Here we support some of the computations that appear in the manuscript (see README).

1. Introduction

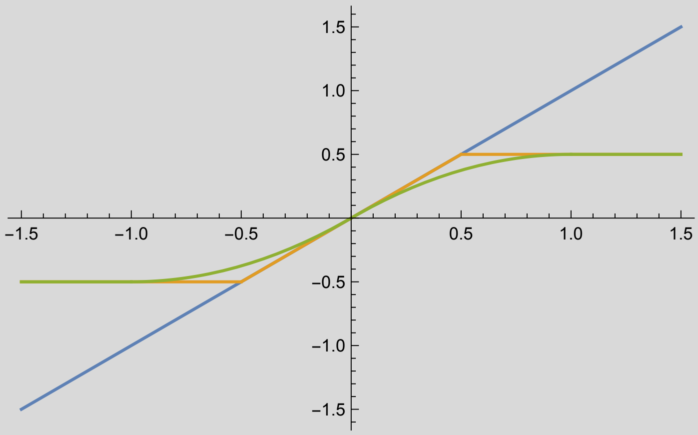
In[1]:=

```
σB₁[t_] := Piecewise[{{-1/2, t <= -1/2}, {t, t <= 1/2}}, 1/2]
           función a trozos

σB₂[t_] := Piecewise[{{-1/2, t <= -1}, {t (1 - Abs[t] / 2), t <= 1}}, 1/2]
           función a trozos                 valor absoluto

id[t_] := t
```

In[4]:=

```
Plot[{t, σB₁[t], σB₂[t]}, {t, -1.5, 1.5}]
representación gráfica
```

Out[4]=



Refinability

In[5]:=

```
σB₁[t] == 1/2 σB₁[2 t + 1/2] + 1/2 σB₁[2 t - 1/2] // Simplify
                                                    simplifica

σB₂[t] == 1/4 σB₂[2 t + 1] + 1/2 σB₂[2 t] + 1/4 σB₂[2 t - 1] // FullSimplify
                                                                simplifica completa

A = 2;
τ = (A - 1) / 2;
id[t] == Sum[1/(2 A) id[2 t + τ - l], {l, 0, A - 1}] // FullSimplify
         suma                                          simplifica completamente
```

Out[5]=

```
True
```

Out[6]=

```
True
```

Out[7]=

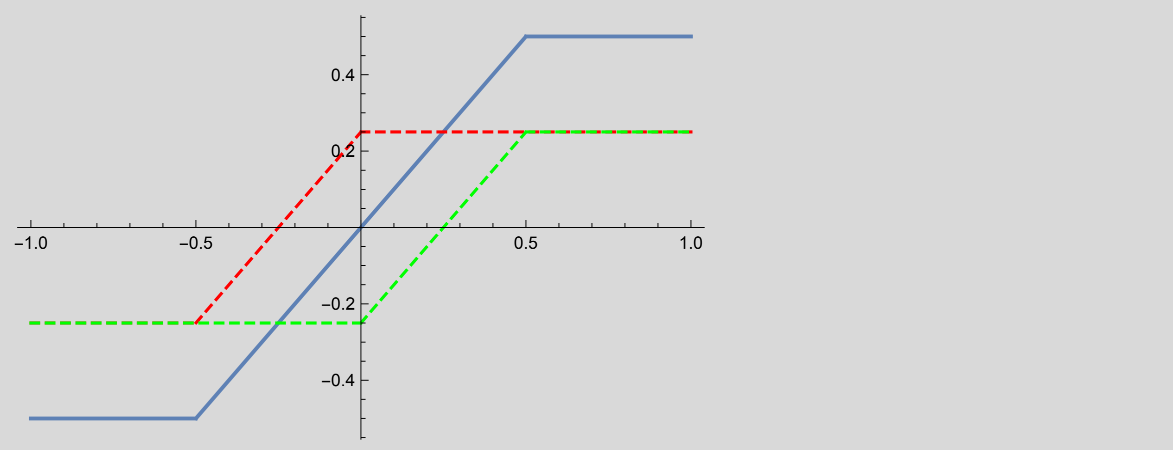
```
True
```

Figure 1

Refinability property of σB_1:

In[8]:=
```
p1 = Plot[σB₁[t], {t, -1, 1}, PlotStyle → Thick];
       ⌊representación gráfica          ⌊estilo de repre··· ⌊grueso
p2 = Plot[1 / 2 σB₁[2 t + 1 / 2], {t, -1, 1}, PlotStyle → {Dashed, Red}];
       ⌊representación gráfica                        ⌊estilo de represe·· ⌊rayado   ⌊rojo
p3 = Plot[1 / 2 σB₁[2 t - 1 / 2], {t, -1, 1}, PlotStyle → {Dashed, Green}];
       ⌊representación gráfica                        ⌊estilo de represe·· ⌊rayado   ⌊verde
Show[p1, p2, p3]
⌊muestra
```

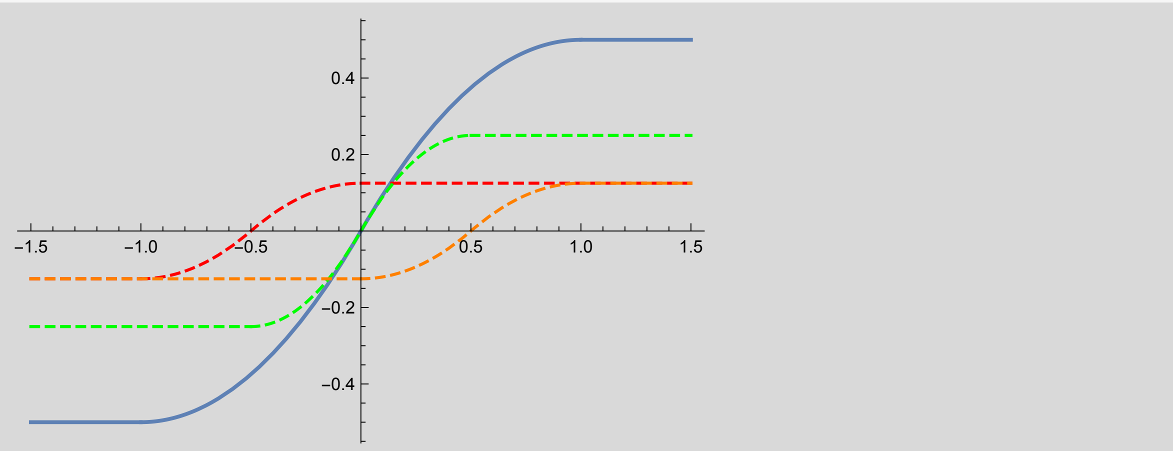
Out[11]=



Refinability property of σB_2 :

In[12]:=
```
p1 = Plot[σB₂[t], {t, -1.5, 1.5}, PlotStyle → Thick];
       ⌊representación gráfica             ⌊estilo de repre··· ⌊grueso
p2 = Plot[{1 / 4 σB₂[2 t + 1]}, {t, -1.5, 1.5}, PlotStyle → {Dashed, Red}];
       ⌊representación gráfica                          ⌊estilo de represe·· ⌊rayado   ⌊rojo
p3 = Plot[{1 / 2 σB₂[2 t]}, {t, -1.5, 1.5}, PlotStyle → {Dashed, Green}];
       ⌊representación gráfica                       ⌊estilo de represe·· ⌊rayado   ⌊verde
p4 = Plot[{1 / 4 σB₂[2 t - 1]}, {t, -1.5, 1.5}, PlotStyle → {Dashed, Orange}];
       ⌊representación gráfica                          ⌊estilo de represe·· ⌊rayado   ⌊naranja
Show[p1, p2, p3, p4]
⌊muestra
```

Out[16]=

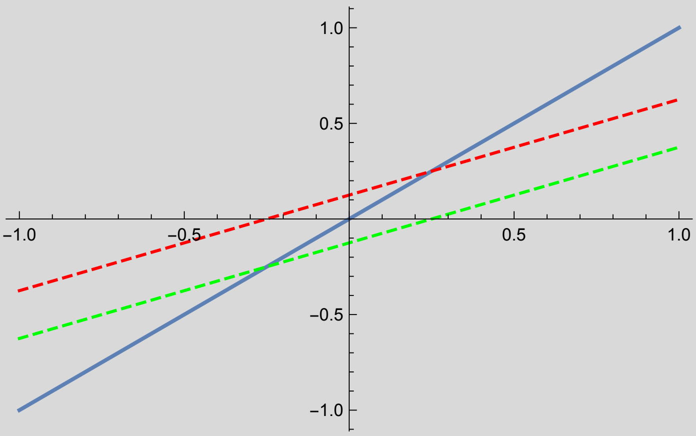


Refinability property of id :

In[17]:=

```
p1 = Plot[id[t], {t, -1, 1}, PlotStyle → Thick];
      representación gráfica          estilo de repre··· grueso
p2 = Plot[1/4 id[2 t + 1/2], {t, -1, 1}, PlotStyle → {Dashed, Red}];
      representación gráfica                      estilo de represe·· rayado  rojo
p3 = Plot[1/4 id[2 t - 1/2], {t, -1, 1}, PlotStyle → {Dashed, Green}];
      representación gráfica                      estilo de represe·· rayado  verde
Show[p1, p2, p3]
muestra
```

Out[20]=



Summing the identity

In[21]:=

```
Assuming[-1/2 <= t <= 1/2, σB₁[t] == t // Simplify]
asumiendo                              simplifica

Assuming[-1/2 <= t <= 1/2, σB₂[t + 1/2] + σB₂[t - 1/2] == t // Simplify]
asumiendo                                                    simplifica
```

Out[21]=

```
True
```

Out[22]=

```
True
```

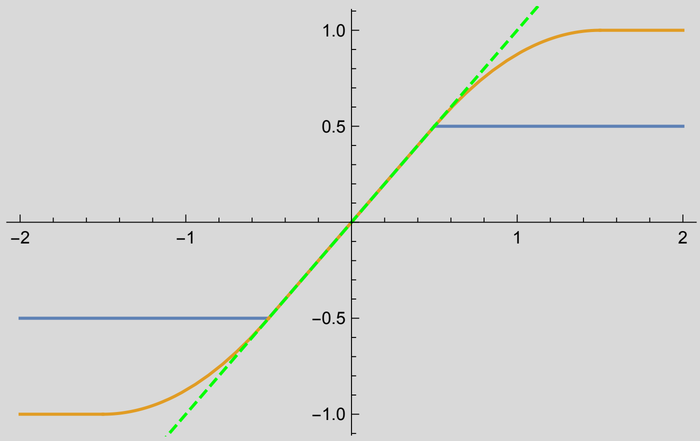
Visual verification:

In[23]:=

```
p1 = Plot[{σB₁[t], σB₂[t + 1/2] + σB₂[t - 1/2]}, {t, -2, 2}];
p2 = Plot[t, {t, -2, 2}, PlotStyle → {Dashed, Green}];
Show[p1, p2]
```

representación gráfica · estilo de represe·· · rayado · verde · muestra

Out[25]=



3.1 B - Spline - based activation functions

Explicit definition

In[26]:=

$$\phi B[d\_][t\_] := \mathrm{Sum}\left[\frac{(-1)^l}{d!}\,\mathrm{Binomial}[d+1, l]\,\mathrm{Max}[t-l, 0]^\wedge d, \{l, 0, d+1\}\right]$$

suma · número binomial · máximo

Definition by convolution

In[27]:=

```
Φ0[t_] := Piecewise[{{1, 0 <= t < 1}}, 0]
ΦB[d_][t_] := If[d > 0 && d ∈ Integers,
  Convolve[ΦB[d - 1][tmp[d]], Φ0[tmp[d]], tmp[d], t], Φ0[t]]
```

función a trozos · si · números enteros · convoluciona

In[29]:=

```
ΦB[0][t] // FullSimplify
```

simplifica completamente

Out[29]=

$$\begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{True} \end{cases}$$

In[30]:=
```
ΦB[1][t] // FullSimplify
ΦB[1][t] == Max[0, 1 - Abs[t - 1]] // FullSimplify
```
simplifica completamente; máximo; valor absoluto; simplifica completa

Out[30]=
$$\begin{cases} 2-t & 1 \le t < 2 \\ t & 0 < t < 1 \\ 0 & \text{True} \end{cases}$$

Out[31]=
```
True
```

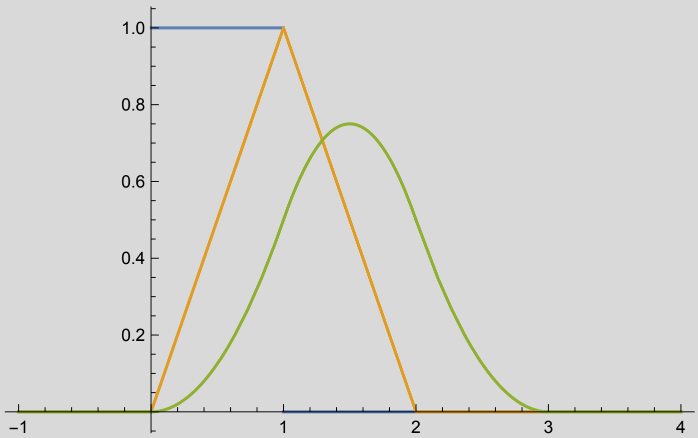
In[32]:=
```
ΦB[2][t] // FullSimplify
```
simplifica completamente

Out[32]=
$$\begin{cases} \frac{t^2}{2} & 0 < t \le 1 \\ -\frac{3}{2} - (-3+t)\, t & 1 < t < 2 \\ \frac{1}{2} (-3+t)^2 & 2 \le t < 3 \\ 0 & \text{True} \end{cases}$$

In[33]:=
```
ΦB[3][t] // FullSimplify
```
simplifica completamente

Out[33]=
$$\begin{cases} \frac{t^3}{6} & 0 < t \le 1 \\ \frac{2}{3} - \frac{1}{2} (-2+t)^2\, t & 1 < t \le 2 \\ -\frac{1}{6} (-4+t)^3 & 3 \le t < 4 \\ -\frac{22}{3} + \frac{1}{2} t \left(20 + (-8+t)\, t\right) & 2 < t < 3 \\ 0 & \text{True} \end{cases}$$

In[34]:=
```
Plot[{ΦB[0][t], ΦB[1][t], ΦB[2][t]}, {t, -1, 4}]
```
representación gráfica

Out[34]=


Check that both definitions coincide:

In[35]:=
```
Table[ΦB[d][t] - ϕB[d][t], {d, 1, 3}] // FullSimplify
tabla                                   simplifica completamente
```
Out[35]=
```
{0, 0, 0}
```

Refinability

In[36]:=
```
Table[
tabla
 ϕB[d][t] - Sum[2^(-d) Binomial[d + 1, l] ϕB[d][2 t - l], {l, 0, d + 1}] //
            suma       número binomial
  FullSimplify,
  simplifica completamente
 {d,
  1,
  2}]
```
Out[36]=
```
{0, 0}
```

Sum

In[37]:=
```
d = 1; n = 10;
Assuming[-n + 1 ≤ t ≤ n + 1, 1 == Sum[ϕB[d][t - i], {i, -n, n}] // FullSimplify]
asumiendo                         suma                          simplifica completam
Assuming[-n + 1 ≤ t ≤ n + 1,
asumiendo
 t - (d + 1)/2 == Sum[i ΦB[d][t - i], {i, -n, n}] // FullSimplify]
                  suma                              simplifica completamente
```
Out[38]=
```
True
```
Out[39]=
```
True
```

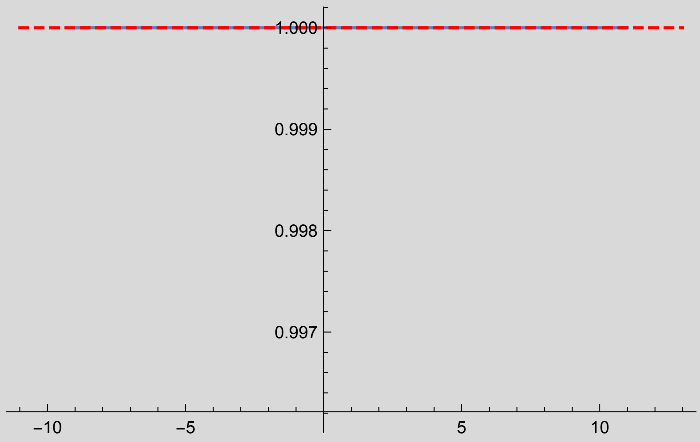
In[40]:=
```
d = 2; n = 10;
Assuming[-n + 2 ≤ t ≤ n + 1, 1 == Sum[ΦB[d][t - i], {i, -n, n}] // FullSimplify]
asumiendo                         suma                          simplifica completam
Assuming[-n + 2 ≤ t ≤ n + 1,
asumiendo
 t - (d + 1)/2 == Sum[i ΦB[d][t - i], {i, -n, n}] // FullSimplify]
                  suma                              simplifica completamente
```
Out[41]=
```
True
```
Out[42]=
```
True
```

Visual verification :

In[43]:=

```
n = 10;
p1 = Plot[Sum[ϕB[1][t - i], {i, -n, n}], {t, -n - 1, n + 3}];
       repr···  suma
p2 = Plot[1, {t, -n - 1, n + 3}, PlotStyle -> {Dashed, Red}];
       representación gráfica       estilo de represen··  rayado   rojo
Show[p1, p2]
muestra
```

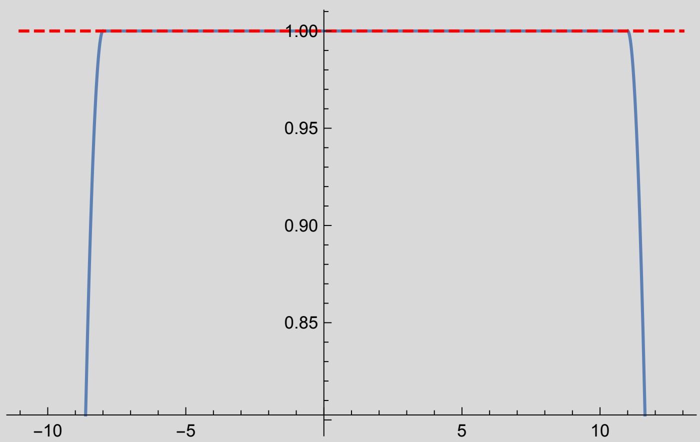
Out[46]=



In[47]:=

```
n = 10;
p1 = Plot[Sum[ϕB[2][t - i], {i, -n, n}], {t, -n - 1, n + 3}];
       repr···  suma
p2 = Plot[1, {t, -n - 1, n + 3}, PlotStyle -> {Dashed, Red}];
       representación gráfica       estilo de represen··  rayado   rojo
Show[p1, p2]
muestra
```

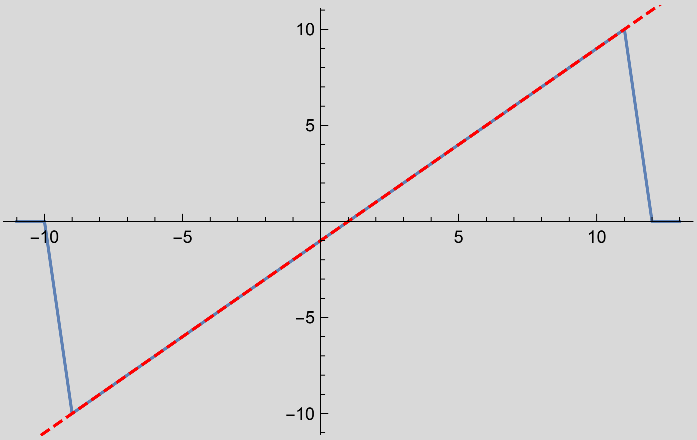
Out[50]=



In[51]:=

```
d = 1; n = 10;
p1 = Plot[Sum[i ϕB[d][t - i], {i, -n, n}], {t, -n - 1, n + 3}];
      repr··· suma
p2 = Plot[t - (d + 1)/2, {t, -n - 1, n + 3}, PlotStyle -> {Dashed, Red}];
      representación gráfica          estilo de represen··· rayado   rojo

Show[p1, p2]
muestra
```

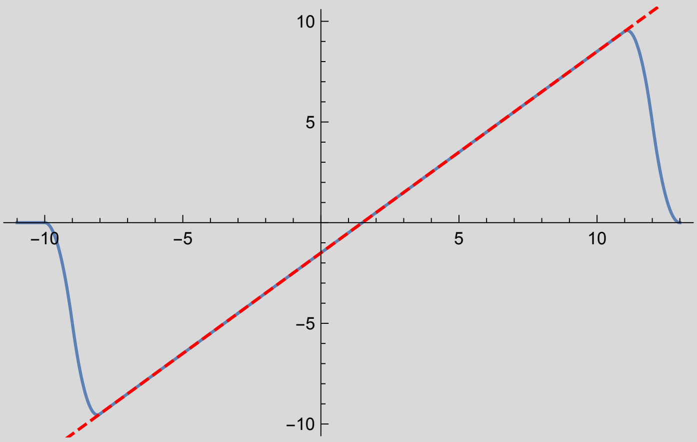
Out[54]=



In[55]:=

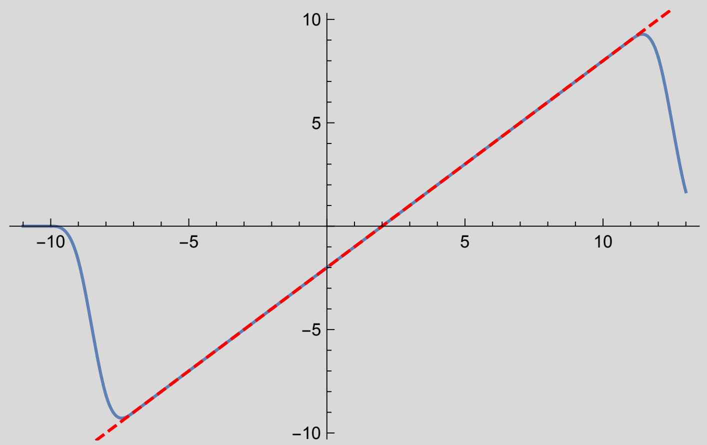
```
d = 2; n = 10;
p1 = Plot[Sum[i ϕB[d][t - i], {i, -n, n}], {t, -n - 1, n + 3}];
      repr··· suma
p2 = Plot[t - (d + 1)/2, {t, -n - 1, n + 3}, PlotStyle -> {Dashed, Red}];
      representación gráfica          estilo de represen··· rayado   rojo

Show[p1, p2]
muestra
```

Out[58]=



In[59]:=
```
d = 3; n = 10;
p1 = Plot[Sum[i ϕB[d][t - i], {i, -n, n}], {t, -n - 1, n + 3}];
      repr··· suma
p2 = Plot[t - (d + 1) / 2, {t, -n - 1, n + 3}, PlotStyle -> {Dashed, Red}];
      representación gráfica                      estilo de represen·· rayado   rojo
Show[p1, p2]
muestra
```

Out[62]=



Theorem 10

(a)

In[63]:=
```
σB[d_][t_] := Piecewise[
                  función a trozos
  {{-1/2, t <= -d/2}, {-1/2 + Sum[ϕB[d][t + d/2 - m], {m, 0, d - 1}], t < d/2}}, 1/2]
                               suma
```

In[64]:=
```
d = 1; σB₁[t] == σB[d][t] // FullSimplify
                             simplifica completa
d = 2;
σB₂[t] == σB[d][t] // FullSimplify
                      simplifica completamente
```

Out[64]=
```
True
```

Out[65]=
```
True
```

(c)

In[66]:=
```
Table[
 σB[d][t] == Sum[2^-d Binomial[d, l] σB[d][2 t + d/2 - l], {l, 0, d}] //
  FullSimplify,
 {d,
  1,
  3}]
```

Out[66]=
```
{True, True, True}
```

(d)

In[67]:=
```
Table[Assuming[-(B - d + 1)/2 <= t <= (B - d + 1)/2,
  t == Sum[σB[d][t + (B - 1)/2 - l], {l, 0, B - 1}] // FullSimplify],
 {B, d, d + 2},
 {d, 1, 3}]
```

Out[67]=
```
{{True, True, True}, {True, True, True}, {True, True, True}}
```

(e)

In[68]:=
```
Table[
 σB[d][t] - (-1/2 + 1/d! Sum[(-1)^l Binomial[d, l] Max[t + d/2 - l, 0]^d, {l, 0, d}]) //
  FullSimplify,
 {d,
  1,
  3}]
```

Out[68]=
```
{0, 0, 0}
```

(f)

In[69]:=
```
Assuming[t ∈ Reals, Table[
  σB[d + 1][t] == Integrate[σB[d][s], {s, t - 1/2, t + 1/2}] // FullSimplify,
  {d, 1, 3}]]
```
asumiendo · número·· · tabla · integra · simplifica completame

Out[69]=
```
{True, True, True}
```

(g)

In[70]:=
```
Assuming[t ∈ Reals, Table[
  σB[d]'[t] == ΦB[d - 1][t + d/2] // FullSimplify,
  {d, 1, 3}]]
```
asumiendo · número·· · tabla · simplifica completame

Out[70]=
```
{t < -1/2 || -1/2 < t < 1/2 || 2 t > 1, True, True}
```

Proposition 11

In[71]:=
```
Assuming[-1/2 < σB[1][t] < 1/2 // FullSimplify, σB[1]'[t] == 1 // FullSimplify]
```
asumiendo · simplifica completamente · simplifica completam

Out[71]=
```
True
```

In[72]:=
```
Assuming[-1/2 < σB[2][t] <= 0 // FullSimplify,
 σB[2]'[t] == Sqrt[1 + 2 σB[2][t]] // FullSimplify]
```
asumiendo · simplifica completamen · simplifica completam

Out[72]=
```
True
```

In[73]:=
```
Assuming[0 < σB[2][t] < 1/2 // FullSimplify,
 σB[2]'[t] == Sqrt[1 - 2 σB[2][t]] // FullSimplify]
```
asumiendo · simplifica completamente · simplifica completam

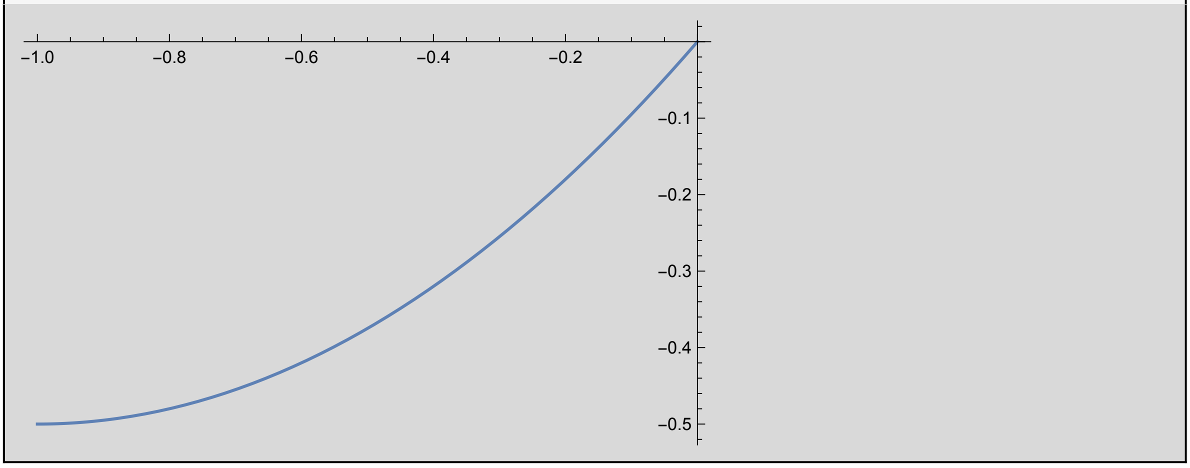
Out[73]=
```
True
```

Assistance for the proof of the case d=2

If $t \in (-1,0]$, then $z = t(1+t/2) \in (-1/2,0]$:

In[74]:=
```
Plot[t (1 + t / 2), {t, -1, 0}]
```
representación gráfica

Out[74]=



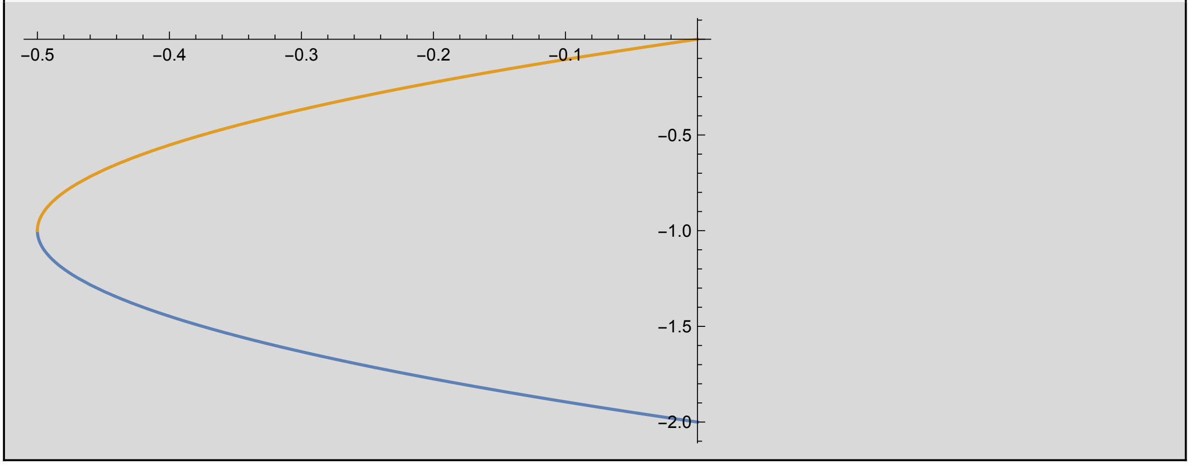
In[75]:=
```
Solve[z == t (1 + t / 2), t]
```
resuelve

Out[75]= $\{\{t \to -1 - \sqrt{1+2z}\}, \{t \to -1 + \sqrt{1+2z}\}\}$

For $z \in (-1/2,0]$, $t = -1 + \sqrt{1+2z} \in (-1,0]$:

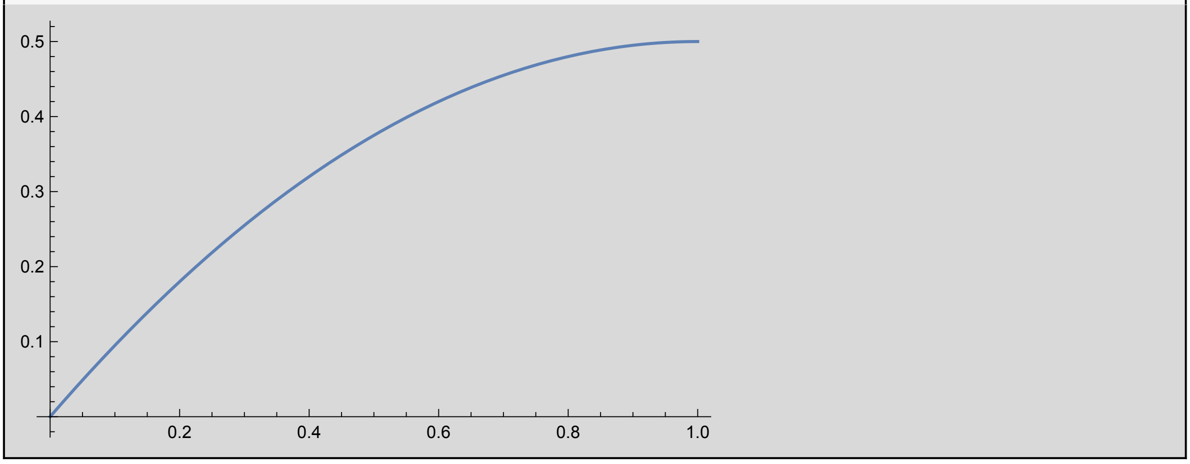
In[76]:=
```
Plot[{-1 - √(1 + 2 z), -1 + √(1 + 2 z)}, {z, -1 / 2, 0}]
```
representación gráfica

Out[76]=



If $t \in (0,1)$, then $z = t\,(1 - t/2) \in (0,1/2)$:

In[77]:=
```
Plot[t (1 - t / 2), {t, 0, 1}]
```
representación gráfica

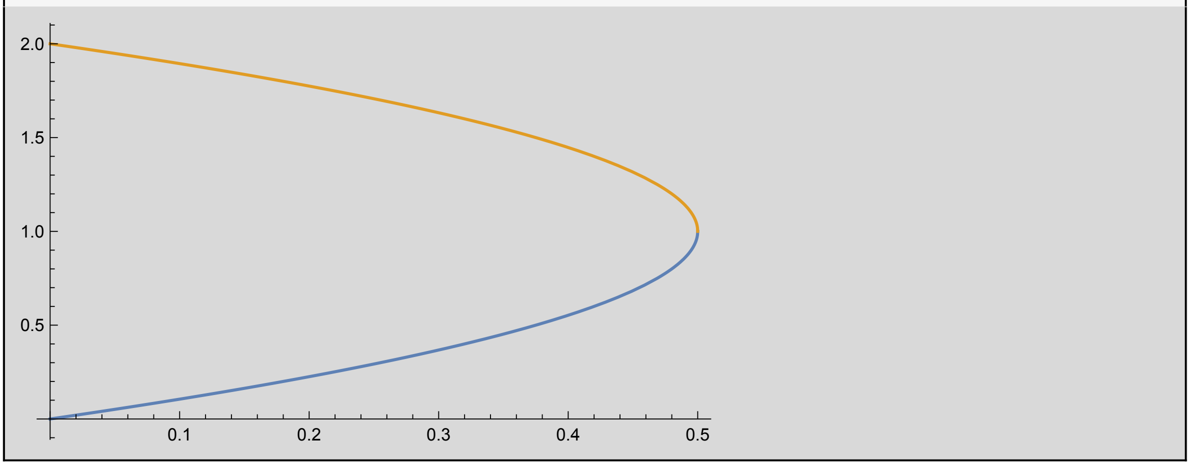
Out[77]=


In[78]:=
```
Solve[z == t (1 - t / 2), t]
```
resuelve

Out[78]= $\{\{t \to 1 - \sqrt{1 - 2z}\}, \{t \to 1 + \sqrt{1 - 2z}\}\}$

For $z \in (0,1/2)$, $t = 1 - \sqrt{1-2z} \in (0,1)$:

In[79]:=
```
Plot[{1 - √(1 - 2 z), 1 + √(1 - 2 z)}, {z, 0, 1 / 2}]
```
representación gráfica

Out[79]=


Theorem 13

(c)

In[80]:=
```
Table[
tabla
 σB[d][t] - σB[d][t - 1] == ΦB[d][t + d / 2] // FullSimplify,
                                             simplifica completame
 {d, 1, 3}]
```

Out[80]=
```
{True, True, True}
```

(d)

In[81]:=
```
Assuming[t ∈ Reals,
asumiendo  números reales
 Table[ϕB[d][t] == ϕB[d][d + 1 - t] // FullSimplify,
 tabla                                simplifica completame
  {d, 1, 3}]]
```

Out[81]=
```
{True, True, True}
```

In[82]:=
```
Assuming[t ∈ Reals,
asumiendo  números reales
 Table[-σB[d][t] == σB[d][-t] // FullSimplify,
 tabla                          simplifica completame
  {d, 1, 3}]]
```

Out[82]=
```
{True, True, True}
```