Let $y^0 \in R^n \setminus \mathcal{A}$ be a fixed vector. Define the following function

$$\mathcal{F}(y) = f(y) - \langle \nabla f(y^0), y \rangle,$$

and similarly to the paper [BS2015] we define

$$\tilde{h}(x,y) = h(x,y) - \langle \nabla f(y^0), x \rangle.$$

It is clear that $x \mapsto \tilde{h}(x,y)$ is still quadratic function with associated matrix $L(y)I$. Therefore, we can write

$$\tilde{h}(x,y) = \tilde{h}(y,y) + \langle \nabla_x \tilde{h}(y,y), x-y \rangle + L(y)\|x-y\|^2$$

$$= \tilde{\mathcal{F}}(y) + \langle 2\nabla f(y) - \nabla f(y^0), x-y \rangle + L(y)\|x-y\|^2. \quad ①$$

On the other hand, from [BS2015, Lemma 3.1 (ii), Page 7] we have that

$$\tilde{h}(x,y) = h(x,y) - \langle \nabla f(y^0), x \rangle \geqslant 2f(x) - f(y) - \langle \nabla f(y^0), x \rangle$$

$$= 2\tilde{\mathcal{F}}(x) - \tilde{\mathcal{F}}(y) + \langle \nabla f(y^0), y - x \rangle, \quad ②$$

where the last inequality follows from the definition of $\mathcal{F}$.

Combining ① and ② yields

$$2\tilde{\mathcal{F}}(x) \leq 2\mathcal{F}(y) + 2\langle \nabla f(y) - \nabla f(y^0), x-y \rangle + L(y)\|x-y\|^2$$

$$= 2\mathcal{F}(y) + 2\langle \nabla \tilde{\mathcal{F}}(y), x-y \rangle + L(y)\|x-y\|^2.$$

Dividing the inequality by 2 and we get

$$\tilde{\mathcal{F}}(x) \leq \tilde{\mathcal{F}}(y) + \langle \nabla \tilde{\mathcal{F}}(y), x-y \rangle + \frac{L(y)}{2}\|x-y\|^2. \quad ③$$

It is clear that the optimal point of $\tilde{\mathcal{F}}$ is y^0 since $\nabla \tilde{\mathcal{F}}(y^0) = 0$, therefore from ③ we obtain

$$\tilde{\mathcal{F}}(y^0) \leq \tilde{\mathcal{F}}\left(y - \frac{1}{L(y)} \nabla \tilde{\mathcal{F}}(y)\right) \leq \mathcal{F}(y) + \left\langle \nabla \tilde{\mathcal{F}}(y), -\frac{1}{L(y)} \nabla \tilde{\mathcal{F}}(y) \right\rangle + \frac{L(y)}{2}\left\|\frac{1}{L(y)} \nabla \tilde{\mathcal{F}}(y)\right\|^2$$

$$= \tilde{f}(y) - \frac{1}{2L(y)} \|\nabla \tilde{f}(y)\|^2 .$$

Thus, using the definition of $\tilde{f}$ and the fact that $\nabla \tilde{f}(y) = \nabla f(y) - \nabla f(y^\circ)$, yields that

$$f(y^\circ) \leq f(y) + \langle \nabla f(y^\circ), y^\circ - y \rangle - \frac{1}{2L(y)} \|\nabla f(y) - \nabla f(y^\circ)\|^2 .$$

Now, following the same arguments we can show that

$$f(y) \leq f(y^\circ) + \langle \nabla f(y), y - y^\circ \rangle - \frac{1}{2L(y^\circ)} \|\nabla f(y) - \nabla f(y^\circ)\|^2$$

and combining these two inequalities yields that

$$\left(\frac{1}{2L(y^\circ)} + \frac{1}{2L(y)} \right) \|\nabla f(y) - \nabla f(y^\circ)\|^2 \leq \langle \nabla f(y^\circ) - \nabla f(y), y^\circ - y \rangle ,$$

that is

$$\|\nabla f(y) - \nabla f(y^\circ)\| \leq \frac{2L(y^\circ)L(y)}{L(y^\circ) + L(y)} \|y^\circ - y\| ,$$

for all $y^\circ, y \in \mathbb{R}^n \setminus \mathcal{A}$.

You need to use this result in Page 12 when I wrote (*) instead of your argument.