Dealing with Hard Problems

CS16: Introduction to Data Structures & Algorithms
Spring 2020

Outline

- Seating Arrangements
- Problem hardness
- P, NP, NP-Complete, NP-Hard
- Dealing with hard problems
    - Problem translation
    - Genetic Algorithms
    - Approximations
- Travling Salesman Problem

Seating Arrangement Problem

- Your dating algorithms worked!
- You need to plan the seating arrangements for a wedding

Seating Arrangement Problem

- Constraints / goals
  - **k** tables
  - **n** people
  - Avoid antagonistic pairs (exes, rivals, etc) sitting at the same table
  - Maximise overall happiness

Quantifications of Pair-wise Happiness

- Assume each pair of people (A, B) has an associated ‘compatibility score’
  - for **friends** comp(A, B) = **10**
  - for **couples** comp(A, B) = **50**
  - for **antagonistic pairs** comp(A, B) = **-500**
- These values are known ahead of time

Quantifications of Table-wise Happiness

- Sum all the compatibility scores for each pair at the table

$$H\left(table\right) = \sum_{pair \in table} comp(pair)$$

Quantification of Total Happiness

- Utilitarian Approach:

$$Total\_H_{utilitarian} = \sum_{t \in tables} H(table)$$

- Egalitarian Approach:

$$Total\_H_{egalitarian} = \min_{t \in tables} H(t)$$

- Many more options!

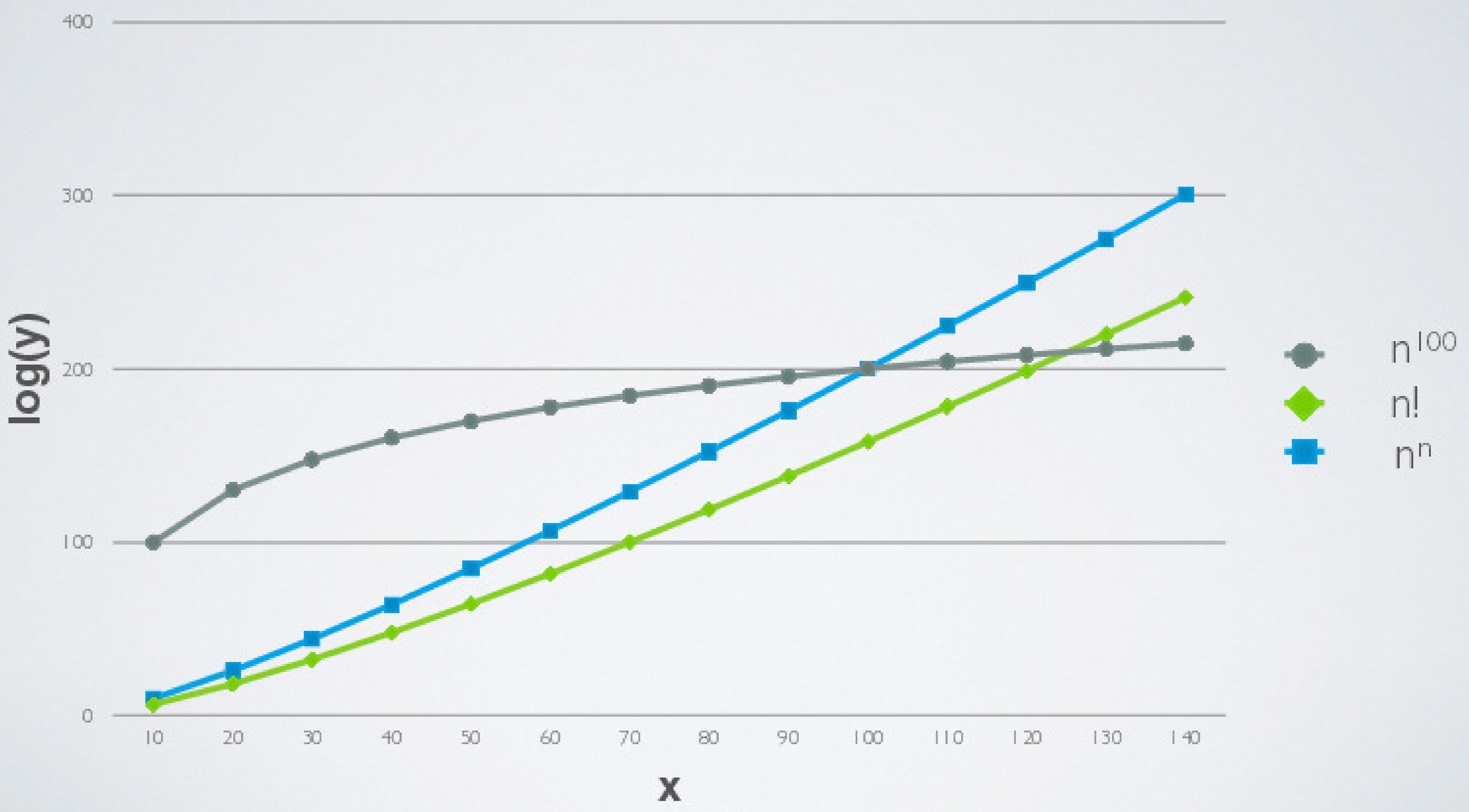
This seems hard

- Could we just try permutations and comparing scores?
- With 60 people, 60! permutations to test
  - 8.32×10^{81}
  - ouch
- This doesn't necessarily mean that the problem *is* hard, however

Defining Problem Hardness

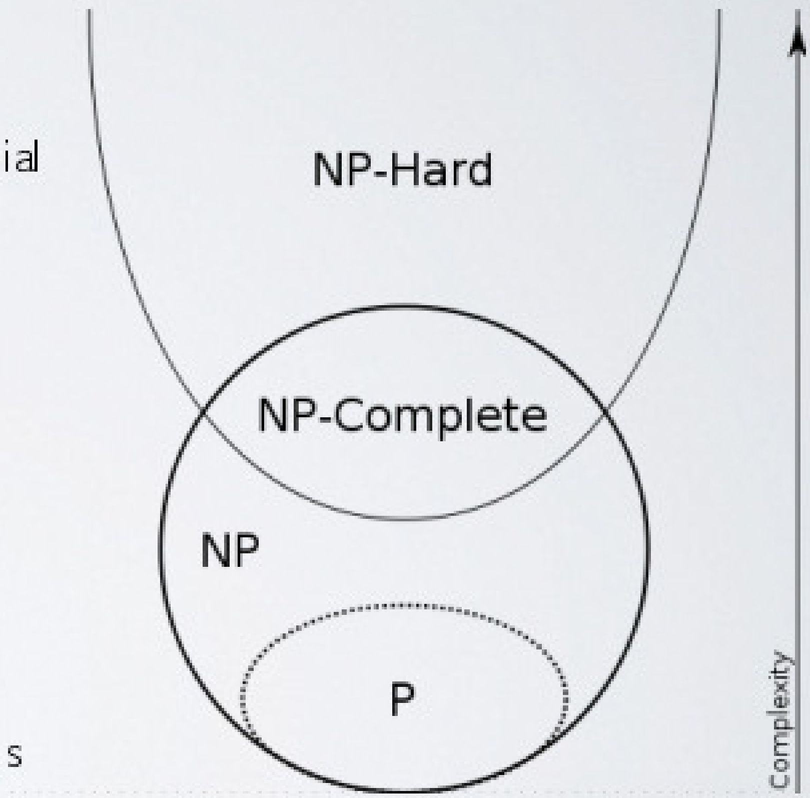
- Hardness of problem is defined by the runtime of the best solution
  - A bad sorting algorithm *could* be $O(n!)$, but sorting in general isn't considered hard, because we have fast algorithms to solve it
- Polynomial Runtimes
  - $O(n)$, $O(n^2)$, $O(n^{500})$
  - Problems with these solutions are **tractable**
- Super-Polynomial Runtimes
  - $O(n!)$, $O(2^n)$, $O(n^n)$
  - Problems with these solutions are **intractable**

Exponential vs. Polynomial Growth Rates



Categories of Hardness

- NP
  - The set of problems for which we can verify the correctness of a solution in polynomial time
- P
  - A subset of NP, where the problem is solvable in polynomial time
- NP-Complete
  - "The hardest problems in NP"
  - Solution is checkable in polynomial time
  - not known whether there exist any polynomial time algorithms to solve them
- NP-Hard
  - Problems that are "at least as hard as the hardest problems in NP"
  - Don't necessarily have solutions that are checkable in polynomial time



Back to our seating arrangement

- To get an intuition as to how hard our problem is, let's see if we can convert it into a problem that has already been proven to be in NP, P, NP-Complete, or NP-Hard
- But… where to start?

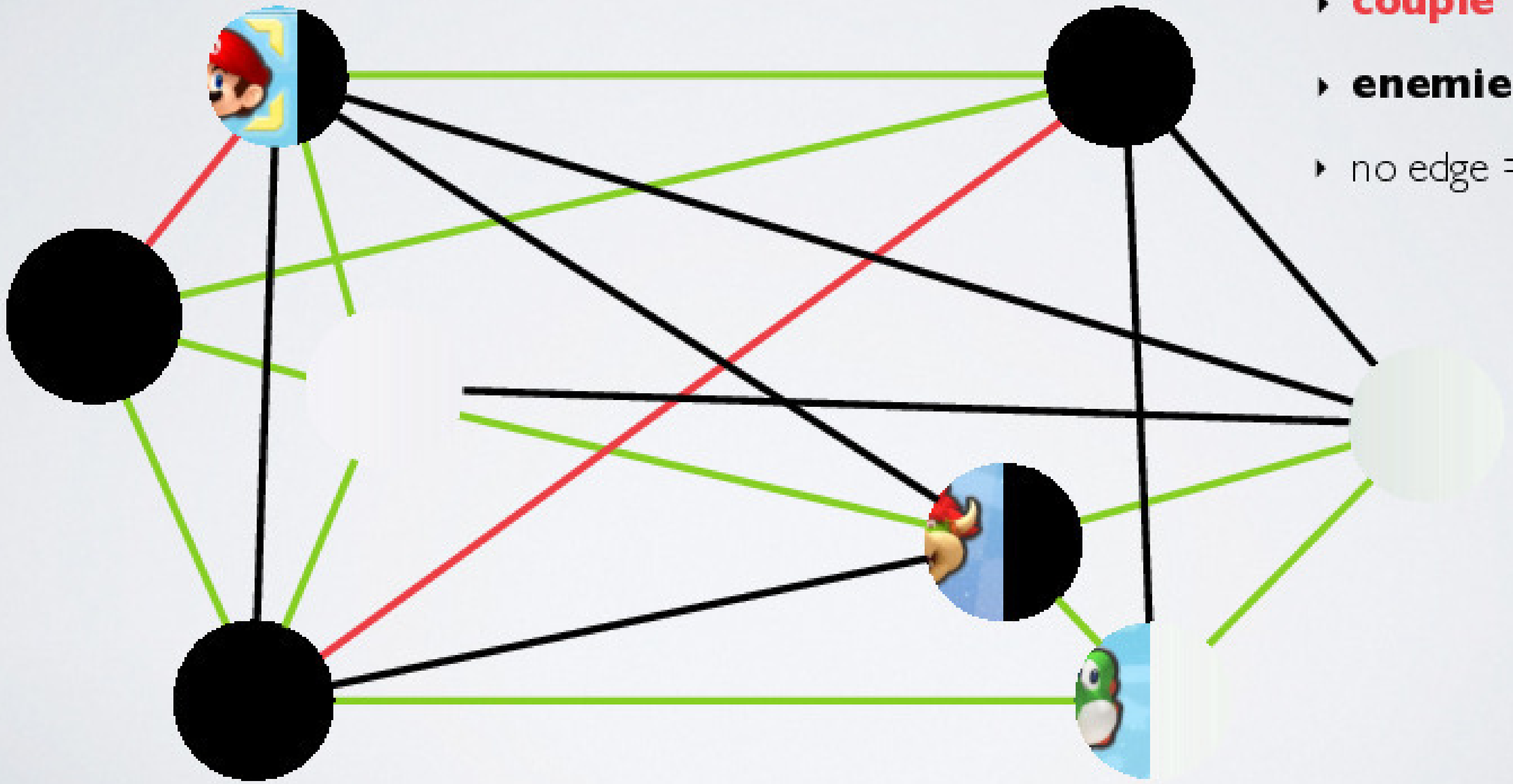
Constraint Relaxation

- See if you can solve an ‘easier’ version of the problem, by removing some of the properties that make the problem hard
- In real life
    - “what would you do if you could not fail?”
    - “which job would you take if they all paid equally?”

Let’s avoid disaster

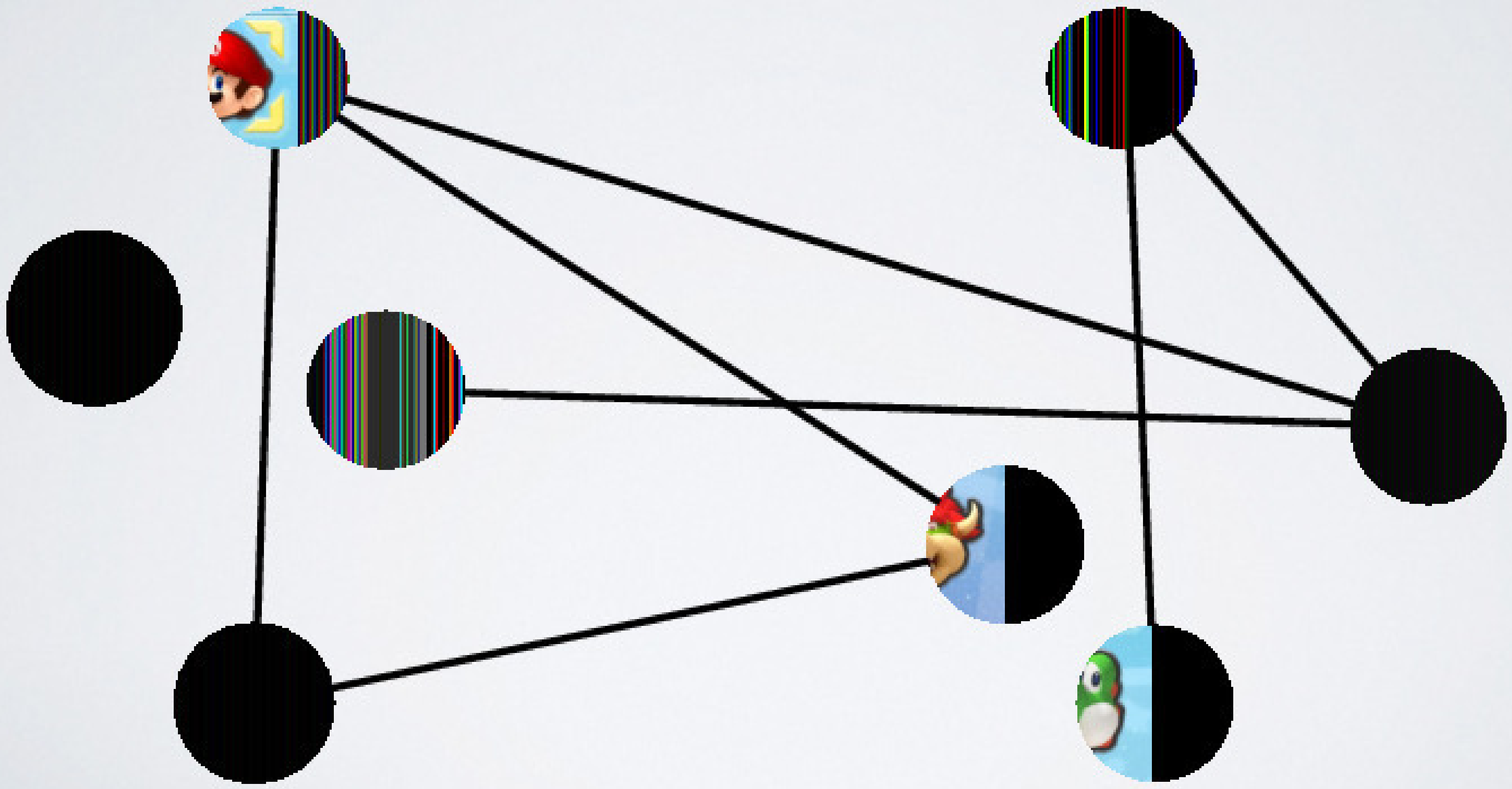
- Constraints / goals
    - # of tables
    - # of people
    - Avoid antagonistic pairs (exes, rivals, etc)
    - ~~Maximise overall happiness~~
- Hopefully, having no tables with antagonistic pairs will put in the right direction for maximising overall happiness

Relationships as a graph



- edge key:
  - **friends**
  - **couple**
  - **enemies**
  - no edge = no prefs

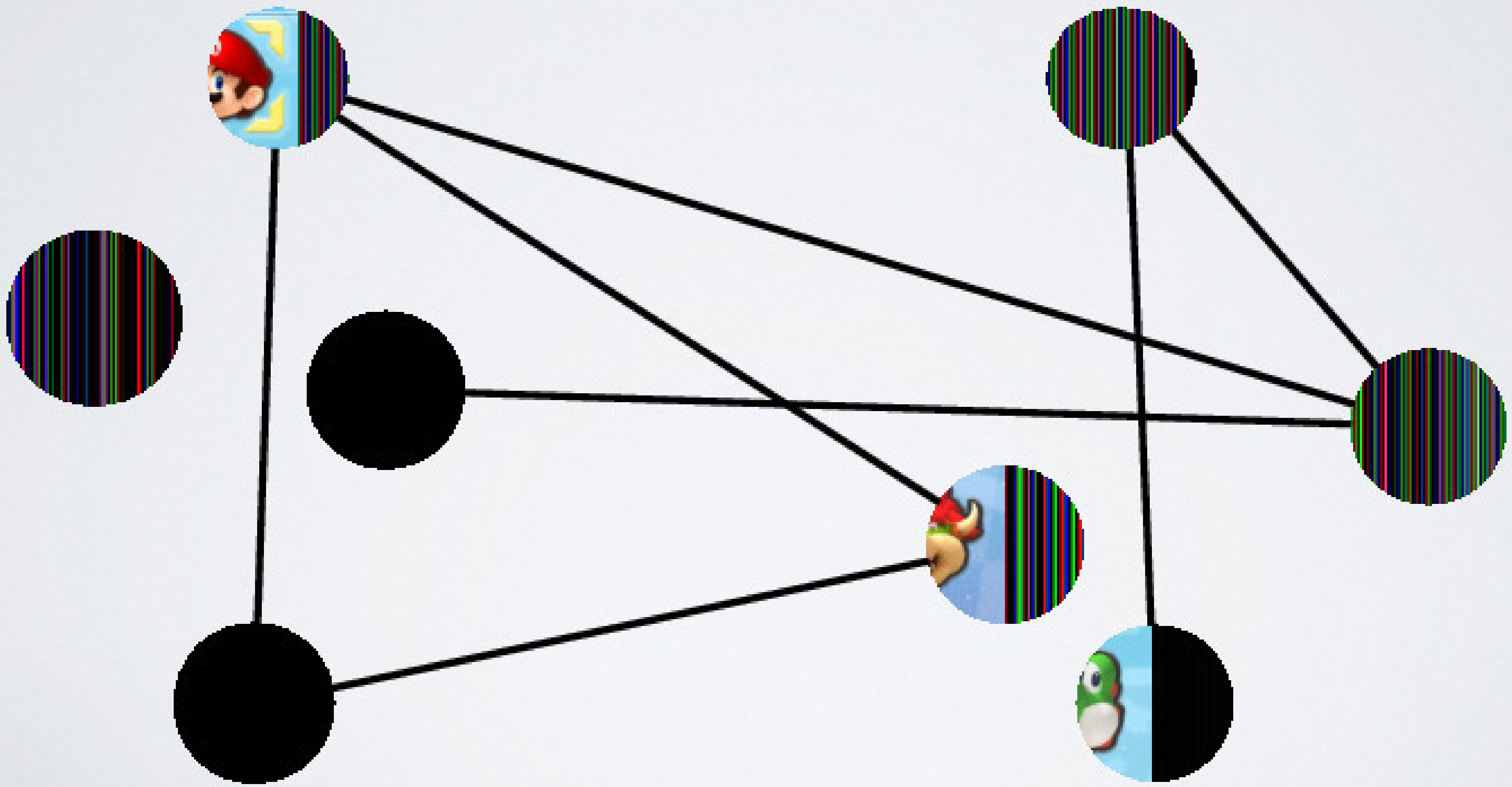
An Antagonism graph



Translating the problem

- Now, we have these antagonistic relationships represented as a graph!
- Question is no longer:
  - Can we avoid antagonistic pairs (exes, rivals, etc) sitting at the same table, given **n** people and **k** tables?
- Instead:
  - Use colours to represent different tables, so:
  - Could we assign 1 of **k** colours to each node in the antagonism graph, such that no two nodes that share an edge have the same colour?

An Antagonism graph



Lecture Activity 3

Try out the Graph k-colouring problem!

2 Mins….

Lecture Activity 3

Try out the Graph k-colouring problem!

1 Min….

Lecture Activity 3

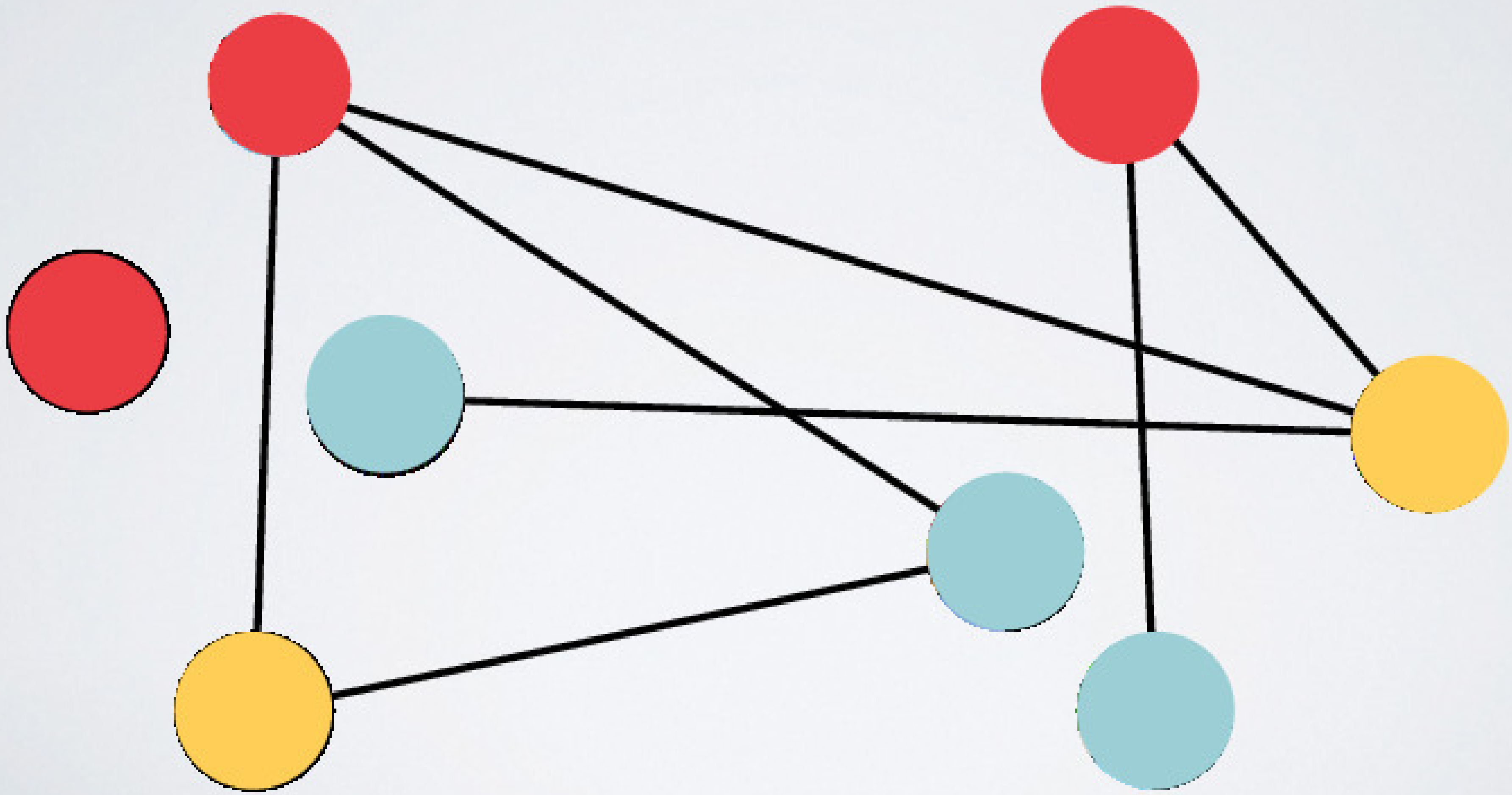
Try out the Graph k-colouring problem!

0 Mins….

Lecture Activity 3 Answers

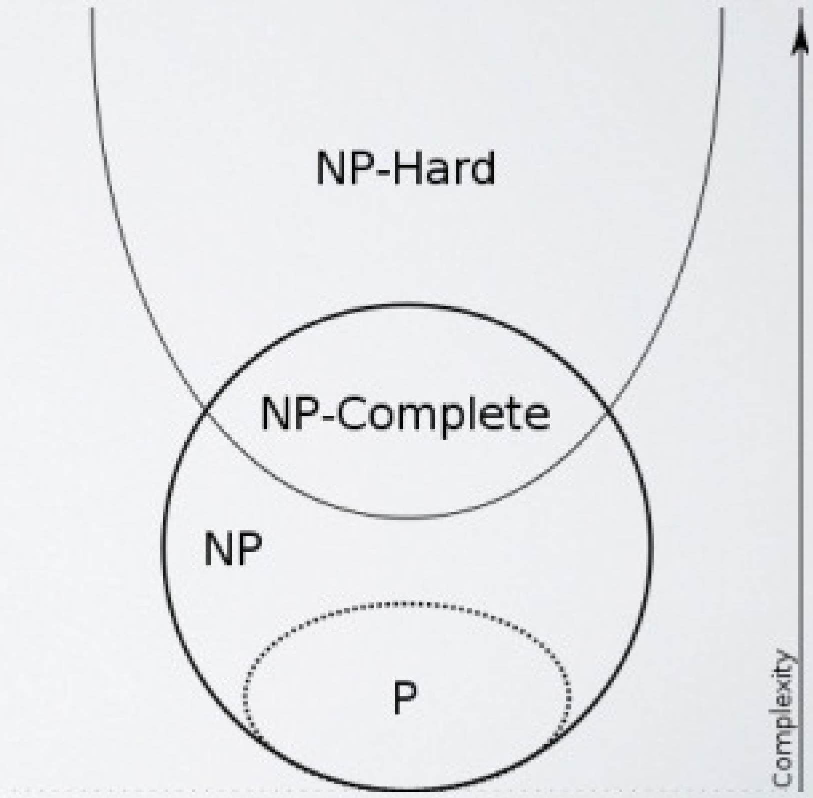
Answers!

Graph colouring example



Graph k-colouring

- Generally, the problem of determining whether nodes in a graph can be coloured using up to **k** separate colours, such that no two adjacent vertices share a colour
- This is NP-Complete!
- And thus, even this much easier version of the problem is **very hard**



Are we screwed?

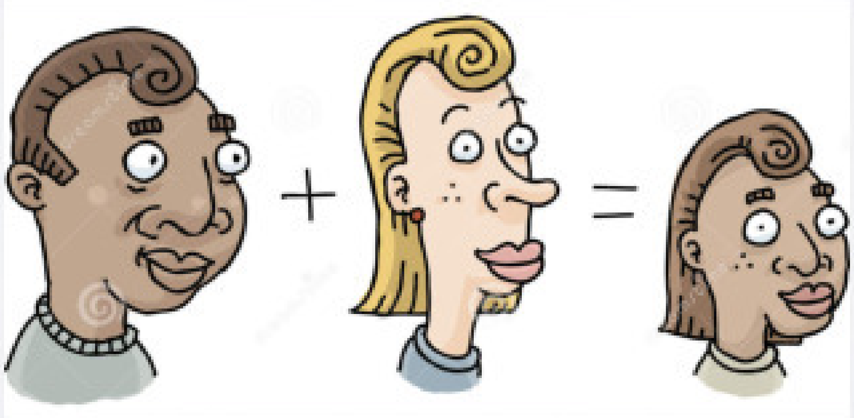
- The best algorithms to solve the graph k-colorability problem take O(2.445^n) time and space
- With 60 guests, 2.445^{60} = ~450 billion
  - which isn't *that* bad
  - Modern computers can handle ~3 billion 'operations' / sec, so this would take more than a couple minutes, probably less than 15
- But we've still only avoided the worst case!

Genetic Algorithms

- A form of ‘guess and check’, using a number of possible solutions to a problem
- Inspired the process of evolution

Biology Review

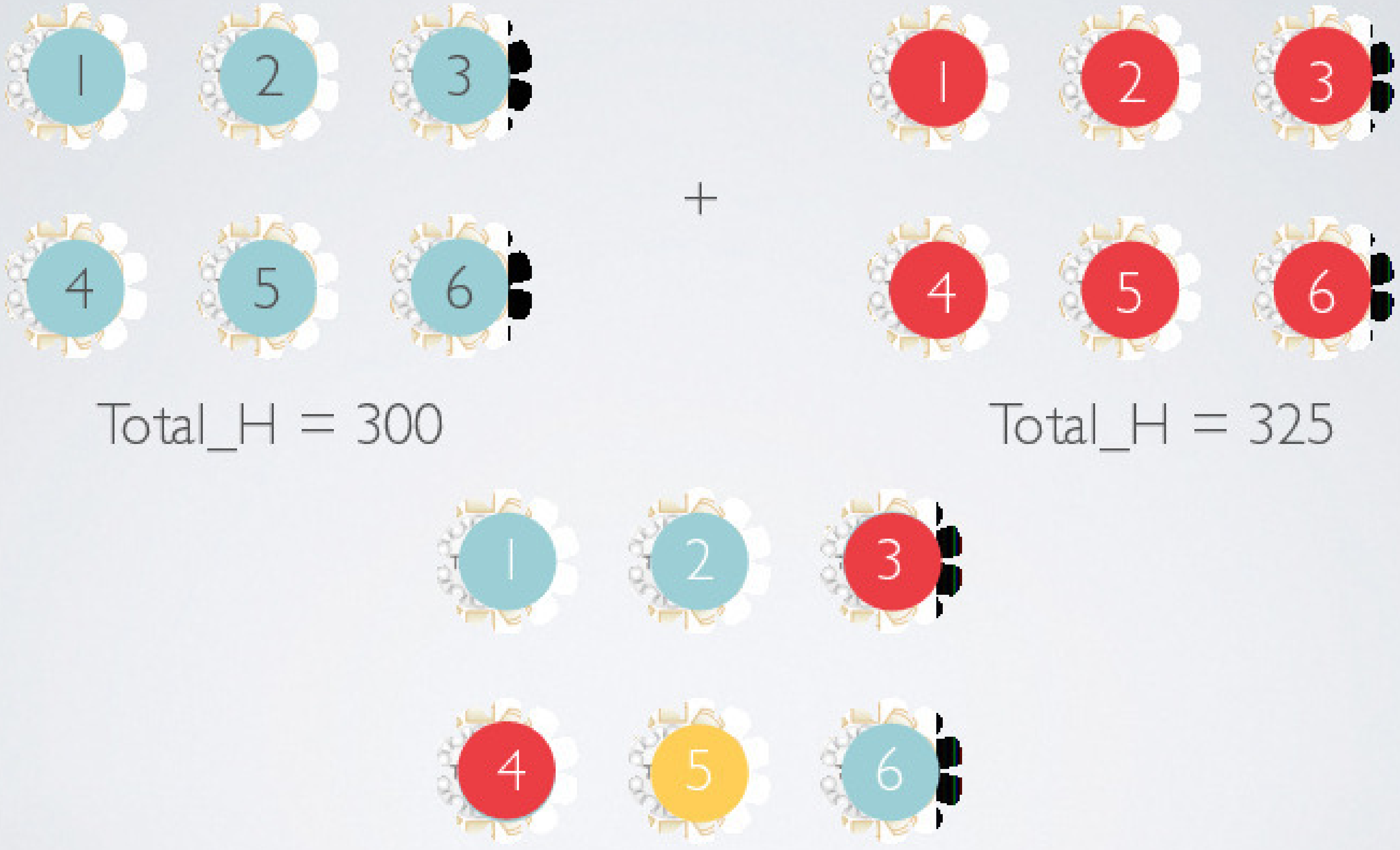
- All organisms are made up of **genes**, where genes (or a combination many genes) interact to produce our **phenotype**, the expression of those genes
- We are all a combination of a mix of our parents genes, and some random mutations



Evolution via Sexual Reproduction, broadly

- There exist an initial population of organisms within a species
- The ‘sexually fit’ organisms reproduce
  - Take some genes of parent A, some of parent B
  - add some random noise
  - this new collection of genes is a new specimen, AB’
- Older + less fit parts of populations die off, leaving the survivors to repeat the reproduction process

Solution Mating



Total_H = 300

Total_H = 325

Total_H = 400

High-Level Genetic Algorithm Pseudocode

```
function geneticAlgo(opt_seed_sols):
  solution_set = opt_seed_sols or || randomly generated initial population of solutions
  init_size = size(solution_set, threshold, time_limit)

  while True:
      new_gen = []
      for some number of iterations:
        A, B = 2 solutions from solution_set, drawn at random
        AB' = a new solution that combines properties of A and B
        randomlyMutate(AB')
        new_gen.append(AB')
      solution_set.addAll(new_gen)
      rank solutions in solution_set based on 'fitness'
      remove all but init_size many best solutions from solution_set
      if best(solution_set) > threshold or time_limit has passed:
        break
  return highest ranking solution from solution_set
```

Genetic Algorithms

- If seeded with 'good' solutions for the initial population of solutions, output is guaranteed to be at least as good as the best of the initial solutions
- Can come up with unexpected solutions
- Tend to do really well!