A Novel Class of Globally Convergent Algorithms For Clustering Problems

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Research conducted under the supervision of
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Develop and analyze two globally convergent center-based clustering algorithms each with a different distance-like function.

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- Numerical results of the proposed algorithms.

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Problem Reformulation

- Using the fact that

$$\min_{1 \le l \le k} u_l = \min \left\{ \langle u, v \rangle : v \in \Delta \right\},$$

where $\Delta := \left\{ u \in \mathbb{R}^d : \sum_{l=1}^{d} u_l = 1, \ u \ge 0 \right\}$ is the simplex in $\mathbb{R}^k$, problem (P_0) can be transformed into

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- Replacing the constrain $w^i \in \Delta$ by adding the indicator function $\delta_\Delta(\cdot)$ results in
$$(P_2) \qquad \min_{x \in \mathbb{R}^{nk}, w \in \mathbb{R}^{km}} \left\{ \sum_{i=1}^{m} \left(\langle w^i, d^i(x) \rangle + \delta_\Delta(w^i) \right) \right\},$$
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We are interested in two important cases for clustering:

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- In both cases the problem is <u>nonconvex</u>.

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- In both cases the problem is nonconvex.
- We want to exploit the special structure to devise simple schemes. Attractive approach is via alternating minimization.

Alternating Minimization

Very simple idea (goes back to Gauss-Seidel),

$$w^{k+1} \in \operatorname{argmin}_{w \in \Delta^m} \sigma\left(w, x^k\right), \quad x^{k+1} \in \operatorname{argmin}_{x \in \mathbb{R}^{nk}} \sigma\left(w^{k+1}, x\right)$$

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- These algorithms are not known to globally converge to stationary (critical) point.
- Here we follow the recent algorithm (PALM) of (Bolte-Sabach-Teboulle (2014)) to tackle these issues and devise simple schemes.

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Assumption 1.

(i) *$f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g : \mathbb{R}^m \to \overline{\mathbb{R}}$ are proper and lsc functions.*

(ii) *$H : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is a C^1 function.*

(iii) *Partial gradients of H are Lipshitz continuous: $H(\cdot, y) \in C^{1,1}_{L(y)}$ and likewise $H(x, \cdot) \in C^{1,1}_{L}(x)$.*

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Let $\sigma : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper and lsc function. Given $x \in \mathbb{R}^n$ and $t > 0$, the proximal map defined by:

$$\mathrm{prox}_t^{\sigma}(x) := \text{argmin}\left\{\sigma(u) + \frac{t}{2}\|u - x\|^2 : \; u \in \mathbb{R}^n\right\}.$$

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$$\widetilde{\sigma}(x^{k+1}, y) = \left\langle y - y^k, \nabla_y H(x^{k+1}, y^k) \right\rangle + \tfrac{d_k}{2} \left\| y - y^k \right\|^2 + g(y).$$

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1. Initialization: start with any $\left(x^0, y^0\right) \in \mathbb{R}^n \times \mathbb{R}^m$.
2. For each $k = 0, 1, \ldots$ generate a sequence $\left\{ \left(x^k, y^k\right) \right\}_{k \in \mathbb{N}}$:
   - 2.1. Take $\gamma_1 > 1$, set $c_k = \gamma_1 L_1 \left(y^k\right)$ and compute

     $$x^{k+1} \in \operatorname{argmin} \left\{ \widehat{\sigma}(x, y^k) : \ x \in \mathbb{R}^n \right\} = \operatorname{prox}^f_{c_k} \left(x^k - {c_k}^{-1} \nabla_x H \left(x^k, y^k \right) \right).$$

   - 2.2. Take $\gamma_2 > 1$, set $d_k = \gamma_2 L_2 \left(x^{k+1}\right)$ and compute

     $$y^{k+1} \in \operatorname{argmin} \left\{ \widetilde{\sigma}(x^{k+1}, y) : \ y \in \mathbb{R}^m \right\} = \operatorname{prox}^g_{d_k} \left(y^k - {d_k}^{-1} \nabla_y H \left(x^{k+1}, y^k \right) \right).$$

KPALM Algorithm for Clustering with the Squared Norm Distance

Recalling the Clustering Problem for the squared norm distance:

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- Thus, here $H(w,x)$ is $C^{1,1}$ and fits the optimization model.

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- The function $w \mapsto H(w, x)$, for fixed x, is linear and therefore there is no need to linearize it as suggested in PALM.

$$w^i(t+1) = \arg\min_{w^i \in \Delta}\left\{\langle w^i, d^i(x(t))\rangle + \frac{\alpha_i(t)}{2}\|w^i - w^i(t)\|^2\right\}.$$

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- The function $x \mapsto H(w,x)$, for fixed w, is quadratic and convex. Hence, <u>there is no need</u> to add a proximal term as suggested in PALM.

$$x(t+1) = \text{argmin}\left\{H(w(t+1),x) \mid x \in \mathbb{R}^{nk}\right\}.$$

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(1) Initialization: $z(0) = (w(0), x(0)) \in \Delta^m \times \mathbb{R}^{nk}$.

(2) General step ($t = 0, 1, \ldots$):

(2.1) Cluster assignment: choose certain $\alpha_i(t) > 0$, $i = 1, 2, \ldots, m$, and compute

$$w^i(t+1) = P_\Delta\left(w^i(t) - \frac{d^i(x(t))}{\alpha_i(t)}\right).$$

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$$x^l(t+1) = \frac{\sum_{i=1}^m w_l^i(t+1)a^i}{\sum_{i=1}^m w_l^i(t+1)}.$$

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- Setting $\alpha_i(t) = 0$ yields the popular k-means algorithm.

Convergence Analysis of KPALM

We will use the methodology developed in (Bolte-Sabach-Teboulle (2014)).

Definition 1.

Let $\sigma : \mathbb{R}^d \to (-\infty, +\infty]$ be a proper and lower semicontinuous function. A sequence $\left\{z^k\right\}_{k\in\mathbb{N}}$ is called a gradient-like descent sequence for σ if for all $k \in \mathbb{N}$ the following two conditions hold:

(C1) Sufficient decrease property: There exists a positive scalar ρ_1 such that

$$\rho_1 \left\|z^{k+1} - z^k\right\|^2 \leq \sigma\left(z^k\right) - \sigma\left(z^{k+1}\right).$$

(C2) A subgradient lower bound for the iterates gap:

- $\left\{z^k\right\}_{k\in\mathbb{N}}$ is bounded.
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$$\left\|w^{k+1}\right\| \leq \rho_2 \left\|z^{k+1} - z^k\right\|, \; w^{k+1} \in \partial\sigma\left(z^{k+1}\right).$$

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Theorem 2 (Bolte-Sabach-Teboulle (2014)).

Let $\sigma : \mathbb{R}^d \to (-\infty, \infty]$ be a proper, lower semicontinuous and semi-algebraic function with $\inf \sigma > -\infty$, and assume that $\left\{z^k\right\}_{k\in\mathbb{N}}$ is a gradient-like descent sequence for σ. If $\omega\left(z^0\right) \subset \mathrm{crit}(\sigma)$ then the sequence $\left\{z^k\right\}_{k\in\mathbb{N}}$ convergences to a critical point z^* of σ.

Semi-Algebraic Functions

Definition 3.

(i) A subset S of $\mathbb{R}^n$ is a real semi-algebraic set if there exists a finite number of real polynomial functions $g_{ij}, h_{ij} : \mathbb{R}^n \to \mathbb{R}$ such that

$$S = \bigcup_{j=1}^{p} \bigcap_{i=1}^{q} \left\{ u \in \mathbb{R}^n : \; g_{ij}(u) = 0 \text{ and } \; h_{ij}(u) < 0 \right\}.$$

(ii) A function $\sigma : \mathbb{R}^n \to \overline{\mathbb{R}}$ is called semi-algebraic if its graph

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- Composition of semi-algebraic functions.

KPALM Analysis - $H \in C^1$

- Semi-Algebraicity: H is a weighted sum of squared Euclidean norms hence semi-algebraic, and Δ is semi-algebraic set, thus $\delta_\Delta(\cdot)$ is semi-algebraic, and in turn σ since it is sum of these functions.

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(ii) *For all* $t \in \mathbb{N}$ *there exists* $\underline{\beta} > 0$ *such that* $2 \min_{1 \leq l \leq k} \sum_{i=1}^m w_l^i(t) := \beta(w(t)) \geq \underline{\beta}$.

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Denote $\underline{\alpha} = \min_{1\le i\le m} \underline{\alpha_i}$ and $\overline{\alpha} = \max_{1\le i\le m} \overline{\alpha_i}$.

Global Convergence of KPALM

Suppose that Assumption 2 holds true and let $\{z(t)\}_{t\in\mathbb{N}}$ be the sequence generated by KPALM.

To apply the general methodology of [BST] we proved the following results:

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Proposition 1 (Sufficient decrease property).

There exists $\rho_1 > 0$ *such that*

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For each $t \in \mathbb{N}$ define

$$\gamma(t) := \left(\left(d^i(x(t)) - d^i(x(t-1)) - \alpha_i(t-1)(w^i(t) - w^i(t-1)) \right)_{i=1,2,\ldots,m}, \mathbf{0} \right).$$

Then $\gamma(t) \in \partial\sigma(z(t))$ and there exists $\rho_2 > 0$ such that

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Theorem 4 (KPALM convergence result).

The sequence $\{z(t)\}_{t\in\mathbb{N}}$ *converges to a critical point of* σ.

ε-KPALM Algorithm for Clustering with the Norm Distance

Recalling the Clustering Problem for the Euclidean norm distance:

$$(P) \qquad min\left\{\sigma(z) := H(w,x) + G(w) \mid z := (w,x) \in \mathbb{R}^{km} \times \mathbb{R}^{nk}\right\},$$

$$H(w,x) = \sum_{i=1}^{m} H^i(w,x) = \sum_{i=1}^{m} \langle w^i, d^i(x) \rangle = \sum_{i=1}^{m} \sum_{l=1}^{k} w_l^i \left\| x^l - a^i \right\|.$$

- <u>H is not smooth, hence PALM cannot be applied as is</u>. Therefore approximating:

$$H_\varepsilon(w,x) = \sum_{l=1}^{k} H_\varepsilon^l(w,x) = \sum_{l=1}^{k} \sum_{i=1}^{m} w_l^i \left(\|x^l - a^i\|^2 + \varepsilon^2 \right)^{1/2},$$

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- Also $d_\varepsilon^i(x)$ is the smoothed version of $d^i(x)$, whose $1 \le l \le k$ coordinate is $\left(\|x^l - a^i\|^2 + \varepsilon^2\right)^{1/2}$.

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- With respect to w we apply the same step as in KPALM:

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Motivated by the recent work on Weber problem (Beck-Sabach(2015)), we develop several auxiliary results. We define the function $f_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ by

$$f_\varepsilon(x) = \sum_{i=1}^{m} v_i \left(\|x - a^i\|^2 + \varepsilon^2 \right)^{1/2},$$

$v^i \in \mathbb{R}_+$ fixed weights. We also need the auxiliary function $h_\varepsilon : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by

$$h_\varepsilon(x, y) = \sum_{i=1}^{m} \frac{v_i \left(\|x - a^i\|^2 + \varepsilon^2 \right)}{\left(\|y - a^i\|^2 + \varepsilon^2 \right)^{1/2}}.$$

Finally we introduce the following modulus, $L_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ defined by

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Key Properties for Sufficient Decrease and Subgradient Bound Proofs

Lemma 5.

For all $y, z \in \mathbb{R}^n$ the following statement holds true

$$\|\nabla f_\varepsilon(y) - \nabla f_\varepsilon(z)\| \leq \frac{2L_\varepsilon(z)L_\varepsilon(y)}{L_\varepsilon(z) + L_\varepsilon(y)}\|z - y\|.$$

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Let $\{z(t)\}_{t\in\mathbb{N}}$ be the sequence generated by ε-KPALM.

Proposition 3 (Bounds for L_ε^l).

(i) *Denote* $d_{\mathcal{A}} = \mathrm{diam}(Conv(\mathcal{A}))$. *For all* $t \in \mathbb{N}$ *and* $l = 1, 2, \ldots, k$ *we have*

$$L_\varepsilon^l(w(t+1), x(t)) \geq \frac{\underline{\beta}}{\left(d_{\mathcal{A}}{}^2 + \varepsilon^2\right)^{1/2}}.$$

(ii) *For all* $t \in \mathbb{N}$ *and* $l = 1, 2, \ldots, k$ *we have*

$$L_\varepsilon^l(w(t+1), x(t)) \leq \frac{m}{\varepsilon}.$$

Key Properties for Sufficient Decrease and Subgradient Bound Proofs

Lemma 5.

For all $y, z \in \mathbb{R}^n$ the following statement holds true

$$\|\nabla f_\varepsilon(y) - \nabla f_\varepsilon(z)\| \leq \frac{2L_\varepsilon(z)L_\varepsilon(y)}{L_\varepsilon(z) + L_\varepsilon(y)}\|z - y\|.$$

Let $\{z(t)\}_{t\in\mathbb{N}}$ be the sequence generated by ε-KPALM.

Proposition 3 (Bounds for L_ε^l).

(i) *Denote $d_{\mathcal{A}} = \mathrm{diam}(Conv(\mathcal{A}))$. For all $t \in \mathbb{N}$ and $l = 1, 2, \ldots, k$ we have*

$$L_\varepsilon^l(w(t+1), x(t)) \geq \frac{\beta}{\left({d_{\mathcal{A}}}^2 + \varepsilon^2\right)^{1/2}}.$$

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$$L_\varepsilon^l(w(t+1), x(t)) \leq \frac{m}{\varepsilon}.$$

Proposition 4 (Sufficient Decrease w.r.t. x).

For all $t \in \mathbb{N}$, we have

$$H_\varepsilon(w(t+1), x(t+1)) \leq H_\varepsilon(w(t+1), x(t)) + \langle \nabla_x H_\varepsilon(w(t+1), x(t)), x(t+1) - x(t)\rangle + \sum_{l=1}^{k} \frac{L_\varepsilon^l(w(t+1), x(t))}{2}\|x^l(t+1) - x^l(t)\|^2.$$

Global Convergence of ε-KPALM

Suppose that Assumption 2 holds true and let $\{z(t)\}_{t\in\mathbb{N}}$ be the sequence generated by ε-KPALM.

Proposition 5 (Sufficient decrease property).

There exists $\rho_1 > 0$ *such that*

$$\rho_1 \|z(t+1) - z(t)\|^2 \leq \sigma_\varepsilon(z(t)) - \sigma_\varepsilon(z(t+1)), \quad \forall\, t \in \mathbb{N}.$$

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Proposition 6 (Subgradient lower bound for the iterates gap).

For each $t \in \mathbb{N}$ *define*

$$\gamma(t) := \left(\left(d^i(x(t)) - d^i(x(t-1)) - \alpha_i(t-1)(w^i(t) - w^i(t-1)) \right)_{i=1,2,\ldots,m}, \nabla_x H_\varepsilon(z(t)) \right).$$

Then $\gamma(t) \in \partial\sigma_\varepsilon(z(t))$ *and there exists* $\rho_2 > 0$ *such that*

$$\|\gamma(t+1)\| \leq \rho_2 \|z(t+1) - z(t)\|, \quad \forall\, t \in \mathbb{N}.$$

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Theorem 6 (ε-KPALM convergence result).

The sequence $\{z(t)\}_{t\in\mathbb{N}}$ converges to a critical point of σ_ε.

Numerical Results: The Squared Norm

- Initialization issues: randomly picking k data points as starting centers vs. k-means++.

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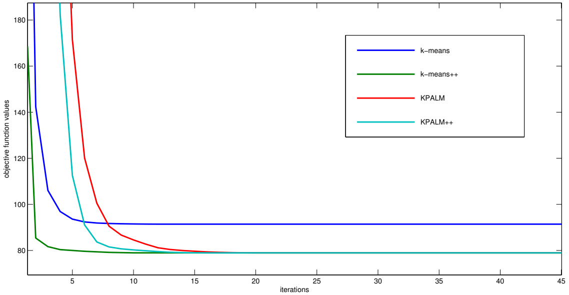
- Comparison of the objective function values performed on the Iris dataset for squared norm algorithms



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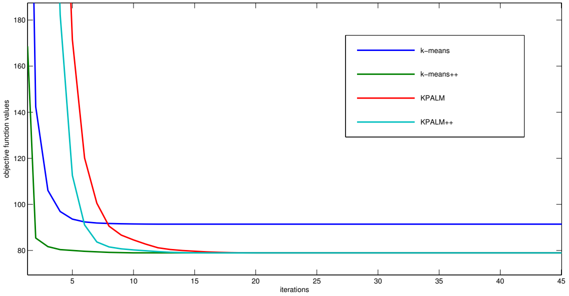

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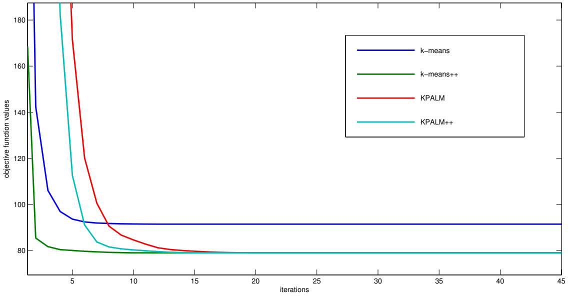
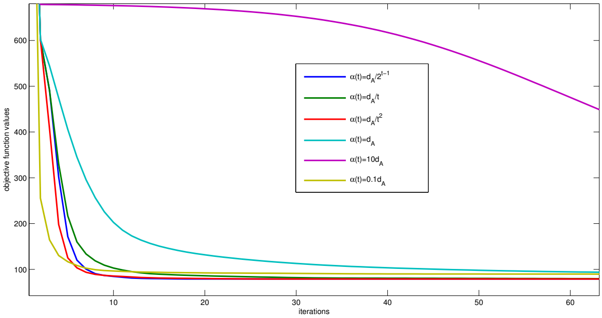


- In the squared Euclidean setting, KPALM achieves lower objective function values than k-means. When using a more sophisticated initialization step, such as the one in k-means++, then k-means++ and KPALM++ achieve similar objective function values.
- k-means needs less number of iterations then KPALM to reach a certain precision.

Numerical Results: The Squared Norm

- Comparison of the objective function values performed on the Iris dataset for KPALM algorithm with different α parameter updates.



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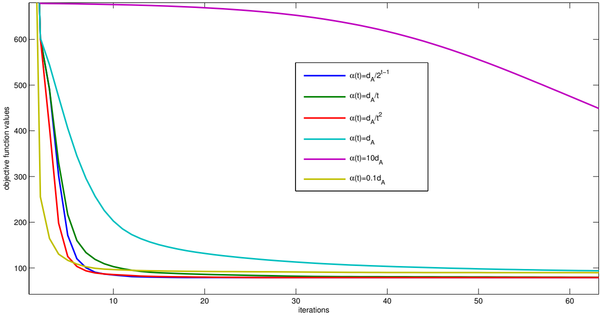
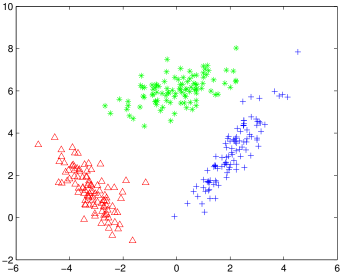


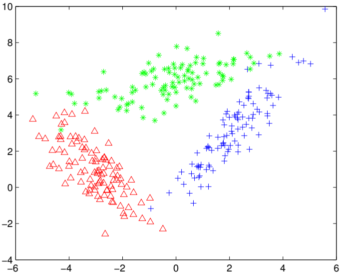
- It is preferable to use dynamic update of $\alpha(t)$ parameter to achieve a faster convergence, both in KPALM and ϵ-KPALM. Example for suitable choices can be $\alpha(t) = \mathrm{diam}(\mathcal{A})/t$ and $\alpha(t) = \mathrm{diam}(\mathcal{A})/2^t$.

Numerical Results: Squared Norm Algorithms vs. ε-KPALM

- Generated two synthetic datasets, each 300 points



(a) Dense Gaussians



(b) Sparse Gaussians

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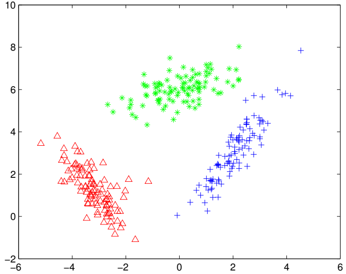

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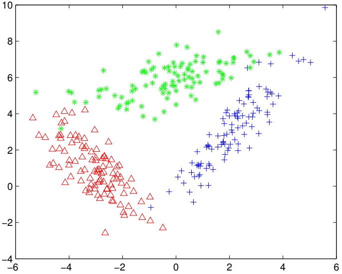
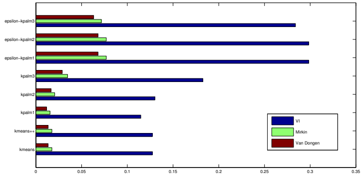


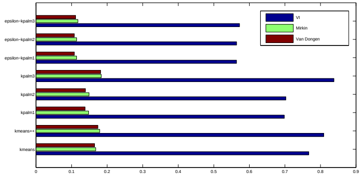
(b) Sparse Gaussians

- Compared the resulting clustering using metrics, such as Variation of Information defined by: $VI(\mathcal{C}^1, \mathcal{C}^2) := -\sum_{i,j} r_{i,j}\left[\log(r_{i,j}/p_i) + \log(r_{i,j}/q_j)\right]$, where $\mathcal{C}^1 = \left\{C_1^1, C_2^1, \ldots C_k^1\right\}$ and $\mathcal{C}^2 = \left\{C_1^2, C_2^2, \ldots C_l^2\right\}$ are two clusterings of $\mathcal{A}$, and $m = |\mathcal{A}|, \quad p_i = |C_i^1|/m, \quad q_j = |C_j^2|/m, \quad r_{i,j} = |C_i^1 \cap C_j^2|/m.$

Numerical Results: Squared Norm Algorithms vs. ε-KPALM

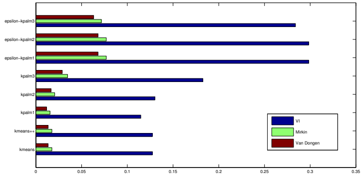


(a) Dense Gaussians metrics comparison



(b) Sparse Gaussians metrics comparison

Numerical Results: Squared Norm Algorithms vs. ε-KPALM



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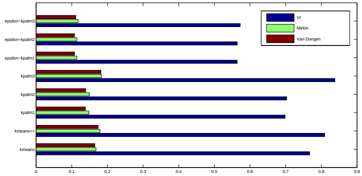
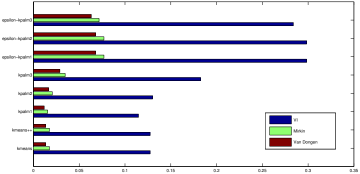


(b) Sparse Gaussians metrics comparison

- When the convex hulls of the desired clusters are mutually exclusive, algorithms which solve the clustering problem with the squared Euclidean distance are preferable to ε-KPALM.

Numerical Results: Squared Norm Algorithms vs. ε-KPALM



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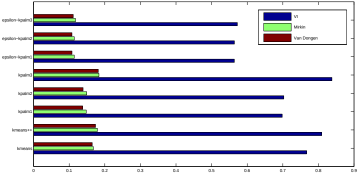

(b) Sparse Gaussians metrics comparison

- When the convex hulls of the desired clusters are mutually exclusive, algorithms which solve the clustering problem with the squared Euclidean distance are preferable to ε-KPALM.
- In datasets with outliers, the clustering obtained with ε-KPALM is more similar to the desired clustering, in terms of clustering metrics, than the clusterings obtained via the squared Euclidean algorithms. Therefore, as expected, for data with outliers, the choice of a norm instead of the squared norm is a more natural choice, and the ε-KPALM algorithm appears to be a promising algorithm to handle such data.

Thank you
for your attention!

KPALM Sufficient Decrease Proof

Since $x \mapsto H(w, x) = \sum_{l=1}^{k} \sum_{i=1}^{m} w_l^i \left\| x^l - a^i \right\|^2$ is C^2, and it Hessian is given by

$$\nabla_{x^j} \nabla_{x^l} H(w, x) = \begin{cases} 0 & \text{if } j \neq l, \quad 1 \leq j, l \leq k, \\ 2 \sum_{i=1}^{m} w_l^i & \text{if } j = l, \quad 1 \leq j, l \leq k, \end{cases}$$

then it is strongly convex with parameter $\beta(w)$, whenever $\beta(w) = 2 \min_{1 \leq l \leq k} \sum_{i=1}^{m} w_l^i > 0$.

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then it is strongly convex with parameter $\beta(w)$, whenever $\beta(w) = 2 \min_{1 \leq l \leq k} \sum_{i=1}^{m} w_l^i > 0$.

Assumption 2(ii) ensures that $x \mapsto H(w(t),x)$ is strongly convex with parameter $\beta(w(t))$, hence

$$\begin{aligned} & H(w(t+1),x(t)) - H(w(t+1),x(t+1)) \geq \\ & \qquad \geq \langle \nabla_x H(w(t+1),x(t+1)), x(t) - x(t+1) \rangle + \frac{\beta(w(t))}{2} \|x(t) - x(t+1)\|^2 \\ & \qquad = \frac{\beta(w(t))}{2} \|x(t+1) - x(t)\|^2 \\ & \qquad \geq \frac{\beta}{2} \|x(t+1) - x(t)\|^2, \end{aligned}$$

where the equality follows from $\nabla_x H(w(t+1),x(t+1)) = 0$.

KPALM Sufficient Decrease Proof-Contd.

From the w update step we derive

$$\begin{aligned} H^i(w(t+1), x(t)) + \frac{\alpha_i(t)}{2}\|w^i(t+1) - w^i(t)\|^2 = \\ &= \langle w^i(t+1), d^i(x(t))\rangle + \frac{\alpha_i(t)}{2}\|w^i(t+1) - w^i(t)\|^2 \\ &\leq \langle w^i(t), d^i(x(t))\rangle + \frac{\alpha_i(t)}{2}\|w^i(t) - w^i(t)\|^2 \\ &= \langle w^i(t), d^i(x(t))\rangle = H^i(w(t), x(t)). \end{aligned}$$

Summing the last inequality over all $1 \leq i \leq m$ yields

$$\frac{\alpha}{2}\|w(t+1) - w(t)\|^2 \leq H(w(t), x(t)) - H(w(t+1), x(t))$$

KPALM Sufficient Decrease Proof-Contd.

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$$
\begin{aligned}
H^i(w(t+1), x(t)) + \frac{\alpha_i(t)}{2}\|w^i(t+1) - w^i(t)\|^2 = & \\
&= \langle w^i(t+1), d^i(x(t))\rangle + \frac{\alpha_i(t)}{2}\|w^i(t+1) - w^i(t)\|^2 \\
&\leq \langle w^i(t), d^i(x(t))\rangle + \frac{\alpha_i(t)}{2}\|w^i(t) - w^i(t)\|^2 \\
&= \langle w^i(t), d^i(x(t))\rangle = H^i(w(t), x(t)).
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Summing the last inequality over all $1 \leq i \leq m$ yields

$$\frac{\underline{\alpha}}{2}\|w(t+1) - w(t)\|^2 \leq H(w(t), x(t)) - H(w(t+1), x(t))$$

Set $\rho_1 = \frac{1}{2}\min\left\{\underline{\alpha}, \underline{\beta}\right\}$, by combining the sufficient decrease in x and w variables we get

$$
\begin{aligned}
&\rho_1\|z(t+1) - z(t)\|^2 = \rho_1\left(\|w(t+1) - w(t)\|^2 + \|x(t+1) - x(t)\|^2\right) \leq \\
&\quad \leq [H(w(t), x(t)) - H(w(t+1), x(t))] + [H(w(t+1), x(t)) - H(w(t+1), x(t+1))] \\
&\quad = H(z(t)) - H(z(t+1)) = \sigma(z(t)) - \sigma(z(t+1)).
\end{aligned}
$$

KPALM Subgradient Lower Bound The Iterates Gap Proof

Recalling the steps of KPLAM algorithm:

$$w^i(t+1) = \arg\min_{w^i \in \Delta} \left\{ \langle w^i, d^i(x(t)) \rangle + \frac{\alpha_i(t)}{2} \|w^i - w^i(t)\|^2 \right\} \tag{1}$$

$$x(t+1) = \text{argmin} \left\{ H(w(t+1), x) \mid x \in \mathbb{R}^{nk} \right\} \tag{2}$$

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The subgradient of σ is given by

$$\partial \sigma = \nabla H + \partial G = \left(\left(\nabla_{w^i} H^i + \partial_{w^i} \delta_\Delta \right)_{i=1,2,\ldots,m}, \nabla_x H \right).$$

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Evaluating the last relation at $z(t+1)$ and using (2)

$$\begin{aligned} \partial \sigma(z(t+1)) &= \left(\left(d^i(x(t+1)) + \partial_{w^i} \delta_\Delta (w^i(t+1)) \right)_{i=1,2,\ldots,m}, \nabla_x H(z(t+1)) \right) \\ &= \left(\left(d^i(x(t+1)) + \partial_{w^i} \delta_\Delta (w^i(t+1)) \right)_{i=1,2,\ldots,m}, \mathbf{0} \right). \end{aligned}$$

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The optimality condition of $w^i(t+1)$ (see (1)), implies that there exists $u^i(t+1) \in \partial \delta_\Delta(w^i(t+1))$ such that

$$d^i(x(t)) + \alpha_i(t) \left(w^i(t+1) - w^i(t) \right) + u^i(t+1) = \mathbf{0}.$$

KPALM Subgradient Lower Bound The Iterates Gap Proof-Contd.

Setting $\gamma(t+1) := \left(\left(d^i(x(t+1)) + u^i(t+1) \right)_{i=1,2,\ldots,m}, \mathbf{0} \right) \in \partial\sigma(z(t+1))$.

KPALM Subgradient Lower Bound The Iterates Gap Proof-Contd.

Setting $\gamma(t+1) := \left(\left(d^i(x(t+1)) + u^i(t+1) \right)_{i=1,2,\ldots,m}, \mathbf{0} \right) \in \partial\sigma(z(t+1))$.

$$\begin{aligned}
\|\gamma(t+1)\| &\leq \sum_{i=1}^{m} \left\| d^i(x(t+1)) - d^i(x(t)) - \alpha_i(t) \left(w^i(t+1) - w^i(t) \right) \right\| \\
&\leq \sum_{i=1}^{m} \left\| d^i(x(t+1)) - d^i(x(t)) \right\| + \sum_{i=1}^{m} \alpha_i(t) \left\| w^i(t+1) - w^i(t) \right\| \\
&\leq \sum_{i=1}^{m} 4M \|x(t+1) - x(t)\| + m\overline{\alpha} \|z(t+1) - z(t)\| \\
&\leq m(4M + \overline{\alpha}) \|z(t+1) - z(t)\|,
\end{aligned}$$

where the third inequality follows from the inequality

$$\|d^i(x(t+1) - d^i(x(t))\| \leq 4M\|x(t+1) - x(t)\|, \quad \forall\, i = 1, 2, \ldots, m,\ t \in \mathbb{N},$$

with $M = \max_{1 \leq i \leq m} \|a^i\|$ and the result follows with $\rho_2 = m(4M + \overline{\alpha})$.

Properties of the Auxiliary Functions h_ε and f_ε

Lemma 7.

The following properties of h_ε hold.

(i) *For any $y \in \mathbb{R}^n$,*

$$h_\varepsilon(y, y) = f_\varepsilon(y).$$

(ii) *For any $x, y \in \mathbb{R}^n$,*

$$h_\varepsilon(x, y) \geq 2f_\varepsilon(x) - f_\varepsilon(y).$$

(iii) *For any $x, y \in \mathbb{R}^n$,*

$$f_\varepsilon(x) \leq f_\varepsilon(y) + \langle \nabla f_\varepsilon(y), x - y \rangle + \frac{L_\varepsilon(y)}{2} \|x - y\|^2.$$

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Lemma 8.

For all $y, z \in \mathbb{R}^n$ the following statement holds true

$$\|\nabla f_\varepsilon(y) - \nabla f_\varepsilon(z)\| \leq \frac{2L_\varepsilon(z)L_\varepsilon(y)}{L_\varepsilon(z) + L_\varepsilon(y)} \|z - y\|.$$

Preliminaries for Sufficient Decrease Proof

Proposition (Bounds for L_ε^l).

Let $\{z(t)\}_{t\in\mathbb{N}}$ be the sequence generated by ε-KPALM.

(i) *Denote $d_{\mathcal{A}} = \mathrm{diam}(Conv(\mathcal{A}))$. For all $t \in \mathbb{N}$ and $l = 1, 2, \ldots, k$ we have*

$$L_\varepsilon^l(w(t+1), x(t)) \geq \frac{\beta}{\left(d_{\mathcal{A}}{}^2 + \varepsilon^2\right)^{1/2}}.$$

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$$L_\varepsilon^l(w(t+1), x(t)) \leq \frac{m}{\varepsilon}.$$

Proof.

(i) Using $\underline{\beta}$ as in Assumption 2(ii) and the fact that $x^l(t) \in Conv(\mathcal{A})$, we obtain

$$L_\varepsilon^l(w(t+1), x(t)) = \sum_{i=1}^{m} \frac{w_l^i(t+1)}{\left(\|x^l(t) - a^i\|^2 + \varepsilon^2\right)^{1/2}} \geq \frac{\sum_{i=1}^{m} w_l^i(t+1)}{\left(d_{\mathcal{A}}{}^2 + \varepsilon^2\right)^{1/2}} \geq \frac{\underline{\beta}}{\left(d_{\mathcal{A}}{}^2 + \varepsilon^2\right)^{1/2}},$$

where the first inequality follows from the fact that $\left\|x^l(t) - a^i\right\| \leq d_{\mathcal{A}}$.

(ii) Since $w(t+1) \in \Delta^m$ we have

$$L_\varepsilon^l(w(t+1), x(t)) = \sum_{i=1}^{m} \frac{w_l^i(t+1)}{\left(\|x^l(t) - a^i\|^2 + \varepsilon^2\right)^{1/2}} \leq \sum_{i=1}^{m} \frac{1}{\varepsilon} = \frac{m}{\varepsilon}.$$

Preliminaries for Sufficient Decrease Proof

Proposition (Sufficient Decrease w.r.t. x).

Let $\{z(t)\}_{t\in\mathbb{N}}$ be the sequence generated by ε-KPALM. Then, for all $t \in \mathbb{N}$, we have

$$H_\varepsilon(w(t+1), x(t+1)) \le H_\varepsilon(w(t+1), x(t)) + \langle \nabla_x H_\varepsilon(w(t+1), x(t)), x(t+1) - x(t)\rangle + \sum_{l=1}^{k} \frac{L_\varepsilon^l(w(t+1), x(t))}{2} \|x^l(t+1) - x^l(t)\|^2.$$

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Proof.

By definition of f_ε, using the weights $v_i = w_l^i(t+1)$, we obtain

$$H_\varepsilon^l(w(t+1), x(t)) = f_\varepsilon(x^l(t)).$$

Therefore, by applying Lemma 7(iii) with $x = x^l(t+1)$ and $y = x^l(t)$, we get

$$H_\varepsilon^l(w(t+1), x(t+1)) \le H_\varepsilon^l(w(t+1), x(t)) + \left\langle \nabla_{x^l} H_\varepsilon^l(w(t+1), x(t)), x(t+1) - x(t) \right\rangle + \frac{L_\varepsilon^l(w(t+1), x(t))}{2} \|x^l(t+1) - x^l(t)\|^2.$$

Summing the last inequality over $l = 1, 2, \ldots, k$, yields

$$H_\varepsilon(w(t+1), x(t+1)) \le H_\varepsilon(w(t+1), x(t)) + \sum_{l=1}^{k} \left\langle \nabla_{x^l} H_\varepsilon(w(t+1), x(t)), x^l(t+1) - x^l(t) \right\rangle + \sum_{l=1}^{k} \frac{L_\varepsilon^l(w(t+1), x(t))}{2} \|x^l(t+1) - x^l(t)\|^2.$$

The result follows by replacing the last term with the following compact form

$$\sum_{l=1}^{k} \left\langle \nabla_{x^l} H_\varepsilon(w(t+1), x(t)), x^l(t+1) - x^l(t) \right\rangle = \langle \nabla_x H_\varepsilon(w(t+1), x(t)), x(t+1) - x(t) \rangle .$$

ε-KPALM Sufficient Decrease Proof

With respect to w ε-KPALM sufficient decrease is similar to that of KPALM, which yields

$$\frac{\alpha}{2}\|w(t+1)-w(t)\|^2 \leq H_\varepsilon(w(t),x(t)) - H_\varepsilon(w(t+1),x(t)). \tag{1}$$

Applying Proposition 4 with the center update step in ε-KPALM we get for all $t \in \mathbb{N}$ that

$$\begin{aligned} H_\varepsilon(w(t+1),x(t)) - H_\varepsilon(w(t+1),x(t+1)) &\geq \sum_{l=1}^{k} \frac{L_\varepsilon^l(w(t+1),x(t))}{2}\|x^l(t+1)-x^l(t)\|^2 \\ &\geq \frac{\beta}{\left({d_\mathcal{A}}^2+\varepsilon^2\right)^{1/2}} \sum_{l=1}^{k} \|x^l(t+1)-x^l(t)\|^2 \\ &\geq \frac{\beta}{\left({d_\mathcal{A}}^2+\varepsilon^2\right)^{1/2}} \|x(t+1)-x(t)\|^2, \qquad (2) \end{aligned}$$

where the second inequality follows from Proposition 3(i).

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&\ge \frac{\underline{\beta}}{\left({d_\mathcal{A}}^2+\varepsilon^2\right)^{1/2}} \sum_{l=1}^{k}\|x^l(t+1)-x^l(t)\|^2 \\
&\ge \frac{\underline{\beta}}{\left({d_\mathcal{A}}^2+\varepsilon^2\right)^{1/2}}\|x(t+1)-x(t)\|^2, \qquad (2)
\end{aligned}$$

where the second inequality follows from Proposition 3(i).
Set $\rho_1 = \frac{1}{2}\min\left\{\underline{\alpha}, \underline{\beta}/\left(d_\mathcal{A}^2+\varepsilon^2\right)^{1/2}\right\}$. Summing (1) and (2) yields

$$\begin{aligned}
\rho_1\|z(t+1)-z(t)\|^2 &= \rho_1\left(\|w(t+1)-w(t)\|^2 + \|x(t+1)-x(t)\|^2\right) \\
&\le [H_\varepsilon(w(t),x(t)) - H_\varepsilon(w(t+1),x(t))] + [H_\varepsilon(w(t+1),x(t)) - H_\varepsilon(w(t+1),x(t+1))] \\
&= H_\varepsilon(z(t)) - H_\varepsilon(z(t+1)) \\
&= \sigma_\varepsilon(z(t)) - \sigma_\varepsilon(z(t+1)).
\end{aligned}$$

ε-KPALM Subgradient Lower Bound The Iterates Gap Proof

Repeating the steps of the proof in the case of KPALM yields that

$$\gamma(t+1) := \left(\left(d_\varepsilon^i(x(t+1)) + u^i(t+1) \right)_{i=1,\ldots,m}, \nabla_x H_\varepsilon(w(t+1), x(t+1)) \right) \in \partial \sigma_\varepsilon(z(t+1)), \qquad (3)$$

where $u^i(t+1) \in \partial \delta_\Delta(w^i(t+1)),\ i = 1, 2, \ldots, m$.

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Writing the optimality condition of the cluster assignment step yields

$$d_\varepsilon^i(x(t)) + \alpha_i(t) \left(w^i(t+1) - w^i(t) \right) + u^i(t+1) = \mathbf{0}. \tag{4}$$

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Plugging (4) into (3), and taking the norm yields

$$\begin{aligned} \|\gamma(t+1)\| &\leq \sum_{i=1}^{m} \|d_\varepsilon^i(x(t+1)) - d_\varepsilon^i(x(t)) - \alpha_i(t) \left(w^i(t+1) - w^i(t) \right) \| + \|\nabla_x H_\varepsilon(w(t+1), x(t+1))\| \\ &\leq \sum_{i=1}^{m} \|d_\varepsilon^i(x(t+1)) - d_\varepsilon^i(x(t))\| + \sum_{i=1}^{m} \alpha_i(t) \|w^i(t+1) - w^i(t)\| + \|\nabla_x H_\varepsilon(w(t+1), x(t+1))\| \\ &\leq \frac{m d_{\mathcal{A}}}{\varepsilon} \|x(t+1) - x(t)\| + \overline{\alpha}\sqrt{m} \|w(t+1) - w(t)\| + \|\nabla_x H_\varepsilon(w(t+1), x(t+1))\|, \end{aligned}$$

where the last inequality follows from the inequality

$$\|d_\varepsilon^i(x) - d_\varepsilon^i(y)\| \leq \frac{d_{\mathcal{A}}}{\varepsilon} \|x - y\|, \text{ whenever } x^l, y^l \in Conv(\mathcal{A}) \quad \forall\, 1 \leq l \leq k,$$

and the fact that $\overline{\alpha} = \max_{1 \leq i \leq m} \overline{\alpha_i}$.

ε-KPALM Subgradient Lower Bound The Iterates Gap Proof-Contd.

It is sufficient to show that $\|\nabla_x H_\varepsilon(w(t+1), x(t+1))\| \leq c\|x(t+1) - x(t)\|$, for some constant $c > 0$.

ε-KPALM Subgradient Lower Bound The Iterates Gap Proof-Contd.

It is sufficient to show that $\|\nabla_x H_\varepsilon(w(t+1), x(t+1))\| \leq c\|x(t+1) - x(t)\|$, for some constant $c > 0$. First, note that for fixed $w \in \Delta^m$ and any $x \in \mathbb{R}^{nk}$ the following relation holds

$$\nabla_{x^l} H_\varepsilon(w, x) = \nabla f_\varepsilon\left(x^l\right), \quad \forall\, l = 1, 2, \ldots, k. \tag{5}$$

Now, for all $l = 1, 2, \ldots, k$, we have

$$\begin{aligned}
\nabla_{x^l} H_\varepsilon(w(t+1), x(t+1)) &= \nabla_{x^l} H_\varepsilon(w(t+1), x(t+1)) - \nabla_{x^l} H_\varepsilon(w(t+1), x(t)) + \nabla_{x^l} H_\varepsilon(w(t+1), x(t)) \\
&= \nabla_{x^l} H_\varepsilon(w(t+1), x(t+1)) - \nabla_{x^l} H_\varepsilon(w(t+1), x(t)) + L_\varepsilon^l(w(t+1), x(t))\left(x^l(t) - x^l(t+1)\right),
\end{aligned}$$

where the last equality follows from center update step.

ε-KPALM Subgradient Lower Bound The Iterates Gap Proof-Contd.

It is sufficient to show that $\|\nabla_x H_\varepsilon(w(t+1), x(t+1))\| \leq c\|x(t+1) - x(t)\|$, for some constant $c > 0$. First, note that for fixed $w \in \Delta^m$ and any $x \in \mathbb{R}^{nk}$ the following relation holds

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Now, for all $l = 1, 2, \ldots, k$, we have

$$\begin{aligned}
\nabla_{x^l} H_\varepsilon(w(t+1), x(t+1)) &= \nabla_{x^l} H_\varepsilon(w(t+1), x(t+1)) - \nabla_{x^l} H_\varepsilon(w(t+1), x(t)) + \nabla_{x^l} H_\varepsilon(w(t+1), x(t)) \\
&= \nabla_{x^l} H_\varepsilon(w(t+1), x(t+1)) - \nabla_{x^l} H_\varepsilon(w(t+1), x(t)) + L_\varepsilon^l(w(t+1), x(t))\left(x^l(t) - x^l(t+1)\right),
\end{aligned}$$

where the last equality follows from center update step.Therefore,

$$\begin{aligned}
\|\nabla_x H_\varepsilon(w(t+1), x(t+1))\| &\leq \sum_{l=1}^{k} \|\nabla_{x^l} H_\varepsilon(w(t+1), x(t+1))\| \qquad (6) \\
&\leq \sum_{l=1}^{k} L_\varepsilon^l(w(t+1), x(t))\|x^l(t+1) - x^l(t)\| + \sum_{l=1}^{k} \|\nabla_{x^l} H_\varepsilon(w(t+1), x(t+1)) - \nabla_{x^l} H_\varepsilon(w(t+1), x(t))\| \\
&\leq \frac{m}{\varepsilon} \sum_{l=1}^{k} \|x^l(t+1) - x^l(t)\| + \sum_{l=1}^{k} \kappa^l(t)\|x^l(t+1) - x^l(t)\|,
\end{aligned}$$

where the last inequality follows from Proposition 3(ii) and Lemma 8 combined with the (5) observation using

$$\kappa^l(t) = \frac{2L_\varepsilon^l(w(t+1), x(t))L_\varepsilon^l(w(t+1), x(t+1))}{L_\varepsilon^l(w(t+1), x(t)) + L_\varepsilon^l(w(t+1), x(t+1))}, \quad l = 1, 2, \ldots, k.$$

ε-KPALM Subgradient Lower Bound The Iterates Gap Proof-Contd.

From Proposition 3(ii) we obtain that

$$\kappa^l(t) = \frac{2}{\frac{1}{L^l_\varepsilon(w(t+1),x(t))} + \frac{1}{L^l_\varepsilon(w(t+1),x(t+1))}} \leq \frac{2}{\frac{\varepsilon}{m} + \frac{\varepsilon}{m}} = \frac{m}{\varepsilon}.$$

ε-KPALM Subgradient Lower Bound The Iterates Gap Proof-Contd.

From Proposition 3(ii) we obtain that

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Hence, from (6), we have

$$\begin{aligned}\|\nabla_x H_\varepsilon(w(t+1), x(t+1))\| &\leq \frac{2m}{\varepsilon} \sum_{l=1}^{k} \|x^l(t+1) - x^l(t)\| \\ &\leq \frac{2m\sqrt{k}}{\varepsilon} \|x(t+1) - x(t)\|.\end{aligned}$$

ε-KPALM Subgradient Lower Bound The Iterates Gap Proof-Contd.

From Proposition 3(ii) we obtain that

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Therefore, setting $\rho_2 = \frac{md_{\mathcal{A}}}{\varepsilon} + \overline{\alpha}\sqrt{m} + \frac{2m\sqrt{k}}{\varepsilon}$, yields the final result

$$\|\gamma(t+1)\| \leq \rho_2 \|z(t+1) - z(t)\| \text{ with } \gamma(t+1) \in \partial\sigma_\varepsilon(z(t+1)).$$