The statements (2), (3) and (4) are straightforward to check by arguing as in their analogues, namely Corollary 3.2.5, Lemma 3.2.7 and Lemma 4.2.3. The statement (5) is an analogue of the inequality (4.15), and can be verified by arguing as in the proof of Proposition 4.4.10; in fact, we immediately find

$$\deg(\mathcal{V}^\vee \otimes \mathcal{W}^{>d})^{\geq 0} \geq \deg(\mathcal{V}^\vee \otimes \mathcal{O}(d)^{\mathrm{rk}(\mathcal{W}^{>d})})^{\geq 0}$$

as the inequality (4.15) holds verbatim for vector bundles on $\mathbb{P}^1$, and also find

$$\mathrm{rk}(\mathcal{V}^\vee \otimes \mathcal{W}^{>d})^{\geq 0} \geq \mathrm{rk}(\mathcal{V}^\vee \otimes \mathcal{O}(d)^{\mathrm{rk}(\mathcal{W}^{>d})})^{\geq 0}$$

by a similar argument that considers the x-coordinates of HN vectors. $\square$

Remark. The function χ does not admit an analogue of Lemma 3.2.8. However, this won't be a problem for us. Indeed, we used Lemma 3.2.8 only once in the proof of Theorem 4.1.1 for reduction to the case of integer slopes. For Theorem A.1.1, we don't need such a reduction step as vector bundles on $\mathbb{P}^1$ only have integer slopes.

Now we present another proof of Theorem A.1.1.

Proposition A.3.4. *Theorem A.1.1 follows from Proposition A.3.2 and Lemma A.3.3.*

Proof. We can reformulate Theorem A.1.1 as an analogue of Theorem 4.1.1 using vector bundles $\mathcal{E}^{\leq \mu}$ and $\mathcal{F}^{\leq \mu}$ for $\mu \in \mathbb{Q}$. Then the necessity part of Theorem A.1.1 becomes an analogue of Proposition 4.3.2, which is a formal consequence of the Harder-Narasimhan theory for vector bundles on $\mathbb{P}^1$ as noted in the proof of Proposition A.3.2. It remains to show that the sufficiency part of Theorem A.1.1 follows from Proposition A.3.2 and Lemma A.3.3. Assume that for each $j = 1, \cdots, s$, we have either $b_j \geq a_{j+1}$ or $b_i = a_i$ for all $i = 1, \cdots, j$. As noted already, our assumption precisely means that we have $\mathrm{rk}(\mathcal{E}^{\leq \mu}) \geq \mathrm{rk}(\mathcal{F}^{\leq \mu})$ for each $\mu \in \mathbb{Q}$ with equality if and only if $\mathcal{E}^{\leq \mu}$ and $\mathcal{F}^{\leq \mu}$ are isomorphic. We wish to show that $\mathcal{E}$ and $\mathcal{F}$ satisfy the conditions of Proposition A.3.2. This is essentially an analogue of Proposition 4.3.5, after arguing as in Lemma 4.3.4 to add an assumption that $\mathcal{E}$ and $\mathcal{F}$ have no common slopes. Our proof of Proposition 4.3.5 relies only on the Harder-Narasimhan theory for vector

bundles on $\mathcal{X}$ and some basic properties of the degree function such as Corollary 3.2.5, Lemma 3.2.7, Lemma 4.2.3 and some inequalities in the proof of Proposition 4.4.10. In the context of Proposition A.3.2, the function χ takes the role of the degree function in Proposition 4.3.5 and has analogous properties as summarized in Lemma A.3.3. Therefore we can verify the conditions of Proposition A.3.2 for $\mathcal{E}$ and $\mathcal{F}$ by only using the Harder-Narasimhan theory for vector bundles on $\mathbb{P}^1$ and Lemma A.3.3. □

Remark. Our argument in this subsection suggests that Theorem 4.1.1 (and Theorem A.1.1) should extend to any Harder-Narasimhan categories where certain moduli spaces of morphisms exist with locally spectral topological spaces that admit a nice dimension formula as in Proposition 3.1.2 or Proposition A.3.1 with some nice algebraic properties as in Lemma A.3.3.

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