Scientific Programming with Mathematica

Problems 1

1. Let F denote the function

$$F(x) \;=\; 5^{5\mathrm{i}x}\,\frac{\Gamma(1-5\mathrm{i}x)}{\Gamma^5(1-\mathrm{i}x)}$$

and consider the integral

$$I(\delta) \;=\; \int_{-\infty}^{\infty} \mathrm{d}x\,\frac{x}{1-\mathrm{e}^{-2\pi x}}\,\mathrm{re}\,(F(x))\,\cos(\delta x)\;,$$

with δ, real.

We want to show that, as $\delta \to 0$,

$$I(\delta) \;=\; \frac{\sqrt{5}}{4\pi^2}\,\log\delta + C + \mathcal{O}(\delta)\;,$$

with C a constant that we wish to compute.

Use the function Series to show that, as $x \to \infty$,

$$F(x) \;\sim\; -\frac{\sqrt{5}}{4\pi^2 x^2}$$

[In order to simplify the result you should first use Normal *and then* Simplify*. You can, of course, check this using Stirling's formula.]*

Plot $\mathrm{re}\,(F(x))$ for x in the range (-3,3) and $\mathrm{re}\,(F(x)) + \frac{\sqrt{5}}{4\pi^2 x}$ for x in the range (1,3).

By breaking up the range of integration into $(-\infty,\,1)$ and $(1,\,\infty)$ and making use of the integral

$$J(\delta) \;=\; -\frac{\sqrt{5}}{4\pi^2}\int_1^{\infty}\frac{\mathrm{d}x}{x}\,\cos(\delta x)\;,$$

which you should evaluate, compute C numerically.

Once you have computed C, compute it again, to higher precision, using the option WorkingPrecision $\to$ 40 for NIntegrate. Increase WorkingPrecision until you have C correct to 100 figures.

[Warning: There seems to be an unresolved bug that particularly affects the Mac front end. If you use WorkingPrecision $\to$ n for large n*, you should also set* AccuracyGoal *to about* n/2*, or Mathematica may crash, without warning.]*

2. Let f and g denote the functions

$$f(x) \;=\; \sin(x^2) + \sin^2(x) \; ; \quad g(x) \;=\; \exp\left(\frac{(5-x)^2}{10}\right) \; .$$

Make a simultaneous plot of f and g for x in the range (2,8), say. How many (real) solutions are there to the equation $f(x) = g(x)$? Find the roots.

[If you use the function FindRoot *you may care to set* WorkingPrecision $\rightarrow$ 80.*]*

3. The function

$$\mathsf{rmat} := \mathsf{RandomReal}[\{-1,1\},\{8,8\}]$$

generates a 8×8 matrix of random numbers, in the range $(-1,1)$. Generate from this two *symmetric* random matrices, A and B with elements in the range $(-1,1)$. Let $M(t)$ be the one parameter family of matrices

$$M(t) \;=\; tA + (1-t)B \; .$$

Plot, simultaneously, the eight eigenvalues of $M(t)$ as a function of t for the range $(-1,1)$. Probably, some of the curves will appear to cross. Do they in fact do so? (If no two curves appear to cross, repeat with different values for A and B).

[For a consistent ordering of the eigenvalues, you may care to use Sort[Eigenvalues[M[t]]]. *]*