Minimum Spanning Trees: Prim-Jarnik & Kruskal

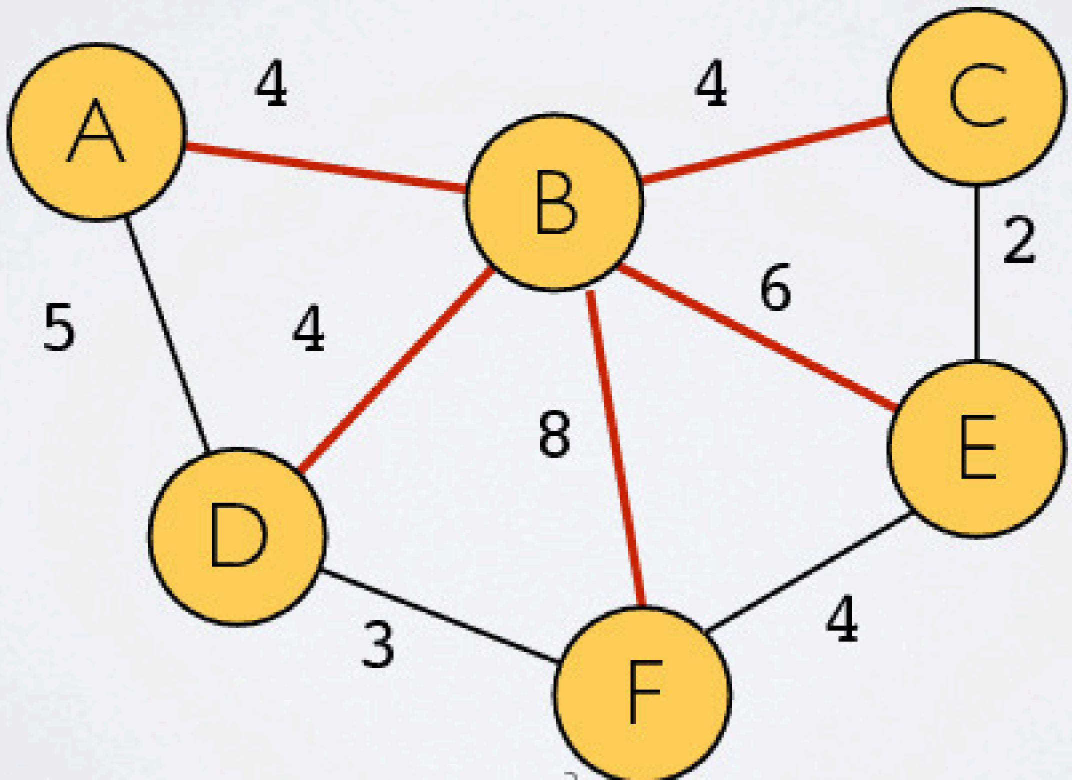
CS16: Introduction to Data Structures & Algorithms
Spring 2020

Outline

- Minimum Spanning Trees
- Prim-Jarnik Algorithm
  - Analysis
  - Proof of Correctness
- Kruskal's Algorithm
  - Union-Find
  - Analysis
  - Proof of Correctness

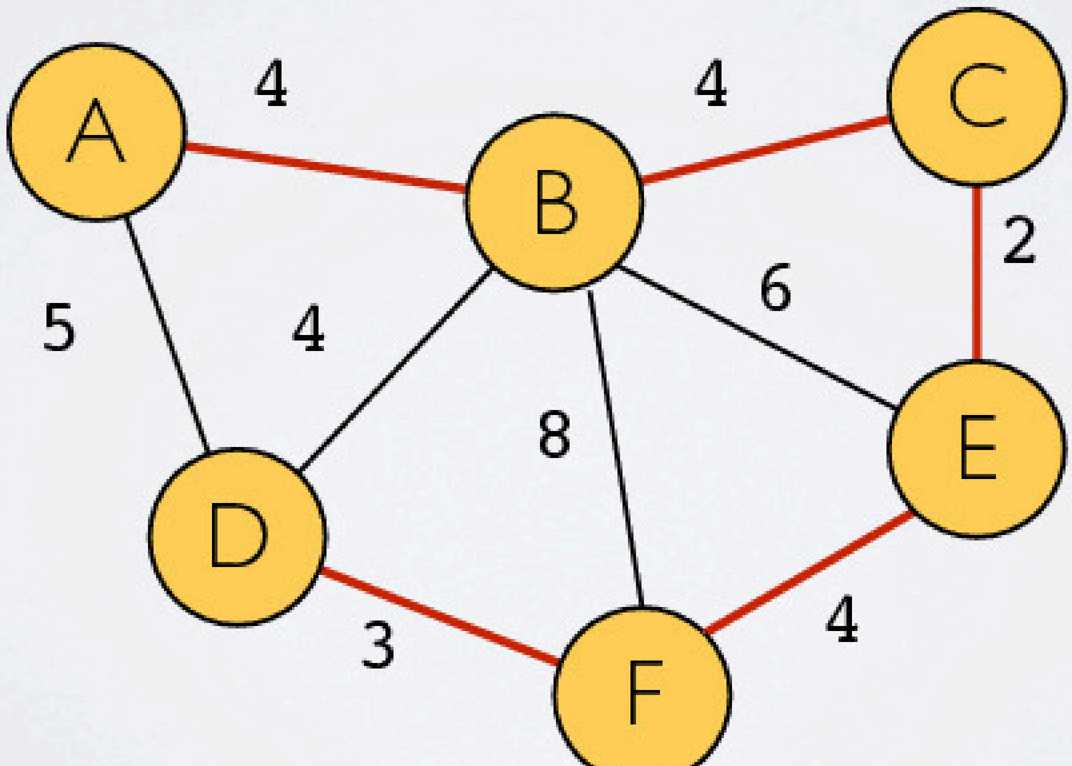
Spanning Trees

- A **spanning tree** of a graph is
  - edge subset forming a tree that spans every vertex



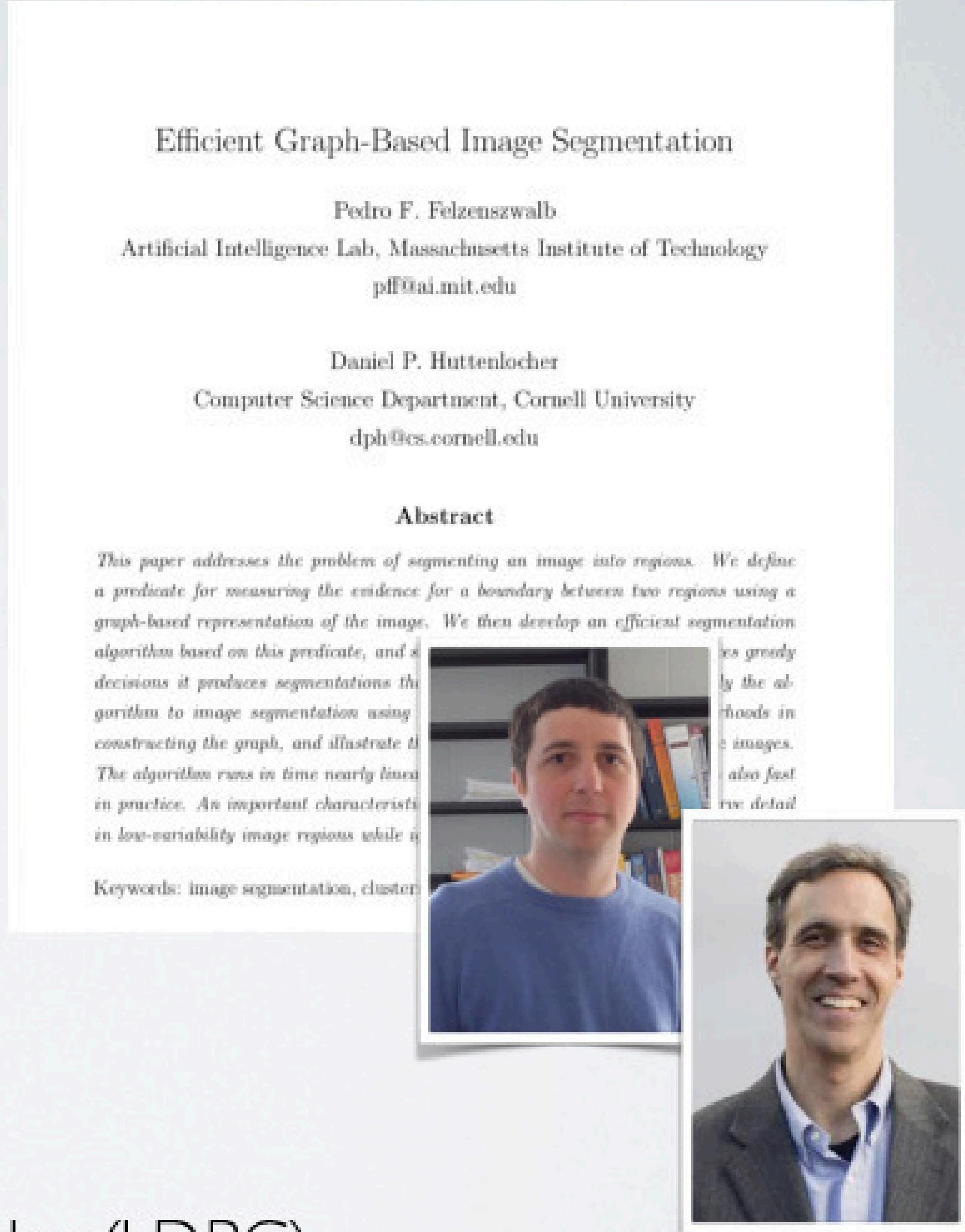
Minimum Spanning Trees

- A **minimum spanning tree** (MST) is
  - spanning tree with minimum total edge weight



Applications

- Networks
  - electric
  - computer
  - water
  - transportation
- Computer vision
  - Facial recognition
  - Handwriting recognition
  - **Image segmentation**
- Low-density parity check codes (LDPC)



Efficient Graph-Based Image Segmentation

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Abstract

This paper addresses the problem of segmenting an image into regions. We define a predicate for measuring the evidence for a boundary between two regions using a graph-based representation of the image. We then develop an efficient segmentation algorithm based on this predicate, and s… …es greedy decisions it produces segmentations th… …ly the algorithm to image segmentation using … …hoods in constructing the graph, and illustrate t… …images. The algorithm runs in time nearly line… …also fast in practice. An important characteristi… …ve detail in low-variability image regions while i…

Keywords: image segmentation, cluster…

Minimum Spanning Tree Algos

- Prim-Jarnik Algorithm

PRÁCE
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SVAZEK VI., SPIS 4. 1930 SIGNATURA: F 50
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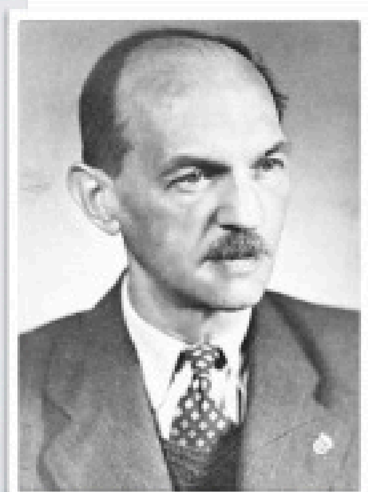


Shortest Connection Networks And Some Generalizations

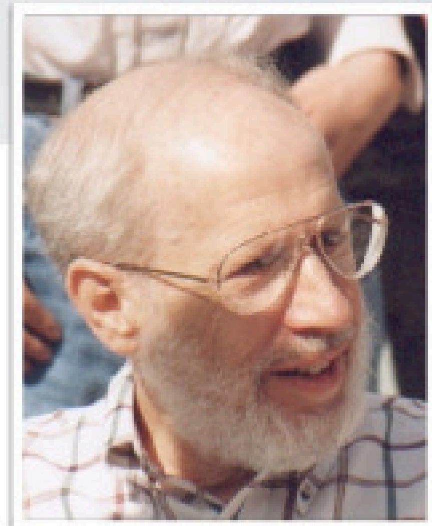
By R. C. PRIM

(Manuscript received May 8, 1957)

The basic problem considered is that of interconnecting a given set of terminals with a shortest possible network of direct links. Simple and practical procedures are given for solving this problem both graphically and computationally. It develops that these procedures also provide solutions for a much broader class of problems, containing other examples of practical interest.

Minimum Spanning Tree Algos

- Kruskal's algorithm (1956)



ON THE SHORTEST SPANNING SUBTREE OF A GRAPH AND THE TRAVELING SALESMAN PROBLEM

JOSEPH B. KRUSKAL, JR.

Several years ago a typewritten translation (of obscure origin) of [1] raised some interest. This paper is devoted to the following theorem: If a (finite) connected graph has a positive real number attached to each edge (the *length* of the edge), and if these lengths are all distinct, then among the spanning[1] trees (German: Gerüst) of the graph there is only one, the sum of whose edges is a minimum; that is, the shortest spanning tree of the graph is unique. (Actually in [1] this theorem is stated and proved in terms of the "matrix of lengths" of the graph, that is, the matrix $\|a_{ij}\|$ where a_{ij} is the length of the edge connecting vertices i and j. Of course, it is assumed that $a_{ij}=a_{ji}$ and that $a_{ii}=0$ for all i and j.)

The proof in [1] is based on a not unreasonable method of constructing a spanning subtree of minimum length. It is in this construction that the interest largely lies, for it is a solution to a problem (Problem 1 below) which on the surface is closely related to one version (Problem 2 below) of the well-known traveling salesman problem.

Minimum Spanning Tree Algos

- Karger-Klein-Tarjan (1995)

A Randomized Linear-Time Algorithm to Find Minimum Spanning Trees

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PHILIP N. KLEIN

Brown University, Providence, Rhode Island

AND

ROBERT E. TARJAN

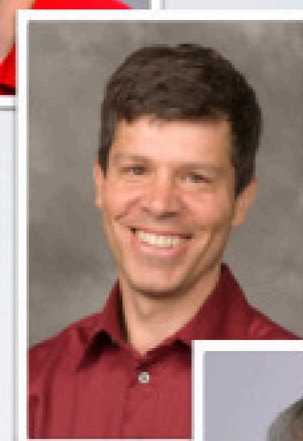
Princeton University and NEC Research Institute, Princeton, New Jersey

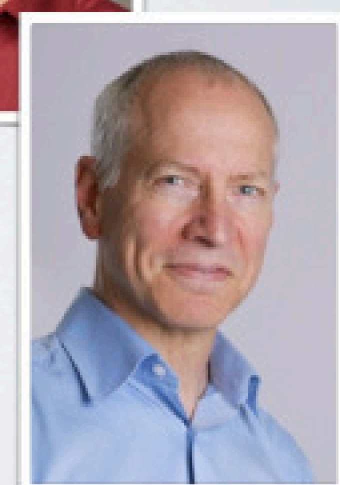
Abstract. We present a randomized linear-time algorithm to find a minimum spanning tree in a connected graph with edge weights. The algorithm uses random sampling in combination with a recently discovered linear-time algorithm for verifying a minimum spanning tree. Our computational model is a unit-cost random-access machine with the restriction that the only operations allowed on edge weights are binary comparisons.

Categories and Subject Descriptors: F.2.2 **[Analysis of Algorithms and Problem Complexity]:** Nonnumerical Algorithms and Problems—*computations on discrete structures*; G.2.2 **[Discrete Mathematics]:** Graph Theory—*graph algorithms, network problems, trees*; G.3 **[Probability and Statistics]:** *probabilistic algorithms (including Monte Carlo)*; I.5.3 **[Pattern Recognition]:** Clustering

General Terms: Algorithms

Additional Key Words and Phrases: Matroid, minimum spanning tree, network, randomized algorithm

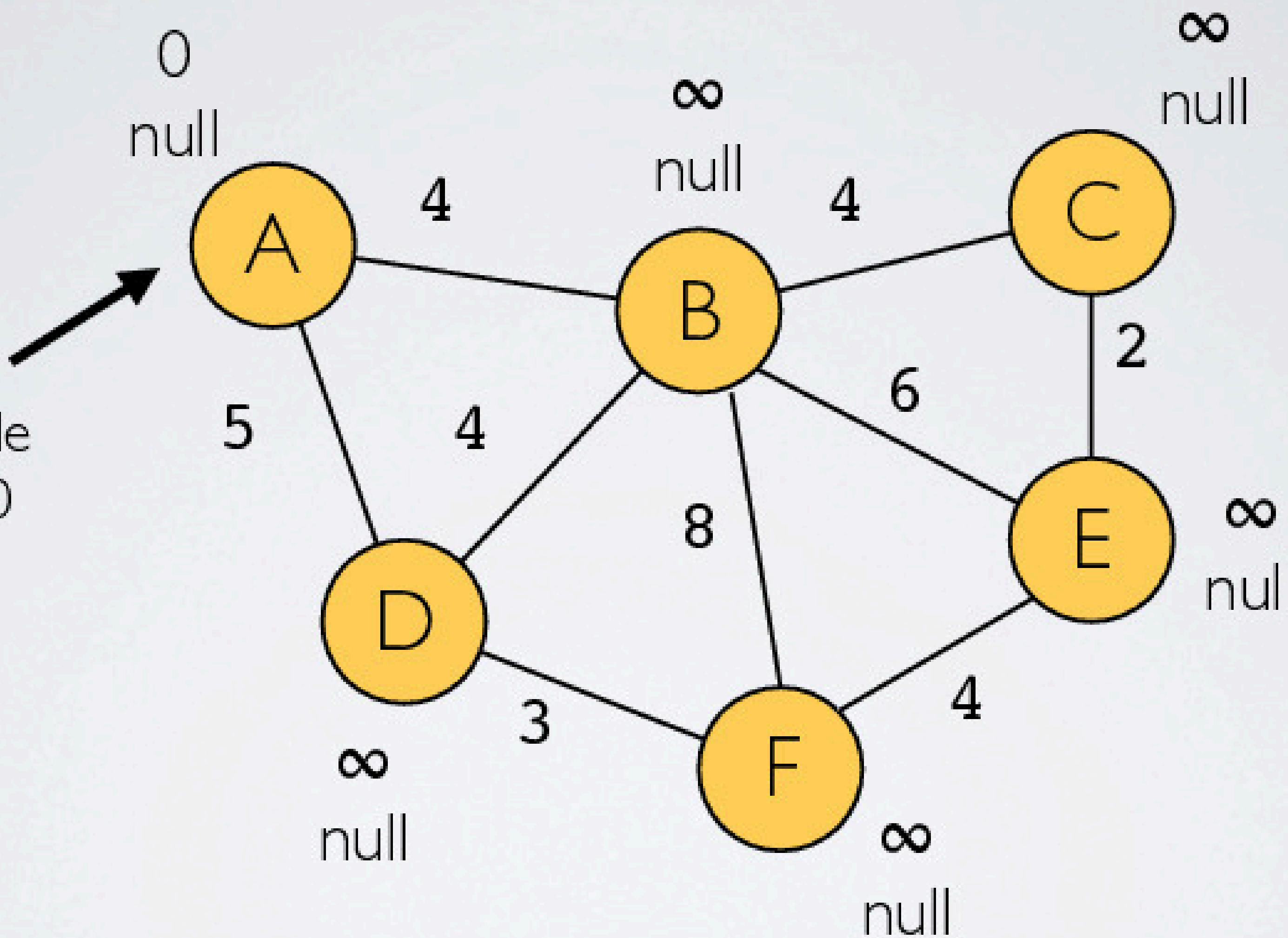




Prim-Jarnik Algorithm

- Traverse **G** starting at any node
  - Maintain priority queue of nodes
  - set priority to weight of the cheapest edge that connects them to MST
- Un-added nodes start with priority ∞
- At each step
  - Add the node with lowest cost to MST
  - Update ("relax") neighbors as necessary
- Stop when all nodes added to MST

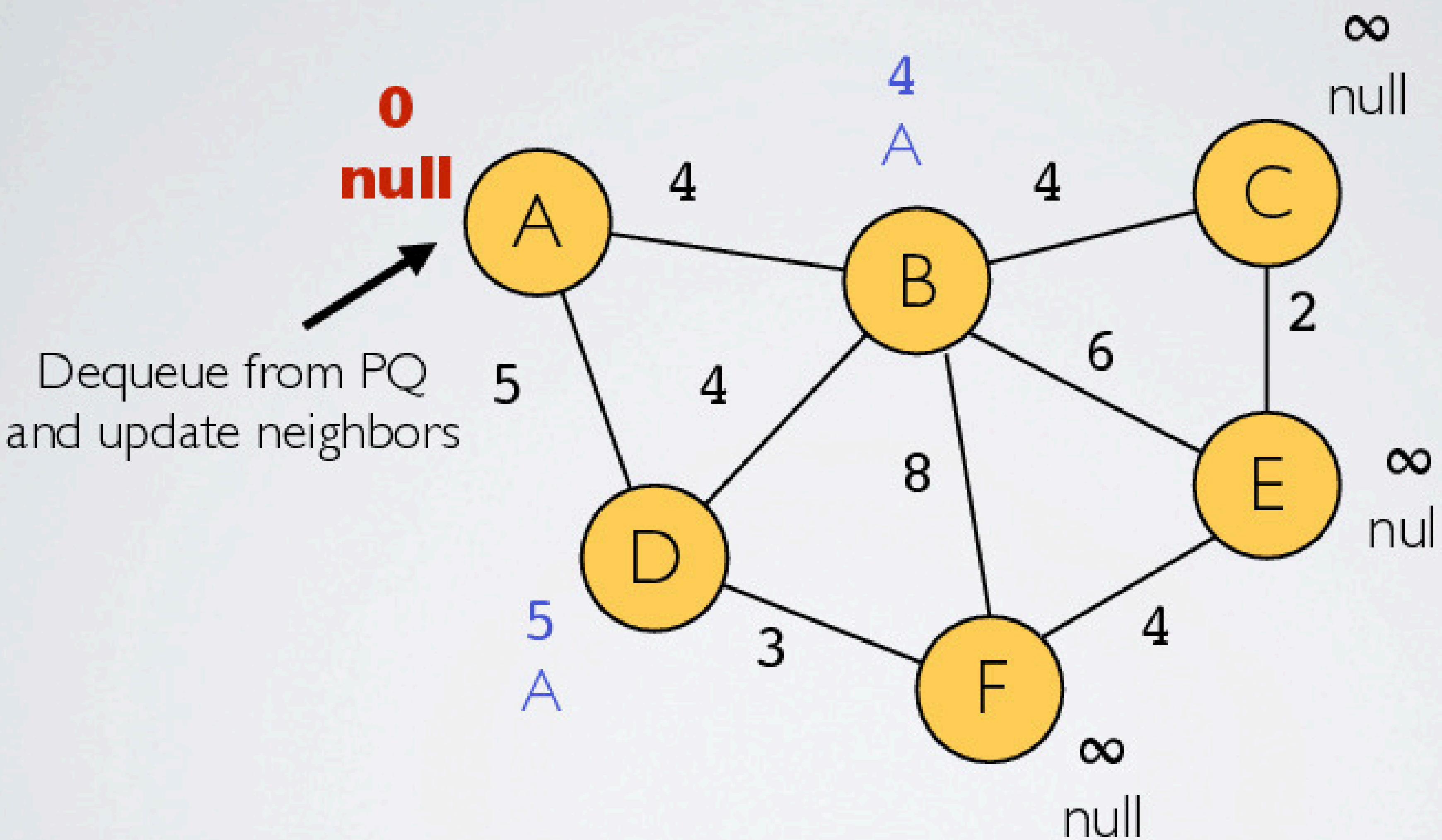
Example



Random node
set to cost 0

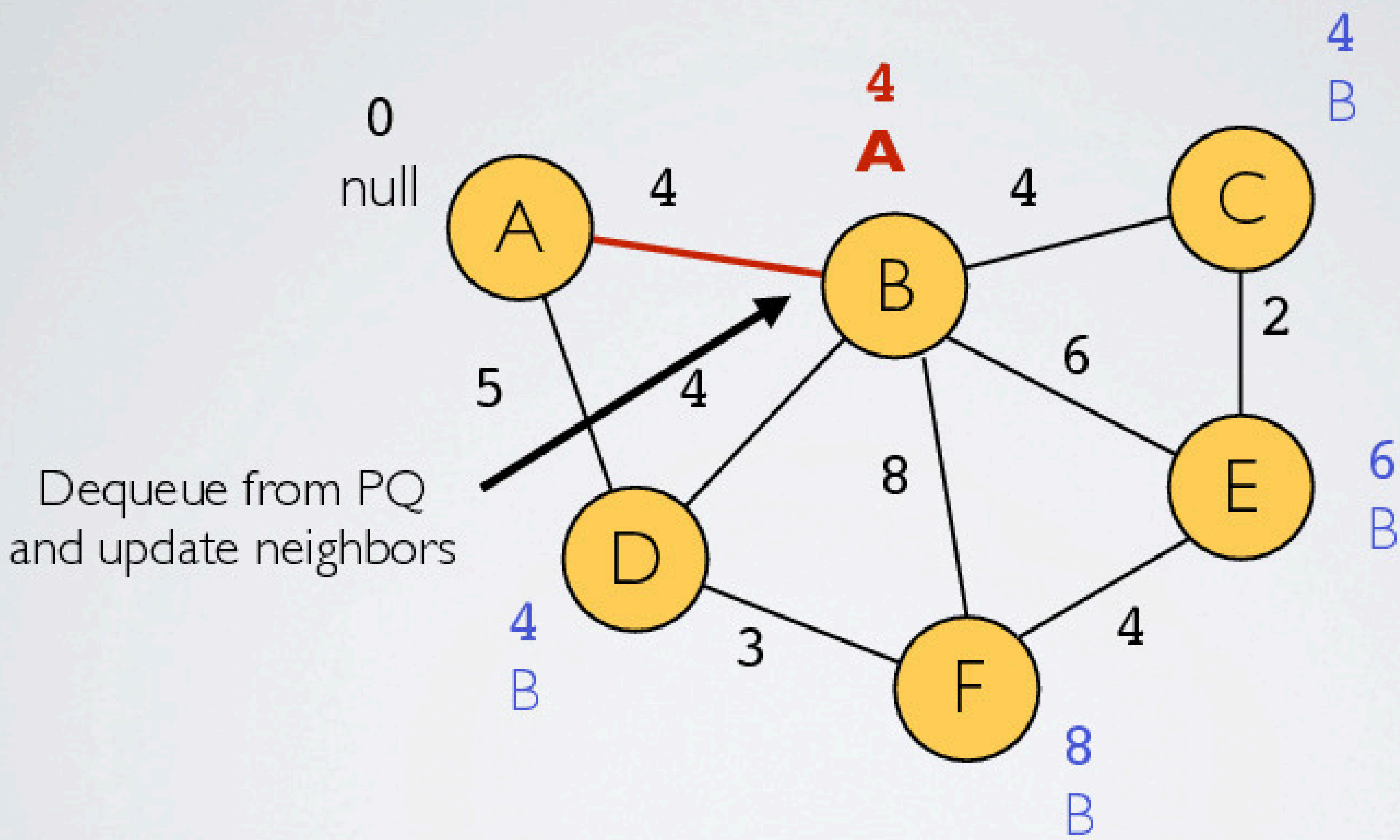
PQ = [(0,A),(∞,B),(∞,C),(∞,D),(∞,E),(∞,F)]

Example



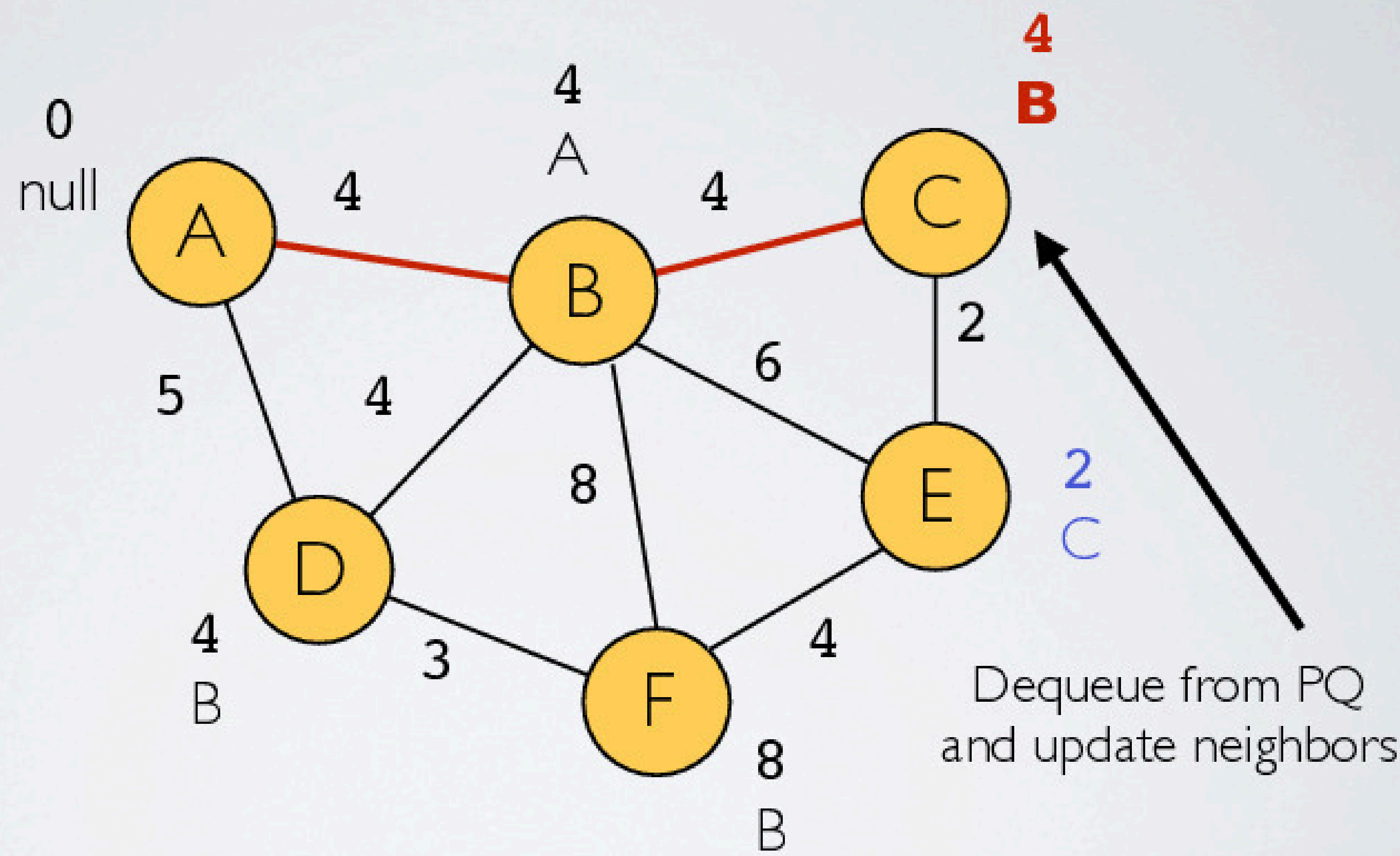
PQ = [(4,B),(5,D),(∞,C),(∞,E),(∞,F)]

Example



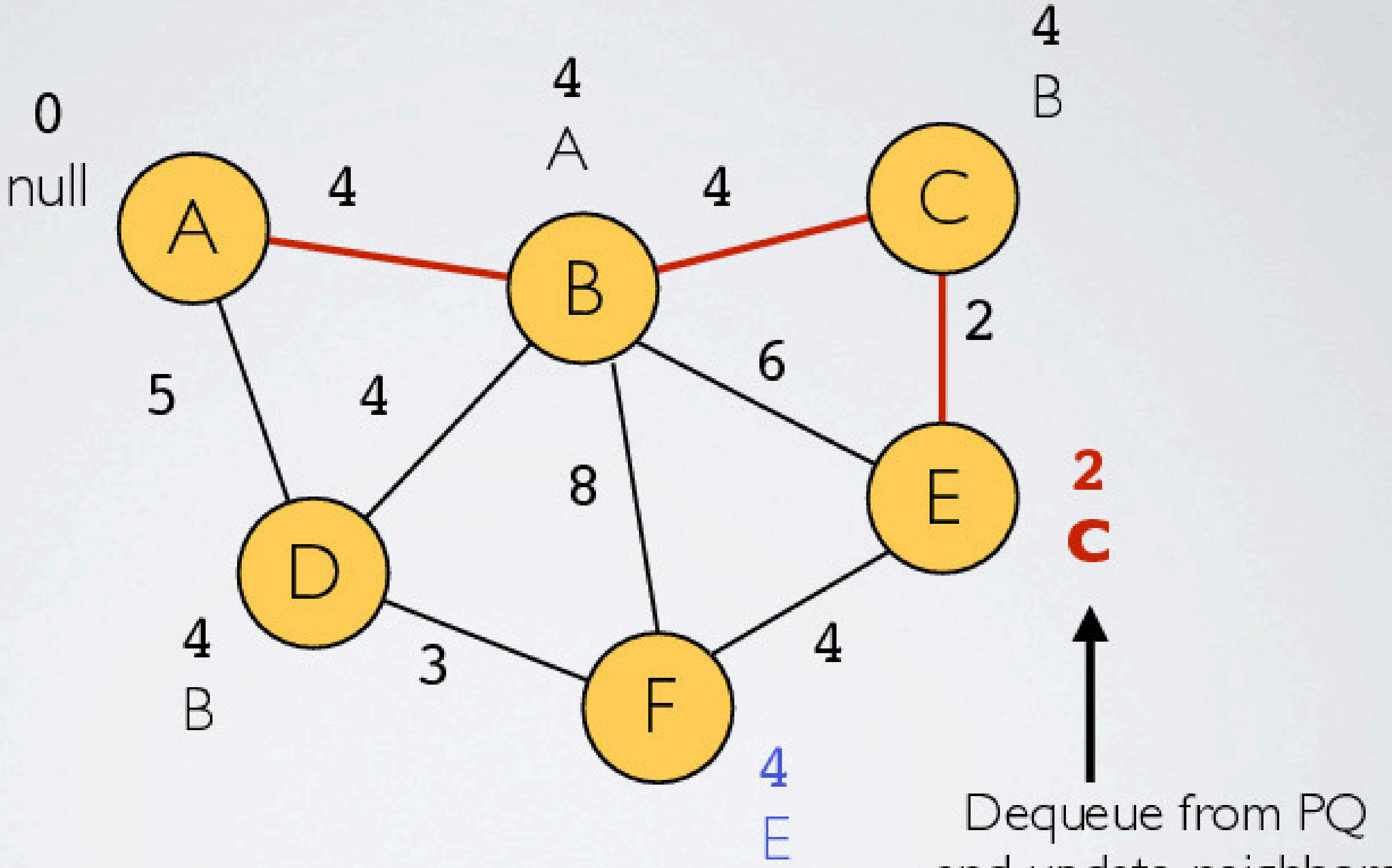
PQ = [(4,C),(4,D),(6,E),(8,F)]

Example



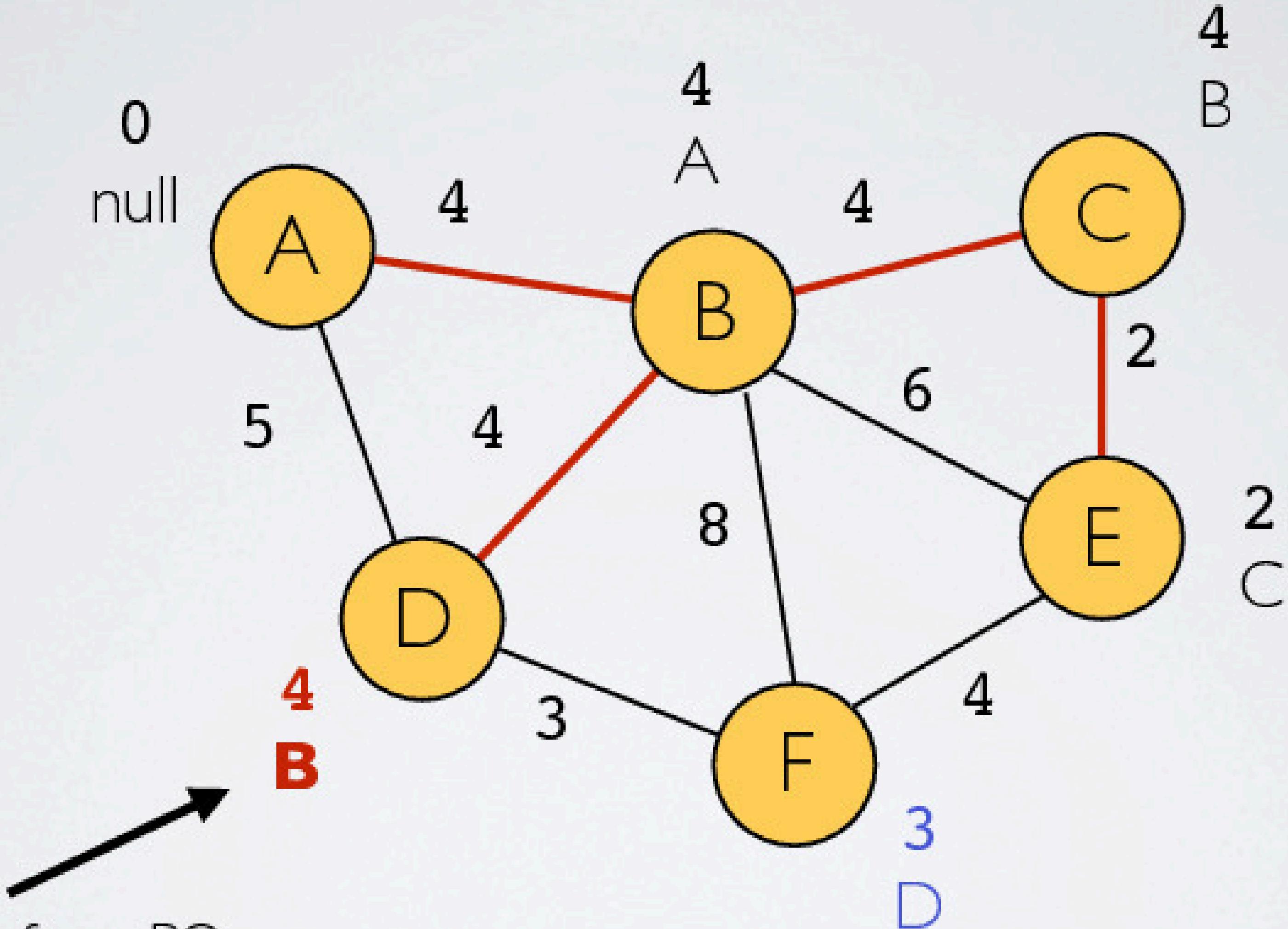
PQ = [(2,E),(4,D),(8,F)]

Example



`PQ = [(4,D),(4,F)]`

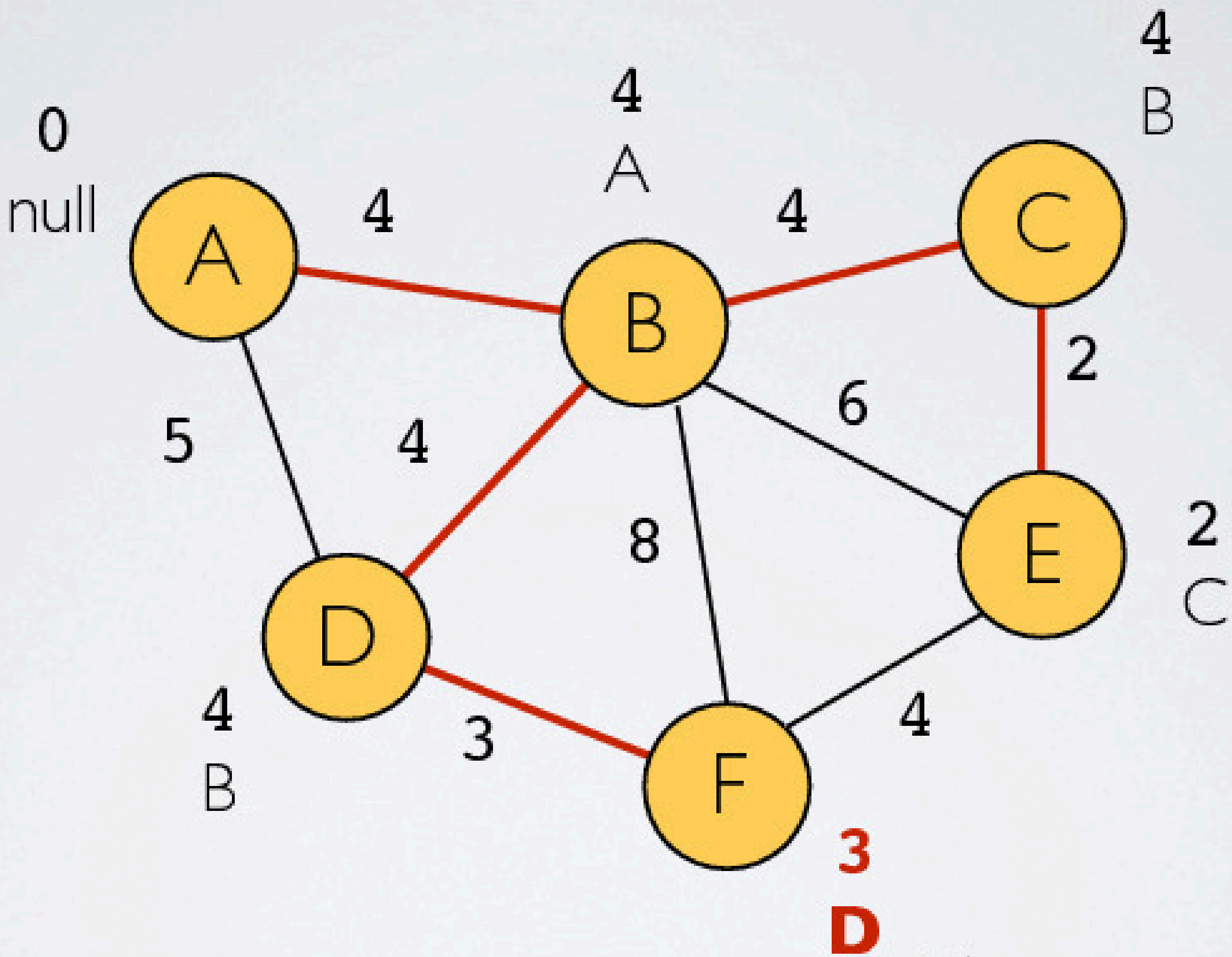
Example



Dequeue from PQ
and update neighbors

`PQ = [(3,F)]`

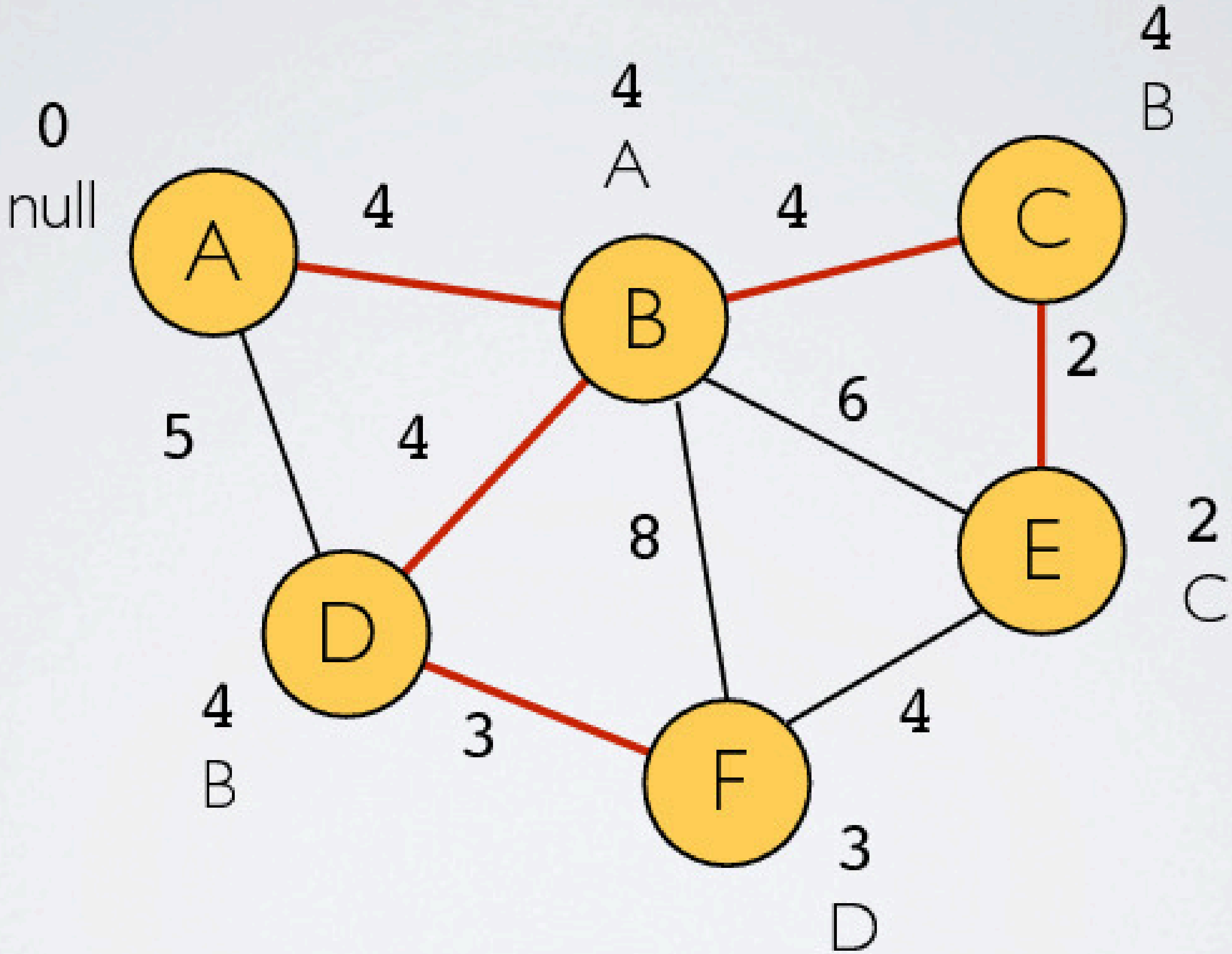
Example




```
PQ = [  ]
```

Dequeue from PQ and update neighbors

Example



Pseudo-code

```
function prim(G):
   // Input: weighted, undirected graph G with vertices V
   // Output: list of edges in MST
   for all v in V:
      v.cost = ∞
      v.prev = null
   s = a random v in V // pick a random source s
   s.cost = 0
   MST = []
   PQ = PriorityQueue(V) // priorities will be v.cost values
   while PQ is not empty:
      v = PQ.removeMin()
      if v.prev != null:
         MST.append((v, v.prev))
      for all incident edges (v,u) of v such that u is in PQ:
         if u.cost > (v,u).weight:
            u.cost = (v,u).weight
            u.prev = v
            PQ.decreaseKey(u, u.cost)
  return MST
```

Simulate Prim-Jarnik

```
function prim(G):
   // Input: weighted, undirected graph G with vertices V
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         if u.cost > (v,u).weight:
            u.cost = (v,u).weight
            u.prev = v
            PQ.decreaseKey(u, u.cost)
  return MST
```

3 min

Activity #1

Simulate Prim-Jarnik

```
function prim(G):
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3 min

Activity #1

Simulate Prim-Jarnik

```
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         if u.cost > (v,u).weight:
            u.cost = (v,u).weight
            u.prev = v
            PQ.decreaseKey(u, u.cost)
  return MST
```

2 min

Activity #1

Simulate Prim-Jarnik

```
function prim(G):
    // Input: weighted, undirected graph G with vertices V
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            if u.cost > (v,u).weight:
                u.cost = (v,u).weight
                u.prev = v
                PQ.decreaseKey(u, u.cost)
  return MST
```

1 min

Activity #1

Simulate Prim-Jarnik

```
function prim(G):
   // Input: weighted, undirected graph G with vertices V
   // Output: list of edges in MST
   for all v in V:
      v.cost = ∞
      v.prev = null
   s = a random v in V // pick a random source s
   s.cost = 0
   MST = []
   PQ = PriorityQueue(V) // priorities will be v.cost values
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      if v.prev != null: //guarantees we don't add (s, s.prev)
         MST.append((v, v.prev))
      for all incident edges (v,u) of v such that u is in PQ:
         if u.cost > (v,u).weight:
            u.cost = (v,u).weight
            u.prev = v
            PQ.decreaseKey(u, u.cost)
 return MST
```

0 min

Activity #1

Runtime of Prim-Jarnik

Activity #2

2 min

Runtime of Prim-Jarnik

Activity #2

2 min

Runtime of Prim-Jarnik

Activity #2

1 min

Runtime of Prim-Jarnik

Activity #2

0 min

Runtime Analysis

- Decorating nodes with distance and previous pointers is `O(|V|)`
- Putting nodes in PQ is `O(|V|log|V|)` (really `O(|V|)` since ∞ priorities)
- While loop runs `|V|` times
  - removing vertex from PQ is `O(log|V|)`
  - So `O(|V|log|V|)`
- For loop (in while loop) runs `|E|` times ***in total***
  - Replacing vertex's key in the PQ is `log|V|`
  - So `O(|E|log|V|)`
- Overall runtime
  - `O(|V| + |V|log|V| + |V|log|V| + |E|log|V|)`
    - `= O((|E| + |V|)log|V|)`

Proof of Correctness

- Common way of proving correctness of greedy algos
  - show that algorithm is always correct at every step
- Best way to do this is by induction
  - tricky part is coming up with the right invariant

Inductive invariant for Prim

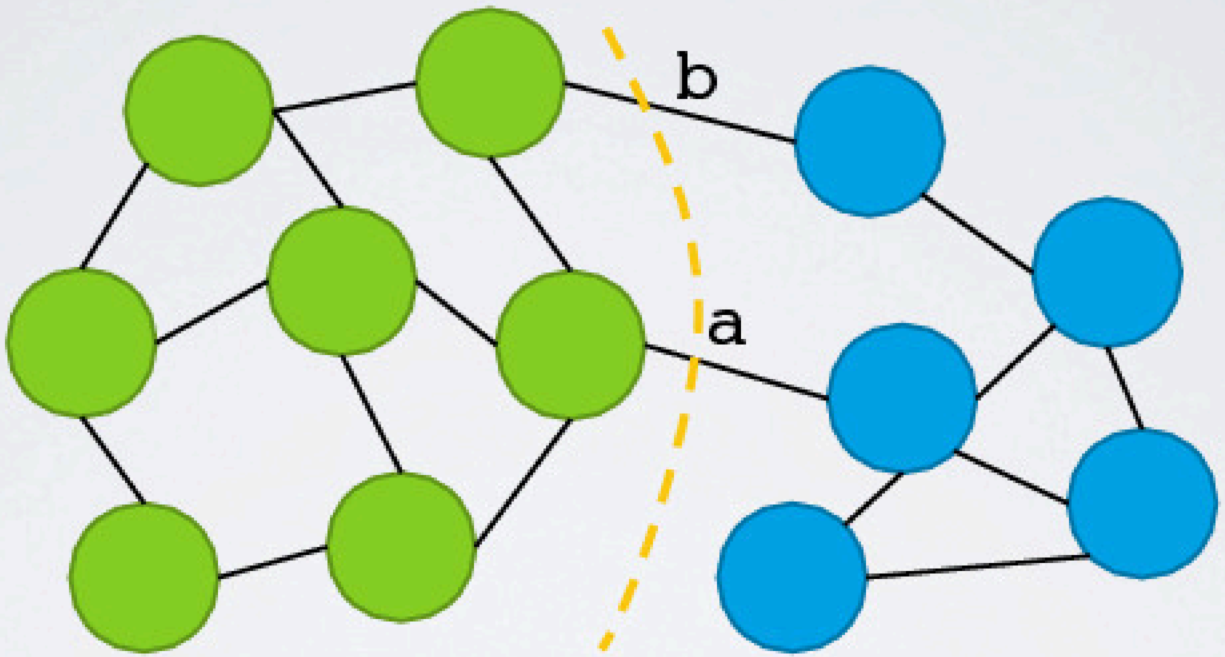
- Want an invariant `P(n)`, where `n` is number of edges added so far
- Need to have:
  - `P(0)` *[base case]*
  - `P(n)` implies `P(n + 1)` *[inductive case]*
  - `P(size of MST)` implies correctness

Inductive invariant for Prim

- Want an invariant `P(n)`, where `n` is number of edges added so far
- Need to have:
  - `P(0)` *[base case]*
  - `P(n)` implies `P(n + 1)` *[inductive case]*
  - `P(size of MST)` implies correctness
- `P(n)`= first `n` edges added by Prim are a subtree of *some* MST

Graph Cuts

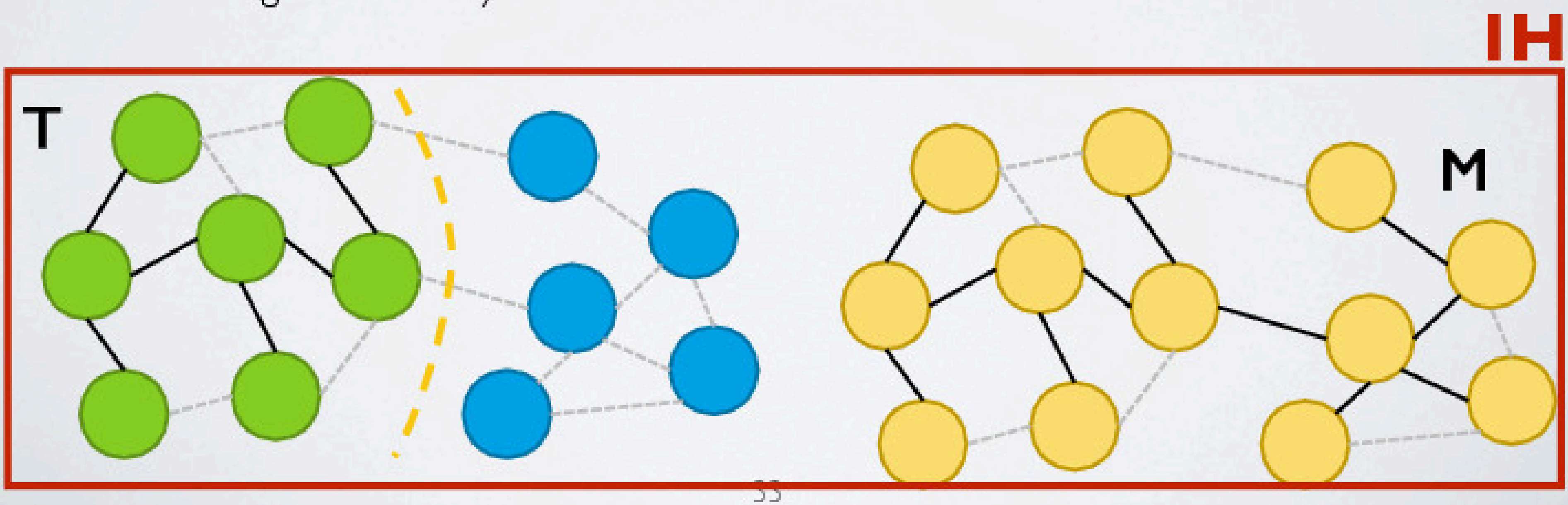
- A cut is any partition of the vertices into two groups



- Here G is partitioned in 2
  - with edges b and a joining the partitions

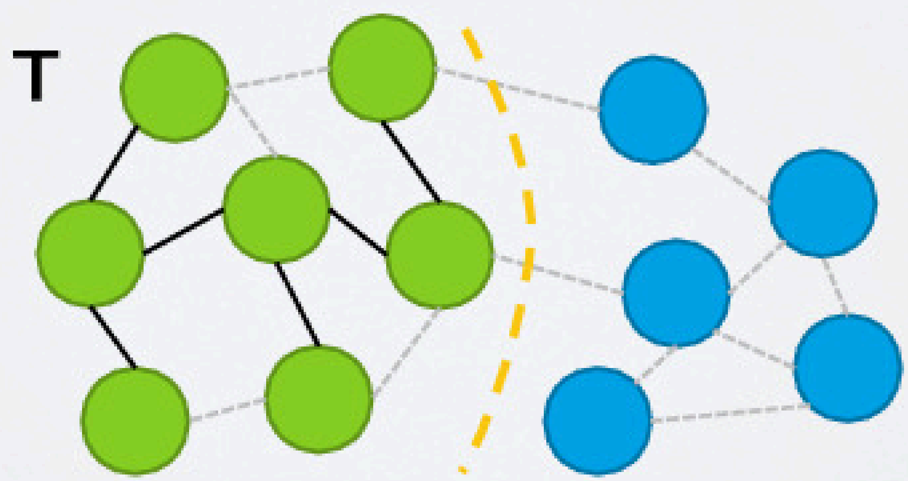
Proof of Correctness

- `P(n)`
  - first `n` edges added by Prim are a subtree of some MST
- Base case when `n=0`
  - no edges have been added yet so `P(0)` is trivially true
- Inductive Hypothesis
  - first `k` edges added by Prim form a tree `T` which is subtree of some MST `M`



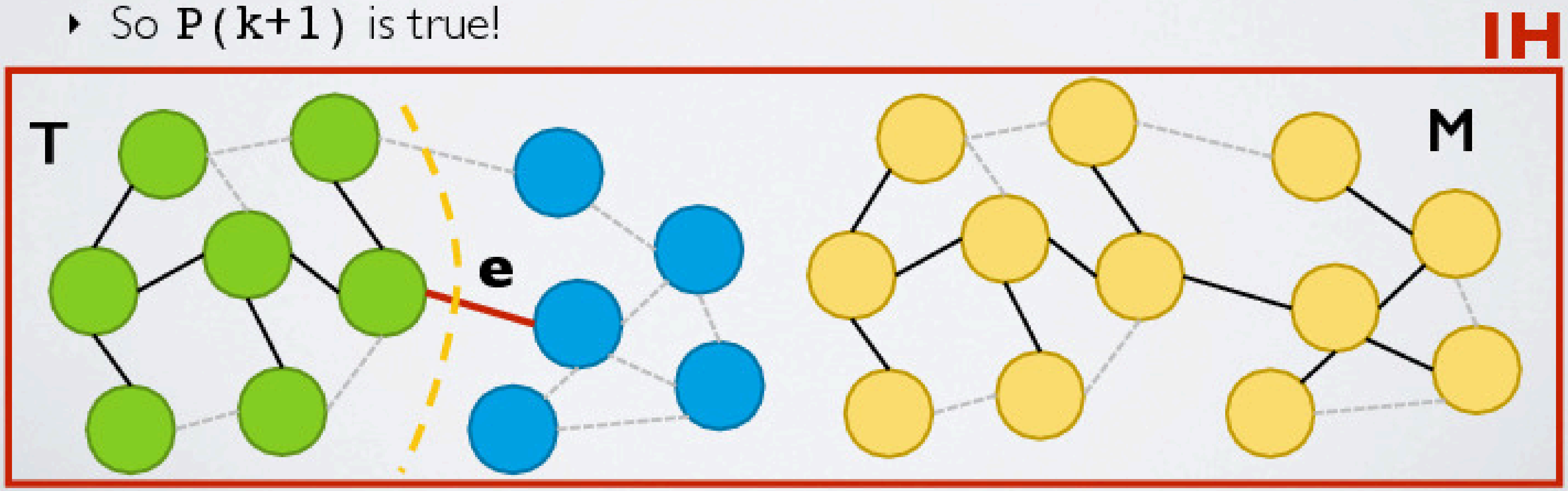
Proof of Correctness

- Inductive Step
  - Let `e` be the `(k+1)`th edge that is added
  - `e` will connect `T` (green nodes) to an unvisited node (one of blue nodes)
  - We need to show that adding `e` to `T`
    - forms a subtree of some MST `M'`
    - (which may or may not be the same MST as `M`)



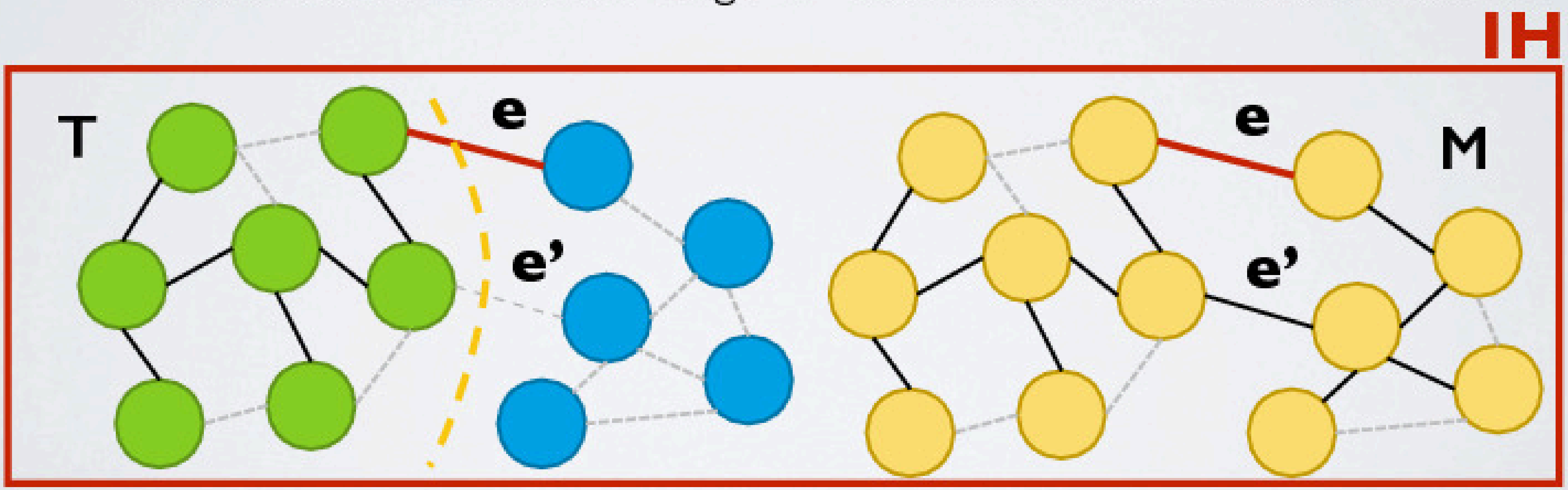
Proof of Correctness

- Two cases
  - `e` is in original MST `M`
  - `e` is not in `M`
- Case `1`: `e` is in `M`
  - there exists an MST that contains first `k+1` edges
  - So `P(k+1)` is true!



Proof of Correctness

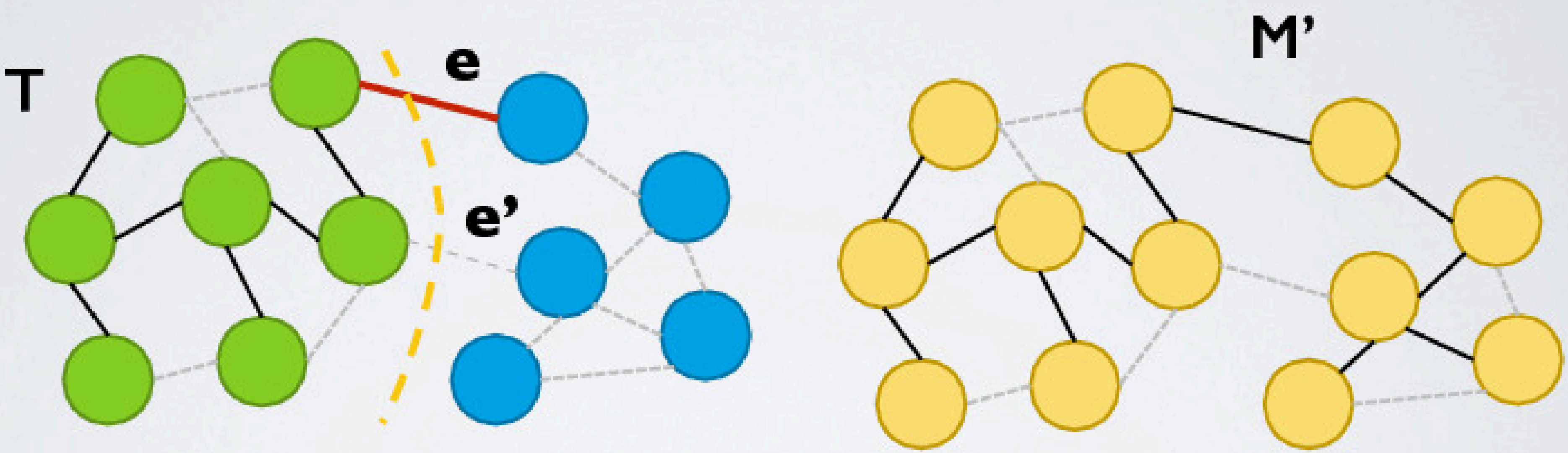
- Case 2: `e` is not in `M`
  - if we add `e=(u,v)` to `M` then we get a cycle
  - why? since `M` is span. tree there must be path from `u` to `v` w/o `e`
  - so there must be another edge `e'` that connects `T` to unvisited nodes



  - We know `e.weight ≤ e'.weight` because Prim chose `e` first

Proof of Correctness

- So if we add **`e`** to **`M`** and remove **`e'`**
  - we get a new MST **`M'`** that is no larger than **`M`** and contains **`T`** & **`e`**



- **`P(k+1)`** is true
  - because **`M'`** is an MST that contains the first **`k+1`** edges added by Prim's

Proof of Correctness

- Since we have shown
  - `P(0)` is true
  - `P(k+1)` is true assuming `P(k)` is true (for both cases)
  - The first `n` edges added by Prim form a subtree of some MST

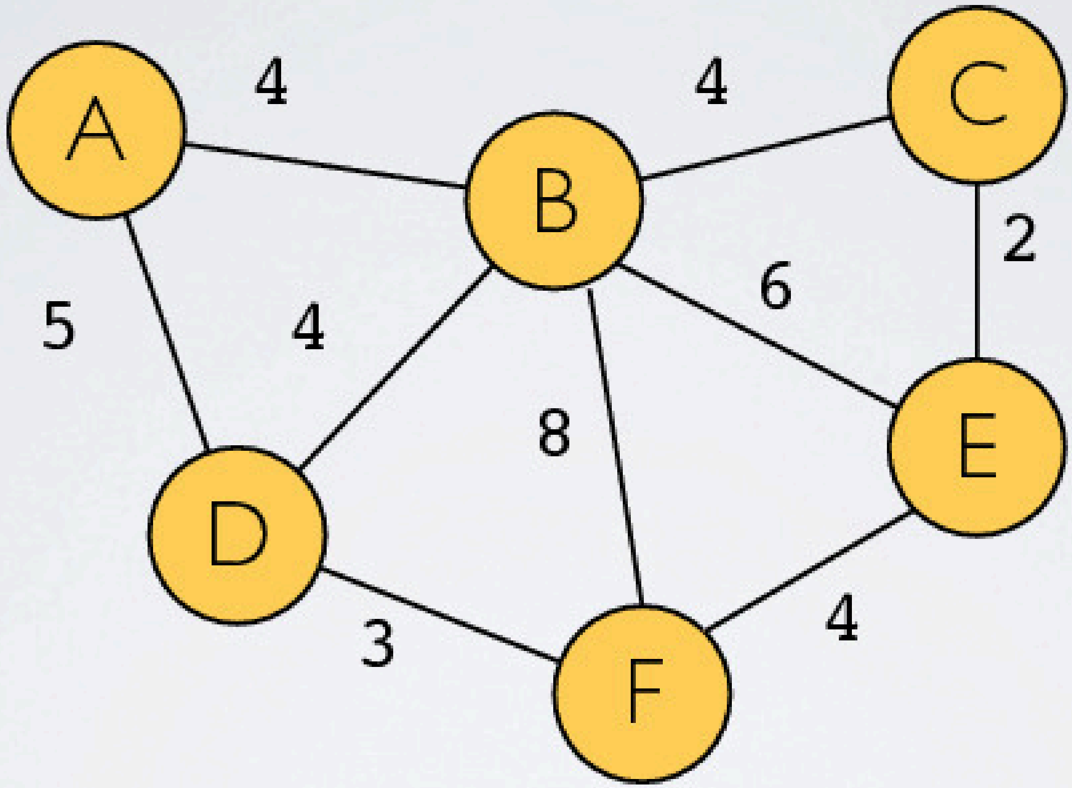
Outline

- Minimum Spanning Trees
- Prim-Jarnik Algorithm
  - Analysis
  - Proof of Correctness
- Kruskal's Algorithm
  - Union-Find
  - Analysis

Kruskal's Algorithm

- Sort edges by weight in ascending order
- For each edge in sorted list
  - If adding edge does not create cycle…
  - …add it to MST
- Stop when you have gone through all edges

Example




```
edges = [(C,E),(D,F),(B,C),(E,F),(B,D),(A,B),(A,D),(B,E),(B,F)]
```

Simulate Kruskal

Activity #3

2 min

Simulate Kruskal

Activity #3

2 min

Simulate Kruskal

Activity #3

1 min

Simulate Kruskal

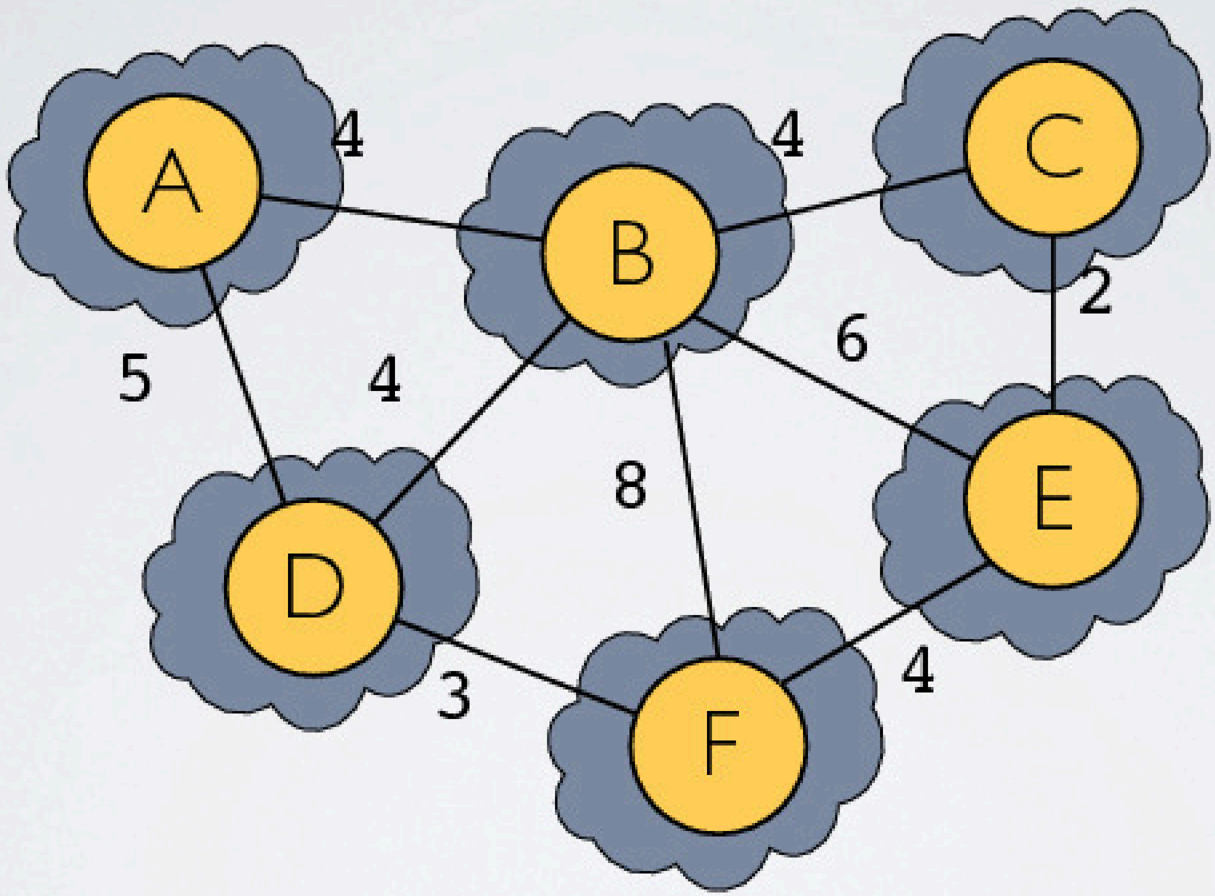
Activity #3

0 min

Kruskal

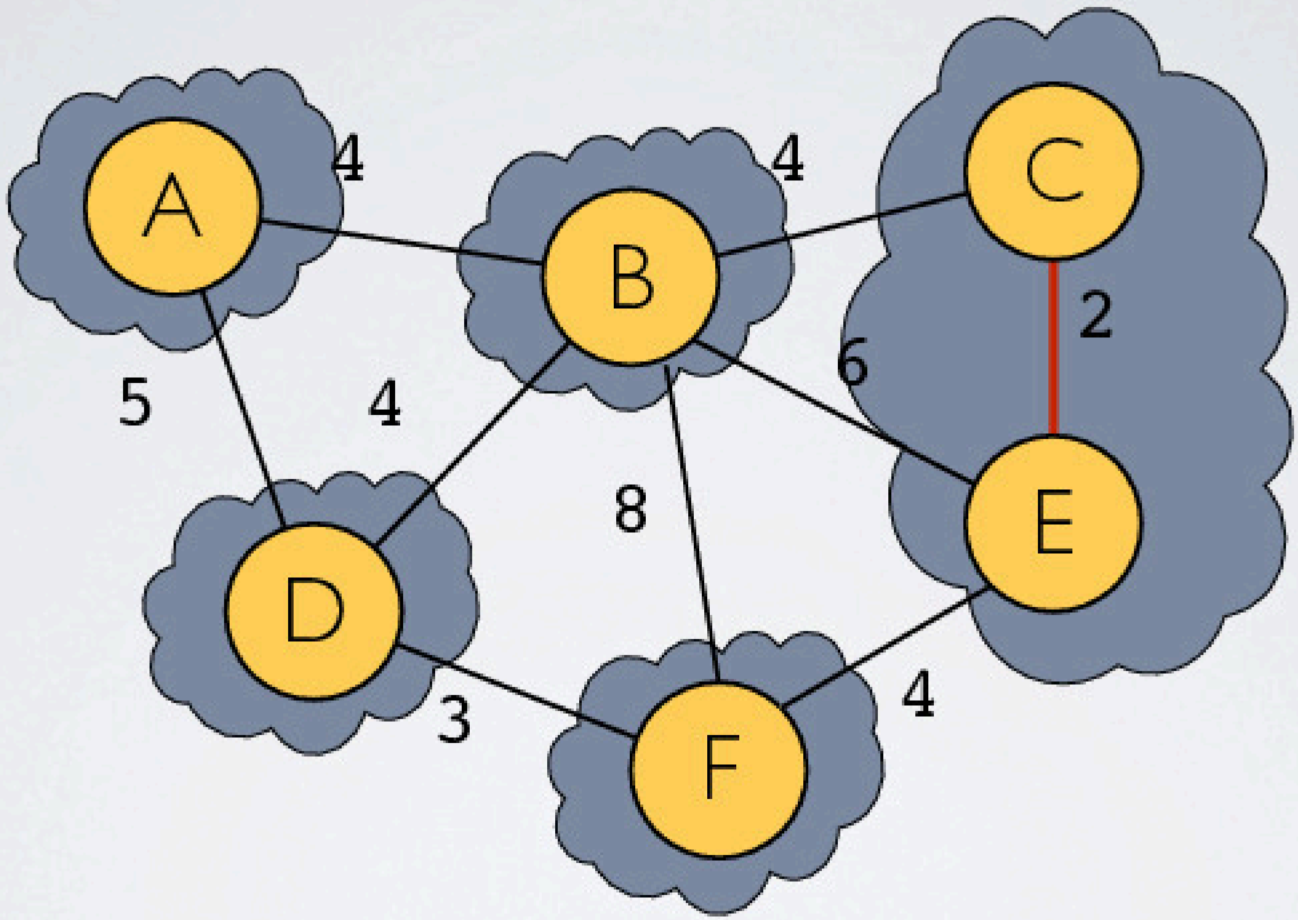
- How can we tell if adding edge will create cycle?
- Start by giving each vertex its own "cloud"
- If both ends of lowest-cost edge are in same cloud
  - we know that adding the edge will create a cycle!
- When edge is added to MST
  - merge clouds of the endpoints

Example



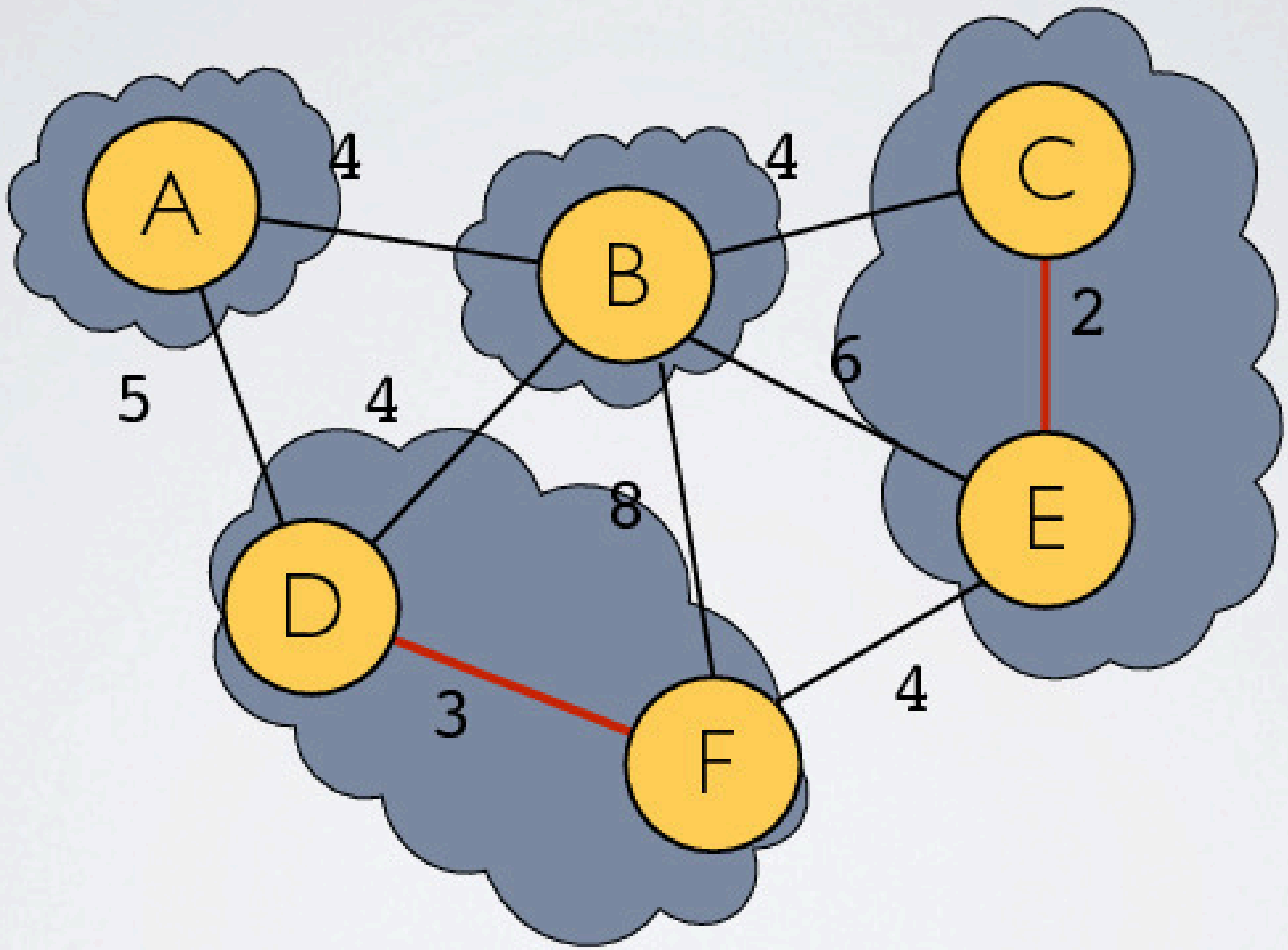

```
edges = [(C,E),(D,F),(B,C),(E,F),(B,D),(A,B),(A,D),(B,E),(B,F)]
```

Example



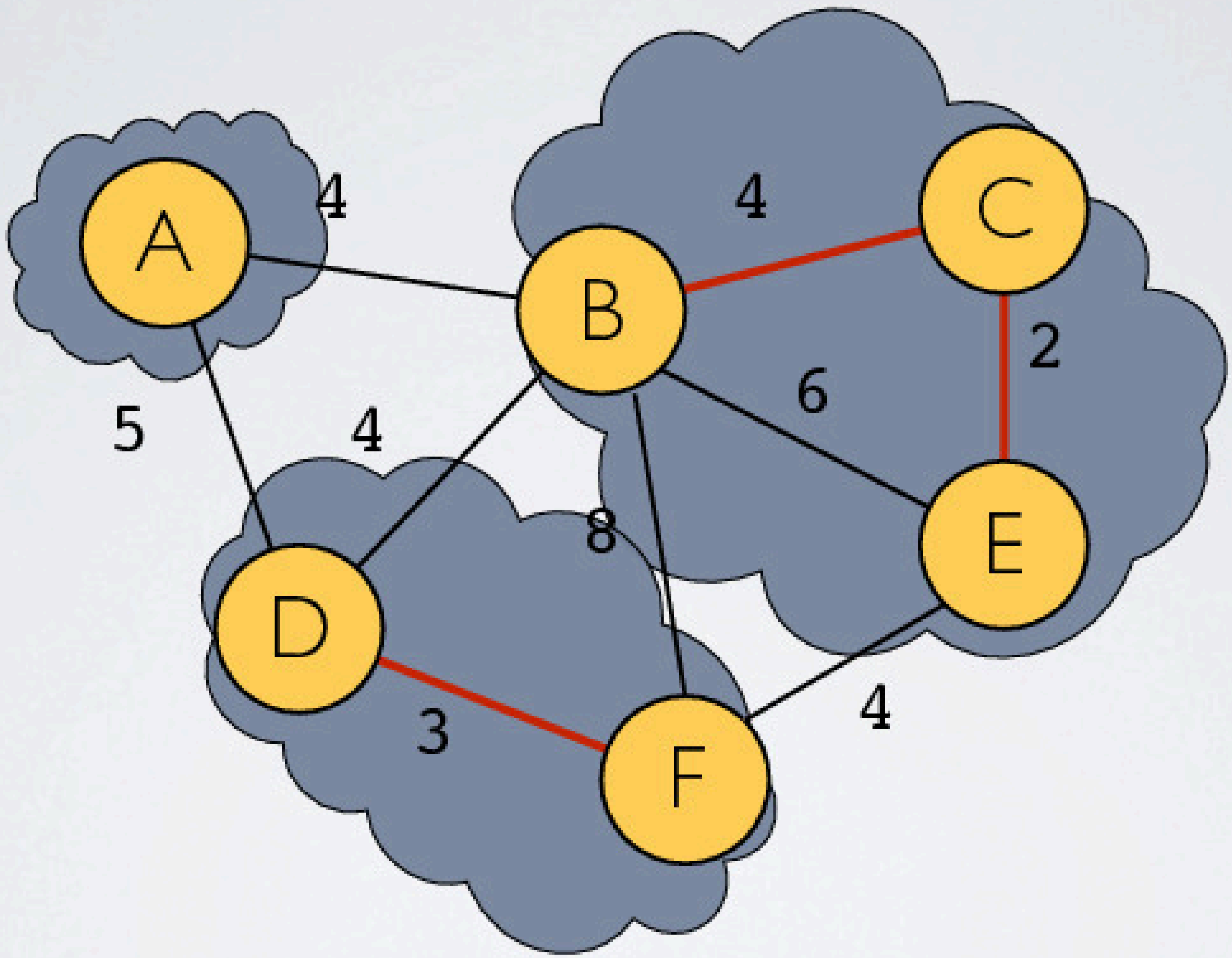

```
edges = [(D,F),(B,C),(E,F),(B,D),(A,B),(A,D),(B,E),(B,F)]
```

Example



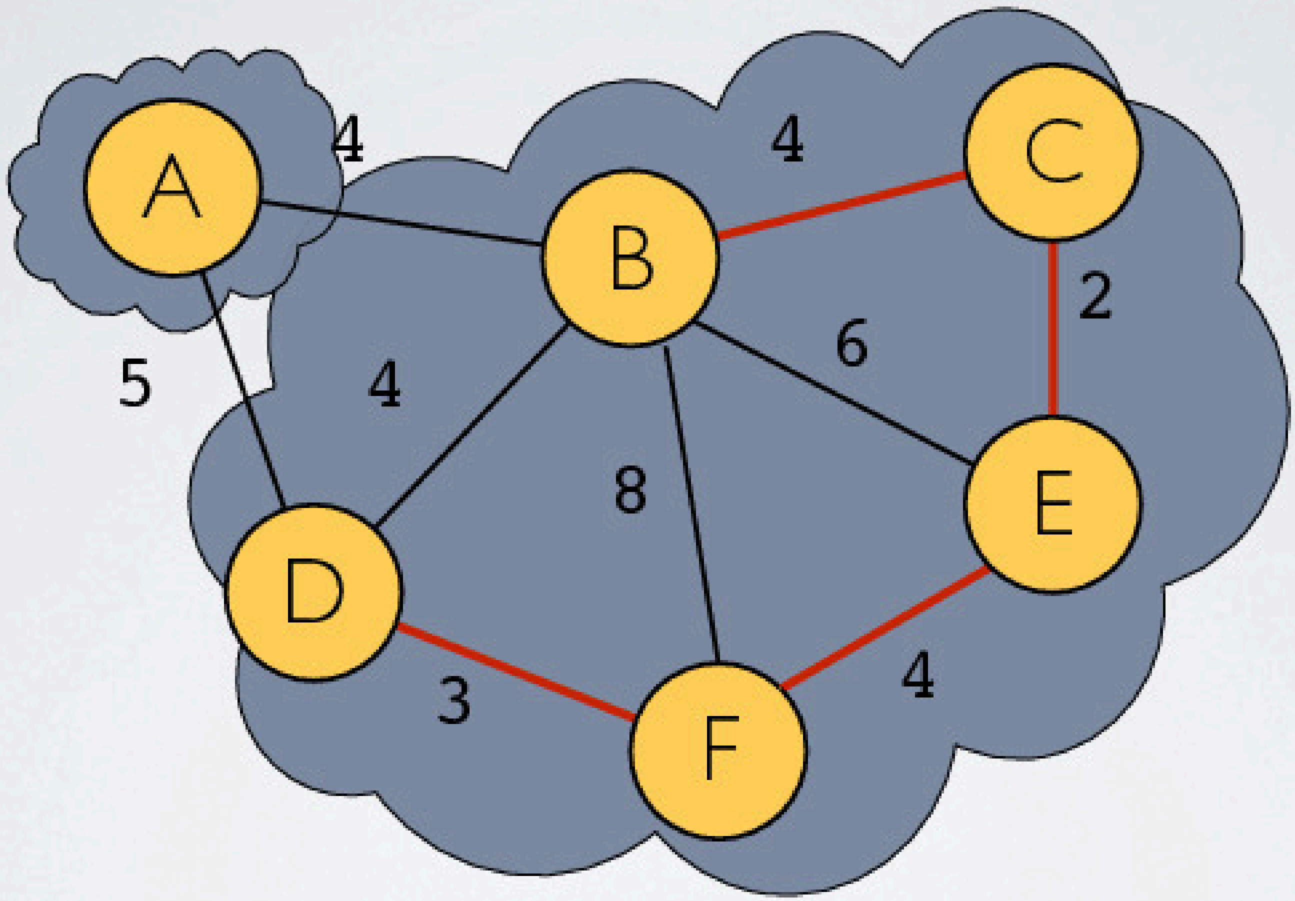
`edges = [(B,C),(E,F),(B,D),(A,B),(A,D),(B,E),(B,F)]`

Example



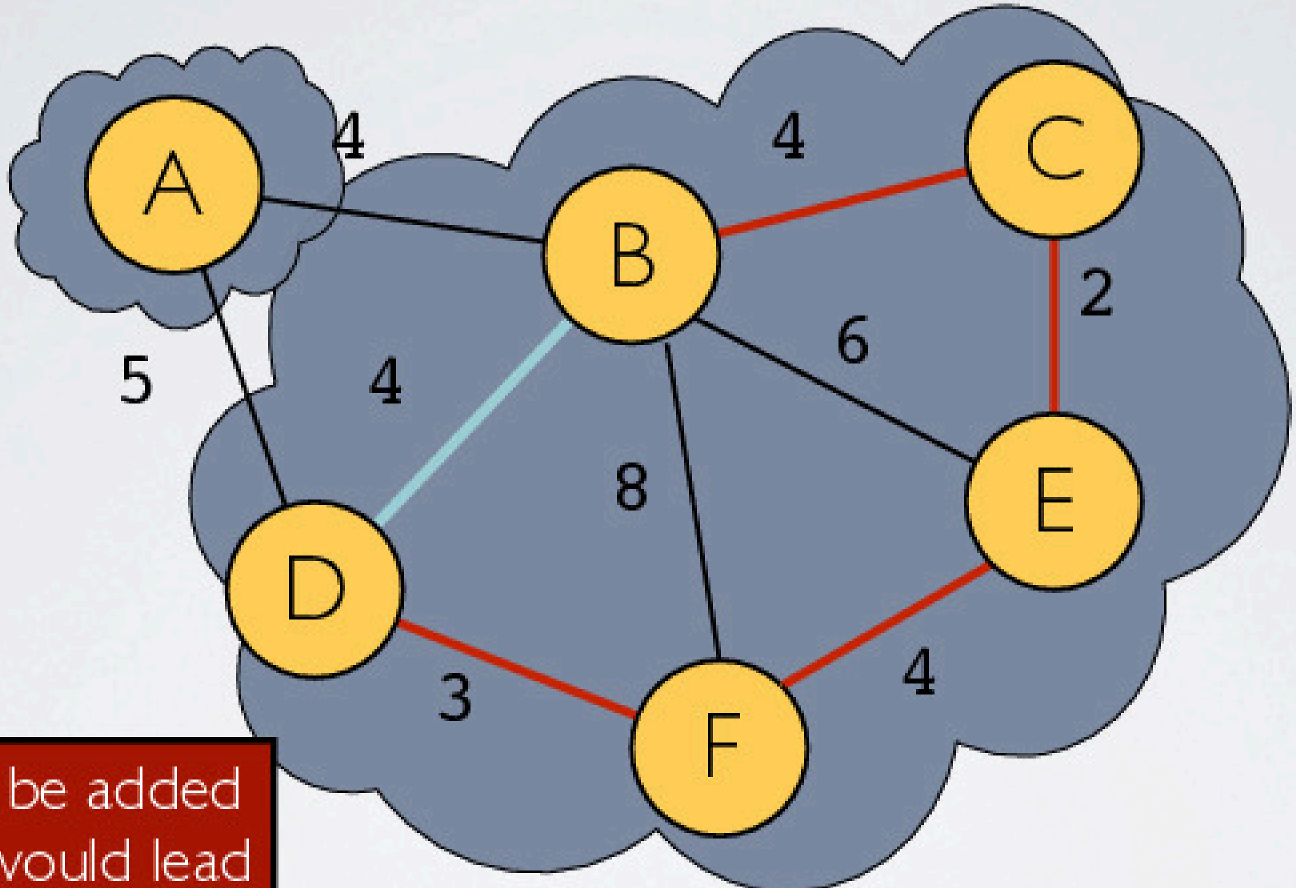

```
edges = [(E,F),(B,D),(A,B),(A,D),(B,E),(B,F)]
```

Example




```
edges = [(B,D),(A,B),(A,D),(B,E),(B,F)]
```

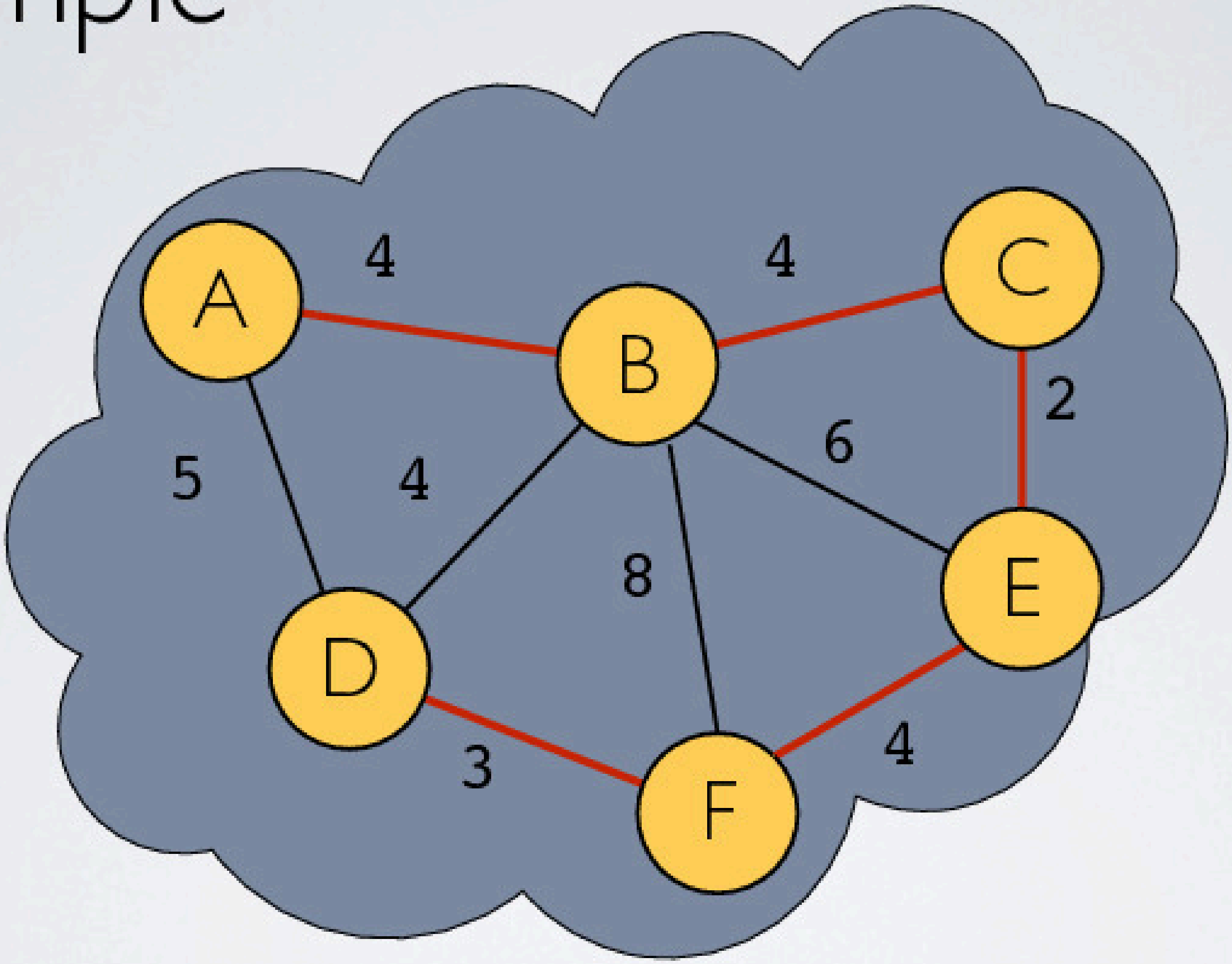
Example



BD cannot be added because it would lead to a cycle

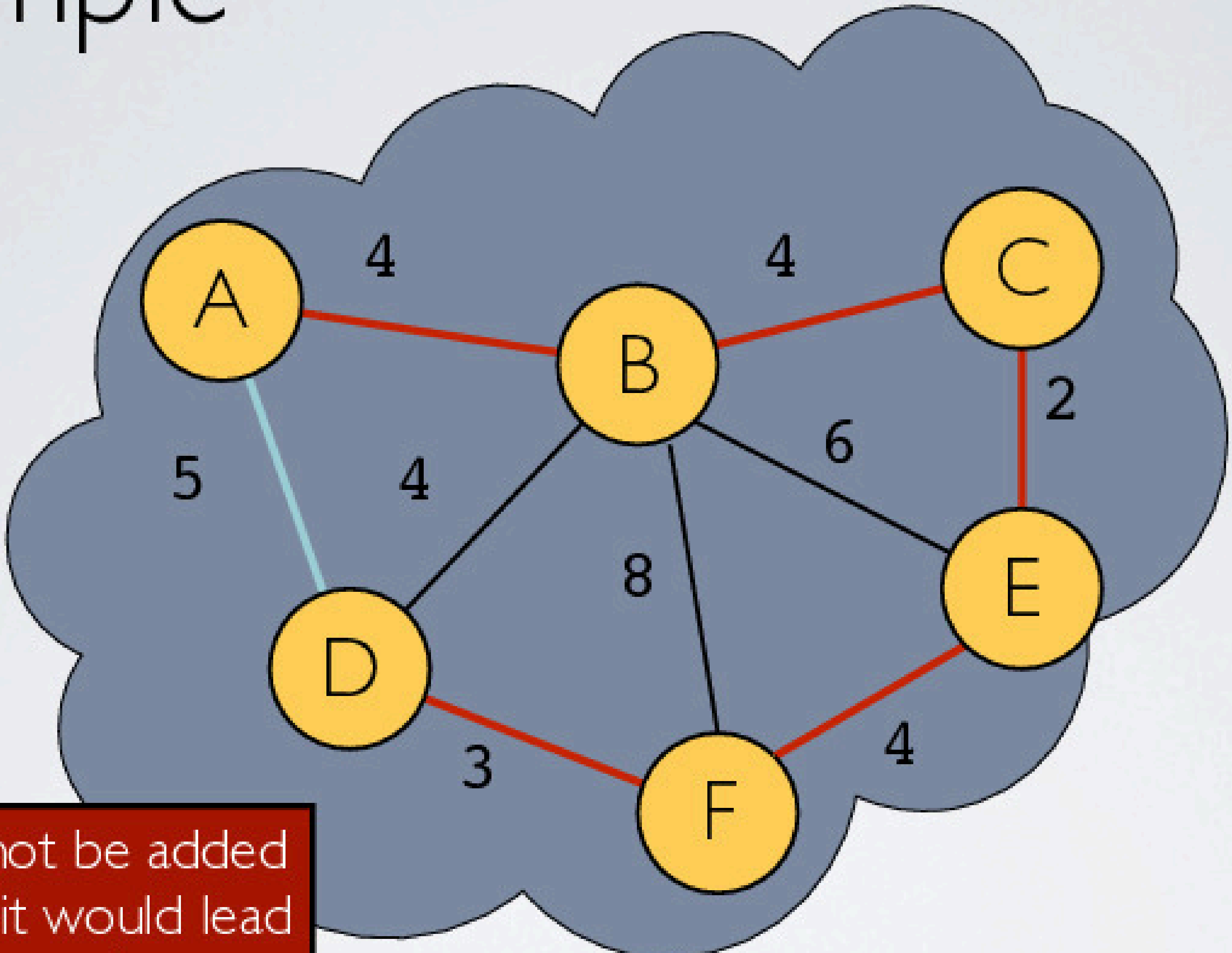
```
edges = [(A,B),(A,D),(B,E),(B,F)]
```

Example




```
edges = [(A,D),(B,E),(B,F)]
```

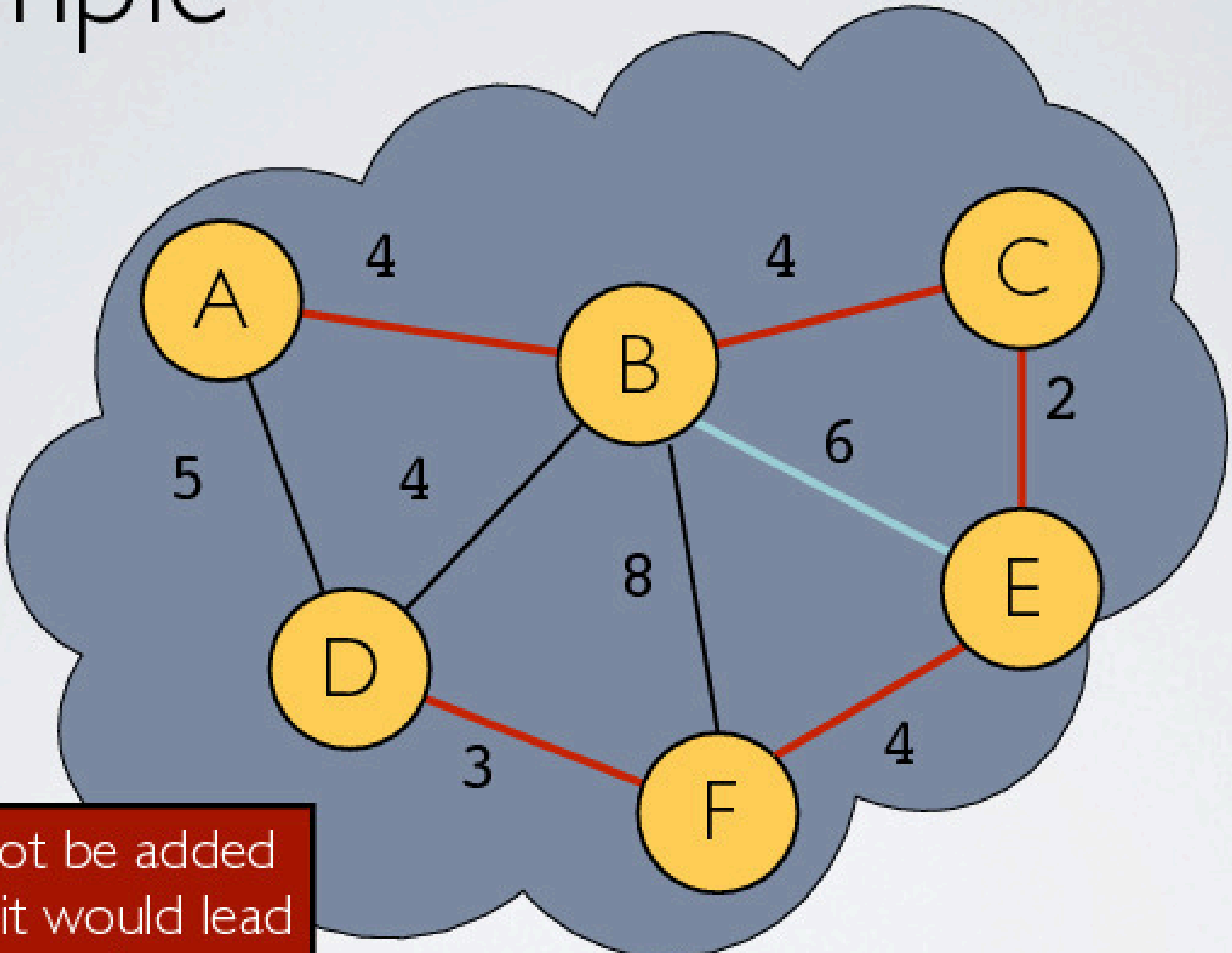
Example



AD cannot be added because it would lead to a cycle

```
edges = [(B,E),(B,F)]
```

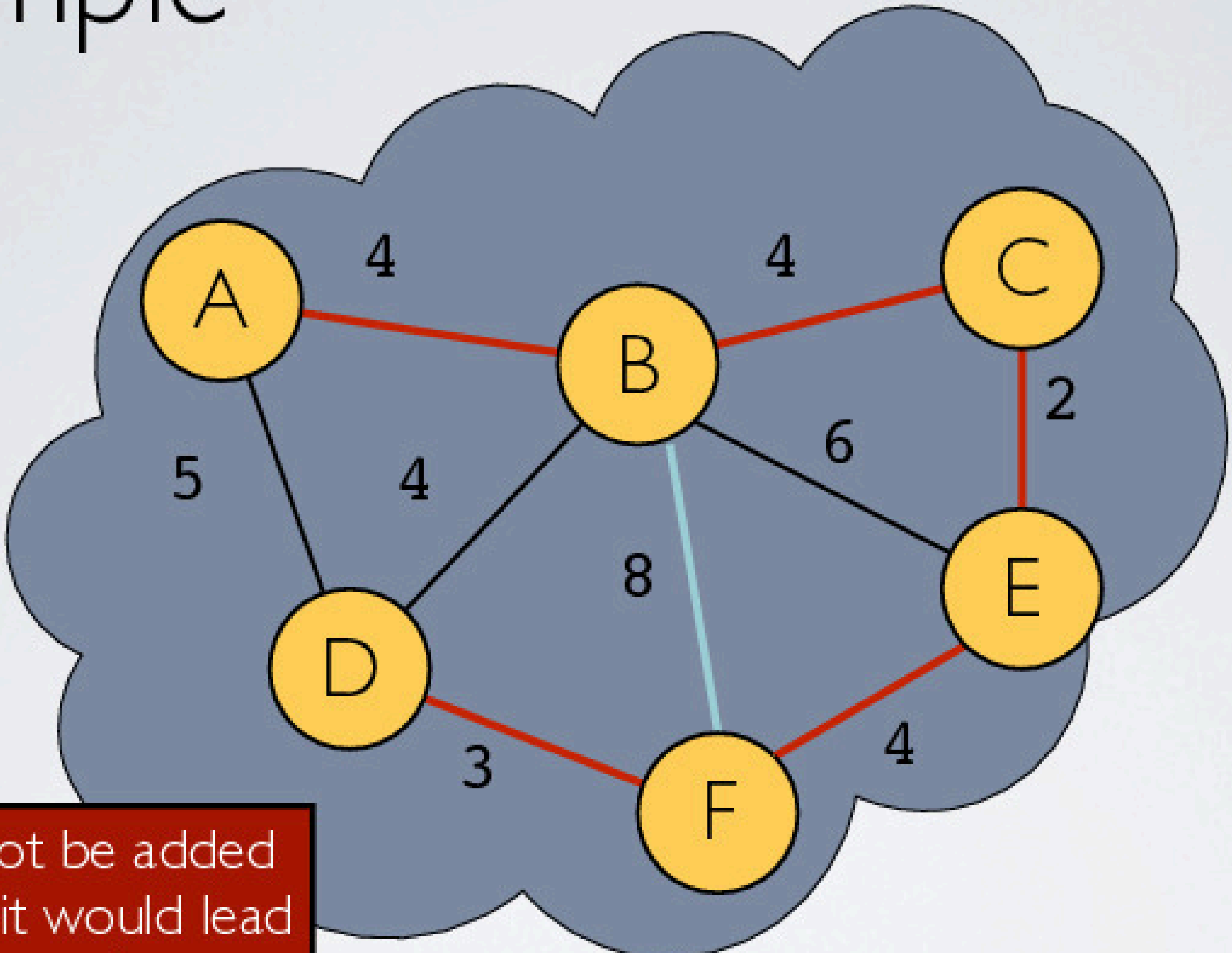
Example



BE cannot be added because it would lead to a cycle

```
edges = [(B,F)]
```

Example



BF cannot be added because it would lead to a cycle

```
edges = [ ]
```

Kruskal Pseudo-Code

```
function kruskal(G):
   // Input: undirected, weighted graph G
   // Output: list of edges in MST
   for vertices v in G:
      makeCloud(v) // put every vertex into it own set
   MST = []
  Sort all edges
   for all edges (u,v) in G sorted by weight:
      if u and v are not in same cloud:
         add (u,v) to MST
         merge clouds containing u and v
   return MST
```

Merging Clouds (Naive way)

- Assign each vertex a different number
  - that represents its initial cloud
- To merge clouds of **`u`** and **`v`**
  - Find all vertices in each cloud
  - Figure out which of the clouds is smaller
  - Redecorate all vertices in smaller cloud w/ bigger cloud's number

Merging Clouds (Naive way)

- Finding all vertices in `u` & `v`'s clouds is `O(|V|)`
  - because we have to iterate through each vertex…
  - …and check if its cloud number matches `u` or `v`'s cloud number
- Figuring out smaller cloud is `O(1)`
  - as long as we keep track of cloud size as we find vertices in them
- Changing cloud numbers of nodes in smaller cloud is `O(|V|)`
  - because smallest cloud could be as big as `|V|/2` vertices
- Total runtime to merge clouds
  - `O(|V| + 1 + |V|) = O(|V|)`

Runtime of Naive Kruskal

- Finding all vertices in `u` & `v`'s clouds is `O(|V|)`
  - because we have to iterate through each vertex…
  - …and check if its cloud number matches `u` or `v`'s cloud number
- Figuring out smaller cloud is `O(1)`
  - as long as we keep track of cloud size as we find vertices in them
- Changing cloud numbers of vertices in smaller cloud is `O(|V|)`
  - because cloud could be as big as `|V|/2` vertices
- Merge Runtime
  - `O(|V|) + O(1) + O(|V|) = O(|V|)`

Activity #4

2 min

Runtime of Naive Kruskal

- Finding all vertices in `u` & `v`'s clouds is `O(|V|)`
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  - …and check if its cloud number matches `u` or `v`'s cloud number
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Activity #4

2 min

Runtime of Naive Kruskal

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- Merge Runtime
  - `O(|V|) + O(1) + O(|V|) = O(|V|)`

Activity #4

1 min

Runtime of Naive Kruskal

- Finding all vertices in `u` & `v`'s clouds is `O(|V|)`
  - because we have to iterate through each vertex…
  - …and check if its cloud number matches `u` or `v`'s cloud number
- Figuring out smaller cloud is `O(1)`
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- Changing cloud numbers of vertices in smaller cloud is `O(|V|)`
  - because cloud could be as big as `|V|/2` vertices
- Merge Runtime
  - `O(|V|) + O(1) + O(|V|) = O(|V|)`

Activity #4

0 min

Kruskal Runtime w/ Naive Clouds

```
function kruskal(G):
   // Input: undirected, weighted graph G
   // Output: list of edges in MST
   for vertices v in G:                              O(|V|)
      makeCloud(v)
   MST = []
   Sort all edges                                    O(|E|log|E|)
   for all edges (u,v) in G sorted by weight:        O(|E|)
      if u and v are not in same cloud:
         add (u,v) to MST
         merge clouds containing u and v             O(|V|)
   return MST
```

Kruskal Runtime

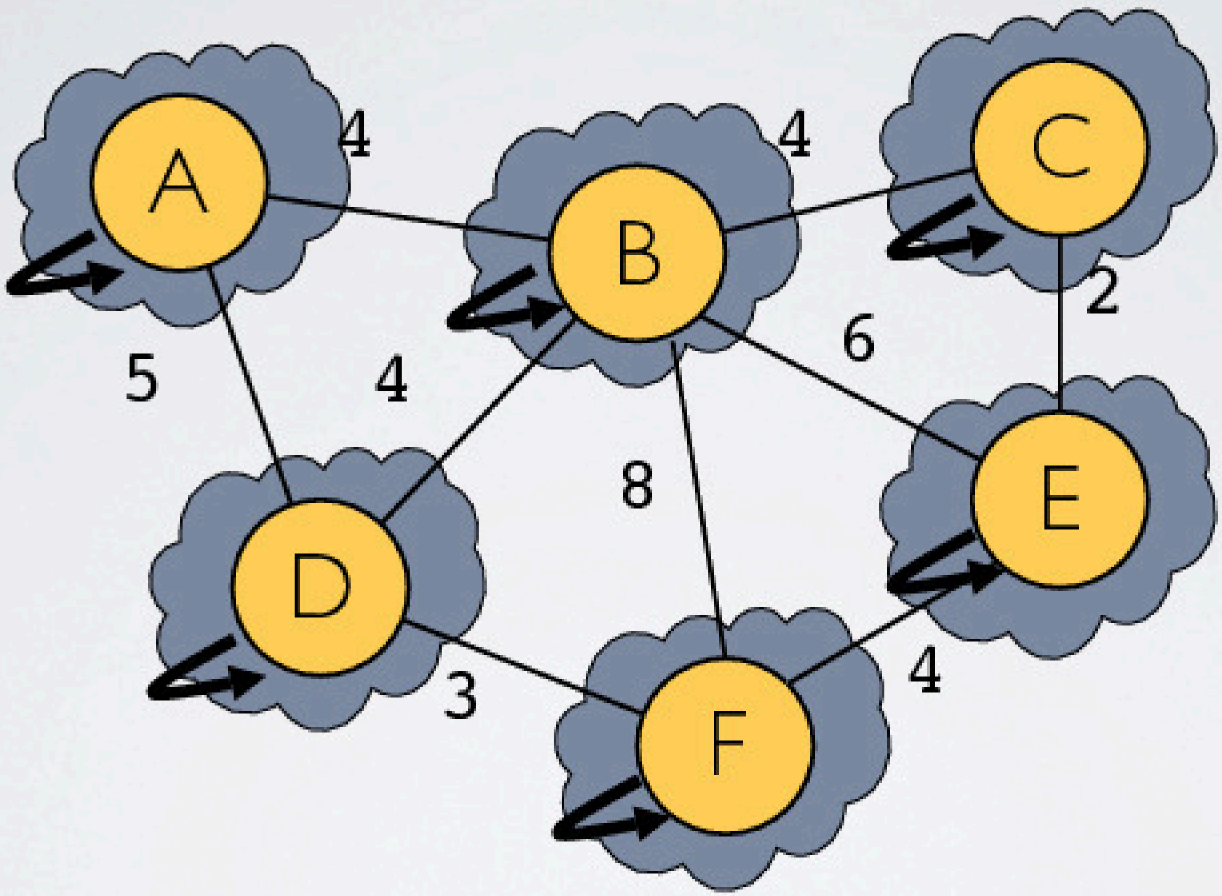
- $O(|V|)$ for iterating through vertices
- $O(|E|\log|E|)$ for sorting edges
- $O(|E|\times|V|)$ for iterating through edges and merging clouds naively
- $O(|V|+|E|\log|E|+|E|\times|V|)$
  - $= O(|E|\times|V|) = O(|V|^2\times|V|) = O(|V|^3)$
- Can we do better?

since $|E| \leq |V|^2$

Union-Find

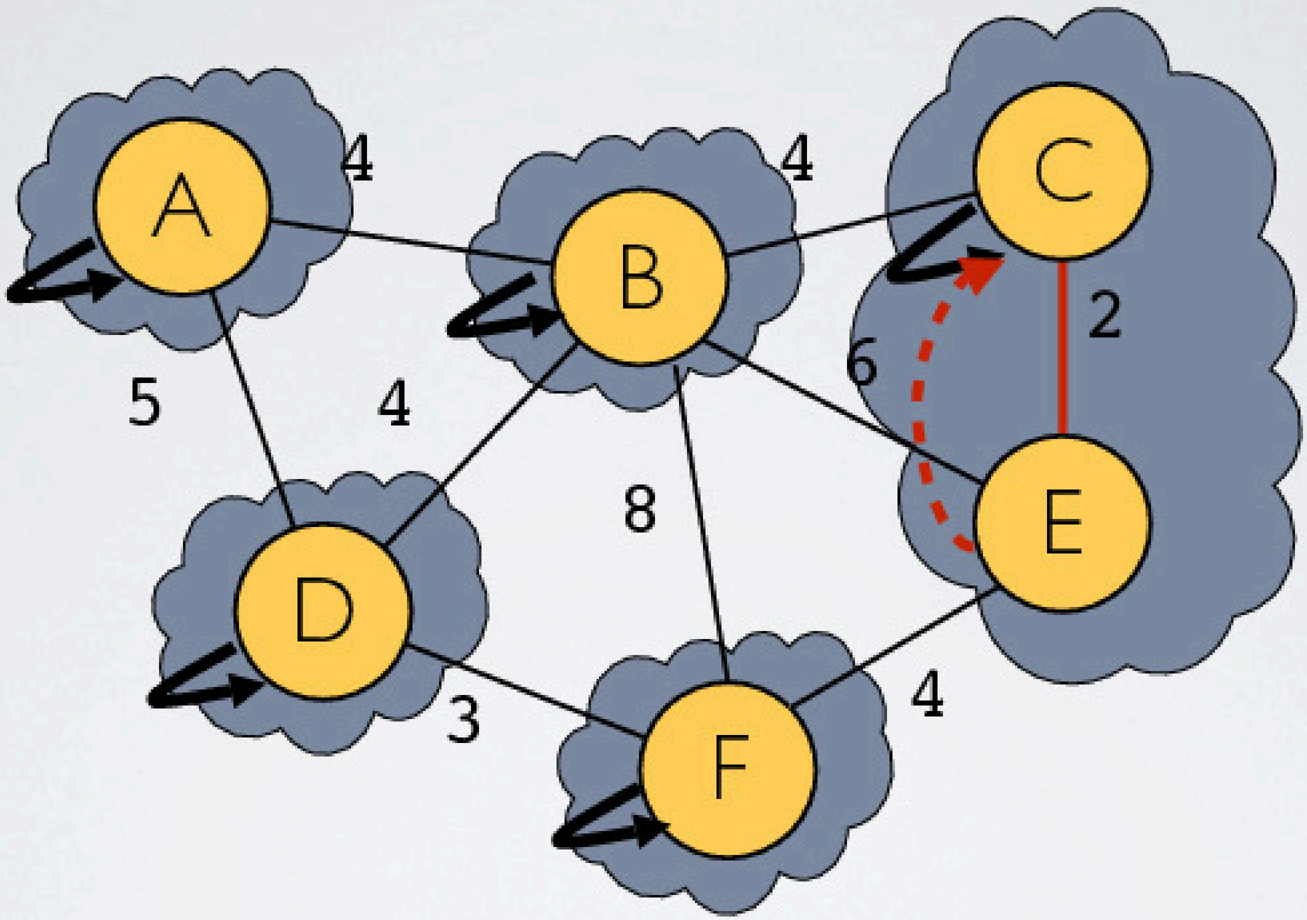
- Let's rethink notion of clouds
  - instead of labeling vertices w/ cloud numbers
  - think of clouds as small trees
- Every vertex in these trees has
  - a parent pointer that leads up to root of the tree
  - a rank that measures how deep the tree is

Example



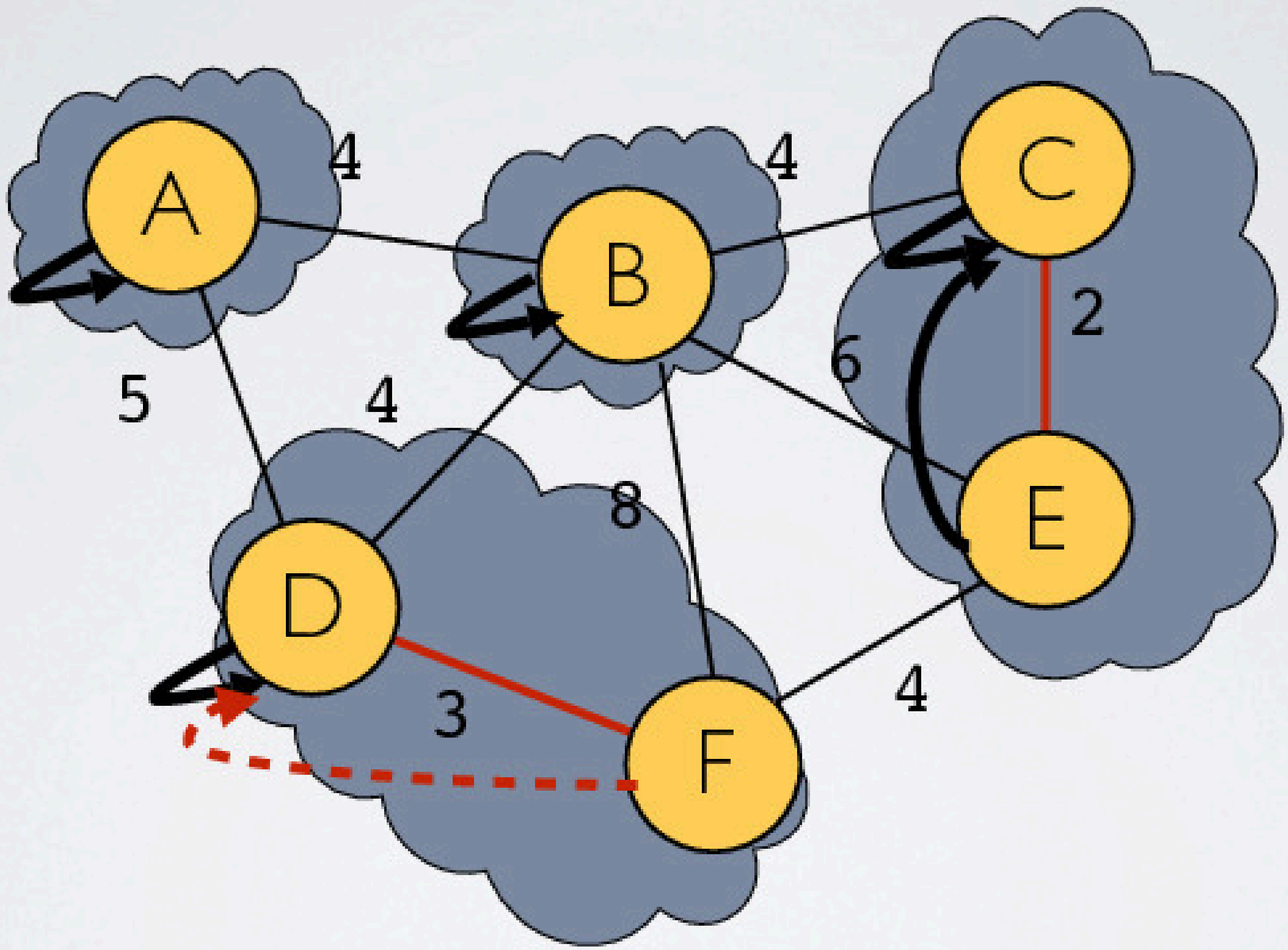

```
edges = [(C,E),(D,F),(B,C),(E,F),(B,D),(A,B),(A,D),(B,E),(B,F)]
```

Example



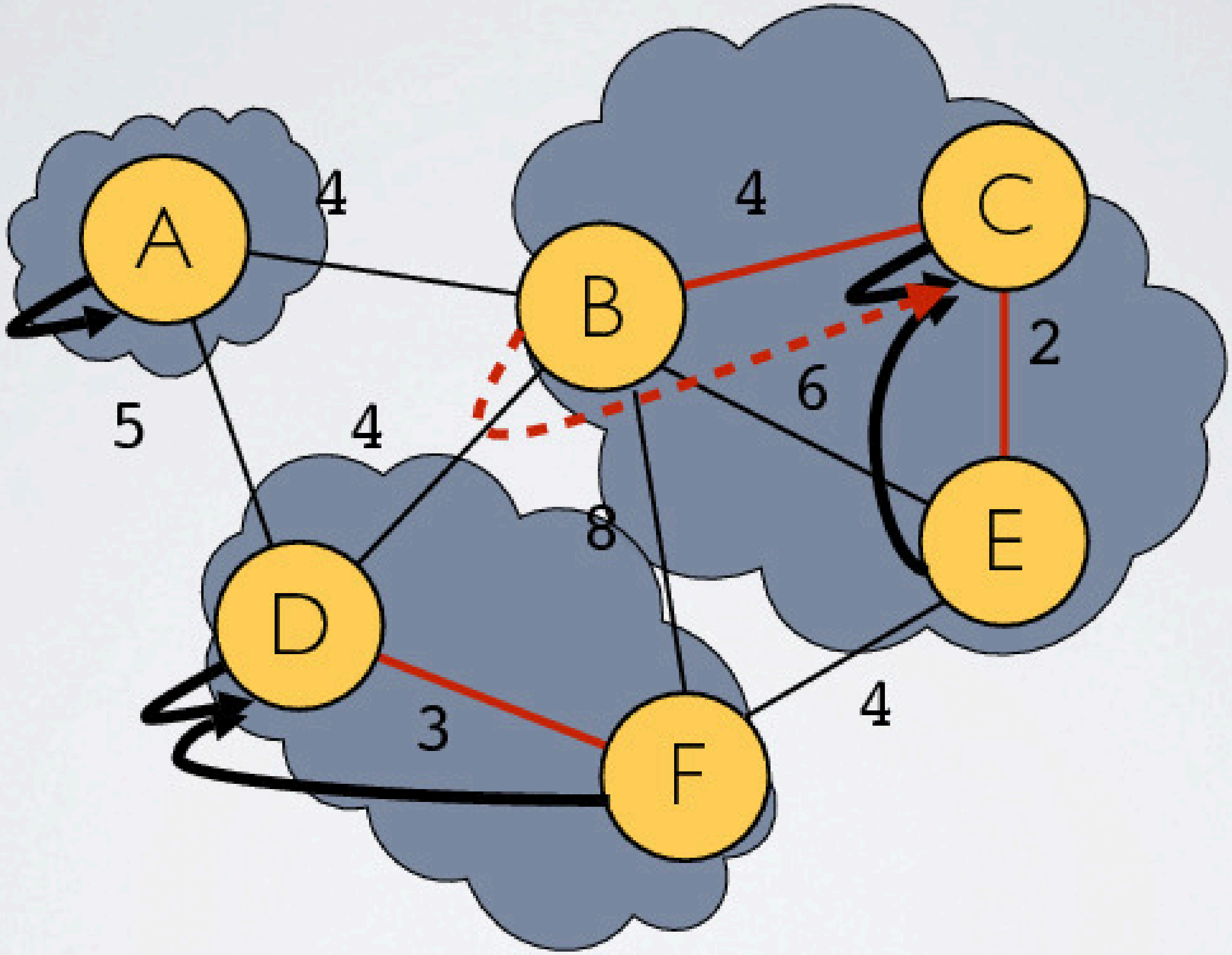

```
edges = [(D,F),(B,C),(E,F),(B,D),(A,B),(A,D),(B,E),(B,F)]
```

Example



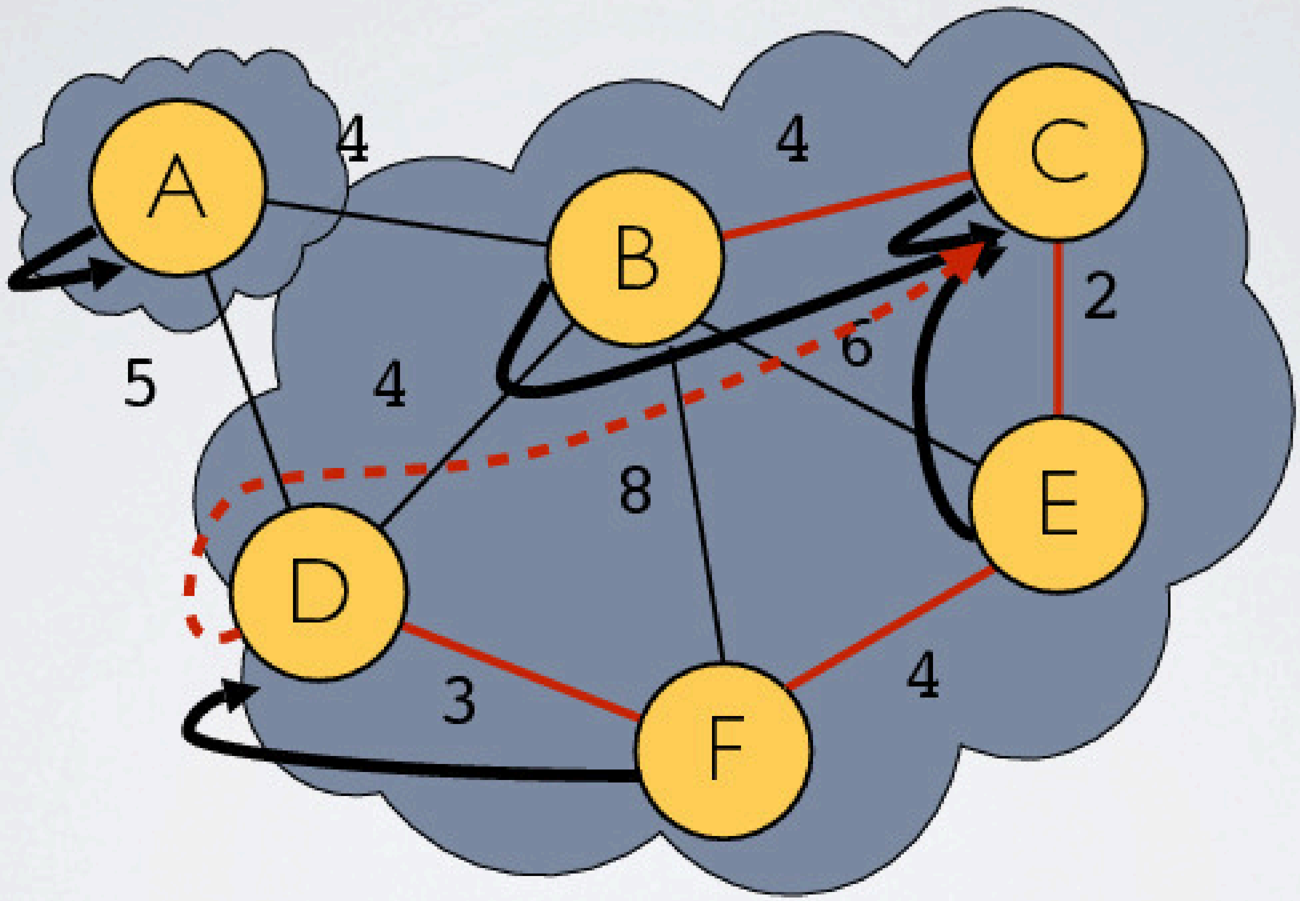

```
edges = [(B,C),(E,F),(B,D),(A,B),(A,D),(B,E),(B,F)]
```

Example



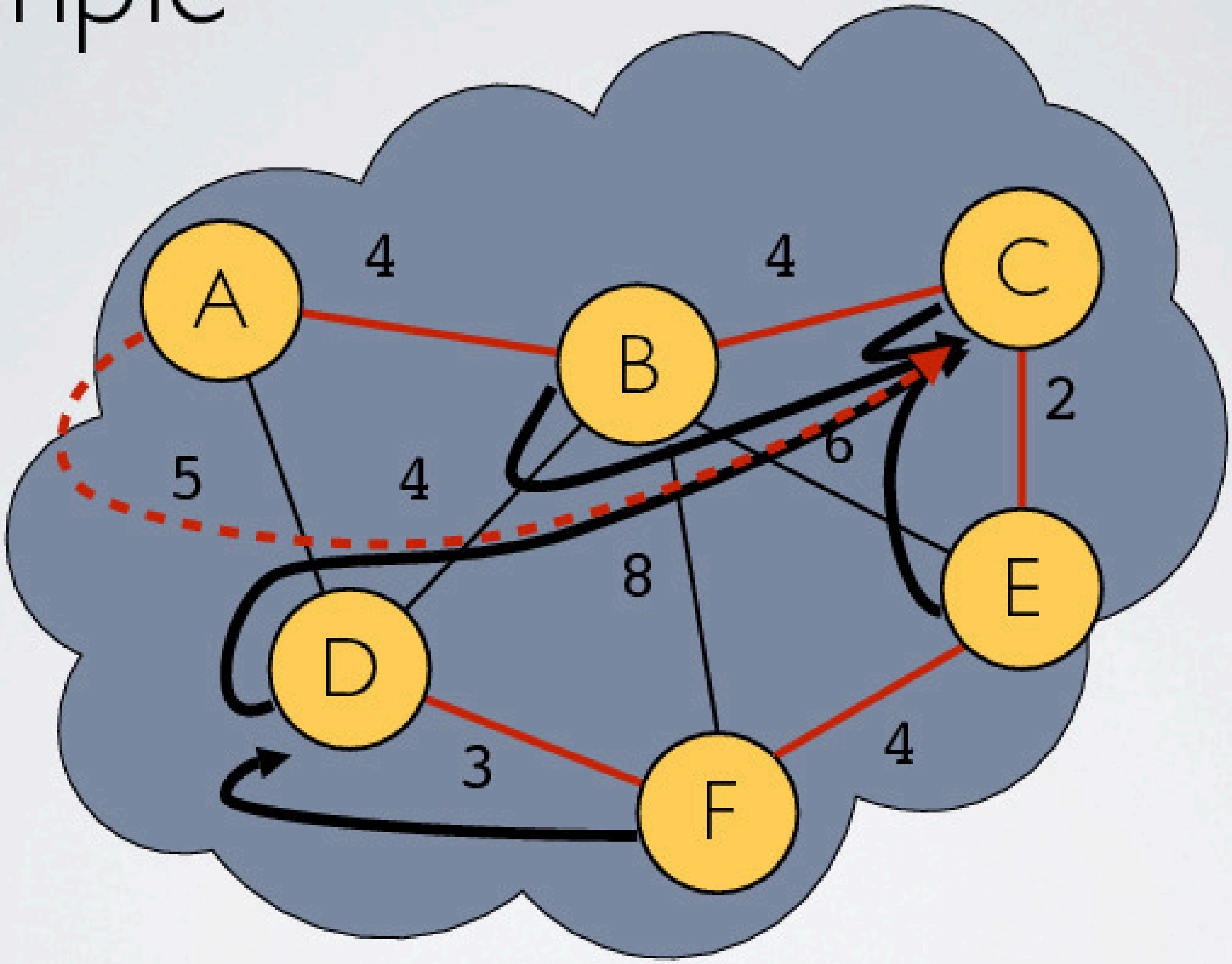

```
edges = [(E,F),(B,D),(A,B),(A,D),(B,E),(B,F)]
```

Example



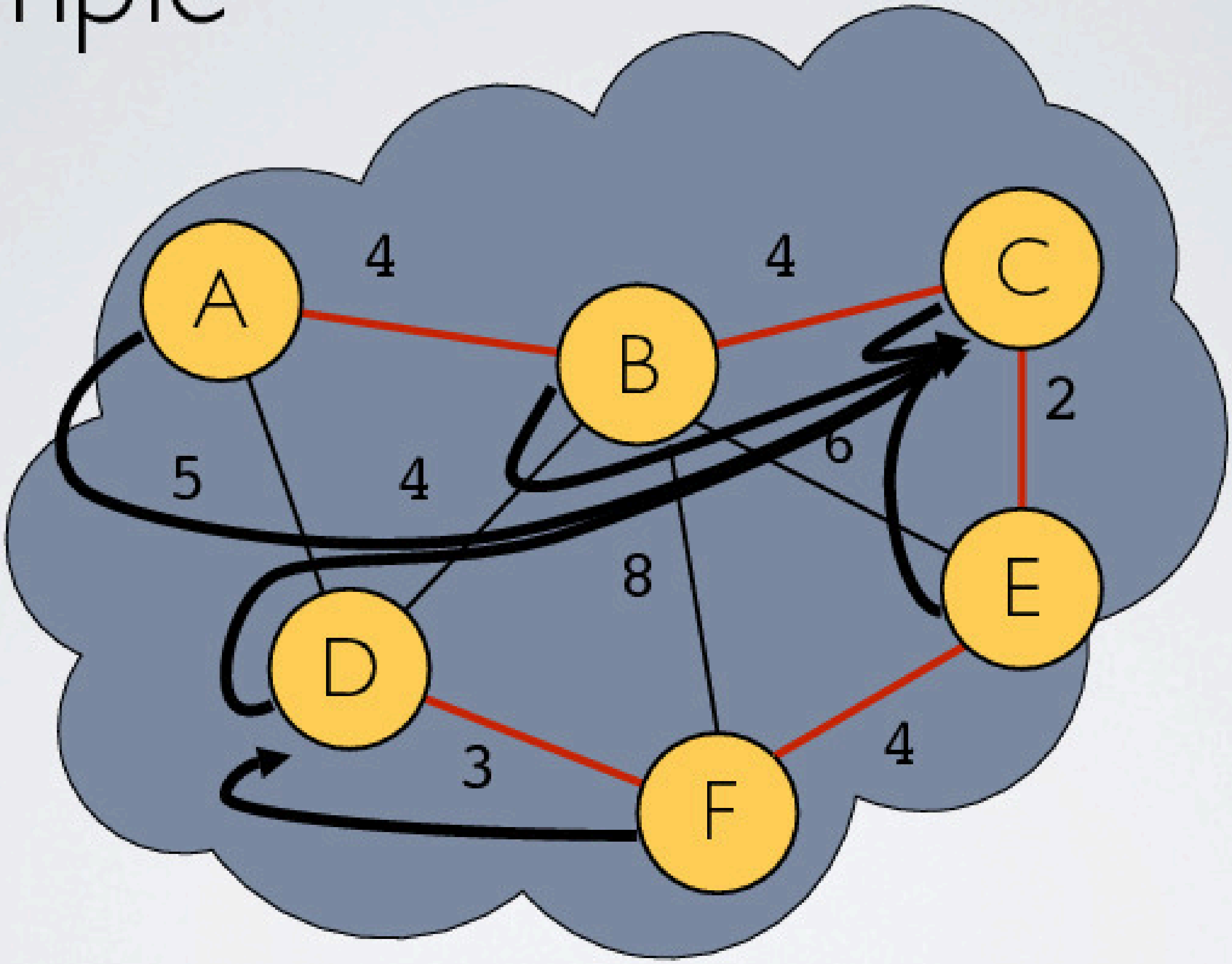

```
edges = [(B,D),(A,B),(A,D),(B,E),(B,F)]
```

Example




```
edges = [(A,D),(B,E),(B,F)]
```

Example



`edges = [(A,D),(B,E),(B,F)]`

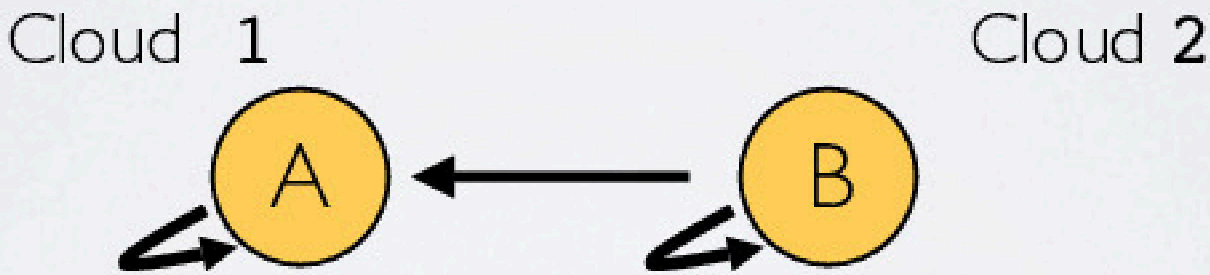
Implementing Union-Find

- At start of Kruskal
  - every node is put into own cloud

```
// Decorates every vertex with its parent ptr & rank
function makeCloud(x):
    x.parent = x
    x.rank = 0
```

Implementing Union-Find

- Suppose `A` is in cloud `1` and `B` is in cloud `2`
- Instead of relabeling `B` as cloud `1` make `B` point to `A`
  - Think of this as the union of two clouds



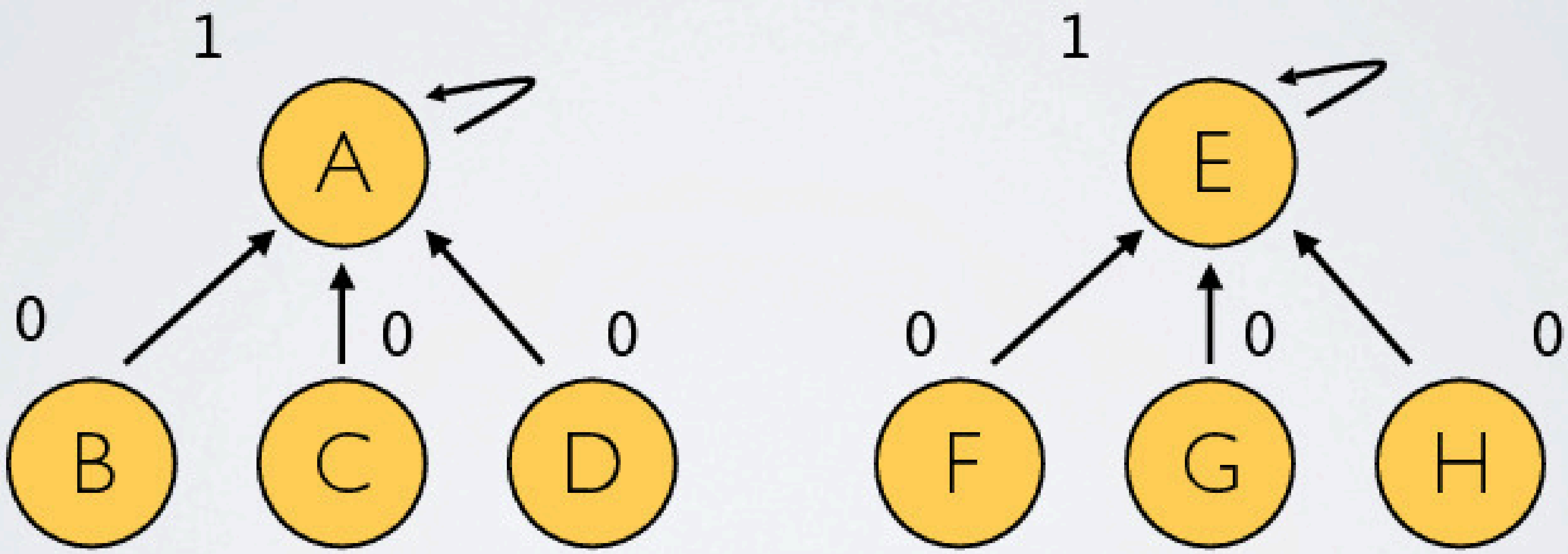
  - Given two clouds which one should point to the other?

Implementing Union-Find

- We use the rank to decide
  - make lower-ranked root point to higher-ranked root
  - then update rank
- How do we update ranks?
  - For clouds of size 1 root always has rank 0
  - For clouds of size larger than 1 we increment rank only when merging clouds of same rank

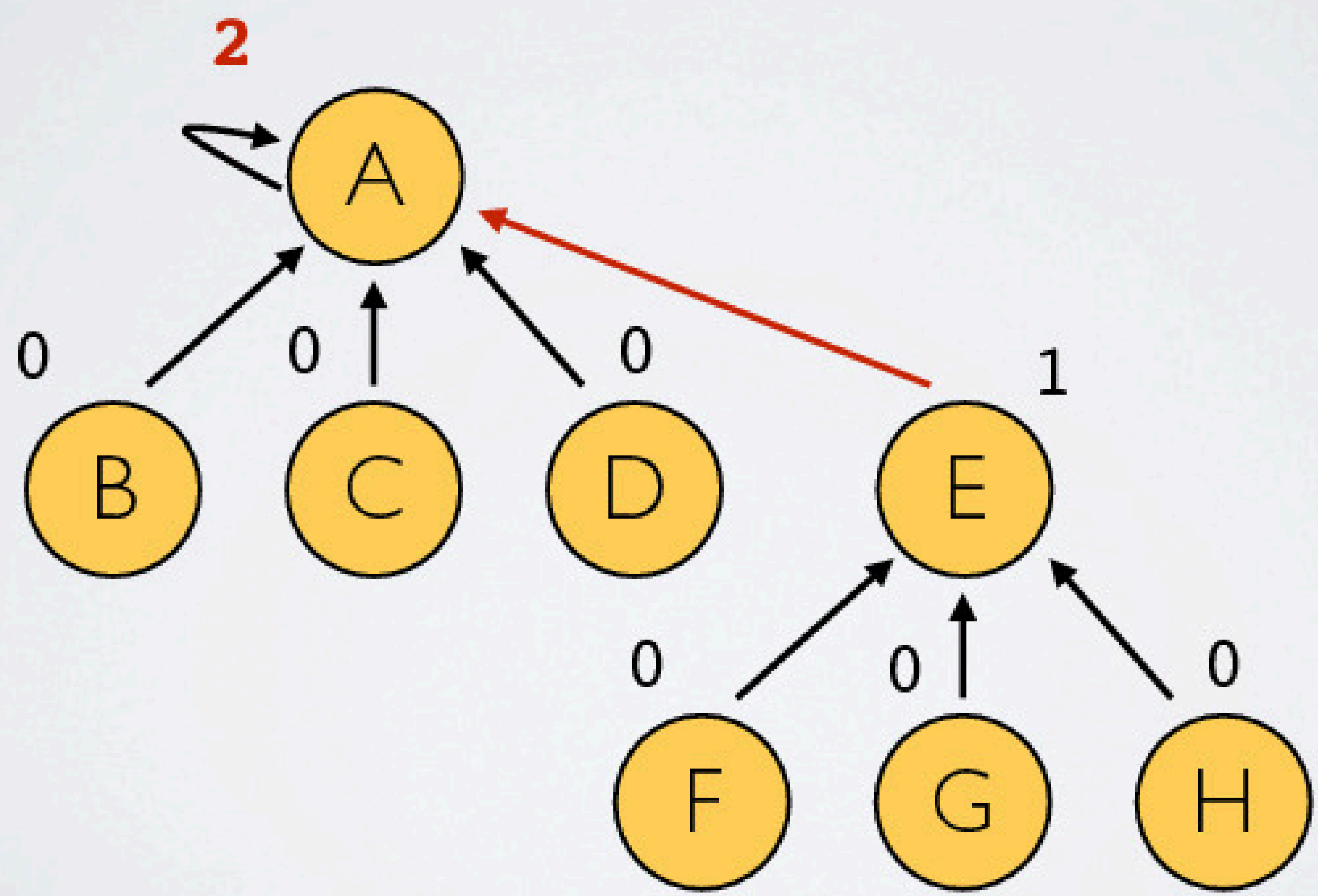
Implementing Union-Find

- Merging trees with same rank



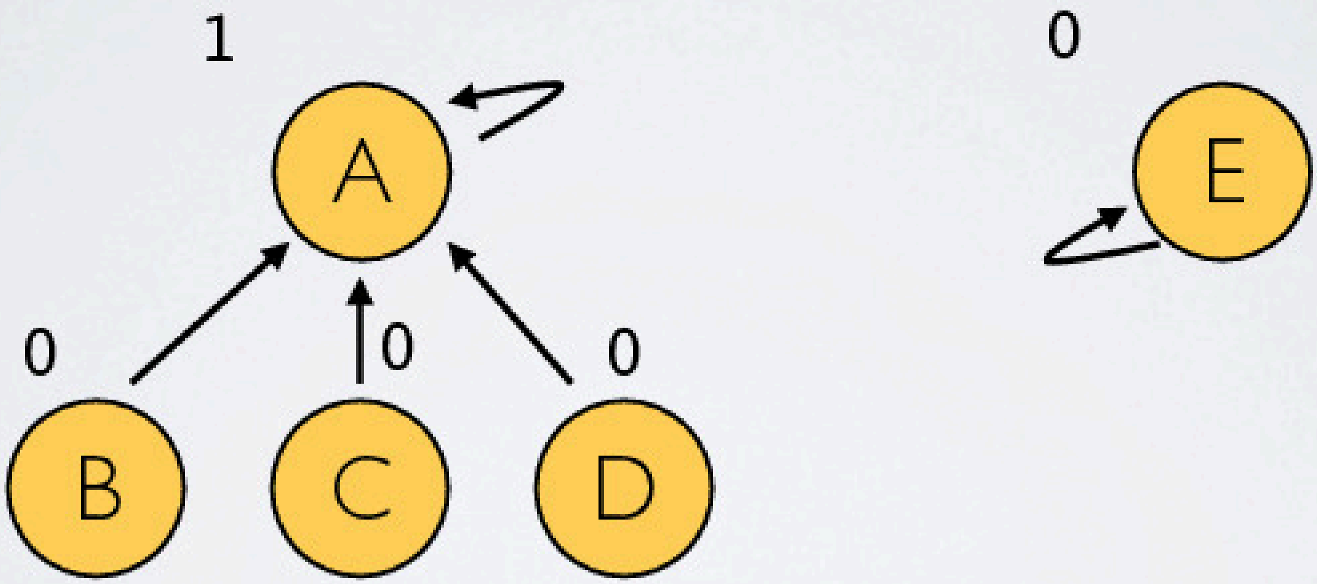
Implementing Union-Find

- Merging trees with same rank



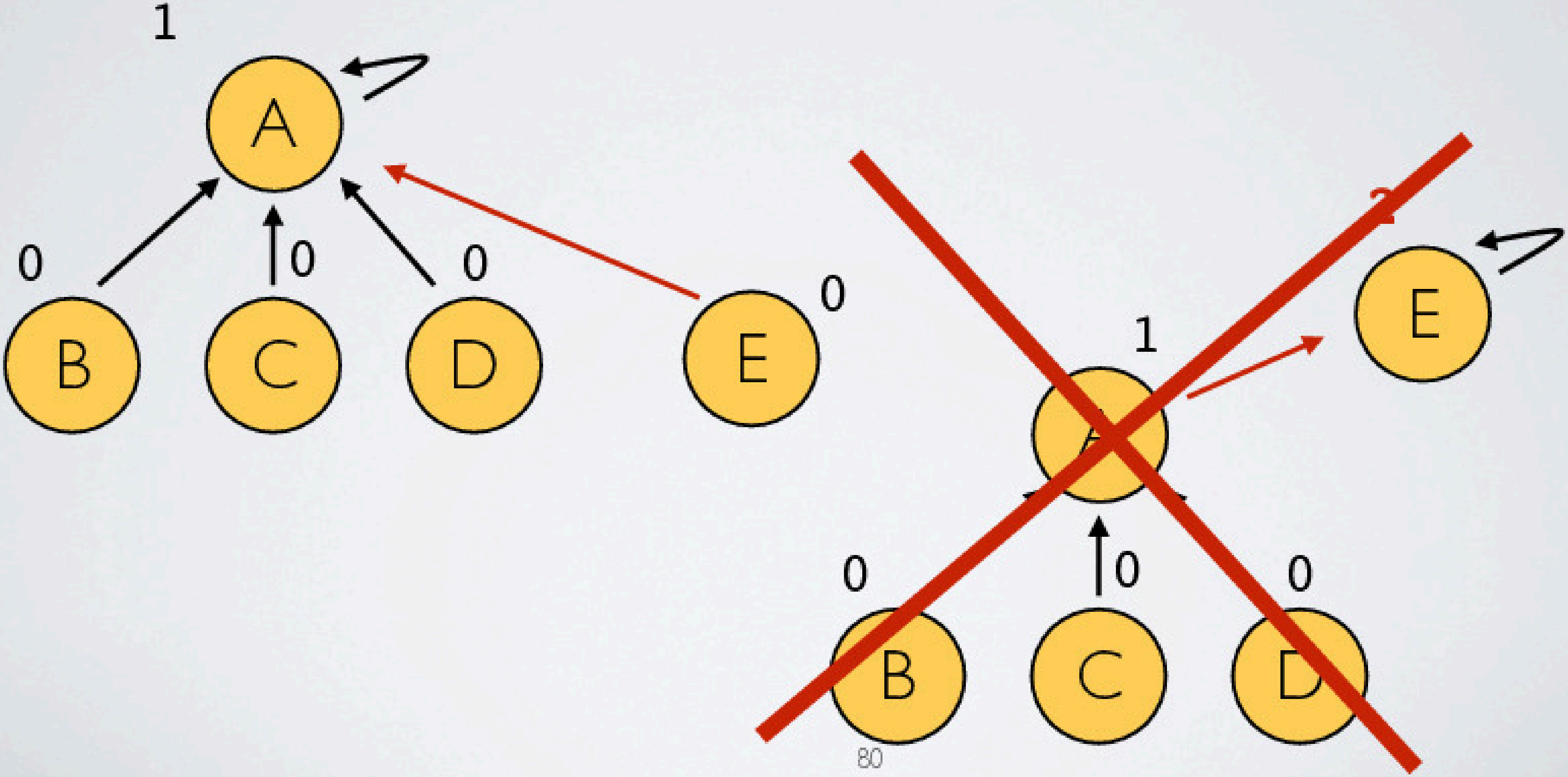
Implementing Union-Find

- Merging trees with different ranks



Implementing Union-Find

- Merging trees with different ranks



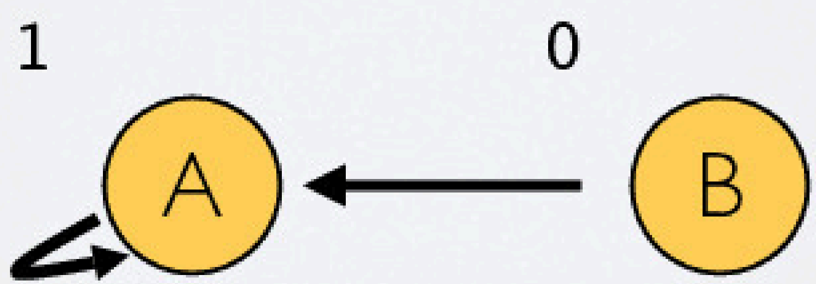
Implementing Union-Find

```
// Merges two clouds, given the root of each cloud
function union(root1, root2):
   if root1.rank > root2.rank:
      root2.parent = root1
   elif root1.rank < root2.rank:
      root1.parent = root2
   else:
      root2.parent = root1
      root1.rank++
```

Implementing Union-Find

- To find the cloud of **B**
  - follow **B**'s parent pointer all the way up to root

```
// Finds the cloud of a given vertex
function find_root(x):
   while x.parent != x:
      x = x.parent
   return x
```

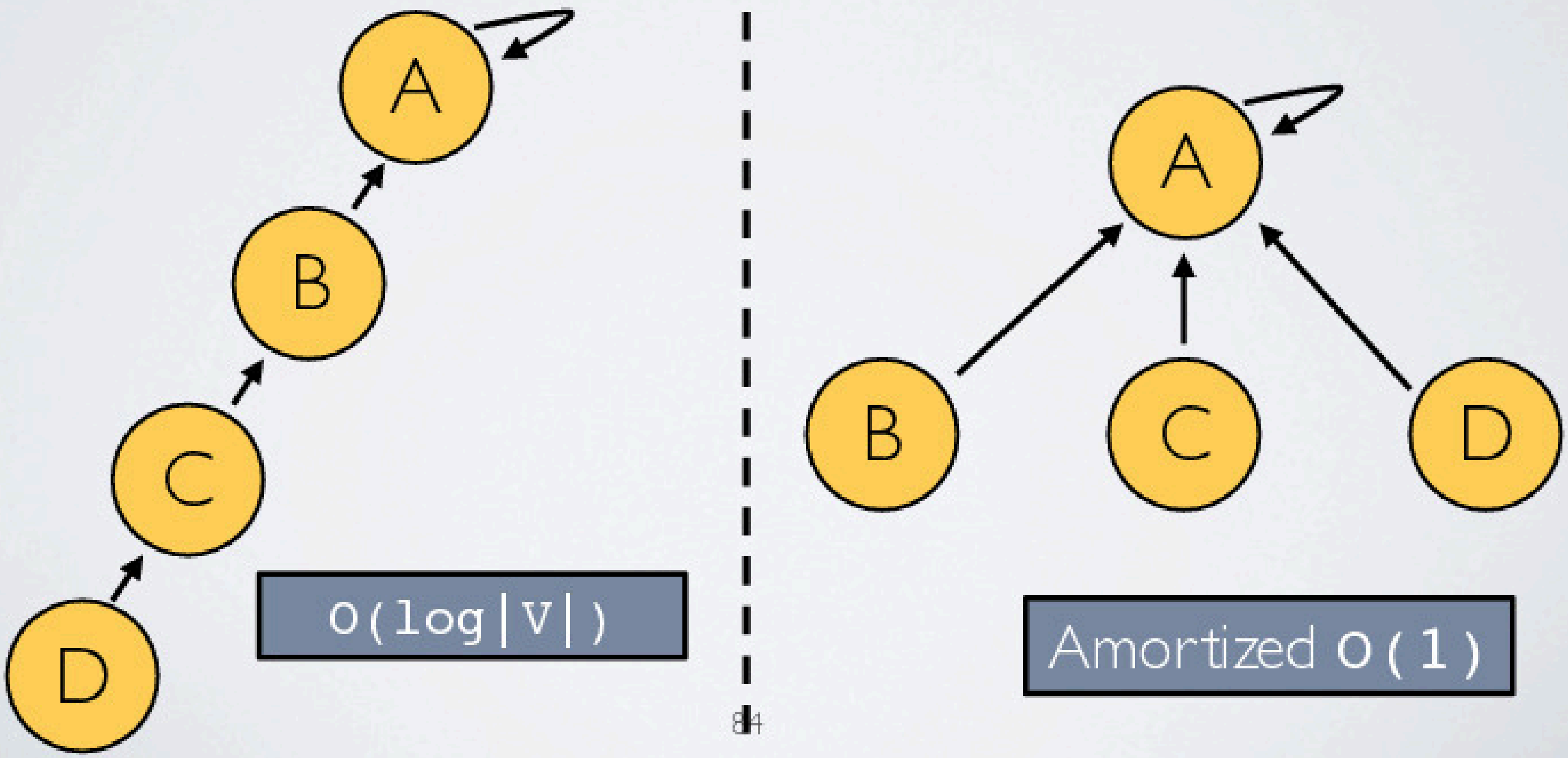


Path Compression

- This approach to implementing **`find`** runs in
  - **`O(log|V|)`**
  - not obvious to see why and proof beyond CS16
- We can bring this down to amortized **`O(1)`**
  - with path compression…
  - …a way of flattening the structure of the tree…
  - …whenever **`find()`** is used on it

Path Compression

- Instead of traversing up tree every time **`D`**'s cloud is asked for
  - We only search for **`D`**'s root once
  - As we follow chain of parents to **`A`** we set parents of **`D`** & **`C`** to **`A`**



Path Compression Pseudo-code

```
function find_root(x):
    if x.parent != x:
        x.parent = find_root(x.parent)
    return x.parent
```

Runtime of Kruskal w/ Path Compression

Activity #5

1 min

Runtime of Kruskal w/ Path Compression

Activity #5

1 min

Runtime of Kruskal w/ Path Compression

Activity #5

0 min

Runtime of Kruskal w/ Path Compression

```
function kruskal(G):
   // Input: undirected, weighted graph G
   // Output: list of edges in MST
   for vertices v in G:                              ← O(|V|)
      makeCloud(v)
   MST = []
   Sort all edges                                    ← O(|E|log|E|)
   for all edges (u,v) in G sorted by weight:        ← O(|E|)
      if u and v are not in same cloud:
         add (u,v) to MST
         merge clouds containing u and v             ← O(1) amortized
   return MST
```

Kruskal Runtime

- $O(|V|)$ for iterating through vertices
- $O(|E|\log|E|)$ for sorting edges
- $O(|E|\times 1)$ for iterating through edges and merging clouds with path compression
- $O(|V|+|E|\log|E|+|E|\times 1)$
  - $= O(|V|+|E|\log|E|)$
- $O(|V|+|E|\log|E|)$ better than $O(|V|^3)$

Readings

- Dasgupta Section 5.1
  - Explanations of MSTs
  - and both algorithms discussed in this lecture