A Multivariable Approach to The Squeeze Theorem

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This paper serves as an extension to my first paper on the subject, "Justice for the Squeeze Theorem". In it, I reviewed the concepts behind the Squeeze Theorem and proved it for both sequences and functions in $\mathbb{R}$. I concluded my paper with a discussion on possible future examinations into more applications and forms of the Squeeze Theorem, one of which being a higher-order proof of the function form. We will accomplish that today by proving the Squeeze Theorem in $\mathbb{R}^3$ and evaluating its implications.

The main difficulty in this proof was the conceptual side. The Squeeze Theorem in $\mathbb{R}$ is fairly intuitive, but shifting this idea into higher dimensions quickly became challenging and truly forced me to reexamine what I knew about $\delta - \varepsilon$ proofs and even Calculus 3. I spent a very long time trying several different unsuccessful approaches before finally finding a definition to guide me and drawing everything out on a whiteboard.

We will use Kosmala's Definition 10.2.1, the limit of a function of two variables in $\mathbb{R}^3$:

> For $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$, where (a,b) is an accumulation point of D, $\lim_{\substack{x\to a\\ y\to b}} f(x,y) = L$ iff $\forall$ $\varepsilon > 0$ $\exists \delta > 0$ s.t. $|f(x,y) - L| < \varepsilon$, whenever (x,y) $\in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$.

For the Squeeze Theorem in a 3D setting, we will have three surfaces stacked ontop of each other, and in close proximity on the z axis (given x and y inputs) at a given point; if this were not the case, the surfaces would fall out of the ε-bounds and a limit could not be defined.

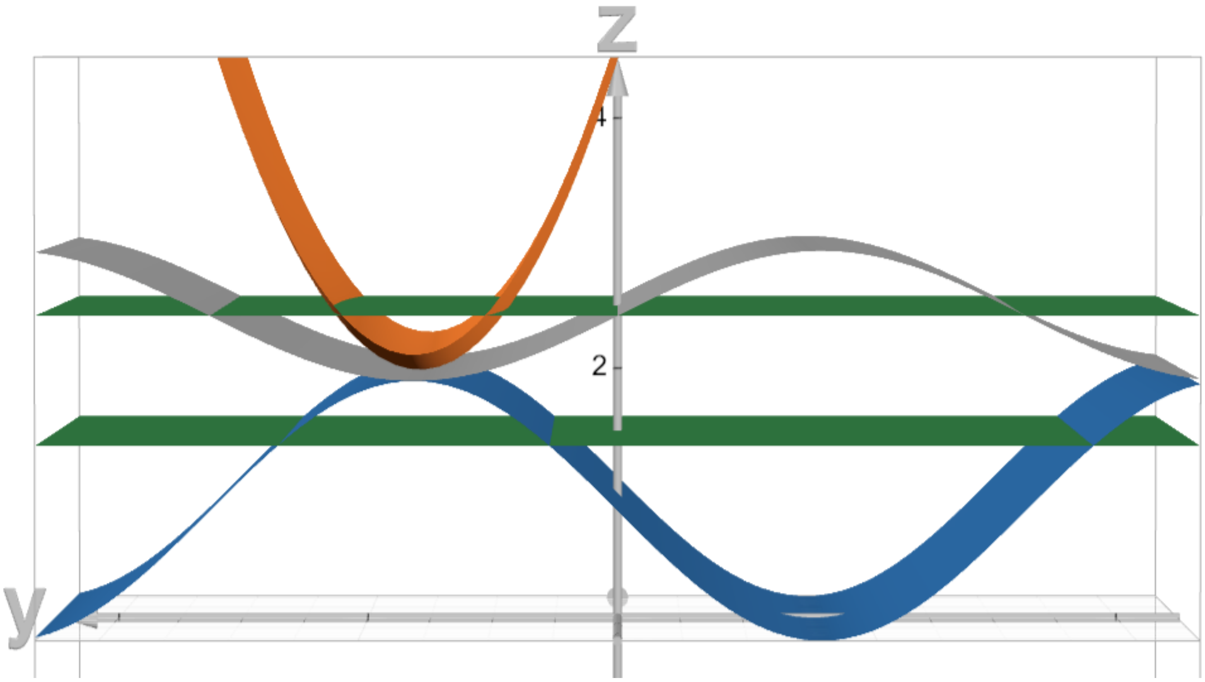

Figure 1: The graphs of $z = (0.1x)^2 + (y - 1.57)^2 + 2$ (orange), $z = -\frac{1}{2}sin(y) + \frac{5}{2}$ (gray) , $z = sin(y) + 1$ (blue), with arbitrary ε bounds (green) established. Here, we can see that all three surfaces fall within the ε bounds on the left-hand side, but not the right-hand side. So, the limit can only be established on the left-hand side.

Prior to this, I had attempted to use methods that more-or-less simply increased the *size* of the proof, but did not account for its *shape*. Regardless of the proof method I attempted, my source of failure was that I was simply adding more variables to a still-flat setting.

Recall Abbott's Definition 4.2.1, the functional limit in $\mathbb{R}$:

> Let $f : A \to \mathbb{R}$, and let c be a limit point of the domain A. We say that $\lim_{x \to c} f(x) = L$ provided that, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ (and $x \in A$), it follows that $|f(x) - L| < \varepsilon$

Abbott's $\varepsilon - \delta$ definition of a limit in $\mathbb{R}$ uses a single input to create a δ-neighborhood on the graph of the function. The y-values of this mapping must land within the established bounds of L's ε-neighborhood in order to define the limit. Kosmala's $\varepsilon - \delta$ definition of a limit in $\mathbb{R}^3$ uses two input variables from $\mathbb{R}^2$ to create a δ-neighborhood that takes the shape of a ring, or δ-disk. The δ-disk maps to the surface of the graph of the function, and the z-values of this mapping must sit inside L's established ε-neighborhood in order to define the limit.

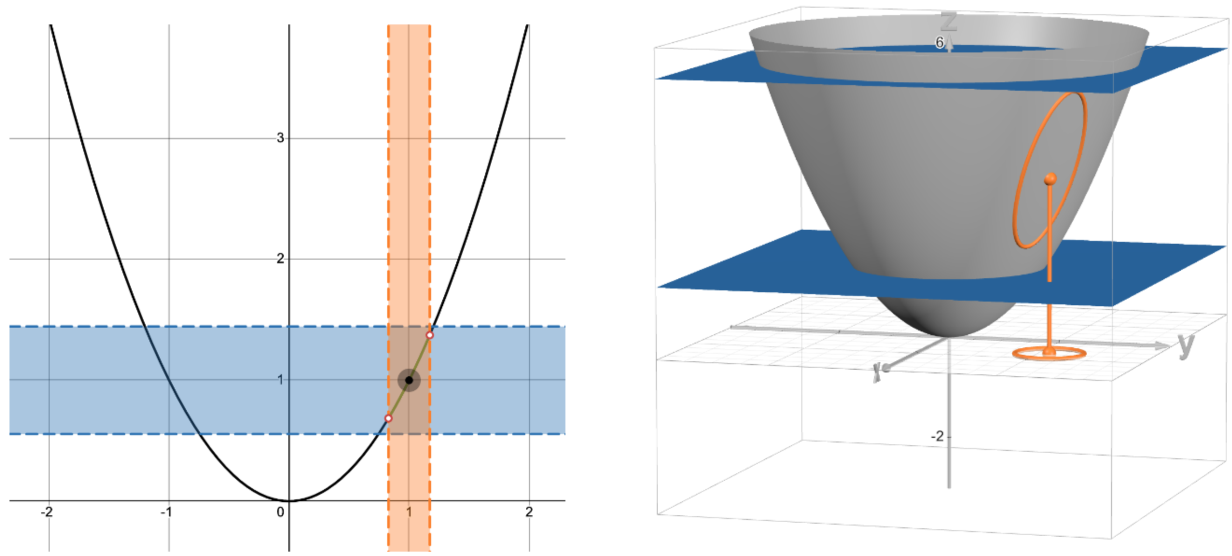

Figure 2: The $\varepsilon - \delta$ limits of $y = x^2$ in $\mathbb{R}$ (left), and $z = x^2 + y^2$ in $\mathbb{R}^3$ (right), where the ε conditions are represented in blue and the δ conditions are shown in orange. We can see the similar mapping and $\varepsilon - \delta$ interactions for both conditions, but in their own distinct forms. For $y = x^2$ in $\mathbb{R}^2$ (with inputs from $\mathbb{R}$), this occurs in a flat setting and generates corresponding flat regions. For $z = x^2 + y^2$ in $\mathbb{R}^3$, the δ-disk creates a cylindrical region surrounding the limit point.

With all of this in hand, I was ready to return to my proof. My initial proof for the function form utilized the triangle inequality, but the $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ term in Kosmala's definition steered me away from that method for $\mathbb{R}^3$. To find a method that was less prone to algebra disasters, I started by returning to $\mathbb{R}$ once again. After finding a more streamlined method to prove the Squeeze Theorem, I believed I would be able to transfer into $\mathbb{R}^3$. All I had left to do was to craft a definition for the Squeeze Theorem in $\mathbb{R}^3$ and use my method to prove it.

Theorem: Squeeze Theorem for Functions in $\mathbb{R}$. For $g, f, h : A \to \mathbb{R}$ $\forall$ x, c $\in$ an interval; A if $g(x) \leq f(x) \leq h(x)$ and $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} f(x) = L$ also.

Proof. We will refer to Abbott's Definition 4.2.1, the definition of a Functional Limit, throughout this proof: For $f : A \to \mathbb{R}$, where c is a limit point of the domain A, $\lim_{x\to c} f(x) = L$ provided that, $\forall\ \varepsilon > 0$, $\exists$ a $\delta > 0$ s.t. whenever $0 < |x - c| < \delta$ (and $x \in A$), it follows that $|f(x) - L| < \varepsilon$.

For our proof, we WTS $\exists\ \delta > 0$ s.t. $|f(x) - L| < \varepsilon$ when $0 < |x - c| < \delta$. Let $\varepsilon > 0$ be given and arbitrary.

By our given and Definition 4.2.1, we know that for $\lim_{x\to c} g(x) = L$, $\exists\ \delta_g > 0$ s.t. $|g(x) - L| < \varepsilon\ \forall\ 0 < |x - c| < \delta_g$. We can rewrite this as $-\varepsilon < g(x) - L < \varepsilon = L - \varepsilon < g(x) < L + \varepsilon\ \forall\ 0 < |x - c| < \delta_g$.

Likewise, we know that for $\lim_{x\to c} h(x) = L$, $\exists$ a $\delta_h > 0$ s.t. $|h(x) - L| < \varepsilon\ \forall\ 0 < |x - c| < \delta_h$. We can rewrite this as $-\varepsilon < h(x) - L < \varepsilon = L - \varepsilon < g(x) < L + \varepsilon\ \forall\ 0 < |x - c| < \delta_h$.

Now, let $\delta_f = \min\{\delta_g, \delta_h\}$. We can use what we have just shown to assert $L - \varepsilon < g(x) \leq f(x) \leq h(x) < L + \varepsilon\ \forall\ 0 < |x - c| < \delta_f$. This also implies $L - \varepsilon < f(x) < L + \varepsilon = -\varepsilon < f(x) - L < \varepsilon = |f(x) - L| < \varepsilon\ \forall\ 0 < |x - c| < \delta_f$, which matches our Definition 4.2.1. Thus, we have shown that for $g(x) \leq f(x) \leq h(x)$ where $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$, $\lim_{x\to c} h(x)$ is also L. ◻

Claim. **Theorem: Squeeze Theorem for Surfaces in $\mathbb{R}^3$.** For $g, f, h : D \subseteq R^2 \to R$ $\forall$ (x,y), (a,b) $\in D$; A if $g(x, y) \leq f(x, y) \leq h(x, y)$ and $\lim_{\substack{x\to a\\ y\to b}} g(x, y) = \lim_{\substack{x\to a\\ y\to b}} h(x, y) = L$, then $\lim_{\substack{x\to a\\ y\to b}} f(x, y) = L$ also.

Proof. For this proof, we will use Kosmala's Definition 10.2.1. Let $\varepsilon > 0$ be given and arbitrary.

According to our claim and Definition 10.1, $|g(x, y) - L| < \varepsilon = -\varepsilon < g(x, y) - L < \varepsilon = L - \varepsilon < g(x, y) < L + \varepsilon\ \forall\ 0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta_g$.

Also, $|h(x, y) - L| < \varepsilon = -\varepsilon < h(x, y) - L < \varepsilon = L - \varepsilon < h(x, y) < L + \varepsilon\ \forall\ 0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta_H$.

Let $\delta_f = \min\{\delta_g, \delta_h\}$. We can now say $L - \varepsilon < g(x, y) \leq f(x, y) \leq h(x, y) < L + \varepsilon\ \forall\ 0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta_f$. This also implies $L - \varepsilon < f(x, y) < L + \varepsilon\ \forall\ 0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta_f$.

This matches Definition 10.1. We have now shown that when $g(x, y) \leq f(x, y) \leq h(x, y)$ and $\lim_{\substack{x\to a\\ y\to b}} g(x, y) = \lim_{\substack{x\to a\\ y\to b}} h(x, y) = L$, $\lim_{\substack{x\to a\\ y\to b}} f(x, y)$ is also L. ◻

We have now successfully proved the Squeeze Theorem in a multivariable setting, $\mathbb{R}^3$. I chose to work in $\mathbb{R}^3$ largely because it was still somewhat comprehensible and I had a limit definition readily available. However, I am interested to see if the method I used transfers beyond just $\mathbb{R}$ and $\mathbb{R}^3$, and if it can be applied to any higher-order setting. This would, of course, require the derivation or location of a new limit definition in each dimension. I am also interested to see if the Squeeze Theorem in $\mathbb{R}^3$ can be proved without this method and using more exclusively higher-order techniques; to rephrase that, I'd like to see how complex this proof can become. At this point I am also simply interested in learning more about what the Squeeze Theorem in higher dimensions may be and what it would look like.

In any dimension, there is still more to explore about the Squeeze Theorem. As previously mentioned, I am still interested in exploring the Squeeze Theorem's application to Unit Circle proofs, Sequences of Functions, the Fundamental Theorem of Calculus, real-world applications, and much, much more. For the time being, however, that will have to wait.

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Sources: Abbott Understanding Analysis, Kosmala A Friendly Introduction to Analysis Single and Multivariable, Desmos
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