Problems:

seriousPossibilists

Problem 1: (Junior Balkan MO Shortlist 2022)

Let a, b, and c be positive real numbers such that $a+b+c=1$. Prove the following inequality

$$a\left(\frac{b}{a}\right)^{\frac{1}{3}}+b\left(\frac{c}{b}\right)^{\frac{1}{3}}+c\left(\frac{a}{c}\right)^{\frac{1}{3}} \leq ab+bc+ca+\frac{2}{3}.$$

Anastasija Trajanova, Macedonia

Solution 1:

We write $\sum_{\text{cyc}}$ to mean the sum where we cycle through the n variables in the problem.
For example, $\sum_{\text{cyc}} a^2 = a^2+b^2+c^2$ if there are 3 variables in a problem.
Now, rewrite the inequality as

$$\sum_{\text{cyc}} a^{2/3}b^{1/3} - \sum_{\text{cyc}} ab \leq \frac{2}{3}$$

Let $f(a,b,c)$ denote the left hand side $\implies$

$$\frac{\partial^2}{\partial a^2} f = \frac{\partial}{\partial a}\left(\frac{2}{3}a^{\frac{-1}{3}}b^{\frac{1}{3}}+\frac{1}{3}a^{\frac{-2}{3}}c^{\frac{2}{3}}-b-c\right)$$

$$= -\frac{2}{9}\cdot\frac{b^{\frac{1}{3}}}{a^{\frac{4}{3}}}-\frac{2}{9}\cdot\frac{c^{\frac{2}{3}}}{a^{\frac{5}{3}}} \leq 0 \implies f \text{ is concave in each of } a,b,c.$$

Hence, f is maximised when $a=b=c$ (pushing the variables together). Together with $a+b+c=1$, this implies $a=b=c=\frac{1}{3} \implies$ maximum value of $f = f(\frac{1}{3},\frac{1}{3},\frac{1}{3}) = \frac{2}{3}$, as required. ■

Solution 2:

Claim:

$$3ab+a+\frac{1}{3} \geq 3(a^2b)^{\frac{1}{3}}.$$

Proof: By AM-GM. ■
Then we have

$$a\left(\frac{b}{a}\right)^{\frac{1}{3}}+b\left(\frac{c}{b}\right)^{\frac{1}{3}}+c\left(\frac{a}{c}\right)^{\frac{1}{3}} = \sum_{\text{cyc}}(a^2b)^{\frac{1}{3}} \leq \sum_{\text{cyc}}\frac{3ab}{3}+\sum_{\text{cyc}}\frac{a}{3}+\frac{3}{9} = ab+bc+ca+\frac{2}{3}. \quad ■$$

Problem 2: (Junior Balkan MO Shortlist 2023)

Let a,b,c,d be positive real numbers with $abcd=1$. Prove that

$$\sqrt{\frac{a}{b+c+d^2+a^3}}+\sqrt{\frac{b}{c+d+a^2+b^3}}+\sqrt{\frac{c}{d+a+b^2+c^3}}+\sqrt{\frac{d}{a+b+c^2+d^3}} \leq 2$$

Solution:

We write $\sum_{\text{cyc}}$ to mean the sum where we cycle through the n variables in the problem.
For example, $\sum_{\text{cyc}} a^2 = a^2+b^2+c^2$ if there are 3 variables in a problem.
Claim:

$$a^2+b^2+c^2+d^2 \geq a+b+c+d \iff \sum_{\text{cyc}} a^2 \geq (abcd)^{\frac{1}{4}}(a+b+c+d) \text{ (since } abcd=1)$$

Proof: Muirhead's inequality finishes, since $(2,0,0,0)$ majorizes $(\frac{5}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$. ■
Now note that by QM-AM:

$$\sum_{\text{cyc}}\sqrt{\frac{a}{b+c+d^2+a^3}} \leq 4\cdot\sqrt{\frac{\sum_{\text{cyc}}\frac{a}{b+c+d^2+a^3}}{4}} \implies$$

it is sufficient to prove

$$\sum_{\text{cyc}}\frac{a}{b+c+d^2+a^3} \leq 1.$$

By Cauchy-Schwarz:

$$(b+d+c^2+a^3)\left(b+d+1+\frac{1}{a}\right) \geq (a+b+c+d)^2 \implies \sum_{\text{cyc}} \frac{a}{b+c+d^2+a^3} \leq \frac{\sum_{\text{cyc}} a\left(b+d+1+\frac{1}{a}\right)}{(a+b+c+d)^2}$$

$$\iff \sum ab \text{ (the pairwise product of terms)} + a+b+c+d+4 \leq (a+b+c+d)^2$$

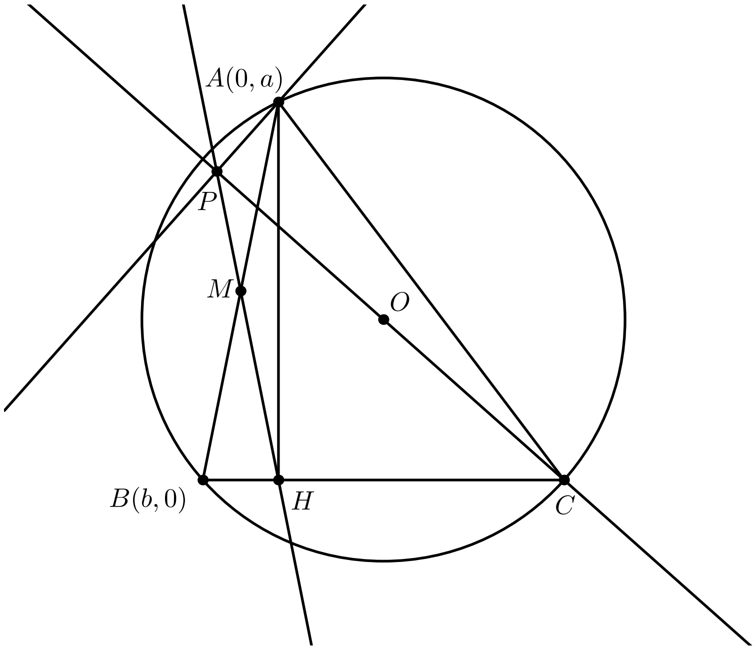
which reduces to proving $a^2+b^2+c^2+d^2 \geq a+b+c+d$ after application of AM-GM and using the fact $abcd=1$ to prove $(a+c)(b+d) \geq 4$. ■

Problem 3: (Tournament Of Towns - Spring 2018 Junior A-Level)

Let O be the center of the circumscribed circle of the triangle ABC. Let AH be the altitude in this triangle, and let P be the base of the perpendicular drawn from point A to the line CO. Prove that the line HP passes through the midpoint of the side AB.

Egor Bakaev

Solution:



We present two solutions:
Cartesian coordinate bash: (not recommended)
Shift the triangle so that B, C lie on the $x-$axis and A lies on the $y-$axis. The line AB has equation $\frac{y}{x-b} = \frac{-a}{b} \implies y = \frac{-a}{b}x + a$. Hence the perpendicular bisector of segment AB has the form $y = \frac{b}{a}x + c$ for an arbitrary constant c. Since the perpendicular bisector of the line passes through $(\frac{b}{2}, \frac{a}{2})$, c turns out to be $\frac{a^2-b^2}{2a}$.
Hence the perpendicular bisector of segment AB is $y = \frac{b}{a}x + \frac{a^2-b^2}{2a}$. Note that $x = \frac{b+c}{2}$ is the perpendicular bisector of BC. The point of intersection of both the perpendicular bisectors, i.e the circumcenter, turns out to be:

$$O = \left(\frac{b+c}{2}, \frac{a^2+bc}{2a}\right).$$

Now, the equation of line $CO =$

$$\frac{y}{x-c} = \frac{a^2+bc}{a(b-c)} \implies y = x\left(\frac{a^2+bc}{a(b-c)}\right) + c\left(\frac{a^2+bc}{a(c-b)}\right)$$

Let this line be ℓ_1. We are ready to solve for the coordinate of P:
Any line perpendicular to line CO has the form:

$$y = x\left(\frac{a(c-b)}{a^2+bc}\right) + p \text{ for an arbitrary constant } p \text{ and since } (0,a) \text{ lies on the line through } P,\ p = a.$$

Let this line be ℓ_2. Solving ℓ_1, ℓ_2 gives us the coordinates of P. Hence:

$$\frac{ax(c-b)}{a^2+bc} - \frac{a^2x - bcx}{a(b-c)} = \frac{a^2c+bc^2}{a(c-b)} - a \implies x = \frac{b(a^2+c^2)(a^2+bc)}{(ab-ac)^2+(a^2+bc)^2}$$

$$\implies \text{(from } \ell_2\text{) } y = \frac{b(a^2+c^2)(ac-ab)}{(ab-ac)^2+(a^2+bc)^2} + a \implies \frac{y}{x} = \frac{ac-ab}{a^2+bc} + \frac{a}{x}$$

Note that $\frac{a}{x} =$

$$\frac{a(ab-ac)^2 + a(a^2+bc)^2}{b(a^2+c^2)(a^2+bc)} = \frac{a^5 + a^3b^2 + a^3c^2 + ab^2c^2}{a^4b + a^2b^2c + a^2bc^2 + b^2c^3} = \frac{a\left(a^2+b^2\right)\left(a^2+c^2\right)}{b\left(a^2+bc\right)\left(a^2+c^2\right)} = \frac{a\left(a^2+b^2\right)}{b\left(a^2+bc\right)}$$

$$\implies \frac{y}{x} = \frac{a\left(a^2+b^2\right)}{b\left(a^2+bc\right)} + \frac{ac-ab}{a^2+bc} = \frac{a\left(a^2+b^2\right) + (ac-ab)\,b}{b\left(a^2+bc\right)} = \frac{a^3+abc}{b\left(a^2+bc\right)} = \frac{a}{b}.$$

Hence if the coordinates of $P = (h,k)$, then $\frac{h}{k} = \frac{b}{a}$. Note that $H = (0,0) \implies$ the equation of line $HP = \frac{y}{x} = \frac{h}{k}$ if $P = (h,k) \implies$ equation of $HP = \frac{y}{x} = \frac{b}{a}$. But then the midpoint of $AB = \left(\frac{b}{2}, \frac{a}{2}\right)$ clearly lies on this line. ■

Synthetic Solution:
Let $HP \cap AB$ at M.
Note that $\angle AOB = 2\angle ABC \implies \angle OCA = 90^\circ - \angle ABC$. But $AH \perp BC \implies \angle BAH = 90^\circ - \angle ABC$. Also, $AP \perp PC$ and $AH \perp BC \implies APHC$ is cyclic. $\implies \angle PCA = \angle PHA = 90^\circ - \angle ABC$. Hence $\angle BAH = \angle PHA \implies AM = MH$. Also, $\angle MHB = 90^\circ - (90^\circ - \angle ABC) = \angle ABC \implies MH = MB = AM$. Hence, M is the midpoint of AB, as required. ■

Problem 4: (Balkan MO Shortlist 2021)

Denote by $f(n)$ the largest prime divisor of n. Let $a_{n+1} = a_n + f(a_n)$ be a recursively defined sequence of integers with $a_1 = 2$. Determine all natural numbers m such that there exists some $i \in \mathbb{N}$ with $a_i = m^2$.

Nikola Velov, North Macedonia

Solution:

The sequence is $2, 4, 6, 9, 12, 15, 20, 25, 30, 35, 42, 49, 56, 63, 70, 77, 88, 99, 110, 121, 132, 143, 156, \dots$

Claim: Let $\{p_1 = 2, p_2 = 3, p_3 = 5, \dots\}$ be the set of primes in ascending order. Define 1 to be the zeroth prime number. Then the sequence is composed of p_i - chains of the form $p_i(p_{i-1}), p_i(p_{i-1}+1) \dots p_i^2, \dots p_i(p_{i+1}-2), p_i(p_{i+1}-1)$
Proof: We proceed by induction. Clearly, the claim holds for $p = 2, 3$. Then assume it holds for the p_i^{th}- chain. For the p_{i+1}^{th}- chain, the first term is $p_{i+1} \cdot p_i$. The following terms are clear by the recursion in the problem since $f(p_{i-1}), f(p_{i-1}+1), \dots f(p_{i+1}-1) < p_{i+1}$. And hence the chain ends at the term $p_{i+1}(p_{i+2}-1)$ (since $f(p_{i+2}) = p_{i+2}$), upon which the next p_{i+1}^{th} chain starts. The next chain starts immediately and is of the same form, again by the recursion. ■

Claim: The only squares that appear in the sequence are squares of prime numbers.
Proof: Let $v_p(x)$ denote the highest power of p that divides x. We work on the p_i^{th} - chain. Note that all prime squares occur, since the p_i^{th} - chain contains $p_i^2\ \forall\ i$. We note that no squares of composite numbers appear by noting that $v_{p_i}(x)\ \forall\ x$ in the chain is 1 unless $x = p_i^2$, since by Bertrand's postulate, $p_i < p_{i+1} < 2p_i$. Hence only prime squares occur. ■