PROBLEMS:

seriousPossibilists

Problem 1: (2025 USAJMO)

Let $\mathbb{Z}$ be the set of integers, and let $f\colon \mathbb{Z} \to \mathbb{Z}$ be a function. Prove that there are infinitely many integers c such that the function $g\colon \mathbb{Z} \to \mathbb{Z}$ defined by $g(x) = f(x) + cx$ is not bijective.
Note: A function $g\colon \mathbb{Z} \to \mathbb{Z}$ is bijective if for every integer b, there exists exactly one integer a such that $g(a) = b$.

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Solution:

Define $h(x) = f(x+1) - f(x)$. Set $c = -h(x)$ for some $x \Rightarrow$

$$g(x+1) = f(x) - h(x) \cdot x = f(x) + f(x+1)x - f(x)x = f(x+1)x - (x-1)f = g(x).$$

Hence $g(x)$ is not bijective. Assume that $h(x)$ does not take infinitely many values over $\mathbb{Z}$, otherwise the problem is now trivial. Let m be the minimum of $h(x)$ over all values of x. Set $c = |m| + 2$ and note that

$$g(x+1) = g(x) + h(x) + c \geq g(x) + m + c \geq g(x) + 2 \ \forall\ x$$

and hence $g(x)$ is not bijective. ■

Problem 2: (Mexican National Olympiad 2022)

Let n be a positive integer. In an $n \times n$ garden, a fountain is to be built with 1×1 platforms covering the entire garden. Ana places all the platforms at a different height. Afterwards, Beto places water sources in some of the platforms. The water in each platform can flow to other platforms sharing a side only if they have a lower height. Beto wins if he fills all platforms with water. Find the least number of water sources that Beto needs to win no matter how Ana places the platforms.

Solution:

We claim that the answer is

$$\left\lceil \frac{n^2}{2} \right\rceil.$$

We first prove that $\left\lceil \frac{n^2}{2} \right\rceil$ is necessary. Consider a checkerboard colouring, with $\left\lceil \frac{n^2}{2} \right\rceil$ white squares, the rest of the squares being black. Let each white square have a taller platform than the each of the black squares. Clearly, each of the white squares needs to have a fountain for Beto to win.
We bow prove that $\left\lceil \frac{n^2}{2} \right\rceil$ is sufficent. Each fountain can bve placed in such a way as to create $\left\lfloor \frac{n^2}{2} \right\rfloor$ 2×1 regions, with a 1×1 cell having its own is fountain if n id odd. Each region has a higher and lower platform, hence all regions are filled. ■

Problem 3: (Canada National Olympiad 2025)

Determine all positive integers a, b, c, p, where p and $p + 2$ are odd primes and

$$2^a p^b = (p+2)^c - 1.$$

Solution:

It is well known that $2^k - 1$ is prime $\implies$ k is prime, and $2^k + 1$ is prime $\implies$ k is a power of 2.
Taking the equation under (mod $p+1$) $\implies$ $p + 1 \mid (-1)^b \cdot 2^a \implies p = 2^m - 1 \implies m$ is prime. $p + 2 = 2^m + 1 \implies m$ is a power of 2 or $m = 0$, where the latter is absurd. The only prime power of 2 is 2, which means $p = 3$. Then the equation reduces to

$$2^a 3^b = 5^c - 1 \quad a, b, c \in \mathbb{Z}^+$$

Since $3 \mid 5^c - 1$, c is even. Let $c = 2c'$. Then the equation is

$$2^a 3^b = 25^{c'} - 1$$

By Lifting The Exponent Lemma (using the standard notation of $v_p(n)$ to denote the highest power of p that divides n for prime p), we get $v_3(c') + 1 = v_3(c) + 1 = b = \implies 3^{b-1} \mid c$ and $v_2(c') + 3 = v_2(c) - 1 + 3 = v_2(c) + 2 = a \implies 2^{a-2} \mid c$. Since $3^{b-1}, 2^{a-2}$ are coprime, $2^{a-2}3^{b-1} \mid c \implies 2^{a-2}3^{b-1} \leq c$.
Now

$$5^{2^{a-2}} \geq 2^a + 1 \iff a \geq 2 \text{ and } 5^{3^{b-1}} > 3^b \iff b \geq 1$$

Hence

$$5^c \geq 5^{2^{a-2}+3^{b-1}} > 2^a 3^b + 3^b > 2^a 3^b + 1 \;\forall\; a > 2 \text{ and } b > 1$$

which leaves us with a single case outlined below.
Equality can thus only hold in the original problem if $b = 1 \implies a = 3$, which yields the solution set

$$(a, b, c, p) = \{(3, 1, 2, 3)\}.$$

■

Alternate: Get $p = 3$ as before. Now, by Zsigmondy's theorem, $c \leq 2$ (since 3 is a divisor), which finishes immediately and gives the above solution. ■

Problem 4: (APMO 2017)

We call a 5-tuple of integers arrangeable *if its elements can be labeled a, b, c, d, e in some order so that $a - b + c - d + e = 29$. Determine all 2017-tuples of integers $n_1, n_2, ..., n_{2017}$ such that if we place them in a circle in clockwise order, then any 5-tuple of numbers in consecutive positions on the circle is* arrangeable.

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Solution:

Let $m_1, m_2, \ldots, m_{2017}$ be an such a tuple and let $m_i' = m_i - 29$. Then for consecutive m_i' where $i \in \{1, 2, 3, 4, 5\}$,

$$\sum m_i' = 0 \implies m_1' - m_2' + m_3' - m_4' + m_5' = m_2' - m_3' + m_4' - m_5' + m_6' = 0 \implies m_1 \equiv m_6 \pmod 2.$$

We can argue similarly to show that $m_i' \equiv m_{i+5}' \pmod 2$ for all i. Since $\gcd(5, 2017) = 1$,

$$m_1' \equiv m_2' \equiv \cdots \equiv m_{2017}' \pmod 2 \text{ and since } m_1' - m_2' + m_3' - m_4' + m_5' = 0, m_1' \equiv m_2' \equiv \cdots \equiv m_{2017}' \equiv 0 \pmod 2$$

Let $m_i'' = \frac{m_i'}{2}$. Clearly, $m_i'' \in \mathbb{Z}$ since all m_i' are even. Note that any 5-tuple in consecutive positions is still *arrangeable*, since

$$m_i'' - m_{i+1}'' + m_{i+2}'' - m_{i+4}'' + m_{i+5}'' = \frac{m_1' - m_2' + m_3' - m_4' + m_5'}{2} = 0.$$

Hence, the integers $m_1'' + 29, m_2'' + 29, \ldots, m_{2017}'' + 29$ satisfy the conditions of the problem. But we now fall into infinite descent, since then all of m_i'' are divisible by 2, and so will m_i''', and so on, ad infinitum. Hence, the only possibility is $m_1' = m_2' = \cdots = m_{2017}' = 0$ and the only tuple that satisfies the conditions of the problem is $n_1 = n_2 = \cdots = n_{2017} = 29$. ■