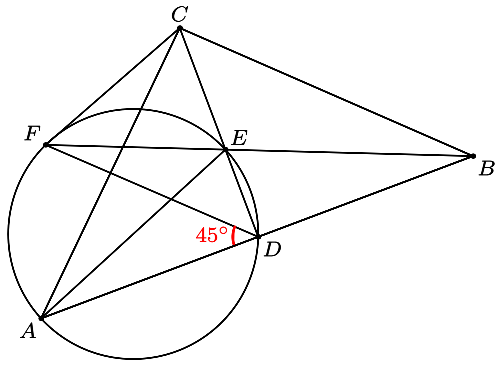
Problems:

Problem 1: (EGMO 2015)

Let $\triangle ABC$ be an acute-angled triangle, and let D be the foot of the altitude from C. The angle bisector of $\angle ABC$ intersects CD at E and meets the circumcircle ω of triangle $\triangle ADE$ again at F. If $\angle ADF = 45^\circ$, show that CF is tangent to ω.

(Luxembourg)

Solution:



Start by using the sine rule in $\triangle CBF$ to get $\frac{\sin \angle CBE}{CF} = \frac{\sin \angle CFE}{BC}$.

Also, $BC = 2\sin\angle FBA \cos\angle FBA = 2\sin\angle CBE \cos\angle CBE$.
Apply the sine rule in $\triangle DCF$ and simplify to get:

$$\frac{CD}{CF} = (\sin\angle CBE + \cos\angle CBE)\cos\angle CFE + (\cos\angle CBE - \sin\angle CBE)\sin\angle CFE$$

and from the first two equations, we can write CD, CF in terms of the RHS. After doing so and simplifying, we get

$$(\sin\angle CBE + \cos\angle CBE)\cos\angle CFE = (\sin\angle CBE + \cos\angle CBE)\sin\angle CFE \implies \angle CFE = 45^\circ.$$

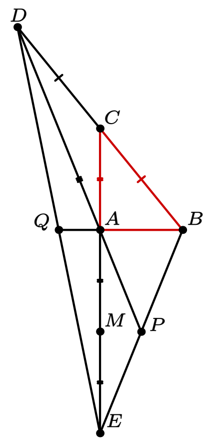
Hence, by AST, we are done. ∎

Problem 2: (EGMO 2013)

The side BC of the triangle ABC is extended beyond C to D so that $CD = BC$. The side CA is extended beyond A to E so that $AE = 2CA$. Prove that, if $AD = BE$, then the triangle ABC is right-angled.

(David Monk, United Kingdom)

Solution:

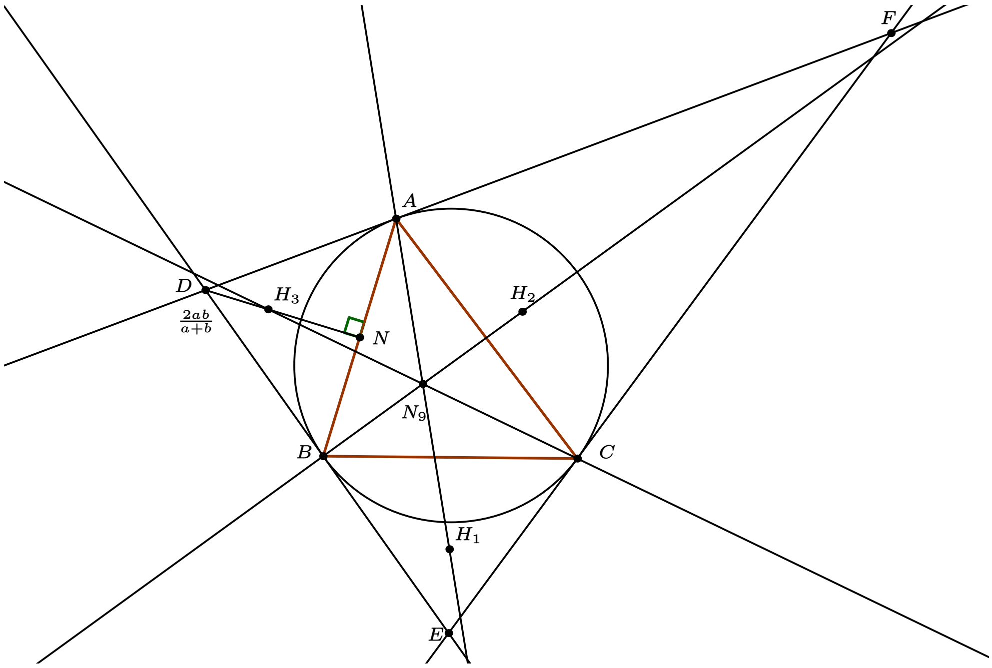


$DA \cap BE = P$ and Menelaus theorem on $\triangle EBC$ with DA as transversal implies $BP = EP$. Also, A divides EC in a $2:1$ ratio $\Rightarrow A$ is the centroid of $\triangle EBD$. Hence, $AP = \frac{AD}{2} = \frac{BE}{2}$ since $AD = BE \Rightarrow AP = BP = PE \Rightarrow \angle BAE = \angle BAC = 90°$. ■

Problem 3:

In $\triangle ABC$, let D be the intersection of tangents to (ABC) at A, B. Similarly, let E be the intersection of tangents to (ABC) at B, C and let F be the intersection of tangents to (ABC) at A, C. Let H_1 be the orthocentre of $\triangle EBC$, let H_2 be the orthocentre of $\triangle FAC$, and let H_3 be the orthocentre of $\triangle DBC$. Prove that AH_1, BH_2 and CH_3 concur and find the point of concurrence.

Solution:



We proceed by complex numbers.

Claim:

The point of concurrence is the nine-point center of $\triangle ABC$ (denoted N_9).

Proof: Note that it is sufficient to prove that H_3, N_9, C are collinear. The other cases follow similarly. Let A, B, C lie on the unit circle and denote them by $a, b, c \in \mathbb{C}$ respectively. Then D is $\frac{2ab}{a+b}$.
Now, let $x \in \mathbb{C}$ be the orthocentre(H_3). Then, by the perpendicularity criterion,

$$\frac{x(a+b) - 2ab}{(b+a)(b-a)} = \frac{ab\overline{x}(a+b) - 2ab}{(b-a)(a+b)} \Rightarrow x = \overline{x} \cdot ab \quad ...(1) \quad \text{since } DH_3 \perp AB.$$

Also,

$$\frac{(a-x)(a+b)}{ab - b^2} + \frac{\frac{1}{a} - \overline{x}}{\frac{2}{a+b} - \frac{1}{b}} = 0 \quad ...(2) \quad \text{since } AH_3 \perp DB.$$

Now (1) in (2) after simplification yields $x = a + b$. Now $H_3 = x = a + b, N_9 = \frac{1}{2}(a + b + c), C = c$ are collinear by collinearity criterion. Hence, AH_1, BH_2, CH_3 are concurrent and pass through N_9. ■

Problem 4: (INMO 2025)

Euclid has a tool called *splitter* which can only do the following two types of operations:

- Given three non-collinear marked points X, Y, Z, it can draw the line which forms the interior angle bisector of $\angle XYZ$.
- It can mark the intersection point of two previously drawn non-parallel lines.

Suppose Euclid is only given three non-collinear marked points A, B, C in the plane. Prove that Euclid can use the *splitter* several times to draw the center of the circle passing through A, B, and C.

(Shankhadeep Ghosh)

Solution:

We use the following well-known **lemma**:

The Incenter-Excenter Lemma

Let $\triangle ABC$ have incenter I and $A-$ excenter I_A. Then II_ABC is cyclic with the midpoint of arc BC (in circumcircle of $\triangle ABC$) as its center.

(The proof is just angle-chasing and is left as a note at the end of the solution).

Lemma 1

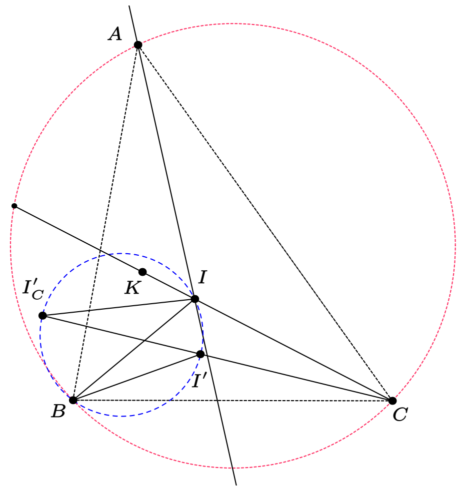
Given 3 marked points X, Y, Z, we can construct the incenter of $\triangle XYZ$.

Proof: Just mark the intersection of angle bisectors. ■

Lemma 2

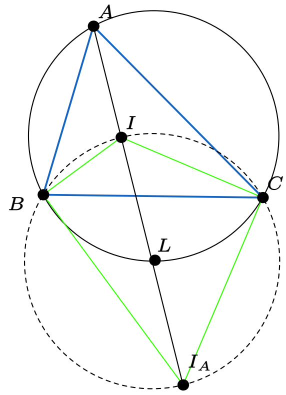
Given concyclic points A, B, C, D, Euclid can mark the center of circle $ABCD$.

Proof: Draw the angle bisector of $\angle BAC$ and $\angle BDC$ and let them intersect at X to get the midpoint of arc BC. Then angle bisector of $\angle BXC$ is the perpendicular bisector of BC. Similarly, draw another perpendicular bisector and intersect them at O. Then O is the center of circle $ABCD$. ■



Now, using **Lemma 1**, mark I, the incenter of $\triangle ABC$ and further mark I', the incenter of $\triangle BIC$. Now, let the angle bisector of $\angle BAI$ and intersect CI at K. Then let CI' intersect the angle bisector of $\angle KIB$ at I'_C. Hence, I'_C is the $C-$excenter of $\triangle BIC$. By the **incenter-excenter lemma**, $I'I'_CIB$ is cyclic and hence the center (the midpoint of arc IB) can be marked, by **Lemma 2**. Let it be O' (not shown in figure).
Now we have concyclic points $O'ICB$ and again we mark the center, which is the midpoint of arc BC. (again, by **incenter-excenter lemma**). Let this point be L. Again, $LBAC$ are concyclic and we can mark the center (by **Lemma 2**) which is the circumcenter of $\triangle ABC$, as required. ■

The Incenter-Excenter Lemma:



Let $A = \angle BAC$, $B = \angle CBA$, $C = \angle ACB$, and note that A, I, L are collinear (as L is on the angle bisector). We are going to show that $LB = LI$, the other cases being similar.
Note that $\angle BIL = \frac{A}{2} + \frac{B}{2}$, by exterior angle on $\triangle AIB$.
But $\angle LBI = \angle LBC + \angle CBI = \frac{A}{2} + \frac{B}{2} \Rightarrow \angle LBI = \angle LIB \Rightarrow LI = LB$. The other cases follow similarly, hence $LI = LB = LC = LI_A$, as required. ■