Notations:

$$H_\ell(w,x) = \sum_{i=1}^{m} w_\ell^i \|x^\ell - a^i\|$$

$$L_\ell(w,x) = \begin{cases} \displaystyle\sum_{i=1}^{m} \frac{w_\ell^i}{\|x^\ell - a^i\|} \\ \displaystyle\sum_{\substack{i=1 \\ i\neq j}}^{m} \frac{w_\ell^i}{\|x^\ell - a^i\|} \end{cases}$$

$$R_\ell^j(w,x) = \sum_{\substack{i=1 \\ i\neq j}}^{m} w_\ell^i \frac{x^\ell - a^i}{\|x^\ell - a^i\|}$$

$$t_\ell^j(w,x) = \frac{1}{L_\ell(w,x)}\left(\|R_\ell^j(w,x)\| - w_\ell^j\right)$$

Now we can define the starting point of the algorithm. Let $w(0) \in \Delta m$. Let

$$p_\ell = p_\ell(w(0)) := \operatorname{argmin}\{H_\ell(w(0), a^i) : i = 1, 2, \ldots, m\} \quad \forall \ell = 1, 2, \ldots, K.$$

Then, choose

$$x^\ell(0) = a^{p_\ell} + t_\ell^{p_\ell}(w(0), a^{p_\ell}) \cdot d_\ell^{p^\ell}(w(0), a^{p_\ell}),$$

where

$$d_\ell^j(w,x) := -\frac{R_\ell^j(w,x)}{\|R_\ell^j(w,x)\|}.$$

Therefore, we have that

$$\Psi(z(0)) = H(w(0), x(0)) < \min\{H(w(0), \tilde{a})\},$$ where $\tilde{a} \in \mathbb{R}^{nK}$ which consist any permutation of $(a^1, \ldots, a^m)^K$

Now, thanks to your Proposition 4.1 we have that

$$\Psi(z(t)) < \min\{H(w(0), \tilde{a})\},$$

which means that Assumption 3 holds true.

Now, you should prove the following result:

Lemma. Let $X(0)$ be chosen as explaind before. Then, for all $(w, X) \in \Delta^m \times \mathbb{R}^{nk}$ satisfying $H_\ell(w, X) \leq H_\ell(w(0), X(0))$, the inequality

$$\|x^\ell - a^i\| \geq \frac{H_\ell(w(0), a^i) - H_\ell(w(0), X(0))}{\sum_{i=1}^{m} w_\ell^i(0)}$$

holds true for all $\ell = 1, 2, \dots, k$ and $i = 1, 2, \dots, m$.