Spherical model on the Cayley tree

1978 J. Phys. A: Math. Gen. 11 1119

(http://iopscience.iop.org/0305-4470/11/6/022)

J. Phys. A: Math. Gen., Vol. 11, No. 6, 1978. Printed in Great Britain

Spherical model on the Cayley tree

W M X Zimmer and G M Obermair
Fachbereich Physik, Universität Regensburg, 8400 Regensburg, Germany

Received 9 November 1977

Abstract. We investigate the phase transition of the spherical model on the Cayley tree. There is only one transition temperature T_c; the specific heat shows a finite discontinuity at T_c; there is no spontaneous magnetisation; the susceptibility remains finite, but shows a cusp at T_c with infinite slope on the high temperature side. These results differ significantly from spherical model properties on a lattice and from the recent Ising model calculations on the Cayley tree with its phase transition of continuous order.

1. Introduction

Statistical mechanics on a Cayley tree has recently met with some attention: this topology is simple (and unrealistic) enough to allow exact solutions for a number of problems.

A Cayley tree (cf figure 1) is distinguished by the fact that there are no closed paths coming back to a point and that all bulk points have the same number of nearest neighbours. If we call this coordination of the bulk points $k+1$, then $k=1$ corresponds to the linear chain and for $k \geq 2$ we have real trees.

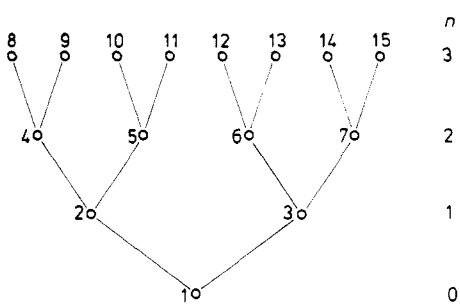

Figure 1. Cayley tree $k=2$, $L=3$. Values of i are shown next to the open circles.

Due to the fact that there are no closed loops on the tree, the number of points grows as k^n where n runs over all 'generations' $n=0, 1, \ldots, L$ and where the Lth generation represents the surface. Therefore the fraction of points on the surface tends towards $(k-1)/k$ for $L \to \infty$.

In particular the Ising model on the Cayley tree has been investigated by a number of authors: Eggarter (1974), Matsuda (1974), Müller-Hartmann and Zittartz (1974, 1975), von Heimburg and Thomas (1974) and Falk (1975). They found some surprising results: the Cayley tree as the basis of a statistical system in its own right is not identical with the Bethe lattice of Domb (1960); instead it exhibits a new type of phase transition which has been called 'phase transition of continuous order': the specific heat is completely regular and is identical with the corresponding expression for a linear chain; there is no spontaneous magnetisation, but the susceptibility and all higher derivatives of the free energy with respect to the field diverge at a series of infinitely many different temperatures.

Stimulated by these remarkable results, we found it worthwhile to look at the behaviour of other models on the Cayley tree.

In this paper we study the mean spherical model (MSM) on a Cayley tree. First we give a short review of the MSM as introduced by Lewis and Wannier (1952, 1953) and Yan and Wannier (1965) on the basis of the spherical model of Berlin and Kac (1952) (cf the review article by Joyce 1972).

In §§ 3 and 4 we calculate the partition function for the tree topology and solve the spherical condition which yields the relevant non-analytic temperature dependence of the Lagrange parameter. This leads to the discussion of the thermodynamic properties of our model and the character of its phase transition, including the distribution of the mean spin length over the tree in §§ 5 and 6.

2. The mean spherical model (MSM)

The MSM is made up of N scalar 'spin' variables $-\infty<\sigma_i<\infty$, with the restriction that the thermal average of the spin length $1/N\langle\Sigma_i\sigma_i^2\rangle=1$; this is called the spherical condition.

Therefore the MSM is described by the following effective Hamiltonian

$$-\beta\mathscr{H}=K\sum_{\langle i,j\rangle}\sigma_i\sigma_j-S\sum_i\sigma_i^2+b\sum_i\sigma_i \tag{1}$$

where the first term is the exchange interaction, $K=J/2k_BT$.

The second term is a Lagrange term for the total length of the spins, where S must be determined in such a way that $\langle\Sigma_i\sigma_i^2\rangle=N$.

The third term is the Zeeman term with b proportional to the magnetic field B: $b=\beta\mu B$.

With this Hamiltonian we get the partition function

$$Q_N=\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}\mathrm{d}\sigma_1\ldots\mathrm{d}\sigma_N\,\mathrm{e}^{-\beta\mathscr{H}\{\sigma_i\}}. \tag{2}$$

To determine the Lagrange parameter S which we can consider as a function of the effective coupling $S=S(K,b)$, we have to solve the spherical condition defined by

$$\left\langle\sum_i\sigma_i^2\right\rangle=N=-\frac{\partial}{\partial S}\ln Q_N. \tag{3}$$

In order to calculate the partition function we define a matrix $\mathbf{A}$ and vectors $\boldsymbol{\sigma}$, $\mathbf{1}$ by

$$-\beta\mathscr{H}=:-K\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{A}\boldsymbol{\sigma}+b\mathbf{1}^{\mathrm{T}}\boldsymbol{\sigma} \tag{4}$$

$\mathbf{A}$ means the $(N\times N)$ matrix coordinated to the bilinear form in the spin variables σ_i and all vectors are elements of $\mathbb{R}^N$.

With this matrix **A**, we can calculate the partition function in closed form and we get (cf appendix 1)

$$Q_N = \left(\frac{\pi}{K}\right)^{N/2} (\det \mathbf{A})^{-1/2} \exp\left(\frac{b^2}{4K}\sum_i \frac{\det \mathbf{A}_i}{\det \mathbf{A}}\right) \tag{5}$$

where $\mathbf{A}_i$ denotes the matrix obtained from **A** by replacing the ith row by

$$\mathbf{1}^{\mathrm{T}} = (1, \ldots, 1) \in \mathbb{R}^N.$$

Subsequently we get for the internal energy per site in zero field $U = \langle -J\, \Sigma_{ij}\, \sigma_i\sigma_j\rangle$. With our definitions and the spherical condition one obtains

$$\begin{aligned} u := & -\mathcal{J}\frac{\partial}{\partial K}\left(\frac{1}{N}\ln Q_N\right) - \mathcal{J}\frac{S(K)}{K} \\ = & \,\mathcal{J}\left(\frac{1}{4K} - z(K)\right) \qquad \text{where } z(K) = \frac{S(K)}{K}; \end{aligned} \tag{6}$$

for the specific heat per site

$$c_{H=0} := \frac{\partial}{\partial T} u = \frac{1}{2}k_B\left(1 + 4K^2\frac{\partial}{\partial K}z(K)\right); \tag{7}$$

for the magnetisation per site

$$m := \frac{\partial}{\partial b}\left(\frac{1}{N}\ln Q_N\right) = \frac{b}{2KN}\sum_i \frac{\det \mathbf{A}_i}{\det \mathbf{A}}; \tag{8}$$

and finally the isothermal susceptibility per site becomes

$$\chi_T := \beta\mu^2\frac{\partial m}{\partial b} = \beta\mu^2\left(\frac{1}{2KN}\sum_i \frac{\det \mathbf{A}_i}{\det \mathbf{A}} + \frac{\partial m}{\partial z}\frac{\partial z(K, b)}{\partial b}\right). \tag{9}$$

From equations (6)–(9) we can see that the most important thermodynamic variable is the reduced Lagrange parameter $z(K, b) = S(K, b)/K$. The value of this parameter is to be chosen in such a way that it solves the spherical condition (3) which, taking (5) into account, becomes

$$2K = \frac{1}{N}\left(\frac{\partial}{\partial z}\ln \det \mathbf{A} - \frac{b^2}{2K}\frac{\partial}{\partial z}\sum_i \frac{\det \mathbf{A}_i}{\det \mathbf{A}}\right). \tag{10}$$

3. Calculation of the partition function

After this review of the MSM, we consider at first the term det **A** in the partition function (5).

On the Cayley tree **A** assumes the following appearance (figure 2) for the simple numbering of the sites i indicated in figure 1.

Though we need only calculate the determinant of **A** it is always of interest to know the eigenvalues of this matrix and therefore we compute at first the roots of

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

To do this, we use elementary transformations to bring the symmetric matrix $\mathbf{A} - \lambda\mathbf{I}$ to a triangular form of the appearance seen in figure 3.

$$
\begin{array}{c|cc|cccc|cccccccc}
\hline
z & -1 & -1 & & & & & & & & & & & & \\
\hline
-1 & z & & -1 & -1 & & & & & & & & & & \\
-1 & & z & & & -1 & -1 & & & & & & & & \\
\hline
 & -1 & & z & & & & -1 & -1 & & & & & & \\
 & -1 & & & z & & & & & -1 & -1 & & & & \\
 & & -1 & & & z & & & & & & -1 & -1 & & \\
 & & -1 & & & & z & & & & & & & -1 & -1 \\
\hline
 & & & -1 & & & & z & & & & & & & \\
 & & & -1 & & & & & z & & & & & & \\
 & & & & -1 & & & & & z & & & & & \\
 & & & & -1 & & & & & & z & & & & \\
 & & & & & -1 & & & & & & z & & & \\
 & & & & & -1 & & & & & & & z & & \\
 & & & & & & -1 & & & & & & & z & \\
 & & & & & & -1 & & & & & & & & z \\
\hline
\end{array}
$$

Figure 2. Matrix **A** for $k=2$, $L=3$; $z=S/K$.

$$
\begin{array}{c|cc|cccc|cccccccc}
\hline
a_4 & & & & & & & & & & & & & & \\
\hline
-1 & a_3 & & & & & & & & & & & & & \\
-1 & & a_3 & & & & & & & & & & & & \\
\hline
 & -1 & & a_2 & & & & & & & & & & & \\
 & -1 & & & a_2 & & & & & & & & & & \\
 & & -1 & & & a_2 & & & & & & & & & \\
 & & -1 & & & & a_2 & & & & & & & & \\
\hline
 & & & -1 & & & & a_1 & & & & & & & \\
 & & & -1 & & & & & a_1 & & & & & & \\
 & & & & -1 & & & & & a_1 & & & & & \\
 & & & & -1 & & & & & & a_1 & & & & \\
 & & & & & -1 & & & & & & a_1 & & & \\
 & & & & & -1 & & & & & & & a_1 & & \\
 & & & & & & -1 & & & & & & & a_1 & \\
 & & & & & & -1 & & & & & & & & a_1 \\
\hline
\end{array}
$$

Figure 3. Triangular matrix **B**.

In the diagonal of this triangular matrix **B** we have now terms a_l, where l corresponds to the $(L+1-l)$th generation of the tree (see figure 1).

In detail we get from the appropriate subtractions of multiples of the lower lines of **A** from lines further above:

$$a_1 = z-\lambda \qquad a_l=(z-\lambda)-\frac{k}{a_{l-1}} \qquad l \geqslant 2. \tag{11}$$

By multiplying this continued fraction with products $A_{l-1}=a_1 \,.\, a_2 \ldots a_{l-1}$ we get a difference equation for these products

$$A_l=(z-\lambda)A_{l-1}-kA_{l-2} \tag{12}$$

which can be solved in an elementary manner and yields the following eigenvalues

$$\lambda(l, r) = z - 2\sqrt{k}\cos\frac{r\pi}{l+1} \qquad r = 1, \ldots, l;\, l = 1, \ldots, L+1. \tag{13}$$

These eigenvalues are identical to that of the MSM on a set of open linear chains of lengths $l+1$ with an effective exchange constant $J\sqrt{k}$. There are many independent eigenmodes on the tree corresponding to oscillations of different regions of the tree; these eigenvalues show a degeneracy depending on l:

$$\text{degeneracy} = \begin{cases} (k-1)k^{L-l} & l \leq L \\ 1 & l = L+1. \end{cases} \tag{14}$$

Hence we get for det $\mathbf{A}$ the following expression

$$\det \mathbf{A} = \prod_{l=1}^{L} \prod_{r=1}^{l} \left(z - 2\sqrt{k}\cos\frac{r\pi}{l+1}\right)^{(k-1)k^{L-l}} \prod_{r=1}^{L+1} \left(z - 2\sqrt{k}\cos\frac{r\pi}{L+2}\right). \tag{15}$$

We first carry out the finite product (cf Gradsteyn and Ryzhik 1965).

$$\begin{aligned} A_l &:= \prod_{r=1}^{l} \left(z - 2\sqrt{k}\cos\frac{r\pi}{l+1}\right) \\ &= (\sqrt{k})^l \frac{\sinh(l+1)y}{\sinh y}, \qquad \text{where } \cosh y = \frac{z}{2\sqrt{k}}. \end{aligned} \tag{16}$$

With (16) we obtain

$$\det \mathbf{A} = \prod_{l=1}^{L} \left((\sqrt{k})^l \frac{\sinh(l+1)y}{\sinh y} \right)^{(k-1)k^{L-l}} (\sqrt{k})^{L+1} \frac{\sinh(L+2)y}{\sinh y}. \tag{17}$$

If L grows beyond all limits we finally get

$$(\det \mathbf{A})^{1/N} = \prod_{l=1}^{\infty} \left((\sqrt{k})^l \frac{\sinh(l+1)y}{\sinh y} \right)^{(k-1)^2(1/k)^{l+1}}. \tag{18}$$

In order to obtain the partition function (5) in a magnetic field we now have to calculate

$$\text{Chi} := \sum_i \frac{\det \mathbf{A}_i}{\det \mathbf{A}}. \tag{19}$$

The procedure to do this is essentially the same as that applied in the preceding part of this section: after transforming the matrix $\mathbf{A}_i$ to a triangular matrix which can be reduced to inhomogeneous difference equation and solved by elementary techniques.

Because this procedure is a little bit lengthy, we give only the final result:

$$\begin{aligned} \text{Chi} = \frac{k^{L+1}}{k+1-z} \sum_{l=1}^{L+1} \left(\frac{1}{k}\right)^l \Bigg\{ (\sqrt{k})^l \frac{\sinh y}{\sinh(l+1)y} + \frac{1}{\sqrt{k}} \frac{\sinh ly}{\sinh(l+1)y} - 1 + \frac{1}{\sqrt{k}} \frac{\sinh ly}{\sinh(l+1)y} \\ \times \left[\left(\frac{1}{\sqrt{k}}\right)^{L+1-l} \frac{\sinh y}{\sinh(L+2-l)y} + \sqrt{k} \frac{\sinh(L+1-l)y}{\sinh(L+2-l)y} - 1 \right] \Bigg\} \end{aligned} \tag{20}$$

which, in the limit $L \to \infty$, reduces to

$$\frac{1}{N}\text{Chi} = \frac{k-1}{k+1-z} \left[e^{-y} \sum_{l=1}^{\infty} \left(\frac{1}{k}\right)^l \frac{\sinh ly}{\sinh(l+1)y} + \sum_{l=1}^{\infty} \left(\frac{1}{\sqrt{k}}\right)^l \frac{\sinh y}{\sinh(l+1)y} - \frac{1}{k-1} \right]. \tag{21}$$

We have made sure that those terms of (20) that vanish in the thermodynamic limit do not contain even infinitely weak singularities.

A careful examination moreover reveals (cf appendix 2) that the apparent pole of Chi at $z=k+1$ has a vanishing residue, in other words, that Chi is analytic for all $z>2\sqrt{k}$, or, equivalently, $y>0$.

4. The spherical condition

According to the construction of the MSM the partition function and the thermodynamic properties have to be considered under the constraint of the spherical condition (3) and (10). With the results (18) and (21) of the previous section we obtain from (10) the spherical condition for our model

$$\frac{2K}{(k-1)^2}=\frac{\mathrm{d}}{\mathrm{d}z}\sum_{l=1}^{\infty}\left(\frac{1}{k}\right)^{l+1}\ln\frac{\sinh(l+1)y}{\sinh y}-\frac{b^2}{2(k-1)K}\frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{k+1-z}$$
$$\times\left[\mathrm{e}^{-y}\sum_{l=1}^{\infty}\left(\frac{1}{k}\right)^{l}\frac{\sinh ly}{\sinh(l+1)y}+\sum_{l=1}^{\infty}\left(\frac{1}{\sqrt{k}}\right)^{l}\frac{\sinh ly}{\sinh(l+1)y}-\frac{1}{k-1}\right]$$

with $y=y(z)=\cosh^{-1}(z/2\sqrt{k})=\cosh^{-1}(S/2\sqrt{k}K)$.

As in all MSM calculations one is interested in that region of parameters where the right-hand side of the spherical condition might have singularities of some sort with respect to the variable $z=S/K$ (Lagrange parameter). Since we have made sure that these terms are, in our case, analytic for all $z>2\sqrt{k}$ (or $y>0$), we have to concentrate on the region $z\gtrsim 2\sqrt{k}$; as z approaches $2\sqrt{k}$ the lowest eigenvalue of the effective Hamiltonian (1) vanishes according to equation (13); thermodynamic stability, however, requires all eigenvalues to be positive; therefore, z is restricted to the interval $(2\sqrt{k},\infty)$.

Keeping this in mind, we evaluate (22) and find

$$\frac{4\sqrt{k}K}{(k-1)^2}=\sum_{l=1}^{\infty}\left(\frac{1}{k}\right)^{l+1}\frac{(l+1)\cosh(l+1)y-\operatorname{cotanh}y\sinh(l+1)y}{\sinh y\sinh(l+1)y}$$
$$-\frac{b^2}{2(k-1)K}\left(\frac{\mathrm{e}^{-y}}{(k+1-z)\sinh y}\sum_{l=1}^{\infty}\left(\frac{1}{k}\right)^{l}\frac{\sinh ly}{\sinh(l+1)y}-R(y)\right). \tag{23}$$

Here we have explicitly given all terms that can give rise to singularities; the remaining $R(y)$ varies slowly and continuously everywhere.

Expanding (23) about $b=0$ and $y=0$, i.e. where the Lagrange parameter z approaches its limiting value $2\sqrt{k}$, we get

$$K=K_c-A_c^{-1}\left(\frac{z}{2\sqrt{k}}-1\right)+\frac{b^2}{K}F_c\left(\frac{z}{2\sqrt{k}}-1\right)^{-1/2}. \tag{24}$$

Here K_c, A_c, F_c depend only on k:

$$K_c=\frac{1}{12\sqrt{k}}\frac{3k-1}{k-1}$$

$$A_c^{-1} = \frac{1}{360\sqrt{k}} \frac{45k^3 - 25k^2 + 35k - 7}{(k-1)^3}$$

$$F_c = \frac{1}{8\sqrt{k}} \frac{k-1}{k+1-2\sqrt{k}} \left(\frac{k}{k-1} - k \ln \frac{k}{k-1} \right).$$

Equation (24) determines the temperature dependence of z; as we lower T we approach a critical temperature $T_c = J/2k_B K_c$ where, for $b \to 0$, $z \to 2\sqrt{k}$ from above. In this case we get

$$(z/2\sqrt{k}) - 1 = A_c(K_c - K) \qquad K < K_c. \tag{25}$$

We see that in the zero field case the reduced Langrange parameter approaches its limiting value 1 linearly as we lower the temperature. This behaviour is in contrast to the three-dimensional lattice model, where the analogous Lagrange parameter $S/6K$ approaches its critical value quadratically.

For finite magnetic field z remains above $2\sqrt{k}$ for all temperatures; if $b \to 0$ for $T < T_c$, $z/2\sqrt{k}$ can approach 1 in such a way that (24) is fulfilled for all temperatures.

This behaviour is sketched in figure 4.

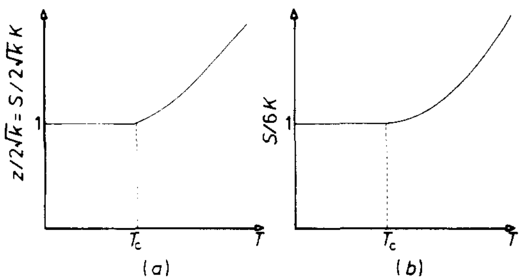

Figure 4. Solution of the spherical equation (*a*) Cayley tree; (*b*) three-dimensional lattice.

5. The thermodynamic properties

According to (25) the slope of $z(T)$ is a discontinuous function of the temperature $T \sim 1/K$. This has to be contrasted with the three-dimensional lattice model where $z(T)$ has a continuous slope and only its second derivative is a discontinuous function of temperature. So we have the remarkable result that our model has a zero field phase transition which is even stronger than that on the three-dimensional lattice, whereas the Ising model on the Cayley tree in zero field exhibits no phase transition of any finite order.

Due to the discontinuity of $\mathrm{d}z/\mathrm{d}T$ the specific heat, calculated from (7), has a finite step at T_c.

This behaviour is sketched in figure 5.

Thus the phase transition is more pronounced on the tree than on a regular three-dimensional lattice. An analogous result was found by Rauh (1976) for a three-dimensional lattice with layerwise disorder, for which the specific heat also shows a finite step. This is another argument in favour of the supposition that a Cayley tree has model properties not too different from those of disordered systems, a

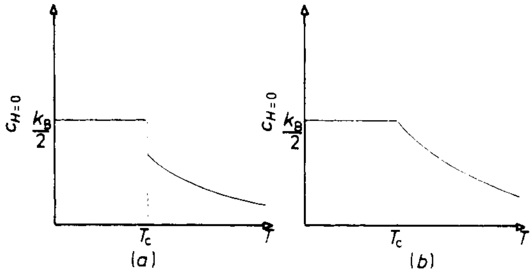

Figure 5. Zero field specific heat per site. (*a*) Cayley tree; (*b*) three-dimensional lattice.

supposition based on the fact that the tree, likes a disordered system, has structural short-range order but in some sense no structural long-range order.

Let us now look at the field dependent properties of our model. From (8) and (21) we get the magnetisation m per site. If we expand m about $y = 0$ we obtain for small b

$$m = \frac{b}{K}\{C_0 - C_1[(z/2\sqrt{k}) - 1]^{1/2}\} \tag{26}$$

where C_0, C_1 are numbers depending only on k:

$$C_0 = \frac{1}{2}\frac{k-1}{k+1-2\sqrt{k}}\left(\sqrt{k}\ln\frac{\sqrt{k}}{\sqrt{k}-1} - k\ln\frac{k}{k-1}\right)$$

$$C_1 = \frac{k}{k-1} - k\ln\frac{k}{k-1}.$$

From (26) we see directly that our model, just like the Ising model on the Cayley tree, has no spontaneous magnetisation.

According to (9) the zero field susceptibility contains the derivative of the Lagrange parameter with respect to the field. It can be shown that this derivative goes to zero for vanishing field. Hence we get

$$\chi_T = 2\mu^2\{C_0 - C_1[(z/2\sqrt{k}) - 1]^{1/2}\}. \tag{27}$$

The zero field susceptibility of our model remains finite for all temperatures and shows a cusp at the critical temperature as sketched in figure 6; its slope, however, has a $(T - T_c)^{-1/2}$ infinity as T approches T_c from above.

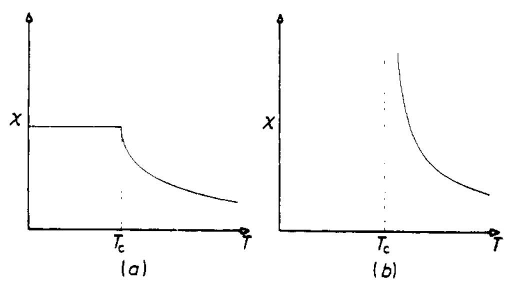

Figure 6. Zero field susceptibility. (*a*) Cayley tree; (*b*) three-dimensional lattice.

Below the critical temperature both the specific heat and the susceptibility are independent of temperature. This is the well known artefact of the spherical model, which, independent of the underlying topology, is due to the pinning of the Lagrange parameter below T_c.

6. Spin distribution

Finally we would like to know, whether, due to the peculiar topology of the Cayley tree, our result might not be produced by a pathological distribution of the spin length over the tree. The spin length distribution is determined by energy minimisation; we therefore expect the surface spins with only one bond to be at least somewhat shorter than those in the interior.

We define an ascending generation number l which denotes the surface of the tree with $l=1$ and its root, in the thermodynamic limit, with $l=\infty$.

All k^{L+1-l} sites of a given generation l of the tree are equivalent. Therefore it is sufficient to calculate the mean spin length of one generation.

For this purpose we add a local Lagrange term $S_l\,\Sigma_{i\in\{l\}}\sigma_i^2$ to our Hamiltonian (1) where $\Sigma_{i\in\{l\}}$ means the summation only over the sites of the lth generation.

We then obtain the mean length of one spin in this generation from

$$\langle\sigma_i^2\rangle_{i\in\{l\}}=\frac{k^l}{k-1}\frac{\partial}{\partial S_l}\left(\frac{1}{N}\ln Q_N^l\right)\bigg|_{S_l=0} \tag{28}$$

where Q_N^l is the partition function including the local Lagrange term.

Equation (28) can be evaluated with the methods of § 3 to give

$$\langle\sigma_i^2\rangle_{i\in\{l\}}=\frac{(k-1)k^l}{2K}\left[\frac{\sinh(2l)y}{2\sqrt{k}\sinh y}\sum_{n=l}^{\infty}\left(\frac{1}{k}\right)^{n+1}-\frac{\cosh(2l)y}{2\sqrt{k}\sinh y}\sum_{h=l}^{\infty}\left(\frac{1}{k}\right)^{n+1}\operatorname{cotanh}(n+1)y\right]. \tag{29}$$

We restrict our considerations to the mean spin length at the critical temperature, where $y=0$, and we obtain

$$\langle\sigma_i^2\rangle_{i\in\{l\}}=\frac{(k-1)}{2\sqrt{k}K_c}k^l\sum_{n=l}^{\infty}\left(\frac{1}{k}\right)^{n+1}\left(l+\frac{l^2}{n+1}\right). \tag{30}$$

For $l=1$, corresponding to the surface, we have

$$\langle\sigma_i^2\rangle_{i\in\{1\}}=\frac{6(k-1)^2}{3k-1}\left(\frac{k}{k-1}-k\ln\frac{k}{k-1}\right) \tag{31}$$

and for $l\to\infty$, $\langle\sigma_i^2\rangle_{i\in\{\infty\}}$ goes to

$$\langle\sigma_i^2\rangle_{i\in\{\infty\}}=\frac{6(k-1)^2}{3k-1}\frac{k}{k-1}. \tag{32}$$

For $k=2$ the numerical values are

$$\langle\sigma_i^2\rangle_{i\in\{1\}}=0{\cdot}735 \qquad \langle\sigma_i^2\rangle_{i\in\{\infty\}}=2,4.$$

This proves that the phase transition is not accompanied by a breakdown of the spin length distribution.

Appendix 1

The partition function Q_N is, according to (2) and (4),

$$Q_N = \int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{\sigma} \exp(-K\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{A}\boldsymbol{\sigma} + b\mathbf{1}^{\mathrm{T}}\boldsymbol{\sigma}) \qquad \text{where } \mathrm{d}\boldsymbol{\sigma} = \mathrm{d}\sigma_1, \ldots \mathrm{d}\sigma_N. \tag{A.1}$$

The symmetric matrix $\mathbf{A}$ can be transformed to a diagonal matrix $\boldsymbol{\lambda}$ by an orthogonal matrix $\mathbf{U}$

$$\mathbf{U}^{-1}\mathbf{A}\mathbf{U} = \boldsymbol{\lambda} \qquad \text{with } \mathbf{U}^{-1}\mathbf{U} = \mathbf{I}.$$

Therefore we have

$$\begin{aligned} Q_N &= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{y} \exp(-K\boldsymbol{y}^{\mathrm{T}}\boldsymbol{\lambda}\boldsymbol{y} + b\boldsymbol{w}^{\mathrm{T}}\boldsymbol{y}) \\ &= \prod_{i=1}^{N} \int_{-\infty}^{\infty} \mathrm{d}y_i \exp(-K\lambda_i y_i^2 + bw_i y_i) \end{aligned}$$

where $\boldsymbol{y} = \mathbf{U}^{-1}\boldsymbol{\sigma}$; $\boldsymbol{w}^{\mathrm{T}} = \mathbf{1}^{\mathrm{T}}\mathbf{U}$. Hence we get after integration

$$Q_N = \left(\frac{\pi}{K}\right)^{N/2} \left(\prod_{i=1}^{N} \lambda_i\right)^{-1/2} \exp\left(\frac{b^2}{4K} \sum_{i=1}^{N} \frac{w_i^2}{\lambda_i}\right). \tag{A.2}$$

We keep in mind that $\Pi_{i=1}^{N} \lambda_i = \det \boldsymbol{\lambda} = \det \mathbf{A}$ and write

$$\sum_{i=1}^{N} \frac{w_i^2}{\lambda_i} = \boldsymbol{w}^{\mathrm{T}}\boldsymbol{\lambda}^{-1}\boldsymbol{w} = \mathbf{1}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{1} = \sum_{i,j} a_{ij}^{-1} = \frac{\Sigma_{i,j} C_{ji}}{\det \mathbf{A}}$$

where C_{ji} is the jith cofactor of $\det \mathbf{A}$. If we define $\mathbf{A}_i$ as a matrix obtained from $\mathbf{A}$ by replacing the ith row by $\mathbf{1}^{\mathrm{T}} = (1, \ldots, 1) \in \mathbb{R}^N$ we get finally $\Sigma_{i,j=1}^{N} C_{ji} = \Sigma_{i=1}^{N} \det \mathbf{A}_i$. This result into (A.2) yields equation (5) of the text.

Appendix 2

To prove that the apparent pole of Chi at $z = k + 1$ has a vanishing residue we show that the numerator on the right-hand side of (21) vanishes term by term for $z = k + 1$ or $y = \ln\sqrt{k}$:

$$e^{-y} \sinh ly + (\sqrt{k})^l \sinh y - \sinh (l+1)y = 0.$$

Indeed this expression gives

$$\frac{1}{\sqrt{k}}\left(\frac{(\sqrt{k})^l - (\sqrt{k})^{-l}}{2}\right) + (\sqrt{k})^l\left(\frac{\sqrt{k} - (\sqrt{k})^{-1}}{2}\right) - \left(\frac{(\sqrt{k})^{(l+1)} - (\sqrt{k})^{-(l+1)}}{2}\right)$$

which vanishes for all values of l and therefore the right-hand side of (21) is also analytic for $z = k + 1$.

References

Berlin T H and Kac M 1952 *Phys. Rev.* B **86** 821–35
Domb C 1960 *Adv. Phys.* **9** 245–361

Eggarter T P 1974 *Phys. Rev. B* **9** 2989–92
Falk H 1975 *Phys. Rev.* B **12** 5184–9
Gradsteyn I S and Ryzhik I M 1965 *Table of Integrals, Series and Products* (New York: Academic) 4th edn p 34(1.395.2)
von Heimburg J and Thomas H 1974 *J. Phys. C: Solid St. Phys* **7** 3433–43
Joyce G S 1972 *Phase Transitions and Critical Phenomena* eds C Domb and M S Green (New York: Academic) vol. 2 pp 375–442
Lewis H W and Wannier G H 1952 *Phys. Rev.* B **88** 682–3
——1953 *Phys. Rev.* B **90** 1131
Matsuda H 1974 *Prog. Theor. Phys.* **51** 1053–63
Müller-Hartmann E and Zittartz J 1974 *Phys. Rev. Lett.* **33** 893–7
——1975 *Z. Phys.* B **22** 59–67
Rauh A 1976 *Phys. Rev.* B **13** 2080–91
—— 1976 *Phys. Rev.* B **14** 1997–2004
Yan C C and Wannier G H 1965 *J. Math. Phys.* **6** 1833–8