PROBLEMS:

seriousPossibilists

Problem 1: (Putnam 2024)

Determine all positive integers n for which there exists positive integers a, b, and c satisfying

$$2a^n + 3b^n = 4c^n.$$

Solution:

The answer is only $n = 1$, which clearly has solutions (consider (1,2,2)).
Scale so that $\gcd(a, b, c) = 1$. If $n = 2$, we get

$$2a^2 \equiv c^2 \pmod 3 \implies 3 \mid a, c$$

Write $a' = \frac{a}{3}$ and $c' = \frac{c}{3} \implies 3 \mid b \quad \hookrightarrow\hookleftarrow$.
By taking everything modulo 2,, b is even. Let $b = 2b'$. Plugging in and dividing both sides by 2 gives us

$$a^n + 2^{n-1} \cdot 3b'^n = 2c^n.$$

Again taking modulo 2 gives us a is even. Let $a = 2a'$. Plugging in and dividing both sides by 2 again gives us

$$2^{n-1}a'^n + 2^{n-2} \cdot 3b'^n = c^n.$$

Taking modulo 2 once again gives c is even. $\quad \hookrightarrow\hookleftarrow$ ■

Problem 2: (INMO 2025)

Euclid has a tool called splitter which can only do the following two types of operations:
• *Given three non-collinear marked points X, Y, Z it can draw the line which forms the interior angle bisector of $\angle XYZ$.*
• *It can mark the intersection point of two previously drawn non-parallel lines.*
Suppose Euclid is only given three non-collinear marked points A, B, C in the plane.
Prove that Euclid can use the splitter several times to draw the centre of circle passing through A, B and C.

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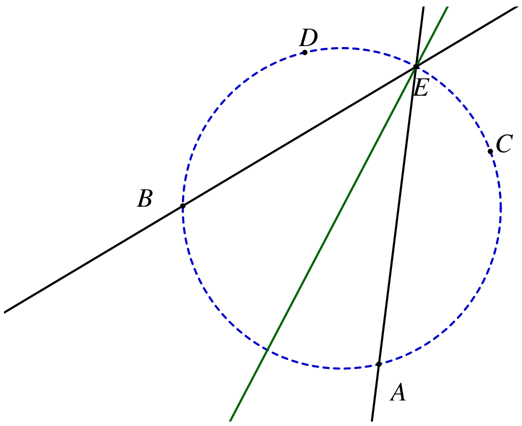
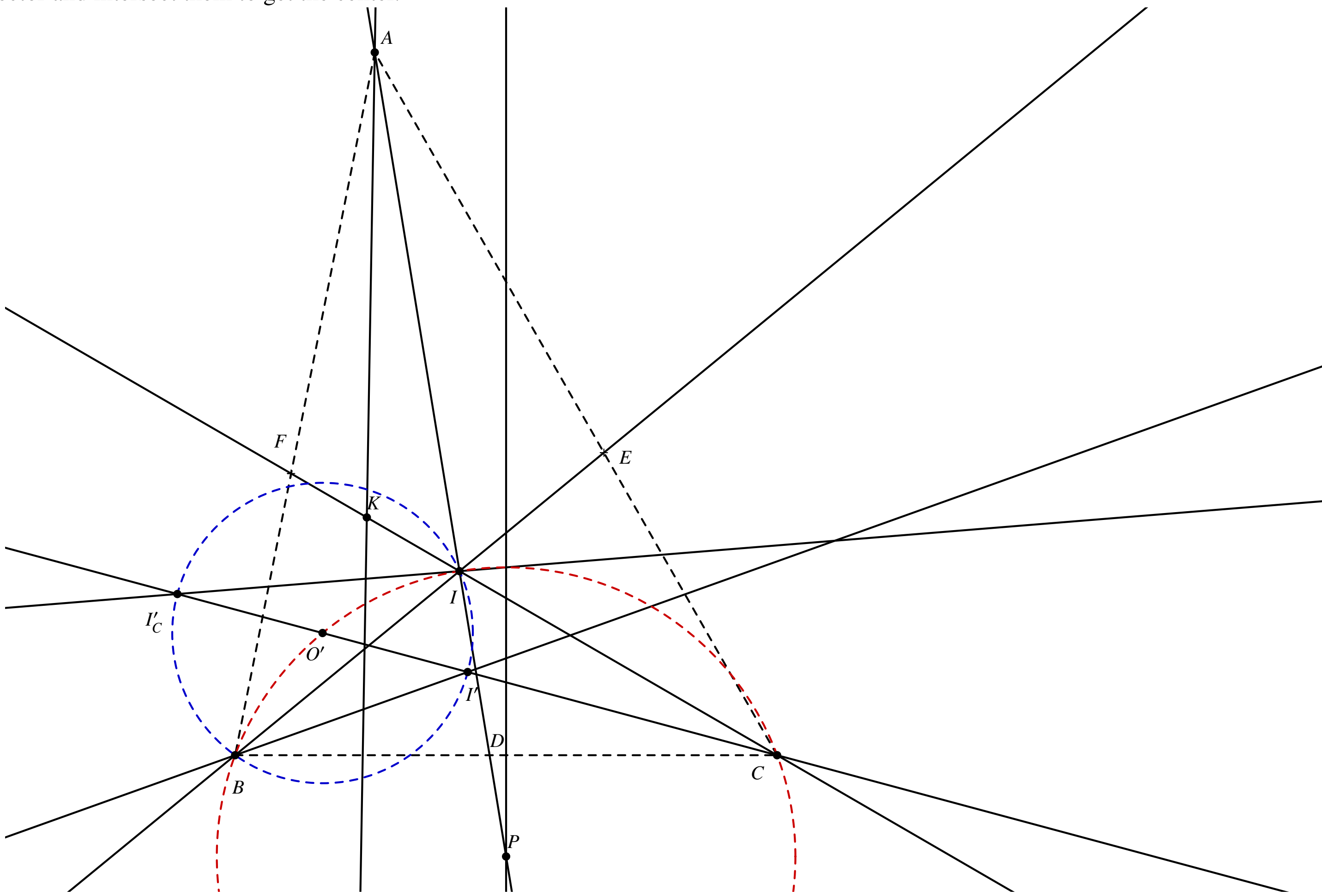
Solution:

We will use the incenter-excenter lemma: which states that if ABC be a triangle with incenter I, $A-$excenter I_A, and L the midpoint of arc BC, IBI_AC is a cyclic quadrilateral with center L.
The proof is left as an exercise. We also claim that using the *splitter*, Euclid can mark the center of a circle passing through 4 concyclic points.



Proof: Consideer concyclic points $ABCD$. Draw the angle bisectors of $\angle DBC$ and $\angle BAC$ and intersect them at E. Then then angle bisector of $\angle BEC$ (highlighted in green) is the perpendicular bisector of BC. Similarly, draw another perpendicular bisector and intersect them to get the center.



We are ready to construct the center of circle ABC.

- First draw angle bisectors of angles $\angle BAC$, $\angle CBA$, $\angle BCA$ and let them intersect the (imaginary) segments BC, AC, AB at D, E, F respectively. Mark I, the incenter.
- Now mark the incenter of $\triangle BIC$, by drawing angle bisector of $\angle IBC$, $\angle ICB$. Label it I'.
- Draw the angle bisector of $\angle BAI$. Let it intersect CI at K.
- Draw the angle bisector of $\angle BIK$ to get the $C-$excenter of $\triangle BIC$. Label it I'_C.
- By the incenter excenter lemma, $II'BI'_C$ is cyclic. Mark its center O'. O' lies on the circumcircle of $\triangle BIC$, by the incenter-excenter lemma. Mark the center of $O'ICB$. Let it be P.
- We draw the angle bisector of BPC to get the perpendicular bisector of BC. Simialrly, construct another perpendicular bisector. Intersect them to get the center of circle ABC.

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Problem 3: (Canada National Olympiad 2024)

Initially, three non-collinear points, A, B, and C, are marked on the plane. You have a pencil and a double-edged ruler having width 1. *Using them, you may perform the following operations:*

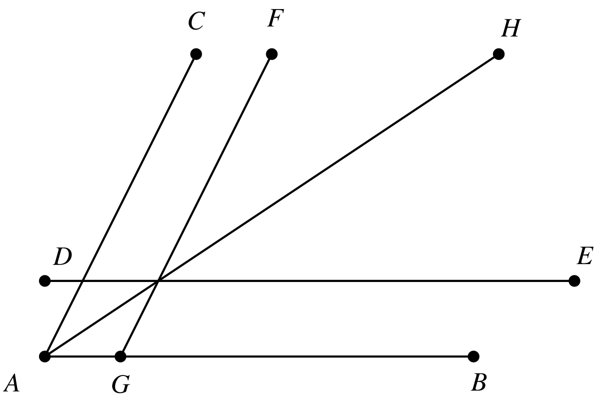
- *Mark an arbitrary point in the plane.*
- *Mark an arbitrary point on an already drawn line.*
- *If two points P_1 and P_2 are marked, draw the line connecting P_1 and P_2.*
- *If two non-parallel lines l_1 and l_2 are drawn, mark the intersection of l_1 and l_2.*

- *If a line l is drawn, draw a line parallel to l that is at distance 1 away from l (note that two such lines may be drawn).*

Prove that it is possible to mark the orthocenter of ABC using these operations.

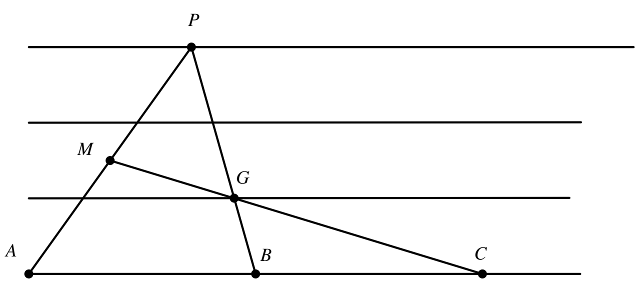
Solution:

Claim 1: We can construct the internal angle bisector of $\angle BAC$ for marked points ABC.



Proof: Given $\overrightarrow{AB}, \overrightarrow{AC}$, draw $\overrightarrow{FG}, \overrightarrow{ED}$ at a distance of 1. Thic creates a rhombus, and the diagonal of the rhombus is clearly the internal angle bisector.

Claim 2: Given points A, B on a drawn line, we can mark C such that $AB = BC$ using the pencil and straightedge. (i, e, we can reflect points over points.)
Proof:



Repeatedly draw lines at a distance 1 as shown and take an arbitrary point P. G divides PB in the ratio $2:1$. Mark M, the midpoint of AP. Then G act as a centroid and hence $\overrightarrow{MG} \cap \overrightarrow{AB}$ is the required point C.

Since we can also mark the intersection of drawn lines, by Claim 1, we can do everything the *splitter* can in INMO 2025/3. (Problem and solution above). Hence we can mark the circumcenter and the centroid (we can draw perpendicular bisectors of sides using the *splitter* and hence mark midpoints). Now we invoke the Euler line. Let H, G, O be the orthocentre, centroid, and circumcenter respectively. Note that G, O are already marked, and that $GH = 2GO$. Since G lies between H, O, we can apply Claim 2 twice to get H. $\blacksquare$