PENDULUM SYSTEM ANALYSIS

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Contents

Introduction

Objectives

In this report, I will analyze the pendulum system. I will analyze the following cases:

A. Simple pendulum: a mass on a string that is fixed to a point. The only forces exerted on the mass are the tension on the string and gravity.
B. Simple pendulum with air resistance: the case above except air resistance is taken into account. I will assume that the mass is a sphere for this report. However, drag coefficients can be altered to simulate different objects.
C. Rod pendulum: a rod with mass that is fixed at one end.
D. Rod pendulum with air resistance: the case above except air resistance is taken into account.

Notation

Here, I will describe the notation system I will use throughout this report. If a capital letter is bolded, it represents a dimension (e.g., **M** represents mass). Double-struck letters will be used for substitutions.

List of Variables

Variable	Description	Variable	Description
θ	Angle between string/rod and vertical line passing through the anchor point	τ or T	Period of system
m	Mass of sphere/rod	v	Linear Velocity
g	Acceleration of gravity	ρ	Fluid density
ℓ	Length of string/rod	C_D	Drag coefficient
a	Linear acceleration	Ā	Cross-sectional area
$\omega = \frac{d\theta}{dt} = \dot{\theta}$	Rotational velocity	r	Roots
$\alpha = \frac{d^2\theta}{d\theta^2} = \ddot{\theta}$	Rotational acceleration	i	Square root of -1
t	Time	θ_0	Initial angle
Sol	Solutions in the context of differential equations	$\mathcal{P}$	Arbitrary periodic function
k	Number of parameters	dims	Number of dimensions
κ	Dimensionless group	$\hat{t}$	Time divided by time scale (period)
$\vec{\tau}$	Torque	I	Moment of inertia
CM	Center of Mass	Fg or F_g	Force of gravity
λ	$\sqrt{\frac{4g}{\ell}}$	F_D or F_d	Drag force

Table 1: Variables and their definitions.

Scenario A: Simple Pendulum

Diagram

The following is a sketch of the system.

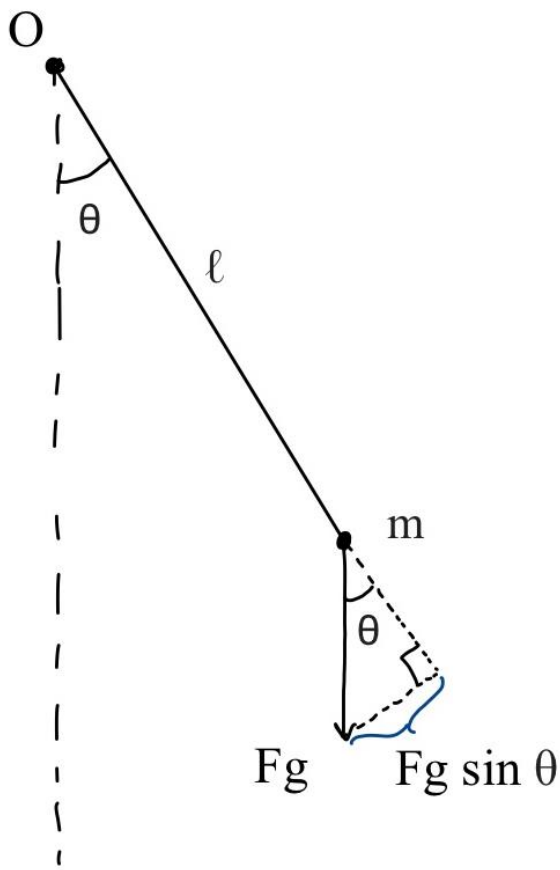


Figure 1: Sketch of simple pendulum. ℓ is the string length. m is the mass of the pendulum. g is the acceleration of gravity. θ is the angle between the pendulum and a line going through O in the upwards direction. Fg is the force of gravity.

Oscillation Period: Dimensional Analysis

Here, I will use dimensional analysis to determine the time period of the system. First, I will start by determining the number of dimensionless groups. Then, I will solve for the time period.

I will now determine the number of dimensionless groups. The four variables used to model the system are mass, gravity, length of the string, and time. Buckingham-π theorem states that the number of dimensionless groups is equal to:

$$k - dims$$

Where k is the number of variables, and dims is the number of dimensions [1]. Mass has one dimension: mass.

$$\text{Mass} \doteq \mathbf{M}$$

Acceleration of gravity has the following dimensions [10]. Trigonometric functions are dimensionless [4]:

$$g \sin\theta \doteq \frac{\boldsymbol{L}}{\boldsymbol{T}^2} = \mathbf{L}\mathbf{T}^{-2}$$

The length of the string has one dimension: length.

$$\ell \doteq \mathbf{L}$$

Time has one dimension: time.

$$t \doteq \mathbf{T}$$

We have four variables and three dimensions. Thus, there is one dimensionless group. I will call this group κ.

Now, I will find the period using dimensional analysis. The period of a system is the time required to complete one cycle [14]. Thus, the period is in units of time. To find which parameters need to be combined, I will multiply the mass, gravitational acceleration, and length of the string's dimensions to an unknown power.

$$Period = T \doteq [\mathbf{T}] \doteq \kappa[Mass]^a[Gravity\ Acceleration]^b[String\ Length]^c \tag{1}$$

$$[\mathbf{T}] \doteq \kappa[\mathbf{M}^{\mathbf{1}}]^a[\mathbf{L}^1\mathbf{T}^{-2}]^b[\mathbf{L}^1]^c \tag{2}$$

Solving for the exponents results in:

$$T \doteq \kappa[\mathbf{M}]^0[\mathbf{L}^1\mathbf{T}^{-2}]^{-\frac{1}{2}}[\mathbf{L}^1]^{\frac{1}{2}} \doteq \kappa g^{-\frac{1}{2}}\ell^{\frac{1}{2}} = \kappa\sqrt{\frac{\ell}{g}} \tag{3}$$

Dimensional analysis alone cannot find the dimensionless group κ specifically. (A more detailed derivation can be found in the appendix under Derivation 1.)

Ordinary Differential Equation and Linearization

Here, I will create an ordinary differential equation which will relate θ to time. Then, I will linearize the equation.

We know all of the forces being exerted on the mass. Thus, we can equate it to Newton's second law [5]:

$$F = ma = m\alpha\ell = mg\sin\theta \tag{4}$$

Which can be re-written into:

$$\frac{F}{m} = a = \alpha\ell = \frac{mg\sin\theta}{m}$$

We define the angular acceleration as the second derivative of the angle with respect to time. We can build a differential equation after dividing out m and ℓ, then rearranging:

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta = 0$$

To linearize the function, I will replace the sine with theta. This is because for small thetas, the sine of theta is very close to theta. Thus:

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\theta = 0 \tag{5}$$

We have arrived at our linearized ordinary differential equation. A detailed derivation can be found in the appendix (derivation 2).

Oscillation Period: Analytical Solution of Ordinary Differential Equation

Here, I will find the analytical solution for the ordinary differential equation. The analytical solution will be used to determine the period.

First, I will rearrange the equation:

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\theta = 0$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\theta$$

Its characteristic equation is [14]:

$$r^2 = -\frac{g}{\ell}$$

By taking the square root of both sides, we can find its roots:

$$r_1 = i\sqrt{\frac{g}{\ell}} \qquad\qquad r_2 = -i\sqrt{\frac{g}{\ell}}$$

Since these roots have imaginary components, the solutions derived from each root is [14]:

$$\theta(t) = A\cos\left(\sqrt{\frac{g}{\ell}}t\right) + B\sin\left(\sqrt{\frac{g}{\ell}}t\right) \tag{6}$$

We are given the following initial conditions:

$$\theta = \theta_0$$

$$\dot{\theta} = 0$$

We can take the derivative of (6) to get:

$$\dot{\theta}(t) = -\sqrt{\frac{g}{\ell}}A\sin\left(\sqrt{\frac{g}{\ell}}t\right) + \sqrt{\frac{g}{\ell}}B\cos\left(\sqrt{\frac{g}{\ell}}t\right) \tag{7}$$

If we plug in our initial velocity, we find that B is 0:

$$\dot{\theta}(0) = -\sqrt{\frac{g}{\ell}}A\sin\left(\sqrt{\frac{g}{\ell}}0\right) + \sqrt{\frac{g}{\ell}}B\cos\left(\sqrt{\frac{g}{\ell}}0\right) = \sqrt{\frac{g}{\ell}}B = 0$$

Now that we have B, we can find A by using the initial position:

$$\theta(t) = A\cos\left(\sqrt{\frac{g}{\ell}}t\right) = \theta_0$$

$$\theta(0) = A\cos\left(\sqrt{\frac{g}{\ell}}0\right) = A = \theta_0$$

We find that A is the initial position.

$$\theta(t) = \theta_0\cos\left(\sqrt{\frac{g}{\ell}}t\right)$$

Our final motion equation is periodic because the cosine is periodic. Since we know that for any periodic function [7]:

$$\mathcal{P}(t+T)=\mathcal{P}(t)$$

Therefore:

$$\theta(t+T)=\theta_0\cos\left(\sqrt{\frac{g}{\ell}}(t+T)\right)=\theta_0\cos\left(\sqrt{\frac{g}{\ell}}(t)\right)=\theta(t)$$

We can solve for T, which is our period:

$$T=2\pi\sqrt{\frac{\ell}{g}} \tag{8}$$

Unlike the period we found through dimensional analysis, we can find the dimensionless group. The dimensionless group is 2π, which has no dimensions. A more detailed derivation can be found in the appendix (derivation 3).

Numerical Solution via Euler's Method

Timestep

Before I begin, I will find an appropriate timestep for Euler's method. A timestep that is too big will result in an inaccurate plot [6]. The following is a plot showing the angular position of the pendulum system. The mass starts at an initial angle of $\pi/6$ radians with no angular velocity. The string length is 2 meters. I will use the period of the system as the timescale. I divided time by the timescale so that time is dimensionless in the plot. The system will be simulated for three periods.

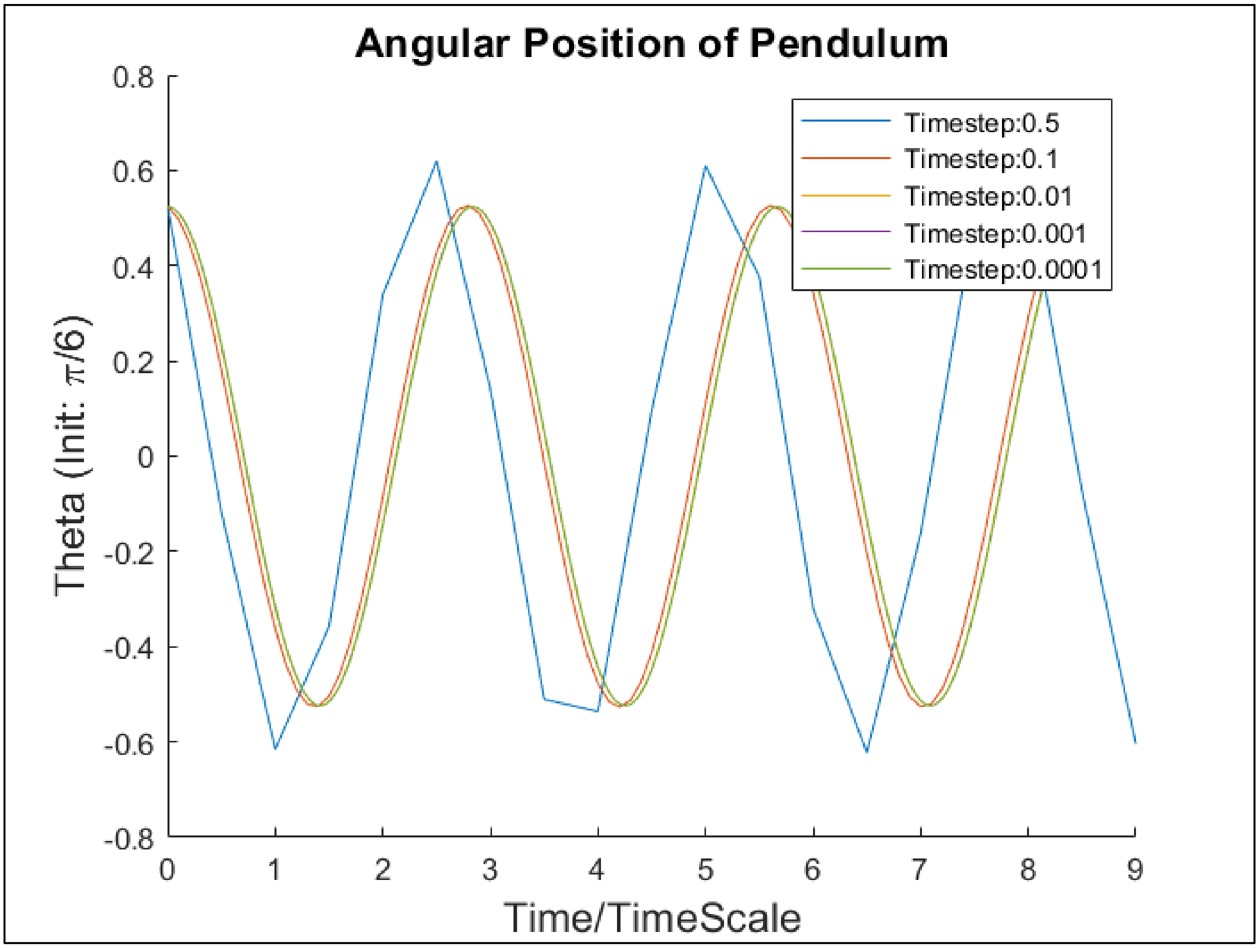


Figure 2: A graph that shows the performance of Euler's method with different timesteps. Smaller timesteps result in smoother functions.

From the graph, we can see that a large timestep results in jagged results. Pendulums smoothly oscillate, so a smaller timestep will work better. For the rest of this report, I will use a timestep of .0001 for maximum accuracy. The small timestep does not significantly impact the time needed to return the results.

Numerical Solution

The following is a graph showing the angle with respect to time for various initial angles. The string length was set to 2, and gravity was set to 9.81.

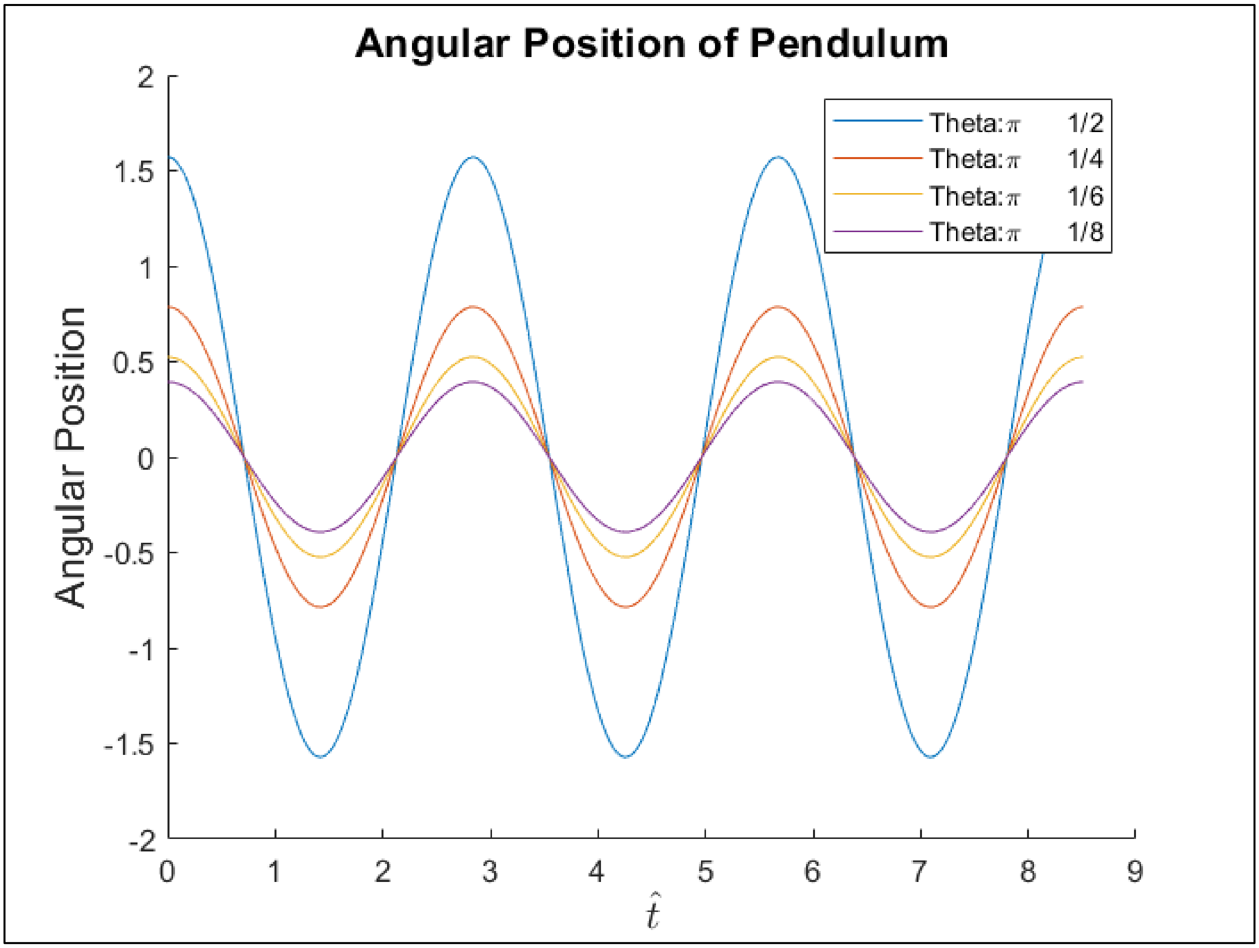


Figure 3: A graph showing the motion of a simple pendulum at various initial angles. We see that the period does not change based on the initial angle.

We see that the amplitude of the function increases as the initial angle increases. From the data, we can also confirm that the period found through our ODE is correct. Using MATLAB commands, I can find the indices where there are local maxima. By definition, the period is the length of time required for the pendulum to complete one cycle [14]. The period can be determined by finding the time elapsed between two local maxima (or minima) next to each other. For a string length of two meters, we get:

```
numerically_derived_period =

    2.8370
```

Figure 4: The period found by finding the difference in time between two local maxima next to each other.

Which is what we get if we plug in values into the period formula we derived:

$$2\pi\sqrt{\frac{\ell}{g}} = 2\pi\sqrt{\frac{2}{9.81}} \approx 2.8370$$

Scenario B: Simple Pendulum with Air Resistance

Diagram

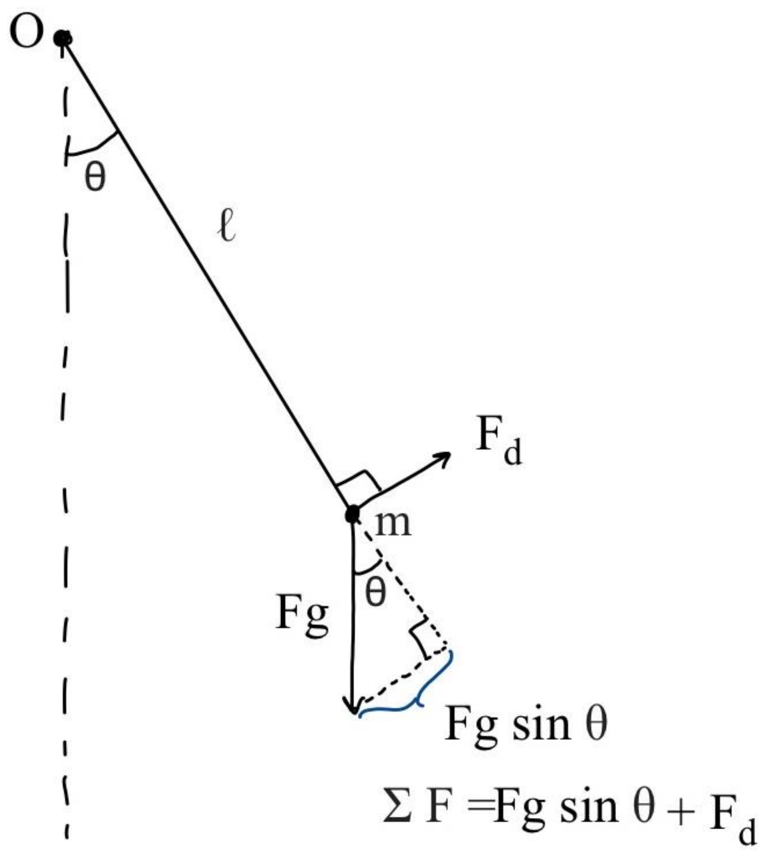


Figure 5: Sketch of pendulum system with air resistance. O is the anchor point. θ is the angle between the string and a line perpendicular to the direction of gravity. Fg is the force of gravity. F_d is the drag force. m is the mass of the sphere. ℓ is the length of the string. ΣF is the total force being exerted on the sphere. The sphere itself has a radius of a.

Oscillation Period: Dimensional Analysis

The computation is like the dimensional analysis for case A. The only difference is the total force, which will alter the total acceleration. Here, the total force is the drag force plus the gravitational force:

$$\Sigma F = -F_g - F_D \tag{9}$$

Since:

$$F_g = mg\sin\theta \ , F_D = \frac{1}{2}\rho v^2 C_D \bar{A} \tag{10}$$

We can plug (10) into (9) to get:

$$\Sigma\mathrm{F} = -mg\sin\theta - \frac{1}{2}\rho v^2 C_D \bar{A}$$

Thus, the total acceleration is:

$$\mathrm{a} = \frac{\Sigma\mathrm{F}}{m} = \frac{mg\sin\theta - \frac{1}{2}\rho v^2 C_D \bar{A}}{m}$$

Using the same steps used in scenario A for dimensional analysis:

$$Period = T \doteq [\mathbf{T}] \doteq \kappa[Mass]^a[Total\ Acceleration]^b[String\ Length]^c$$

$$T \doteq \kappa[\mathbf{M}]^0[\mathbf{L}^1\mathbf{T}^{-2}]^{-\frac{1}{2}}[\mathbf{L}^1]^{\frac{1}{2}} \doteq \kappa\left.\frac{mg\sin\theta - \frac{1}{2}\rho v^2 C_D \bar{A}}{m}\right.^{-\frac{1}{2}} \ell^{\frac{1}{2}} = \kappa\sqrt{\frac{\ell}{\frac{mg\sin\theta - \frac{1}{2}\rho v^2 C_D \bar{A}}{m}}}$$

$$= \kappa\sqrt{\frac{\ell m}{mg\sin\theta - \frac{1}{2}\rho v^2 C_D \bar{A}}} \tag{11}$$

Therefore, the period for this system is:

$$T = \kappa\sqrt{\frac{\ell m}{mg\sin\theta - \frac{1}{2}\rho v^2 C_D \bar{A}}}$$

Ordinary Differential Equation and Linearization

Again, I will start with Newton's second law [5]. I will equate it to the sum of the drag force and gravitational force.

$$\mathrm{F} = \mathrm{ma} = -\mathrm{F_D} - \mathrm{F_g}$$

The drag and gravitational force are negative because it works in the opposite direction of motion. This can be rearranged into:

$$\frac{F}{m} = \frac{ma}{m} = \ell\alpha = \ell\frac{d^2\theta}{dt^2} = \ell\frac{-\mathrm{F_D} - \mathrm{F_g}}{m} \tag{12}$$

We know that:

$$F_D = \frac{\rho C_D \bar{A}}{2} v^2 \text{ }, F_g = mg \sin\theta$$

For simplification, the velocity will not be squared because for small velocities, the velocity squared is significantly less than the velocity itself. As done in the previous scenarios, the sine will be replaced with theta because for small thetas, the sine of theta is close to theta. Thus:

$$F_D = \frac{\rho C_D \bar{A}}{2m} v = \frac{\rho C_D \bar{A}}{2m} \ell\omega = \frac{\rho C_D \bar{A}}{2m} \ell \frac{d\theta}{dt} \text{ }, F_g = mg\theta \tag{13}$$

We can plug in (13) into (12). We get:

$$\frac{d^2\theta}{dt^2} + \frac{\rho C_D \bar{A}}{2m}\frac{d\theta}{dt} + \frac{g}{\ell}\theta = 0 \tag{14}$$

We can simplify (14) by collecting all constant variables for the drag force into one variable. We get:

$$\frac{d^2\theta}{dt^2} = \frac{-g\theta}{\ell} - \mathbb{D}v \tag{15}$$

Where:

$$\mathbb{D} = \frac{\rho C_D \bar{A}}{2m}$$

(See derivation 5 for details.)

Oscillation Period: Analytical Solution of Ordinary Differential Equation

Here, I will determine the period of the system based on the ODE representing the pendulum's motion. To reiterate, the ODE is:

$$\frac{d^2\theta}{dt^2} = \frac{-g\theta}{\ell} - \mathbb{D}v$$

If we take the characteristic function, we see that the roots have a complex components [14]:

$$r_{1,2} = \frac{-\mathbb{D} \pm i\sqrt{\frac{4g}{\ell} - \mathbb{D}^2}}{2} \tag{16}$$

Since we have complex roots, I can build the following general solution [14]:

$$\theta(t) = Ae^{\frac{-\mathbb{D}t}{2}}\left(\cos\left(\frac{t\sqrt{\frac{4g}{\ell} - \mathbb{D}^2}}{2}\right)\right) + Be^{\frac{-\mathbb{D}t}{2}}\left(\sin\left(\frac{t\sqrt{\frac{4g}{\ell} - \mathbb{D}^2}}{2}\right)\right) \tag{17}$$

Which will be simplified as:

$$\theta(t) = Ae^{\mathbb{F}t}(\cos(\mathbb{E}t)) + Be^{\mathbb{F}t}(\sin(\mathbb{E}t)) \tag{18}$$

We can modify (18) so we only have one trigonometric function [14]:

$$\theta(t) = Ce^{\mathbb{F}t}(\cos(\mathbb{E}t + \psi)) \tag{19}$$

Where ψ is some angle, and C is a constant. The frequency of a trigonometric function is altered by mutiplying the input variable with a coefficient [2]. Thus, we know that $\mathbb{E}$ is the frequency of the system. I will solve for $\mathbb{E}$ to get the frequency of the system. We know that 2π divided by frequency is the period [14]. I will use this relation to derive the period. If we plug in the first and second derivatives of into the original ODE, we find that $\mathbb{E}$ is (see derivation 6 for details):

$$\mathbb{E} = \frac{\sqrt{-\mathbb{D}^2 + \frac{4g}{\ell}}}{2} \tag{20}$$

Since the period is 2π divided by the angular frequency, the period of the system is:

$$T = 2\pi\sqrt{\frac{4\ell}{4g - \mathbb{D}^2\ell}} \tag{21}$$

We can simplify (21) to get:

$$T = 2\pi\sqrt{\frac{4\ell}{4g - \mathbb{D}^2\ell}} = 4\pi\sqrt{\frac{\ell}{4g - \mathbb{D}^2\ell}} \tag{22}$$

(22) shows that the dimensionless group is no longer 2π. The dimensionless group now is 4π. (See derivation 6 for details.)

Numerical Solution via Euler's Method

Instead of plotting different values for each parameter in the drag force equation, I will instead alter $\mathbb{D}$. By doing so, I can show how different values for each parameter collectively alter the behavior of the system.

From (20), we can see that the behavior changes when:

$$\mathbb{D} = \sqrt{\frac{4g}{\ell}} \tag{23}$$

In my code, I wrote $\mathbb{D}$ as a collection of parameters from the drag force equation. When $\mathbb{D}$ exceeds the value in (23), the value of the roots found in the characteristic function will change. I will call this value λ. To simulate different values of $\mathbb{D}$, I will multiply λ by a coefficient and equate it to $\mathbb{D}$. In the following sections, I will examine the behavior of the system when $\mathbb{D}$ is above, below, and at λ.

Above λ

Below is the behavior of the system when $\mathbb{D}$ is $1.5 \times \lambda$:

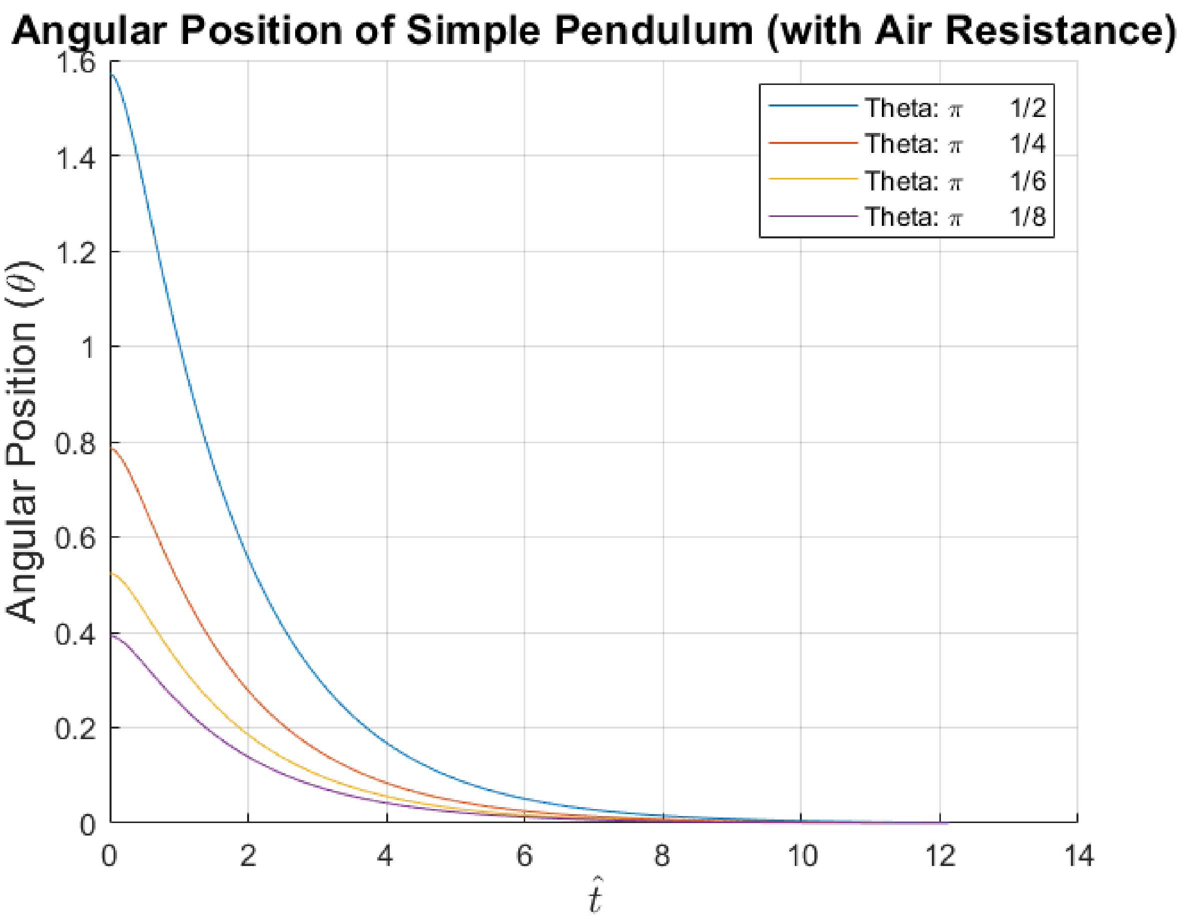


Figure 6: Behavior of a pendulum system with air resistance. Here, the x-axis is the dimensionless time. $\mathbb{D}$ is larger than λ.

Here, the pendulum gradually falls back to the lowest point possible. There is no periodic behavior.

Below λ

Below is the behavior of the system when $\mathbb{D}$ is $0.5 \times \lambda$:

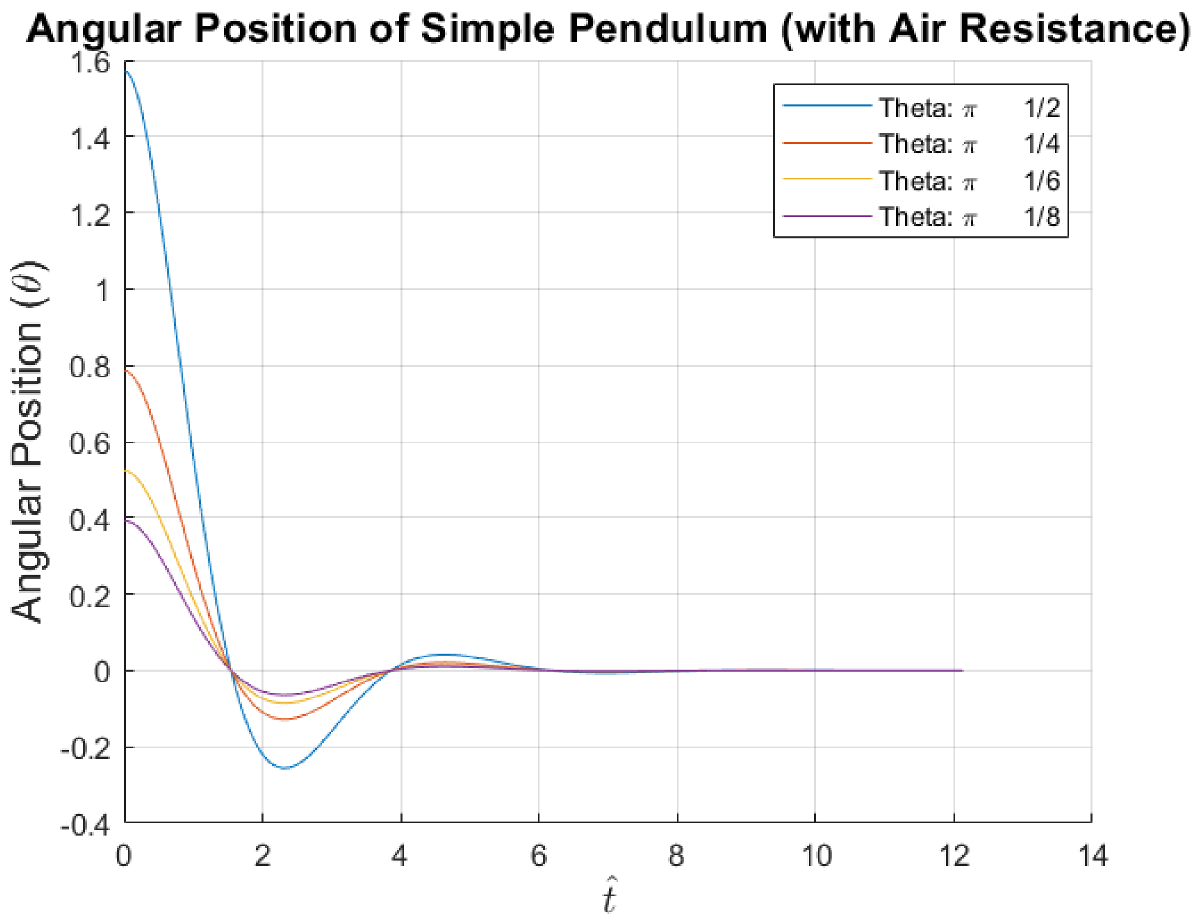


Figure 7: Behavior of a pendulum system with air resistance, where the x-axis is dimensionless time. Here, $\mathbb{D}$ is smaller than λ. The system shows periodic behavior.

Here, the pendulum swings back and forth until settling down at the lowest point possible. Therefore, the system exhibits periodic behavior. Since there exists local minima, I can numerically check the period of the system.

```
numerically_derived_period

    4.6324
```

Figure 8: This is the period calculated by taking two minima close to each other calculating the distance between them.

This is close to the period we found through the analytical solution of the ordinary differential equation. For the graph above, $\mathbb{D}$ is around 1.5660, ℓ is 4, and g is 9.81:

$$T = 4\pi\sqrt{\frac{\ell}{4g - \mathbb{D}^2\ell}} = 4\pi\sqrt{\frac{4}{(4 \times 9.81) - (4 \times 1.566^2)}} \approx 4.6328$$

At λ

Below is the behavior of the system when $\mathbb{D}$ is exactly λ:

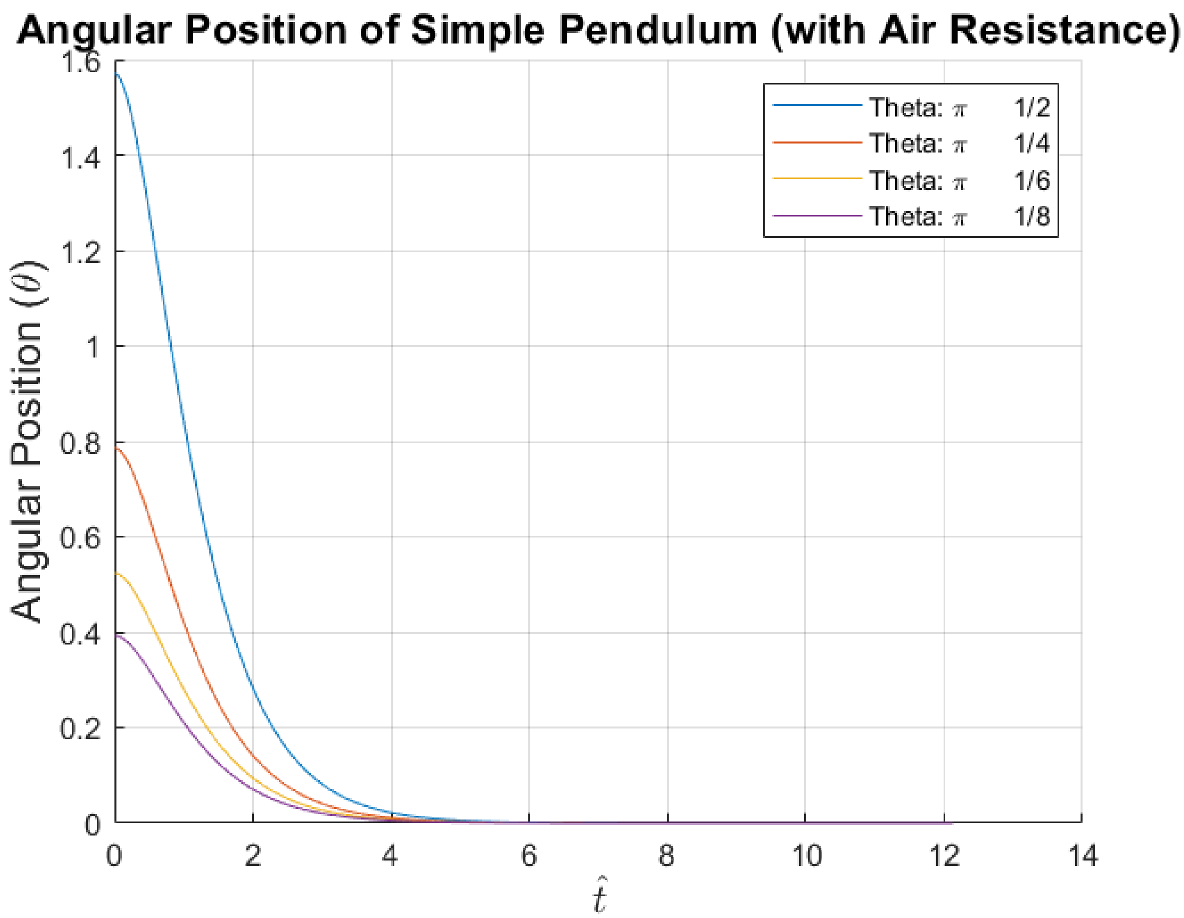


Figure 9:Behavior of a pendulum system with air resistance, where the x-axis is dimensionless time. Here, $\mathbb{D}$ is exactly λ.

Here, the pendulum gradually goes down to the lowest point possible. There is no periodic behavior here.

Effect of Drag on the System

Below are two graphs: the graph on the left is a system that does not experience air resistance. The graph on the right is a system that experiences air resistance.

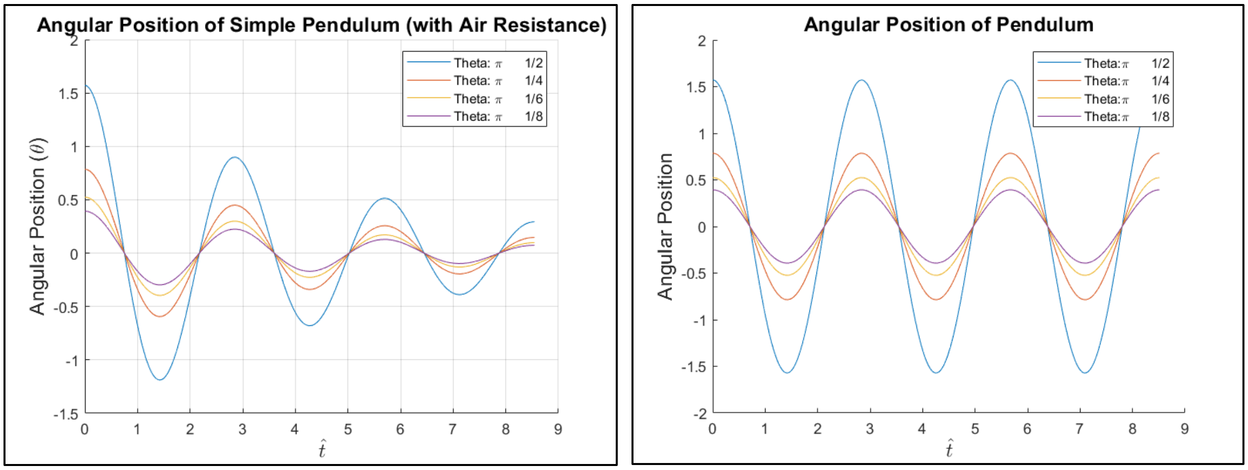


Figure 10: Two graphs showing pendulum behavior. The left experiences air resistance. The right experiences no air resistance. From the graph, we see that air resistance gradually decreases the maximum height the pendulum can reach.

From the graphs, the air resistance gradually decreases the amplitude of the function at an exponential rate. Therefore, the maximum angle the pendulum decreases as time increases. The average length of the period also decreases:

```
avg_period =

    4.0439
```

Figure 11: The average period in the system with air resistance. It is calculated by taking the average distance between local minima next to each other. Here, $\mathbb{D}$ is 0.3927.

```
avg_period =

    4.0121
```

Figure 12: The average period in the system without air resistance. It is calculated by taking the average distance between local minima next to each other.

Scenario C: Rod Pendulum

Diagram

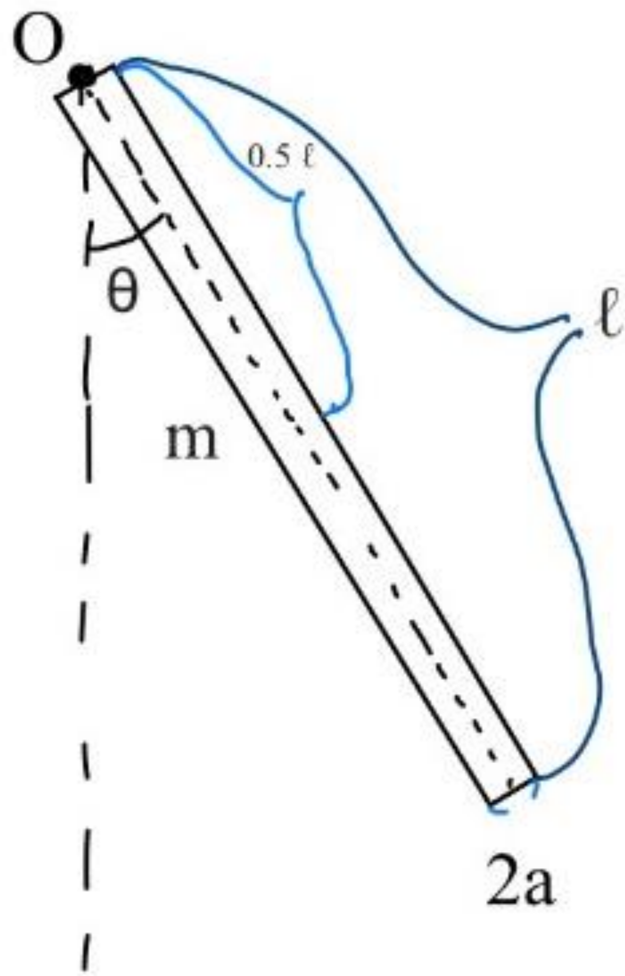


Figure 13: Sketch of the rod pendulum. O is the anchor point, m is the mass of the rod, 2a is the diameter of the rod, θ is the angle between the rod and a line perpendicular to the ground, and ℓ is the length of the rod.

To analyze the system, we can simplify the model. We can change the rod system into a spherical pendulum system. Based on the scenario, the only force acting upon the system is the force of gravity. The force of gravity is exerted on the center of gravity [3]. The center of gravity of an object is also the center of mass for the object [3]. The rod has a uniform mass. Therefore, the center of mass for the rod is at half of its length. The force of gravity is exerted at this point. Therefore, we can model the rod as a point with the same mass, but the string length is half of the length of the rod.

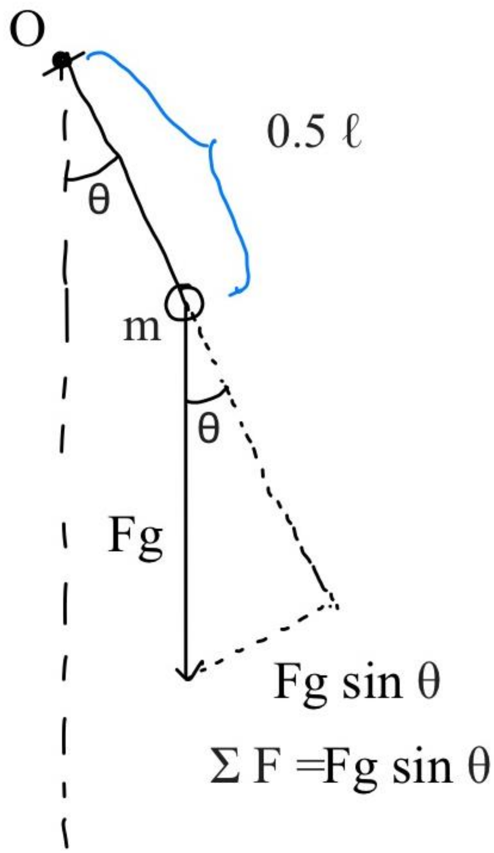


Figure 14: Simplification of Scenario C. O is the anchor point. ℓ is the string length. θ is the angle between the rod and a line parallel to the direction of gravity. m is mass. g is the acceleration due to gravity. Fg is the force of gravity.

Therefore, the analysis of this system will be similar to the analysis for scenario A.

Oscillation Period: Dimensional Analysis

In the previous section, I simplified the rod system into a more familiar simple pendulum system. The dimensional analysis is similar to the dimensional analysis conducted in scenario A . The only difference is that the string length is half as long. I can manipulate (3) to find the period for this system. The period of the pendulum is:

$$T \doteq \kappa[\mathbf{M}]^{0}[\mathbf{L}^{1}\mathbf{T}^{-2}]^{-\frac{1}{2}}[\mathbf{L}^{1}]^{\frac{1}{2}} \doteq \kappa\sqrt{\frac{\ell}{g\sin\theta}} \Rightarrow \kappa\sqrt{\frac{\frac{\ell}{2}}{g\sin\theta}} = \kappa\sqrt{\frac{\ell}{2g\sin\theta}} \Longrightarrow \kappa\sqrt{\frac{\ell}{2g}} \tag{24}$$

The sine function was removed because it is dimensionless [4]. (See derivation 7 for a detailed dimensional analysis for this system.)

Ordinary Differential Equation and Linearization

The ordinary differential equation for this system will be similar to the ordinary differential equation in scenario A. The only difference is that the string is half as long. Therefore, using (5) and halving the string length, we get:

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\theta = 0 \Longrightarrow \frac{d^2\theta}{dt^2} + \frac{g}{\frac{\ell}{2}}\theta = \frac{d^2\theta}{dt^2} + \frac{2g}{\ell}\theta = 0 \tag{25}$$

(See derivation 8 for details.)

Oscillation Period: Analytical Solution of Ordinary Differential Equation

Again, since we simplified the scenario, we can use the same solution we found in scenario A and divide ℓ by 2. Thus, using (8):

$$T = 2\pi\sqrt{\frac{\ell}{g}} \Rightarrow T = 2\pi\sqrt{\frac{\ell}{2g}}$$

(See derivation 9 for details.)

Numerical Solution via Euler's Method

Below is a graph showing the motion of the pendulum for three periods. Here, the string length was set to 2, and gravity was set to 9.81:

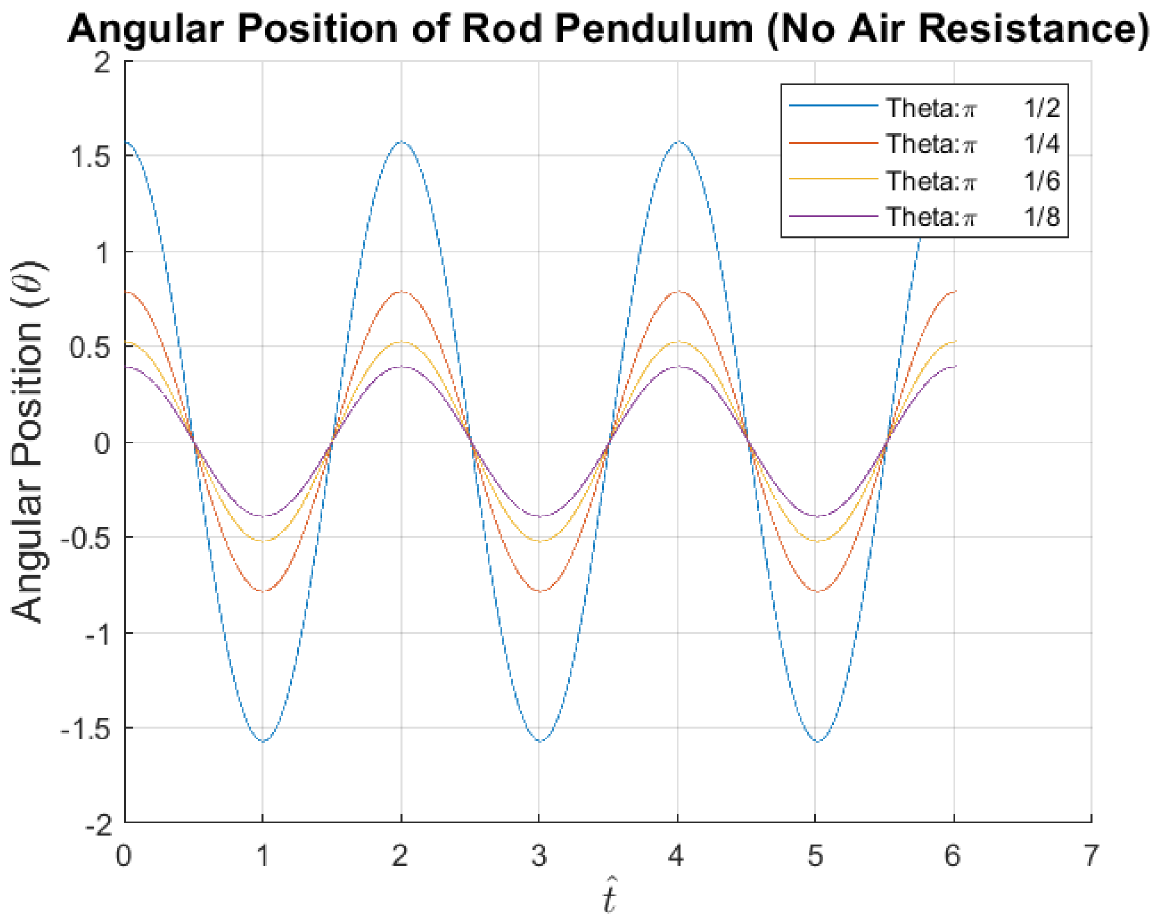


Figure 15: Graph showing the angular position of the pendulum as dimensionless time increases. There is no air resistance in this system.

The behavior for this scenario is similar to the behavior seen in scenario A. The pendulum oscillates without its amplitude decreasing. The initial position does not alter the periodicity of the system. Furthermore, the period found by finding the time between two local minima (or maxima) next to each other is:

```
numerically_derived_period

        2.0061
```

Figure 16: This is the numerically derived period of the system. The period was found by taking the distance between two local minima immediately next to each other.

The value found through computation is the same as the period found via the ordinary differential equation:

$$2\pi\sqrt{\frac{\ell}{2g}} = 2\pi\sqrt{\frac{2}{2\times 9.81}} \approx 2.0061$$

Scenario D: Rod Pendulum with Air Resistance

Diagram

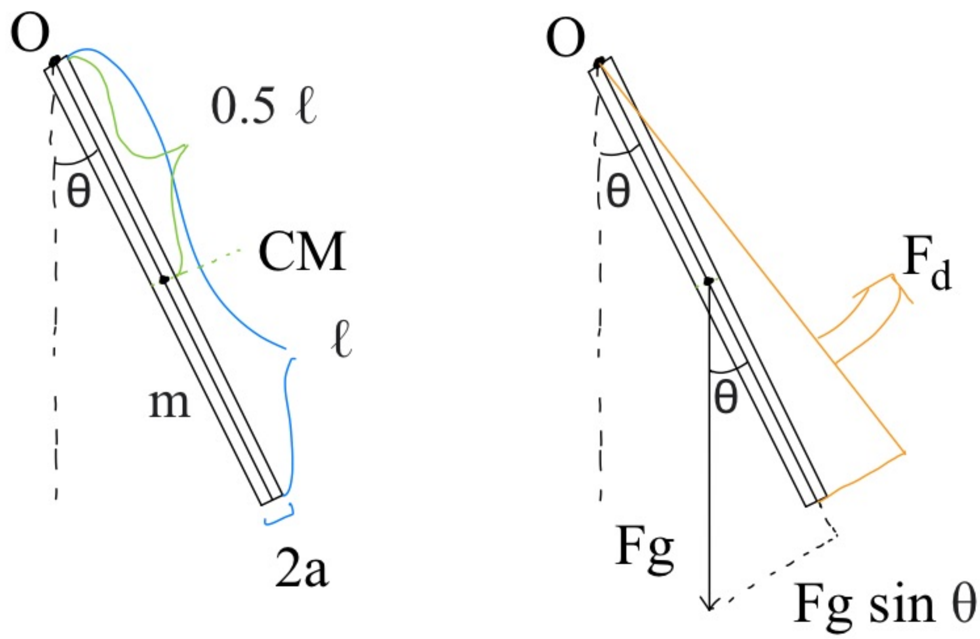


Figure 17: A sketch of a rod pendulum with air resistance. Here, O is the anchor point. θ is the angle between the rod and a line perpendicular to the direction of gravity. m is the mass of the rod. CM is the center of mass. 2a represents the diameter of the rod. ℓ is the length of the rod. Fg is the force of gravity. F_d is the drag force

Here, I will use torque to derive the total acceleration.

Drag Torque Derivation

Here, I will derive the total drag force on the system. In the previous scenarios, the drag force was related to the angular velocity of the pendulum. The angular velocity is the radians covered per unit time [10]. The linear velocity is the angular velocity multiplied by the distance to the anchor point [10]:

$$v = \ell\omega$$

Therefore, the velocity at each distinct point on the rod is different. Therefore, the drag force is different at each distinct point on the rod. Thus, the scenario requires a separate method of derivation for the drag force because each segment of the rod experiences a different velocity.

Below is a sketch of the rod:

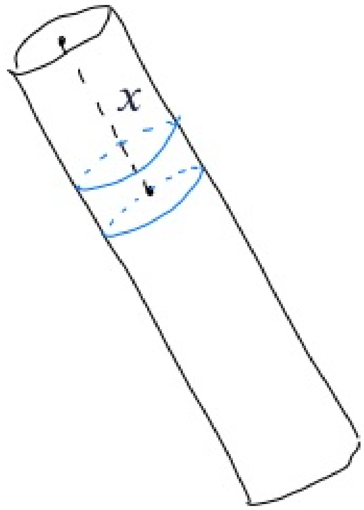


Figure 18: A sketch of the rod. Here, x represents the length from the anchor point to the center of the disk.

I will take a small, disk-shaped part of the rod. The small part of the rod has the same angular velocity and diameter as the rest of the rod. However, the distance from the anchor point is different. Therefore, the objective will be to build an integral that covers the entire length of the rod.

The length of the small part of the rod is represented by dx. Therefore, after linearization, the drag force on the disk can be represented by:

$$\frac{1}{2}\rho x\omega C_D(2adx)$$

To get the total drag on the rod, I can integrate the drag force on the small portion of the rod from the length of the rod to zero (anchor point):

$$\Sigma|\vec{\tau}_D| = \int_0^{\ell}\frac{1}{2}\rho x\omega C_D 2adx$$

By computing the integral, we get:

$$\Sigma|\vec{\tau}_D| = \frac{1}{2}\rho\omega C_D a\ell^2 = \frac{1}{2}\rho C_D a\ell^2\frac{d\theta}{dt} \tag{26}$$

(See derivation 10 for details.)

Gravitational Torque

As stated previously, gravity is exerted on the center of mass of the rod. Therefore, we can use similar computation used in scenario C.

Using the formula for torque [13]:

$$\overrightarrow{\tau_g} = (Distance\ from\ force\ exertion\ point\ to\ anchor\ point) \times F_g$$

$$\overrightarrow{\tau_g} = \frac{\ell}{2} \times mg$$

Since we are taking the cross product, the value of the torque is the length from the force exertion point to the anchor point times the perpendicular component of the gravitational force [13]:

$$\left|\overrightarrow{\tau_g}\right| = \frac{\ell}{2} mg \sin\theta$$

Thus, the torque exerted by gravity is:

$$\left|\vec{\tau}_g\right| = \frac{\ell mg \sin\theta}{2}$$

After linearization:

$$\left|\vec{\tau}_g\right| = \frac{\ell mg\theta}{2} \tag{27}$$

(See derivation 11.)

Moment of Inertia Derivation

Here, I will derive the moment of inertia for this system. This step is necessary because I am using torque. Torque is [13]:

$$\vec{\tau} = \vec{r} \times \vec{F} = I\vec{\alpha}$$

Dividing the total torque by the moment of inertia, we get the total angular acceleration necessary for deriving the ordinary differential equation. Below is a sketch of a rod that spins around the center of mass:

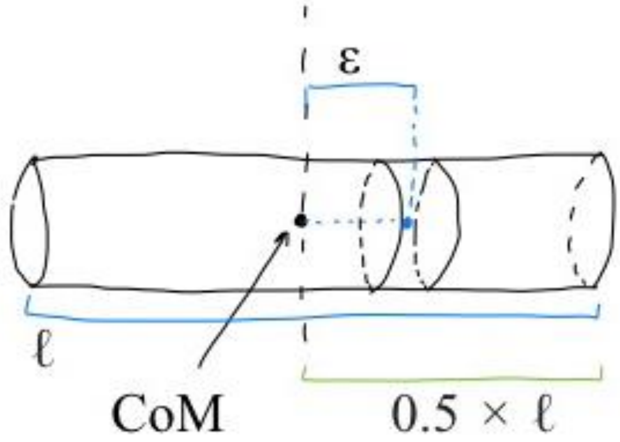


Figure 19: Sketch of a rod spinning around its center of mass. ℓ is the total length of the rod. CoM is the center of mass. ε is the distance to the small bit of the rod from the center of mass.

The basic formula for the moment of inertia is [9]:

$$\int \varepsilon^2 dm \tag{28}$$

Where ε is a small distance away from the center of mass. We need to change units here because we want to integrate from the center of mass to the ends of the rod. The mass in the small portion of the rod is:

$$dm = \frac{m_b}{\ell} d\varepsilon \tag{29}$$

Where m_b is the mass of the small bit of the rod. We can substitute (28) into (27):

$$\int \varepsilon^2 dm \Rightarrow \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \varepsilon^2 \frac{m_b}{\ell} d\varepsilon$$

Since the function is even, I can alter the limits of integration:

$$\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \varepsilon^2 \frac{m_b}{\ell} d\varepsilon = 2 \int_{0}^{\frac{\ell}{2}} \varepsilon^2 \frac{m_b}{\ell} d\varepsilon \tag{30}$$

After computing (30), the moment of inertia for a rod spinning around its center of mass is:

$$\frac{1}{12} m\ell^2 = I_{rod\ center}$$

Using the parallel axis theorem [9], the moment of inertia of a rod spinning at its end is:

$$I = I_{rod\ center} + m_{\Sigma}{\ell_{btwn}}^2$$

(31)

Where m_{Σ} is the total mass of the system, ℓ_{btwn} is the distance between the center of mass and the end of the rod. By plugging in values into (31), we find:

$$I = I_{rod\ center} + m_{\Sigma}{\ell_{btwn}}^2 = \frac{1}{12}m\ell^2 + m\frac{\ell^2}{2^2} = \frac{m\ell^2}{3}$$

Thus, the moment of inertia for this system is:

$$I = \frac{m\ell^2}{3}$$

(32)

(See derivation 12.)

Oscillation Period: Dimensional Analysis

Using similar steps for dimensional analysis in scenario B, the period should be:

$$T \doteq \kappa[\mathbf{M}]^0[\mathbf{L}^1\mathbf{T}^{-2}]^{-\frac{1}{2}}[\mathbf{L}^1]^{\frac{1}{2}} \doteq \kappa(Total\ Acceleration)^{-\frac{1}{2}}(\ell)^{\frac{1}{2}}$$

(33)

The total acceleration is:

$$\alpha = \frac{\Sigma\vec{\tau}}{I} = \frac{-|\vec{\tau}_g| - \Sigma|\vec{\tau}_D|}{I} = \frac{-\frac{\ell}{2}mg\theta - \frac{1}{2}\rho\omega C_D a\ell^2}{\frac{1}{3}m\ell^2}$$

(34)

We can plug in (34) into (33) to get the period:

$$T = \kappa\sqrt{\frac{\ell}{\frac{-\frac{\ell}{2}mg\theta - \frac{1}{2}\rho\omega C_D a\ell^2}{\frac{1}{3}m\ell^2}}}$$

(See derivation 13.)

Ordinary Differential Equation and Linearization

Here, I will derive the ordinary differential equation for the system. Since I have the total torque, I can use:

$$\Sigma\vec{\tau} = I\vec{\alpha}$$

We found the total acceleration of the system in (34). Thus, the ordinary differential equation is:

$$\frac{d^2\theta}{dt^2} = \vec{\alpha} = \frac{\Sigma\vec{\tau}}{I} = \frac{-\frac{\ell mg\sin\theta}{2} - \frac{\rho C_D a\ell^2}{2}\frac{d\theta}{dt}}{\frac{m\ell^2}{3}} \tag{35}$$

(See derivation 13 for finding the total acceleration.)

Oscillation Period: Analytical Solution of Ordinary Differential Equation

The process of finding the period analytically from the ordinary differential equation is similar to the process for scenario B. Using the characteristic function, (35) is shown to have complex roots [14]:

$$\mathrm{r}_{1,2} = \frac{-\mathbb{G} \pm \sqrt{\mathbb{G}^2 - 4\mathbb{H}}}{2} = \frac{-\mathbb{G} \pm \mathrm{i}\sqrt{4\mathbb{H} - \mathbb{G}^2}}{2}$$

Where:

$$\mathbb{G} = \frac{\frac{3}{2}\rho C_D a}{m}, \mathbb{H} = \frac{3g}{2\ell}$$

Given this information, we can construct the following solution:

$$\theta(t) = Ce^{\mathbb{J}t}(\cos(\mathbb{K}t + \psi)) \tag{36}$$

Where:

$$\mathbb{J} = \frac{-\mathbb{G}}{2}, \mathbb{K} = \frac{\sqrt{4\mathbb{H} - \mathbb{G}^2}}{2}$$

C is some constant. Since $\mathbb{K}$ alters the frequency of the trigonometric function in (36), $\mathbb{K}$ is the frequency. Since:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\mathbb{K}} \tag{37}$$

We can use this relation to find the period [14]. Solving for $\mathbb{K}$ results in:

$$\mathbb{K} = \frac{\sqrt{4\mathbb{H} - \mathbb{G}^2}}{2} \tag{38}$$

Therefore, we can plug in (38) into (37) to get the period:

$$T = \frac{2\pi}{\mathbb{K}} = \frac{2\pi}{\frac{\sqrt{4\mathbb{H} - \mathbb{G}^2}}{2}} = \frac{4\pi}{\sqrt{4\mathbb{H} - \mathbb{G}^2}}$$

(See derivation 14.)

Numerical Solution via Euler's Method

Below is a graph showing the rod pendulum's motion under air resistance. Here, $\mathbb{G}$ is 0.75, and $\mathbb{H}$ is roughly 0.3575:

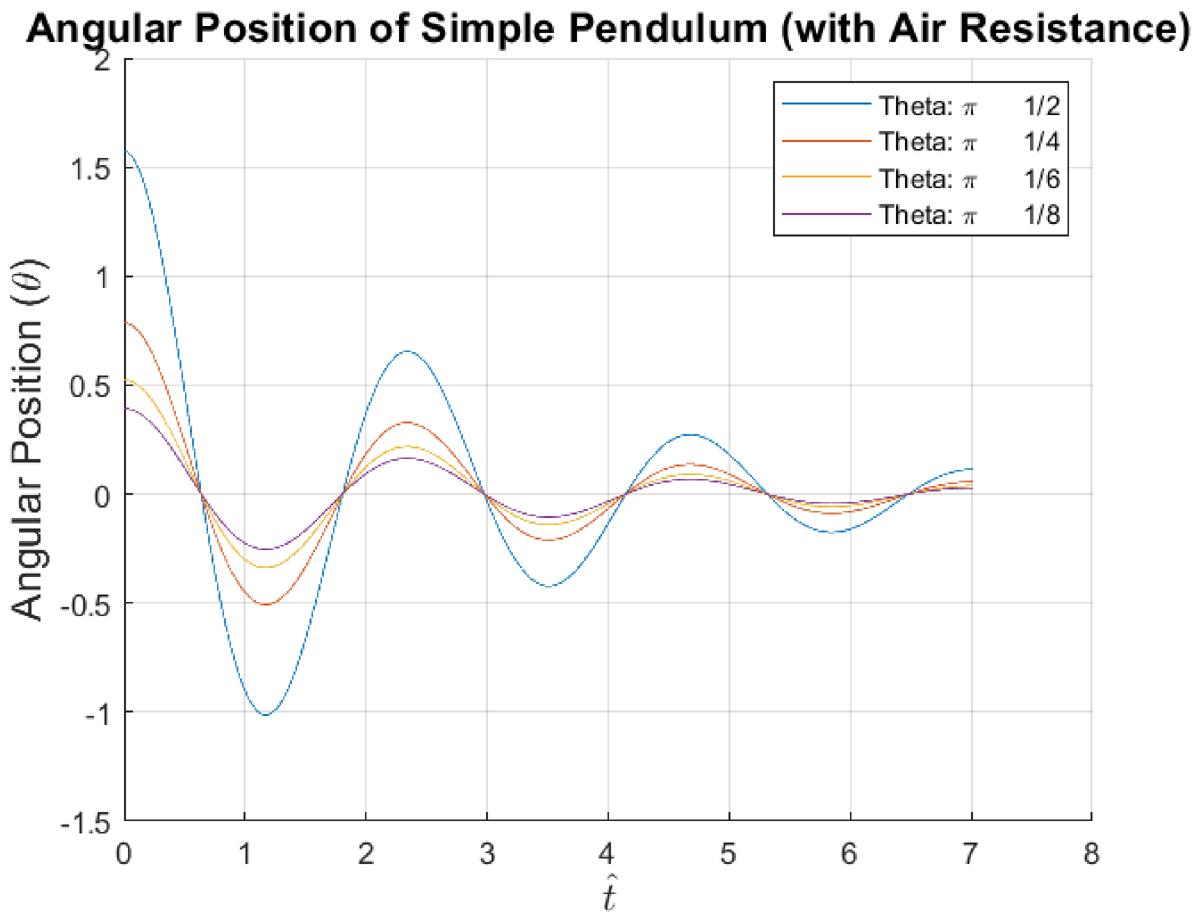


Figure 20: Graph showing motion of rod pendulum under air resistance.

Here, we see that the system shows periodic behavior. As time progresses, the drag force exponentially decreases the maximum height the rod can reach.

For the period of the system, however, there seems to be an inaccuracy when comparing the analytical solution to the numeric solution. Using the same method of using the distance between minima, the period is:

```
numerically_derived_period

     2.3387
```

Figure 21: The period of the rod pendulum that experiences air resistance. Here, the distance between local minima immediately next to each other was used to find the period.

The numerically derived period was consistent for all pairs of minima close to each other, and for different initial angles. Therefore, the initial angle does not change the period of the system. The numerically derived period is similar to the period we found through analytically solving the ordinary differential equation:

$$\mathrm{T} = \frac{4\pi}{\sqrt{4\mathbb{H} - \mathbb{G}^2}} = \frac{4\pi}{\sqrt{4(.3575) - (.75)^2}} = 2.3389$$

Appendix

Derivations

Derivation 1

$$[T] \doteq [M]^a \; [L T^{-2}]^b \; [L]^c \, K$$

$$\doteq [M^a] \; [L^b T^{-2b}] \; [L^c] \, K$$

$$\doteq [M^a] \; [L^b L^c] \; [T^{-2b}] \, K$$

T has no mass or length. Only time

$$M^a = M^0 \quad \Big| \quad L^b L^c = L^0 \quad \Big| \quad T^{-2b} = T^1$$

$$a = 0$$

$$b + c = 0$$

$$-\frac{1}{2} + c = 0$$

$$c = \frac{1}{2}$$

$$-2b = 1$$

$$b = -\frac{1}{2}$$

$$[T] \doteq [M]^a \; [L T^{-2}]^b \; [L]^c \, K$$

$$\doteq [M]^0 \; [L T^{-2}]^{-1/2} \; [L]^{1/2} \, K$$

$$T = K \, g^{-1/2} \, \ell^{1/2} = \frac{\ell^{1/2}}{g^{1/2}} K = \sqrt{\frac{\ell}{g}} \, K$$

Derivation 2

Newton's 2nd : $F = ma$ ①

Angular acceleration: $a = \alpha l$ ②

Total force : $mg \sin\theta$ ③

① + ② + ③ : $F = ma$

$$F = m\alpha l = -mg\sin\theta$$

$$\alpha l = -g\sin\theta$$

$$\alpha = \frac{-g\sin\theta}{l} \quad ④$$

Alternative way of representing α: $\alpha = \frac{d^2\theta}{dt^2}$ ⑤

④ + ⑤ : $\frac{d^2\theta}{dt^2} = \alpha = \frac{-g\sin\theta}{l}$

$$\frac{d^2\theta}{dt^2} = \frac{-g\sin\theta}{l}$$

$$\frac{d^2\theta}{dt^2} + \frac{g\sin\theta}{l} = 0 \quad ⑥$$

Linearize: For small θ, $\sin\theta \approx \theta$ ⑦:

⑥ + ⑦ : $\frac{d^2\theta}{dt^2} + \frac{g\sin\theta}{l} = 0$

: $\frac{d^2\theta}{dt^2} + \frac{g\theta}{l} = 0$

Derivation 3

ODE: $$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\theta$$

Characteristic function: $$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\theta$$

$$r^2 = -\frac{g}{\ell}$$

Roots:

$$r^2 = -\frac{g}{\ell}$$

$$\sqrt{r^2} = \sqrt{-\frac{g}{\ell}}$$

$$r_1 = i\sqrt{\frac{g}{\ell}} \qquad\qquad r_2 = -i\sqrt{\frac{g}{\ell}}$$

Solution form: for $r_1 = i\lambda$, $r_2 = -i\lambda$:

$$Sol_{1,2} = \cos(\lambda t) \pm i\sin(\lambda t)$$

$$r_1 = i\sqrt{\frac{g}{\ell}} \Rightarrow Sol_1 = \cos\left(t\sqrt{\frac{g}{\ell}}\right) + i\sin\left(t\sqrt{\frac{g}{\ell}}\right)$$

$$r_2 = -i\sqrt{\frac{g}{\ell}} \Rightarrow Sol_2 = \cos\left(t\sqrt{\frac{g}{\ell}}\right) - i\sin\left(t\sqrt{\frac{g}{\ell}}\right)$$

General Solution:

$$\theta(t) = A\cos\left(t\sqrt{\tfrac{g}{l}}\right) + B\sin\left(t\sqrt{\tfrac{g}{l}}\right)$$

Derivative w.r.t. t:

$$\theta(t) = A\cos\left(t\sqrt{\tfrac{g}{l}}\right) + B\sin\left(t\sqrt{\tfrac{g}{l}}\right)$$

$$\dot{\theta}(t) = \frac{d\theta}{dt} = -\sqrt{\frac{g}{l}}\,A\sin\left(t\sqrt{\tfrac{g}{l}}\right) + \sqrt{\frac{g}{l}}\,B\cos\left(t\sqrt{\tfrac{g}{l}}\right)$$

Since $\dot{\theta} = 0$ when $t = 0$:

$$\dot{\theta}(t) = -\sqrt{\frac{g}{l}}\,A\sin\left(t\sqrt{\tfrac{g}{l}}\right) + \sqrt{\frac{g}{l}}\,B\cos\left(t\sqrt{\tfrac{g}{l}}\right) = 0$$

$$\dot{\theta}(0) = -\sqrt{\frac{g}{l}}\,A\sin\left(0\cdot\sqrt{\tfrac{g}{l}}\right) + \sqrt{\frac{g}{l}}\,B\cos\left(0\cdot\sqrt{\tfrac{g}{l}}\right) = 0$$

$$= -\sqrt{\frac{g}{l}}\,A\sin(0) + \sqrt{\frac{g}{l}}\,B\cos(0) = 0$$

$$= \sqrt{\frac{g}{l}}\,B = 0$$

$$\Downarrow$$

$$B = 0$$

$$\theta(t) = A\cos\left(t\sqrt{\tfrac{g}{l}}\right) + 0\cdot\sin\left(t\sqrt{\tfrac{g}{l}}\right)$$

$$= A\cos\left(t\sqrt{\tfrac{g}{l}}\right)$$

Since $\theta(0) = \theta_0$:

$$\theta(t) = A\cos\left(t\sqrt{\tfrac{g}{l}}\right)$$

$$\theta(0) = A\cos\left(0\cdot\sqrt{\tfrac{g}{l}}\right) = \theta_0$$

$$= A = \theta_0$$

$$\Downarrow$$

$$\theta(t) = \theta_0\cos\left(t\sqrt{\tfrac{g}{l}}\right)$$

$\theta(t)$ is periodic because cosine has a period of 2π.

Since for any periodic function $\mathcal{P}$:

$$\mathcal{P}(t) = \mathcal{P}(t+T)$$
$$\mathcal{P}(0) = \mathcal{P}(T)$$

Thus:

$$\theta(t) = \theta(t+T)$$
$$\theta_0 \cos\left(t\sqrt{\tfrac{g}{l}}\right) = \theta_0 \cos\left((t+T)\sqrt{\tfrac{g}{l}}\right)$$
$$\cos\left(t\sqrt{\tfrac{g}{l}}\right) = \cos\left(t\sqrt{\tfrac{g}{l}} + T\sqrt{\tfrac{g}{l}}\right)$$
$$\cos(0) = \cos\left(T\sqrt{\tfrac{g}{l}}\right) = \cos(2\pi)$$
$$\cos^{-1}\left(\cos\left(T\sqrt{\tfrac{g}{l}}\right)\right) = \cos^{-1}\left(\cos(2\pi)\right)$$
$$T\sqrt{\frac{g}{l}} = 2\pi$$
$$T = 2\pi\sqrt{\frac{l}{g}}$$

Derivation 4

$$\sum \text{Force} = ma = mg\sin\theta - F_D = mg\sin\theta - \frac{1}{2}\rho v^2 C_D A$$

$$a_B = \frac{mg\sin\theta - \frac{1}{2}\rho v^2 C_D A}{m}$$

Buckingham – π:

(# Parameters) – (# of Dimensions)

$\left| \{ \text{Mass, Acceleration from total forces, String length, time} \} \right| - \left| \{ \text{Mass, Length, Time} \} \right|$

$4 - 3$

1 Dimensionless group K

$$\text{Period} \doteq [T] \doteq [m]^a [a_B]^b [l]^c K \doteq [M]^a [LT^{-2}]^b [L]^c K$$

$$\doteq [M^a] [L^b T^{-2b}] [L^c] K$$

$$\doteq [M^a] [L^b L^c] [T^{-2b}] K$$

$a = 0$ | $b + c = 0$, $-\frac{1}{2} + c = 0$, $c = \frac{1}{2}$ | $-2b = 1$, $b = -\frac{1}{2}$

$$\doteq [M]^a [LT^{-2}]^b [L]^c K = K [LT^{-2}]^{-1/2} [L]^{1/2}$$

$$\text{Period} = \tau = K a_B^{-\frac{1}{2}} l^{\frac{1}{2}}$$

$$= \frac{l^{1/2}}{a_B^{1/2}} K$$

$$= K \sqrt{\frac{l}{\frac{mg\sin\theta - \frac{1}{2}\rho v^2 C_D A}{m}}}$$

$$= K \sqrt{\frac{l m}{mg\sin\theta - \frac{1}{2}\rho v^2 C_D A}}$$

Derivation 5

Newton's 2nd: $F = ma = m\alpha \ell = F_g + F_D$ ①

Force gravity: $F_g = -mg\sin\theta$ ②

Drag Force: $-\frac{1}{2}\rho v^2 C_D \bar{A}$ ③

① + ② + ③: $m\alpha\ell = -mg\sin\theta - \frac{1}{2}\rho v^2 C_D \bar{A}$

Divide by $m\ell$ on both sides:

$$\alpha = \frac{-mg\sin\theta - \frac{1}{2}\rho v^2 C_D \bar{A}}{m\ell}$$

$\alpha = \frac{d^2\theta}{dt^2}$

$$\frac{d^2\theta}{dt^2} = \frac{-mg\sin\theta - \frac{1}{2}\rho v^2 C_D \bar{A}}{m\ell}$$

$$\frac{d^2\theta}{dt^2} + \frac{mg\sin\theta}{m\ell} + \frac{\frac{1}{2}\rho v^2 C_D \bar{A}}{m\ell} = 0$$

$$\frac{d^2\theta}{dt^2} + \underbrace{\frac{g\sin\theta}{\ell}}_{\substack{\sin\theta \approx \theta \\ \text{for small } \theta}} + \underbrace{\frac{\rho C_D \bar{A}}{2m\ell} v^2}_{\substack{\text{For small } v: \\ v^2 << v \\ \text{e.g., } v = .01,\ v^2 = .0001}} = 0$$

$$\frac{d^2\theta}{dt^2} + \frac{g\sin\theta}{\ell} + \frac{\rho C_D \bar{A}}{2m\ell} v = 0$$

$v = r\omega$

$$\frac{d^2\theta}{dt^2} + \frac{g\sin\theta}{\ell} + \frac{\rho C_D \bar{A}}{2m\cancel{\ell}} \cancel{\ell}\omega = 0$$

$\sin\theta \approx \theta$ for small θ, $\mathbb{D} = \frac{\rho C_D \bar{A}}{2m}$

$$\frac{d^2\theta}{dt^2} + \frac{g\theta}{\ell} + \mathbb{D}\frac{d\theta}{dt} = 0$$

Derivation 6

$$\frac{d^2\theta}{dt^2} + \frac{g\,\theta}{\ell} + \mathbb{D}\,\frac{d\theta}{dt} = 0$$

Characteristic equation: $r^2 + \frac{g}{\ell} + \mathbb{D}r = 0$

Roots: $\dfrac{-b \pm \sqrt{b^2-4ac}}{2a} = \dfrac{-\mathbb{D} \pm \sqrt{\mathbb{D}^2 - \frac{4g}{\ell}}}{2} = r_{1,2}$

Under-damped: $\mathbb{D}^2 - \frac{4g}{\ell} < 0$:

$$r_1 = \frac{-\mathbb{D}}{2} + \frac{i\sqrt{\frac{4g}{\ell} - \mathbb{D}^2}}{2}$$

General solution: $\Theta(t) = A e^{(\text{Real part of root})\,t}\left(\cos\left(t\,(\text{Complex part of root})\right)\right) + B e^{(\text{Real part of root})\,t}\left(\cos\left(t\,(\text{Complex part of root})\right)\right)$

$$= A e^{-\frac{\mathbb{D}t}{2}} \cos\left(\frac{t\sqrt{\frac{4g}{\ell} - \mathbb{D}^2}}{2}\right) + B e^{-\frac{\mathbb{D}t}{2}} \sin\left(\frac{t\sqrt{\frac{4g}{\ell} - \mathbb{D}^2}}{2}\right)$$

$$\mathbb{F} = \frac{-\mathbb{D}}{2}, \quad \mathbb{E} = \frac{\sqrt{\frac{4g}{\ell} - \mathbb{D}^2}}{2}$$

$$= A e^{\mathbb{F}t} \cos\left(t\mathbb{E}\right) + B e^{\mathbb{F}t} \sin\left(t\mathbb{E}\right)$$

$\cos(\beta) + \sin(\beta) = \cos(\beta + \psi)$ (ψ is some angle)

$$\Theta(t) = C e^{\mathbb{F}t} \cos(t\mathbb{E} + \psi)$$

$$\frac{d\theta}{dt} = \frac{d\left(C e^{\mathbb{F}t}\cos(t\mathbb{E}+\psi)\right)}{dt} = C\left(\mathbb{F}e^{\mathbb{F}t}\cos(t\mathbb{E}+\psi) - e^{\mathbb{F}t}\,\mathbb{E}\sin(t\mathbb{E}+\psi)\right)$$

Finding $\frac{d^2\theta}{dt^2}$: $C\Big(\underbrace{\mathbb{F}e^{\mathbb{F}t}\cos(t\mathbb{E}+\psi)}_{\downarrow\text{ Derivative:}} - \mathbb{E}\underbrace{e^{\mathbb{F}t}\sin(t\mathbb{E}+\psi)}_{\downarrow\text{ Derivative}}\Big)$

$$\mathbb{F}e^{\mathbb{F}t}\cos(t\mathbb{E}+\psi) - e^{\mathbb{F}t}\mathbb{E}\sin(t\mathbb{E}+\psi) \qquad \mathbb{F}e^{\mathbb{F}t}\sin(t\mathbb{E}+\psi) + e^{\mathbb{F}t}\mathbb{E}\cos(t\mathbb{E}+\psi)$$

$$= C\mathbb{F}^2e^{\mathbb{F}t}\cos(t\mathbb{E}+\psi) - C\mathbb{F}e^{\mathbb{F}t}\mathbb{E}\sin(t\mathbb{E}+\psi) - C\mathbb{E}\mathbb{F}e^{\mathbb{F}t}\sin(t\mathbb{E}+\psi) - C\mathbb{E}^2e^{\mathbb{F}t}\cos(t\mathbb{E}+\psi)$$

Plug in: $\frac{d^2\theta}{dt^2} + \frac{g\theta}{l} + \mathbb{D}\frac{d\theta}{dt} = 0$

$$C\mathbb{F}^2e^{\mathbb{F}t}\cos(t\mathbb{E}+\psi) - C\mathbb{F}e^{\mathbb{F}t}\mathbb{E}\sin(t\mathbb{E}+\psi) - C\mathbb{E}\mathbb{F}e^{\mathbb{F}t}\sin(t\mathbb{E}+\psi) - C\mathbb{E}^2e^{\mathbb{F}t}\cos(t\mathbb{E}+\psi) + \frac{g}{l}Ce^{\mathbb{F}t}\cos(t\mathbb{E}+\psi)$$
$$+\mathbb{D}C\left(\mathbb{F}e^{\mathbb{F}t}\cos(t\mathbb{E}+\psi) - e^{\mathbb{F}t}\mathbb{E}\sin(t\mathbb{E}+\psi)\right) = 0$$

$\downarrow$ Divide by C:

$$\mathbb{F}^2e^{\mathbb{F}t}\cos(t\mathbb{E}+\psi) - \mathbb{F}e^{\mathbb{F}t}\mathbb{E}\sin(t\mathbb{E}+\psi) - \mathbb{E}\mathbb{F}e^{\mathbb{F}t}\sin(t\mathbb{E}+\psi) - \mathbb{E}^2e^{\mathbb{F}t}\cos(t\mathbb{E}+\psi) + \frac{g}{l}e^{\mathbb{F}t}\cos(t\mathbb{E}+\psi)$$
$$+\mathbb{D}\left(\mathbb{F}e^{\mathbb{F}t}\cos(t\mathbb{E}+\psi) - e^{\mathbb{F}t}\mathbb{E}\sin(t\mathbb{E}+\psi)\right) = 0$$

$\downarrow$ Divide by $e^{\mathbb{F}t}$

$$\mathbb{F}^2\cos(t\mathbb{E}+\psi) - \mathbb{F}\mathbb{E}\sin(t\mathbb{E}+\psi) - \mathbb{E}\mathbb{F}\sin(t\mathbb{E}+\psi) - \mathbb{E}^2\cos(t\mathbb{E}+\psi) + \frac{g}{l}\cos(t\mathbb{E}+\psi)$$
$$+\mathbb{D}\left(\mathbb{F}\cos(t\mathbb{E}+\psi) - \mathbb{E}\sin(t\mathbb{E}+\psi)\right) = 0$$

$\downarrow$ Plug $\mathbb{F}$ back in: $\mathbb{F} = -\frac{\mathbb{D}}{2}$

$$\frac{\mathbb{D}^2}{4}\cos(t\mathbb{E}+\psi) + \underbrace{\frac{\mathbb{D}}{2}\mathbb{E}\sin(t\mathbb{E}+\psi) + \mathbb{E}\frac{\mathbb{D}}{2}\sin(t\mathbb{E}+\psi)}_{\text{Combine:}} - \mathbb{E}^2\cos(t\mathbb{E}+\psi) + \frac{g}{l}\cos(t\mathbb{E}+\psi)$$
$$+\mathbb{D}\left(-\frac{\mathbb{D}}{2}\cos(t\mathbb{E}+\psi) - \mathbb{E}\sin(t\mathbb{E}+\psi)\right) = 0$$

$$\frac{\mathbb{D}^2}{4}\cos(t\mathbb{E}+\psi) + \mathbb{D}\mathbb{E}\sin(t\mathbb{E}+\psi) - \mathbb{E}^2\cos(t\mathbb{E}+\psi) + \frac{g}{l}\cos(t\mathbb{E}+\psi) + \underbrace{\mathbb{D}\left(-\frac{\mathbb{D}}{2}\cos(t\mathbb{E}+\psi) - \mathbb{E}\sin(t\mathbb{E}+\psi)\right)}_{\text{Expand}} = 0$$

$$\frac{\mathbb{D}^2}{4}\cos(t\mathbb{E}+\psi) + \underbrace{\mathbb{D}\mathbb{E}\sin(t\mathbb{E}+\psi)}_{\text{Eliminate}} - \mathbb{E}^2\cos(t\mathbb{E}+\psi) + \frac{g}{l}\cos(t\mathbb{E}+\psi) - \frac{\mathbb{D}^2}{2}\cos(t\mathbb{E}+\psi) \underbrace{- \mathbb{D}\mathbb{E}\sin(t\mathbb{E}+\psi)}_{\text{Eliminate}} = 0$$

$$\frac{\mathbb{D}^2}{4}\cos(t\mathbb{E}+\psi) - \mathbb{E}^2\cos(t\mathbb{E}+\psi) + \frac{g}{l}\cos(t\mathbb{E}+\psi) - \frac{\mathbb{D}^2}{2}\cos(t\mathbb{E}+\psi) = 0$$

$$\frac{\mathbb{D}^2}{4}\cos(t\mathbb{E}+\psi) - \mathbb{E}^2\cos(t\mathbb{E}+\psi) + \frac{g}{l}\cos(t\mathbb{E}+\psi) - \frac{\mathbb{D}^2}{2}\cos(t\mathbb{E}+\psi) = 0$$

Divide by $\cos(t\mathbb{E}+\psi)$

$$\frac{\mathbb{D}^2}{4} - \mathbb{E}^2 + \frac{g}{l} - \frac{\mathbb{D}^2}{2} = 0$$

Rearrange

$$-\frac{\mathbb{D}^2}{4} + \frac{g}{l} = \mathbb{E}^2$$

Mult. by 4

$$-\mathbb{D}^2 + \frac{4g}{l} = 4\mathbb{E}^2$$

Take square root

$$\sqrt{-\mathbb{D}^2 + \frac{4g}{l}} = 2\mathbb{E}$$

$$\sqrt{-\mathbb{D}^2 + \frac{4g}{l}} = 2\mathbb{E}$$

Divide by 2

$$\frac{\sqrt{-\mathbb{D}^2 + \frac{4g}{l}}}{2} = \mathbb{E}$$

$$\text{Period} = T = \frac{2\pi}{\omega} = \frac{2\pi}{\mathbb{E}} = \frac{2\pi}{\frac{\sqrt{-\mathbb{D}^2 + \frac{4g}{l}}}{2}}$$

Mult. by 2 on both ends

$$= \frac{4\pi}{\sqrt{\frac{4g}{l} - \mathbb{D}^2}}$$

$$\frac{4\pi}{\sqrt{\left(\frac{4g}{l} - \mathbb{D}^2\right)\frac{4}{4}}}$$

$$= \frac{4\pi}{\sqrt{\frac{16g}{4l} - \frac{\mathbb{D}^2 4l}{4l}}}$$

$$= \frac{4\pi}{\sqrt{\frac{16g - \mathbb{D}^2 4l}{4l}}}$$

$$= 4\pi\sqrt{\frac{4l}{16g - \mathbb{D}^2 4l}}$$

$$= 4\pi\sqrt{\frac{4l}{4(4g - \mathbb{D}^2 l)}}$$

$$= \frac{4\pi}{2}\sqrt{\frac{4l}{4g - \mathbb{D}^2 l}}$$

$$= 2\pi\sqrt{\frac{4l}{4g - \mathbb{D}^2 l}}$$

Derivation 7

Buckingham - Π :

(# Parameters) $-$ (# of dimensions)

$$\left|\{\text{Mass, Acceleration from total forces, String length, time}\}\right| - \left|\{\text{Mass, Length, Time}\}\right|$$

$$4 - 3$$

1 Dimensionless group K

$$\text{Period} \doteq [T] \doteq [m]^a \, [a_B]^b \, \left[\tfrac{\ell}{2}\right]^c K \doteq [M]^a \, [L T^{-2}]^b \, [L]^c K$$

$$\doteq [M^a] \, [L^b T^{-2b}] \, [L^c] K$$

$$\doteq [M^a] \, [L^b L^c] \, [T^{-2b}] K$$

$$a = 0 \quad \left|\ \begin{aligned} b + c &= 0 \\ -\tfrac{1}{2} + c &= 0 \\ c &= \tfrac{1}{2} \end{aligned}\ \right| \quad \begin{aligned} -2b &= 1 \\ b &= -\tfrac{1}{2} \end{aligned}$$

$$\doteq [M]^a \, [L T^{-2}]^b \, [L]^c K \doteq K \, [L T^{-2}]^{-1/2} \, [L]^{1/2}$$

$$\text{Period} = \tau = K \, a_B^{-\frac{1}{2}} \left(\frac{\ell}{2}\right)^{\frac{1}{2}}$$

$$= \frac{\left(\frac{\ell}{2}\right)^{1/2}}{a_B^{1/2}} K$$

$$= K \sqrt{\frac{\ell}{2 \, \frac{\cancel{m} g \sin\theta}{\cancel{m}}}}$$

$$= K \sqrt{\frac{\ell}{2 g \sin\theta}}$$

Derivation 8

Newton's 2nd : $F = ma$ ①

Angular acceleration: $a = \alpha \frac{\ell}{2}$ ②

Total force : $-mg \sin \theta$ ③

① + ② + ③ : $F = ma$

$$F = m\alpha\frac{\ell}{2} = -mg\sin\theta$$

$$\alpha\frac{\ell}{2} = -g\sin\theta$$

$$\alpha = \frac{-2g\sin\theta}{\ell} \quad ④$$

Alternative way of representing α:

$$\alpha = \frac{d^2\theta}{dt^2} \quad ⑤$$

④ + ⑤ :

$$\frac{d^2\theta}{dt^2} = \alpha = -\frac{2g\sin\theta}{\ell}$$

$$\frac{d^2\theta}{dt^2} = -\frac{2g\sin\theta}{\ell}$$

$$\frac{d^2\theta}{dt^2} + \frac{2g\sin\theta}{\ell} = 0 \quad ⑥$$

Linearize: For small θ, $\sin\theta \approx \theta$ ⑦:

⑥ + ⑦ :

$$\frac{d^2\theta}{dt^2} + \frac{2g\sin\theta}{\ell} = 0$$

$$: \frac{d^2\theta}{dt^2} + \frac{2g\theta}{\ell} = 0$$

Derivation 9

ODE: $$\frac{d^2\theta}{dt^2} = -\frac{2g}{\ell}\theta$$

Characteristic function: $$\frac{d^2\theta}{dt^2} = -\frac{2g}{\ell}\theta$$

$$r^2 = -\frac{2g}{\ell}$$

Roots:

$$r^2 = -\frac{2g}{\ell}$$

$$\sqrt{r^2} = \sqrt{-\frac{2g}{\ell}}$$

$$r_1 = i\sqrt{\frac{2g}{\ell}} \qquad\qquad r_2 = -i\sqrt{\frac{2g}{\ell}}$$

Solution form: for $r_1 = i\lambda$, $r_2 = -i\lambda$:

$$Sol_{1,2} = \cos(\lambda t) \pm i\sin(\lambda t)$$

$$r_1 = i\sqrt{\frac{2g}{\ell}} \Longrightarrow Sol_1 = \cos\left(t\sqrt{\frac{2g}{\ell}}\right) + i\sin\left(t\sqrt{\frac{2g}{\ell}}\right)$$

$$r_2 = -i\sqrt{\frac{2g}{\ell}} \Longrightarrow Sol_2 = \cos\left(t\sqrt{\frac{2g}{\ell}}\right) - i\sin\left(t\sqrt{\frac{2g}{\ell}}\right)$$

General Solution:

$$\theta(t) = A\cos\left(t\sqrt{\frac{2g}{\ell}}\right) + B\sin\left(t\sqrt{\frac{2g}{\ell}}\right)$$

Derivative w.r.t. t:

$$\theta(t) = A\cos\left(t\sqrt{\frac{2g}{\ell}}\right) + B\sin\left(t\sqrt{\frac{2g}{\ell}}\right)$$

$$\dot{\theta}(t) = \frac{d\theta}{dt} = -\sqrt{\frac{2g}{\ell}}\,A\sin\left(t\sqrt{\frac{2g}{\ell}}\right) + \sqrt{\frac{2g}{\ell}}\,B\cos\left(t\sqrt{\frac{2g}{\ell}}\right)$$

Since $\dot{\theta} = 0$ when $t = 0$:

$$\dot{\theta}(t) = -\sqrt{\frac{2g}{\ell}}\,A\sin\left(t\sqrt{\frac{2g}{\ell}}\right) + \sqrt{\frac{2g}{\ell}}\,B\cos\left(t\sqrt{\frac{2g}{\ell}}\right) = 0$$

$$\dot{\theta}(0) = -\sqrt{\frac{2g}{\ell}}\,A\sin\left(0\cdot\sqrt{\frac{2g}{\ell}}\right) + \sqrt{\frac{2g}{\ell}}\,B\cos\left(0\cdot\sqrt{\frac{2g}{\ell}}\right) = 0$$

$$= -\sqrt{\frac{2g}{\ell}}\,A\sin(0) + \sqrt{\frac{2g}{\ell}}\,B\cos(0) = 0$$

$$= \sqrt{\frac{2g}{\ell}}\,B = 0$$

$$\Downarrow$$

$$B = 0$$

$$\theta(t) = A\cos\left(t\sqrt{\frac{2g}{\ell}}\right) + 0\cdot\sin\left(t\sqrt{\frac{2g}{\ell}}\right)$$

$$= A\cos\left(t\sqrt{\frac{2g}{\ell}}\right)$$

Since $\theta(0) = \theta_0$:

$$\theta(t) = A\cos\left(t\sqrt{\frac{2g}{\ell}}\right)$$

$$\theta(0) = A\cos\left(0\cdot\sqrt{\frac{2g}{\ell}}\right) = \theta_0$$

$$= A = \theta_0$$

$$\Downarrow$$

$$\theta(t) = \theta_0\cos\left(t\sqrt{\frac{2g}{\ell}}\right)$$

$\theta(t)$ is periodic because cosine has a period of 2π.

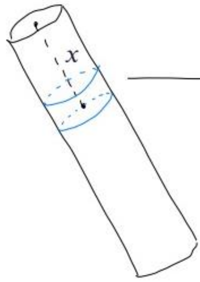
Since for any periodic function $\mathcal{P}$:

$$\mathcal{P}(t) = \mathcal{P}(t+T)$$
$$\mathcal{P}(0) = \mathcal{P}(T)$$

Thus:

$$\theta(t) = \theta(t+T)$$
$$\theta_0 \cos\left(t\sqrt{\frac{2g}{l}}\right) = \theta_0 \cos\left((t+T)\sqrt{\frac{2g}{l}}\right)$$
$$\cos\left(t\sqrt{\frac{2g}{l}}\right) = \cos\left(t\sqrt{\frac{2g}{l}} + T\sqrt{\frac{2g}{l}}\right)$$
$$\cos(0) = \cos\left(T\sqrt{\frac{2g}{l}}\right) = \cos(2\pi)$$
$$\cos^{-1}\left(\cos\left(T\sqrt{\frac{2g}{l}}\right)\right) = \cos^{-1}(\cos(2\pi))$$
$$T\sqrt{\frac{2g}{l}} = 2\pi$$
$$T = 2\pi\sqrt{\frac{l}{2g}}$$

Derivation 10

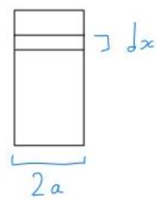


Velocity here:
$V = x\omega$

Drag Force: $F_D = \frac{1}{2}\rho v^2 C_D \bar{A}$

$V^2 \longrightarrow V$ (b/c for small V, $V^2 << V$)

$\bar{A} :=$ Cross-sectional area:



$$F_D = \frac{1}{2}\rho v C_D 2a\, dx$$

$$= \frac{1}{2}\rho x\omega C_D 2a\, dx$$

Total τ from Drag:
$$\int_0^{\ell} \frac{1}{2}\rho x\omega C_D 2a\, dx$$

$$= \frac{1}{2}\rho\omega C_D 2a \int_0^{\ell} x\, dx$$

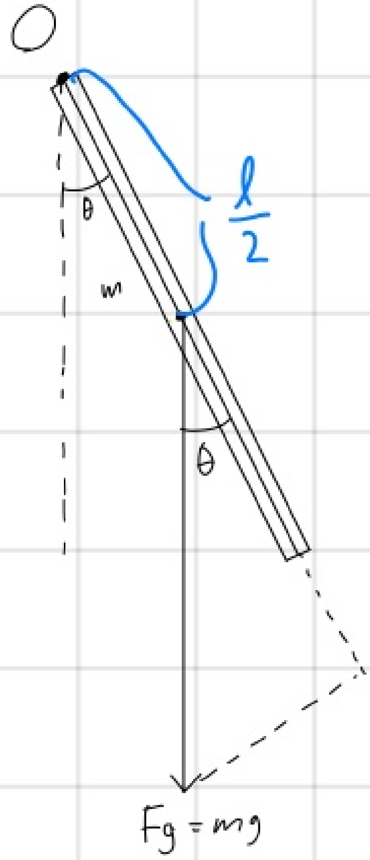
$$= \frac{1}{2}\rho\omega C_D 2a \left(\frac{x^2}{2} \Big|_0^{\ell} \right)$$

$$= \frac{1}{2}\rho\omega C_D 2a \frac{\ell^2}{2}$$

$$= \frac{1}{2}\rho C_D 2a \frac{\ell^2}{2}\omega$$

$$= \frac{1}{2}\rho C_D 2a \frac{\ell^2}{2} \frac{d\theta}{dt}$$

Derivation 11

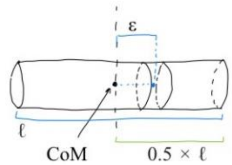


$$\left|\vec{\tau}_{F_g}\right| = \left|\vec{r} \times \vec{F}_g\right| = r \cdot F_{g\perp} = \frac{l}{2} mg \sin\theta$$

$\sin\theta \approx \theta$ for small θ:

$$\left|\vec{\tau}_{F_g}\right| = \frac{l}{2} mg\,\theta$$

Derivation 12



General formula for I : $I = \int_0^M r^2\, dm$

Change of variables to integrate over length of rod:

Density of rod: $\rho_r = \frac{m}{l}$

$m = l\rho_r$

$dm = dl\,\rho_r = \frac{m}{l}\, dl$

$$= \int_{-\frac{l}{2}}^{l/2} r^2 \frac{m}{l}\, dr$$

Alter limits of integration because r^2 is even function

$$= \frac{m}{l}\, 2 \int_0^{l/2} r^2\, dr$$

Integrate (Reverse power rule)

$$= \frac{2m}{l}\left(\frac{r^3}{3} \Big|_0^{l/2} \right)$$

$$= \frac{2m}{l} \cdot \frac{\left(\frac{l}{2}\right)^3}{3}$$

$$= \frac{2m\,l^3}{l \cdot 8 \cdot 3} = \frac{2m\,l^3}{24\,l} = \frac{1}{12} m l^2$$

Parallel axis theorem: $I = I_o + m_z\, l_{\text{btm}}^{\ 2} = \frac{1}{12} m\, l^2 + m\left(\frac{l}{2}\right)^2$

$$= \frac{1}{12} m l^2 + \frac{m l^2}{4}$$

$$= \left(\frac{1}{12} + \frac{1}{4}\right) m l^2$$

$$= \left(\frac{1}{12} + \frac{3}{12}\right) m l^2$$

$$= \left(\frac{4}{12}\right) m l^2$$

$$= \frac{1}{3} m l^2$$

Derivation 13

$$|\tau| = I\alpha = |\tau \text{ from gravity} + \tau \text{ from drag}|$$

$$-\frac{\ell}{2} mg\theta - \frac{1}{2}\rho C_D a \ell^2 \frac{d\theta}{dt} = \frac{1}{3} m\ell^2 \frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} = \frac{-\frac{\ell}{2} mg\theta - \frac{1}{2}\rho C_D a \ell^2 \frac{d\theta}{dt}}{\frac{1}{3} m\ell^2}$$

Buckingham – π:

(# Parameters) – (# of Dimensions)

$\left|\{\text{Mass, Acceleration from total forces, String length, time}\}\right| - \left|\{\text{Mass, Length, Time}\}\right|$

$4 - 3$

1 Dimensionless group K

$$\text{Period} \doteq [T] \doteq [m]^a [a]^b [\ell]^c K \doteq [M]^a [LT^{-2}]^b [L]^c K$$

$$\doteq [M^a] [L^b T^{-2b}] [L^c] K$$

$$\doteq [M^a] [L^b L^c] [T^{-2b}] K$$

$$a = 0 \quad \left| \begin{array}{c} b + c = 0 \\ -\frac{1}{2} + c = 0 \\ c = \frac{1}{2} \end{array} \right| \begin{array}{c} -2b = 1 \\ b = -\frac{1}{2} \\ \end{array}$$

$$\doteq [M]^a [LT^{-2}]^b [L]^c K \doteq K [LT^{-2}]^{-1/2} [L]^{1/2}$$

$$\text{Period} = \tau = K a_B^{-\frac{1}{2}} \ell^{\frac{1}{2}}$$

$$= \frac{\ell^{1/2}}{a_B^{1/2}} K$$

$$= K \sqrt{\frac{\ell}{\dfrac{-\frac{\ell}{2} mg\theta - \frac{1}{2}\rho C_D a \ell^2 \frac{d\theta}{dt}}{\frac{1}{3} m\ell^2}}}$$

Derivation 14

$$\frac{d^2\theta}{dt^2} = \frac{-\frac{1}{2}\rho C_D a\, l^2 \frac{d\theta}{dt} - \frac{l}{2} mg\sin\theta}{\frac{1}{3} m l^2}$$

$$= \frac{-\frac{3}{2}\rho C_D a\, l^2 \frac{d\theta}{dt}}{m l^2} - \frac{\frac{3l}{2} mg\sin\theta}{m l^2}$$

$$= \frac{-\frac{3}{2}\rho C_D a\, \cancel{l^2} \frac{d\theta}{dt}}{m \cancel{l^2}} - \frac{\frac{3\cancel{l}}{2} \cancel{m} g\sin\theta}{\cancel{m} l^{\cancel{2}}}$$

$$= \frac{-\frac{3}{2}\rho C_D a \frac{d\theta}{dt}}{m} - \frac{3g\theta}{2l}$$

$$G = \frac{\frac{3}{2}\rho C_D a}{m}, \quad H = \frac{3g}{2l}$$

$$\frac{d^2\theta}{dt^2} = -G\frac{d\theta}{dt} - H\theta$$

Characteristic func: $r^2 = -Gr - H$

$$r^2 + Gr + H = 0$$

Roots: $\frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-G \pm \sqrt{G^2-4H}}{2} = \frac{-G \pm i\sqrt{4H-G^2}}{2}$

$\downarrow$ Roots $\in \mathbb{C}$:

$$\theta(t) = Ae^{(\text{real part})t}\left(\cos((\text{real complex})t)\right) + Be^{(\text{real part})t}\left(\sin((\text{real complex})t)\right)$$

$$\theta(t) = Ae^{-\frac{G}{2}t}\cos\left(t\left(\frac{\sqrt{4H-G^2}}{2}\right)\right) + Be^{-\frac{G}{2}t}\sin\left(t\left(\frac{\sqrt{4H-G^2}}{2}\right)\right)$$

$$J = -\frac{G}{2}, \quad K = \frac{\sqrt{4H-G^2}}{2}$$

$$\theta(t) = Ae^{Jt}\cos(tK) + Be^{Jt}\sin(tK)$$

$$= Ce^{Jt}\cos(tK - \psi)$$

$$\dot{\theta} = C\left(Je^{Jt}\cos(tK-\psi) - e^{Jt}K\sin(tK-\psi)\right)$$

$$\ddot{\theta} = C\left(\underbrace{Je^{Jt}\cos(tK-\psi)}_{Je^{Jt}\cos(tK-\psi) - Ke^{Jt}\sin(tK-\psi)} - K\underbrace{e^{Jt}\sin(tK-\psi)}_{Je^{Jt}\sin(tK-\psi) + Ke^{Jt}\cos(tK-\psi)}\right)$$

$$\ddot{\theta} = CJ\left(Je^{Jt}\cos(tK-\psi)\right) - CJ\left(Ke^{Jt}\sin(tK-\psi)\right) - CKJe^{Jt}\sin(tK-\psi) - CKKe^{Jt}\cos(tK-\psi)$$

$$\frac{d^2\theta}{dt^2} = -G\frac{d\theta}{dt} - H\theta$$

$$\ddot{\theta} = -G\dot{\theta} - H\theta$$

$$CJ\left(Je^{Jt}\cos(tK-\psi)\right) - CJ\left(Ke^{Jt}\sin(tK-\psi)\right) - CKJe^{Jt}\sin(tK-\psi) - CKKe^{Jt}\cos(tK-\psi) = -GC\left(Je^{Jt}\cos(tK-\psi) - e^{Jt}K\sin(tK-\psi)\right) - HCe^{Jt}\cos(tK-\psi)$$

$$J\left(Je^{Jt}\cos(tK-\psi)\right) - J\left(Ke^{Jt}\sin(tK-\psi)\right) - KJe^{Jt}\sin(tK-\psi) - K^2e^{Jt}\cos(tK-\psi) = -G\left(Je^{Jt}\cos(tK-\psi) - e^{Jt}K\sin(tK-\psi)\right) - H\,e^{Jt}\cos(tK-\psi)$$

$$\mathbb{J}\left(\mathbb{J}e^{\mathbb{J}t}\cos(t\mathbb{K}-\psi)\right)-\mathbb{J}\left(\mathbb{K}e^{\mathbb{J}t}\sin(t\mathbb{K}-\psi)\right)-\mathbb{K}\mathbb{J}e^{\mathbb{J}t}\sin(t\mathbb{K}-\psi)-\mathbb{K}^2e^{\mathbb{J}t}\cos(t\mathbb{K}-\psi)=-\mathbb{G}\left(\mathbb{J}e^{\mathbb{J}t}\cos(t\mathbb{K}-\psi)-e^{\mathbb{J}t}\mathbb{K}\sin(t\mathbb{K}-\psi)\right)-\mathbb{H}\,e^{\mathbb{J}t}\cos(t\mathbb{K}-\psi)$$

$$\mathbb{J}^2\cos(\mathbb{K}t-\psi)-\mathbb{J}\mathbb{K}\sin(t\mathbb{K}-\psi)-\mathbb{K}\mathbb{J}\sin(t\mathbb{K}-\psi)-\mathbb{K}^2\cos(t\mathbb{K}-\psi)=-\mathbb{G}\mathbb{J}\cos(t\mathbb{K}-\psi)+\mathbb{G}\mathbb{K}\sin(t\mathbb{K}-\psi)-\mathbb{H}\cos(t\mathbb{K}-\psi)$$

$$-\frac{\mathbb{G}}{2}=\mathbb{J}$$

$$\mathbb{J}^2\cos(\mathbb{K}t-\psi)-\mathbb{J}\mathbb{K}\sin(t\mathbb{K}-\psi)-\mathbb{K}\mathbb{J}\sin(t\mathbb{K}-\psi)-\mathbb{K}^2\cos(t\mathbb{K}-\psi)=-\mathbb{G}\mathbb{J}\cos(t\mathbb{K}-\psi)+\mathbb{G}\mathbb{K}\sin(t\mathbb{K}-\psi)-\mathbb{H}\cos(t\mathbb{K}-\psi)$$

$$\frac{\mathbb{G}^2}{4}\cos(\mathbb{K}t-\psi)+\frac{\mathbb{K}\mathbb{G}}{2}\sin(t\mathbb{K}-\psi)+\frac{\mathbb{K}\mathbb{G}}{2}\sin(t\mathbb{K}-\psi)-\mathbb{K}^2\cos(t\mathbb{K}-\psi)=\frac{\mathbb{G}^2}{2}\cos(t\mathbb{K}-\psi)+\mathbb{G}\mathbb{K}\sin(t\mathbb{K}-\psi)-\mathbb{H}\cos(t\mathbb{K}-\psi)$$

$$\frac{\mathbb{G}^2}{4}\cos(\mathbb{K}t-\psi)+\mathbb{K}\mathbb{G}\sin(t\mathbb{K}-\psi)-\mathbb{K}^2\cos(t\mathbb{K}-\psi)=\frac{\mathbb{G}^2}{2}\cos(t\mathbb{K}-\psi)+\mathbb{G}\mathbb{K}\sin(t\mathbb{K}-\psi)-\mathbb{H}\cos(t\mathbb{K}-\psi)$$

$$\frac{\mathbb{G}^2}{4}\cos(\mathbb{K}t-\psi)-\mathbb{K}^2\cos(t\mathbb{K}-\psi)=\frac{\mathbb{G}^2}{2}\cos(t\mathbb{K}-\psi)-\mathbb{H}\cos(t\mathbb{K}-\psi)$$

$$\frac{\mathbb{G}^2}{4}-\mathbb{K}^2=\frac{\mathbb{G}^2}{2}-\mathbb{H}$$

$$-\frac{\mathbb{G}^2}{4}+\mathbb{H}=\mathbb{K}^2$$

$$-\mathbb{G}^2+4\mathbb{H}=4\mathbb{K}^2$$

$$\sqrt{-\mathbb{G}^2+4\mathbb{H}}=2\mathbb{K}$$

$$\mathbb{K}=\frac{\sqrt{4\mathbb{H}-\mathbb{G}^2}}{2}$$

$$T=\frac{2\pi}{\omega}=\frac{2\pi}{\mathbb{K}}=\frac{4\pi}{\sqrt{4\mathbb{H}-\mathbb{G}^2}}$$

Code

Timescale

```
%% Simple Pendulum Euler %%
clc
clear
%State Variables
g=9.81; %accerleration due to gravity
string_length=2; %string length

%Initial conditions
init_theta=[pi/6];%initial angular position
angular_velocity=0; %angular velocity

%Time step and total time
dt_array=[.5 .1 .01 .001 .0001]; %various timesteps
%total_time=10; %Total time
plt=figure(1);
legend_names=[];
hold on
timescale=2*pi*sqrt(string_length/g);
mult=3;
total_time=mult*timescale;
for theta=init_theta
    for dt=dt_array
        %Arrays for plotting and string concatenation for building labels
        t_array=zeros(ceil(total_time/dt),1);
        angular_velocity_array=zeros(ceil(total_time/dt),1);
        angular_velocity_array(1)=angular_velocity;
        theta_array=zeros(ceil(total_time/dt),1);
        theta_array(1)=theta;
        theta_label='Theta: ';
        theta_string=string(theta);
        dt_label=' Timestep: ';
        dt_string=string(dt);
        space=' ';
        new_label=strcat(dt_label,dt_string);
        legend_names=[legend_names new_label];
        %Forward Euler's Method
        for k=1:ceil(total_time/dt)
            angular_velocity_array(k+1)=angular_velocity_array(k)-((g*theta_array(k))/string_length)*dt; %k+1th
angular velocity
            theta_array(k+1)=theta_array(k)+angular_velocity_array(k+1)*dt; %k+1th angular position
            t_array(k+1)=t_array(k)+dt; %time
        end
        plot(t_array,theta_array)
    end
end
%Plot data
title('Angular Position of Pendulum','FontSize',14)
xlabel('Time/TimeScale','FontSize',14)
ylabel('Theta (Init: \pi/6)','FontSize',14)
legend(legend_names)

%Numerically confirm the period
max_logical=islocalmax(theta_array);
```

```
max_index=find(max_logical==1);
numerically_derived_period=t_array(max_index(2))-t_array(max_index(1))
iter_length=length(max_index);
differences=[];
for k=2:iter_length
   diff=t_array(max_index(k))-t_array(max_index(k-1));
   differences=[differences diff]
end
```

Scenario A

```
%% Simple Pendulum Euler %%
clc
clear
%State Variables
g=9.81; %accerleration due to gravity
string_length=2; %string length

%Initial conditions
init_theta=[pi/2 pi/4 pi/6 pi/8];%initial angular position
angular_velocity=0; %angular velocity

%Time step and total time
dt_array=[.0001]; %various timesteps
plt=figure(1);
legend_names=[];
hold on
timescale=2*pi*sqrt(string_length/g);
mult=3;
total_time=mult*timescale;
for theta=init_theta
    for dt=dt_array
        %Arrays for plotting and string concatenation for building labels
        adj_time=floor(total_time/dt);
        t_array=zeros(adj_time,1);
        angular_velocity_array=zeros(adj_time,1);
        angular_velocity_array(1)=angular_velocity;
        theta_array=zeros(adj_time,1);
        theta_array(1)=theta;
        theta_label='Theta:\pi';
        theta_string=string(rats(theta/pi));

        new_label=strcat(theta_label,theta_string);
        legend_names=[legend_names new_label];
        %Forward Euler's Method
        for k=1:total_time/dt
            angular_velocity_array(k+1)=angular_velocity_array(k)-((g*theta_array(k))/string_length)*dt; %k+1th
angular velocity
            theta_array(k+1)=theta_array(k)+angular_velocity_array(k+1)*dt; %k+1th angular position
            t_array(k+1)=t_array(k)+dt; %time
        end
        plot(t_array,theta_array)
    end
end
%Plot data
title('Angular Position of Simple Pendulum (No Air Resistance)','FontSize',14)
%xlabel('Time/TimeScale','FontSize',14)
xlabel('$$\hat{t}$$','Interpreter','Latex','FontSize',14)
ylabel('Angular Position (\theta)','FontSize',14)
legend(legend_names)
grid on
hold off

%Numerically confirm the period
max_logical=islocalmax(theta_array);
max_index=find(max_logical==1);
```

```
numerically_derived_period=t_array(max_index(2))-t_array(max_index(1))
iter_length=length(max_index);
differences=[];
for k=2:iter_length
   diff=t_array(max_index(k))-t_array(max_index(k-1));
   differences=[differences diff]
end
```

Scenario B

```
%% Simple Pendulum Euler %%
clc
clear
cla()
%State Variables / Initial Conditions
g=9.81; %accerleration due to gravity
string_length=2*2; %string length
mass=1; %Mass of ball on string
roh=1;%Fluid density
Drag_coeff=1;%Drag Coefficient
mass_radius=.5;%Radius of ball
angular_velocity=0; %angular velocity

cross_sec_area=(mass_radius^2)*pi;
init_theta=[pi/2 pi/4 pi/6 pi/8];%initial angular position
vel_D=(roh*Drag_coeff*pi*mass_radius*mass_radius)/(2*mass);
vel_D=sqrt(4*g/string_length)*.5;

%Period from calculations
period=2*pi*sqrt(((4*string_length)/((4*g)-
(((roh*Drag_coeff*cross_sec_area/(2*mass))^2)*string_length))))
period=4*pi*sqrt(string_length/((4*g)-((vel_D^2)*string_length)))

%Time step and total time
dt_array=[ .0001 ]; %various timesteps
plt=figure(1);
legend_names=[];
hold on
sub_D=(roh*Drag_coeff*cross_sec_area)/(2*mass);
timescale=2*pi*sqrt((4*string_length)/((4*g)-((sub_D^2)*string_length)));
mult=3;
total_time=mult*timescale;
for theta=init_theta
    for dt=dt_array
        %Arrays for plotting and string concatenation for building labels
        adj_time=floor(total_time/dt);
        t_array=zeros(adj_time,1);
        theta_array=zeros(adj_time,1);
        angular_velocity_array=zeros(adj_time,1);
        angular_acc_array=zeros(adj_time,1);

        angular_acceleration=((vel_D*angular_velocity)-((g*theta)/string_length));

        theta_array(1)=theta;
        angular_acc_array(1)=angular_acceleration;
        angular_velocity_array(1)=angular_velocity;

        theta_label='Theta: \pi';
        theta_string=string(rats(theta/pi));
        new_label=strcat(theta_label,theta_string);
        legend_names=[legend_names ; new_label];
        %Forward Euler's Method
        for k=1:(adj_time)
```

```
            angular_acc_array(k+1)=-(angular_velocity_array(k)*vel_D)-
((g*(theta_array(k)))/string_length);

angular_velocity_array(k+1)=angular_velocity_array(k)+(angular_acc_array(k))*dt; %k+1
th angular velocity

theta_array(k+1)=theta_array(k)+(angular_velocity_array(k+1)*dt)+(.5*(angular_acc_arr
ay(k+1)*dt*dt)); %k+1th angular position
            t_array(k+1)=t_array(k)+dt; %time
        end
        plot(t_array,theta_array)
    end
end
%Plot data
title('Angular Position of Simple Pendulum (with Air Resistance)','FontSize',14)
%xlabel('Time/TimeScale','FontSize',14)
xlabel('$$\hat{t}$$','Interpreter','Latex','FontSize',14)
ylabel('Angular Position (\theta)','FontSize',14)
legend(legend_names)

grid on
hold off
drawnow
%Numerically confirm the period
max_logical=islocalmax(theta_array);
max_index=find(max_logical==1);
numerically_derived_period=t_array(max_index(2))-t_array(max_index(1))
iter_length=length(max_index);
differences=[];
for k=2:iter_length
    diff=t_array(max_index(k))-t_array(max_index(k-1));
    differences=[differences diff];
end
avg_period=mean(differences)
```

Scenario C

```
%% Simple Pendulum Euler %%
clc
clear
%State Variables
g=9.81; %accerleration due to gravity
string_length=2/2; %string length

%Initial conditions
init_theta=[pi/2 pi/4 pi/6 pi/8];%initial angular position
angular_velocity=0; %angular velocity

%Time step and total time
dt_array=[.0001]; %various timesteps
plt=figure(1);
legend_names=[];
hold on
timescale=2*pi*sqrt(string_length/g);
mult=3;
total_time=mult*timescale;
for theta=init_theta
    for dt=dt_array
        %Arrays for plotting and string concatenation for building labels
        adj_time=floor(total_time/dt);
        t_array=zeros(adj_time,1);
        angular_velocity_array=zeros(adj_time,1);
        angular_velocity_array(1)=angular_velocity;
        theta_array=zeros(adj_time,1);
        theta_array(1)=theta;
        theta_label='Theta:\pi';
        theta_string=string(rats(theta/pi));

        new_label=strcat(theta_label,theta_string);
        legend_names=[legend_names new_label];
        %Forward Euler's Method
        for k=1:total_time/dt
            angular_velocity_array(k+1)=angular_velocity_array(k)-
((g*theta_array(k))/string_length)*dt; %k+1th angular velocity
            theta_array(k+1)=theta_array(k)+angular_velocity_array(k+1)*dt; %k+1th
angular position
            t_array(k+1)=t_array(k)+dt; %time
        end
        plot(t_array,theta_array)
    end
end
%Plot data
title('Angular Position of Rod Pendulum (No Air Resistance)','FontSize',14)
%xlabel('Time/TimeScale','FontSize',14)
xlabel('$$\hat{t}$$','Interpreter','Latex','FontSize',14)
ylabel('Angular Position (\theta)','FontSize',14)
legend(legend_names)
grid on
hold off

%Numerically confirm the period
```

```
max_logical=islocalmax(theta_array);
max_index=find(max_logical==1);
numerically_derived_period=t_array(max_index(2))-t_array(max_index(1))
iter_length=length(max_index);
differences=[];
for k=2:iter_length
    diff=t_array(max_index(k))-t_array(max_index(k-1));
    differences=[differences diff]
end
```

Scenario D

```
%% Simple Pendulum Euler %%
clc
clear
cla()
%State Variables
g=9.81; %accerleration due to gravity
string_length=2; %string length
mass=1; %Mass of ball on string
roh=1;%Fluid density
Drag_coeff=1;%Drag Coefficient
mass_radius=.5;%Radius of ball
cross_sec_area=(mass_radius^2)*pi;

%Initial conditions
init_theta=[pi/2 pi/4 pi/6 pi/8];%initial angular position
angular_velocity=0; %angular velocity
vel_G=((1.5*roh*Drag_coeff*mass_radius)/(mass));
theta_H=(3*g)/(2*string_length);

%Period from calculations
freq=sqrt((4*theta_H)-(vel_G^2))/2;
period=(2*pi)/freq

%Time step and total time
dt_array=[ .0001 ]; %various timesteps
plt=figure(1);
legend_names=[];
hold on
mult=3;
total_time=mult*period;
for theta=init_theta
    for dt=dt_array
        %Arrays for plotting and string concatenation for building labels
        adj_time=floor(total_time/dt);
        t_array=zeros(adj_time,1);
        theta_array=zeros(adj_time,1);
        angular_velocity_array=zeros(adj_time,1);
        angular_acc_array=zeros(adj_time,1);

        angular_acceleration=((vel_G*angular_velocity)-((theta_H*theta)));

        theta_array(1)=theta;
        angular_acc_array(1)=angular_acceleration;
        angular_velocity_array(1)=angular_velocity;

        theta_label='Theta: \pi';
        theta_string=string(rats(theta/pi));
        new_label=strcat(theta_label,theta_string);
        legend_names=[legend_names ; new_label];
        %Forward Euler's Method
        for k=1:(adj_time)
            angular_acc_array(k+1)=-(angular_velocity_array(k)*vel_G)-
((theta_H*(theta_array(k))));
```

```
angular_velocity_array(k+1)=angular_velocity_array(k)+(angular_acc_array(k))*dt; %k+1
th angular velocity

theta_array(k+1)=theta_array(k)+(angular_velocity_array(k+1)*dt)+(.5*(angular_acc_arr
ay(k+1)*dt*dt)); %k+1th angular position
            t_array(k+1)=t_array(k)+dt; %time
        end
        plot(t_array,theta_array)
        %Numerically confirm the period
        max_logical=islocalmax(theta_array);
        max_index=find(max_logical==1);
        numerically_derived_period=t_array(max_index(2))-t_array(max_index(1))
        iter_length=length(max_index);
        differences=[];
        for k=2:iter_length
            diff=t_array(max_index(k))-t_array(max_index(k-1));
            differences=[differences diff];
        end
    end
end
%Plot data
title('Angular Position of Simple Pendulum (with Air Resistance)','FontSize',14)
%xlabel('Time/TimeScale','FontSize',14)
xlabel('$$\hat{t}$$','Interpreter','Latex','FontSize',14)
ylabel('Angular Position (\theta)','FontSize',14)
legend(legend_names)
grid on
hold off

%Numerically confirm the period
max_logical=islocalmax(theta_array);
max_index=find(max_logical==1);
numerically_derived_period=t_array(max_index(2))-t_array(max_index(1))
iter_length=length(max_index);
differences=[];
for k=2:iter_length
    diff=t_array(max_index(k))-t_array(max_index(k-1));
    differences=[differences diff];
end
avg_period=mean(differences)
```

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