12.5. Equations of lines and planes
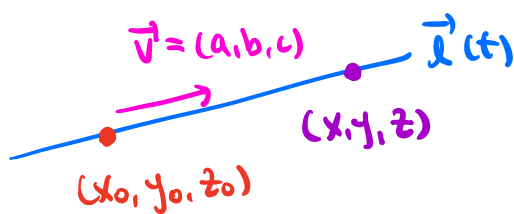
<u>Note</u> In order to specify a line in $\mathbb{R}^3$, you need one of the following data:

- two points on the line
- a point and a direction vector.

★ <u>Thm</u> The line through the point (x_0, y_0, z_0) with a direction vector $\vec{v} = (a, b, c)$ is parametrized by

$$\vec{\ell}(t) = (x_0 + at, \; y_0 + bt, \; z_0 + ct), \quad t \in \mathbb{R}.$$

$$= (x_0, y_0, z_0) + t(a, b, c)$$

Explanation:
$\vec{v}$ is parallel to $(x - x_0, y - y_0, z - z_0)$
$\Rightarrow t(a, b, c) = (x - x_0, y - y_0, z - z_0)$

<u>Rmk</u> There are (infinitely) many different ways to parametrize a given line, depending on the choice of a point and a direction vector.

e.g. The x-axis is parametrized by

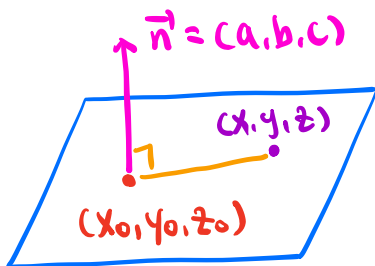$$\vec{\ell_1}(t) = (t, 0, 0) \quad = (0, 0, 0) + t(1, 0, 0)$$

$$\vec{\ell_2}(t) = (1 + 2t, 0, 0) = (1, 0, 0) + t(2, 0, 0)$$

<u>Note</u> In order to specify a plane in $\mathbb{R}^3$, you need one of the following data:

- three points on the plane
- a point and a <u>normal</u> vector
$$\underset{\text{perpendicular}}{\parallel}$$

<u>Thm</u> The plane through the point (x_0, y_0, z_0) with a normal vector $\vec{n} = (a, b, c)$ is given by the following equation:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$\vec{n} = (a, b, c)$

(x, y, z)

(x_0, y_0, z_0)

Explanation:

$\vec{n}$ is orthogonal to $(x - x_0, y - y_0, z - z_0)$

$\Rightarrow (a, b, c) \cdot (x - x_0, y - y_0, z - z_0) = 0$

<u>Rmk</u> (1) The above equation can also be written as

$$ax + by + cz = d \quad \text{with} \quad d = -ax_0 - by_0 - cz_0.$$

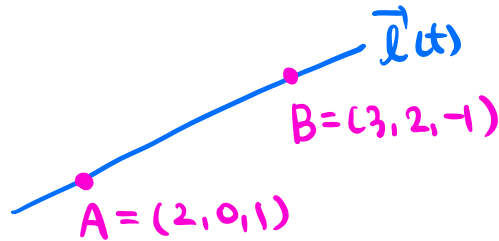(2) The coefficients of the equation depend on the choice of a normal vector.

e.g. The xy-plane is given by

$$\begin{cases} z = 0 \iff 0 \cdot x + 0 \cdot y + 1 \cdot z = 0 & (\vec{n} = (0, 0, 1)) \\ 3z = 0 \iff 0 \cdot x + 0 \cdot y + 3 \cdot z = 0 & (\vec{n} = (0, 0, 3)) \end{cases}$$

Ex Parametrize the following lines.

(1) The line through $A = (2, 0, 1)$ and $B = (3, 2, -1)$
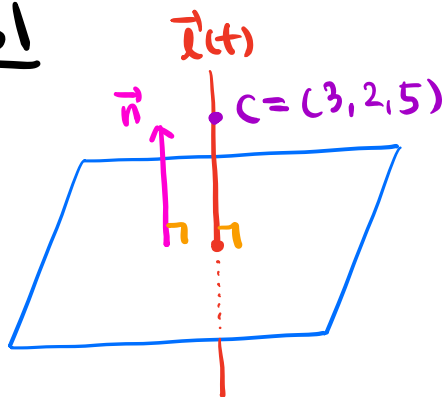
Sol A direction vector is $\vec{AB} = (1, 2, -2)$

$$\leadsto \boxed{\vec{\ell}(t) = (2+t, \ 0+2t, \ 1-2t), \quad t \in \mathbb{R}}$$

Note You can use the point $B = (3, 2, -1)$ to get a different parametrization:

$$\vec{\ell}(t) = (3+t, \ 2+2t, \ -1-2t), \quad t \in \mathbb{R}$$

(2) The line through $C = (3, 2, 5)$ which is perpendicular to the plane $2x - y + 3z = 5$.
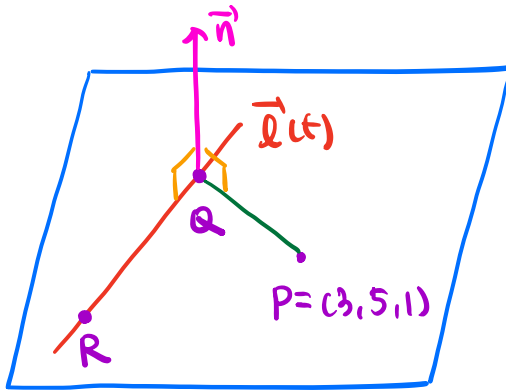
Sol

A direction vector is given by a normal vector

$$\vec{n} = (2, -1, 3)$$

$$\leadsto \boxed{\vec{\ell}(t) = (3+2t, \ 2-t, \ 5+3t), \quad t \in \mathbb{R}}$$

Ex Find an equation of the plane which passes through $P = (3, 5, 1)$ and contains the line $\vec{\ell}(t) = (4-t, 2t-1, -3t)$.

Sol

Choose two points on $\vec{\ell}(t)$:

$Q = \vec{\ell}(0) = (4, -1, 0)$, $R = \vec{\ell}(1) = (3, 1, -3)$.

A normal vector $\vec{n}$ should be perpendicular to both $\vec{QP}$ and $\vec{QR}$

$$\Rightarrow \vec{n} = \vec{QP} \times \vec{QR} = (-1, 6, 1) \times (-1, 2, -3)$$
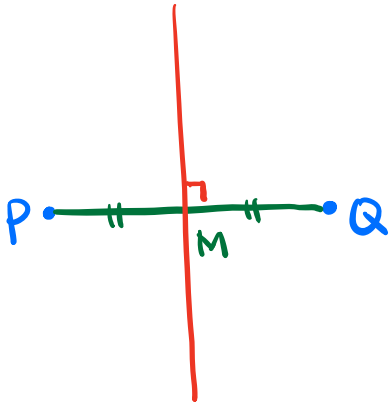
$$= (-20, -4, 4)$$

$$\rightsquigarrow \boxed{-20(x-3) - 4(y-5) + 4(z-1) = 0}$$

Note You can also use the point Q or R to get the same equation.

Ex Find the set of all points which are equidistant from $P = (2, 1, 3)$ and $Q = (4, -3, 1)$.
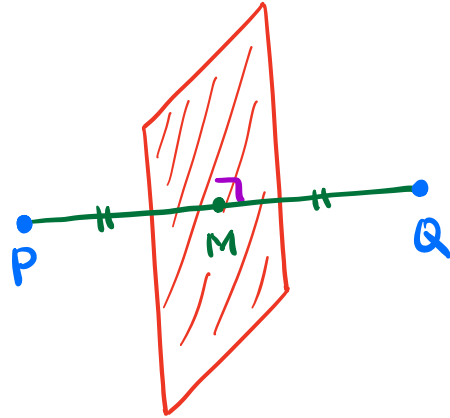
Sol We get hints from a similar problem in $\mathbb{R}^2$.

- In $\mathbb{R}^2$:

- In $\mathbb{R}^3$:

equidistant line

equidistant plane

The midpoint between P and Q is

$$M = \left(\frac{2+4}{2}, \frac{1-3}{2}, \frac{3+1}{2} \right) = (3, -1, 2)$$

A normal vector is given by $\vec{PQ} = (2, -4, -2)$

$$\rightsquigarrow \boxed{2(x-3) - 4(y+1) - 2(z-2) = 0}$$

Rmk You can also use the distance formula.

Distance from P = Distance from Q

$$\Rightarrow \sqrt{(x-2)^2 + (y-1)^2 + (z-3)^2} = \sqrt{(x-4)^2 + (y+3)^2 + (z-1)^2}$$

You can simplify this equation to get the same answer.