Nonequilibrium critical dynamics of ferromagnetic spin systems

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Abstract. We use simple models (the Ising model in one and two dimensions, and the spherical model in arbitrary dimension) to put to the test some recent ideas on the slow dynamics of nonequilibrium systems. In this review the focus is on the temporal evolution of two-time quantities and on the violation of the fluctuation-dissipation theorem, with special emphasis given to nonequilibrium critical dynamics.

Prologue

The aim of this review is to summarise recent works devoted to the dynamics of ferromagnetic spin systems after a quench from infinite temperature to their critical temperature.

The initial impetus for such an investigation was the desire to put to the test, on simple models, some recent ideas on the slow dynamics of nonequilibrium systems (aging of two-time quantities and violation of the fluctuation-dissipation theorem). By simple models we mean models with no quenched disorder, with, for some of them at least, the virtue of being solvable. Here we address the case of ferromagnetic spin systems, such as the Ising model in one and two dimensions, and the spherical model in arbitrary dimension. Urn models are also simple enough to serve the same purpose. They are the subject of another review in this volume [1].

During the course of this investigation we realised the interest of posing the same questions for nonequilibrium critical dynamics [2, 3].

1. The fluctuation-dissipation theorem and its violation

Consider a generic spin system evolving at constant temperature from a disordered initial configuration.

Let s and t, with $s < t$, be two successive instants of time, and $\tau = t - s$, their difference. Denoting by $\sigma(t)$ the spin at time t, we consider the correlation

$$C(t, s) = \langle \sigma(s)\sigma(t) \rangle,$$

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and the local response to a time-dependent external magnetic field $H(t)$

$$R(t,s) = \frac{\delta\langle\sigma(t)\rangle}{\delta H(s)}.$$

At equilibrium, that is when the waiting time s is large compared to the equilibration time $\tau_{\rm eq}$, these functions are stationary. They only depend on the time difference τ:

$$C(s,t) = C_{\rm eq}(\tau),$$
$$R(t,s) = R_{\rm eq}(\tau),$$

and are related by the fluctuation-dissipation theorem (for a simple presentation see e.g. [4]):

$$R_{\rm eq}(\tau) = -\frac{1}{T}\frac{\mathrm{d}C_{\rm eq}(\tau)}{\mathrm{d}\tau}.$$

This situation is typical of the high-temperature regime (e.g. $T > T_c$ for a ferromagnet), where $\tau_{\rm eq}$ is small.

In experiments or simulations, instead of measuring $R(t,s)$, one considers the integrated response, i.e., either the thermoremanent magnetisation of the system at time t, $M_{\rm TRM}(t,s)$, obtained after applying a small magnetic field h, constant between $t=0$ and s; or the zero-field-cooled magnetisation $M_{\rm ZFC}(t,s)$, where now h is constant between s and t. Defining the reduced integrated response $\rho(t,s)$ by

$$\rho(t,s) = \frac{T}{h}M(t,s),$$

we thus have

$$\rho_{\rm TRM}(t,s) = T\int_0^s \mathrm{d}u\, R(t,u), \tag{1.1}$$
$$\rho_{\rm ZFC}(t,s) = T\int_s^t \mathrm{d}u\, R(t,u).$$

At equilibrium, using the fluctuation-dissipation theorem, we have

$$\rho_{\rm TRM}(t,s) = \int_0^{C(\tau)} \mathrm{d}C = C(\tau),$$
$$\rho_{\rm ZFC}(t,s) = \int_{C(\tau)}^1 \mathrm{d}C = 1 - C(\tau),$$

thus a plot of ρ against C is given by a straight line of slope $+1$ ($\rho_{\rm TRM}$) or -1 ($\rho_{\rm ZFC}$), as soon as s is large enough.

At low temperature (below T_c for a ferromagnet), $\tau_{\rm eq}$ is either very large or infinite. In the scaling regime where $1 \ll s \sim t \ll \tau_{\rm eq}$, aging takes place, i.e., C and R are no longer stationary, and the fluctuation-dissipation theorem does not hold. The question is therefore to determine the relationship between C and R, if any. This can be done by defining the fluctuation-dissipation ratio $X(t,s)$ by [5, 6, 7]

$$R(t,s) = \frac{X(t,s)}{T}\frac{\partial C(t,s)}{\partial s}.$$

Assume that, in the scaling regime, all the time dependence of R can be parameterised by C. Or, in other words, that C acts as a clock for R. That is, for $1 \ll s \sim t$,

$$X(t,s) \approx X(C(t,s)). \tag{1.2}$$

As a consequence, we have

$$\rho_{\mathrm{TRM}}(t,s) \approx \int_0^{C(t,s)} \mathrm{d}C\, X(C),$$
$$\rho_{\mathrm{ZFC}}(t,s) \approx \int_{C(t,s)}^1 \mathrm{d}C\, X(C).$$

Hence, in a plot of ρ against C, the slope at a given point is given by $\pm X(C)$.

This behaviour has been observed in a number of instances. In particular, a census of the different cases of spin systems hitherto studied shows the existence of three main types of behaviour at low temperature (for a summary, see [8], and references therein). For domain-growth models, $X(C)$ is discontinuous in C, taking a first value equal to 1, and a second one equal to zero [9, 10, 11] (see discussion in section 2). For spin-glass models with p-spin interactions, $X(C)$ is still discontinuous but the second value is non-zero. Finally, for continuous spin-glass models, $X(C)$ is a non-trivial curve [5].

In the present review we show that, at $T = T_c$, non-trivial statements can be formulated on the same issue. Hereafter we specialise to ferromagnetic spin systems. We take as representatives the Ising model in one and two dimensions, and the spherical model in arbitrary dimension. The Hamiltonian describing these models reads

$$E(t) = -J\sum_{(i,j)} \sigma_i(t)\sigma_j(t) - \sum_i H_i(t)\sigma_i(t),$$

where the first sum runs over pairs of neighbouring sites.

For the Ising model, $\sigma_i = \pm 1$, and the (non-conserved Glauber) dynamics is governed by the heat-bath rule:

$$\mathcal{P}(\sigma_i(t+\mathrm{d}t) = \pm 1) = \frac{1}{2}\left(1 \pm \tanh \beta h_i(t)\right),$$

where the local field reads $h_i = \sum_j \sigma_j + H_i$, the sum running over the neighbours of site i.

For the spherical model, σ_i is a real number with the constraint $\sum_i \sigma_i^2 = N$, where N is the number of spins [12, 13, 14]. The dynamics is governed by the Langevin equation [15]

$$\frac{\mathrm{d}\sigma_i}{\mathrm{d}t} = -\frac{\partial E}{\partial \sigma_i} - \lambda(t)\sigma_i + \eta_i(t).$$

In the right side, $\lambda(t)$ is a Lagrange multiplier ensuring the constraint, and $\eta_i(t)$ is a Gaussian white noise with correlation $\langle \eta_i(t)\eta_j(t')\rangle = \delta_{ij}\delta(t-t')$.

In both cases, at time $t = 0$, the system is in a disordered initial configuration (e.g. corresponding to equilibrium at infinite temperature).

2. Aging below T_c: low-temperature coarsening

We first describe in more detail the behaviour of correlation and response at low temperature, for a generic ferromagnetic model such as the spherical model or the 2D Ising model, evolving at constant temperature after a quench from $T = \infty$ down to $T < T_c$. We defer the discussion of the 1D Ising model to section 4.

In such a situation, domains of opposite sign grow, with a characteristic size $L(t) \sim t^{1/z}$, where $z = 2$ is the growth exponent [16, 17].

In a first regime ($1 \sim \tau \ll s$), dynamics is stationary. Correlations decay from $C(s,s) = 1$, to the plateau value

$$q_{\rm EA} = \lim_{\tau\to\infty}\lim_{s\to\infty} C(s+\tau, s) = M_{\rm eq}^2,$$

where $M_{\rm eq}$ is the equilibrium magnetisation. Though the system becomes stationary, it is still coarsening, and therefore does not reach thermal equilibrium. However the fluctuation-dissipation theorem holds, and $X = 1$.

In the scaling regime where s and t are simultaneously large ($1 \ll s \sim t$), with arbitrary ratio $x = t/s$, aging takes place, and correlations behave as [17]

$$C(t,s) \approx M_{\rm eq}^2\, f_C\left(\frac{t}{s}\right). \tag{2.1}$$

For small temporal separations ($\tau \ll s$, or $x \to 1$), we have $f_C(x) \to 1$, implying $C(t,s) \to M_{\rm eq}^2$. In other words, equation (2.1) describes the departure from the plateau value $M_{\rm eq}^2$. For well-separated times ($1 \ll s \ll t$, or $x \gg 1$) $f_C(x)$ decays algebraically as

$$f_C(x) \approx A_C\, x^{-\lambda/z},$$

where λ is the autocorrelation exponent [18]. As a consequence, we have

$$\frac{\partial C(t,s)}{\partial s} \approx \frac{M_{\rm eq}^2}{s}\, f_{C'}\left(\frac{t}{s}\right),$$

with $f_{C'}(x) \approx A_{C'}\, x^{-\lambda/z}$, at large x.

In the same regime it is reasonable to make the scaling assumption (see discussion below)

$$R(t,s) \approx s^{-1-a}\, f_R\left(\frac{t}{s}\right), \tag{2.2}$$

with an unknown exponent $a > 0$, and with again the decay at large x

$$f_R(x) \approx A_R\, x^{-\lambda/z}. \tag{2.3}$$

We have therefore

$$X(t,s) \approx \frac{s^{-a}}{M_{\rm eq}^2} T \frac{f_R(t/s)}{f_{C'}(t/s)} \approx \frac{s^{-a}}{M_{\rm eq}^2} T \frac{A_R}{A_{C'}}.$$

The fluctuation-dissipation ratio thus vanishes in the scaling regime, irrespective of the ratio t/s.

For instance, for the spherical model, the equilibrium magnetisation reads

$$M_{\rm eq}^2 = 1 - \frac{T}{T_c}$$

and the correlation $C(t,s)$ is given by (2.1) with

$$f_C(x) = \left(\frac{4x}{(x+1)^2}\right)^{D/4},$$

hence the autocorrelation exponent $\lambda = D/2$. The response is given, in the scaling regime, by [19]

$$R(t,s) \approx (4\pi)^{-D/2}\left(\frac{t}{s}\right)^{D/4}(t-s)^{-D/2}, \tag{2.4}$$

which is in agreement with the form (2.2), with scaling function

$$f_R(x) = (4\pi)^{-D/2} x^{D/4} (x-1)^{-D/2},$$

and the exponent $a = D/2 - 1$.

For the 2D Ising model, the exponent $\lambda \approx 1.25$ [18] is only known numerically. This is also the case of the scaling functions f_C and f_R [20]. The latter work is compatible with $a = 1/2$, as predicted in Refs. [11, 21], where it is argued that the integrated response scales as $\rho(t,s) \sim L(s)^{-1} g\left(L(t)/L(s)\right)$ for soft spin models with non-conserved dynamics.

In summary, for a ferromagnetic spin system [10, 15, 19, 11],

- for short times ($\tau \ll s$), such that $C(t,s) > M_{\rm eq}^2$, the fluctuation-dissipation theorem holds, and $X = 1$;
- for long times ($\tau \sim s$), such that $C(t,s) < M_{\rm eq}^2$, the fluctuation-dissipation theorem does not hold, and $X(t,s) \to 0$ independently of the ratio t/s.

Note that we have

$$\frac{\mathrm{d}X(C)}{\mathrm{d}C} = \delta\left(C - M_{\rm eq}^2\right),$$

in agreement with the static interpretation of $X(C)$ in terms of the distribution of overlaps $P(q)$ [22].

3. Aging at T_c: critical coarsening

The system is now quenched from $T = \infty$ down to T_c.

In such circumstances, spatial correlations develop in the system, just as in the critical state, but only over a length scale which grows like t^{1/z_c}, where z_c is the dynamic critical exponent. On scales smaller than t^{1/z_c} the system looks critical, while on larger scales the system is still disordered. For instance, the equal-time correlation function $C_r(t) = \langle \sigma_0(t)\sigma_r(t)\rangle$ scales as

$$C_r(t) \approx |r|^{-2\beta/\nu}\, g\left(\frac{r}{t^{1/z_c}}\right),$$

where β and ν are the usual static exponents. (A summary of the values of exponents is given in the Table.) The scaling function $g(y)$ goes to a constant as $y \to 0$, while it falls off very rapidly when $y \to \infty$.

The same temporal regimes, as defined in the previous section, are to be considered. However, their physical interpretation is slightly different, since the order parameter $M_{\rm eq}^2$ vanishes and symmetry between the phases is restored.

In the first regime ($\tau \ll s$), the system again becomes stationary, so that the fluctuation-dissipation holds.

In the scaling regime ($\tau \sim s$), temporal correlations behave as¶

$$C(t,s) \approx s^{-a_c}\, f_C\left(\frac{t}{s}\right), \qquad a_c = 2\beta/\nu z_c = (D-2+\eta)/z_c. \tag{3.1}$$

It is instructive to relate this behaviour to that observed for $T < T_c$, namely, $C(t,s) \approx M_{\rm eq}^2\, f_C\left(t/s\right)$. The passage from one formula to the other one is done by

¶ For simplicity we use the same notation f_C, f_R, etc. for the scaling functions appearing in this section, though they are different from those appearing in the previous section. We use the same convention for the amplitudes A_C, A_R, etc.

noticing that in the critical region one has $M_{\rm eq} \sim |T - T_c|^\beta \sim \xi_{\rm eq}^{-\beta/\nu}$. Replacing $\xi_{\rm eq}$ by s^{1/z_c} implies the replacement of $M_{\rm eq}^2$ by $s^{-2\beta/\nu z_c} \sim s^{-(D-2+\eta)/z_c}$.

At large time separations ($x \gg 1$) we have (see [23] for a derivation in the case of the so-called model A [24])

$$f_C(x) \approx A_C\, x^{-\lambda_c/z_c},$$

where λ_c is the critical autocorrelation exponent [25], related to the initial-slip critical exponent Θ_c [23] by $\lambda_c = D - z_c\Theta_c$.

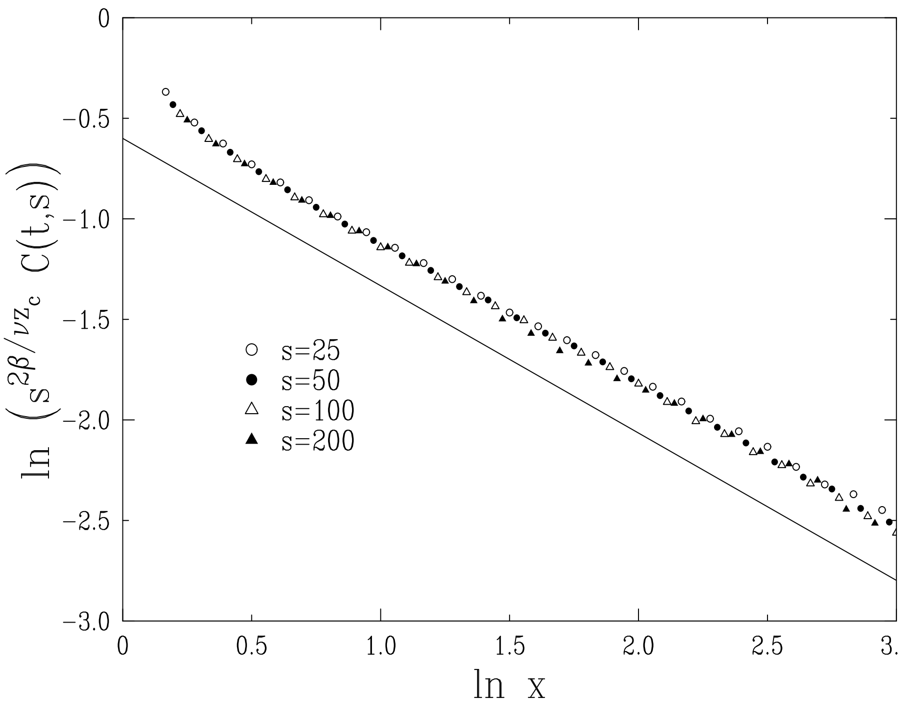

Figure 1. Log-log plot of the critical autocorrelation function $C(t, s)$ of the two-dimensional Ising model, against time ratio $x = t/s$, for several values of the waiting time s. Data are multiplied by $s^{2\beta/\nu z_c}$, in order to demonstrate collapse into the scaling function $f_C(x)$ of eq. (3.1). Straight line: exponent $-\lambda_c/z_c \approx -0.73$ of the fall-off at large x. (After ref. [2].)

As a consequence of (3.1), we have

$$\frac{\partial C(t,s)}{\partial s} \approx s^{-1-a_c}\, f_{C'}\left(\frac{t}{s}\right),$$

with the decay $f_{C'}(x) \approx A_{C'}\, x^{-\lambda_c/z_c}$ at large x.

In the scaling regime, the response function behaves as

$$R(t,s) \approx s^{-1-a_c}\, f_R\left(\frac{t}{s}\right), \tag{3.2}$$

and, for large temporal separations,

$$f_R(x) \approx A_R\, x^{-\lambda_c/z_c}. \tag{3.3}$$

(See [23] for a derivation of (3.2) and (3.3) in the case of model A.) Note the similarity of (3.2) and (3.3) with (2.2) and (2.3), respectively. The scaling form (3.2) of the response implies

$$\rho_{\mathrm{TRM}}(t,s) \approx s^{-a_c}\, f_\rho\left(\frac{t}{s}\right),$$

with, as $x \gg 1$, $f_\rho(x) \approx A_\rho x^{-\lambda_c/z_c}$.

We finally obtain the fluctuation-dissipation ratio

$$X(t,s) \approx T_c \frac{f_R(t/s)}{f_{C'}(t/s)} = \mathcal{X}\left(\frac{t}{s}\right),$$

and, at large temporal separations,

$$X_\infty = \lim_{s\to\infty}\lim_{t\to\infty} X(t,s) = \lim_{x\to\infty}\mathcal{X}(x) = T_c\frac{A_R}{A_{C'}} = T_c\frac{A_\rho}{A_C}.$$

The last equality is equivalent to saying that, for $1 \ll s \ll t$,

$$\rho_{\mathrm{TRM}}(t,s) \approx X_\infty C(t,s).$$

The limit fluctuation-dissipation ratio X_∞ can thus be measured as the slope near the origin of the $C - \rho_{\mathrm{TRM}}$ plot. The scaling function $\mathcal{X}(x)$, and in particular the amplitude ratio X_∞, are universal, in the sense that they neither depend on initial conditions nor on the details of the dynamics [2, 3].

In the scaling regime, neither $\rho_{\mathrm{TRM}}(t,s)$ nor $X(t,s)$ are functions of $C(t,s)$. Instead, $X(t,s)$ and $s^{a_c}\rho(t,s)$ are functions of $x = t/s$, which is in contrast with the situations where equation (1.2) holds, and further described in section 1.

We now illustrate the results presented above. For the spherical model (see the Table for the value of exponents), the two-time correlation function reads

$$C(t,s) \approx s^{-(D/2-1)} f_C(x),$$

where

$$f_C(x) = \begin{cases} T_c \dfrac{4(4\pi)^{-D/2}}{(D-2)(x+1)} x^{1-D/4}(x-1)^{1-D/2} & 2 < D < 4, \\[2ex] T_c \dfrac{2(4\pi)^{-D/2}}{D-2}\Big((x-1)^{1-D/2} - (x+1)^{1-D/2}\Big) & D > 4. \end{cases}$$

Thus

$$\lambda_c = \begin{cases} 3D/2 - 2 & 2 < D < 4, \\ D & D > 4. \end{cases}$$

Similarly, the response function behaves as

$$R(t,s) \approx s^{-D/2} f_R(x),$$

where the scaling function $f_R(x)$ reads

$$f_R(x) = \begin{cases} (4\pi)^{-D/2} x^{1-D/4}(x-1)^{-D/2} & 2 < D < 4, \\[2ex] (4\pi)^{-D/2}(x-1)^{-D/2} & D > 4. \end{cases}$$

Finally

$$X_\infty = \begin{cases} 1 - 2/D & 2 < D < 4, \\ 1/2 & D > 4. \end{cases}$$

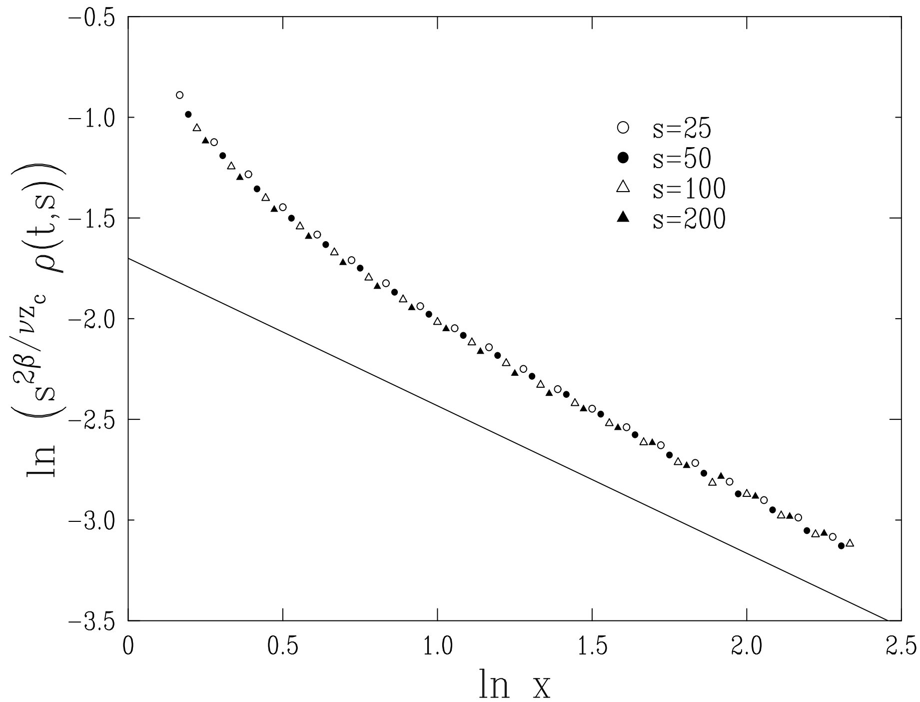

Figure 2. Log-log plot of the critical integrated response function $\rho_{\rm TRM}(t,s)$ of the two-dimensional Ising model, against time ratio $x = t/s$, for several values of the waiting time s. Data are multiplied by $s^{2\beta/\nu z_c}$, in order to demonstrate collapse into the scaling function $f_\rho(x)$. Straight line: exponent $-\lambda_c/z_c \approx -0.73$ of the fall-off at large x. (After ref. [2].)

For the 2D Ising model $\lambda_c \approx 1.59$ [25] is only known numerically. Figures 1 and 2 show numerical determinations of the scaling functions f_C and f_ρ [2]. In two dimensions, we have $X_\infty \approx 0.26$, and a preliminary study leads to $X_\infty \approx 0.40$ in three dimensions [2].

The above discussion can be summarised as follows:

- For short times ($\tau \ll s$), such that $C(t,s) \gg s^{-2\beta/\nu z_c}$, the fluctuation-dissipation theorem holds, and $X = 1$.
- For long times ($\tau \sim s$), such that $C(t,s) \sim s^{-2\beta/\nu z_c}$, the fluctuation-dissipation theorem does not hold. The fluctuation-dissipation ratio $X(t,s)$ is given by the scaling function $\mathcal{X}(t/s)$, such that $\mathcal{X}(x) \to X_\infty$ as $x \to \infty$.

This is the critical counterpart of the behaviour of $X(t,s) = X(C)$ for $T < T_c$, summarised at the end of section 2.

A last comment is in order. At thermal equilibrium, for a ferromagnetic system at criticality, the relationship between magnetic field and magnetisation, $h \sim M_{\rm eq}^\delta$, is nonlinear. Therefore linear-response theory, used above to extract the response of the system, only holds for a magnetic field small compared to the scale $h_0 \sim s^{-\beta\delta/\nu z_c} \sim s^{-(D+2-\eta)/2z_c}$.

4. One-dimensional Ising model at $T = 0$

The one-dimensional Ising model is special in the sense that its critical temperature T_c is zero. Hence the low-temperature phase does not exist.

Another peculiarity of the model stems from the fact that the magnetisation exponent β is equal to zero. As a consequence, at criticality (i.e., at $T = 0$), there is no temporal prefactor in the expression of $C(t,s)$ (or equivalently, no spatial prefactor in that of $C_r(t)$). Indeed, let us recall that, at criticality, for a generic ferromagnetic model, we had

$$\begin{aligned} C_r(t) &\approx |r|^{-2\beta/\nu}\, g\left(\frac{r}{t^{1/z_c}}\right) \\ C(t,s) &\approx s^{-2\beta/\nu z_c}\, f_C\left(\frac{t}{s}\right). \end{aligned}$$

For the 1D Ising model at zero temperature we have

$$\begin{aligned} C_r(t) &\approx \operatorname{erfc}\left(\frac{|r|}{2t^{1/2}}\right) \\ C(t,s) &\approx \frac{2}{\pi}\arctan\left(\frac{2s}{t-s}\right)^{\frac{1}{2}}. \end{aligned} \tag{4.1}$$

The latter formulas are compatible with the former ones, taking into account that $\beta = 0$ for the 1D Ising model. Otherwise stated, the absence of an anomalous dimension implies that $C(t,s)$ is not small in the critical region, in contrast to the generic cases considered in the previous section.

From (4.1) we obtain

$$f_{C'}(x) = \frac{x}{\pi(x+1)}\sqrt{\frac{2}{x-1}}.$$

The critical temperature T_c being equal to zero, we define the dimensionless response function

$$\tilde{R}(t,s) = T\frac{\delta\langle\sigma(t)\rangle}{\delta H(s)}.$$

In the scaling region ($1 \ll s \sim t$), this function is found to behave as

$$\tilde{R}(t,s) \approx s^{-1}\, f_{\tilde{R}}\left(\frac{t}{s}\right),$$

where

$$f_{\tilde{R}}(x) = \frac{1}{\pi\sqrt{2(x-1)}}.$$

This again is compatible with the generic case, with $\beta = 0$.

The reduced magnetisation $\rho_{\mathrm{TRM}}(t,s)$ and the fluctuation-dissipation ratio $X(t,s)$ can be computed explicitly. Both quantities only depend on t/s, or equivalently on C, in the scaling regime. One finds, in this regime [3, 26],

$$\rho_{\mathrm{TRM}}(C) = \frac{\sqrt{2}}{\pi}\arctan\left(\frac{1}{\sqrt{2}}\tan\frac{\pi C}{2}\right),$$

while X is more simply written in terms of the ratio $x = t/s$ as

$$X(t,s) = \frac{f_{\tilde{R}}(x)}{f_{C'}(x)} = \frac{x+1}{2x}.$$

We note once again that the fact that $\beta = 0$ implies no dependence in s in these quantities. Finally, the last equation implies for the limiting ratio

$$X_\infty = \lim_{s\to\infty} \lim_{t\to\infty} X(t,s) = \frac{1}{2}.$$

5. Discussion

At criticality, for the generic cases of the spherical model and of the 2D Ising model, $X(t,s)$ is not a function of $C(t,s)$. It is instead a function of the ratio $x = t/s$, or equivalently of $s^{2\beta/\nu z_c} C(t,s) = f_C(x)$. In this last representation, the value of X at the origin is equal to X_∞. Then the fluctuation-dissipation ratio increases and reaches the limit value 1 when the abscissa $f_C(x)$ goes to infinity, that is, for $x \to 1$, where the fluctuation-dissipation theorem holds.

Is the amplitude ratio X_∞ related to equilibrium quantities? This remains an interesting open question. More generally, do the above results on the fluctuation-dissipation ratio admit a static interpretation, e.g. in terms of the distribution of overlaps $P(q)$ [22]? Strictly speaking, the existence of a non-trivial X_∞ should imply the presence of an unexpected discrete component in $P(q)$. We mention a recent work on related matters [27], where the finite-size behaviour of $P(q)$ for the 2D X-Y model is related to the finite-time behaviour of $\rho(t,s)$.

A recent analysis [20], based on conformal invariance, predicts the following form of the response function

$$R(t,s) = r_0 (t-s)^{-A} \left(\frac{t}{s}\right)^{-B}, \tag{5.1}$$

without predicting the values of exponents appearing in the right side. This prediction should hold for a large class of systems. We note in particular that, for the spherical model, equations (2.2) and (3.2), together with the explicit forms of the scaling functions $f_R(x)$, given in sections 2 and 3, confirm this prediction, which is also verified by numerical computations on the Ising model in two and three dimensions [20]. The analytical results for the 1D Ising model given in section 4 do not, however, satisfy the prediction (5.1).

Finally it is worth adding a few words on the comparison between the results reviewed here and those reviewed in ref. [1] for urn models. For the zeta urn model, the situation at criticality is in all aspects similar to that of a generic ferromagnetic model, as described in section 3. However the prediction (5.1) is not fulfilled by this model. In the low-temperature phase, the results obtained for the zeta urn model do not fall in the framework reviewed in section 2, valid for a coarsening system. Finally, the results obtained for the backgammon model at $T = 0$ are rather different from the generic behaviour of a ferromagnetic model. A natural explanation of the discrepancy between urn models and ferromagnetic models is that in the former case the system is rather subject to condensation than to coarsening.

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Table 1. Static and dynamical exponents of the ferromagnetic spherical model, and of the Ising model in one and two dimensions. First group: usual static critical exponents η, β, and ν (equilibrium). Second group: zero-temperature dynamical exponents z and λ (coarsening below T_c). Third group: dynamic critical exponents z_c, λ_c, and Θ_c (nonequilibrium critical dynamics).

exponent	spherical ($2 < D < 4$)	spherical ($D > 4$)	2D Ising	1D Ising
η	0	0	1/4	1
β	1/2	1/2	1/8	0
ν	$1/(D-2)$	1/2	1	1
z	2	2	2	
λ	$D/2$	$D/2$	≈ 1.25	
z_c	2	2	≈ 2.17	2
λ_c	$3D/2-2$	D	≈ 1.59	1
Θ_c	$1-D/4$	0	≈ 0.19	0

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