Justice for The Squeeze Theorem

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The Squeeze Theorem is a concept that is usually first introduced in a student's Calculus 1 course, and, due to its limit-based foundations, tends to be the first theorem they interact with. In most circumstances it is discussed as a niche case, accompanied by an in-class example or practice problem, and never mentioned again afterwards. The Calculus 1 setting may not harbor the necessity or general interest for a deeper dive into the subject, but the Squeeze Theorem offers far more theory and variety than can be discussed in a short 15 minutes of class lecture. My objective in this paper is to not only explore the depth of the Squeeze Theorem, but to also appreciate its elegance and beauty.

The Squeeze Theorem (or Sandwich Theorem) is a tool that allows us to determine the limit of a sequence or function at a location by squeezing, or sandwiching, it between two similar sequences or functions with known, equivalent limits at that location. By sandwiching the center sequence or function from above and below, we are able to trap its limit between the limits of the two outer sequences or functions, and can then assert that it also has the same limit at that location. As such, we are provided with a method to determine the limit of a sequence or function at a location even if we are not able to easily compute it. Fascinatingly, this can be used to determine limits at a specific point or end behavior.

In this paper we will investigate the formal definition of both versions of the Squeeze Theorem as is introduced in an analysis setting, prove them, and explore examples of each. I hope to lead readers past the point of quick intuition, yet rather to deep understanding.

We will start with our first form, The Squeeze Theorem for Sequences.

Theorem: Squeeze Theorem for Sequences. If $x_n \leq y_n \leq z_n \, \forall \, n \in \mathbb{N}$, and if $\lim_{n\to c} x_n = \lim_{n\to c} z_n = L$, then $\lim_{n\to c} y_n = L$ also, where c is any limit point or infinity.

Proof. Recall the definition for Convergence of a Sequence (Abbott Definition 2.2.3), which states that a sequence (a_n) converges to a real number a if, $\forall$ positive ε, $\exists$ an $\hat{n} \in \mathbb{N}$ s.t. whenever $n \geq \hat{n}$, it follows that $|a_n - a| < \varepsilon$. Let $\varepsilon > 0$ be given. We can now use this to say that $\exists \, \hat{n} \in \mathbb{N} \, \forall \, n > \hat{n}$ for both $|x_n - L| < \varepsilon$ and $|z_n - L| < \varepsilon$. Consider,

$$\begin{aligned} -\varepsilon &< |x_n - L| < \varepsilon \\ = L - \varepsilon &< x_n < L + \varepsilon \end{aligned} \qquad\qquad \begin{aligned} -\varepsilon &< |z_n - L| < \varepsilon \\ = L - \varepsilon &< z_n < L + \varepsilon \end{aligned}$$

Using our given $x_n \leq y_n \leq z_n$, we can write $L - \varepsilon < x_n \leq y_n \leq z_n < L + \varepsilon$. This implies $-\varepsilon < y_n - L < \varepsilon$, which $= |y_n - L| < \varepsilon$, matching the definition of $\lim_{n\to\infty} y_n = L$. We have now shown that given $x_n \leq y_n \leq z_n$, where $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = L$, $\lim_{n\to\infty} y_n$ is also L. This is our Squeeze Theorem for Sequences.

□

An example of this, and of an end-behavior scenario, is for the sequence $(a_n) = -1^n(\frac{1}{n!})$. Choose $(b_n) = \frac{1}{n!}$ and $(c_n) = -\frac{1}{n!}$ as our sandwich sequences. It is clear that $\lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} c_n = 0$, but it is less obvious what $\lim_{n\to\infty} a_n$ should be. We know $c_n \leq a_n \leq b_n$, and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 0$. We can squeeze (a_n) between (b_n) and (c_n) to get $0 \leq \lim_{n\to\infty} a_n \leq 0$, and thereby conclude that $\lim_{n\to\infty} a_n$ must also be 0. This is depicted in Figure 1 below, where (b_n) is represented by the orange dots, (c_n) is represented by the blue dots, and (a_n) is represented by the larger black rings. We can see that (a_n) remains trapped between the two sandwich sequences, and they all approach the same limit towards infinity.

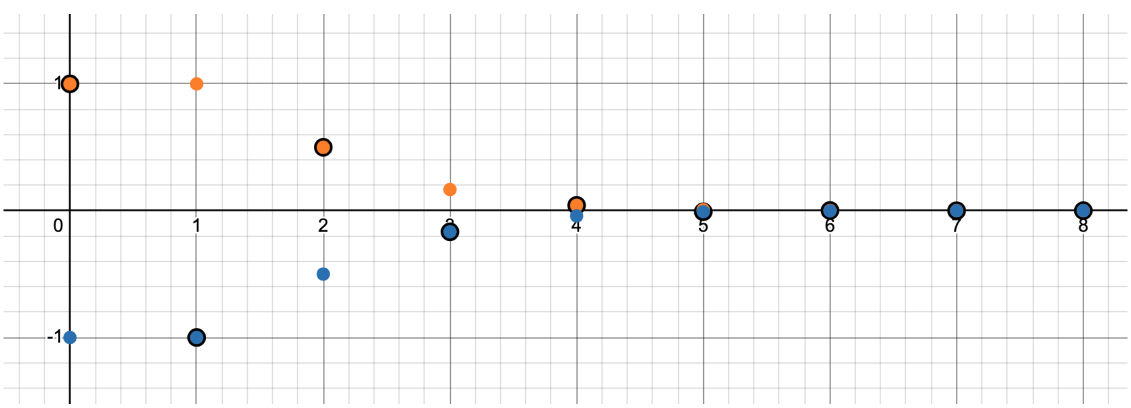

Figure 1: The Squeeze Theorem for Sequences: The limit as $(a_n) = -1^n(\frac{1}{n!})$ (black) approaches infinity is determined to be 0 by using $(b_n) = \frac{1}{n!}$ (orange) and $(c_n) = -\frac{1}{n!}$ (blue) as sandwich sequences, each of which also have a limit of 0 approaching infinity.

Now we will investigate our second form, The Squeeze Theorem for Functions.

Theorem: Squeeze Theorem for Functions. For a given f, g, and h where $f(x) \leq g(x) \leq h(x)\ \forall\ x \in A$, if $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$ at a limit point $c \in A$, then $\lim_{x\to c} g(x) = L$ also.

Proof. Refer to the definition of a functional limit (Abbott Definition 4.2.1), which states that $\lim_{x\to c} f(x) = L$ provided that $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ s.t. whenever $0 < |x-c| < \delta$, it follows $|f(x) - L| < \varepsilon$. By this definition, we will establish a δ_f s.t. $0 < |x - c| < \delta_f$ and $|f(x) - L| < \frac{\varepsilon}{3}$. Also establish a δ_h s.t. $0 < |x-c| < \delta_h$ and $|h(x) - L| < \frac{\varepsilon}{3}$. (Note that we have let ε be > 0 and arbitrary.) We WTS: $|g(x) - L| < \frac{\varepsilon}{3}$. Rewrite this $|g(x) - L|$ and apply the Triangle inequality. Then continue to simplify using the given information:

$$|g(x) - f(x) + f(x) - L| \leq |g(x) - f(x)| + |f(x) - L| \leq |h(x) - f(x)| + |f(x) - L|$$

$$= |h(x) - L + L - f(x)| + |f(x) - L| \leq |h(x) - L| + |L - f(x)| + |f(x) - L|$$
$$= |h(x) - L| + 2|f(x) - L|$$

Now we have $|g(x) - L| \leq |h(x) - L| + 2|f(x) - L| < \frac{\varepsilon}{3} + 2(\frac{\varepsilon}{3}) = \varepsilon$. By the functional definition of the limit, this shows that $\lim_{x\to c} g(x) = L$ for our given conditions. □

An example of this at a given limit point is for the function $g(x) = x^2 sin\frac{1}{x}$ at (0,0). As depicted in Figure 2, the function (black) oscillates wildly as it approaches the origin and is undefined at that point. To resolve this, we can choose similar functions with known limits at the point to sandwich $g(x)$ between. Use $h(x) = x^2$ (orange) for the upper side, and $f(x) = -x^2$ (blue) for the lower side. According to Abbott's Definition 4.2.1 we know $\lim_{n\to 0} h(x) = \lim_{n\to 0} f(x) = 0$. The sandwich of $g(x)$ between $f(x)$ and $h(x)$ goes as follows: $-x^2 \leq x^2 sin\frac{1}{x} \leq x^2$, where $\lim_{n\to 0} f(x) \leq \lim_{n\to 0} g(x) \leq \lim_{n\to 0} h(x) = 0 \leq \lim_{n\to 0} g(x) \leq 0$. So, we can conclude $\lim_{n\to 0} g(x)$ is also 0.

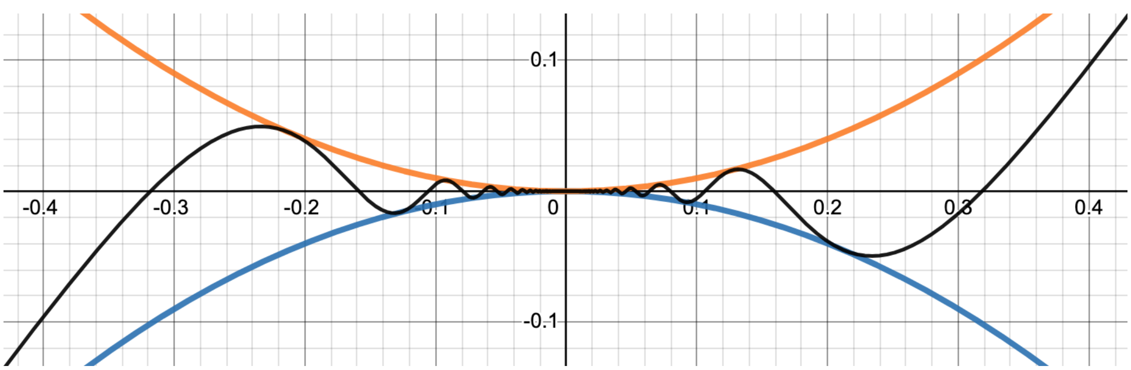

Figure 2: The Squeeze Theorem for Functions: The value of $g(x) = x^2 sin\frac{1}{x}$'s (black) limit approaching 0 is determined to be 0 by squeezing it between $h(x) = x^2$ (orange) and $f(x) = -x^2$ (blue), two functions with known limits of 0 at that point.

Now, consider Abbott's Definition 6.2.1.: For each $n \in \mathbb{N}$, let f_n be a function defined on a set $A \subseteq \mathbb{R}$. The sequence (f_n) of functions converges pointwise on A to a function f if, for all $x \in A$, the sequence of real numbers $f_n(x)$ converges to $f(x)$.

With this, it is clear to see that the next step of this discussion is to begin considering how sequences of functions can be applied to the Squeeze Theorem. And as a Calculus concept, we can of course consider working in more dimensions. The Squeeze Theorem also has notable (or at least interesting) geometric applications in relation to the Unit Circle and various trigonometric proofs that merit further investigation in the future. As is always the case in math, the methods discussed in this paper are far from the only way to prove each concept and there are an endless number of examples and real-world applications yet to explore. However, I will currently leave all of these practices up to the reader.

We have now reviewed the complete definition and proofs for the The Squeeze Theorem for both sequences and functions, viewed examples that apply to specific limit points or general end behavior, and even considered further expansions of this concept. This quick investigation is purely a depiction of the Squeeze Theorem's charm; we have managed to arrive at the same mathematical principal despite navigating entirely separate environments and have discovered a surprising amount of versatility. This type of wonder is not completely unique in the math world, but for some reason we fail to appreciate it whenever the time comes to discuss the Squeeze Theorem. Not only does this leave us bored, it starves us from the joy of understanding which is perhaps the biggest crime of all.

Special shout out to Dr.Shelley B. Rohde Poole for sparking my interest by opening my eyes to new examples of The Squeeze Theorem that I had never seen before, and to Dr.John Ethier for wrongfully slandering both analysis and the Squeeze Theorem (he is a great professor though) and setting me on a war path to shout its praises to the world.

Sources: Abbott Understanding Analysis, Openstax Calculus Volume 1, Desmos
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