Getting Real About Our Sandwiches: Single and Multivariable Approaches to The Squeeze Theorem

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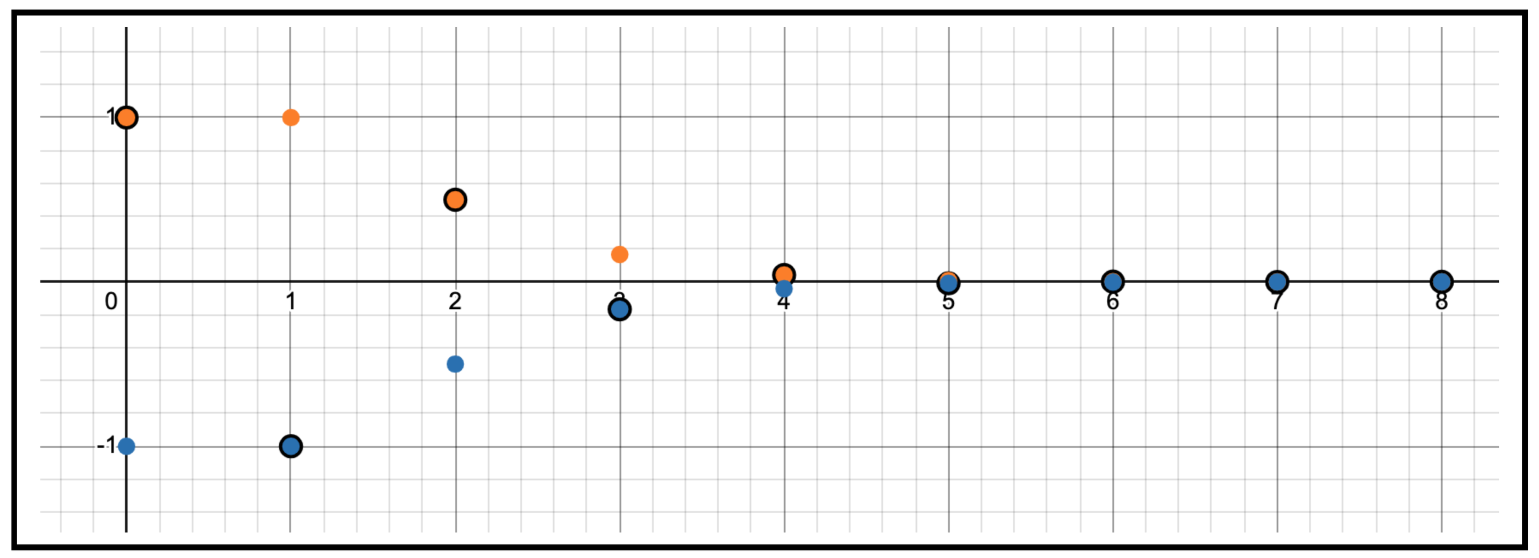
In mathematics, the Squeeze Theorem (or Sandwich Theorem) is a tool that allows us to determine the limit of a sequence or function at a given location by trapping it between two similar sequences or functions. By doing this, we can restrict its limit from above and below with known, equivalent limits, and can therefore assert that it must also have the same limit. This provides us with a method to determine the limit of a sequence or function even if we are not able to easily compute it. The Squeeze Theorem is usually brushed of quickly, but as we will see, it has so much to offer!

Squeeze Theorem for Sequences: For sequences x_n, y_n, z_n, if $x_n \leq y_n \leq z_n$ $\forall\ n \in \mathbb{N}$ and $\lim_{n\to c} x_n = \lim_{n\to c} z_n = L$ where c is any limit point, then $\lim_{n\to c} y_n = L$ also.

For the sequences $(x_n) = -\frac{1}{n!}$ (blue), $(y_n) = -1^n(\frac{1}{n!})$ (black), and $(z_n) = \frac{1}{n!}$ (orange), it is not particularly easy to infer the end behavior of (y_n), but the end behavior of (x_n) and (z_n) are well known. Our squeeze goes as follows: If $(x_n) \leq (y_n) \leq (z_n)$, and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = 0$, then we have $0 \leq \lim_{n\to\infty} y_n \leq 0$ and can conclude that $\lim_{n\to\infty} y_n$ is also 0.



Proof. Recall Abbott's definition for Convergence of a Sequence (Definition 2.2.3), which states that a sequence (a_n) converges to a real number a if, $\forall$ positive ε, $\exists$ an $\hat{n} \in \mathbb{N}$ s.t. whenever $n \geq \hat{n}$, it follows that $|a_n - a| < \varepsilon$. Let $\varepsilon > 0$ be given. We can now use this to say that $\exists\ \hat{n} \in \mathbb{N}\ \forall\ n > \hat{n}$ for both $|x_n - L| < \varepsilon$ and $|z_n - L| < \varepsilon$. Consider,

$$-\varepsilon < |x_n - L| < \varepsilon \qquad\qquad -\varepsilon < |z_n - L| < \varepsilon$$
$$= L - \varepsilon < x_n < L + \varepsilon \qquad\qquad = L - \varepsilon < z_n < L + \varepsilon$$

Using our given $x_n \leq y_n \leq z_n$, we can write $L - \varepsilon < x_n \leq y_n \leq z_n < L + \varepsilon$. This implies $-\varepsilon < y_n - L < \varepsilon$, which is $|y_n - L| < \varepsilon$, and matches the definition of $\lim_{n\to\infty} y_n = L$. We have now shown that given $x_n \leq y_n \leq z_n$, when $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = L$, $\lim_{n\to\infty} y_n$ is also L. This is the Squeeze Theorem for Sequences. □

Thanks to Dr.Diane Davis for talking through the conceptual aspects of this with me, helping me build my graphs, and checking over my proofs; Dr.John Carter for teaching me real analysis and supporting my deep dive into this subject; Dr.Shelley B. Rohde Poole for sparking my interest by opening my eyes to new examples of The Squeeze Theorem that I had never seen before; Dr.John Ethier for wrongfully slandering both analysis and the Squeeze Theorem and setting me on a path to shout its praises to the world; and Dr. Brendan Fry for first teaching me about the Squeeze Theorem. Much Love to the MSU Denver Math Department and thank you for all that you do!

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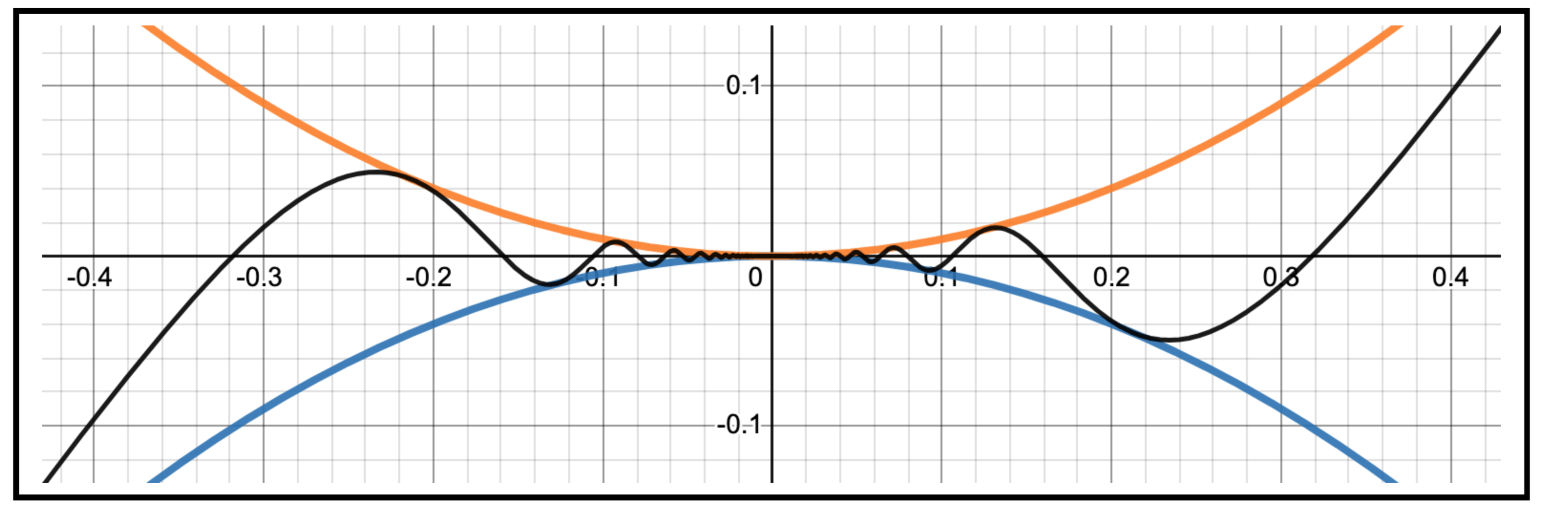
Squeeze Theorem for Functions in $\mathbb{R}$: For $g, f, h : A \to \mathbb{R}\ \forall\ x, c \in A$, if $g(x) \leq f(x) \leq h(x)$ and $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$ where c is any limit point, then $\lim_{x\to c} h(x) = L$ also.

Consider the functions $g(x) = -x^2$ (blue), $f(x) = x^2 sin\frac{1}{x}$ (black), and $h(x) = x^2$ (orange) at the point (0,0). As $f(x)$ approaches (0,0), it oscillates wildly and its limit at the point cannot be easily resolved. To overcome this, we can use $g(x)$ and $h(x)$ to sandwich the it from above and below: For $g(x) \leq f(x) \leq h(x)$ where $\lim_{n\to 0} g(x) = \lim_{n\to 0} h(x) = 0$, $0 \leq \lim_{n\to 0} f(x) \leq 0$, so we can conclude $\lim_{n\to 0} f(x)$ is also 0.

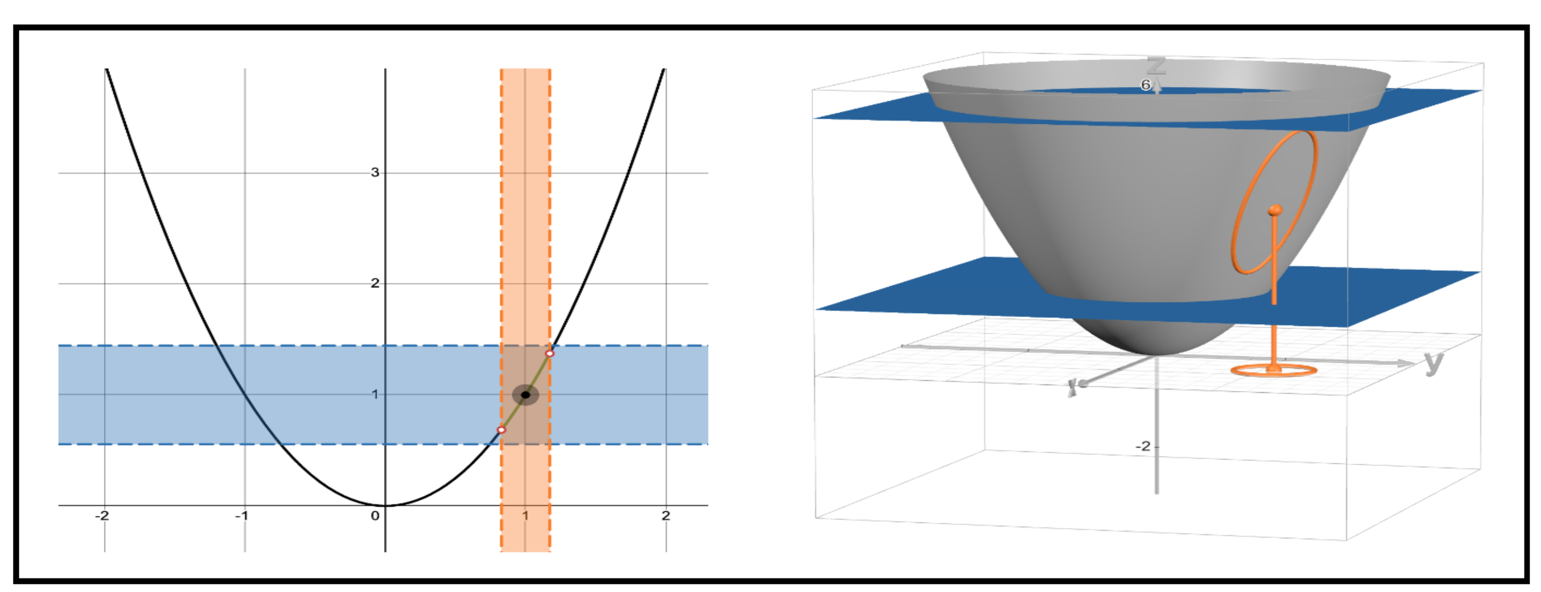


Proof. Let $\varepsilon > 0$ be given and arbitrary. Refer to the definition of a functional limit (Abbott Definition 4.2.1), which states that $\lim_{x\to c} f(x) = L$ provided that $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ s.t. whenever $0 < |x - c| < \delta$, it follows $|f(x) - L| < \varepsilon$. By this definition, we will establish a δ_g s.t. $0 < |x - c| < \delta_g$ and $|g(x) - L| < \frac{\varepsilon}{3}$. Also establish a δ_h s.t. $0 < |x - c| < \delta_h$ and $|h(x) - L| < \frac{\varepsilon}{3}$. We WTS: $|f(x) - L| < \frac{\varepsilon}{3}$. Rewrite this $|f(x) - L|$ and apply the Triangle Inequality. Then continue to simplify using the given information:
$|f(x) - g(x) + g(x) - L| \leq |f(x) - g(x)| + |g(x) - L| \leq |h(x) - g(x)| + |g(x) - L|$
$= |h(x) - L + L - g(x)| + |g(x) - L| \leq |h(x) - L| + |L - g(x)| + |g(x) - L|$
$= |h(x) - L| + 2|g(x) - L|$
Now we have $|f(x) - L| \leq |h(x) - L| + 2|g(x) - L| < \frac{\varepsilon}{3} + 2(\frac{\varepsilon}{3}) = \varepsilon$. This matches our definition, and shows that $\lim_{x\to c} f(x) = L$ for our given conditions. This is the Squeeze Theorem for Functions. □

Now, what if we jumped into a 3D space?

Consider Kosmala's definition for the limit of a function of two variables (Definition 10.2.1): For $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$, where (a,b) is an accumulation point of D, $\lim_{\substack{x\to a\\ y\to b}} f(x,y) = L$ iff $\forall\ \varepsilon > 0\ \exists > 0$ s.t. $|f(x,y) - L| < \varepsilon$, whenever (x,y) $\in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$.

Abbott's $\varepsilon - \delta$ definition of a limit in $\mathbb{R}$ uses a single input to create a δ-neighborhood on the graph of the function. The y-values of this mapping must land within the established bounds of L's ε-neighborhood in order to define the limit. Kosmala's $\varepsilon - \delta$ definition of a limit in $\mathbb{R}^3$ uses two input variables from $\mathbb{R}^2$ to create a δ-neighborhood that takes the shape of a ring, or δ-disk. The δ-disk maps onto the surface of the graph of the function, and the z-values of this mapping must sit inside L's established ε-neighborhood in order to define the limit. How would that look?

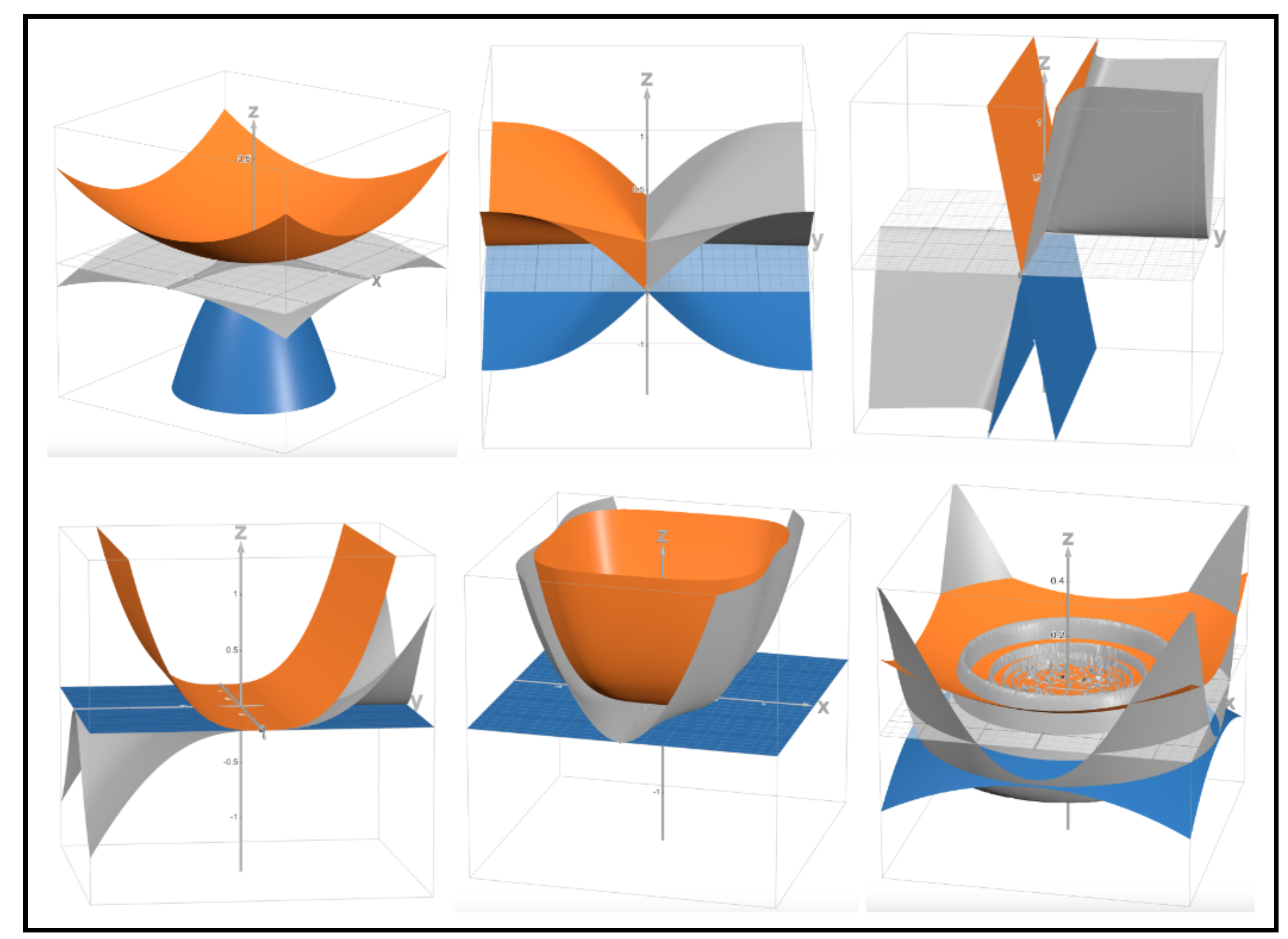


The $\varepsilon - \delta$ limits of $y = x^2$ in $\mathbb{R}$ (left), and $z = x^2 + y^2$ in $\mathbb{R}^3$ (right), where the δ conditions are represented in orange and the ε conditions are represented in blue are shown. We can see the similar mapping and $\varepsilon - \delta$ interactions for both conditions, but in their own distinct forms. For $y = x^2$, this occurs in a flat setting and generates corresponding flat regions. For $z = x^2 + y^2$, the δ-disk creates a cylindrical region surrounding the limit point.

Claim. **Squeeze Theorem for Surfaces in $\mathbb{R}^3$:** For $g, f, h : D \subseteq \mathbb{R}^2 \to \mathbb{R}\ \forall\ (x,y), (a,b) \in D$, if $g(x,y) \leq f(x,y) \leq h(x,y)$ and $\lim_{\substack{x\to a\\ y\to b}} g(x,y) = \lim_{\substack{x\to a\\ y\to b}} h(x,y) = L$ where (a,b) is any accumulation point, then $\lim_{\substack{x\to a\\ y\to b}} f(x,y) = L$ also.

Proof. Let $\varepsilon > 0$ be given and arbitrary.
According to our claim and Kosmala's Definition 10.2.1, $|g(x,y) - L| < \varepsilon = -\varepsilon < g(x,y) - L < \varepsilon = L - \varepsilon < g(x,y) - L < L + \varepsilon\ \forall\ 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta_g$.
Also, $|h(x,y) - L| < \varepsilon = -\varepsilon < h(x,y) - L < \varepsilon = L - \varepsilon < h(x,y) - L < L + \varepsilon$ $\forall\ 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta_h$.
Let $\delta_f = \min\{\delta_g, \delta_h\}$. We can now say $L - \varepsilon < g(x,y) \leq f(x,y) \leq h(x,y) < L + \varepsilon\ \forall\ 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta_f$. This also implies $L - \varepsilon < f(x,y) < L + \varepsilon$ $\forall\ 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta_f$.
This matches Kosmala's Definition 10.2.1. We have now shown that when $g(x,y) \leq f(x,y) \leq h(x,y)$ and $\lim_{\substack{x\to a\\ y\to b}} g(x,y) = \lim_{\substack{x\to a\\ y\to b}} h(x,y) = L$, $\lim_{\substack{x\to a\\ y\to b}} f(x,y)$ is also L. This is the Squeeze Theorem for functions in $\mathbb{R}^3$. □

Increasing dimensions makes things really exciting visually. For each of these examples of the Squeeze Theorem in 3D, the center surface $f(x)$ is grey, the bottom $g(x)$ is blue, and top $h(x)$ is orange.



We have now reviewed the complete definition and proofs for the The Squeeze Theorem for sequences, functions in $\mathbb{R}$, and surfaces in $\mathbb{R}^3$. We have even seen examples that apply to specific limit points and end behavior, but still have yet to explore its applications in subjects like sequences of functions, trigonometric proofs, the Fundamental Theorem of Calculus, even higher dimensions, or simply other proof methods and more examples. It's clear that the Squeeze Theorem offers a surprising amount of variety, significance, and wonder; it deserves to be both celebrated and loved.