Here we support some of the computations that appear in the paper (see README).

Finite difference operator:

```
In[ ]:= Nabla[ξ_][i_] := ξ[i + 1] - ξ[i]
```

Proposition 6 (Annihilation property)

```
In[ ]:= α = Nabla[ξ][i - 1] / Nabla[ξ][i];
β = Nabla[ξ][i + 1] / Nabla[ξ][i];
```

$$A = \frac{\beta + 1}{\alpha (\alpha + \beta + 2)};$$

$$B = \frac{\alpha + 1}{\beta (\alpha + \beta + 2)};$$

```
(*Π2 basis*)
FF = {(1 + 0 #) &, (#) &, (#^2) &};
(*Check annihilation for each basis element*)
Table[F = FF[[i]];
  Print[A (F[ξ[i]] - F[ξ[i - 1]]) - (F[ξ[i + 1]] - F[ξ[i]]) + B (F[ξ[i + 2]] - F[ξ[i + 1]]) == 0 // Simplify]
  , {i, 1, Length[FF]}];
```

```
True
True
True
```

Definition 7 (Lagrange subdivision rule)

In[ ]:=

$$a[-1][\alpha\_, \beta\_] := -\frac{3 (4 \beta + 3)}{64 (\alpha ^\wedge 2 + \alpha) (\alpha + \beta + 1)}$$

$$a[0][\alpha\_, \beta\_] := \frac{3 (4 \alpha + 1) (4 \beta + 3)}{64 \alpha (\beta + 1)}$$

$$a[1][\alpha\_, \beta\_] := \frac{(4 \alpha + 1) (4 \beta + 3)}{64 \beta (\alpha + 1)}$$

$$a[2][\alpha\_, \beta\_] := -\frac{3 (4 \alpha + 1)}{64 (\beta ^\wedge 2 + \beta) (\alpha + \beta + 1)}$$

```
Λ[0][α_, β_][f_] := Sum[a[j][α, β] × f[j], {j, -1, 2}]
Λ[1][α_, β_][f_] := Sum[a[1 - j][β, α] × f[j], {j, -1, 2}]
```

Proposition 8 (Reproduction of Π_3^n)

Λ[0], Λ[1] defined in Definition 7

In[ ]:=
```
α = Nabla[ξ][i - 1] / Nabla[ξ][i];
β = Nabla[ξ][i + 1] / Nabla[ξ][i];
(* Π_3^1 basis*)
FF = {(1 + 0 #) &, (#) &, (# ^ 2) &, (# ^ 3) &};
(*Check the reproduction of each basis element*)
Table[F = FF[[i]];
  Print[F[3 / 4 ξ[i] + 1 / 4 ξ[i + 1]] == Λ[0][α, β][F[ξ[i + #]] &] // Simplify];
  Print[F[1 / 4 ξ[i] + 3 / 4 ξ[i + 1]] == Λ[1][α, β][F[ξ[i + #]] &] // Simplify]
  , {i, 1, Length[FF]}];
```

```
True
True
True
True
True
True
True
True
```

Remark 9 (Coefficients sum 1)

a[j] defined in Definition 7

In[ ]:=
```
Clear[α, β]
Sum[a[j][α, β], {j, -1, 2}] // Simplify
```

Out[ ]=
```
1
```

Remark 10 (Case α=β=1)

Λ[0], Λ[1] defined in Definition 7

In[ ]:=
```
Λ[0][1, 1][f] // Simplify
Λ[1][1, 1][f] // Simplify
```

Out[ ]= $\frac{1}{128}(-7\,f[-1]+105\,f[0]+35\,f[1]-5\,f[2])$

Out[ ]= $\frac{1}{128}(-5\,f[-1]+35\,f[0]+105\,f[1]-7\,f[2])$

Definition 11 (New subdivision rules)

Λ[0], Λ[1] defined in Definition 7

In[ ]:=
```
Clear[A, B, α, β]
num[δ_] := Det[{{Dot[δ[-1], δ[0]], Dot[δ[-1], δ[1]]}, {Dot[δ[1], δ[0]], Dot[δ[1], δ[1]]}}]
den[δ_] := Det[{{Dot[δ[-1], δ[-1]], Dot[δ[-1], δ[1]]}, {Dot[δ[-1], δ[1]], Dot[δ[1], δ[1]]}}]
A[δ_] := If[num[δ] × den[δ] == 0, 1/2, Abs[num[δ]/den[δ]]]
B[δ_] := A[δ[-#] &]
```

$$\alpha[k\_][A\_, B\_] := \mathrm{Max}\left[(1+2^{-k}\rho)^{-1}, \mathrm{Min}\left[1+2^{-k}\rho, \frac{1}{A+\frac{\sqrt{A\,(A+1)\,B\,(B+1)}}{B+1}}\right]\right]$$

```
β[k_][A_, B_] := α[k][B, A]
Ψ[i_][k_][f_] := Λ[i][α[k][A[Nabla[f]], B[Nabla[f]]], β[k][A[Nabla[f]], B[Nabla[f]]]][f]
```

Lemma 14 (Grid refinement)

Suppose that $x \in \left[(1+\rho)^{-1}, 1+\rho\right]$. Since $f(t)=1+\frac{1}{2}(t-1)$ is an increasing function, then $f(x) \in [f((1+\rho)^{-1}), f(1+\rho)]$. Simplifying the interval:

In[ ]:= $\left\{1+\frac{1}{2}\left((1+\rho)^{-1}-1\right), 1+\frac{1}{2}(1+\rho-1)\right\}$ // FullSimplify

Out[ ]= $\left\{\frac{2+\rho}{2+2\,\rho}, \frac{2+\rho}{2}\right\}$

This interval is contained in $\left[(1+\rho/2)^{-1}, 1+\rho/2\right]$, since

In[ ]:= Assuming$\left[\rho \geq 0, \left\{\frac{2+\rho}{2+2\,\rho} \geq (1+\rho/2)^{-1}, \frac{2+\rho}{2} \leq 1+\rho/2\right\}\right.$ // FullSimplify$\left.\right]$

Out[ ]= {True, True}

Theorem 15 (Reproduction of Π_2^η)

First, observe that the function $g(\alpha,\beta) = \left(\frac{\beta+1}{\alpha(\alpha+\beta+2)}, \frac{\alpha+1}{\beta(\alpha+\beta+2)}\right)$, $\alpha,\beta>0$, is bijective:

In[ ]:=

```
g[α_, β_] := {(β + 1)/(α (α + β + 2)), (α + 1)/(β (α + β + 2))}
Reduce[α1 > 0 && β1 > 0 && α2 > 0 && β2 > 0 && g[α1, β1] == g[α2, β2]] // FullSimplify
```

Out[ ]=

$$\beta 2 > 0 \;\&\&\; \beta 1 == \beta 2 \;\&\&\; \alpha 2 > 0 \;\&\&\; \alpha 1 == \frac{-((1+\alpha 2)\,\beta 1\,(2+\beta 1)) + (2+\alpha 2)\,\beta 2 + \beta 2^2}{(1+\alpha 2)\,\beta 1 - \beta 2\,(2+\alpha 2+\beta 2)}$$

The latter can be more simplified using $\beta 1 == \beta 2$:

In[ ]:=

```
α1 == (-((1 + α2) β1 (2 + β1)) + (2 + α2) β2 + β2^2)/((1 + α2) β1 - β2 (2 + α2 + β2)) /. {β2 → β1} // Simplify
```

Out[ ]=

$\alpha 1 == \alpha 2$

Hence, it is bijective for $\alpha,\beta>0$. Its inverse is $g^{-1}(A,B) = \left(\frac{1}{A+\frac{\sqrt{A(A+1)B(B+1)}}{B+1}}, \frac{1}{B+\frac{\sqrt{A(A+1)B(B+1)}}{A+1}}\right)$:

In[ ]:=

```
g1[A_, B_] := {1/(A + Sqrt[A (A+1) B (B+1)]/(B+1)), 1/(B + Sqrt[A (A+1) B (B+1)]/(A+1))}
Assuming[α > 0 && β > 0, Apply[g1, g[α, β]] // FullSimplify]
Assuming[A > 0 && B > 0, Apply[g, g1[A, B]] // FullSimplify]
```

Out[ ]=

$\{\alpha, \beta\}$

Out[ ]=

$\{A, B\}$