ON NONEMPTINESS OF NEWTON STRATA IN THE B_{dR}^+-GRASSMANNIAN FOR GL_n

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ABSTRACT. We study the Newton stratification in the B_{dR}^+-Grassmannian for GL_n associated to an arbitrary (possibly nonbasic) element of $B(\mathrm{GL}_n)$. Our main result classifies all nonempty Newton strata in an arbitrary minuscule Schubert cell. For a large class of elements in $B(\mathrm{GL}_n)$, our classification is given by some explicit conditions in terms of Newton polygons. For the proof, we proceed by induction on n using a previous result of the author that classifies all extensions of two given vector bundles on the Fargues-Fontaine curve.

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1. INTRODUCTION

1.1. Motivation and main result.

The B_{dR}^+-Grassmannian is an analogue of the affine Grassmannian in p-adic geometry. It was introduced by Caraiani-Scholze [CS17] to study the cohomology of certain Shimura varieties, and also used by Scholze-Weinstein [SW20] as a crucial tool for the construction of local Shimura varieties. In addition, it played a fundamental role in the work of Fargues-Scholze [FS21] on the geometrization of the local Langlands correspondence via the geometric Satake equivalence for p-adic groups.

The main objective of this paper is to study a natural stratification of the B_{dR}^+-Grassmannian known as the *Newton stratification*, which we briefly describe now. Let us fix a connected reductive group G over a finite extension E of $\mathbb{Q}_p$. We write Gr_G for the B_{dR}^+-Grassmannian for G, and $\mathrm{Gr}_{G,\mu}$ for the Schubert cell associated to a dominant cocharacter μ of G. For an complete algebraically closed extension C of E, we have

$$\mathrm{Gr}_G(C) = G(B_{\mathrm{dR}})/G(B_{\mathrm{dR}}^+) \quad \text{and} \quad \mathrm{Gr}_{G,\mu}(C) = G(B_{\mathrm{dR}}^+)\mu(t)^{-1}G(B_{\mathrm{dR}}^+)/G(B_{\mathrm{dR}}^+)$$

where B_{dR} is the p-adic de Rham period ring with valuation ring B_{dR}^+, residue field C and a fixed uniformizer t. The Cartan decomposition for G induces a decomposition

$$\mathrm{Gr}_G = \bigsqcup_{\mu \in X_*(T)^+} \mathrm{Gr}_{G,\mu}$$

where $X_*(T)^+$ denotes the set of all dominant cocharacters of G. Moreover, each Schubert cell $\mathrm{Gr}_{G,\mu}$ is related to the (diamond of the) p-adic flag variety $\mathscr{F}\ell(G,\mu)$ via a natural Bialynicki-Birula map

$$\mathrm{BB}_\mu : \mathrm{Gr}_{G,\mu} \longrightarrow \mathscr{F}\ell(G,\mu),$$

which is an isomorphism if μ is minuscule. In order to define the Newton stratification on Gr_G and its Schubert cells, we consider the stack Bun_G of G-bundles on the Fargues-Fontaine curve X. By the result of Fargues [Far20], the topological space $|\mathrm{Bun}_G|$ of Bun_G is in natural bijection with the set $B(G)$ of Frobenius-conjugacy classes of elements of $G(\breve{E})$, where $\breve{E}$ as usual denotes the p-adic completion of the maximal unramified extension of E. Given an element $b \in B(G)$, we write $\mathcal{E}_b$ for the corresponding G-bundle on X. The theorem of Beauville-Laszlo [BL95] implies that a G-bundle on the Fargues-Fontaine curve is specified by the gluing data of the trivial G-bundles on $\mathrm{Spec}\,(B_{\mathrm{dR}}^+)$ and $X - \infty$, where ∞ is a fixed closed point on X with residue field C and completed local ring B_{dR}^+. If we fix $b \in B(G)$, for every point $x \in \mathrm{Gr}_G(C)$ we can modify the gluing data for $\mathcal{E}_b$ by x to obtain a new G-bundle $\mathcal{E}_{b,x}$. We thus obtain a map

$$\mathrm{Newt}_b : \mathrm{Gr}_G(C) \longrightarrow B(G)$$

which maps each $x \in \mathrm{Gr}_G(C)$ to the element $b' \in B(G)$ corresponding to $\mathcal{E}_{b,x}$. For each Schubert cell $\mathrm{Gr}_{G,\mu}$, the Newton stratification associated to $b \in B(G)$ is a decomposition into subdiamonds

$$\mathrm{Gr}_{G,\mu} = \bigsqcup_{b' \in B(G)} \mathrm{Gr}_{G,\mu,b}^{b'}$$

where $\mathrm{Gr}_{G,\mu,b}^{b'}(C)$ is the preimage of b' in $\mathrm{Gr}_{G,\mu}(C)$ under the map Newt_b.

The Newton stratification of minuscule Schubert cells was originally introduced in the aforementioned work of Caraiani-Scholze [CS17] as a key tool for studying the fibers of the Hodge-Tate period map. It has also been used as a pivotal tool for studying the p-adic period domain by many authors, such as Chen-Fargues-Shen [CFS21], Shen [She23], Chen [Che22], Viehmann [Vie24], Nguyen-Viehmann [NV23], and Chen-Tong [CT22].

For the trivial element $b = 1$, a result of Rapoport [Rap18] shows that the Newton stratum $\mathrm{Gr}_{G,\mu,b}^{b'}$ is nonempty if and only if b' is an element of the set $B(G,-\mu)$ defined by Kottwitz [Kot85]. When b is basic, meaning that $\mathcal{E}_b$ is semistable, Chen-Fargues-Shen [CFS21] and Viehmann [Vie24] extends the result of Rapoport to parametrize all nonempty Newton strata by a generalized Kottwitz set. However, for a general element $b \in B(G)$, no explicit parametrization is known for nonempty Newton strata in an arbitrary Schubert cell.

In order to explain our main result, which classifies all nonempty Newton strata in the Schubert cell $\mathrm{Gr}_{G,\mu}$ for $G = \mathrm{GL}_n$ and a minuscule cocharacter μ, we need to set up some notations. Let us recall that, as observed by Kottwitz [Kot85], the set $B(\mathrm{GL}_n)$ is naturally

identified with the set of concave polygons on the interval $[0, n]$ with rational slopes and integer breakpoints, where a polygon refers to a continuous piecewise linear function whose graph passes through the origin. Given an element $b \in B(\mathrm{GL}_n)$, we write $\nu(b)$ for the corresponding polygon and often regard it as a tuple of rational numbers $(\nu_1(b), \cdots, \nu_n(b))$ where $\nu_i(b)$ denotes the slope of $\nu(b)$ on the interval $[i-1, i]$. We may also represent the dominant cocharacter μ of GL_n as an n-tuple of descending integers $(\mu_1, \cdots, \mu_n)$ and regard it as a concave polygon on $[0, n]$ whose slope on $[i-1, i]$ is μ_i.

Given two arbitrary elements b, $b' \in B(\mathrm{GL}_n)$, our main result gives an inductive criterion for the nonemptiness of the Newton stratum $\mathrm{Gr}_{\mathrm{GL}_n, \mu, b}^{b'}$. Let us provide a brief description of the inductive criterion here and refer the readers to Theorem 3.1.12 for a precise statement. If b is basic, meaning that $\nu(b)$ is a line segment, the desired classification is given by the aforementioned results of Chen-Fargues-Shen [CFS21] and Viehmann [Vie24]. If b is not basic, we have unique elements $a \in B(\mathrm{GL}_m)$ and $c \in B(\mathrm{GL}_{n-m})$ for some integer m such that $\nu(a)$ and $\nu(c)$ together form a partition of $\nu(b)$ with $\nu(a)$ being the line segment in $\nu(b)$ of maximum slope. The key observation for our main result is that $\mathrm{Gr}_{\mathrm{GL}_n, \mu, b}^{b'}$ is not empty if and only if there exist $a' \in B(\mathrm{GL}_m)$ and $c' \in B(\mathrm{GL}_{n-m})$ with the following properties:

(i) The Newton strata $\mathrm{Gr}_{\mathrm{GL}_m, \mu_1, a}^{a'}$ and $\mathrm{Gr}_{\mathrm{GL}_{n-m}, \mu_2, c}^{c'}$ are not empty for some minuscule cocharacters μ_1 of GL_m and μ_2 of GL_{n-m}.

(ii) The vector bundle $\mathcal{E}_{b'}$ arises as an extension of $\mathcal{E}_{c'}$ by $\mathcal{E}_{a'}$; in other words, there exists a short exact sequence of vector bundles

$$0 \longrightarrow \mathcal{E}_{a'} \longrightarrow \mathcal{E}_{b'} \longrightarrow \mathcal{E}_{c'} \longrightarrow 0.$$

The cocharacters μ_1 and μ_2 in the property (i) are uniquely determined by a' and c'. Moreover, the property (i) imposes explicit bounds on the slopes in $\nu(a')$ and $\nu(c')$, and consequently yields a finite list of candidates for (a', c'). For each candidate, we can check the property (ii) by a previous result of the author [Hon22]. Then for each candidate with the property (ii), we can inductively proceed to check the property (i); indeed, if $\nu(b)$ has r distinct slopes, then $\nu(c)$ has $r-1$ distinct slopes while $\nu(a)$ is a line segment by construction.
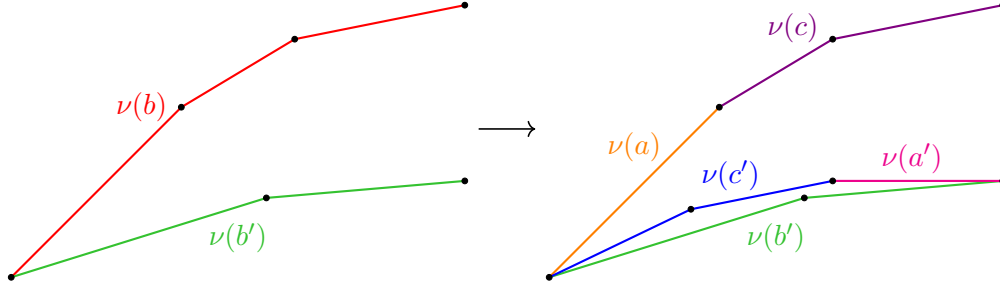

FIGURE 1. Illustration of the inductive criterion

For a concrete example, we illustrate how our inductive criterion shows the nonemptiness of the stratum $\mathrm{Gr}_{\mathrm{GL}_8, \mu, b}^{b'}$ with

$$\nu(b) = \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5} \right),$$

$$\nu(b') = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, 0, 0, 0 \right),$$

$$\mu = (1, 1, 1, 1, 0, 0, 0, 0).$$

The elements $a \in B(\mathrm{GL}_3)$ and $c \in B(\mathrm{GL}_5)$ are given by

$$\nu(a) = \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right) \quad \text{and} \quad \nu(c) = \left(\frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}\right).$$

We apply the inductive criterion with $a' \in B(\mathrm{GL}_3)$ and $c' \in B(\mathrm{GL}_5)$ given by

$$\nu(a') = \left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right) \quad \text{and} \quad \nu(c') = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0\right).$$

Indeed, the nonemptiness of the stratum $\mathrm{Gr}_{\mathrm{GL}_8, \mu, b}^{b'}$ follows from the following statements:

- $\mathcal{E}_{b'}$ arises as an extension of $\mathcal{E}_{c'}$ by $\mathcal{E}_{a'}$.
- $\mathrm{Gr}_{\mathrm{GL}_3, \mu_1, a}^{a'}$ with $\mu_1 = (1,1,1)$ and $\mathrm{Gr}_{\mathrm{GL}_5, \mu_2, c}^{c'}$ with $\mu_2 = (1,0,0,0,0)$ are not empty.

For the second statement, we note that a and c are basic for $\nu(a)$ and $\nu(c)$ being line segments.

A special case of our main result reduces to a noninductive criterion as follows:

Theorem 1.1.1. *Let μ be a minuscule dominant cocharacter of $G = \mathrm{GL}_n$ represented by an n-tuple with entries 0 and 1. Given two arbitrary elements $b, b' \in B(\mathrm{GL}_n)$ such that the difference between any two distinct slopes in $\nu(b)$ is greater than 1, the Newton stratum $\mathrm{Gr}_{\mathrm{GL}_n, \mu, b}^{b'}$ is nonempty if and only if the following conditions are satisfied:*

(i) The polygon $\nu(b')$ lies below the polygon $\nu(b) + \mu^*$ with the same endpoints, where μ^* denotes the unique dominant cocharacter of GL_n in the conjugacy class of μ^{-1}.

(ii) We have inequalities

$$\nu_i(b') \leq \nu_i(b) \leq \nu_i(b') + 1 \quad \text{for } i = 1, \cdots, n.$$

(iii) For each breakpoint of $\nu(b)$, there exists a breakpoint of $\nu(b')$ with the same x-coordinate.
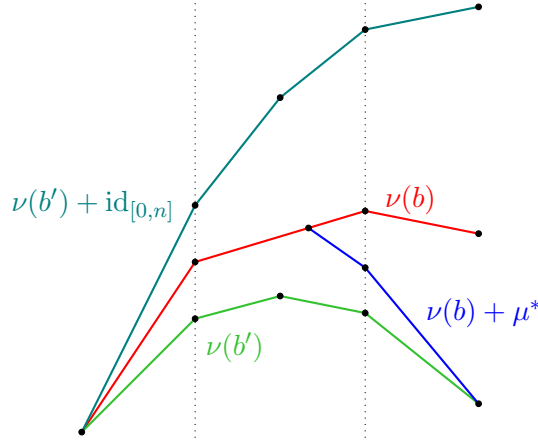

FIGURE 2. Illustration of the conditions in Theorem 1.1.1

The condition (i) is in fact equivalent to having b' in the generalized Kottwitz set considered by Chen-Fargues-Shen [CFS21] and Viehmann [Vie24]. When b is basic, the condition (i) also implies the conditions (ii) and (iii). Hence when b is basic Theorem 1.1.1 agrees with the aforementioned result of Chen-Fargues-Shen [CFS21] and Viehmann [Vie24].

The hypothesis on the cocharacter μ having entries 0 and 1 is insignificant; indeed, without this assumption we still get a similar statement by a simple reduction technique as stated in Proposition 3.1.6. On the other hand, the hypothesis on the slopes in $\nu(b)$ is crucial. For the general case, the conditions (i) and (ii) are still necessary but not sufficient.

1.2. Outline of the proof.

Given a vector bundle $\mathcal{E}$ on the Fargues-Fontaine curve X, its *minuscule effective modification at ∞* of degree d refers to an injective bundle map $\mathcal{E}' \hookrightarrow \mathcal{E}$ whose cokernel is the skyscraper sheaf at ∞ with value $C^{\oplus d}$. The Newton stratum $\mathrm{Gr}_{\mathrm{GL}_n, \mu, b}^{b'}$ is not empty if and only if there exists a minuscule effective modification $\mathcal{E}_{b'} \hookrightarrow \mathcal{E}_b$ at ∞. We thus wish to classify all minuscule effective modifications of $\mathcal{E}_b$ at ∞. If b is basic, the desired classification is given by the aforementioned results of Chen-Fargues-Shen [CFS21] and Viehmann [Vie24]. Let us now assume that b is not basic. We can find a direct sum decomposition

$$\mathcal{E}_b \simeq \mathcal{E}_a \oplus \mathcal{E}_c \quad \text{with} \quad a \in B(\mathrm{GL}_m) \text{ and } c \in B(\mathrm{GL}_{n-m})$$

where a is basic such that $\nu(a)$ equals the line segment in $\nu(b)$ of maximum slope. For every minuscule effective modification $\iota : \mathcal{E}_{b'} \hookrightarrow \mathcal{E}_b$ at ∞, the above decomposition extends to a commutative diagram of short exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E}_a & \longrightarrow & \mathcal{E}_b & \longrightarrow & \mathcal{E}_c & \longrightarrow & 0 \\
& & \alpha \big\uparrow & & \beta \big\uparrow & & \gamma \big\uparrow & & \\
0 & \longrightarrow & \mathcal{E}_{a'} & \longrightarrow & \mathcal{E}_{b'} & \longrightarrow & \mathcal{E}_{c'} & \longrightarrow & 0
\end{array}
$$

where α and γ are also minuscule effective modifications at ∞. Conversely, given such a commutative diagram we apply a result of Chen-Tong [CT22] to observe that α and γ can be adjusted so that β is a minuscule effective modification at ∞. Then we use a previous result of the author [Hon22] to classify all vector bundles $\mathcal{E}_{a'}$ and $\mathcal{E}_{c'}$ that fit into such a commutative diagram, and consequently proceed by induction to obtain the desired classification.

1.3. Notations and conventions.

Throughout the paper, we fix the following data:

- E is a finite extension of $\mathbb{Q}_p$.
- C is a complete and algebraically closed extension of E.
- G is a reductive group over E with Borel subgroup B and maximal torus $T \subseteq B$.

We also retain the following notations:

- $\breve{E}$ is the p-adic completion of the maximal unramified extension of E.
- $B(G)$ is the set of Frobenius-conjugacy classes of elements of $G(\breve{E})$.
- $X_*(T)^+$ is the set of all dominant cocharacters of G.

In addition, we use the following standard notations:

- Given a valued field K, we write $\mathcal{O}_K$ for its valuation ring.
- Given a ringed space S, we write $\mathcal{O}_S$ for its structure sheaf.
- Given a perfectoid ring R, we write $R^\flat$ for its tilt and R° for its subring of power bounded elements.
- Given a perfect $\mathbb{F}_p$-algebra A, we write $W(A)$ for the ring of Witt vectors over A.

2. Preliminaries

In this section, we review some basic facts about the B_{dR}^+-Grassmannian and G-bundles on the Fargues-Fontaine curve.

2.1. The B_{dR}^+-Grassmannian.

Proposition 2.1.1 ([Fon82, Proposition 2.4], [KL15, Lemma 3.6.3]). *Let R be a perfectoid algebra over C. There exists a natural surjective homomorphism $W(R^{\circ\flat}) \twoheadrightarrow R^\circ$ whose kernel is a principal ideal of $W(R^{\circ\flat})$.*

Definition 2.1.2. Let R be a perfectoid algebra over C. Choose a generator t of the kernel of the map $W(R^{\circ\flat}) \twoheadrightarrow R^\circ$ in Proposition 2.1.1. We write $B_{\mathrm{dR}}^+(R)$ for the t-adic completion of $W(R^{\circ\flat})[1/p]$, and define the *de Rham period ring* associated to R by $B_{\mathrm{dR}}(R) := B_{\mathrm{dR}}^+(R)[1/t]$.

Proposition 2.1.3 ([Fon82, Proposition 2.17]). *The ring $B_{\mathrm{dR}}(C)$ is a discretely valued field with valuation ring $B_{\mathrm{dR}}^+(C)$ and residue field C.*

We will henceforth write $B_{\mathrm{dR}} := B_{\mathrm{dR}}(C)$ and $B_{\mathrm{dR}}^+ := B_{\mathrm{dR}}^+(C)$. We also fix a uniformizer t of B_{dR} in light of Proposition 2.1.3.

Definition 2.1.4. The B_{dR}^+-*Grassmannian* is the functor Gr_G that associates to each perfectoid affinoid algebra (R, R^+) over C the set of pairs $(\mathcal{E}, \beta)$ consisting of a G-torsor $\mathcal{E}$ over $\mathrm{Spec}\,(B_{\mathrm{dR}}^+(R))$ and a trivialization β of $\mathcal{E}$ over $\mathrm{Spec}\,(B_{\mathrm{dR}}(R))$.

Proposition 2.1.5 ([SW20, Proposition 19.1.2]). *There exists a natural identification*
$$\mathrm{Gr}_G(C) \cong G(B_{\mathrm{dR}})/G(B_{\mathrm{dR}}^+).$$

Remark. In fact, we can naturally identify Gr_G as the étale sheafification of the functor that associates to each perfectoid affinoid algebra (R, R^+) over C the coset $G(B_{\mathrm{dR}}(R))/G(B_{\mathrm{dR}}^+(R))$.

Proposition 2.1.6 ([SW20, Corollary 19.3.4]). *Given $\mu \in X_*(T)^+$, there exists a locally spatial diamond $\mathrm{Gr}_{G,\mu}$ with*
$$\mathrm{Gr}_{G,\mu}(C) = G(B_{\mathrm{dR}}^+)\mu(t)^{-1}G(B_{\mathrm{dR}}^+)/G(B_{\mathrm{dR}}^+).$$

Remark. In this paper, we won't use the language of diamonds in an essential way because we are only interested in the C-valued points of Gr_G and $\mathrm{Gr}_{G,\mu}$.

Definition 2.1.7. Let μ be a dominant cocharacter of G.

(1) We refer to the locally spatial diamond $\mathrm{Gr}_{G,\mu}$ in Proposition 2.1.6 as the *Schubert cell* of Gr_G associated to μ.

(2) We define the *parabolic subgroup of G associated to μ* by
$$P_\mu := \{g \in G : \lim_{t \to 0} \mu(t)g\mu(t)^{-1} \text{ exists}\}.$$

(3) We define the *flag variety* associated to the pair (G, μ) by
$$\mathscr{F}\ell(G, \mu) := G/P_\mu.$$

(4) We define the *Bialynicki-Birula map* associated to μ as the map
$$\mathrm{BB}_\mu : \mathrm{Gr}_{G,\mu}(C) \longrightarrow \mathscr{F}\ell(G, \mu)(C)$$
which associates to $g\mu(t)^{-1}G(B_{\mathrm{dR}}^+) \in \mathrm{Gr}_{G,\mu}(C)$ the parabolic subgroup $\bar{g}P_\mu\bar{g}^{-1}$, where $\bar{g}$ denotes the image of g under the natural map $G(B_{\mathrm{dR}}^+) \to G(C)$.

Proposition 2.1.8 ([CS17, Theorem 3.4.5]). *If μ is a minuscule cocharacter of G, the Bialynicki-Birula map BB_μ is bijective.*

2.2. G-bundles on the Fargues-Fontaine curve.

Definition 2.2.1. Fix a uniformizer π of E and a pseudouniformizer ϖ of $C^\flat$. Let q be the number of elements in the residue field of E.

(1) We set
$$\mathcal{Y} := \mathrm{Spa}(W_{\mathcal{O}_E}(\mathcal{O}_{C^\flat}) \setminus \{|\pi[\varpi]| = 0\},$$
where we write $W_{\mathcal{O}_E}(\mathcal{O}_{C^\flat}) := W(\mathcal{O}_{C^\flat}) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$ for the ring of ramified Witt vectors over $\mathcal{O}_{C^\flat}$ with coefficients in $\mathcal{O}_E$ and the Teichmuller lift $[\varpi]$ of ϖ, and define the *adic Fargues-Fontaine curve* associated to the pair $(E, C^\flat)$ by
$$\mathcal{X} := \mathcal{Y}/\phi^{\mathbb{Z}}$$
where ϕ denotes the automorphism of $\mathcal{Y}$ induced by the q-Frobenius automorphism on $W_{\mathcal{O}_E}(\mathcal{O}_{C^\flat})$.

(2) We define the *schematic Fargues-Fontaine curve* associated to the pair $(E, C^\flat)$ by
$$X := \mathrm{Proj}\left(\bigoplus_{n \geq 0} H^0(\mathcal{Y}, \mathcal{O}_\mathcal{Y})^{\phi = \pi^n} \right).$$

Remark. The definition of the adic Fargues-Fontaine curve relies on the fact that the action of ϕ on $\mathcal{Y}$ is properly discontinuous.

Theorem 2.2.2 ([Ked16, Theorem 4.10], [FF18, Théorème 6.5.2], [KL15, Theorem 8.7.7]). *We have the following statements:*

(1) $\mathcal{X}$ is a Noetherian adic space over E.
(2) X is a Dedekind scheme over E.
(3) There exists an equivalence of the categories of vector bundles on $\mathcal{X}$ and X, induced by pullback along a natural map of locally ringed spaces $\mathcal{X} \longrightarrow X$.

Remark. The scheme X is not a curve in the usual sense as it is not of finite type over E.

In light of the statement (3) in Theorem 2.2.2, we will henceforth identify G-bundles on $\mathcal{X}$ with G-bundles on X.

Definition 2.2.3. Given an element $b \in B(G)$, we define the associated G-bundle $\mathcal{E}_b$ on $\mathcal{X}$ (or on X) by descending along the map $\mathcal{Y} \longrightarrow \mathcal{Y}/\phi^{\mathbb{Z}} = \mathcal{X}$ the trivial G-bundle on $\mathcal{Y}$ equipped with the ϕ-linear automorphism given by b.

Theorem 2.2.4 ([Far20, Théorème 5.1]). *The map $B(G) \longrightarrow H^1_{\text{ét}}(X, G)$ sending b to the isomorphism class of $\mathcal{E}_b$ is a bijection.*

Proposition 2.2.5. *The set of isomorphism classes of isocrystals over $\breve{E}$ and the set of isomorphism classes of vector bundles on X admit a natural bijection which is compatible with direct sums, duals, and ranks.*

Proof. Consider an arbitrary integer $n > 0$. Given $b \in B(\mathrm{GL}_n)$, we write N_b for the isocrystal over $\breve{E}$ with underlying vector space $\breve{E}^{\oplus n}$ and the Frobenius-semilinear automorphism given by b. As observed by Kottwitz [Kot85], there exists a natural bijection between $B(\mathrm{GL}_n)$ and the set of isomorphism classes of isocrystals over $\breve{E}$ of rank n where $b \in B(\mathrm{GL}_n)$ maps to the isomorphism class of N_b. Moreover, Theorem 2.2.4 yields a bijection between $B(\mathrm{GL}_n)$ and the set of isomorphism classes of vector bundles over X of rank n where $b \in B(\mathrm{GL}_n)$ maps to the isomorphism class of $\mathcal{E}_b$. We thus obtain a bijection between the set of isomorphism classes of isocrystals over $\breve{E}$ and the set of isomorphism classes of vector bundles on X. It is straight forward to check that this bijection is compatible with diret sums, duals, and ranks. $\square$

Definition 2.2.6. Let $\mathcal{E}$ be a vector bundle on X. We denote by $N(\mathcal{E})$ the isomorphism class of isocrystals over $\breve{E}$ that corresponds to $\mathcal{E}$ under the bijection in Proposition 2.2.5.

(1) We write $\mathrm{rk}(\mathcal{E})$ for the rank of $\mathcal{E}$, and define the *degree* of $\mathcal{E}$, denoted by $\deg(\mathcal{E})$, to be the degree of $N(\mathcal{E})$.

(2) We define the *Harder-Narasimhan (HN) polygon* of $\mathcal{E}$ by $\mathrm{HN}(\mathcal{E}) := -\mathrm{Newt}(N(\mathcal{E})^{\vee})$, where $\mathrm{Newt}(N(\mathcal{E})^{\vee})$ refers to the Newton polygon of the dual of $N(\mathcal{E})$.

(3) We say that $\mathcal{E}$ is *semistable* of slope λ if $\mathrm{HN}(\mathcal{E})$ is a line segment of slope λ.

Remark. The definition of $\mathrm{HN}(\mathcal{E})$ is in line with the convention that Newton polygons are convex while Harder-Narasimhan polygons are concave. It is also worthwhile to mention that the correct (or usual) definition of semistability should be given in terms of the Harder-Narasimhan formalism for vector bundles on X; in fact, the equivalence of our definition and the correct definition is due to a highly nontrivial result of Fargues-Fontaine [FF18].

Proposition 2.2.7. *Let $\mathcal{E}$ be a vector bundle on X.*

(1) $\mathcal{E}$ admits a direct sum decomposition $\mathcal{E} \simeq \oplus \mathcal{E}_i$ where the $\mathcal{E}_i$'s are semistable vector bundles on X of distinct slopes.

(2) If the $\mathcal{E}_i$'s are arranged in order of descending slope, $\mathrm{HN}(\mathcal{E})$ is given by the concatenation of the polygons $\mathrm{HN}(\mathcal{E}_i)$.

Proof. The assertion is evident by Proposition 2.2.5 and the semisimplicity of isocrystals. $\square$

Remark. The statement (2) implies that the direct summands $\mathcal{E}_i$ are uniquely determined up to permutations.

Definition 2.2.8. Let $\mathcal{E}$ be a vector bundle on X. We refer to the direct sum decomposition $\mathcal{E} \simeq \oplus \mathcal{E}_i$ in Proposition 2.2.7 as the *Harder-Narasimhan (HN) decomposition* of $\mathcal{E}$.

2.3. The Newton stratification of Schubert cells and flag varieties.

For the rest of this paper, we fix a closed point ∞ on X given by the following proposition:

Proposition 2.3.1 ([FF18, Théorèmes 6.5.2 and 7.3.3], [CT22, Remark 1.7]). *There exists a closed point ∞ on X with the following properties:*

(i) $X - \infty$ is the spectrum of a principal domain $B_e \subseteq B_{\mathrm{dR}}$.

(ii) The completed local ring at ∞ is canonically isomorphic to B_{dR}^+.

Remark. A closed point on X corresponds to a characteristic 0 untilt of $C^{\flat}$ (i.e., a perfectoid field K with an isomorphism $K^{\flat} \simeq C^{\flat}$) up to ϕ-equivalences. We may take ∞ to be the closed point on X corresponding to C with the identity map on $C^{\flat}$. The field C alone does not determine ∞ as $C^{\flat}$ has automorphisms which are not ϕ-equivalent to the identity map.

Proposition 2.3.2. *The set $H^1_{\acute{e}t}(X, G)$ is naturally in bijection with the set of isomorphism classes of triples $(\mathcal{E}^{\circ}, \widehat{\mathcal{E}}, \beta)$ where*

- *$\mathcal{E}^{\circ}$ is a G-bundle on $X - \infty$,*
- *$\widehat{\mathcal{E}}$ is a trivial G-bundle on $\mathrm{Spec}\,(B_{\mathrm{dR}}^+)$, and*
- *β is a gluing map of $\mathcal{E}^{\circ}$ and $\widehat{\mathcal{E}}$ over $\mathrm{Spec}\,(B_{\mathrm{dR}})$.*

Proof. Every G-bundle on X becomes trivial after the pullback via the map $\mathrm{Spec}\,(B_{\mathrm{dR}}^+) \to X$ induced by ∞, as noted by Nguyen-Viehmann [NV23, §2.1] and Chen-Tong [CT22, Remark 1.7]. Hence the desired assertion follows from Proposition 2.3.1 and the theorem of Beauville-Laszlo [BL95]. $\square$

Definition 2.3.3. Let $\mathcal{E}$ be a G-bundle on X. A *modification* of $\mathcal{E}$ at ∞ is a G-bundle $\mathcal{E}'$ on X together with an isomorphism between $\mathcal{E}$ and $\mathcal{E}'$ on $X - \infty$.

Example 2.3.4. Consider an element $b \in B(G)$ and a point $x \in \mathrm{Gr}_G(C)$. We may write $x = gG(B_{\mathrm{dR}}^+)$ for some $g \in G(B_{\mathrm{dR}})$ under the identification $\mathrm{Gr}_G(C) \cong G(B_{\mathrm{dR}})/G(B_{\mathrm{dR}}^+)$ noted in Proposition 2.1.5. Now, in light of Proposition 2.3.2 we take a triple $(\mathcal{E}^\circ, \widehat{\mathcal{E}}, \beta)$ corresponding to $\mathcal{E}_b$ and a G-bundle $\mathcal{E}_{b,x}$ on X corresponding to $(\mathcal{E}^\circ, \widehat{\mathcal{E}}, g\beta)$. By construction, $\mathcal{E}_{b,x}$ is naturally a modification of $\mathcal{E}_b$ at ∞.

Definition 2.3.5. Consider an element $b \in B(G)$ and a dominant cocharacter μ of G.

(1) For each $x \in \mathrm{Gr}_G(C)$, we refer to the G-bundle $\mathcal{E}_{b,x}$ constructed in Example 2.3.4 as the *modification of $\mathcal{E}_b$ at ∞ induced by x*.

(2) For each $b' \in B(G)$, we define the associated *Newton stratum with respect to b* in $\mathrm{Gr}_{G,\mu}$ as the subdiamond $\mathrm{Gr}_{G,\mu,b}^{b'}$ of $\mathrm{Gr}_{G,\mu}$ with

$$\mathrm{Gr}_{G,\mu,b}^{b'}(C) = \{x \in \mathrm{Gr}_{G,\mu}(C) : \mathcal{E}_{b,x} \simeq \mathcal{E}_{b'}\}.$$

(3) For each $b' \in B(G)$, we define the associated *Newton stratum with respect to b* in $\mathscr{F}\ell(G,\mu)$ as the subvariety $\mathscr{F}\ell(G,\mu,b)^{b'}$ of $\mathscr{F}\ell(G,\mu)$ such that $\mathscr{F}\ell(G,\mu,b)^{b'}(C)$ is the image of $\mathrm{Gr}_{G,\mu,b}^{b'}(C)$ under the map BB_μ.

Remark. The subdiamond $\mathrm{Gr}_{G,\mu,b}^{b'}$ of $\mathrm{Gr}_{G,\mu}$ is uniquely determined by its set of C-points since $\mathrm{Gr}_{G,\mu}$ is a locally spatial diamond.

2.4. Subsheaves and extensions of vector bundles on the Fargues-Fontaine curve.

Definition 2.4.1. Given two integers n and d with $n > 0$, a *rationally tuplar polygon* of rank n and degree d is the graph $\mathscr{P}$ of a continuous function f with the following properties:

(i) f is defined on $[0, n]$ with $f(0) = 0$ and $f(n) = d$.

(ii) f is linear on $[i-1, i]$ for each $i = 1, \cdots, n$ with a rational slope denoted by $\lambda_i(\mathscr{P})$.

Example 2.4.2. We are particularly interested in the following rationally tuplar polygons:

(1) For every vector bundle $\mathcal{E}$ on X of rank n and degree d, its HN polygon $\mathrm{HN}(\mathcal{E})$ is a rationally tuplar polygon of rank n and degree d.

(2) For $G = \mathrm{GL}_n$ with Borel subgroup B of upper triangular matrices and maximal torus T of diagonal matrices, we regard all dominant cocharacters as rationally tuplar polygons of rank n under the natural identification

$$X_*(T)^+ \cong \{(a_i) \in \mathbb{Z}^n : a_1 \geq a_2 \geq \cdots \geq a_n\}.$$

(3) We write $\overline{d/n}^{(n)}$ for the line segment connecting $(0,0)$ and (n,d), which is a rationally tuplar polygon of rank n and degree d.

Definition 2.4.3. Let $\mathbb{P}_n$ denote the set of rationally tuplar polygons of rank n.

(1) We define the *Bruhat order* $\geq$ on $\mathbb{P}_n$ by writing $\mathscr{P} \geq \mathscr{Q}$ if we have

$$\sum_{i=1}^{j} \lambda_i(\mathscr{P}) \geq \sum_{i=1}^{j} \lambda_i(\mathscr{Q}) \qquad \text{for each } j = 1, \cdots, n$$

with equality for $j = n$.

(2) We define the *slopewise dominance order* $\succeq$ on $\mathbb{P}_n$ by writing $\mathscr{P} \succeq \mathscr{Q}$ if we have $\lambda_i(\mathscr{P}) \geq \lambda_i(\mathscr{Q})$ for each $i = 1, \cdots, n$.

Remark. Intuitively, we have $\mathscr{P} \geq \mathscr{Q}$ if and only if $\mathscr{P}$ lies on or above $\mathscr{Q}$ with the same endpoints, as illustrated by Figure 3.
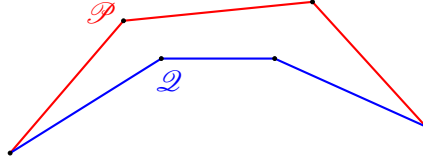
FIGURE 3. Illustration of the Bruhat order

Proposition 2.4.4 ([Hon21, Theorem 1.2.1]). *Let $\mathcal{D}$ and $\mathcal{E}$ be vector bundles on X of rank n. Then $\mathcal{D}$ is a subsheaf of $\mathcal{E}$ if and only if we have $\mathrm{HN}(\mathcal{E}) \succeq \mathrm{HN}(\mathcal{D})$.*

Definition 2.4.5. Given vector bundles $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ on X, we define a $(\mathcal{D}, \mathcal{E}, \mathcal{F})$-*permutation* of $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$ to be a rationally tuplar polygon $\mathscr{P} \geq \mathrm{HN}(\mathcal{E})$ with the following properties:

(i) The tuple $(\lambda_i(\mathscr{P}))$ is a permutation of the tuple $(\lambda_i(\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})))$.

(ii) For each $i = 1, \cdots, \mathrm{rk}(\mathcal{E})$, we have
   - $\lambda_i(\mathscr{P}) < \lambda_i(\mathrm{HN}(\mathcal{E}))$ only if $\lambda_i(\mathscr{P})$ occurs as a slope in $\mathrm{HN}(\mathcal{D})$, and
   - $\lambda_i(\mathscr{P}) > \lambda_i(\mathrm{HN}(\mathcal{E}))$ only if $\lambda_i(\mathscr{P})$ occurs as a slope in $\mathrm{HN}(\mathcal{F})$.
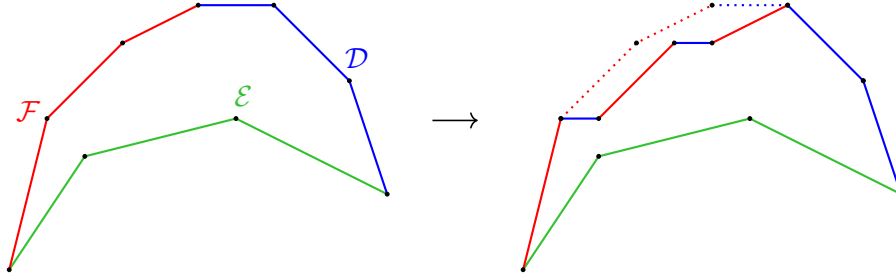
FIGURE 4. Illustration of the conditions in Definition 2.4.5

Proposition 2.4.6 ([FF18, Proposition 5.6.23]). *Given vector bundles $\mathcal{D}$ and $\mathcal{F}$ on X such that the minimum slope in $\mathrm{HN}(\mathcal{D})$ is greater than or equal to the maximum slope in $\mathrm{HN}(\mathcal{F})$, every extension of $\mathcal{F}$ by $\mathcal{D}$ splits.*

Proposition 2.4.7 ([Hon22, Theorem 3.12], [CT22, Proposition 5.3]). *Let $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on X such that there exists a short exact sequence*
$$0 \longrightarrow \mathcal{D} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0.$$
There exists a $(\mathcal{D}, \mathcal{E}, \mathcal{F})$-permutation of $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$.

Proposition 2.4.8 ([Hon22, Theorem 4.4], [CT22, Proposition 5.9]). *Let $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on X. We write the HN decomposition of $\mathcal{F}$ as*
$$\mathcal{F} \simeq \bigoplus_{i=1}^{m} \mathcal{F}_i$$
where the $\mathcal{F}_i$'s are arranged in order of descending slope. There exists a short exact sequence
$$0 \longrightarrow \mathcal{D} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0$$
if and only if there exists a sequence of vector bundles $\mathcal{D} = \mathcal{E}_0, \mathcal{E}_1, \cdots, \mathcal{E}_m = \mathcal{E}$ on X such that the polygon $\mathrm{HN}(\mathcal{E}_{i-1} \oplus \mathcal{F}_i)$ has an $(\mathcal{E}_{i-1}, \mathcal{E}_i, \mathcal{F}_i)$-permutation for each $i = 1, \cdots, m$.

3. NONEMPTY NEWTON STRATA IN MINUSCULE SCHUBERT CELLS FOR GL_n

In this section, we classify all nonempty Newton strata in an arbitrary minuscule Schubert cell for GL_n by studying modifications of vector bundles on the Fargues-Fontaine curve. We first establish in §3.1 an inductive classification for nonempty Newton strata associated to an arbitrary element of $B(\mathrm{GL}_n)$. We then prove in §3.2 some combinatorial lemmas about rationally tuplar polygons and use them in §3.3 to give an explicit classification of all nonempty Newton strata associated to a large class of element of $B(\mathrm{GL}_n)$. Throughout this section, we take dominant cocharacters of GL_n with respect to the standard Borel subgroup of upper triangular matrices and the standard maximal torus of diagonal matrices.

3.1. **An inductive classification of nonempty Newton strata.**

Definition 3.1.1. Given a rationally tuplar polygon $\mathscr{P}$ of rank n, we define its *dual* to be the rationally tuplar polygon $\mathscr{P}^*$ with $\lambda_i(\mathscr{P}^*) = -\lambda_{n+1-i}(\mathscr{P})$ for each $i = 1, \cdots, n$.

Example 3.1.2. We illustrate the notion of duality for the polygons in Example 2.4.2.

(1) For a vector bundle $\mathcal{E}$ on X of rank n, we have $\mathrm{HN}(\mathcal{E})^* = \mathrm{HN}(\mathcal{E}^\vee)$ where $\mathcal{E}^\vee$ denotes the dual bundle of $\mathcal{E}$.

(2) For a dominant cocharacter μ of GL_n, the polygon μ^* represents the unique dominant cocharacter in the conjugacy class of μ^{-1}.

(3) For arbitrary integers d and n, we have $\underline{d/n}^{(n)^*} = \underline{-d/n}^{(n)}$.

Proposition 3.1.3 ([CFS21, Proposition 5.2], [Vie24, Corollary 5.4]). *Let b and b' be elements of $B(\mathrm{GL}_n)$ such that $\mathcal{E}_b$ is semistable. Given a dominant cocharacter μ of GL_n, the Newton stratum $\mathrm{Gr}_{\mathrm{GL}_n,\mu,b}^{b'}$ is nonempty if and only if we have*

$$\nu(b) + \mu^* \geq \nu(b') \tag{3.1}$$

where $\nu(b)$ and $\nu(b')$ respectively denote $\mathrm{HN}(\mathcal{E}_b)$ and $\mathrm{HN}(\mathcal{E}_{b'})$.

Remark. For a reductive group G and a basic element $b \in B(G)$, the results of Chen-Fargues-Shen [CFS21, Proposition 5.2] and Viehmann [Vie24, Corollary 5.4] classify all nonempty newton strata with respect to b in an arbitrary Schubert cell in terms of the Kottwitz map and the Newton map defined by Kottwitz [Kot85]. In our context, their results are translated to Proposition 3.1.3 by the following facts:

(a) An element $b \in B(\mathrm{GL}_n)$ is basic if and only if $\mathcal{E}_b$ is semistable.

(b) The condition involving the Kottwitz map holds for all elements in $B(\mathrm{GL}_n)$.

(c) The condition involving the Newton map is equivalent to the inequality (3.1) as $\nu(b)$ and $\nu(b')$ are identified with the (concave) Newton polygons of b and b'.

Lemma 3.1.4. *Let b be an element of $B(\mathrm{GL}_n)$. For $x = \underline{1}^{(n)}(t)\,\mathrm{GL}_n(B_{\mathrm{dR}}^+) \in \mathrm{Gr}_{\mathrm{GL}_n,\underline{1}^{(n)}}(C)$, we have $\mathrm{HN}(\mathcal{E}_{b,x}) = \mathrm{HN}(\mathcal{E}_b) - \underline{1}^{(n)}$.*

Proof. Let us write the HN decomposition of $\mathcal{E}_b$ as

$$\mathcal{E}_b \simeq \bigoplus_{i=1}^m \mathcal{E}_{b_i} \quad \text{with } b_i \in B(\mathrm{GL}_{n_i}).$$

For each $i = 1, \cdots, m$, we take $x_i := \underline{1}^{(n_i)}(t)\,\mathrm{GL}_{n_i}(B_{\mathrm{dR}}^+) \in \mathrm{Gr}_{\mathrm{GL}_{n_i},\underline{1}^{(n_i)}}(C)$. Then we have $\mathrm{HN}(\mathcal{E}_{b_i,x_i}) \leq \mathrm{HN}(\mathcal{E}_{b_i}) - \underline{1}^{(n_i)}$ by Proposition 3.1.3 and thus find $\mathrm{HN}(\mathcal{E}_{b_i,x_i}) = \mathrm{HN}(\mathcal{E}_{b_i}) - \underline{1}^{(n_i)}$ as $\mathrm{HN}(\mathcal{E}_{b_i}) - \underline{1}^{(n_i)}$ is a line segment. Now the desired assertion follows by the fact that $\mathcal{E}_{b,x}$ is a direct sum of the vector bundles $\mathcal{E}_{b_i,x_i}$. $\qquad \square$

Proposition 3.1.5. *Let μ be a dominant cocharacter of GL_n. For elements $b, b' \in B(\mathrm{GL}_n)$, the Newton stratum $\mathrm{Gr}_{\mathrm{GL}_n,\mu,b}^{b'}$ is not empty if and only if it contains a C-point.*

Proof. The assertion is evident by definition. □

Proposition 3.1.6. *Let μ be a dominant cocharacter of GL_n with nonnegative slopes. For two elements $b, b' \in B(\mathrm{GL}_n)$, we have the following equivalent conditions:*

(i) $\mathrm{Gr}_{\mathrm{GL}_n,\mu,b}^{b'}$ is nonempty.

(ii) $\mathrm{Gr}_{\mathrm{GL}_n,\mu^*,b'}^{b}$ is nonempty.

(iii) $\mathrm{Gr}_{\mathrm{GL}_n,\mu+\underline{1}^{(n)},b}^{\widetilde{b'}}$ is nonempty for $\widetilde{b'} \in B(\mathrm{GL}_n)$ with $\mathrm{HN}(\mathcal{E}_{\widetilde{b'}}) = \mathrm{HN}(\mathcal{E}_{b'}) - \underline{1}^{(n)}$.

Proof. For $x = g\mu(t)G(B_{\mathrm{dR}}^+) \in \mathrm{Gr}_{\mathrm{GL}_n,\mu,b}^{b'}(C)$, we take $x^* := g^{-1}\mu^*(t)G(B_{\mathrm{dR}}^+) \in \mathrm{Gr}_{\mathrm{GL}_n,\mu^*}(C)$ and find $\mathcal{E}_{b',x^*} \simeq \mathcal{E}_b$, thereby deducing that x^* lies in $\mathrm{Gr}_{\mathrm{GL}_n,\mu^*,b'}^{b}(C)$. Similarly, every point in $\mathrm{Gr}_{\mathrm{GL}_n,\mu^*,b'}^{b}(C)$ gives rise to a point in $\mathrm{Gr}_{\mathrm{GL}_n,\mu,b}^{b'}(C)$. Hence by Proposition 3.1.5 we establish the equivalence of the conditions (i) and (ii).

Now it remains to verify the equivalence of the conditions (i) and (iii). For every $x = g\mu(t)G(B_{\mathrm{dR}}^+) \in \mathrm{Gr}_{\mathrm{GL}_n,\mu,b}^{b'}(C)$, we take $\widetilde{x} := g\mu(t)\underline{1}^{(n)}(t)G(B_{\mathrm{dR}}^+) \in \mathrm{Gr}_{\mathrm{GL}_n,\mu+\underline{1}^{(n)}}(C)$ and find $\mathcal{E}_{b,\widetilde{x}} \simeq \mathcal{E}_{\widetilde{b'}}$ by Lemma 3.1.4, thereby deducing that $\widetilde{x}$ lies in $\mathrm{Gr}_{\mathrm{GL}_n,\mu+\underline{1}^{(n)},b}^{\widetilde{b'}}(C)$. Conversely, for every $\widetilde{x} := g\mu(t)\underline{1}^{(n)}(t)G(B_{\mathrm{dR}}^+) \in \mathrm{Gr}_{\mathrm{GL}_n,\mu+\underline{1}^{(n)}}(C)$ we take $x := g\mu(t)G(B_{\mathrm{dR}}^+) \in \mathrm{Gr}_{\mathrm{GL}_n,\mu,b}^{b'}(C)$ and find $\mathcal{E}_{b,x} \simeq \mathcal{E}_{b'}$ by Lemma 3.1.4, thereby deducing that x lies in $\mathrm{Gr}_{\mathrm{GL}_n,\mu,b}^{b'}(C)$. Hence we complete the proof by Proposition 3.1.5. □

Remark. In light of Proposition 3.1.6, for our desired classification it suffices to consider minuscule cocharacters with slopes 0 and 1.

Definition 3.1.7. Let $\mathcal{E}$ be a vector bundle on X of rank n.

(1) Given a dominant cocharacter μ of GL_n, we define an *effective modification* of $\mathcal{E}$ at ∞ of *type* μ to be an injective $\mathcal{O}_X$-module map $\mathcal{E}' \hookrightarrow \mathcal{E}$ whose cokernel is the skyscraper sheaf at ∞ with value $\bigoplus_{i=1}^{n} B_{\mathrm{dR}}^+/t^{\lambda_i(\mu)}B_{\mathrm{dR}}^+$.

(2) We say that an effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞ is *minuscule of degree d* if its type is minuscule of degree d with slopes 0 and 1.

Proposition 3.1.8. *Take a dominant cocharacter μ of GL_n and two elements $b, b' \in B(\mathrm{GL}_n)$.*

(1) If μ has nonnegative slopes, the Newton stratum $\mathrm{Gr}_{\mathrm{GL}_n,\mu,b}^{b'}$ is nonempty if and only if there exists an effective modification $\mathcal{E}_{b'} \hookrightarrow \mathcal{E}_b$ at ∞ of type μ.

(2) If μ is minuscule with slopes 0 and 1, the Newton stratum $\mathrm{Gr}_{\mathrm{GL}_n,\mu,b}^{b'}$ is nonempty if and only if there exists a minuscule effective modification $\mathcal{E}_{b'} \hookrightarrow \mathcal{E}_b$ at ∞.

Proof. As the second statement is a special case of the first statement, it suffices to prove the first statement. If $\mathrm{Gr}_{\mathrm{GL}_n,\mu,b}^{b'}$ is not empty, Proposition 3.1.5 yields a point $x \in \mathrm{Gr}_{\mathrm{GL}_n,\mu,b}^{b'}(C)$, which gives rise to an effective modification $\mathcal{E}_{b,x} \hookrightarrow \mathcal{E}_b$ at ∞ of type μ. Let us now assume for the converse that there exists an effective modification $\mathcal{E}_{b'} \hookrightarrow \mathcal{E}_b$ at ∞ of type μ. Take triples $(\mathcal{E}_b^\circ, \widehat{\mathcal{E}_b}, \beta_b)$ and $(\mathcal{E}_{b'}^\circ, \widehat{\mathcal{E}_{b'}}, \beta_{b'})$ which respectively correspond to $\mathcal{E}_b$ and $\mathcal{E}_{b'}$ under the bijection in Proposition 2.3.2. We may set $\mathcal{E}_b^\circ = \mathcal{E}_{b'}^\circ$ since the map $\mathcal{E}_{b'} \hookrightarrow \mathcal{E}_b$ is an isomorphism on $X - \infty$. Then we conjugate β_b by a suitable element in $\mathrm{GL}_n(B_{\mathrm{dR}}^+)$ to write $\beta_{b'} = g\mu(t)\beta_b$ for some $g \in \mathrm{GL}_n(B_{\mathrm{dR}}^+)$, and in turn find $g\mu(t)\mathrm{GL}_n(B_{\mathrm{dR}}^+) \in \mathrm{Gr}_{\mathrm{GL}_n,\mu,b}^{b'}(C)$ to complete the proof. □

Proposition 3.1.9. *Let $\mathcal{E}$ and $\mathcal{E}'$ be vector bundles on X of rank n. Take a direct sum decomposition*

$$\mathcal{E} \simeq \mathcal{D} \oplus \mathcal{F} \tag{3.2}$$

such that $\mathrm{HN}(\mathcal{D})$ coincides with the line segment of maximal slope in $\mathrm{HN}(\mathcal{E})$. There exists a minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞ if and only if there exist minuscule effective modifications $\mathcal{D}' \hookrightarrow \mathcal{D}$ and $\mathcal{F}' \hookrightarrow \mathcal{F}$ at ∞ with a short exact sequence

$$0 \longrightarrow \mathcal{D}' \longrightarrow \mathcal{E}' \longrightarrow \mathcal{F}' \longrightarrow 0.$$

Proof. The assertion is essentially a result of Chen-Tong [CT22, Proposition 4.6]. Our main observation is that, while the result in loc. cit. for $G = \mathrm{GL}_n$ only concerns the case where $\mathcal{E}$ is semistable, its proof remains valid without the semistability assumption on $\mathcal{E}$. For convenience of the readers, we explain how the result in loc. cit. is translated to the desired assertion.

Let us take $b, b' \in B(\mathrm{GL}_n)$ with $\mathcal{E} \simeq \mathcal{E}_b$ and $\mathcal{E}' \simeq \mathcal{E}_{b'}$. We write r for the rank of $\mathcal{D}$ and P for the standard parabolic subgroup of GL_n with Levi subgroup

$$M := \mathrm{GL}_r \times \mathrm{GL}_{n-r} \subseteq \mathrm{GL}_n.$$

The direct sum decomposition (3.2) corresponds to an element $b_M \in B(M)$ which maps to b under the natural map $B(M) \longrightarrow B(G)$. Let $E(M, b')$ denote the set of all elements $b'_M \in B(M)$ which correspond to a direct sum $\mathcal{D}' \oplus \mathcal{F}'$ for some vector bundles $\mathcal{D}'$ of rank r and $\mathcal{F}'$ of rank $n - r$ such that $\mathcal{E}' \simeq \mathcal{E}_{b'}$ arises as an extension of $\mathcal{F}'$ by $\mathcal{D}'$.

We take μ to be the minuscule dominant cocharacter of GL_n of degree $d := \deg(\mathcal{E}) - \deg(\mathcal{E}')$ with slopes 0 and 1. In addition, we choose an arbitrary element w in the Weyl group of GL_n and denote by μ^w the dominant cocharacter of M whose M-conjugacy class contains the w-conjugate of μ. We have $\mu^w = (\mu_1, \mu_2)$ for some minuscule dominant cocharacters μ_1 of GL_r and μ_2 of GL_{n-r}. We denote the degrees of μ_1 and μ_2 respectively by d_1 and d_2.

Let $\mathscr{F}\ell(\mathrm{GL}_n, \mu)_P^w$ be the subscheme of $\mathscr{F}\ell(\mathrm{GL}_n, \mu)$ given by the P-orbit of P_{μ^w}. The projection to M induces a map

$$\mathrm{pr}_{P,w} : \mathscr{F}\ell(\mathrm{GL}_n, \mu)_P^w \longrightarrow \mathscr{F}\ell(M, \mu^w).$$

The aforementioned result of Chen-Tong [CT22, Proposition 4.6] yields an identity

$$\mathrm{pr}_{P,w}\left(\mathscr{F}\ell(\mathrm{GL}_n, \mu)_P^w \cap \mathscr{F}\ell(\mathrm{GL}_n, \mu, b)^{b'}\right) = \bigsqcup_{b'_M \in E(M, b')} \mathscr{F}\ell(M, \mu^w, b_M)^{b'_M}. \tag{3.3}$$

As both μ and μ^w are minuscule, Proposition 2.1.8 implies that the Newton strata on $\mathrm{Gr}_{\mathrm{GL}_n, \mu}$ and Gr_{M, μ^w} are respectively identified with the Newton strata on $\mathscr{F}\ell(\mathrm{GL}_n, \mu)$ and $\mathscr{F}\ell(M, \mu^w)$. Hence the identity (3.3) shows that for minuscule effective modifications $\alpha : \mathcal{D}' \hookrightarrow \mathcal{D}$ and $\beta : \mathcal{F}' \hookrightarrow \mathcal{F}$ at ∞ of degrees d_1 and d_2 we have the following equivalent conditions:

(i) There exists a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{D} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
& & {\scriptstyle\alpha}\uparrow & & \uparrow & & {\scriptstyle\beta}\uparrow & & \\
0 & \longrightarrow & \mathcal{D}' & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{F}' & \longrightarrow & 0
\end{array} \tag{3.4}$$

with the top row given by the direct sum decomposition (3.2) and the middle vertical arrow being a minuscule effective modification at ∞ (of degree d).

(ii) There exists a short exact sequence

$$0 \longrightarrow \mathcal{D}' \longrightarrow \mathcal{E}' \longrightarrow \mathcal{F}' \longrightarrow 0.$$

Since w is arbitrary, we deduce the desired assertion. $\qquad\square$

Remark. The necessity part of Proposition 3.1.9 is evident as every minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞ gives rise to a commutative diagram (3.4). The main point of Proposition 3.1.9 is the sufficiency part, which is essentially equivalent to the identity (3.3) by Chen-Tong [CT22].

Proposition 3.1.10 ([FF18, §5.5.2.1]). *Let $\mathcal{E}$ be a vector bundle on X. For every minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞, its degree is equal to $\deg(\mathcal{E}) - \deg(\mathcal{E}')$.*

Lemma 3.1.11. *Let $\mathcal{E}$ and $\mathcal{E}'$ be vector bundles on X of rank n such that $\mathcal{E}$ is semistable. Take μ to be the minuscule dominant cocharacter of GL_n of degree $d := \deg(\mathcal{E}) - \deg(\mathcal{E}')$ with slopes 0 and 1. There exists a minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞ if and only if $\mathcal{E}$ and $\mathcal{E}'$ satisfy the following equivalent inequalities:*

$$\mathrm{HN}(\mathcal{E}) + \mu^* \geq \mathrm{HN}(\mathcal{E}') \quad and \quad \mathrm{HN}(\mathcal{E}') + \underline{1}^{(n)} \succeq \mathrm{HN}(\mathcal{E}) \succeq \mathrm{HN}(\mathcal{E}'). \qquad (3.5)$$

Proof. By Proposition 3.1.3, Proposition 3.1.8 and Proposition 3.1.10, there exists a minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞ if and only if $\mathcal{E}$ and $\mathcal{E}'$ satisfy the first inequality in (3.5). If we write λ for the slope of the line segment $\mathrm{HN}(\mathcal{E})$, the polygon $\mathrm{HN}(\mathcal{E}) + \mu^*$ has two distinct slopes λ and $\lambda - 1$. Hence it is not hard to verify the equivalence of the two inequalities in (3.5) by the concavity of HN polygons, thereby deducing the desired assertion. $\qquad \square$

Theorem 3.1.12. *Let μ be a minuscule dominant cocharacter of GL_n with slopes 0 and 1. Consider two arbitrary elements $b, b' \in B(\mathrm{GL}_n)$. Take a direct sum decomposition*

$$\mathcal{E}_b \simeq \mathcal{E}_a \oplus \mathcal{E}_c \quad with \quad a \in B(\mathrm{GL}_r) \text{ and } c \in B(\mathrm{GL}_{n-r})$$

such that $\mathrm{HN}(\mathcal{E}_a)$ coincides with the line segment of maximal slope in $\mathrm{HN}(\mathcal{E}_b)$.

  (1) *If the degree of μ is not equal to $\deg(\mathcal{E}_b) - \deg(\mathcal{E}_{b'})$, the Newton stratum $\mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}$ is empty.*

  (2) *If the degree of μ is equal to $\deg(\mathcal{E}_b) - \deg(\mathcal{E}_{b'})$ the Newton stratum $\mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}$ is nonempty if and only if there exist $a' \in B(\mathrm{GL}_r)$ and $c' \in B(\mathrm{GL}_{n-r})$ with the following properties:*

    (i) *We have $\mathrm{HN}(\mathcal{E}_{a'}) + \underline{1}^{(r)} \succeq \mathrm{HN}(\mathcal{E}_a) \succeq \mathrm{HN}(\mathcal{E}_{a'})$.*

    (ii) *If we write the HN decomposition of $\mathcal{E}_{c'}$ as*

$$\mathcal{E}_{c'} \simeq \bigoplus_{i=1}^{m} \mathcal{F}_i$$

   where $\mathcal{F}_i$ are arranged in order of descending slope, there exists a sequence of vector bundles $\mathcal{E}_{a'} = \mathcal{E}_0, \mathcal{E}_1, \cdots, \mathcal{E}_m = \mathcal{E}_b$ on X such that $\mathrm{HN}(\mathcal{E}_{i-1} \oplus \mathcal{F}_i)$ has an $(\mathcal{E}_{i-1}, \mathcal{E}_i, \mathcal{F}_i)$-permutation for each $i = 1, \cdots, r$.

    (iii) *The Newton stratum $\mathrm{Gr}^{c'}_{\mathrm{GL}_n,\overline{\mu},c}$ is nonempty where $\overline{\mu}$ is a minuscule dominant cocharacter of GL_{n-r} of degree $\overline{d} := \deg(\mathcal{E}_c) - \deg(\mathcal{E}_{c'})$ with slopes 0 and 1.*

Proof. The assertion is straightforward to verify by Proposition 2.4.8, Proposition 3.1.8, Proposition 3.1.9, Proposition 3.1.10, and Lemma 3.1.11. $\qquad \square$

Remark. The elements $a \in B(\mathrm{GL}_r)$ and $c \in B(\mathrm{GL}_{n-r})$ are uniquely determined by the HN decomposition of $\mathcal{E}_b$. In addition, the Schubert cell $\mathrm{Gr}_{\mathrm{GL}_n,\overline{\mu}}$ contains finitely many nonemtpy Newton strata, as easily seen by Proposition 2.4.4 and Proposition 3.1.8. Hence the conditions (i) and (iii) together yield finitely many candidates for $a' \in B(\mathrm{GL}_r)$ and $c' \in B(\mathrm{GL}_{n-r})$. We can thus use Theorem 3.1.12 to inductively classify all nonempty Newton strata in an arbitrary minuscule Schubert cell of $\mathrm{Gr}_{\mathrm{GL}_n}$.

3.2. Concave rationally tuplar polygons.

Definition 3.2.1. Given a rationally tuplar polygon $\mathscr{P}$, we define its *concave rearrangement* to be the rationally tuplar polygon $\widehat{\mathscr{P}}$ such that the tuple $(\lambda_i(\widehat{\mathscr{P}}))$ is the rearrangement of $(\lambda_i(\mathscr{P}))$ in descending order.

Lemma 3.2.2. *For every rationally tuplar polygon $\mathscr{P}$, we have $\widehat{\mathscr{P}} \geq \mathscr{P}$.*

Proof. The assertion is evident by definition. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark. In fact, $\widehat{\mathscr{P}}$ is the maximal rearrangement of $\mathscr{P}$ with respect to the Bruhat order.

Definition 3.2.3. Given two rationally tuplar polygon $\mathscr{P}$ and $\mathscr{Q}$, we define their direct sum $\mathscr{P} \oplus \mathscr{Q}$ to be the concave rearrangement of the concatenation of $\mathscr{P}$ and $\mathscr{Q}$.

Example 3.2.4. Let us record some important examples of direct sums for our purpose.

(1) For two vector bundles $\mathcal{E}$ and $\mathcal{F}$ on X, we have $\mathrm{HN}(\mathcal{E} \oplus \mathcal{F}) = \mathrm{HN}(\mathcal{E}) \oplus \mathrm{HN}(\mathcal{F})$.

(2) For two minuscule dominant cocharacters μ_1 of GL_{n_1} and μ_2 of GL_{n_2} with slopes 0 and 1, their direct sum (as a rationally tuplar polygon) is a minuscule dominant cocharacter of $\mathrm{GL}_{n_1+n_2}$ with slopes 0 and 1.

Lemma 3.2.5. *Given concave rationally tuplar polygons $\mathscr{P}, \mathscr{P}', \mathscr{Q}$ and $\mathscr{Q}'$ with $\mathscr{P} \geq \mathscr{P}'$ and $\mathscr{Q} \geq \mathscr{Q}'$, we have $\mathscr{P} \oplus \mathscr{Q} \geq \mathscr{P}' \oplus \mathscr{Q}'$.*

Proof. Let m and n respectively denote the ranks of $\mathscr{P}$ and $\mathscr{Q}$. Take two sets A and B which form a partition of the set $\{1, \cdots, m+n\}$ with

$$(\lambda_i(\mathscr{P}' \oplus \mathscr{Q}'))_{i \in A} = (\lambda_i(\mathscr{P}')) \quad \text{and} \quad (\lambda_i(\mathscr{P}' \oplus \mathscr{Q}'))_{i \in B} = (\lambda_i(\mathscr{Q}')).$$

Let $\mathscr{R}$ to be the rationally tuplar polygon of rank $m+n$ with

$$(\lambda_i(\mathscr{R}))_{i \in A} = (\lambda_i(\mathscr{P})) \quad \text{and} \quad (\lambda_i(\mathscr{R}))_{i \in B} = (\lambda_i(\mathscr{Q})).$$

Since $\mathscr{P}, \mathscr{P}', \mathscr{Q}$ and $\mathscr{Q}'$ are all concave, the inequalities $\mathscr{P} \geq \mathscr{P}'$ and $\mathscr{Q} \geq \mathscr{Q}'$ together imply $\mathscr{R} \geq \mathscr{P}' \oplus \mathscr{Q}'$. Now we find $\mathscr{P} \oplus \mathscr{Q} = \widehat{\mathscr{R}} \geq \mathscr{R}$ by Lemma 3.2.2 to complete the proof. $\quad$ $\square$

Remark. Lemma 3.2.5 does not hold without the concavity assumption. For example, if we take $\mathscr{P} = \mathscr{Q} = d/r^{(r)}$ for some integers r and d with $r > 0$, for arbitrary nonlinear convex polygons $\mathscr{P}'$ and $\mathscr{Q}'$ of rank r and degree d we do not have $\mathscr{P} \oplus \mathscr{Q} \geq \mathscr{P}' \oplus \mathscr{Q}'$ despite having $\mathscr{P} \geq \mathscr{P}'$ and $\mathscr{Q} \geq \mathscr{Q}'$, as illustrated in Figure 5.
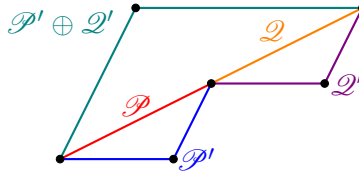

FIGURE 5. A counter example for Lemma 3.2.5 without the concavity assumption

Lemma 3.2.6. *Let $\mathscr{P}$ and $\mathscr{Q}$ be rationally tuplar polygons of rank m and n. For arbitrary rationally tuplar polygons $\mathscr{P}'$ of rank m and $\mathscr{Q}'$ of rank n, we have*

$$(\mathscr{P} \oplus \mathscr{Q}) + (\mathscr{P}' \oplus \mathscr{Q}') \geq (\mathscr{P} + \mathscr{P}') \oplus (\mathscr{Q} + \mathscr{Q}').$$

Proof. We observe that there exist permutations σ and σ' of the set $\{1, \cdots, m+n\}$ with

$$\lambda_i((\mathscr{P} + \mathscr{P}') \oplus (\mathscr{Q} + \mathscr{Q}')) = \lambda_{\sigma(i)}(\mathscr{P} \oplus \mathscr{Q}) + \lambda_{\sigma'(i)}(\mathscr{P}' \oplus \mathscr{Q}') \quad \text{for each } i = 1, \cdots, m+n,$$

and consequently deduce the desired assertion by the concavity of $\mathscr{P} \oplus \mathscr{Q}$ and $\mathscr{P}' \oplus \mathscr{Q}'$. $\quad$ $\square$

3.3. An explicit classification of nonempty Newton strata.

Lemma 3.3.1. *Let $\mathcal{E}$ be a vector bundle on X of rank n. Every minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞ gives rise to a minuscule effective modification $\widetilde{\mathcal{E}} \hookrightarrow \mathcal{E}'$ at ∞ with $\mathrm{HN}(\widetilde{\mathcal{E}}) = \mathrm{HN}(\mathcal{E}) - \underline{1}^{(n)}$.*

Proof. Let μ be the minuscule dominant cocharacter of GL_n of degree $d := \deg(\mathcal{E}) - \deg(\mathcal{E}')$ with slopes 0 and 1. Take elements b, b' and $\widetilde{b}$ in $B(\mathrm{GL}_n)$ with $\mathcal{E} \simeq \mathcal{E}_b$, $\mathcal{E}' \simeq \mathcal{E}_{b'}$ and $\widetilde{\mathcal{E}} \simeq \mathcal{E}_{\widetilde{b}}$. The effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞ yields a point in $\mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}$ by Proposition 3.1.8 and Proposition 3.1.10, and in turn yields a point in $\mathrm{Gr}^{\widetilde{b}}_{\mathrm{GL}_n,\mu^*+\underline{1}^{(n)},b'}$ by Proposition 3.1.6. Hence we obtain a minuscule effective modification $\widetilde{\mathcal{E}} \hookrightarrow \mathcal{E}'$ at ∞ by Proposition 3.1.8 as desired. $\square$

Proposition 3.3.2. *Let $\mathcal{E}$ be a vector bundle on X of rank n. For every minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞, we have*

$$\mathrm{HN}(\mathcal{E}) + \mu^* \geq \mathrm{HN}(\mathcal{E}') \quad and \quad \mathrm{HN}(\mathcal{E}') + \underline{1}^{(n)} \succeq \mathrm{HN}(\mathcal{E}) \succeq \mathrm{HN}(\mathcal{E}')$$

where μ is the minuscule dominant cocharacter of GL_n of degree $d := \deg(\mathcal{E}) - \deg(\mathcal{E}')$ with slopes 0 and 1.

Proof. The second inequality is an immediate consequence of Proposition 2.4.4 and Lemma 3.3.1. Hence it remains to establish the first inequality. Let us write m for the number of distinct slopes in $\mathrm{HN}(\mathcal{E})$ and proceed by induction on m. If $\mathcal{E}$ is semistable, the assertion is evident by Lemma 3.1.11. We henceforth assume that $\mathcal{E}$ is not semistable, so that we have $m > 1$. Take a direct sum decomposition

$$\mathcal{E} \simeq \mathcal{D} \oplus \mathcal{F}$$

such that $\mathrm{HN}(\mathcal{D})$ coincides with the line segment of maximal slope in $\mathrm{HN}(\mathcal{E})$. The numbers of distinct slopes in $\mathrm{HN}(\mathcal{D})$ and $\mathrm{HN}(\mathcal{F})$ are respectively 1 and $m - 1$. Now Proposition 3.1.9 yields minuscule effective modifications $\alpha : \mathcal{D}' \hookrightarrow \mathcal{D}$ and $\beta : \mathcal{F}' \hookrightarrow \mathcal{F}$ at ∞ with a short exact sequence

$$0 \longrightarrow \mathcal{D}' \longrightarrow \mathcal{E}' \longrightarrow \mathcal{F}' \longrightarrow 0. \tag{3.6}$$

Let us denote the types of α and β respectively by μ_1 and μ_2. In a concrete form, we have

$$\mu_1 = \underline{1}^{(d_1)} \oplus \underline{0}^{(n_1-d_1)} \quad and \quad \mu_2 = \underline{1}^{(d_2)} \oplus \underline{0}^{(n_2-d_2)}$$

where we set $n_1 := \mathrm{rk}(\mathcal{D}) = \mathrm{rk}(\mathcal{D}')$, $n_2 := \mathrm{rk}(\mathcal{F}) = \mathrm{rk}(\mathcal{F}')$, $d_1 := \deg(\mathcal{D}) - \deg(\mathcal{D}')$ and $d_2 := \deg(\mathcal{F}) - \deg(\mathcal{F}')$. By the induction hypothesis, the minuscule effective modifications α and β at ∞ respectively yield the inequalities

$$\mathrm{HN}(\mathcal{D}) + \mu_1^* \geq \mathrm{HN}(\mathcal{D}') \quad and \quad \mathrm{HN}(\mathcal{F}) + \mu_2^* \geq \mathrm{HN}(\mathcal{F}').$$

Then by Example 3.2.4, Lemma 3.2.5 and Lemma 3.2.6 we find

$$\mathrm{HN}(\mathcal{E}) + \mu^* = (\mathrm{HN}(\mathcal{D}) \oplus \mathrm{HN}(\mathcal{F})) + (\mu_1^* \oplus \mu_2^*) \geq (\mathrm{HN}(\mathcal{D}) + \mu_1^*) \oplus (\mathrm{HN}(\mathcal{F}) + \mu_2^*) \geq \mathrm{HN}(\mathcal{D}' \oplus \mathcal{F}').$$

In addition, by Proposition 2.4.7 the short exact sequence (3.6) yields the inequality

$$\mathrm{HN}(\mathcal{D}' \oplus \mathcal{F}') \geq \mathrm{HN}(\mathcal{E}').$$

We thus obtain the first inequality, thereby completing the proof. $\square$

Remark. The two inequalities in Proposition 3.3.2 are not equivalent in general, although they are equivalent if $\mathcal{E}$ is semistable as shown in Lemma 3.1.11.

Example 3.3.3. Let us present an example showing that the converse of Proposition 3.3.2 does not hold. Take $\mathcal{E}$ and $\mathcal{E}'$ to be vector bundles on X with

$$\mathrm{HN}(\mathcal{E}) = \underline{4/3}^{(3)} \oplus \underline{3/4}^{(4)} \quad \text{and} \quad \mathrm{HN}(\mathcal{E}') = \underline{1}^{(2)} \oplus \underline{1/3}^{(3)} \oplus \underline{0}^{(2)}.$$

By construction, we have $\mathrm{rk}(\mathcal{E}) = \mathrm{rk}(\mathcal{E}') = 7$, $\deg(\mathcal{E}) = 7$ and $\deg(\mathcal{E}') = 3$. Now for the minuscule dominant cocharacter μ of GL_7 of degree 4 with slopes 0 and 1, we find

$$\mathrm{HN}(\mathcal{E}) + \mu^* \geq \mathrm{HN}(\mathcal{E}') \quad \text{and} \quad \mathrm{HN}(\mathcal{E}') + \underline{1}^{(7)} \succeq \mathrm{HN}(\mathcal{E}) \succeq \mathrm{HN}(\mathcal{E}').$$
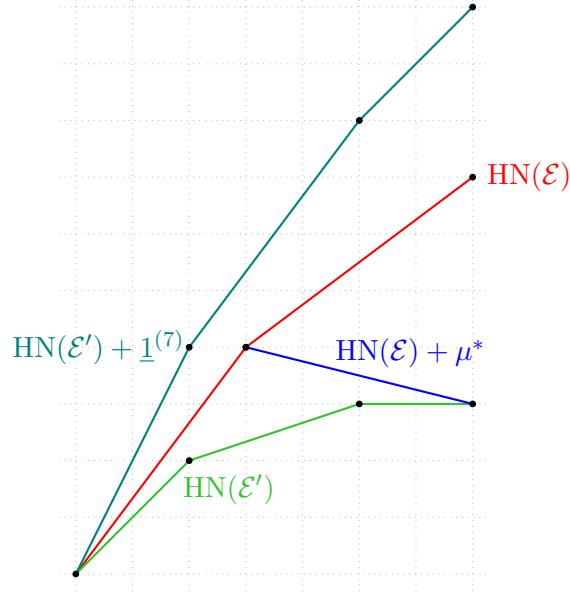

FIGURE 6. A counter example for the converse of Proposition 3.3.2

We wish to show that there does not exist a minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞. Suppose for contradiction that such a modification exists. Take a direct sum decomposition

$$\mathcal{E} \simeq \mathcal{D} \oplus \mathcal{F}$$

with $\mathrm{HN}(\mathcal{D}) = \underline{4/3}^{(3)}$ and $\mathrm{HN}(\mathcal{F}) = \underline{3/4}^{(4)}$. Proposition 3.1.9 yields minuscule effective modifications $\mathcal{D}' \hookrightarrow \mathcal{D}$ and $\mathcal{F}' \hookrightarrow \mathcal{F}$ at ∞ with a short exact sequence

$$0 \longrightarrow \mathcal{D}' \longrightarrow \mathcal{E}' \longrightarrow \mathcal{F}' \longrightarrow 0.$$

Then by Proposition 2.4.7 we obtain a $(\mathcal{D}', \mathcal{E}', \mathcal{F}')$-permutation $\mathscr{P}$ of $\mathrm{HN}(\mathcal{D}' \oplus \mathcal{F}')$. Since we have $\mathscr{P} \geq \mathrm{HN}(\mathcal{E}')$ by construction, we find

$$\lambda_1(\mathscr{P}) \geq \lambda_1(\mathrm{HN}(\mathcal{E}')) = 1 \quad \text{and} \quad \lambda_2(\mathscr{P}) \geq \lambda_2(\mathrm{HN}(\mathcal{E}')) = 1. \tag{3.7}$$

Moreover, as $\mathcal{F}'$ is a subsheaf of $\mathcal{F}$ by construction, Proposition 2.4.4 implies that all slopes in $\mathrm{HN}(\mathcal{F}')$ are less than or equal to $3/4$. We then deduce by (3.7) that $\lambda_1(\mathscr{P})$ and $\lambda_2(\mathscr{P})$ should occur as a slope of $\mathcal{D}'$, and in turn find that the inequalities in (3.7) are in fact equalities. Therefore $\mathrm{HN}(\mathcal{D}')$ must contain the line segment $\underline{1}^{(2)}$, and consequently is given by $\underline{1}^{(2)} \oplus \underline{d}^{(1)}$ for some integer d. Then we have $d = \lambda_i(\mathscr{P})$ for some $i > 2$ and thus find $d \leq \lambda_i(\mathrm{HN}(\mathcal{E}')) \leq 1/3$. On the other hand, since $\mathcal{D}'$ occurs as a minuscule effective modification of $\mathcal{D}$ at C, Proposition 3.3.2 implies $d \geq 1/3$. Now we have a desired contradiction as d is an integer with $d \leq 1/3$ and $d \geq 1/3$.

Proposition 3.3.4. *Let $\mathcal{E}$ and $\mathcal{E}'$ be vector bundles on X of rank n. Denote by μ the minuscule dominant cocharacter of GL_n of degree $d := \deg(\mathcal{E}) - \deg(\mathcal{E}')$ with slopes 0 and 1. Assume that $\mathcal{E}$ satisfies the following property:*

   (∗) *All distinct slopes in $\mathrm{HN}(\mathcal{E})$ differ by more than 1.*

There exists a minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞ if and only if $\mathcal{E}$ and $\mathcal{E}'$ satisfy the following conditions:

   (i) *We have $\mathrm{HN}(\mathcal{E}) + \mu^* \geq \mathrm{HN}(\mathcal{E}')$ and $\mathrm{HN}(\mathcal{E}') + \underline{1}^{(n)} \succeq \mathrm{HN}(\mathcal{E}) \succeq \mathrm{HN}(\mathcal{E}')$.*

   (ii) *For each breakpoint of $\mathrm{HN}(\mathcal{E})$, there exists a breakpoint of $\mathrm{HN}(\mathcal{E}')$ with the same x-coordinate.*
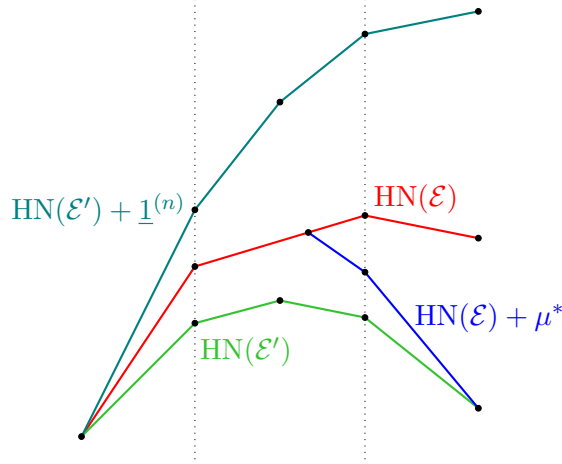

FIGURE 7. Illustration of the conditions in Proposition 3.3.4

Proof. Let us first assume that $\mathcal{E}$ and $\mathcal{E}'$ satisfy the conditions (i) and (ii). We write the HN decomposition of $\mathcal{E}$ as

$$\mathcal{E} \simeq \bigoplus_{i=1}^{m} \mathcal{E}_i \tag{3.8}$$

where the direct summands $\mathcal{E}_i$ are arranged in order of descending slope, and set

$$x_i := \sum_{j=1}^{i} \mathrm{rk}(\mathcal{E}_j) \quad \text{for } i = 0, \cdots, m.$$

By the condition (ii), we get a direct sum decomposition

$$\mathcal{E}' \simeq \bigoplus_{i=1}^{m} \mathcal{E}'_i \tag{3.9}$$

where each $\mathrm{HN}(\mathcal{E}'_i)$ coincides with the restriction of $\mathrm{HN}(\mathcal{E}')$ on the interval $[x_{i-1}, x_i]$. Then by the condition (i) we find

$$\mathrm{HN}(\mathcal{E}'_i) + \underline{1}^{(x_i - x_{i-1})} \succeq \mathrm{HN}(\mathcal{E}_i) \succeq \mathrm{HN}(\mathcal{E}'_i) \quad \text{for } i = 1, \cdots, m.$$

Now for each $i = 1, \cdots, m$, Lemma 3.1.11 yields a minuscule effective modification $\mathcal{E}'_i \hookrightarrow \mathcal{E}_i$ at ∞ as $\mathcal{E}_i$ is semistable. Hence we obtain a minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞ from the direct sum decompositions (3.8) and (3.9).

For the converse, we now assume that there exists a minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞. Since $\mathcal{E}$ and $\mathcal{E}'$ satisfy the condition (i) by Proposition 3.3.2, it remains to establish the condition (ii). We proceed by induction on the number m of distinct slopes in $\mathrm{HN}(\mathcal{E})$. If $\mathcal{E}$ is semistable, the assertion is vacuously true as $\mathrm{HN}(\mathcal{E})$ does not have a breakpoint. We henceforth assume that $\mathcal{E}$ is not semistable, so that we have $m > 1$. Take a direct sum decomposition

$$\mathcal{E} \simeq \mathcal{D} \oplus \mathcal{F} \tag{3.10}$$

such that $\mathrm{HN}(\mathcal{D})$ coincides with the line segment of maximal slope in $\mathrm{HN}(\mathcal{E})$. Let us denote the slope of $\mathrm{HN}(\mathcal{D})$ by λ. By construction, $\mathrm{HN}(\mathcal{F})$ has $m-1$ distinct slopes which are all less than $\lambda - 1$ by the property $(*)$. In addition, we have $\mathrm{HN}(\mathcal{E}') + \underline{1}^{(n)} \succeq \mathrm{HN}(\mathcal{E})$ by Proposition 3.3.2 and thus find

$$\lambda_i(\mathrm{HN}(\mathcal{E}')) \geq \lambda - 1 \quad \text{for } i = 1, \cdots, \mathrm{rk}(\mathcal{D}). \tag{3.11}$$

Now we note by Proposition 3.1.9 that there exist minuscule effective modifications $\mathcal{D}' \hookrightarrow \mathcal{D}$ and $\mathcal{F}' \hookrightarrow \mathcal{F}$ at ∞ with a short exact sequence

$$0 \longrightarrow \mathcal{D}' \longrightarrow \mathcal{E}' \longrightarrow \mathcal{F}' \longrightarrow 0. \tag{3.12}$$

Then we find

$$\lambda_i(\mathrm{HN}(\mathcal{F}')) \leq \lambda_i(\mathrm{HN}(\mathcal{F})) < \lambda - 1 \quad \text{for } i = 1, \cdots, \mathrm{rk}(\mathcal{F}') \tag{3.13}$$

by Proposition 2.4.4, and also obtain a $(\mathcal{D}', \mathcal{E}', \mathcal{F}')$-permutation $\mathscr{P}$ of $\mathrm{HN}(\mathcal{D}' \oplus \mathcal{F}')$ by Proposition 2.4.7. For each $i = 1, \cdots, \mathrm{rk}(\mathcal{D})$, the inequalities (3.11) and (3.13) together imply that $\lambda_i(\mathscr{P})$ occurs as a slope in $\mathrm{HN}(\mathcal{D}')$. Since we have $\mathscr{P} \geq \mathrm{HN}(\mathcal{E}')$ by construction, we find

$$\lambda_i(\mathscr{P}) = \lambda_i(\mathrm{HN}(\mathcal{D}')) = \lambda_i(\mathrm{HN}(\mathcal{E}')) \quad \text{for } i = 1, \cdots, \mathrm{rk}(\mathcal{D})$$

and consequently deduce from the inequalities (3.11) and (3.13) that all slopes in $\mathrm{HN}(\mathcal{D}')$ are greater than all slopes in $\mathrm{HN}(\mathcal{F}')$. Hence the short exact sequence (3.12) induces a direct sum

$$\mathcal{E}' \simeq \mathcal{D}' \oplus \mathcal{F}' \tag{3.14}$$

by Proposition 2.4.6, and consequently yields a breakpoint of $\mathrm{HN}(\mathcal{E}')$ with x-coordinate $\mathrm{rk}(\mathcal{D}') = \mathrm{rk}(\mathcal{D})$. In addition, since we have a minuscule effective modification $\mathcal{F}' \hookrightarrow \mathcal{F}$ at ∞, we find by the induction hypothesis that for every breakpoint of $\mathrm{HN}(\mathcal{F})$ there exists a breakpoint of $\mathrm{HN}(\mathcal{F}')$ with the same x-coordinate. We thus establish the condition (ii) by the direct sum decompositions (3.10) and (3.14), thereby completing the proof. $\square$

Theorem 3.3.5. *Let μ be a minuscule dominant cocharacter of GL_n with slopes 0 and 1. Take two arbitrary elements $b, b' \in B(\mathrm{GL}_n)$ and write $\nu(b) := \mathrm{HN}(\mathcal{E}_b)$ and $\nu(b') := \mathrm{HN}(\mathcal{E}_{b'})$. Assume that b satisfies the following property:*

$(*)$ All distinct slopes in $\nu(b)$ differ by more than 1.

The Newton stratum $\mathrm{Gr}_{\mathrm{GL}_n, \mu, b}^{b'}$ is nonempty if and only if $\nu(b)$ and $\nu(b')$ satisfy the following conditions:

(i) We have $\nu(b) + \mu^* \geq \nu(b')$ and $\nu(b') + \underline{1}^{(n)} \succeq \nu(b) \succeq \nu(b')$.

(ii) For each breakpoint of $\nu(b)$, there exists a breakpoint of $\nu(b')$ with the same x-coordinate.

Proof. The assertion is an immediate consequence of Proposition 3.1.8, Proposition 3.1.10 and Proposition 3.3.4. $\square$

Remark. Theorem 3.3.5 is identical to Theorem 1.1.1. For a non-minuscule cocharacter μ of GL_n with slopes in $[0, d]$, we should be able to get a similar classification theorem with d in place of 1 using the Demazure resolution.

Example 3.3.6. Let us provide an example to show that Proposition 3.3.4 and Theorem 3.3.5 do not hold without assuming (∗). Take $\mathcal{E}$ and $\mathcal{E}'$ to be vector bundles on X with

$$\mathrm{HN}(\mathcal{E}) = \underline{5/4}^{(4)} \oplus \underline{3/4}^{(4)} \quad \text{and} \quad \mathrm{HN}(\mathcal{E}') = \underline{3/5}^{(5)} \oplus \underline{1/3}^{(3)}.$$

Then $\mathrm{HN}(\mathcal{E})$ and $\mathrm{HN}(\mathcal{E}')$ do not have breakpoints with the same x-coordinates. We wish to show that there exists a minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞. Take vector bundles $\mathcal{D}, \mathcal{D}', \mathcal{F}$ and $\mathcal{F}'$ on X with

$$\mathrm{HN}(\mathcal{D}) = \underline{5/4}^{(4)}, \quad \mathrm{HN}(\mathcal{D}') = \underline{1/4}^{(4)}, \quad \mathrm{HN}(\mathcal{F}) = \mathrm{HN}(\mathcal{F}') = \underline{3/4}^{(4)}.$$

By construction, we have a direct sum decomposition

$$\mathcal{E} \simeq \mathcal{D} \oplus \mathcal{F}.$$

In addition, we obtain minuscule effective modifications $\mathcal{D}' \hookrightarrow \mathcal{D}$ and $\mathcal{F}' \hookrightarrow \mathcal{F}$ at ∞ by Lemma 3.1.11, and find a short exact sequence

$$0 \longrightarrow \mathcal{D}' \longrightarrow \mathcal{E}' \longrightarrow \mathcal{F}' \longrightarrow 0$$

by Proposition 2.4.8. Therefore Proposition 3.1.9 yields a minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞ as desired.
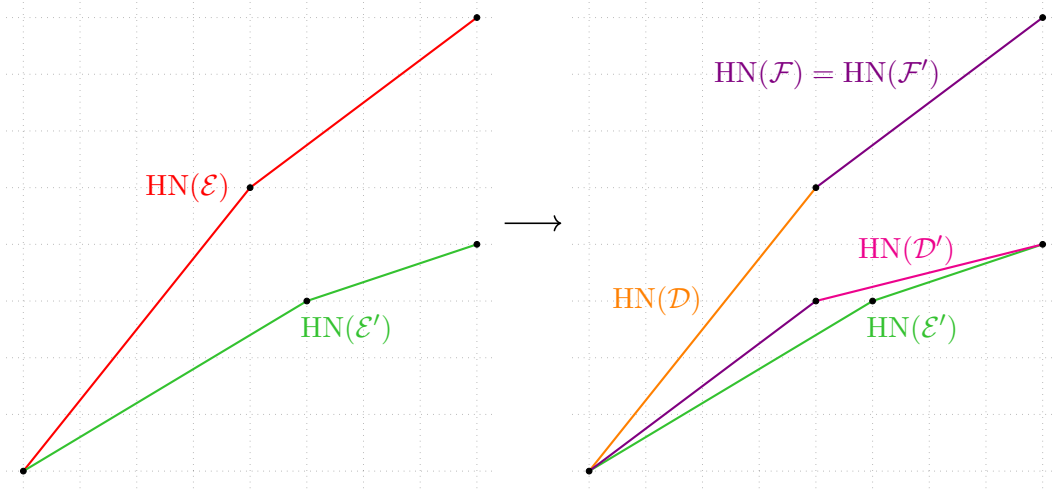

FIGURE 8. Illustration of Example 3.3.6

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