Reformulation for norm case

Shalom Sergui, In the following note, is a reformulation of the nonsmooth case $\|x^l - a^i\|$ which I think is worth to consider. Marc

In ε-KPALM the function $H_\varepsilon(x, \omega)$ does not admit a $\nabla_x H_\varepsilon(x, \omega)$ which is globally Lip, + the minimizer w.r.t x of $x \to H_\varepsilon(x, \omega)$ cannot be done explicit.

Here is an alternative way to handle the term $\sqrt{|x^l - a^i|^2 + \varepsilon^2}$ with the help of an additional minimization step.

It uses the following simple and well known fact.

Lemma 1 For any $\alpha \geq 0$

$$\frac{1}{2} \min_{s \geq 0} \left\{ s\alpha + \frac{1}{s} \right\} = \sqrt{\alpha}$$

Proof This is the arithmetic-Geometric inequality: $\frac{1}{2}\left(s\alpha + \frac{1}{s}\right) \geq \sqrt{s\alpha \cdot \frac{1}{s}} = \sqrt{\alpha}$

$\square$

So using Lemma 1 we can write

$$\sqrt{\|u\|^2 + \varepsilon^2} = \frac{1}{2} \min_{v \geq 0} \left\{ v\left(\|u\|^2 + \varepsilon\right) + \frac{1}{v} \right\}$$

with $v^* = \dfrac{1}{\sqrt{\|u\|^2 + \varepsilon^2}} \leq \dfrac{1}{\varepsilon}$ (*) and hence we can constrained $v \in [0, \frac{1}{\varepsilon}]$.

Equipped with the above we can reformulate by using an additional $v = (v^1, \dots, v^m) \in \mathbb{R}^{km}$.

Thus instead of $H_\varepsilon(\omega, x)$, we now use 3 variables: (ω, x, v)

$$H_\varepsilon(\omega, x, v) = \frac{1}{2}\sum_\ell \sum_i \left\{ v_\ell^i \omega_\ell^i \left(|x^\ell - a^i|^2 + \varepsilon^2 \right) + \frac{\omega_\ell^i}{v_\ell^i} \right\}$$

so that now the x-step can be solved analytically (like in KPALM) and the v-step is also given explicitly

via (*) $\quad v_\ell^i = \dfrac{1}{\sqrt{|x^\ell - a^i| + \varepsilon^2}} \quad \forall i, \ell.$

This way ε-KPALM yields 3 explicit formula via:

$$\begin{cases} \omega^i(t+1) = \operatorname{argmin}\left\{ H_\varepsilon(\omega, x(t), v(t)) : \omega \in \Delta^m \right\} & (1) \\ v^i(t+1) = \operatorname{argmin}\left\{ H_\varepsilon(\omega(t+1), x(t), v) : v \in \left[0, \tfrac{1}{\varepsilon}\right]^m \right\} & (2) \\ x(t+1) = \operatorname{argmin}_x H_\varepsilon(\omega(t+1), x, v(t+1)) & (3) \end{cases}$$

where (1) is similar to (4.3) [just the coefficients change] and (3) is " to (3.6) (with new weights), and (2) is given by (*).