ASYMPTOTIC FREEDOM: AN APPROACH TO STRONG INTERACTIONS

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Abstract:

The discovery of asymptotic freedom has opened up the possibility of extracting new sorts of detailed, dynamical consequences from a strongly interacting quantum field theory. The necessary tools – perturbation theory, the renormalization group, gauge theories, and the operator product expansion – are not new. To anyone familiar with these field theoretic approaches to strong interactions, the novel feature is a simple fact: there is a unique class of theories in which "the origin is an ultraviolet fixed point". But the consequences are so exciting that it seemed appropriate to review these ideas as they reflect on each other. Many important applications of the renormalization group and the operator product expansion to hadronic physics are omitted; the emphasis here is on recent work based on asymptotically free field theories. No doubt, there are some developments so recent that they are not treated in this article.

The discussion of the basic results concerning short distance behavior is informal, but, hopefully, accurate and complete. The specific applications are treated in varying detail.

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1. Introduction

It was twenty years ago, 1954, that Yang and Mills [1] first described a non-Abelian generalization of the gauge symmetry of electrodynamics, a Lagrangian theory symmetric under arbitrary, space-time dependent isospin rotations. In that same year Gell-Mann and Low [2] rediscovered [3] the renormalization group, the connection of renormalizability to scale transformations. Bringing these ideas together has only been possible with the aid of major theoretical advances in the understanding of both subjects, but it produced an exciting surprise: asymptotic freedom.* The excitement is twofold. For the first time, detailed, dynamical calculations can be made for strongly interacting field theories. We can predict observable manifestations of the fundamental fields and short distance structure. The second, but consequent, point is that these calculations offer an explanation for the observed scaling in deep inelastic lepton–hadron scattering.

Whether the currently observed scaling is truly an asymptotic phenomenon is an experimental question. Perhaps a new level of sub-structure will be revealed by higher energies. In any case, asymptotic freedom provides our first quantitative handle on strong interaction dynamics within the context of renormalized quantum field theory. (A further qualification is in order: the analysis of asymptotic freedom is firmly based in perturbation theory – partially summed, stood on its ear, etc., but perturbation theory nonetheless. It is on faith, aesthetics, or intuition that one may choose it as a guide.)

The outstanding challenge remains whether we can predict anything else on the basis of asymptotically free theories, besides properties of the inelastic structure functions. The basic result derived so far is that, in some high momentum domain, the response of any function of the theory to a uniform rescaling of all components of all four-momenta is calculable. But a corresponding experimental rescaling is just not feasible for any physical process. Physical particles are always on their mass shells. One possible program is to isolate the dependence on those invariants which are becoming uniformly large. This can be done for lepton–hadron scattering if we presume to know the structure of the hadronic weak and electromagnetic currents in terms of the fundamental fields. For purely hadronic processes, we will have to face or further cleverly evade the strong coupling and relativistic bound state problems.

As yet, absolutely nothing is known about the particle spectrum of any asymptotically free theory. It is even unclear whether there are particles corresponding directly to the fundamental fields. Rather than regard this as a drawback, many people conjecture that it is an important clue. Perhaps these specific Lagrangians will yield theories with fundamental quarks that simply cannot be produced as asymptotic states. Yet another alternative is that there just is no physically acceptable interpretation.

What is asymptotic freedom?

Any function in a field theory depends on a set of four-momenta, p_i, and on the parameters of the theory, coupling constants, masses, etc., denoted generically by g_i. Replacing the p_i by λp_i, we can ask how things depend on λ, what happens as λ increases from $\lambda = 1$, and if there is anything simple about $\lambda \to \infty$.‡

* However inaccurate or misleading the term may be, it has caught on universally.

‡ In a position space a set of variables, x_i, goes to $\lambda^{-1} x_i$.

The renormalization group analysis [4] allows the λ dependence to be shifted from the momentum variables to scale dependent, effective parameters, $\bar{g}_i(\lambda)$. These functions, $\bar{g}(\lambda)$, are defined as solutions to certain differential equations, which are determined by the structure of the theory. The values of the parameters of the Lagrangian enter only as initial conditions: $\bar{g}_i(\lambda = 1) = g_i$. Hence, it is possible to imagine that the large λ behavior of the $\bar{g}_i$ (and consequently the behavior of any function in the theory) is virtually independent of the specific g_i. Wilson has long emphasized that this is the most powerful approach available to strong interaction dynamics because in some limit the dynamics may become dependent only on the form of the interaction, insensitive to the fact that it has a given strength [5].

The simplest possible behavior is

$$\lim_{\lambda \to \infty} \bar{g}_i(\lambda) = g_i^*$$

where the limit or fixed points g_i^* are finite. (These are generalizations of the Gell-Mann–Low eigenvalue.) An interacting field theory for which all $g_i^* = 0$ is called asymptotically free.

While the existence of non-zero fixed points remains a matter of conjecture,‡ the status of the origin can be investigated perturbatively. From the renormalization group analysis, a perturbation theory in $\bar{g}$ can be constructed. (In practice, it is related but not identical to the conventional expansion in g. The domains of validity, g_i small and $\bar{g}_i$ small, are clearly not identical.) We can then ask: if $\bar{g}_i \ll 1$ for some λ, what is the subsequent evolution as λ increases? Either

$$\lim_{\lambda \to \infty} \bar{g}_i = 0$$

or some of the $\bar{g}_i$ increase until the perturbation expansion is no longer reliable. If indeed $g_i^* = 0$, then perturbations in $\bar{g}$ get better and better as λ increases.

Symanzik was the first to discuss an asymptotically free theory, $\lambda\varphi^4$ with $\lambda < 0$ [7], but such a model is reputedly suspect because, in as much as it is asymptotically free, its spectrum can be shown to be unbounded from below.‡ Typically in other theories, at least one of the $\bar{g}_i$ necessarily increases with λ. Non-Abelian gauge theories are the now famous exception. This discovery involved a simple, straightforward calculation,† essentially the lowest order charge renormalization [11, 12]. Furthermore, no theory without Yang–Mills fields is asymptotically free [13].

If we take hadrons to be composite systems generated by an asymptotically free theory of fundamental quark fields, certain inclusive cross sections can be computed. The electroproduction structure functions are an obvious first example, as they have been the subject of much theoretical activity in recent years [14]. The necessary theoretical tools had been developed prior to the discovery of asymptotic freedom.‡ The operator product expansion suggested by Wilson [16] sep-

‡ There have been many attempts to establish whether there can be such fixed points. The expansion in ϵ, the number of spatial dimensions away from four, provides an infrared fixed point, on the order of ϵ, which lead to a revolution in the theory of phase transitions [6]. But in relativistic quantum field theories our primary tools to date are perturbation expansions in g or $\bar{g}$, and it is unclear whether these expansions, even to all orders, bear any relation to the theories at finite distances away from the origin.

‡ This argument is discussed in ref. [8].

† The calculation was only possible given the recent taming of the renormalized perturbation theory of Yang–Mills fields [9, 10]. The result was in fact presented by 't Hooft at the 1972 Marseille Conference on Yang–Mills Fields but went unpublicized and unpublished.

‡ The analogous calculation for an Abelian gluon theory had been performed in detail [15]; but, as we shall see, it simply does not give a consistent picture for strong interactions or large momenta.

arates the dependence on q^2, p^2 and $p \cdot q$ (where p is the initial hadron momentum and q is the momentum transferred by the electromagnetic current). While p^2 is necessarily fixed, the effects of scaling up q^2 and $p \cdot q$ are calculable in the expansion in $\bar{g}$. In an alternative language, the short distance and lightcone singularities of these theories are the singularities of free field theory modified by calculable powers of logarithms. The free field singularity structure, the field theoretic version of the parton impulse approximation, has been widely conjectured as an explanation for the observed dependence of the structure functions on only the ratio $x = -q^2/(2p \cdot q)$. The logarithms present in asymptotically free theories induce an additional slow dependence on q^2. Very preliminary results on muon deep inelastic scattering at NAL suggest sizeable q^2 dependence [17]. Perhaps we will soon know if there are indeed violations of scaling of the form predicted by asymptotic freedom.

A focus of considerable excitement is the approximately constant e^+e^- cross section. The scaling prediction is that it should drop like $1/s$, where s is the center of mass energy squared, as does the purely electromagnetic cross section. The cross section shows no sign of decrease for $s \approx 25\ \mathrm{GeV}^2$, a value much bigger than the $-q^2 \sim 2$–$4\ \mathrm{GeV}^2$ at which electroproduction scales. While the two processes are related in asymptotically free theories, the observed behaviors are not incompatible.

Whether or not SLAC scaling is an asymptotic phenomenon, the discovery of asymptotic freedom has opened new avenues in strong interaction theory. For example, many aspects of the weak and electromagnetic corrections to hadronic symmetries are now calculable, no longer obscured by strong dynamics. Perhaps the most promising development is that asymptotic freedom can be viewed as an explanation of why strong interactions are strong and suggests a natural unification of strong, electromagnetic and weak interactions within the context of a single gauge theory.

Asymptotic Freedom and the Renormalization Group

2. A first look at the renormalization group equation

We wish to make quantitative sense out of the notion that, as all the momenta in a given problem become large, the values of the physical masses become irrelevant. Clearly, as $q^2 \to \infty$, a function as simple as $q^2 - m^2$ can be expanded about $m = 0$ in powers of m^2/q^2:

$$q^2 - m^2 \to q^2(1 + \mathrm{O}(m^2/q^2)) .$$

So a natural starting point is the study of purely massless theories, deferring till later the effects of finite masses and the question of the validity of expansions in m.‡ (In the physical, massive theories, uniform rescaling of momenta carries the Green's functions off shell. So the infrared divergences of the massless theories are of no immediate concern, as the interest is in off shell functions.)

As is evident from perturbation theory, even if all masses are zero, dimensionless functions can depend on momenta through dimensionless ratios, for instance q^2/M^2, and as $q^2 \to \infty$,

$$q^2/M^2 \to q^2/M^2 .$$

‡ As a matter of taste, unless specifically noted, I will refer to finite, renormalized parameters and equations.

But what is M? M is a parameter not of the Lagrangian but of renormalization. Because of infrared divergences, the Green's functions are not normalizable at zero momenta, their would-be mass shells. So the normalization conditions must refer to some non-zero momentum points, and one dimensionful parameter, M, must be introduced to set the scale for the location of these points.

The magnitude of M is purely a matter of convention. So if the theory is indeed renormalizable, one choice of M is as good as another, and the predictions of the theory are convention independent. It is somewhat subtle how this works out in detail, for M enters in the definitions of all the renormalized parameters as well as appearing explicitly in ratios with momenta. A concise statements of convention independence is that, if the value of M is changed, there exist compensating changes in all the renormalized quantities such that all Green's functions remain unchanged. For concreteness, if Γ is the irreducible, truncated vertex function for a particular set of operators, then

$$\Gamma(p_i, g_i, M) = Z\,\Gamma(p_i, g_i', M') \tag{2.1}$$

where Z is the product of the necessary scaling factors for the set of operators. Since the scaling factors are dimensionless (and taking all the g_i dimensionless), $Z = Z(g, M'/M)$ and $g_i' = g_i'(g, M'/M)$. Expressing the invariance of Γ under changes in M in a differential form yields the renormalization group equation:

$$[M\,\partial/\partial M + \beta_i\,\partial/\partial g_i + \gamma]\;\Gamma = 0\,, \tag{2.2}$$

where $\beta_i = \beta_i(g)$ gives the differential transformation of each g_i, and $\gamma = \gamma(g)$ is the sum of the differential rescalings of each operator in Γ. Eq. (2.2) expresses that for any small change in M there exist appropriate changes in the g_i and appropriate rescalings of operators such that *any* Green's function remains unchanged. The β_i and γ's depend on the theory and not the particular Γ, and they give precise meaning to "appropriate".

This derivation of the renormalization group equation rests on the fact that if (possibly infinite) multiplicative renormalizations of the bare quantities render a theory finite, then further finite renormalizations do not change the predictive content of the theory. Any other statement and implementation of renormalizability equally well generates a similar equation.[‡] In appendix 1, a derivation is sketched based on cut-off independence of the renormalized theory. For non-Abelian gauge theories, higher order perturbative calculations, and the treatment of non-zero masses, the technique of dimensional regularization [18] and renormalization [19] puts the equation in a particularly convenient form [20, 21].

If we know how the theory responds for fixed couplings and momenta to a change in M, then we know how it responds for fixed couplings and M to a rescaling of momenta, by simple dimensional analysis. Thus, the λ dependence of a vertex function, $\Gamma(\lambda p, g, M)$, can be re-expressed, given that Γ satisfies eq. (2.2). If Γ has a (canonical or naive) dimension D, it scales like λ^D times some dimensionless function of $\lambda^2 p_i \cdot p_j/M^2$. For convenience, a logarithmic scaling variable is defined,

$$t = \log\lambda \tag{2.3}$$

so for $D = 0$, $M\,\partial/\partial M$ becomes $-\partial/\partial t$. Now define the scale dependent effective couplings $\bar{g}_i(g, t)$ by

[‡] Plausibility arguments are often presented totally independent of the question of renormalizability. In Wilson's picture [5, 6], the coefficient functions simply describe how the dominant dynamics on one scale are related to dynamics on another scale.

$$\partial \bar{g}_i/\partial t = \beta_i(\bar{g})$$

and

$$\bar{g}_i(g, t=0) = g_i \,. \tag{2.4}$$

These have the property that any function that depends on g_i and t only through $\bar{g}_i$ is a solution of the equation

$$[-\partial/\partial t + \beta_i(g)\,\partial/\partial g_i]\, f(\bar{g}) = 0\,.^{\ddagger} \tag{2.5}$$

In the presence of non-zero γ, a finite overall rescaling is obtained by integrating the differential transformations up to the appropriate t. Putting together all these parts,

$$\Gamma(\lambda p_i, g_i, M) = \lambda^D\, \Gamma(p_i, \bar{g}_i, M)\, \exp\left\{\int_0^t \gamma(\bar{g})\, \mathrm{d}t'\right\}. \tag{2.6}$$

In purely massless theories, this equation is exactly true for any Green's function of multiplicatively renormalizable operators at any and all momenta.

$$\text{If } \lim_{t\to\infty} \bar{g}_i(g, t) = g_i^*$$

and the limit converges sufficiently rapidly in t as determined by $\beta_i(\bar{g}_j \approx g_j^*)$, then, for large λ, the total λ dependence of eq. (2.6) is

$$\Gamma \propto \lambda^{D+\gamma(g^*)}. \tag{2.7}$$

Hence, $\gamma(g^*)$ and more generally the function γ are called anomalous dimensions. (Recall that γ for a particular Γ is a sum of terms, γ_i, where γ_i gives the appropriate differential scaling law for the ith operator in Γ under differential changes in M.) If $g_i^* \neq 0$, there are as yet no techniques for computing in a given theory the g_i^*, the behavior of β_i near g_j^*, the values of $\gamma(g^*)$, or $\Gamma(p, g^*, M)$.

If the theory is asymptotically free, the λ dependence is somewhat more complicated but can be investigated in detail. The complication arises from a very slow approach to limiting values. Typically,

$$\int_0^t \gamma(\bar{g}_i(g, t'))\, \mathrm{d}t'$$

does not converge as $t \to \infty$ but depends logarithmically on t. Furthermore, vertex functions evaluated at some arbitrarily small $\bar{g}_i$ may differ qualitatively from their free field behavior. Eqs. (2.4) and (2.6) can be studied using perturbation expansions in $\bar{g}_i$ (as will be done shortly in detail for a specific model). If for some t, the $\bar{g}_i$ are small enough to justify studying a finite order of an ex-

‡ Whether or not this is obvious, explicit verification can be tricky. To show that $-\partial\bar{g}_i/\partial t + \beta_j(g)\,\partial\bar{g}_i/\partial g_j = 0$, note first that it is true for $t = 0$. Then it is straightforward to show that for all t (no sum on i)

$$(\partial/\partial t)[(\partial\bar{g}_i/\partial t)^{-1} \textstyle\sum_j \beta_j(g)\, \partial\bar{g}_i/\partial g_j] = 0\,,$$

or if $\partial g_i/\partial t = 0$ for some i and some t, then $\bar{g}_i = g_i$ and $\beta_i(g) = 0$, although not necessarily identically.

pansion in $\bar{g}_i$, then $\beta_i(\bar{g})$ can be evaluated approximately, giving the subsequent t evolution of $\bar{g}_i$. $\gamma(\bar{g})$ and $\Gamma(p, \bar{g}, M)$ can be similarly evaluated, and the quality of the approximation gets better with increasing t.

While these functions separately become virtually independent of $g_i = \bar{g}_i(g, t = 0)$ for large enough t, there is always a remnant of small t dynamics in

$$\int_0^t \gamma(\bar{g})\, dt'.$$

If the g_i are strong couplings, then very little is known about $\gamma(\bar{g})$ for small t. However, this ignorance amounts simply to a set of unknown constants – strong interaction renormalizations – one for each operator of the theory. Specifically,

$$\exp\left\{\int_0^t \sum_i \gamma_i(\bar{g})\, dt'\right\} = \left(\prod_j Z_j\right) \exp\left\{\int_{t_0}^t \sum_i \gamma_i(\bar{g})\, dt'\right\}$$

where t_0 is some value of t for which the $\gamma(\bar{g})$ are known perturbatively. (In practice, the Z_i can be factored out of certain predictions, and some are known to be equal to one on general symmetry considerations.) Alternatively, $\bar{g} \to 0$ implies that there exist normalization scales for which g is arbitrarity small. Choosing M to be some such large value amounts to defining $t_0 = 0$. Nothing is lost by such a convention because no explicit computation can be done for the strong domain anyway. Note that this is very different from taking a random Lagrangian, assuming the coupling is small, and calculating scaling behavior in perturbation theory. By assumption, construction, and demonstration, asymptotically free theories are inherently strongly interacting. It is only at very short distances that the effects of interactions become less important. In contrast, if for example an Abelian gluon theory is effectively weakly coupled on some momentum scale, for larger momenta the coupling will be more important, and at smaller momenta it is necessarily less important.

It is already apparent from eq. (2.6) how approximate canonical or free-field scaling is achieved for $g_i^* = 0$. Naive power counting would suggest that

$$\Gamma(\lambda p_i) \propto \lambda^D,$$

which would be the case for $\gamma(g_i) = 0$. But for large λ, this is not so far from true. $\gamma(g_i = 0)$ is zero, so γ in perturbation theory begins proportional to the expansion parameters, which themselves go to zero as λ increases. Thus for sufficiently large λ, the deviation from naive scaling,

$$\exp\left\{\int^{\log \lambda} \gamma(\bar{g})\, dt'\right\},$$

varies slower than λ^ϵ for any $\epsilon > 0$.

How do the $\bar{g}_i$ become small? Defined by eq. (2.4), the $\bar{g}_i$ are the results of integrating a set of coupled, non-linear differential equations whose structure is not known away from the origin of coupling constant space. For simplicity, then, consider a theory with a single coupling constant, g. If $\beta(g)$ were known for all g, then $\bar{g}$ could be generated by integration. A qualitative description requires only a knowledge of the existence and location of the zeros of β and the sign of β in between the successive zeros.‡ For t such that $\beta(\bar{g}(g, t)) < 0$, $\bar{g}$ is a decreasing function of t, and con-

‡ I am in no position to consider seriously the analytic properties of $\beta(g)$.

versely. For asymptotic freedom, the first non-vanishing term in the expansion of $\beta(g)$ in powers of g must be negative. Assuming that there is a finite interval over which $\beta < 0$ assures that there are some values of initial g for which $\bar{g} \to 0$. A further assumption is that this interval contains those strong values of g that are of interest. If $\beta(\bar{g})$ were ever positive, $\bar{g}$ would be increasing there and would either approach the next zero of β or go to infinity, if there are no subsequent zeros. (Yet more zeros would serve as alternate limit points.)

The assumption $\beta(\bar{g}) < 0$ for $0 < \bar{g} < g_{\text{strong}}$ determines nothing about the approach to asymptotia except that it is inevitable. Once $\bar{g}$ is sufficiently small, we can predict its subsequent evolution, but how rapidly it gets that way is determined by how negative β is for strong $\bar{g}$. So all that can be said is that asymptotically free theories are not known to be incompatible with any approach to asymptotia, the region of small $\bar{g}$.

3. The prototype: non-Abelian gauge theory with fermions

The construction of a renormalized perturbation theory for non-Abelian gauge theories has opened up vast new possibilities – witness the renaissance of weak interaction theory. Discussions of local symmetries, Feynman rules, regularization, renormalization, Ward identities, gauge questions, models, and specific calculations are given in several recent reviews (e.g. [22]). So I will assume some familiarity with the techniques used in one loop calculations and assume an awareness that many of the questions of principle have been resolved.

A non-Abelian, semi-simple gauge theory is described by the Lagrangian

$$\mathcal{L} = -\tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \mathrm{i}\bar{\psi}_i \gamma_\mu D^\mu_{ij} \psi_j$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$

$$D^\mu_{ij} = \partial^\mu \delta_{ij} - \mathrm{i} g A^{a\mu} T^a_{ij} \qquad (3.1)$$

and f^{abc} are the structure constants of the group and T^a are the matrices of the group generators in the fermion representation. (The fermions are included here because no practical complication arises from their presence.) Quantizing in a renormalizable gauge leads to the Feynman rules summarized in fig. 1.

The renormalization group equation appropriate to this theory is

$$\left[M \frac{\partial}{\partial M} + \beta(g,\alpha) \frac{\partial}{\partial g} + \delta(g,\alpha) \frac{\partial}{\partial \alpha} + n\gamma_A(g,\alpha) + n'\gamma_\psi(g,\alpha)\right] \Gamma^{n,n'}(p_1 \ldots p_{n+n'}; g, \alpha, M) = 0, \qquad (3.2)$$

where M sets the scale of the normalization points, α is a gauge parameter, the coefficient of the longitudinal vector propagator, δ is just alternative notation for β_α, $\gamma_{A \text{ or } \psi}$ is the anomalous dimension of the vector or fermion field, and $\Gamma^{n,n'}$ is the irreducible, truncated vertex function for n vector and n' Fermi fields.‡

‡ A virtually identical equation can always be derived for the complete, connected Green's functions, $G^{n,n'}$, but the γ's enter with the opposite sign – an endless source of mild confusion. The relation of these equations, many of their properties, and alternate derivations can be elegantly discussed in terms of the generating functionals of $\Gamma^{n,n'}$ and $G^{n,n'}$, $\Gamma(A_\mu, \psi, \bar{\psi})$ and $G(J_\mu, \bar{\eta}, \eta)$, which are Legendre transforms of each other. These satisfy functional renormalization group equations [8, 23].

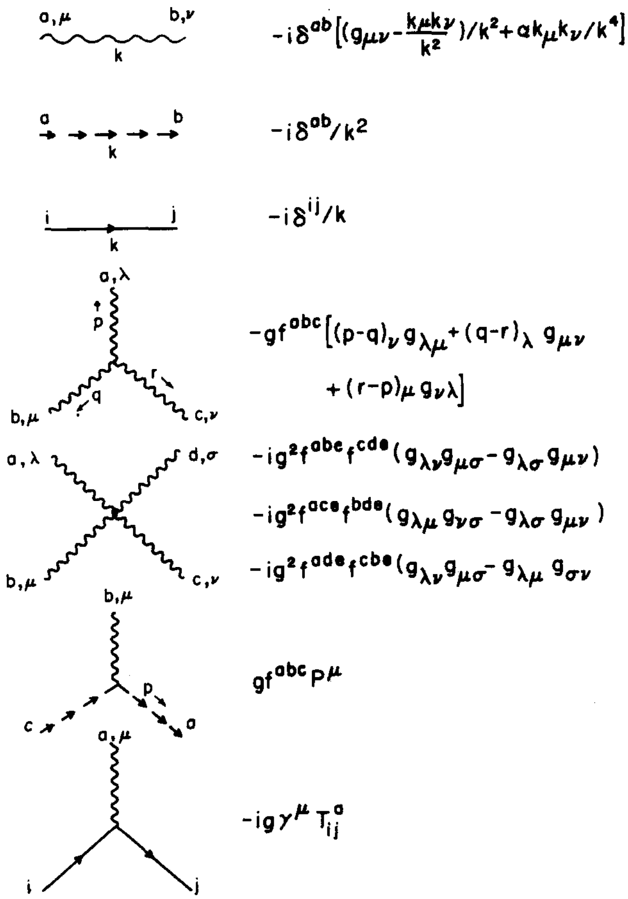


Fig. 1. Feynman rules for a gauge theory with fermions.

Before explicitly constructing the "improved" perturbation expansion to lowest order in $\bar{g}$, it is worth establishing some general features of β and gauge dependence.

One of the simplest Ward identities states that the longitudinal vector propagator (or its inverse) is not renormalized. Explicitly,

$$p_\mu p^\alpha \int \mathrm{d}^4x\, \mathrm{e}^{ipx} \langle 0|\, T(A_\alpha(x)\, A_\nu(0))|0\rangle = \alpha p_\mu p_\nu/p^2 \tag{3.3}$$

to every order in perturbation theory, manifestly independent of g and M. In each order, however, A_μ is renormalized to maintain the value and the derivative of the transverse propagator at some momentum point. Hence, α must be scaled like A_μ^2, in accordance with eq. (3.3). Explicitly, the inverse longitudinal propagator,

$$\Gamma_{\mathrm{L}}^{2,0} = \frac{1}{\alpha}\, p_\mu p_\nu\,,$$

must satisfy eq. (3.2), from which it follows that

$$\delta(g,\alpha) = 2\alpha\gamma_A(g,\alpha)\,. \tag{3.4}$$

Note that if we quantize in Landau gauge, $\alpha = 0$; $\delta(g,0) = 0$, which implies that, under changes in M, g may change, the scale of the fields may change, but α remains zero. So it is consistent to write eq. (3.2) in Landau gauge and do all computations with $\alpha = 0$.

β reflects the response of the coupling constant to a change in M, but a given renormalized coupling constant, g, is defined by a particular normalization prescription, for example the value of a vertex function at a specific momentum configuration. Using a different prescription, we arrive at a different coupling, g'. Typically, a specific value of g corresponds to the same physics as a different value of g'. That value of g' can be computed perturbatively in g,

$$g' = g + ag^3 + \mathrm{O}(g^5)\,. \tag{3.5}$$

(I have used the topological structure of gauge theories. For a scalar self-coupling, the expansion would also contain even powers of g.) $\beta(g)$ is in principle a different function from β', the coefficient function appropriate to changes in g'. But it is simple to demonstrate that the two functions are identical to lowest order in perturbation theory.

$\beta(g)$ is inferred from considering how g changes under a uniform rescaling of the momentum point at which it is defined. If g is defined at some momenta, p_i, let $\bar{g}$ be the analogous coupling defined at $(M/\bar{M})p_i$. In perturbation theory

$$\bar{g} = g - bg^3 \log(\bar{M}/M) + \mathrm{O}(g^5)\,. \tag{3.6}$$

β is defined by the differential transformation:

$$\begin{aligned} &\text{if} \quad M \to \bar{M} = M + \Delta M\,,\\ &\text{then}\ \ g \to \bar{g} = g + \Delta g\\ &\text{and} \quad \Delta g = \beta(g)\,\Delta M/M\,. \end{aligned}$$

From eq. (3.6)

$$\beta(g) = -bg^3 + \mathrm{O}(g^5)\,. \tag{3.7}$$

If g' is defined using different conventions, say at a different momentum configuration, p_i', then under the same rescaling

$$\bar{g}' = g' - b'g'^3 \log(\bar{M}/M) + \mathrm{O}(g'^5)\,. \tag{3.8}$$

It follows analogously that

$$\beta'(g') = -b'g'^3 + \mathrm{O}(g'^5)\,. \tag{3.9}$$

Potentially $b' \neq b$. However, eq. (3.5) implies that there is also an expansion on the new scale

$$\bar{g}' = \bar{g} + \bar{a}\bar{g}^3 + \mathrm{O}(\bar{g}^5)\,, \tag{3.10}$$

and inverting eq. (3.5)

$$g = g' - ag'^3 + \mathrm{O}(g'^5)\,. \tag{3.11}$$

Combining eqs. (3.6), (3.10) and (3.11)

$$\bar{g}' = g' - bg'^3 \log(\bar{M}/M) + (\bar{a} - a)g'^3 + \mathrm{O}(g'^5)\,. \tag{3.12}$$

Comparing eqs. (3.12) and (3.8) we see that $\bar{a} = a$ and $b' = b$. So therefore β and β' are identical in lowest order.

An immediate corollary is that the lowest order β is gauge independent. Imagine that the above g and g' differ also in that they are defined as the values of some vertex function at some p_i in different gauges. The perturbative expansion coefficients may be gauge dependent, but the above argument shows that $b = b'$. (These arguments rest simply on the fact that the expansion of a coupling constant, g', in terms of another, g, where they differ by conventions but both refer to the same term in the Lagrangian, is always $g' = g + \ldots$.)

The renormalization group coefficient functions can be evaluated perturbatively by recognizing that eq. (3.2) is true order by order in perturbation theory. Given some vertex function to a particular order, we can infer some appropriate combination of coefficient functions to that order. So three vertex functions suffice to determine γ_A, γ_ψ and β. Which three is a matter of convenience or interest; whether they are done for $\alpha = 0$ or for general α is a matter of taste; and how renormalizations are prescribed, as argued above, simply does not matter to lowest order.

The γ's are most immediately obtained from the inverse propagators, $\Gamma_{\mathrm{T}}^{2,0}$ and $\Gamma^{0,2}(-p,p;g,\alpha,M)$, which can be normalized to p^2 at some Euclidean $p^2 = -M^2$. To lowest order, from the graphs of figs. 2A and 2B

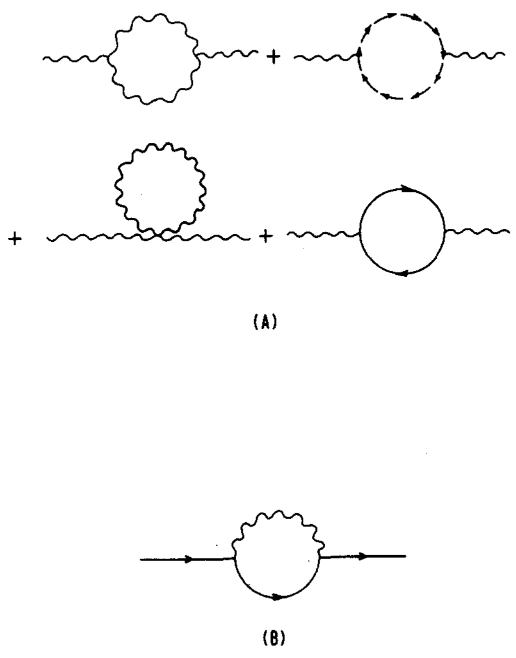


Fig. 2. Lowest order corrections to the A) vector and B) fermion self-energies.

$$\Gamma_T^{2,0} = (-g_{\mu\nu}p^2 + p_\mu p_\nu)\left[1 + \left\{\left(\frac{13}{6} - \frac{\alpha}{2}\right)c_1 - \frac{4}{3}c_2\right\}\frac{g^2}{16\pi^2}\log\frac{-p^2}{M^2}\right] \tag{3.13}$$

$$\Gamma^{0,2} = \not{p}\left(1 - \alpha c_1 \frac{g^2}{16\pi^2}\log\frac{-p^2}{M^2}\right) \tag{3.14}$$

where

$$c_1 \delta_{ab} = f_{acd} f_{bcd}$$

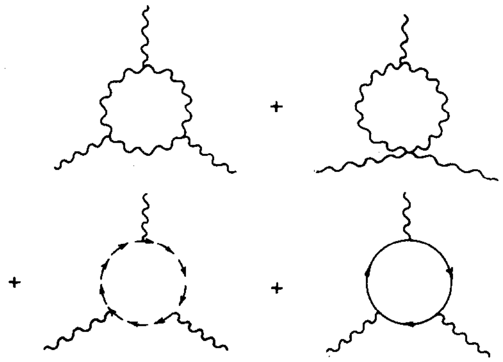
and

$$c_2 \delta_{ab} = \operatorname{tr} T_a T_b .$$

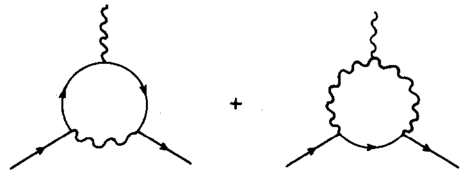
(The c's are numbers if the representations are irreducible; in general, there is a different c for each type of irreducible representation.) Applying the differential operation of eq. (3.2) to $\Gamma_T^{2,0}$ and $\Gamma^{0,2}$ and then evaluating at $p^2 = -M^2$ yields

$$\gamma_A = d_A g^2 + O(g^4) = \frac{1}{16\pi^2}\left[\left(\frac{13}{6} - \frac{\alpha}{2}\right)c_1 - \tfrac{4}{3}c_2\right]g^2 + \ldots , \tag{3.15}$$

$$\gamma_\psi = d_\psi g^2 + O(g^4) = -\frac{1}{16\pi^2}\alpha c_1 g^2 + \ldots . \tag{3.16}$$



(A)



(B)

Fig. 3. Lowest order gauge vertex corrections.

Any three-point function would suffice to determine β to lowest order. For convenience (particularly for $\alpha = 0$) instead of a symmetric Euclidean point ($p_1^2 = p_2^2 = p_3^2$ and $p_1 \cdot p_2 = p_2 \cdot p_3 = p_3 \cdot p_1$) consider $(p_1, p_2, p_3) = (0, -p, p)$ and normalize at $p^2 = -M^2$.‡ The graphs of figs. 3A and 3B lead to

$$\Gamma^{3,0} = f^{abc}(p_\lambda g_{\mu\nu} + p_\mu g_{\nu\lambda} - 2p_\nu g_{\lambda\mu})\, g \left(1 + \left[\frac{17}{12} - \frac{3\alpha}{4}\right] c_1 \frac{g^2}{16\pi^2} \log \frac{-p^2}{M^2}\right) \tag{3.17}$$

$$\Gamma^{1,2} = gT^a \gamma_\mu \left[1 - \left(\frac{3}{4} + \frac{5\alpha}{4}\right) c_1 \frac{g^2}{16\pi^2} \log \frac{-p^2}{M^2}\right]. \tag{3.18}$$

Inserting either of these into eq. (3.2) and using the γ's yields

$$\beta = -bg^3 + \mathrm{O}(g^5) = -\frac{1}{16\pi^2}\left[\frac{11}{3} c_1 - \frac{4}{3} c_2\right] g^3 + \dots . \tag{3.19}$$

So if $c_2 < \frac{11}{4} c_1$, then $\beta(\bar{g})$ is presumably negative for sufficiently small $\bar{g}$, which implies that $\bar{g}$, with appropriate initial conditions, is a decreasing function of t. (For example, if the gauge group is SU(N), $c_1 = N$ and $c_2 = \frac{1}{2} m$ for m fundamental N dimensional representations and $c_2 = N$ for each adjoint, $N^2 - 1$ dimensional representation. So for SU(3), up to sixteen fermion triplets or two octets will leave $\beta < 0$.) Explicitly integrating β in eq. (2.3) gives

$$\bar{g}^2 = \frac{g^2}{1 + 2bg^2 t} + \mathrm{O}(\bar{g}^4)\,. \tag{3.20}^\ddagger$$

The renormalization group improved Green's functions can now be constructed to lowest order in $\bar{g}$ using eq. (2.6). For a specific example, consider the meson inverse propagator:

$$\Gamma_{\mathrm{T}}^{2,0}(-k, k, g, M) = \frac{k^2}{k_0^2}\, \Gamma_{\mathrm{T}}^{2,0}(-k_0, k_0, \bar{g}, M) \exp\left\{2 \int_0^t \gamma_A(\bar{g})\, \mathrm{d}t'\right\} \tag{3.21}$$

where I have let $k = \lambda p$ and $k_0 = p$, so $\lambda^2 = k^2/k_0^2$ and $t = \log \lambda$. Choosing $k_0^2 = -M^2$ puts $\Gamma^{2,0}$ in a simple form because, by definition,

$$\Gamma_{\mathrm{T}}^{2,0}(-k_0, k_0, g, M = (-k_0^2)^{1/2}) = -g^{\mu\nu} k_0^2 + k_0^\mu k_0^\nu\,. \tag{3.22}$$

If the theory is asymptotically free, then there certainly exists a choice of M (or a zero of t) such that $\bar{g}(g, 0) = g$ is perturbatively small. With such a choice, the constant coming from the small t' end of the integral of γ_A is known as well as anything else, so we find

$$\Gamma_{\mathrm{T}}^{2,0} = (-g^{\mu\nu} k^2 + k^\mu k^\nu) \left(1 + bg^2 \log \frac{-k^2}{M^2}\right)^{d_A/2b} + \dots . \tag{3.23}$$

Although the k^2 behavior appears to depend on both g and M, really only one combination of them is an independent variable. This can be exhibited explicitly:

‡ In general, if renormalizations are defined by the values of vertices at certain momentum points, care must be taken to preserve the gauge symmetry as expressed by the Ward identities.

‡ It was this equation that prompted the harangue of appendix 2.

$$\Gamma_T^{2,0} = (-g^{\mu\nu}k^2 + k^\mu k^\nu)\, g^{d_A/b} \bar{g}^{-d_A/b}(1 + O(\bar{g}^2)) \tag{3.24}$$

where $g^{d_A/b}$ is the inevitable multiplicative constant arising from the small t' domain.

While this approximation gets better for larger and larger t, it says nothing about the behavior near the would-be mass shell, $k^2 = 0$. That domain is described by $t \to -\infty$, for which $\bar{g}$ increases either to infinity or to the first zero of β away from the origin.

4. The effects of mass parameters

In theories of massive particles, it is perfectly natural that the normalization conditions be physically motivated and refer to on-shell Green's functions. But if the role of masses as renormalized parameters is kept distinct from their role of setting the scale of momenta, a more general renormalization group equation can be derived, which again expresses convention independence or the arbitrary nature of the scale of momentum, M. This is a modest conceptual and practical advance over the Callan–Symanzik equations [23, 24] because it can be solved formally (in the sense of eq. (2.6)) for any momenta. In such a scheme, the renormalized mass parameters, m, cannot be physical particle masses because that would tie m to M. So they must be what are commonly referred to as intermediate renormalized masses. As usual, various choices of definitions are more or less convenient for various applications.

To construct a simple example, consider theories with a single dimensionless coupling, g, and a single mass, m, with all renormalizations done on a scale set by M. Define a dimensionless variable μ by $\mu = m/M$. Then the particular form of eq. (2.2) is

$$[M\, \partial/\partial M + \beta_g\, \partial/\partial g + \beta_\mu\, \partial/\partial \mu + \gamma]\ \Gamma = 0. \tag{4.1}$$

The coefficient functions depend in general both on g and μ. A function $\bar{\mu}$ and a general solution following eq. (2.6) can be constructed. While μ is regarded here as a generalized coupling constant, μ and β_μ are slightly unusual. Even in free field theory $\beta_\mu \neq 0$. Specifically, for $g = 0$, $\mu M = m$, an invariant quantity. So

$$\beta_\mu(g = 0, \mu) = -\mu\,. \tag{4.2}$$

Although the theory may have been asymptotically free for $\mu = 0$, the situation is now clouded by the fact that $\bar{\mu}$ and $\bar{g}$ are defined by coupled equations. It is plausible that, for small initial $\bar{g}$ and $\bar{\mu}$, the theory is subsequently asymptotically free because $\beta_\mu = -\mu + \ldots$ to lowest order, so $\bar{\mu} \approx \mu\, e^{-t}$. As $t \to \infty$, once $\bar{\mu}$ is ridiculously small, $\beta_g(\bar{g}, \bar{\mu}) \approx \beta_g(\bar{g}, 0) = -b\bar{g}^3 + \ldots$.

Using a definition of g and μ due to Weinberg [25]‡, it is possible to make β_g μ-independent. In particular, $\beta_g(g, \mu) = \beta_g(g, 0)$. $\bar{g}$ is evidently independent of $\bar{\mu}$. If $g \to 0$, then $\bar{\mu} \to 0$‡‡ because, as we shall see,

$$\beta_\mu = -\mu(1 + \gamma_m(g)) = -\mu(1 + O(g^2))\,. \tag{4.3}$$

‡ Weinberg excludes scalar mesons from his discussion because his derivation (analogous to appendix 1) rests on cut-off, Λ, dependence. Recognizing that it is M rather than Λ dependence that enters into the renormalization group equations and that the quadratic mass divergences, once they are subtracted generate no M dependence, it is clear that his technique applies to all renormalizable theories.

‡‡ So it is uniquely for asymptotically free theories that neglecting the effects of mass insertions is justifiable in solving the Callan–Symanzik equations for asymptotic forms. For a comparison of the several versions of renormalization group equations, see refs. [21, 25].

Standard approaches treat the mass terms in the Lagrangian as contributions to the propagators. An alternative, implicit in eq. (4.1), is to treat them as two-point vertices with m as a coupling constant. Weinberg's suggestion is to regard m (or m^2) as a source of a local operator, $\theta = \bar{\psi}(x)\, \psi(x)$ (or $\varphi^\dagger(x)\, \varphi(x)$). The theory is considered and renormalized as strictly massless, but physical Green's functions must include all numbers of mass-operator insertions. Mass renormalizations are defined as follows: if m (or m^2) is a source of θ and, under finite renormalizations of $M \to M'$ and $\theta \to \theta' = Z_\theta\, \theta'$, then m', source of θ', is given by $m' = Z_\theta^{-1} m$. (Remember Z_θ and the corresponding differential γ_θ are defined in the $m = 0$ theory.) So evidently $\beta_g = \beta_g(g) = \beta_g(g, \mu = 0)$ and the γ_m of eq. (4.3) is simply minus γ_θ.

Whether this prescription works and whether the implied renormalization group equation is true are questions of renormalizability, about which I have been uniformly cavalier. But I argue that aside from the Λ^2 scalar mass divergence, whose renormalization is independent of M, all other divergences can be absorbed in multiplicative, logarithmically divergent constants, and, since $\log \Lambda$ is so innocuous, any prescription that accounts for each parameter is as good as any other.

5. Euclidean versus physical momenta

Applications of asymptotic freedom are often prefaced by a restriction to all Euclidean or space-like momenta, but thus far nothing in particular has been said about the sets of momenta which are collectively rescaled. The problem lies not in the renormalization group but in our limited knowledge of the theories to which it is applied. Before we can extract any useful information by these techniques, we have to face questions to which perturbation theory obviously does not apply. But lacking any alternative guides, we can only make plausible hypotheses and investigate their implications.

If a theory is renormalizable for all domains of momentum space, then the renormalization group equation must be true for any momenta, expressing, as always, convention independence. It is used to construct the following relation (symbolically):

$$\Gamma(\lambda p, g, m, M) = \lambda^D\, \Gamma(p, \bar{g}, \bar{m}, M) \exp\left\{ \int_0^{\log\lambda} \gamma(\bar{g})\, \mathrm{d}t' \right\}. \tag{5.1}$$

g and m are parameters of the renormalized Lagrangian, not necessarily related directly to observable quantities, and $\bar{g}$ and $\bar{m}$ presumably go to zeros as λ increases.

What can be said about $\Gamma(p, \bar{g}, \bar{m}, M)$? The standard technique is to apply a straightforward perturbation expansion in the coupling constant, and $\bar{g}$ and $\bar{m}$ are just particular values for g and m. (Often, only the leading non-trivial behavior as $\bar{m} \to 0$ is desired, so it is tempting to expand in m about $m = 0$. But even perturbation theory is not analytic in m. However, all Green's functions exist for $m = 0$ – except for infrared divergences where some momenta or sums of momenta vanish – and, depending on the definition of m, so do their first one or two derivatives with respect to m at $m = 0$ [25][‡].) In perturbation theory, there is no essential difference between the

[‡] For footnote see next page.

Euclidean and physical regions other than the obvious. In particular, behavior in the latter is customarily obtained as the appropriate analytic continuation of the former. So from eq. (5.1) we can obtain the scaling behavior for any momenta for which we choose to trust perturbation theory for small g and m.

A salient feature of perturbation theory is that the particle spectrum and the location of cuts and poles are determined from the outset (with minor exceptions for the approximate schemes used for unstable particles). This facilitates the above-mentioned continuation. In contrast, as emphasized earlier, nothing is known about the spectrum of any asymptotically free theory. Furthermore, the size of the initial g cannot qualitatively affect the nature of the solutions because for arbitrarily small g there exist redefinitions for which the coupling is manifestly strong. Or in other words, even for small g, the effective coupling between the fundamental fields is large near their would-be mass shells. Also, the application to hadronic physics rests on the assumption that the hadrons are collective excitations, not associated with fundamental fields. Since the spectrum is determined by non-perturbative effects, it is not unreasonable that behavior for timelike momenta is radically different from that suggested by perturbation theory. And it is an open question whether or not the hadron spectrum extends indefinitely, in which case there may always be new two-particle thresholds.

One lesson of perturbation theory is that Green's functions for Euclidean momenta are fairly insensitive to details for small couplings and masses. However, for timelike external momenta, unitarity considerations make it obvious that the details of the physical spectrum are always important. So even though we choose to believe that the renormalization group techniques provide the leading behavior in the deep Euclidean region, the continuation of this behavior to physical momenta is not necessarily the leading behavior in the physical region. Such a possibility is totally foreign to perturbation theory, and to provide examples (see section 15) we must look to functions that cannot appear to any finite order. But that, of course, is the point: potentially, there are strong interaction effects which become negligible in the maximally unphysical region but which are always present in the physical region.

In $\Gamma(p, \bar{g}, \bar{m}, M)$, what can be said for fixed p as $\bar{m} \to 0$? It is plausible that nothing much happens for Euclidean p, because that is the case to all orders of perturbation theory. But for timelike p, branch points are moving and thresholds are opening up. The situation is further confused by the fact that some physical masses may depend on m for their scale, while others only on M. And if, in fact, the spectrum does extend indefinitely, there will always be new thresholds.

6. Composite operators

We are often interested in local operators other than the fundamental fields. In various applications to hadron physics, it is the weak and electromagnetic currents rather than the quark fields themselves that are observable. Also, properties of the dominant operators of the operator product expansion will be useful. The renormalization group analysis is easily extended to include such operators, but it is worth taking particular note of some novel features.

‡ Weinberg argues that within a given loop of a non-infrared divergent function only a single propagator will be near its mass shell. So the mass singularities will come from $\int d^4k\,(k^2 - m^2)^{-1}$ or $\int d^4k\,(k - m)^{-1}$ in some domain about $k = 0$. These are once or twice differentiable, respectively, at $m = 0$.

Because a product of renormalized fields at the same space-time point is typically a singular object, care must be taken to define a finite composite operator. A simple procedure is to specify a sufficient number of normalization conditions defining the appropriate subtractions, so that all matrix elements are finite. If the only effect of a change in the scale, M, of all the normalization points is to multiply the operator, $O(x)$, by a scale factor, Z_{o}, then any matrix element involving that operator will satisfy a renormalization group equation, with a factor of γ_{o} for each O. (This will be true if O is multiplicatively renormalizable or if all further necessary subtractions are essentially independent of where they are made.)

With composite operators defined as above, if a change of M is studied in perturbation theory, rather than $O \to Z_{\mathrm{o}} O$, often $O \to ZO + Z'O'$, where O' is yet another operator, but with the same quantum numbers. If there is a finite set O_i which only mix among themselves under changes in M, then this can be described as a matrix rescaling:

$$O_i \to Z_{ij} O_j \,. \tag{6.1}$$

There are linear combinations of O_i which correspond to the eigenvectors of Z_{ij}, which are, in a sense, natural operators to study in the context of scale transformations. But these generally differ from the operators natural to free field theory, which were the obvious starting points for definitions. Furthermore, since in practice Z is determined perturbatively, it is necessary to retain the matrix formalism.

If Γ_i^n is a vertex function of a set of n fundamental fields and one O_i, for example, then

$$[(M\, \partial/\partial M + \beta\, \partial/\partial g + \gamma^n)\, \delta_{ij} + \gamma_{ij}^0]\; \Gamma_j^n = 0\,. \tag{6.2}$$

The general solution becomes

$$\Gamma_i^n(\lambda p, g, M) = \lambda^D \,\exp\left\{\int_0^t \gamma^n \delta_{ij} + \gamma_{ij}^0)\, \mathrm{d}t'\right\} \Gamma_j^n(p, \bar{g}, M)\,. \tag{6.3}$$

Composite operators can have one simplifying feature, namely gauge invariance [26, 27]. Certainly the "observable" operators, such as electromagnetic currents, must be invariant under non-Abelian gauge transformations. Restricting the description of choices of gauge to the parameter α, the weakest possible statement of gauge invariance for matrix elements of gauge invariant operators is

$$[-\,\partial/\partial\alpha + \beta'(g,\alpha)\, \partial/\partial g + \gamma'(g,\alpha)]\; \Gamma_{\mathrm{GI}} = 0\,. \tag{6.4}$$

This states that any apparent dependence on α is present only because of the definition of g and normalization of the operators. The gauge invariant functions Γ_{GI} depend only on a single parameter $g'(g,\alpha)$ defined by

$$\partial g'/\partial\alpha = \beta'(g',\alpha) \tag{6.5}$$

and

$$g'(g, 0) = g\,,$$

the coupling in Landau gauge. This can be folded into the renormalization group equation for Γ_{GI} to give

$$[M\, \partial/\partial M + \bar{\beta}(g,\alpha)\, \partial/\partial g + \bar{\gamma}(g,\alpha)]\; \Gamma_{\mathrm{GI}} = 0 \tag{6.6}$$

where

$$\bar{\beta} = \beta + 2\alpha\, \gamma_A\, \beta' \quad \text{and} \quad \bar{\gamma} = \gamma + 2\alpha\, \gamma_A\, \gamma'. \tag{6.7}$$

The outstanding feature of this equation is that, because of the absence of any $\partial/\partial\alpha$, it is true for any fixed α, independent of the functional dependence on a variable α.

An as yet unanswered question is whether the various $\gamma'(g,\alpha)$ might in fact be identically zero. If they were, then $\Gamma_{\mathrm{GI}}(g,\alpha) = \Gamma_{\mathrm{GI}}(g', 0)$, which is to say that g' could be used as the *definition* of the coupling constant, and the renormalization group equation would become totally gauge invariant:

$$[M\, \partial/\partial M + \beta(g')\, \partial/\partial g' + \gamma(g')]\; \Gamma_{\mathrm{GI}} = 0 \tag{6.8}$$

where $\beta(g') = \beta(g', 0)$, etc.

It is also possible that this discussion of gauge invariance has been totally too naive. The status of gauge invariant operators in non-Abelian theories and the significance of ghost operators have not been investigated in great detail. Specific calculations have been checked against the results in non-covariant but ghost-free axial gauges,‡ where the Ward identities are far simpler [28, 29]. (For instance, Z *does* equal Z_2.) While the naive approach to the anomalous dimension computation of section 10 satisfies several immediate consistency checks, it runs afoul in the applications studied by Kainz, Kummer and Schweda [29]. This whole question certainly deserves further study and clarification.

7. Multiple coupling constants

The generalization to theories with more than one coupling constant is non-trivial. A $\bar{g}_i$ is defined for each g_i from β_i, but each β_i will depend on all the couplings, unless the whole theory factorizes. Conceivably, there are simultaneous zeros of all β_i in coupling constant space, which may be stable or unstable in various directions.

It is our basic hypothesis that the behavior near the origin can be investigated perturbatively. There are some very simple things that can be said about what gauge couplings do; there is a very important theorem concerning what coupling constants cannot do in the absence of non-Abelian gauge fields; and there is a crude and tedious program for investigating the evolution of the $\bar{g}_i$'s in t when they are coupled even to lowest order through their β functions.

Noting that the β function for any gauge coupling, g_i, can be inferred from diagrams of the forms of figs. 2A, 3A and 4, we observe that

$$\beta_i = -b_i\, g_i^3\, [1 + (\text{higher order in all couplings})]. \tag{7.1}$$

As long as all effective couplings are small compared to unity, each non-Abelian coupling will decrease (provided $b_i > 0$), and each Abelian coupling will increase ($b_{\mathrm{Abelian}} < 0$, always), independent of all other couplings. If the effects of the other couplings ever become comparable to one, then the signs of the gauge β's are no longer known.

‡ Axial gauges are defined by the condition $\eta_\mu A^\mu = 0$, with η^μ some fixed four-vector.

The price of asymptotic freedom

Coleman and Gross prove that no renormalizable theory without non-Abelian gauge fields can be asymptotically free [13]. The proof is by enumeration: simple characterizations of all possible one-loop graphs plus some general positivity statements suffice to show that in any such theory at least one effective coupling must grow with t.

No theory containing Abelian gauge fields can be asymptotically free, as demonstrated in the preceding paragraphs. For theories containing only spinless mesons, if the matrix of quartic couplings is positive‡, then at least one of the couplings that is initially positive has a positive derivative. If there are also Yukawa couplings, the hermiticity of the Lagrangian implies that a quadratic form in the Yukawa couplings can be constructed which must increase with t.

Gauge theories with scalar fields

The preceding results do not preclude the possibility of asymptotically free theories involving scalar fields, but rather they imply that non-Abelian gauge fields are a necessary (but not sufficient) ingredient. The status of the origin as an ultraviolet fixed point can again be investigated in perturbation theory. But now, in addition to the gauge couplings, g_i, there are as many scalar quartic couplings, λ_i, as there are independent quartic invariants in the scalar representation. The $\bar{\lambda}_i$ are in general coupled to each other and to the $\bar{g}_i$.

A fairly straightforward analysis turns the problem into an algebraic one, once the lowest order β functions are known. This will be done for the case of a single g and one and then two λ's. (The analysis for more λ's is analogous but tedious and has been carried out in detail for classes of models including up to five of them [30].) The result is that some theories are and some theories are not asymptotically free, but as yet there exists no characterization or criterion other than explicit computation, model by model, for telling which is which.

Yukawa couplings can be similarly investigated but prove to be rather uninteresting in that either they grow with t or they go to zero so rapidly that their presence cannot affect the asymptotic behavior of the other effective couplings. They can, however, affect the form of specific model predictions at finite t.

For several reasons the presence of fundamental scalar fields in theories of strong interactions is undesirable. With unified theories of weak and electromagnetic interactions, the presence of strongly interacting scalars typically allows the violation of parity and strangeness conservation to order α [31]. And from almost any perspective, all hadrons are most naturally described as composite systems of spin $-\frac{1}{2}$ constituents.

Rather, the impetus behind the investigations of scalar fields has been the gnawing question of whether asymptotically free field theories do in fact generate physically acceptable models, or is asymptotic freedom simply an amusing symptom of some horrendous pathology? One means of assuaging such fears would be to produce a theory which is on the one hand asymptotically free and on the other hand completely physically interpretable within the context, say, of perturbation theory ‡. The only way known to get vector particles in perturbation theory from non-Abelian gauge fields is via the Higgs mechanism [8, 22], whereby the vacuum is apparently not symmetric

‡ The quartic form $\lambda_{ijkl}\varphi_i\varphi_j\varphi_k\varphi_l$ must be positive following from the vacuum stability arguments of ref. [8].

‡ The model need not resemble the hadrons, for there strong interaction effects certainly alter the nature of the particle spectrum. The question here is one of consistency.

under gauge group transformations, the particle-field identification is slightly altered, and there is a massive vector meson for each group generator under which the vacuum is not symmetric.

Thus far, no model has been found which is both asymptotically free and, via the Higgs mechanism, has mass terms for all, or all but one, of the gauge fields. The search has been strenuous and fairly systematic but by no means exhaustive and, furthermore, has provided no insight as to why the two phenomena are apparently incompatible.

A method for analyzing gauge theories with a single λ is presented in ref. [32]. β_g receives contributions from scalar loops, but is independent of λ to lowest order: λ first occurs in order $g^3\lambda^2$. Since g can still be defined by the vector–fermion coupling, the lowest order scalar contribution can be inferred from the vector two-point function. Fig. 4 gives

$$-p^2 g^{\mu\nu} + p^\mu p^\nu \left[\ldots - \tfrac{1}{3}\, c_2(s)\, \frac{g^2}{16\pi^2} \log \frac{-p^2}{M^2} \right] \tag{7.2}$$

where $c_2(s)$ is the analogous group theory constant for the scalar representation and the scalars are taken to be complex. (If the scalar fields are hermetian, then their effect is exactly half that given in eq. (7.2).) Thus

$$\beta_g = -bg^3 + \ldots = -\frac{1}{16\pi^2}\left(\tfrac{11}{3} c_1 - \tfrac{4}{3} c_2(f) - \tfrac{1}{3} c_2(s)\right) g^3 + \mathrm{O}(g^5)\,. \tag{7.3}$$

b is positive for a wide variety of models, so $\bar{g}$ is substantially of the same form as before.

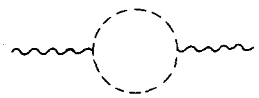


Fig. 4. The first scalar meson contribution to the vector propagator.

β_λ can be deduced from the first order corrections to the scalar two- and four-point functions, indicated in fig. 5. β_λ is therefore always of the form

$$\beta_\lambda = A\lambda^2 + B'\lambda g^2 + Cg^4 + \ldots\,. \tag{7.4}$$

This evidently couples $\bar{\lambda}$ to $\bar{g}$, since $\partial\bar{\lambda}/\partial t = \beta_\lambda(\bar{\lambda}, \bar{g})$. But the problem can be simplified with a change of variables. Define $\bar{R} = \bar{\lambda}/\bar{g}^2$. From eq. (7.4), the evolution of $\bar{R}$ is determined by

$$\partial\bar{R}/\partial t = \bar{g}^2\,[A\bar{R}^2 + B\bar{R} + C] \tag{7.5}$$

where $B = B' + 2b$. (Note that the criteria for validity of eq. (7.4) are $\bar{g}^2 \ll 1$ and $\bar{\lambda} \ll 1$, which translate into $\bar{g}^2 \ll 1$ and $\bar{R} \ll 1/\bar{g}^2$ for eq. (7.5); so $\bar{R}$ need not be small.)

If $\bar{g}$ is approaching zero ($b > 0$) and $\bar{R}$ is bounded as $t \to \infty$, then $\bar{\lambda} \to 0$, and the theory is asymptotically free. At first glance, there appears to be another possibility. What if $\bar{g} \to 0$ and $\bar{\lambda} \to 0$ but $\bar{R} \to \infty$ because $\bar{g}^2$ decrease much faster than $\bar{\lambda}$? This is in fact not possible because if $\bar{g}^2/\bar{\lambda} \to 0$, then the equation governing $\bar{\lambda}$ will eventually become identical, in a perturbative sense, to the equation for a theory with no gauge couplings at all. And, as we have seen, no such theory is asymptotically free. In the particular example at hand, if $\bar{g}^2/\bar{\lambda} \to 0$, eqs. (7.4) and (7.5) are dominated by the A term, and A is always positive.

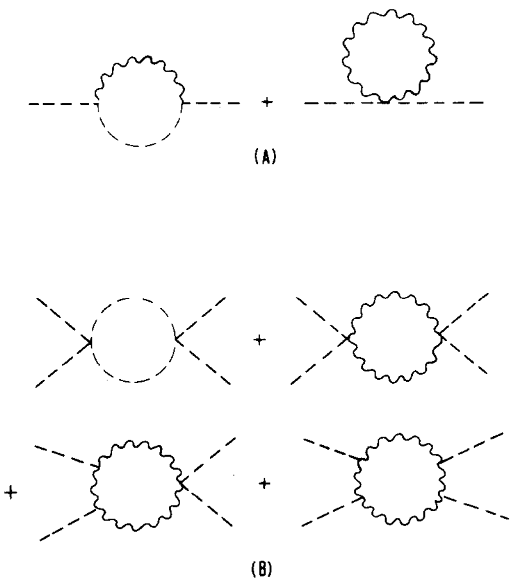


Fig. 5. The one loop scalar A) two-point function and B) four-point function.

From eq. (7.5), $\bar{R}$ will be bounded if it approaches a zero of the quadratic form. Since C is always positive, the two real zeros, if they exist, are either both positive or both negative. The negative solutions are again unsatisfactory because of the vacuum stability problem. So $\bar{\lambda} \to 0$ requires that there be real, positive roots of the quadratic form, r_1 and r_2, and the initial R must satisfy $R \leqslant r_2$ where r_2 is the larger root. If $R < r_2$, then $\bar{R} \to r_1$. (Note that $r_1 = r_2$ is a satisfactory solution.) In terms of A, B and C

$$-\sqrt{4AC} \leqslant B < 0 \,. \tag{7.6}$$

Eq. (7.5) can be solved by quadratures, and an explicit expression for $\bar{\lambda}(g, \lambda, t)$ can be obtained. (The specific expression for $\bar{\lambda}$ is of little practical value since $\bar{\lambda}$ is of order $\bar{g}^2$, and in most applications the first interesting effects of $\bar{\lambda}$ are order $\bar{\lambda}^2$, negligible compared to $\bar{g}^2$.)

The only effect of fermions on this lowest order discussion is through b in $B = B' + b$. While b must be positive so $\beta_g < 0$, the smaller b is, the slower will $\bar{g}$ approach zero; making b as small as possible, with an appropriate choice of fermion representation, maximizes the effects of the gauge couplings in eq. (7.5).

Many one-λ theories satisfy eq. (7.6) with sufficient fermions: for instance, SU($N \geqslant 3$) with one scalar N-tuplet, SU(3) with one scalar octet [28]. But for any such theory known, even the most general vacuum expectation value of the scalar fields is invariant under more than one of the group transformations.

Models with two scalar couplings, λ_1 and λ_2, can be analyzed with an analogous change of variables, $R_{1,2} = \lambda_{1,2}/g^2$. The evolution of $\bar{g}$ will again be essentially the same, and the problem is reduced to finding zeros in R. The equations for $\bar{R}$ to lowest order are of the form

$$\frac{1}{\bar{g}^2}\frac{\partial \bar{R}_1}{\partial t} = A^1_{11}\bar{R}_1^2 + A^1_{12}\bar{R}_1\bar{R}_2 + A^1_{22}\bar{R}_2^2 + B_1\bar{R}_1 + C_1$$
$$\frac{1}{\bar{g}^2}\frac{\partial \bar{R}_2}{\partial t} = A^2_{11}\bar{R}_1^2 + A^2_{12}\bar{R}_1\bar{R}_2 + A^2_{22}\bar{R}_2^2 + B_2\bar{R}_2 + C_2 \; . \tag{7.7}$$

In the real $\bar{R}_1-\bar{R}_2$ plane there will be lines of zeros of $\partial\bar{R}_1/\partial t$ and of $\partial\bar{R}_2/\partial t$. The first problem is to determine if these lines ever intersect: are there any simultaneous zeros of $\partial\bar{R}_1/\partial t$ and $\partial\bar{R}_2/\partial t$? If so, one may ask whether they are stable zeros: as $t\to\infty$, is there any non-trivial set of trajectories of $\bar{R}_1-\bar{R}_2$ that approach the points in question?

The behavior of trajectories near a fixed point is determined by the derivative matrix $(\partial/\partial\bar{R}_i)\,\partial\bar{R}_j/\partial t$. This matrix has real eigenvalues – a general property of any field theory, which is easy to establish in lowest order perturbation theory. The demonstration is as follows: let g_i be any dimensionless coupling constant, gauge, scalar, or Yukawa. From a survey of the possible lowest order graphs, it becomes apparent that β_i has the form of a variation of a scalar, invariant function of all the couplings with respect to g_i. Hence, $\partial\beta_i/\partial g_j$ is symmetric in i and j.

If the eigenvalues of $(\partial/\partial\bar{R}_i)\,\partial\bar{R}_j/\partial t$ are all negative at the zero of $\partial\bar{R}_j/\partial t$, then it is a stable fixed point. If just one eigenvalue is positive, then the presence of any amount of the corresponding linear combination of couplings produces instability; thus, if there are stable trajectories, they are special and constitute a set of measure zero in the full coupling space. In the case of zero eigenvalues, it is the first non-vanishing derivative that is significant.

This analysis has been carried out for a wide variety of models and with several λ's. There are many examples of asymptotic freedom, but none of these has sufficient group structure to allow for the spontaneous breakdown of the whole gauge group.

If all Yukawa couplings, f_i, go to zero due to the presence of the gauge couplings, g_i, then they go to zero with a rational power of t faster than the g_i, themselves. (This will be proven below.) Furthermore, the Yukawa couplings enter the β functions of the scalar quartic couplings, λ_i, with the same powers as g_i. Schematically, suppressing all indices and numerical coefficients,

$$\partial\bar{\lambda}/\partial t = \bar{\lambda}^2 + \bar{g}^2\bar{\lambda} + \bar{f}^2\bar{\lambda} + \bar{g}^4 + \bar{f}^4 + \dots \, . \tag{7.8}$$

All the $\bar{g}_i$ go to zero like t^{-1} as $t\to\infty$. If the $\bar{f}$'s vanish faster, then they are irrelevant to the ultimate stability of the $\bar{\lambda}_i$.

The f's can be assembled into a matrix, f^a_{ij}, where a labels the meson and i and j label the fermions. (It is convenient to organize the particles into the natural, irreducible multiplets.) $\bar{f}$ is governed by an equation of the form

$$\partial\bar{f}/\partial t = P_3(\bar{f}) + P_1(\bar{f}) \tag{7.9}$$

where $P_3(f)$ is a cubic form in f and is precisely the lowest order β_f in the absence of gauge couplings. (See fig. 6.) $P_1(f)$ is a linear form in f; in particular P_1 is quadratic in $\bar{g}_i$ and is diagonal on f:

$$P_1(f) = p_1\cdot f \tag{7.10}$$

where p_1 is a diagonal matrix. ($p_1 \propto 1$ if all the particle representations are irreducible.) Coleman and Gross prove

$$\mathrm{tr}\{h\,P_3(h)\} \geqslant 0 \tag{7.11}$$

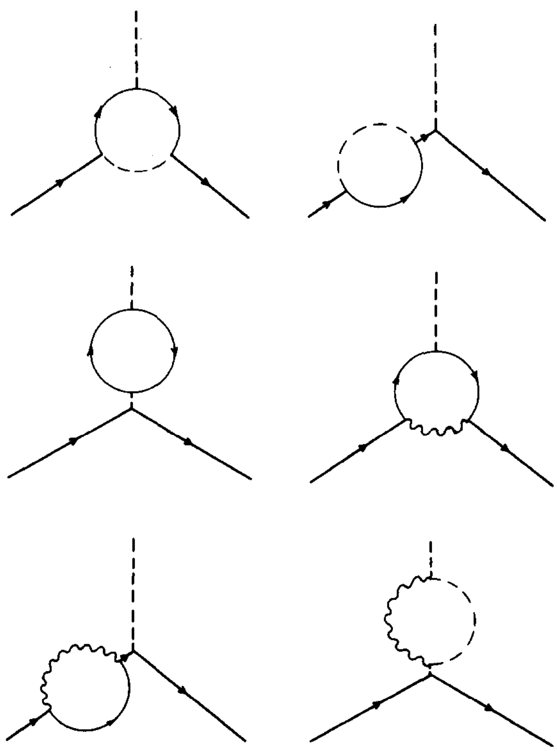


Fig. 6. Renormalization of Yukawa couplings.

for any hermetian h. If the theory is to be asymptotically free, it must be because of the gauge contribution. Define a new matrix by $\bar{R} = (t)^{1/2}\bar{f}$. Then

$$t\,\partial\bar{R}/\partial t = P_3(\bar{R}) + P_1'(\bar{R}) \tag{7.12}$$

where P_1' is a related diagonal functional but without any implicit t dependence. From eq. (7.11), $\bar{R}$ can go to a constant only if the eigenvalues of P_1' are negative. So P_1' negative is a necessary condition for asymptotic freedom. (In particular, it is a condition on the representation content of the theory and is satisfied in a variety of models.) We can further observe that for small enough initial R, eq. (7.12) is dominated by the P_1' term, and hence, for P_1' negative, there are trajectories in R that go to zero like t^{-1} to some power.

The remaining question is: can $\bar{R}$ go to a constant for larger initial values? There are in general other fixed points because eq. (7.12) is cubic, but are they stable? Let R_0 be one such fixed point; then

$$P_3(R_0) + P_1'(R_0) = 0\,. \tag{7.13}$$

Let us explore stability in a particular direction, that given by R_0, itself. Consider the behavior along $R = \lambda R_0$:

$$\mathrm{tr}\{\bar{R}\,[P_3(\bar{R}) + P_1'(\bar{R})]\} = \lambda^2(\lambda^2 - 1)\,\mathrm{tr}\{R_0 P_3(R_0)\}\,. \tag{7.14}$$

From eq. (7.11), for $\lambda > 1$ this is positive, and for $\lambda < 1$ it is negative, so R_0 is unstable in the radial direction and, hence, unstable in general.

8. Temporary freedom

In the previous section, methods were discussed for determining whether the origin is a stable or unstable fixed point as $t \to \infty$. I wish to subdivide the unstable theories into two classes: (1) those in which at least one of the $\bar{g}_i$ always grows with t, and (2) those in which there is some domain in t and some domain of initial values of g_i such that all of the $\bar{g}_i$ decreases in magnitude before ultimately at least one of them increases. I call the models of the second class "temporarily free" and argue that, within the same framework of hypotheses that is constructed around asymptotic freedom, they provide alternative models of strongly interacting fields for which approximate scaling is only a temporary phenomenon. If and when approximate scaling occurs in these models, it must be regarded somewhat as an accident rather than as inevitable, in contrast to asymptotically free theories. And nothing in particular can be said about the ultimate short distance behavior [33].

It is worth recalling the argument used to apply perturbative methods to strong interactions. It is assumed that the β functions of the theory in question are such that the effective couplings, starting at some initial strong values, rapidly become small. Once they are small, the expansion in effective couplings is reasonable. This scheme is consistent only if, once the couplings are small, they continue to decrease in magnitude. Otherwise, they could have never become small in the first place. Approximate scaling is then a consequence of the smallness of the effective couplings.

The logic of temporary freedom is identical. We seek in perturbation theory models for which all couplings decrease in magnitude. Once they are small, their evolution is slow because the β's vanish in zeroth order. Eventually in t, at least one coupling begins to increase, and with increasing speed the couplings move away from the origin.

It is trivial to use the results of Coleman and Gross to determine what types of models are not temporarily free. Abelian couplings are monotone increasing functions of t when all couplings are small. So a temporarily free model of strong interactions can contain no Abelian gauge factors. Theories with spinless mesons only can be temporarily free because the positivity arguments raised previously are no longer applicable; since nothing is known about either the $t \to \infty$ or $t \to -\infty$ limits, there is nothing in principle wrong with the quartic effective couplings being negative for some t, approaching the origin, and then moving away in some direction. Theories of fermions and spinless mesons only are ineligible because it is still true that at least one combination of Yukawa couplings will always grow with t. Theories including non-Abelian gauge fields may be temporarily free. (Note that if an acceptable model is to include fermions, it must also include non-Abelian gauge fields.)

In ref. [33], a particular class of examples is treated in detail. The examples, SU(N) gauge theories with $m \geqslant N - 1$ fundamental scalar representations and any (appropriately small) fermion representation, were chosen because the meson content is sufficient to provide spontaneous breaking of all the gauge symmetries through meson vacuum expectation values. As with any temporarily free theory, these models involve at least two β's that are essentially coupled. (If the problem can be factored into uncoupled β's, the resulting $\bar{g}$'s are necessarily monotonic.) So the effective couplings are only known from approximate, numerical integrations. The results are not terribly enlightening because we have no natural measure of the likelihood of a given trajectory. Of course, the nearer a trajectory passes to the origin, the longer it lingers in that vicinity. For as long as it is there, the predictions of temporary and asymptotic freedom are virtually identical. The deviations come once the trajectory picks up speed and moves far from the origin. Even if we had exact solutions for the $\bar{g}_i$, when in t this occurs depends critically on the particular trajectory or, equivalently, on the initial conditions.

Asymptotically Free Theories of the Strong Interactions

9. Color

For convenience and simplicity, I choose to discuss a particular scheme, the colored quark–gluon model, described below. The choice brings with it certain theoretical prejudices, but the results are sufficiently general to be easily translated into another framework. The alternative possibilities differ on the status given the dynamical gauge group. Perhaps the true vacuum is not invariant under any of the gauge symmetries, which is to say that there is spontaneous symmetry breakdown of a dynamical origin. In this case, no vestige is left of the gauge group as an ordinary symmetry, and there may be particles corresponding to the fundamental fields, but with dynamical masses. Various relations between the physical symmetries (electric and weak charges, SU(3), etc.) and the dynamical gauge symmetries are possible, including the Han–Nambu integral charge assignment to quarks [34], in which case they may have already been observed. In any case, there is no need to fret over the massless gauge fields because, as emphasized earlier, there is no reason to expect that they correspond to massless particles because no reliable statements can be made concerning behavior near their "would-be" mass shells.

In the color scheme, the physical symmetries are symmetries of the quarks which commute with color, and the gluons correspond to the color generators but are otherwise neutral. For example, if the color group is SU(3) and the hadronic symmetry is SU(4), then there are four fermion color triplets: p_i, n_i, λ_i, and p'_i, where i = red, white or blue.‡ This scheme incorporates the color symmetry of the quarks suggested by Gell-Mann [35] and suggests the following bold hypothesis: perhaps the only physical, observable states are color singlets. (The local color gauge symmetry may be exact for all states, or it may be spontaneously broken, perhaps with global color left exact.) Hence, the nonexistence of gluons and quarks as particles are the same phenomenon and the result of non-Abelian dynamics. While no progress has as yet been made by traditional field theorists towards establishing such a result, precisely because it is a strong coupling, collective phenomenon, it may be worth noting that in the meantime an amazing and amusing variety of models of quark confinement have been suggested.‡‡ Curiously, each of these recent analyses makes essential use of gauge fields.

The strong interactions are thus described by the Lagrangian of eq. (3.1) and the Fermi field ψ consists of m fundamental representations of the gauge group, say SU(N). A fermion mass term

‡ This choice of colors is traditional.

‡‡ Just a sampling of these ideas are: the dual vortex string [36, 37], strong coupling [38, 39], charge shielding [40], and the bag [41].

may or may not be present, but the masses would not enter in the following discussions of leading asymptotic behavior. (The mass matrix is of interest in discussions of the physical symmetries and the effects of weak and electromagnetic symmetry breaking [42].) In fact, any mass, vacuum expectation value, or other dimensionful parameter is irrelevant to the discussions of leading short distance behavior.

10. Inelastic structure functions

In electroproduction experiments, the structure functions W_1 and W_2 are measured. These are defined by

$$\frac{1}{2\pi}\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}q\cdot x}\,\langle p|J^\mu(x)\,J^\nu(0)|p\rangle = \left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right)W_1+\frac{1}{m_\mathrm{p}^2}\left(p^\mu-q^\mu\,\frac{p\cdot q}{q^2}\right)\left(p^\nu-q^\nu\,\frac{p\cdot q}{q^2}\right)W_2 \tag{10.1}$$

where the matrix element is the product of two electromagnetic currents taken between hadron target states, spin averaged‡, and m_p is the mass of the proton. The target has four-momentum p, each current transfers momentum q, and $x=-q^2/2p\cdot q$.‡ We are interested in the large q^2 behavior, but p^2 is a fixed mass. Trivial kinematics gives $0<x<1$.

Since only q^2 and $p\cdot q$ can be scaled up, we will separate the various dependences using the operator product expansion of the product of the two currents. Only then can be renormalization group analysis be applied.

The operator product expansion, derived with various degrees of sophistication under various circumstances [16, 43], describes and separates the two outstanding features of what happens when the separation of two operators goes to zero or becomes lightlike. In either case the limit is singular. Furthermore, in the limit, the product includes the effects of virtually any local operator in the theory, restricted only by symmetries and conservation laws. So we write, suppressing all indices,

$$J(x)\,J(0)=\sum_i F_i(x)\,O_i(0) \tag{10.2}$$

where the F_i are c-number functions of x, potentially singular as x or x^2 goes to zero and O_i are the local operators of the theory (with the quantum numbers of JJ). It is really a statement of completeness of the local operators.

Eq. (10.2) becomes useful if we can identify the dominant operators and say something about their coefficient functions. (Remember that x is the Fourier transform variable conjugate to q.) Begin by organizing the O_i according to their internal symmetries, dimension, and spin. The corresponding F_i must have the appropriate dimension and Lorentz indices so that each F_iO_i matches JJ. If all dimensionful parameters were truly irrelevant in the x or x^2 goes to zero limit,

‡ The states are normalized to $\langle p|p'\rangle=2E(2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{p}')$. The natural definitions, using the alternative normalization with an additional $(2m_\mathrm{p})^{-1}$, leads to a difference of a factor of two in the dimensionless W_i.

‡ x will be used both as the scaling variable and as the space-time separation of the two currents: the meaning will be clear from the context.

then the singularities of the F_i would be dictated by dimensional analysis, and the dominant operators would be those with the most singular coefficients. However, the F_i can involve arbitrary functions of the dimensionless variables xM, so the naive argument is fallacious. But the point of all this is that in asymptotically free theories (as will be demonstrated in detail) only $\log(x^2M^2)$ to a calculable power occurs, and the ordering according to dimensional analysis is valid in the short or lightlike distance limit.

It is worth clarifying the distinction between the short and lightlike distance limits, which is simplest in momentum space. In the first case, $q \to \infty$ and the O_i of lowest dimension have F_i with the most powers of q. In the lightcone limit, of interest for electroproduction, both $q^2 \to \infty$ and $p \cdot q \to \infty$. The hadron matrix element of operators of spin n may have terms proportional to $p^{\mu_1} p^{\mu_2} \dots p^{\mu_n}$. These will be contracted with the appropriate number of q's, but $p \cdot q$ is now comparable to q^2. So the dominant terms of the operator expansion will come from operators of lowest dimension-minus-spin, defined as "twist".

A simple example, whose result will be of some use subsequently, may be instructive. Let us study the leading lightcone behavior of the product of two electromagnetic currents for a free proton. The current is

$$J_\mu(x) = \bar{\psi}(x)\, \gamma_\mu\, \psi(x)\,. \tag{10.3}$$

Consider the forward Compton amplitude, averaged over proton spin‡:

$$\mathrm{i} \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}q \cdot x} \langle p | T(J^\mu(x)\, J^\nu(0)) | p \rangle = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) T_1 + \frac{1}{m_\mathrm{p}^2} \left(p^\mu - q^\mu \frac{p \cdot q}{q^2}\right) \left(p^\nu - q^\nu \frac{p \cdot q}{q^2}\right) T_2\,. \tag{10.4}$$

This is described graphically in fig. 7A. Since the protons do not interact, the only connected graphs are those of fig. 7B. Thus, the product JJ has the effect of local operators bilinear in the fields. The bilinear operators of lowest twist (twist-2) that have non-zero, spin-averaged matrix elements are

$$O^n(x) = \frac{\mathrm{i}^{n-1}}{n!}\, [\bar{\psi}\, \gamma^{\mu_1} \partial^{\mu_2} \dots \partial^{\mu_n}\, \psi(x) + \text{permutations of vector indices}]\,. \tag{10.5}$$

The coefficient functions, F^n, can be inferred by comparing a direct computation of the graphs of fig. 7B to the result of inserting the operator product expansion into eq. (10.4). Ignoring terms of order p^2 compared to q^2 or $p \cdot q$, we find for the Compton amplitude

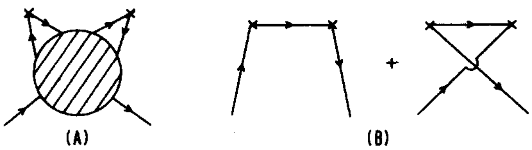


Fig. 7. A) The proton matrix element of two currents and B) the connected graphs in free field theory.

‡ We study the Compton amplitude because it satisfies a renormalization group equation.

$$\sum_{j=0}^{\infty}\left(\frac{1}{x^2}\right)^j\left[-g^{\mu\nu}\frac{2}{x^2}+8\,\frac{p^\mu p^\nu}{q^2}-\frac{4}{x}\,\frac{(p^\mu q^\nu+p^\nu q^\mu)}{q^2}\right]. \tag{10.6}$$

On comparing this result to eq. (10.4),

$$T_1=2\sum_{j=1}^{\infty}\left(\frac{1}{x^2}\right)^j,\qquad T_2=\frac{8m_\mathrm{p}^2}{q^2}\sum_{j=0}^{\infty}\left(\frac{1}{x^2}\right)^j \tag{10.7}$$

Notice that the longitudinal structure function vanishes:

$$T_\mathrm{L}=2m_\mathrm{p}xT_1-\nu T_2=0 \tag{10.8}$$

where $\nu=p\cdot q/m_\mathrm{p}$. This is essentially the Callan–Gross sum rule [44], which is a kinematic result for free fermions.

To identify the coefficient functions, we must write down the operator product expansion in detail, separating the tensor structures corresponding to T_1 and T_2: ‡

$$\mathrm{i}T(J^\mu(x)\,J^\nu(0))=\sum_n\,[-g^{\mu\nu}\mathrm{i}^n\,\partial_{\mu_1}\ldots\partial_{\mu_n}\mathcal{F}_1^n(x^2)+g^\mu_{\mu_1}g^\nu_{\mu_2}\mathrm{i}^{n-2}\,\partial_{\mu_3}\ldots\partial_{\mu_n}\mathcal{F}_2^n(x^2)]\,O^{\mu_1\cdots\mu_n}(0)+\ldots\,. \tag{10.9}$$

Dimensional analysis gives for the Fourier transforms

$$\mathcal{F}_1^n(q^2)=\frac{1}{(-q^2)^n}\,f_1^n\,,\qquad \mathcal{F}_2^n(q^2)=\frac{1}{(-q^2)^{n-1}}\,f_2^n \tag{10.10}$$

where the f_i^n are dimensionless. The spin averaged matrix elements of the O^n are

$$\langle p|O^{\mu_1\cdots\mu_n}(0)|p\rangle=A^n p^{\mu_1}\ldots p^{\mu_n}\,. \tag{10.11}$$

And in this free field theory, $A^n=2$. Thus,

$$T_1=\sum_n\left(\frac{1}{2x}\right)^n f_1^n A^n,\qquad T_2=\frac{m_\mathrm{p}^2}{-q^2}\sum_n\left(\frac{1}{2x}\right)^{n-2}f_2^n A^n=\frac{m_\mathrm{p}}{\nu}\sum_n\left(\frac{1}{2x}\right)^{n-1}f_2^n A^n\,. \tag{10.12}$$

And finally,

$$f_1^n=f_2^n=2\,. \tag{10.13}$$

Actually, this is true for even n. The terms odd in x and n vanish, as follows from parity invariance.

In the colored quark–gluon model, we take the current to be

$$J_\mu(x)=\sum_{i=1}^{m}\,Q_i\,\bar{\psi}_i(x)\,\gamma_\mu\,\psi_i(x) \tag{10.14}$$

where Q_i is the charge in units of e of the ith multiplet, and a sum over colors is implied. There are now three sets (or towers) of twist-2 operators that dominate the proton, spin-averaged matrix elements of the product of two electromagnetic currents. Note that these operators must be invariant under local gauge transformations because both J_μ and $|p\rangle$ certainly are. The first set is

‡ The ... includes not only operators of higher twist but also the three other independent tensor structures. The five are linear combinations of $g^{\mu\nu}$, $\partial^\mu\partial^\nu$, $g^{\mu\mu_1}g^{\nu\mu_2}$, $\partial^\mu g^{\nu\mu_1}$ and $g^{\mu\mu_1}\partial^\nu$. But these others do not contribute to electroproduction.

$$O_a^n = \frac{\mathrm{i}^{n-1}}{n!}\,[\bar\psi\,\lambda^a\,\gamma^{\mu_1} D^{\mu_2}\dots D^{\mu_n}\,\psi + \text{permutations}] \tag{10.15}$$

where λ^a is a representation matrix acting over the space of physical symmetries. (If ψ is a triplet under physical SU(3), O_a^n is the octet in $3\times\bar 3 = 1 + 8$.) There are also singlet quark operators

$$O_\psi^n = \frac{\mathrm{i}^{n-1}}{n!}\,[\bar\psi\,\gamma^{\mu_1} D^{\mu_2}\dots D^{\mu_n}\,\psi + \text{permutations}] \tag{10.16}$$

and lastly there are gluon operators

$$O_A^n = \frac{\mathrm{i}^{n-2}}{2n!}\,[F^{\alpha\mu_1} D^{\mu_2}\dots D^{\mu_{n-1}} F_\alpha^{\mu_n} + \text{permutations}] \tag{10.17}$$

which are also singlets under the physical symmetries. Since O_ψ^n and O_A^n have identical quantum numbers, they will mix under renormalizations, as discussed in section 6. The structure functions of any particular hadronic target will be affected by all three sets, but the effects of O_a^n can be isolated by taking a difference, such as $\nu W_2^{\text{proton}} - \nu W_2^{\text{neutron}}$.

The large q^2 behavior of the coefficient functions is governed by a renormalization group equation. Such an equation can be derived because we already know equations satisfied by any matrix elements of J's and O^n's [45, 15]. Essentially, in eq. (10.2) each $F_i O_i$ must scale like JJ. So $\gamma_{F_i} = 2\gamma_J - \gamma_{O_i}$. Furthermore, since the J's are currents of exact symmetries, they are not renormalized by the strong interactions, which is to say $\gamma_J = 0$ to all orders in g and only begins order e^2, so it will be ignored as a higher order electromagnetic effect. Let us work this out in detail to get the matrix indices correct.

Normalization conditions must be established so that the matrix elements of O^n are finite. Only the terms with the maximal number of p^{μ_i}'s will contribute to the leading behavior because any other tensor structure must include $p^2 g^{\mu_i\mu_j}$. These leading terms are at most logarithmically divergent and so can be normalized at $p^2 = -M^2$ in the following particular cases (to order g^2):

$$\langle 0|\,T(A_a^\mu(-p)\,O_k^{\mu_1\cdots\mu_n}(0)\,A_b^\nu(p))\,|0\rangle_{\text{amputated}} = \delta_{ab}\,g^{\mu\nu}p^{\mu_1}\dots p^{\mu_n}\left[\delta_{Ak} + b_{Ak}^n \log\frac{-p^2}{M^2}\right] + \dots \tag{10.18}$$

$$\langle 0|\,T(\psi(-p)\,O_k^{\mu_1\cdots\mu_n}(0)\,\bar\psi(p))\,|0\rangle_{\text{amputated}} = \sum_l \frac{1}{n}\,\gamma^{\mu_l}p^{\mu_1}\dots p^{\mu_{l-1}}p^{\mu_{l+1}}\dots p^{\mu_n}\left[\delta_{\psi k} + b_{\psi k}^n \log\frac{-p^2}{M^2}\right] + \dots$$

where ... stands for the other tensor structures. The non-singlet operator, O_k^n with $k = a$, is normalized just like O_ψ^n except for the presence of a λ^a on the right side of eq. (10.18). For convenience of notation, define φ_k by $\varphi_A = A^a$ and $\varphi_\psi = \psi$ or $\bar\psi$ and abbreviate eq. (10.18) by

$$\langle\varphi_j(-p)\,O_k^n(0)\,\varphi_j(p)\rangle = \delta_{jk} + b_{jk}^n \log\frac{-p^2}{M^2}\,. \tag{10.19}$$

Write the expansion of two currents as

$$J(x)\,J(0) = \sum_{n,k} F_k^n(x)\,O_k^n(0) \tag{10.20}$$

where n runs over the possible spins and k is the type, $k = A$, ψ or a. Let

$$D = M\,\partial/\partial M + \beta\,\partial/\partial g \tag{10.21}$$

and consider working in a Landau gauge. A φ–φ matrix element of eq. (10.20) satisfies the equation

$$(D+2\gamma_j)\,\langle\varphi_j(-p)\,J(x)\,J(0)\,\varphi_j(p)\rangle = 0 = (D+2\gamma_j)\sum_{n,k} F_k^n(x)\,\langle\varphi_j(-p)\,O_k^n(0)\,\varphi_j(p)\rangle \tag{10.22}$$

because $\gamma_J = 0$. The F_k^n for different n have different tensor structure, so

$$(D+2\gamma_j)\sum_k F_k^n(x)\,\langle\varphi_j(-p)\,O_k^n(0)\,\varphi_j(p)\rangle = 0 \tag{10.23}$$

for each n. The effects of D on O_k^n are given by a matrix anomalous dimension $\gamma_{kk'}^n$:

$$\sum_{k'}\,[(D+2\gamma_j)\,\delta_{kk'} + \gamma_{kk'}^n]\,\langle\varphi_j(-p)\,O_{k'}^n(0)\,\varphi_j(p)\rangle = 0\,. \tag{10.24}$$

Combine eqs. (10.23) and (10.24):

$$\sum_{k,k'}\,[\langle\varphi_j(-p)\,O_{k'}^n(0)\,\varphi_j(p)\rangle\,\{D\,\delta_{kk'} - \gamma_{kk'}^n\}\,F_k^n(x)] = 0\,. \tag{10.25}$$

This is simplified by evaluating it at $p^2 = -M^2$, where the matrix elements are defined to be $\delta_{jk'}$:

$$\sum_k\,\{D\,\delta_{jk} - \gamma_{kj}^n\}\,F_k^n = 0\,. \tag{10.26}$$

From eqs. (10.19) and (10.24),

$$\gamma_{kj}^n = 2b_{jk}^n - 2\gamma_j\,\delta_{jk}\,. \tag{10.27}$$

Thus, the anomalous dimension of the coefficient function F_j^n is minus the transpose of the anomalous dimension of the gauge invariant operator O_k^n and is, therefore, itself gauge invariant.

This can be applied to electroproduction on, following the example of the earlier free field computation. The Lorentz structure is again

$$\begin{aligned} \mathrm{i}T(J^\mu(x)\,J^\nu(0)) = \sum_{n,i}\,[&-g^{\mu\nu}\,\mathrm{i}^n\,\partial_{\mu_1}\cdots\partial_{\mu_n}\,\mathcal{F}_{1,i}^n(x^2,g,M) \\ &+ g_{\mu_1}^\mu\,g_{\mu_2}^\nu\,\mathrm{i}^{n-2}\,\partial_{\mu_3}\cdots\partial_{\mu_n}\,\mathcal{F}_{2,i}^n(x^2,g,M)]\,O_i^{\mu_1\cdots\mu_n}(0) + \dots\,. \end{aligned} \tag{10.28}$$

(Note that the odd terms must vanish once the symmetry in $\mu \leftrightarrow \nu$ is manifest because of the original symmetry of $\mu \leftrightarrow \nu$ and $x \leftrightarrow -x$.) By dimensional analysis

$$\mathcal{F}_{1,i}^n(q^2,g,M) = \frac{1}{(-q^2)^n}\,f_{1,i}^n\left(\frac{-q^2}{M^2},g\right), \qquad \mathcal{F}_{2,i}^n(q^2,g,M) = \frac{1}{(-q^2)^{n-1}}\,f_{2,i}^n\left(\frac{-q^2}{M^2},g\right). \tag{10.29}$$

The dimensionless f's satisfy eq. (10.26), so they are of the form

$$f_{\alpha,i}^n\left(\frac{-q^2}{M^2},g\right) = \sum_j f_{\alpha,j}^n(1,\bar g)\,\exp\left\{-\int_0^t \gamma_{ji}^n(\bar g)\,\mathrm{d}t'\right\}. \tag{10.30}$$

If we define d^n by

$$\gamma^n(g) = d^n g^2 + \mathrm{O}(g^4)\,, \tag{10.31}$$

then

$$f^n_{\alpha,i} \approx \sum_j f^n_{\alpha,j}(1,\bar{g})\,(1+2bg^2t)^{-d^n_{ji}/2b} \approx \sum_j f^n_{\alpha,j}(1,\bar{g})\left[\frac{\bar{g}}{g}\right]^{d^n_{ji}/b}. \tag{10.32}$$

The forward Compton structure functions T_1 and T_2 are defined in eq. (10.4). So much as before,

$$\begin{aligned} T_1(x,q^2) &= \sum_{n,i}\left(\frac{1}{2x}\right)^n f^n_{1,i}\left(\frac{-q^2}{M^2},g\right)A^n_i + \mathrm{O}\left(\frac{m_\mathrm{p}^2}{q^2}\right) \\ T_2(x,q^2) &= \frac{m_\mathrm{p}}{\nu}\left[\sum_{n,i}\left(\frac{1}{2x}\right)^{n-1} f^n_{2,n}\left(\frac{-q^2}{M^2},g\right)A^n_i + \mathrm{O}\left(\frac{m_\mathrm{p}^2}{q^2}\right)\right] \end{aligned} \tag{10.33}$$

where A^n_i is related to the spin averaged proton matrix element of O^n_i:

$$\langle p|O^{\mu_1\cdots\mu_n}|p\rangle = A^n_i\, p^{\mu_1}\ldots p^{\mu_n} + \ldots \tag{10.34}$$

which is in general unknown. The terms of order m_p^2/q^2, where m_p is the proton mass, come from the contributions of operators of higher twist.

Strictly speaking, with the exact f^n's, the $1/q^2$ corrections arise from operators of higher twist which could, in principle, themselves be studied in detail. In practice, however, a systematic expansion is virtually impossible. It is a straightforward task to exhibit the leading behavior by going to first order in $\bar{g}^2$ or $(\log q^2)^{-1}$ and zeroth order in m. With considerable labor, the next corrections of order $\bar{g}^4$ can be included. But the effects of $m \neq 0$ appear almost everywhere, and, because of logarithmic dependence on m, only the first couple orders in m are computable, even in principle. Furthermore, all orders in $(\log q^2)^{-1}$ are formally as important as the first order in $1/q^2$.

The electroproduction structure functions are related by the optical theorem to the absorbtive parts of T_α:

$$W_\alpha(x,q^2) = (1/\pi)\,\mathrm{Im}\,T_\alpha(x,q^2)\,. \tag{10.35}$$

But in the physical region of x, $0<x<1$, the sums of eq. (10.33) clearly diverge. So what is needed is an appropriate analytic continuation in x of eq. (10.33). It is in unraveling this problem that the x-moments of the W's naturally arise.

Eq. (10.33) suffices to define T_α as a function of complex x which is analytic as $|x| \to \infty$. The W_α have support for $0<x<1$ and x real, so T_α will have branch points at $x = \pm 1$ (T_α is even in x); and we can run the cut between the branch points along the real axis. A particular term in the expansion in eq. (10.33) can be isolated by multiplying by an appropriate power of x to produce a simple pole at $x = \infty$, whose residue is picked out by taking a clockwise circular contour integral along $|x| = R \gg 1$:

$$\frac{1}{2\pi\mathrm{i}}\int_\mathrm{c} \mathrm{d}x\, x^{n-1}\,T_1 = \frac{1}{2^n}\sum_i f^n_{1,i}A^n_i, \qquad \frac{1}{2\pi\mathrm{i}}\int_\mathrm{c} \mathrm{d}x\, x^{n-2}\,\nu T_2 = \frac{m_\mathrm{p}}{2^{n-1}}\sum_i f^n_{2,i}A^n_i\,. \tag{10.36}$$

If the contour is shrunk to go just around the cut, the contour integrals are simply four times the integrals of $\mathrm{Im}\, T_\alpha$ from zero to one above the cut:

$$\int_0^1 xW_1(x, q^2)\, x^{n-2}\, \mathrm{d}x = \frac{1}{2^{n+1}} \sum_i f^n_{1,i}\left(\frac{-q^2}{M^2}, g\right) A^n_i$$
$$\int_0^1 \nu W_2(x, q^2)\, x^{n-2}\, \mathrm{d}x = \frac{m_\mathrm{p}}{2^n} \sum_i f^n_{2,i}\left(\frac{-q^2}{M^2}, g\right) A^n_i\,. \tag{10.37}$$

So the moments have logarithmic dependence on q^2, with the power of the log given by $d^n_{ij}/2b$.

The calculation of b^n_{ij} is straightforward. It involves the graphs of fig. 8. The results are [26, 28, 46]

$$b^n_{AA} = \frac{g^2}{16\pi^2}\, c_1 \left[\tfrac{5}{2} - \frac{4}{n(n-1)} - \frac{4}{(n+1)(n+2)} + 4\sum_{j=2}^{n} \frac{1}{j}\right]$$
$$b^n_{A\psi} = \frac{g^2}{16\pi^2}\, c_2 \left[\frac{8}{n+2} + \frac{16}{n(n+1)(n+2)}\right]$$
$$b^n_{\psi A} = \frac{g^2}{16\pi^2}\, c_3 \left[\frac{1}{n+1} + \frac{2}{n(n-1)}\right]$$
$$b^n_{\psi\psi} = \frac{g^2}{16\pi^2}\, c_3 \left[1 - \frac{2}{n(n+1)} + 4\sum_{j=2}^{n} \frac{1}{j}\right] \tag{10.38}$$

where $c_3 1 = T^a T^a$. (c_3 is simply related to c_2, defined by $\mathrm{tr}\,(T^a T^b)$. If n_A is the number of vector fields and n_ψ is the number of Fermi fields, then $c_3 = c_2\, n_A/n_\psi$.) Subtracting off the appropriate number of γ_i's, we find

$$\gamma^n_{AA} = \frac{g^2}{16\pi^2} \left\{2c_1 \left[\tfrac{1}{3} - \frac{4}{n(n-1)} - \frac{4}{(n+1)(n+2)} + 4\sum_{j=2}^{n} \frac{1}{j}\right] + \tfrac{8}{3}\, c_2\right\}$$
$$\gamma^n_{\psi A} = 2b^n_{A\psi}\,, \quad \gamma^n_{A\psi} = 2b^n_{\psi A}\,, \quad \gamma^n_{\psi\psi} = 2b^n_{\psi\psi} \tag{10.39}$$

and since O^n_a does not mix with the other operators,

$$\gamma^n_{aa} = 2b^n_{\psi\psi}\,, \quad \gamma^n_{ai} = \gamma^n_{ia} = 0 \tag{10.40}$$

for $i = \psi, A$.

γ^n_{ij} could be diagonalized. Its eigenvectors correspond to the linear combinations of O^n_i that are multiplicatively renormalized, and its eigenvalues determine the powers of the possible $\log q^2$ dependence. The eigenvalues of γ^n for $n \geqslant 2$ and even are positive and increase with n, with one exception: γ^2 has one zero eigenvalue whose eigenvector is the energy-momentum tensor, which is not renormalized because it is a conserved current. Hence all moments go to zero as $-q^2 \to \infty$, with successive moments decreasing faster, except for the $n = 2$ moments, having terms which approach constants. The values of those constants are calculable because, as $-q^2 \to \infty$, $f^2_{\alpha,i}(1, \bar{g}) \to f^2_{\alpha,i}(1, 0)$, the free field value, and the appropriate A^2 is related to the proton matrix element of the energy-momentum tensor. In particular, $A^2_{\text{energy-momentum}} = 2$,

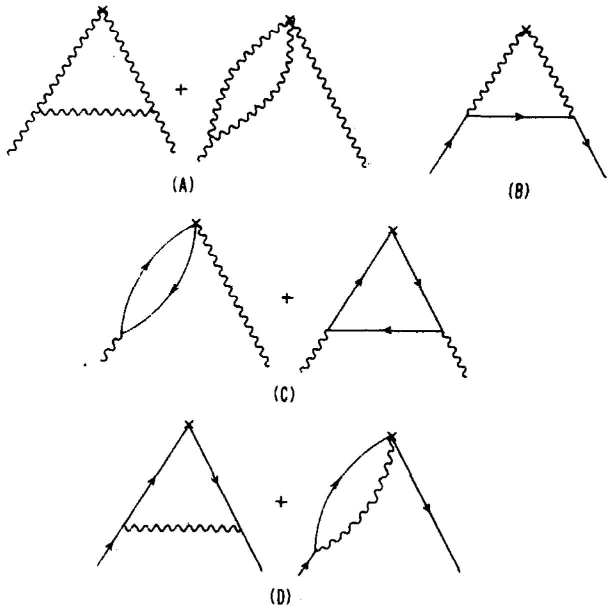


Fig. 8. Graphs entering in the calculation of A) b^n_{AA}, B) $b^n_{\psi A}$, C) $b^n_{A\psi}$, and D) $b^n_{\psi\psi}$.

$$f^2_{1,A}(1,0) = f^2_{2,A}(1,0) = 0 \quad \text{and} \quad f^2_{1,\psi}(1,0) = f^2_{2,\psi}(1,0) = 4\langle Q^2_j\rangle ,$$

where $\langle Q^2_j\rangle$ is the average square quark charge. Diagonalizing eqs. (10.32) and (10.37) and putting in these numers yields

$$\lim_{-q^2\to\infty} \int_0^1 x W_1(x,q^2)\,\mathrm{d}x = \tfrac{1}{2}a$$
$$\lim_{-q^2\to\infty} \int_0^1 \nu W_2(x,q^2)\,\mathrm{d}x = m_{\mathrm{p}} a \tag{10.41}$$

where

$$a = 2\langle Q^2_j\rangle \frac{n_\psi}{2n_A + n_\psi} . \tag{10.42}$$

This is reduced from the free quark result, for which $a = 2\langle Q^2_j\rangle$, reflecting the fact that only a fraction of the total energy-momentum is carried by charged particles.

For a typical quark–gluon model, the eigenvalues of γ^n_{ij} for $n \geqslant 4$ are approximately γ^n_{AA} and $\gamma^n_{\psi\psi}$. Furthermore, $\gamma^n_{AA} \approx c_1/c_3\gamma^n_{\psi\psi}$ and for an SU(N) gauge group with any number of fundamental fermion representations $c_1/c_3 = 2/(1-1/N^2)$.

Asymptotic freedom makes predictions not only for limits as $-q^2 \to \infty$ but also for approximate behavior once $\bar{g}$ is small and mass parameters are negligible [47]. If the approximate scaling observed at SLAC is taken as evidence that (1) $\bar{g}$ is already perturbatively small at $-q^2$ on the order of $(2m_{\mathrm{p}})^2$ and (2) corrections of order $1/q^2$ are negligible, then the results derived so far

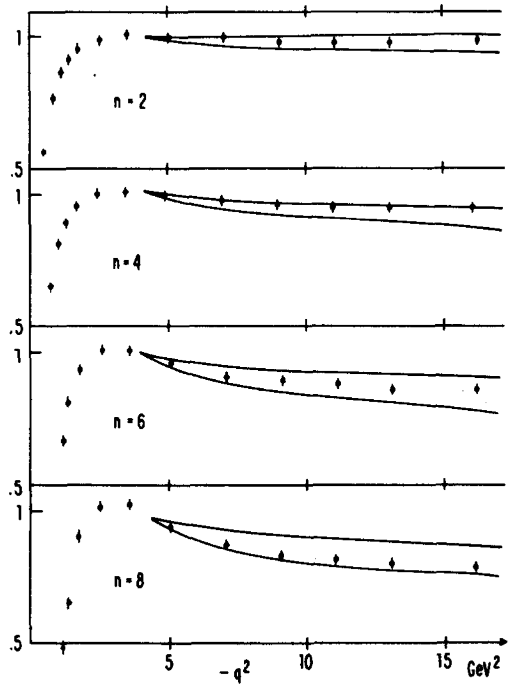


Fig. 9. Plots of $\int_0^1 \nu W_2^{\text{proton}} x^{n-2} \mathrm{d}x$ for $n = 2, 4, 6, 8$; normalized to one at $-q^2 = 4\ \text{GeV}^2$. The solid lines are the maximal and minimal variation for $\bar{g}^2/4\pi|_{-q^2} = 0.1$.

make detailed predictions for the violations of scaling. In fig. 9 the predictions for the first few moments are sketched for the $\mathrm{SU}(3)_{\text{physical}} \times \mathrm{SU}(3)_{\text{color}}$ model. The only free parameter, g, is chosen so that $\bar{g}^2/4\pi = 0.1$ at $-q^2 = 4m_{\mathrm{p}}^2$. Because the A^n are not calculable, the moments are normalized to one at $4m_{\mathrm{p}}^2$, and for each moment, there is a minimum and a maximum possible variation corresponding to the smallest and largest eigenvalues of γ^n. The credibility of the data points [48] may be suspect because they represent a particular choice of the x variable from the whole range of asymptotically equivalent variables. Clearly, over a small range in q^2, the appearance of the data can be drastically altered by using an $x' = x + \mathrm{O}(1/q^2)$. But the moments in x and x' are identical to any order in $\bar{g}$ and zeroth order in $1/q^2$. Fig. 10 includes a prediction for the moments at yet higher q^2's, using the data of fig. 9 to fix the ratios of the A_i^n for each n.

The moment calculation has been emphasized here because it forms a basis for some of the least ambiguous predictions of asymptotic freedom. All the predictions are plagued by the x–x' ambiguity, which can only be resolved with good data over a much larger range in q^2. But a particular difficulty is that, strictly speaking, moments are impossible to measure. For fixed q^2, measurements for $x \to 0$ require the lab energy to go to infinity, and as $x \to 1$, the structure functions themselves vanish rapidly (something like $(1-x)^3$), and are therefore inherently hard to measure. True enough, but perhaps the situation is not so bleak, as long as we have patience. While the $n = 2$ moment may have some sensitivity to $x \to 0$ [49], the $n = 4$ moment certainly does not. And since the structure functions do vanish as $x \to 1$, the signal-to-noise problem there only limits how high a moment can be reasonably studied.

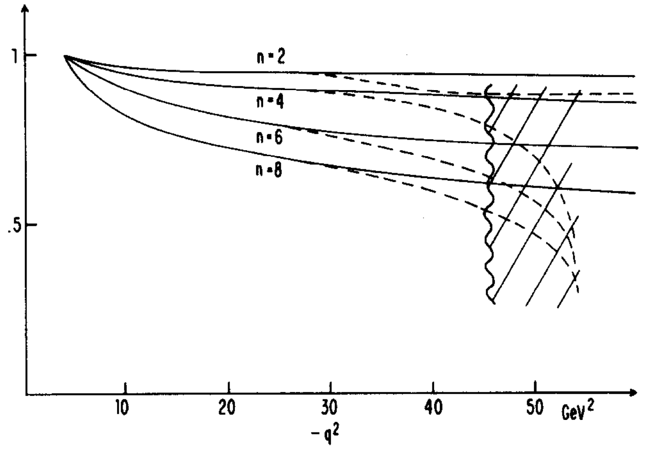


Fig. 10. $\int_0^1 \nu W_2^{\text{proton}} x^{n-2}\, dx$: the solid lines are asymptotic freedom fits to the data of fig. 9; the dashed lines reflect the effects of a particular model with temporarily free scalar fields.

11. Weak currents

The structure functions of deep inelastic neutrino scattering can be discussed analogously [50]. The product of two weak currents is expanded in local operators. The dominant operators on the lightcone are the twist-2 O_a^n, O_ψ^n and O_A^n of section 10. (Note that upon taking spin-averaged matrix elements, the contributions of pseudotensor operators, constructed with an $\epsilon^{\alpha\beta\gamma\delta}$ or a γ_5, are suppressed by $(p\cdot q)^{-3/2}$.) If the currents are described as vector-plus-axial-vector, the vector–vector and axial–axial terms in the current product are associated with structure functions W_1 and W_2 (T_1 and T_2), defined by particular tensor structures, as in eq. (10.1) (eq. (10.4)). The vector–axial cross terms which have the opposite parity, admit a third structure function W_3 (T_3), identified as the coefficient of $\mathrm{i}\epsilon_{\mu\nu\lambda\sigma} p^\lambda q^\sigma$.

An interesting feature of neutrino scattering from the standpoint of asymptotic freedom is that there are several situations in which the structure function moments have a single, leading q^2 dependence for each n – as opposed to the full 3×3 matrix structure. (This is in addition to the fact that as in electroproduction only O_a^n contributes to non-singlet differences, such as $W_i^{\text{proton}} - W_i^{\text{neutron}}$.)

The charged weak currents can be written as

$$J^\mu = J_1^\mu + \mathrm{i} J_2^\mu \tag{11.1}$$

where J_1 and J_2 are hermetian. To analyze the various possibilities, it is convenient to add or subtract the hermetian conjugate current product; that is, consider

$$\mathrm{i}\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}qx} \langle p|\, T(J^\mu(x)\, J^\nu(0)^\dagger \pm J^\mu(x)^\dagger J^\nu(0))\,|p\rangle \tag{11.2}$$

whose absorptive parts give the ν-plus-$\bar\nu$ and the ν-minus-$\bar\nu$ structure functions, respectively. The first observation is that the difference can be expressed

$$J^\mu(x)\,J^\nu(0)^\dagger - J^\mu(x)^\dagger\,J^\nu(0) = -2\mathrm{i}\,(J_1^\mu(x)\,J_2^\nu(0) - J_2^\mu(x)\,J_1^\nu(0))\,. \tag{11.3}$$

But the singlet operators, O_ψ^n and O_A^n, contribute equally to any product $J_i^\mu(x)\,J_j^\nu(0)$, for $i, j = 1, 2$. Hence, O_a^n is the only twist-2 operator of interest in the light cone expansion of eq. (11.3), and the moments of any $W_i^{\nu-\bar\nu}$ $(i = 1, 2, 3)$ off any particular target are governed by γ_{aa}^n (as in eq. (10.37)).

It is a simple exercise in dimensional analysis (analogous to the considerations of eqs. (10.28)–(10.33)) to determine that O_i^n governs the $n-2$ moment of xW_3, just as xW_1 and νW_2. Eq. (11.2) has a symmetry under simultaneous $\mu \to \nu$ and $x \to -x$: it goes into ± itself in the two cases, respectively. This transformation suffices to determine in which cases even or odd spin operators contribute. One finds that, for xW_1 and νW_2, only even n occurs in the $\nu + \bar\nu$ combination (and in general the singlet operators mix and contribute) and odd n occurs for $\nu - \bar\nu$. For xW_3 it is the other way around: even n corresponds to $\nu - \bar\nu$ and odd n to $\nu + \bar\nu$. This has the consequence that $xW_3^{\nu \text{ or } \bar\nu}$ has a single q^2 dependence for its moments, as follows: For odd n, O_ψ^n and O_A^n do not mix because they transform oppositely under charge conjugation. And since O_A^n is not present in the free field expansion of $JJ^\dagger + J^\dagger J$, its effects must be down by at least $\bar g^2$ compared to O_ψ^n. Both O_ψ^n and O_a^n contribute to the leading behavior but their anomalous dimensions are identical. For even n, only O_a^n is present, but the functional dependence of γ_{aa}^n on n is the same for n even or odd. In summary,

$$\int_0^1 x\,W_3(x, q^2)\,x^{n-2}\,\mathrm{d}x \approx (\bar g)^{d^n/b} B^n \tag{11.4}$$

where

$$d^n = \frac{c_3}{8\pi^2}\left[1 - \frac{2}{n(n+1)} + 4\sum_{j=2}^{n} \frac{1}{j}\right] \tag{11.5}$$

and B^n are constants, into which various predicted and unpredictable factors have been absorbed.

12. Sum rules

Sum rules which follow from exact symmetries of the interactions are as true as ever. For example, from current algebra, the Adler sum rule [51]

$$\int_0^1 \frac{\mathrm{d}x}{x}\,(\nu W_2^{\nu\mathrm{n}}(x, q^2) - \nu W_2^{\nu\mathrm{p}}(x, q^2)) = 2 \tag{12.1}$$

is presumably true for all q^2 (assuming the current commutators are not altered in the interacting theory).

There are a host of sum rules which follows from free field (or "leading" lightcone) behavior [52]. These can be established, moment for moment, as asymptotic statements, true in the $-q^2 \to \infty$ limit but typically receive corrections of order $\bar g^2$. This is because the moments, eq. (10.37), are proportional to $f_{\alpha,i}^n(-q^2/M^2, g)$, which in turn (from eq. (10.30)) are proportional to $f_{\alpha,i}^n(1, \bar g)$. The latter can be expanded in $\bar g$:

$$f_{\alpha,i}^n(1, \bar g) = f_{\alpha,i}^n(1, 0) + \dots\,. \tag{12.2}$$

So to lowest order in $\bar{g}$, the coefficient functions have the Lorentz, SU(3), etc. properties of free field theory. Hence, combinations chosen to vanish in free field theory will now vanish, moment by moment, in zeroth order in $\bar{g}^2$. An example of this type is the Callan–Gross sum rule, which becomes

$$\int_0^1 (2m_p x W_1 - \nu W_2)\, x^{n-2}\, dx = m_p\, \bar{g}^2 \sum_{i,j} h_j^n \left[\frac{\bar{g}}{g}\right]^{d_{ji}^n/b} A_i^n \tag{12.3}$$

where the numbers h_j^n can be determined from the second order forward Compton amplitude. This expression still contains the uncalculable hadron matrix elements A_i^n. However, the same constants appear in the moments of νW_2. But the presence of the three operators (the nontrivial sum over i and j) makes it difficult to eliminate the ambiguity of the A_i^n. Only for structure functions with a single q^2 dependence of its moments, such as p–n or $\nu - \bar{\nu}$, we can write

$$\int_0^1 (2m_p x W_1 - \nu W_2)\, x^{n-2}\, dx \Big/ \int_0^1 \nu W_2\, x^{n-2}\, dx = h^n \bar{g}^2 . \tag{12.4}$$

These h^n have been computed to be [53]

$$h^n = \frac{1}{16\pi^2} \frac{4c_3}{n+1} . \tag{12.5}$$

13. Temporary moments

The preceding discussions carry over for theories that are temporarily free for the domain in q^2 in which all effective couplings are small. There are some obvious distinctions, however. Nothing is known about $-q^2 \to \infty$. There is the choice in any particular model whether the scalar fields are electrically charged, although the observed smallness of the longitudinal structure functions suggests that they are not. Also, there are now scalar operators of twist-2 that mix with the vector operators to order $\bar{g}^2$, but their effects are not unlike those of the fermion operators. (They are described in detail in ref. [33].)

There are also contributions to the γ^n matrix that are second order in the scalar self-couplings. At any given q^2, these may or may not be comparable to $\bar{g}^2$. But since $\bar{g}^2$ decreases with t and at least one of the $\bar{\lambda}$'s eventually becomes large, these $\bar{\lambda}_i \bar{\lambda}_j$ terms will dominate for sufficiently large q^2. When some $\bar{\lambda}_i \bar{\lambda}_j$ is no longer small compared to unity, the whole perturbation expansion is no longer valid. In detail, if the anomalous dimension of the scalar field is of the form

$$\gamma_\varphi = -\sum_{i,j} d_{ij} \lambda_i \lambda_j + \ldots \tag{13.1}$$

as determined from the graph of fig. 11A, then the order $\lambda_i \lambda_j$ part of $\gamma_{\varphi\varphi}^n$ (from fig. 11B) is

$$\gamma_{\varphi\varphi}^n = \left[2 - \frac{12}{n(n+1)}\right] \sum_{i,j} d_{ij} \lambda_i \lambda_j + \ldots . \tag{13.2}$$

As the $\bar{\lambda}_i$ increase in magnitude, the structure function moments will decrease at a greater and greater rate as governed by

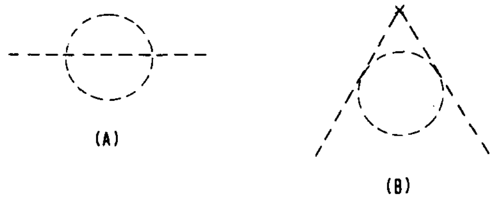


Fig. 11. The $O(\lambda^2)$ contributions to A) γ_φ and B) $\gamma^n_{\varphi\varphi}$.

$$\exp\left\{-\int_0^t \gamma^n_{\varphi\varphi}(\bar{\lambda})\, \mathrm{d}t'\right\}.$$

When, in t, this factor dominates depends totally on the initial λ_i or rather on the particular trajectory. Fig. 10 suggests the effects of a possible trajectory. The dashed lines reflect the scalar meson contributions. Note that for this trajectory, for $-q^2 \gtrsim 45\ \mathrm{GeV}^2$, the effective couplings have gotten so large that the perturbative expressions are no longer trustworthy.

14. Inverting the moments

Moments are essentially Mellin transforms, and, by continuing the index n to complex values, they can be formally inverted [15], as follows: if

$$M_n(q^2) = \int_0^1 \mathrm{d}x\, x^{n-2} F(x, q^2) \tag{14.1}$$

then

$$F(x, q^2) = \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}n\, x^{-n-1} M_n(q^2) \tag{14.2}$$

where the contour goes to the right of any singularities of M_n. This is easily verified by inserting eq. (14.2) into the right-hand side of eq. (14.1), doing the x integral first, and then completing the n contour in the right half-plane. If we require regularity as $n \to \infty$, the continuation from the exact positive, even moments is unique. But the inversion of asymptotic approximations of the moments via the continuation in n is certainly suspect. Although we may believe the leading behavior in q^2 that has been computed for each moment, the assemblage of this series of moments does not necessarily indicate the analytic properties in n which dominate the behavior of the structure functions as functions of x for large q^2. While it may be a straightforward procedure to continue the leading results of perturbation theory,‡ it is unclear whether it is in the last bit reasonable.

‡ The sum $\Sigma_{j=1}^n 1/j$ is the digamma function, up to a constant, and has the following useful integral representation:

$$\sum_{j=1} \frac{1}{j} = \int_0^1 \frac{\mathrm{d}x}{(1-x)} (1 - x^n) = \int_0^\infty \mathrm{e}^{-z}\, \mathrm{d}z\, (1 - \mathrm{e}^{-z})^{-1} (1 - \mathrm{e}^{-nz})$$

which defines the analytic continuation.

Once we accept the philosophy of simply following our noses, the key observation for the inversion program is that, while we do not know how to compute the A^n from dynamics, they can be inferred from a knowledge of the structure functions at a given four-momentum, q_0^2, if q_0^2 corresponds to $\bar{g}$ small [54]. Strictly speaking, this is only true if each moment is governed by a single operator and, hence, has a single A^n. In the matrix case, input data at two or three reference momenta would be necessary, and the whole analysis would be far more complicated. So let us assume that the moments of the structure function in question, some $F(x, q^2)$, have a single dominant dependence for each n.

Any proton–neutron or neutrino–antineutrino structure function, say νW_2, satisfies this condition, and so does xW_3. In addition, it has been argued that any structure function in the two extreme x limits is dominated by a single set of operators. The limiting behavior as $x \to 1$ depends only on the q^2 behavior of the moments as $n \to \infty$. Since all high moments decrease with q^2, for large n and large q^2 the dominant behavior comes from the moments that decrease the slowest [55, 56]. These are governed by $\gamma^n_{\psi\psi}$. In the limit $x \to 0$, the leading behavior is dominated by the nearest singularity in n. One of the singlet eigenvalues of γ^n_{ij} has a pole at $n = 1$, which means an essential singularity in M_n at $n = 1$ for any F receiving singlet contributions [49].

The inversion can be effected as follows. It is convenient to introduce the variables Λ^2 and s: for $\bar{g}^2$ small, let

$$\bar{g}^2 = \{b \log(-q^2/\Lambda^2)\}^{-1}; \tag{14.3}$$

so that

$$\Lambda^2 = M^2 \exp(-1/bg^2). \tag{14.4}$$

Λ^2 determines the momentum scale on which $\bar{g}^2$ becomes large. Let q_0^2 be the reference momentum at which some structure function, $F(x, q^2)$, is presumed to be known. (The scheme is to calculate the moments of $F(x, q_0^2)$, extrapolate them in q^2, and then reconstruct F.) Define s by

$$s = \log\left(\frac{\log(-q^2/\Lambda^2)}{\log(-q_0^2/\Lambda^2)}\right). \tag{14.5}$$

Then the moments are of the form

$$M_n(s) = \exp(-a_n s)\, M_n(s=0)\, (1 + \mathrm{O}(\bar{g}^2)). \tag{1.46}$$

Eqs. (14.1), (14.2) and (1.46) can be combined to give (with a change of integration variable) [57]

$$F(x, s) = \int_x^1 \frac{\mathrm{d}x'}{x'}\, F\left(\frac{x}{x'}, s = 0\right) T(x', s) \tag{14.7}$$

where

$$T(x', s) = \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}n\, \exp(-a_n s)\, x'^{-n-1}. \tag{14.8}$$

In section 10, a_n was determined from $\gamma^n_{\psi\psi}$ to be

$$a_n = G\left(-3 - \frac{2}{n} + \frac{2}{n+1} + 4\sum_{j=1}^{n} \frac{1}{j}\right) \tag{14.9}$$

where $G = 6c_3/(11c_1 - 4c_2)$ (= 4/25 in the SU(4) × SU(3) model).

T has not been evaluated exactly. Approximations can be made which apply to the two extreme x limits (as shall be done subsequently). However, eq. (14.6) implies that

$$\mathrm{d}M_n(s)/\mathrm{d}s = -a_n\, M_n(s)\,. \tag{14.10}$$

Eqs. (14.1), (14.2) and (14.10) yield a differential equation for the violations of scaling:

$$\frac{\mathrm{d}F(x,s)}{\mathrm{d}s} = \int_x^1 \frac{\mathrm{d}x'}{x'}\, F\left(\frac{x'}{x}, s\right) \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}n\,(-a_n)\, x'^{-n-1}\,. \tag{14.11}$$

The relevant integrals over n can be done explicitly to give [58]

$$\frac{\mathrm{d}F(x,s)}{\mathrm{d}s} = G\left\{(3+4\log(1-x))\, F(x,s) + \int_x^1 \mathrm{d}x'(2-2x')\, F\left(\frac{x}{x'}, s\right) + 4\int_x^1 \frac{\mathrm{d}x'}{1-x'}\left[x' F\left(\frac{x}{x'}, s\right) - F(x,s)\right]\right\}\,. \tag{14.12}$$

Fig. 12 shows the results of a numerical integration [59] of eq. (14.12) with $G = 4/25$ for $F(x,0) = 4x^{1/2}(1-x)^3$. (This function is a good fit to $\nu W_2^{\mathrm{p-n}}$ and is consistent with xW_3.) If $\Lambda^2 = 1\ \mathrm{GeV}^2$ and $-q_0^2 = 15\ \mathrm{GeV}^2$, then the values of s plotted, $s = -0.69, -4.3, 0.0, 0.55$ and 2.2 correspont to $-q^2$ of 4, 6, 15, 110 and $4 \times 10^{10}\ \mathrm{GeV}^2$, respectively. Note that Λ^2 must be determined experimentally before a value of s can be associated with a particular q^2. Note also that a serious confrontation of these predictions with experiment will require a verification that the $O(m^2/q^2)$ corrections are indeed unimportant [59].

Explicit forms for $F(x,s)$ can be derived in the two extreme x domains. For $x \to 0$, a_n may be taken to be of the form

$$a_n \approx -A/(n-1) + B\,. \tag{14.13}$$

Near $n = 1$, approximate M_n by

$$M_n(s=0) \approx \mathcal{M}(n-1)\, K^{A/(n-1)} \tag{14.14}$$

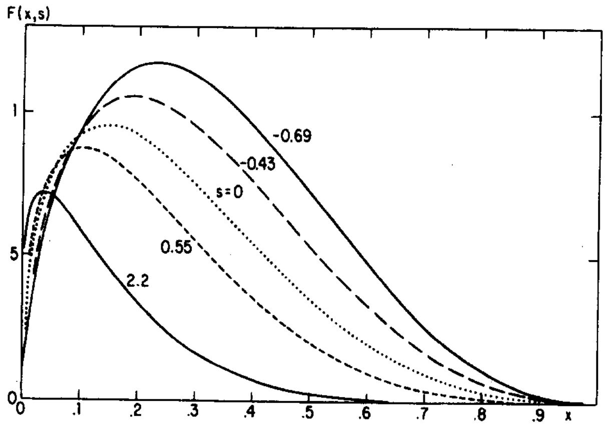


Fig. 12. The evolution in s of $F(x, s=0) = 4x^{1/2}(1-x)^3$.

where K is some constant and $\mathcal{M}$ is some meromorphic function, which is to say that the dominant of M_n at any t is the essential singularity at $n = 1$, and $\mathcal{M}$ reflects the possible presence of arbitrary poles, cuts and zeros. Using the approximations of eqs. (14.11) and (14.12), eq. (14.2) can be estimated by the method of steepest descent for large $\log x$:

$$F(x,s) \approx \frac{1}{2\sqrt{\pi}}\, e^{-Bs}\, \mathcal{M}\left(\sqrt{\frac{z}{-\log x}}\right) \left(\frac{z}{-\log^3 x}\right)^{1/4} \exp\{2\sqrt{-z \log x}\} \tag{14.15}$$

where $z = A(s + \log K)$. For fixed q^2 in terms of the variable ν, this represents a growth as $\nu \to \infty$ like $\exp\sqrt{\log \nu}$. ‡

$x \to 1$ is determined by the high n moments, for which

$$a_n \approx A + B \log n\,. \tag{14.16}$$

A simple way to get a handle on the $F(x \approx 1, s)$ is to approximate F by

$$F(x,s) \approx a(s)\,(1-x)^{\alpha(s)} \tag{14.17}$$

since F is known to vanish as $x \to 1$. In this case

$$M_n(s) = \int_0^1 a(s)\,(1-x)^{\alpha(s)}\, x^{n-2}\, dx = a(s)\, \frac{(n-2)!\,\Gamma(\alpha(s)+1)}{\Gamma(n+\alpha(s))}\,. \tag{14.18}$$

Using Stirling's formula and eqs. (14.4) and (14.14), we can compare large n moments for $s = 0$ and s and determine that

$$\alpha(s) = \alpha(0) + Bs$$

$$a(s) = e^{-As}\, \Gamma(\alpha(0)+1)/\Gamma(\alpha(s)+1)\,. \tag{14.19}$$

It has been suggested that this result is testable not only by precision measurements of the threshold behavior of inelastic structure functions but also via the q^2 dependence of elastic, electromagnetic form factors [55, 56]. The two phenomena are directly connected under the hypotheses of local duality of Bloom and Gilman [60].

15. e^+e^- annihilation

It is commonly assumed that at high energies the e^+e^- total cross section, $\sigma_{e^+e^-}$, is dominated by lowest order electromagnetic, one photon processes. With the observation of a roughly constant $\sigma_{e^+e^-}$ [61], even this has come under close scrutiny again. (One possible consistency check is the isotopic spin predictions for hadron distributions.) But taking this for granted, a scaling prediction ($\sigma_{e^+e^-} \sim 1/s$, where s is the four-momentum squared, q^2, of the virtual photon) can be derived with ease [62, 63].

$\sigma_{e^+e^-}$ is proportional to the *vacuum* expectation of the product of two currents and, hence, the absorbtive part of the two current correlation function. Since there is only one momentum, q^2,

‡ It is frivolous and irresponsible (but so irresistable) to remark that perhaps asymptotic freedom accounts for the rising proton–proton total cross section. The tenuous connection is made by the suggestion that both processes are dominated by pomeron exchange.

this can be thought of as a short distance problem, $x \to 0$ or $q \to \infty$. The operator of lowest dimension contributing to the vacuum to vacuum amplitude is the operator 1. Naturally, 1 has no anomalous dimension, so the leading behavior scales canonically. (This behavior, then is common to any theory with free field or "almost" free field short distance singularities.) As we shall see, the zeroth order coefficient function of 1 is proportional to $\Sigma_j Q_j$.

It is the order $\bar{g}^2$ corrections to the coefficient function that will now be computed. The most direct derivation begins by noting that the two current correlation function is also the inverse photon propagator, $D^{-1}(q^2)$, which can be approximated in "improved" perturbation theory. The function that must be computed is γ_{photon}, to first order in e^2 and g^2. Just as for γ_A, γ_{photon} can be inferred from the perturbation expansion of $D^{-1}(q^2)$. Fig. 13A represents the order e^2

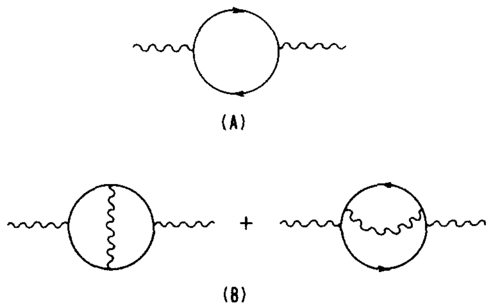


Fig. 13. The photon propagator A) to zeroth order in $\bar{g}$ and B) to $O(\bar{g}^2)$.

contribution of charged quark loops, and fig. 13B gives the lowest order strong corrections to the loops. These diagrams differ from the electromagnetic vacuum polarization only by traces over the coupling matrices. One finds γ_{photon} to be of the form

$$\gamma_{\text{photon}} = -ae^2 \sum_j Q_j^2 (1 + Bg^2) + \dots . \tag{15.1}$$

The improved D^{-1} is

$$D^{-1}(q^2) = q^2 \exp\left\{2 \int_0^t \gamma_{\text{photon}}(\bar{e}, \bar{g})\, \mathrm{d}t'\right\}. \tag{15.2}$$

Since $\beta_e > 0$, $\bar{e}$ is an increasing function, but for the momentum range of interest $\bar{e}$ is well approximated by e. Furthermore, since we are working to first order in e^2, the exponential of eq. (15.2) can be expanded to be

$$D^{-1}(q^2) = q^2 \left(1 - 2ae^2 \sum_j Q_j^2 \int_0^{\frac{1}{2}\log(-q^2/M^2)} (1 + B\bar{g}^2)\, \mathrm{d}t'\right). \tag{15.3}$$

If q^2 is taken timelike, we can derive the cross section from the absorbtive part:

$$\sigma_{e^+e^- \to \text{hadrons}} = \frac{4\pi}{3q^2} \left(\frac{e^2}{4\pi}\right)^2 \sum_j Q_j^2 \,[1 + B\text{“}\bar{g}^2\text{”}] \tag{15.4}$$

where $B = 3\,c_3/16\pi^2$ and

$$\text{“}\bar{g}^2\text{”} = \frac{g^2}{1 + bg^2 \log(q^2/M^2)} . \tag{15.5}$$

This result is hard to circumvent. The inclusion of scalar gluons and Yukawa couplings does not alter the result that the scaling limit is approached from above [64]. But more importantly, the corrections to the scaling limit are of the order of the effective couplings, whose smallness we inferred from electroproduction scaling. There appear to be only two alternatives: either asymptotic freedom (or any other picture with quasi-free constituents) has nothing to do with either of these processes, at least at present momenta, or the analysis of timelike q^2 is indeed more complicated than spacelike $-q^2$ – a possibility that was raised in section 5.

A mild version of the first alternative is that scaling is only a temporary phenomenon, valid for $q^2 \lesssim 15\ \text{GeV}^2$, and the effective couplings become important again at shorter distances [33, 65]. In contrast to the structure function moments, which drop off as the effective couplings grow, $\sigma_{e^+e^-}$ or $D^{-1}(q^2)$ could be enhanced by stronger couplings. The difference arises because the moments are determined by Wilson coefficient functions whose anomalous dimensions are negative; while the leading operator for e^+e^- annihilation has no anomalous dimension, and the interaction corrections to its coefficient function may be positive.

If we wish to maintain the validity of asymptotic freedom, it is a matter of taste where one puts the blame. I prefer, for consistency, to believe that the sizes of the effective couplings are determined by the same scale, whether timelike or spacelike, and it is purely non-perturbative, strong interaction effects in the Green's functions, themselves, which die out in the spacelike region but persist for timelike momenta. (See section 5 for a further discussion.) Examples of such behavior are easy to construct. If the photon propagator contains terms proportional to

$$\exp\{-a\sqrt{-q^2 + b}\} ,$$

those terms die exponentially for $q^2 < b$, while their imaginary parts oscillate indefinitely for $q^2 > b$. This example is extreme; it suggests that in the roughly constant $\sigma_{e^+e^-}$ we may only be observing the first quarter wave. But such a possibility cannot be ruled out on any general grounds, although it cannot occur in perturbation theory. (Of course, that is just the point.) We could imagine functions (damped oscillations) which die in both regions, but at different rates, in which case scaling behavior may eventually set in for $q^2 > 0$. In these examples, eq. (15.4) is, in the long run, true in the mean, but I would be as shocked as anyone to see the subsequent drop that would be needed to compensate for the current behavior.

16. Corrections to hadronic symmetries

The development of renormalizable, unified theories of weak and electromagnetic interactions has prompted a renewed interest in current algebra and the origin and nature of approximate hadronic symmetries.‡ Asymptotic freedom provides motivation for taking seriously a particular model or class of models for the strong interactions. With specific candidates for *the* Lagrangian

‡ There is already a wealth of literature on the subject. The results discussed here follows ref. [42] closely. Other references may be found in ref. [42] and any review of recent progress in weak and electromagnetic theory.

of elementary particle interactions, a wealth of questions can be investigated. As yet, none of the results have been so spectacular as to gain universal acceptance for one scheme or another, but there is a certain elegance to the colored quark–gluon models. I shall briefly describe some of the outstanding features and then discuss one of the results that follows specifically from asymptotic freedom.

The theory begins with a quark–gluon model, with a color gauge group $\mathcal{C}$. (SU(3) is suggested for $\mathcal{C}$ on the basis of the hadron spectrum [35].) The quarks come in several irreducible multiplets which, in the absence on any other interactions, would be identical. Let $\mathcal{S}$ be the group of symmetry transformations among the multiplets. Hence the generators of $\mathcal{S}$ commute with those of $\mathcal{C}$; the colored gluons are $\mathcal{S}$ singlets; and, the fermions can be placed in a matrix, where $\mathcal{C}$ transformations mix the rows and $\mathcal{S}$ mixes the columns. $\mathcal{S}$ includes the observed hadronic symmetries (presumably isotopic spin, SU(3), charm, etc.); $\mathcal{S}$ also includes $\mathcal{W}$, the gauge group of the weak and electromagnetic interactions. The Lagrangian is completed by adding the weak and electromagnetic gauge bosons, whose couplings are of order e, and the leptons and weak, Higgs scalars, which, like the photon and weak (W) bosons, are $\mathcal{C}$ neutral.

Even to zeroth order in e^2, some symmetries of $\mathcal{S}$ can be broken by large vacuum expectation values of weakly coupled scalar fields. The net effect, to $O(e^0)$ is to alter the quark mass matrix, giving various color multiplets different masses. The color interactions cannot alter the $\mathcal{S}$ charges of a state. So any quantum number expressible in terms of a number of quarks-minus-antiquarks (summed over color) is conserved by the color dynamics; in particular, that includes charge, baryon number, strangeness and charm. The quark mass matrix may possess some symmetries such as isotopic spin; this, too, would then be a symmetry of the total theory to $O(e^0)$. If $\mathcal{C}$ is non-chiral, then parity will also be conserved.

If the distinction is made between $O(e^2)$ effects and truly weak effects of $O(e^2/m_W^2)$, where m_W is the W-boson mass (or $O(e^2 m_p^2/m_W^2)$ where we expect $m_p^2/m_W^2 \sim 10^{-3}$), then the $O(e^2)$ effects can be characterized completely and computed explicitly as corrections to the strong interaction ($O(e^0)$) Lagrangian. The e^2 corrections to any matrix element are of the form

$$\delta\langle A|B\rangle = \int \kappa^3\, \mathrm{d}\kappa\, \mathcal{F}^{AB}_{\alpha\beta}(\kappa) \left[\frac{1}{\kappa^2 - \mu_W^2}\right]_{\alpha\beta} \tag{16.1}$$

where

$$\mathcal{F}^{AB}_{\alpha\beta}(\kappa) = \int \mathrm{d}\Omega_k\, \mathrm{e}^{ikx}\, \mathrm{d}^4x\, \langle A|T(J^\alpha_\mu(x)\, J^{\beta\mu}(0))|B\rangle \tag{16.2}$$

and J^α is a $\mathcal{W}$ current and μ_W is the $\mathcal{W}$ gauge boson mass matrix‡. There are two ways in which $\delta\langle A|B\rangle$ may not be suppressed by a factor of m_W^{-2}. First, one of the $\mathcal{W}$ mesons is the photon, which corresponds to a zero eigenvalue of $(\mu_W^2)_{\alpha\beta}$. Its effects can be conveniently separated as follows:

$$\frac{1}{k^2} = \frac{1}{k^2 - \Lambda^2} + \frac{-\Lambda^2}{k^2(k^2 - \Lambda^2)}, \tag{16.3}$$

μ_W^2 can be replaced by a $\mu_W'^2$ which is identical except that the photon is given a mass Λ^2 on the

‡ There are also "tadpole" terms, or corrections to the scalar field vacuum expectation values, which simply induce corrections to the quark mass matrix.

order of m_W^2, and a purely photonic term which is effectively cut off for $\kappa^2 \gtrsim \Lambda^2$. (Each part will have a Λ^2 dependence, but there will be a cancellation.) Note that this second term conserves parity, strangeness, and charm.

The only way that the $\mu_W'^2$ term may not be suppressed by m_W^{-2} is if $\mathcal{F}_{\alpha\beta}^{AB}(\kappa)$ does not vanish sufficiently rapidly as $\kappa \to \infty$, in particular if $\mathcal{F}$ vanishes no faster than $1/\kappa^2$. Power counting determines that in the direction averaged short distance expansion of the two currents only Lorentz scalar operators of dimension four or less can be significant. Color gluon operators are irrelevant because they must be $\mathcal{W}$ singlets. The quark operators, $\bar{\psi}\psi$ and $\bar{\psi}\gamma^\mu D_\mu \psi$, are related by the field equations; so we need consider only one of them. $\bar{\psi}\psi$ is simplest. The coefficient function $F_{ij}^{\alpha\beta}(\kappa)$ of $\bar{\psi}_i \psi_j$ is of the form

$$F_{ij}^{\alpha\beta}(\kappa, g, m) = F_{ij}^{\alpha\beta}(\kappa_0, \bar{g}, \bar{m}) \left(\frac{\kappa}{\kappa_0}\right)^{-1} \exp\left\{-\int_{\kappa_0}^{\kappa} \gamma_{\bar{\psi}\psi}(\bar{g}) \frac{\mathrm{d}\kappa'}{\kappa'}\right\} \tag{16.4}$$

where κ_0 is a normalization point. But consider now the m dependence of $F_{ij}^{\alpha\beta}(\kappa, g, m)$, where m is the $O(e^0)$ quark mass matrix. If m were zero, chiral invariance would imply $F = 0$. If $m \neq 0$, the chiral transformations of the theory imply that F is odd in m. And it can be argued that in any order of perturbation theory, as $m \to 0$, $F \to m\, \partial F/\partial m|_{m=0}$. So

$$F(\kappa, g, m) \approx \bar{m} \frac{\partial}{\partial m} F(\kappa_0, \bar{g}, m)|_{m=0} \left(\frac{\kappa}{\kappa_0}\right)^{-1} \exp\left\{-\int_{\kappa_0}^{\kappa} \gamma_{\bar{\psi}\psi}(\bar{g}) \frac{\mathrm{d}\kappa'}{\kappa'}\right\}. \tag{16.5}$$

But by construction (see section 4)

$$\bar{m}(\kappa) = m \exp\left\{\int_{\kappa_0}^{\kappa} [-1 + \gamma_{\bar{\psi}\psi}(\bar{g})] \frac{\mathrm{d}\kappa'}{\kappa'}\right\}. \tag{16.6}$$

So upon inserting eq. (16.6) into eq. (16.5) the $\gamma_{\bar{\psi}\psi}$'s cancel ‡. Consequently, eq. (16.5) has a simple limit as $\kappa \to \infty$; in particular we can take $\bar{g} \to 0$:

$$F(\kappa, g, m) \to m \frac{\partial}{\partial m} F(\kappa_0, 0, m)|_{m=0} \left(\frac{\kappa}{\kappa_0}\right)^{-2}. \tag{16.7}$$

But this implies that the strong interactions have absolutely no effect on the $O(e^2)$ corrections because we can set $g = 0$ in eq. (16.1), where only the $\kappa \to \infty$ domain can possibly contribute to $O(e^2)$ as opposed to $O(e^2/m_W^2)$.

Hence, the electromagnetic effects are computed with a cut-off and the non-electromagnetic $O(e^2)$ effects can be computed in perturbation theory, totally ignoring the color interactions. The latter effects can be expressed as corrections to the Lagrangian – in particular, to the quark mass matrix – because only operators of dimension four or less were important in the operator product expansion. (Note, it can be shown that the divergent part of eq. (16.1) has the symmetries of the $O(e^2)$ Lagrangian and is, therefore, an unobservable renormalization.) Current algebra techniques must now be applied, using the shift in the quark mass matrix as a symmetry breaking perturbation.

‡ Had we dealt only with the operator $\bar{\psi}\gamma^\mu D_\mu \psi$, an explicit calculation would have given $\gamma_{\bar{\psi}\gamma^\mu D_\mu \psi} = 0$ to lowest order in g^2 [29].

17. Unified field theories

This whole picture can be turned around and regarded as an explanation of why strong interactions are strong. It is apparently not a question of inherent strength, since there is no invariant meaning to the size of a non-Abelian gauge coupling. But rather, it depends on the length or momentum scale at which the effective coupling has a particular magnitude. This suggests the following elegant possibility [66]: perhaps the strong, electromagnetic and weak interactions are described by a single, simple gauge group $\mathcal{G}$, with a single gauge coupling g. $\mathcal{G}$ suffers a first level of spontaneous breakdown – due to vacuum expectation values of explicit scalar fields – and as a result some of the gauge bosons acquire masses, but a subgroup remains unbroken: $\mathcal{C} \times \mathcal{W}$, where $\mathcal{C}$ is identified as the color gauge group and $\mathcal{W}$ is the weak and electromagnetic gauge group. Subsequently, $\mathcal{W}$ is itself spontaneously broken, leaving massive W-bosons and the photon. We imagine that $\mathcal{C}$ is unbroken; so the color interactions are strong in the sense that colored fields are strongly coupled at long distances ‡.

All particles are fit into $\mathcal{G}$ multiplets, and $\mathcal{G}$ transformations connect leptons and hadrons. Proton decay (a phenomenon typical of these unified theories) is mediated by the vector mesons that acquired mass when $\mathcal{G} \to \mathcal{C} \times \mathcal{W}$. So it is hypothesized that these masses (and the relevant vacuum expectation values) are enormous, to agree with the observed baryon conservation.

Georgi and Glashow [66] offer a particular model. They begin by assuming $\mathcal{C}$ is an SU(3) and $\mathcal{W}$ is an SU(2) × U(1). A long chain of plausible hypotheses leads them to a unique $\mathcal{G}$, SU(5). The particular embedding of SU(2) × U(1) in the simple SU(5) gives a determination of the Weinberg angle.

There are several pieces of the unified theory that are known: the fine structure constant $e^2/4\pi$, the Fermi weak coupling $G \sim e^2/m_W^2$, the color effective coupling $\bar{g}_c$ at SLAC momentum transfers ‡, and the measured limit on the stability of the proton. These numbers must be related, somehow. Recently, a tentative approach has been suggested [67]. It is proposed that we follow the evolution of the effective couplings for the various subgroups of $\mathcal{G}$. If μ is a mass typical of the superheavy mesons that mediate proton decay, a crude estimate indicates that $\mu > 10^{15}$ GeV (based on the present experimental lower limit of 10^{30} years for the life of the proton [68]. Since μ, m_W and m_p are decades apart, a direct application of perturbation theory would be suspect. However, all of the *effective* couplings are small in the range of interest (i.e. down to $O(m_p^2)$). So perhaps a renormalization group analysis can be applied.

But the theory contains masses which are certainly not negligible. So we invoke the following important theorem [69]. In a renormalizable theory ‡, if the fields can be categorized as light and heavy, then for momenta comparable to the light masses the heavy fields are negligible in the following sense: there exists an effective Lagrangian, obtained by deleting the heavy fields from the

‡ It is perhaps unnecessary to presume the ultimate disposition of $\mathcal{C}$. $\mathcal{C}$ may have a natural scale set by its own spontaneous breakdown. But it would have to be a scale somewhat lighter/longer than that set by the nucleon, to make sense out of SLAC scaling, and a scale where the effective color coupling has gotten larger.

‡ It is premature to take a particular determination of g_i too seriously. If SLAC scaling is an asymptotic phenomenon, then there is certainly an upper bound: at $-q^2 \sim 4\ \mathrm{GeV}^2$, the effective expansion parameter is certainly less than one. But it is not clear whether that parameter is $\bar{g}_c^2/4\pi$, $\bar{g}_c^2/4\pi^2$, or something else. The dubious fig. 9 suggests $\bar{g}_c^2/4\pi|_{-q^2=4\,\mathrm{GeV}^2} \approx 0.1$. Eq. (12.3) offers a different determination: somewhing like $\bar{g}_c^2/4\pi^2|_{-q^2=4\,\mathrm{GeV}^2} \sim 0.2$. One may hope that the value is more like the larger so that the violations of scaling discussed in section 14 will be observable.

‡ There are certain restrictions on the allowed couplings of heavy scalars.

original Lagrangian; however, the parameters of the theory will receive finite renormalizations that are functions of mass ratios and all the original couplings.

If M characterizes the momentum scale of interest, for $M \ll \mu$ the effective gauge couplings of $\mathcal{C}$ and $\mathcal{W}$ evolve independently, as determined by their effective Lagrangians. At $M \sim m_p$, the couplings differ by an order of magnitude, but for $M > \mu$ they must all be identical. Therefore μ can be estimated by requiring that all effective couplings are equal at $M = \mu$. The details of the SU(5) model give $\mu = 10^{18\pm 1}$ GeV. (The simple renormalization group analysis is suspect when masses are not negligible, in particular when $M \sim O(\mu)$. However, in that domain, straightforward perturbation theory can be applied because $\log M/\mu$ is not large, or rather that $\bar{g}(\mu) \log M/\mu$ is still small. And in perturbation theory the change in effective couplings is a higher order effect. So to a first approximation, the couplings just do not change for $M \sim \mu$.)

18. Conclusion

The discovery of theories whose effective couplings decrease at shorter distances has opened up new possibilities for the application of field theoretic techniques to the problems of strong interaction dynamics. It is suggested that, while the interactions are strong on the scale set by some physical dimensions, say the mass of the pion, at shorter distances the interactions become less important and can be adequately described by the appropriate perturbation expansion in the effective couplings. An immediate consequence of this picture is approximate scaling in short distance phenomena.

There are many other plausible explanations for the observed scaling. Even within the context of the renormalization group analysis there are some alternatives. First, there is the old idea of non-zero fixed points. It is difficult to say anything precise, one way or the other, but that in itself is, perhaps, a defect. Another possibility is really very close to the ideas that have been developed here. Perhaps scaling is an intermediate phenomenon. Some masses are negligible because they are small, while others are negligible because they are very large compared to the scale currently being probed.

One general point that is often the source of some confusion is the distinction between "improved" and "unimproved" perturbation theory. In these strong interaction models, there always exist normalization conventions in which the coupling constants are apparently weak. But the unimproved expansion in these couplings, although superficially similar, is fundamentally different from the improved expansion in the effective couplings. This is most apparent upon asking what are the criteria for validity of the expansion of what happens at yet shorter distances. In the first case it is impossible to compare momenta that are very different on a logarithmic scale, while in the improved expansion only the effective couplings need be small.

Other applications are in progress. One obvious question is the validity of the observed approximate scaling laws in fixed angle hadron scattering and elastic form factors. Asymptotic freedom and the operator product expansion determine the short distance behavior of Bethe–Salpeter wave functions [70], which is a start on hadron scattering problem.

Of course, there remains a great challenge: what about not-so-short distances? Speculations abound, and many people are attacking these problems on many fronts. Perhaps it is not so ridiculous to imagine that someday soon it will all be understood.

A1. A derivation of the renormalization group equation

A discussion relating a cut-off bare theory to the corresponding renormalized theory makes clear the connection of the renormalization group functions to perhaps more familiar quantities. For simplicity I will only consider purely massless theories, with both bare and renormalized mass parameters equal to zero ‡. Multiplicative renormalizability implies that for every unrenormalized operator there exists a cut-off dependent scale factor which renders any unrenormalized Green's function cut-off independent when expressed in terms of renormalized parameters. Denoting the momentum cut-off by Λ and unrenormalized quantities by a zero subscript:

$$Z(g_0, \Lambda/M)\, \Gamma_0(p, g_0, \Lambda) = \Gamma(p, g, M)$$

and $g_i = g_i(g_0, \Lambda/M)$. (In dimensionless functions, only the ratio Λ/M can occur.) Taking the variation $M\,\partial/\partial M$ for fixed g_0 and Λ, noting that Γ_0 does not while the g do vary with M, and juggling a little yields an equation of the form

$$[M\,\partial/M + \beta_i\,\partial/\partial g_i + \gamma]\;\Gamma(p, g, M) = 0$$

where in particular

$$\beta_i = M(\partial g_i/\partial M)_{g_0,\Lambda} = -\Lambda(\partial g_i/\partial\Lambda)_{g_0,M}$$

and

$$\gamma = -Z^{-1} M(\partial Z/\partial M)_{g_0,\Lambda} = Z^{-1}\Lambda(\partial Z/\partial\Lambda)_{g_0,M}\,.$$

Since Γ and $M(\partial\Gamma/\partial M)_g$ are Λ independent when expressed in terms of g, so are β and γ. And since they are dimensionless, they cannot depend on M. (Note, as always, that Z is the product of the rescalings of each operator, and that γ, consequently, is a sum of logarithmic derivatives.)

A2. Some historical comments

The lowest order formula for $\bar{g}$, eq. (3.20), is not only similar but exactly analogous to a formula from quantum electrodynamics,

$$\bar{e}^2 = e^2 \Big/ \left\{1 - \frac{e^2}{12\pi^2}\log\frac{-q^2}{m^2} + \ldots\right\},$$

where $\bar{e}^2$ is commonly called the invariant charge, identified by

$$e^2 D(q^2) = \bar{e}^2/q^2$$

and $D(q^2)$ is the photon propagator. It can be obtained from the electrodynamic renormalization group equation [2] or from a leading log summation to all orders in perturbation theory of the photon propagator [71]. The direct connection between $\bar{e}^2$ and $D(q^2)$ is a consequence of the U(1) gauge structure, specifically the $Z_1 = Z_2$ Ward identity, which is just not true for non-Abelian theories.

‡ Scalar mesons are excluded by this restriction. But since the quadratic mass divergence is totally momentum independent, it can be subtracted or renormalized once and for all at any M and never enter again into any discussion of scale transformations.

While it may not always be possible to identify graph by graph a sum effectively performed by going from perturbation theory in g to the improved expansion in $\bar{g}$, such a sum is indeed implicit. Starting with a perturbative approximation, the renormalization group machinery essentially adds those higher order terms necessary to make the approximation explicitly invariant under finite renormalizations (changes in M). The domain of validity is naturally extended. In straight forward perturbation theory we are restricted to not only small g^2 but also small $g^2 \log p^2/M^2$, where p^2 is a typical momentum. The improved expansion combines these into the single restriction that $\bar{g}^2$ be small. This possibility is familiar from the standard discussions of QED. For momenta so large that $e^2 \log p^2/m^2$ is no longer small, perturbation theory breaks down. Leading logs can be summed, but the sum of the sub-dominant logs is found to be comparable to the leading terms. This is reflected in the fact that $\bar{e}^2 \to \infty$ for large t. However, the infrared divergences of perturbation theory can be properly interpreted by suitable infinite summations. The renormalization group offers an equivalent approach to the infrared problem [4], and its success rests on the fact that the infrared, soft photon limit corresponds to $t \to -\infty$, and hence $\bar{e}^2 \to 0$.

At this point it is difficult to resist commenting on why asymptotic freedom was really no surprise at all. With only one coupling constant, $\beta(\bar{g})$ is either negative or positive for small $\bar{g}$. So the theory is either asymptotically free or infrared stable, meaning that soft radiation is in some sense summable. But, as opposed to soft photons or gravitons, soft Yang–Mills mesons defied summation by several of physics' leading lights. So either these people were missing something, or non-Abelian gauge theories were asymptotically free. (Furthermore, to date virtually nothing is known about non-trivial solutions to the classical Yang–Mills field equations, which, if understood, would certainly bear on the long wavelength problems.)

While it first appears that this is a totally facetious argument because the presence of additional fermions can alter the sign of β, such is not the case. Unequivocally, a large enough number of fermions can destroy asymptotic freedom, but consider the infrared question more closely. By a solution to the infrared problem, what is sought is an honest, on-shell, particle interpretation for massive fermion fields with massless gauge particles, whose description may become complicated at low energies. But fermion masses are certainly not negligible near the mass shells. However, as shown in section 4, non-zero masses can be treated in the renormalization group equation as generalized coupling constants. But now, with masses, it is no longer a one parameter theory, and the absence of asymptotic freedom does not imply infrared stability. A stronger argument is provided by a recent specific result: the infrared singularities of a gauge theory with massive fermions in perturbation theory are the same as those of the same gauge theory with no fermions at all (as opposed to the theory with massless fermions [69]) ‡. So the infrared problem never gets better with the addition of fermions.

Admittedly the exactly massless theory remains an enigma. Either it is the reddest of herrings or, by virtue of a totally non-perturbative mechanism, it is certainly a most aesthetically appealing Lagrangian for a theory of elementary particle interactions. For aside from the attraction of unifying particle interactions by a single (or a few) gauge principles, the exactly massless feature implies the determination of the fine structure constant (or at least one of the several dimensionless cou-

‡ This is true for QED in an amusing way. The addition of massive fermions renders the theory of photons only, a free field theory known for all momenta, incalculable at ultra-high momenta. However, near zero photon momentum, all purely photonic Green's functions have the same behavior in both theories: $D(q^2) \approx 1/q^2$ and all higher functions vanish.

plings) via dimensional transmutation [32], a phenomenon totally different from the Gell-Mann–Low eigenvalue condition. The massless theory is apparently described by the parameters g and M, but only one of these or one combination, e.g. $\bar{g}(g, M)$, is a free parameter. If, through a non-perturbative process, the theory in fact describes massive particles, only M can determine the scale of the masses. So in this case M or one of the particle masses, and not $\bar{g}$, is the natural choice of the independent variable. And now the values of all dimensionless quantities are fixed by the dynamics. Having a mass as the only free parameter is almost like no free parameter at all. (With several gauge couplings, in general only one of them will be transmuted into a mass.)

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References

[1] C.N. Yang and R. Mills, Phys. Rev. 96 (1954) 191.
[2] M. Gell-Mann and F. Low, Phys. Rev. 95 (1954) 1300.
[3] E. Stueckelberg and A. Petermann, Helv. Phys. Acta 5 (1953) 499.
[4] N.N. Bogoliubov and D.V. Shirkov, Introduction to the Theory of Quantized Fields (Interscience, New York, 1959).
[5] K. Wilson, Phys. Rev. D3 (1971) 1818.
[6] K. Wilson and J. Kogut, Phys. Reports 12 (1974) 75.
[7] K. Symanzik, Lett. Nuovo Cimento 6 (1973) 77.
[8] S. Coleman, Secret Symmetry, in: Proc. 1973 Intern. Summer School of Physics "Ettore Majorana".
[9] G. 't Hooft, Nucl. Phys. B33 (1971) 1973.
[10] B.W. Lee and J. Zinn-Justin, Phys. Rev. D5 (1972) 3121.
[11] D. Gross and F. Wilczek, Phys. Rev. Letters 26 (1973) 1343.
[12] H.D. Politzer, Phys. Rev. Letters 26 (1973) 1346.
[13] S. Coleman and D. Gross, Phys. Rev. Letters 31 (1973) 851.
[14] C. Callan, in: Proc. Intern. School of Physics "Enrico Fermi" Course LIV (Academic Press, 1972).
[15] N. Christ, B. Hasslacher and A. Muller, Phys. Rev. D6 (1972) 3543.
[16] K. Wilson, Phys. Rev. 179 (1969) 1499.
[17] W. Chen, talk given at the APS meeting, Chicago, February 1974.
[18] G. Bollini and J. Giambiagi, Phys. Letters 40B (1972) 566.
[19] G. 't Hooft and M. Veltman, Nucl. Phys. B44 (1972) 189.
[20] G. 't Hooft, Nucl. Phys. B61 (1973) 455.
[21] M. Holwerda, W. Van Neerven and R. Van Royen, The Callan–Symanzik equation and Dimensional Regularization, University of Nijmegen preprint, November 1973.
[22] E.S. Abers and B.W. Lee, Phys. Reports 9 (1973) 1.
[23] K. Symanzik, Comm. Math. Phys. 18 (1970) 227.
[24] C. Callan, Phys. Rev. D2 (1970) 1541.
[25] S. Weinberg, Phys. Rev. D8 (1973) 3497.
[26] H. Georgi and H.D. Politzer, Phys. Rev. D9 (1974) 416.
[27] W. Caswell and F. Wilczek, Phys. Letters B49 (1974) 291.
[28] D. Gross and F. Wilczek, Phys. Rev. D8 (1973) 3633.
[29] W. Kainz, W. Kummer and M. Schweda, Non-Abelian Gauge Theories for Hadrons and Radiative Corrections, Institute for Theoretical Physics, Technische Hochschule Wien, February 1974.
[30] T.P. Cheng, E. Eichten and L.F. Li, Higgs Phenomena in Asymptotically Free Gauge Theories, SLAC preprint, November 1973.

[31] S. Weinberg, Phys. Rev. Letters 31 (1973) 494.
[32] S. Coleman and E. Weinberg, Phys. Rev. D7 (1973) 1888.
[33] H.D. Politzer and G. Ross, Nucl. Phys. B, to be published.
[34] Y. Nambu and M.Y. Han, Three Triplets, Paraquarks, and Colored Quarks, University of Chicago preprint, September 1973.
[35] H. Fritzsch and M. Gell-Mann, in: Proc. XVI Intern. Conf. on High Energy Physics, Chicago–Baravia, Ill., 1973 (NAL, Batavia, 1973).
[36] H. Nielsen and P. Olesen, Nucl. Phys. B61 (1973) 45.
[37] Y. Nambu, Quarks, Strings and Gauge Fields, talk presented at the Johns Hopkins Workshop on Current Problems in High Energy Particle Theory, January 1974.
[38] D. Amati and M. Testa, Phys. Letters 48B (1974) 227.
[39] K. Wilson, Confinement of Quarks, Cornell preprint, February 1974.
[40] J. Kogut and L. Susskind, Vacuum Polarization and the Absence of Free Quarks in Four Dimensions, Cornell preprint, February 1974.
[41] A. Chodos et al., A New Extended Model of Hadrons, MIT preprint, March 1974.
[42] S. Weinberg, Phys. Rev. D8 (1973) 605, 4482.
[43] W. Zimmerman, in: Lectures on Elementary Particles and Quantum Field Theory (MIT Press, 1970).
[44] C. Callan and D. Gross, Phys. Rev. Letters 22 (1969) 156.
[45] K. Symanzik, Comm. Math. Phys. 23 (1971) 49.
[46] D. Gross and F. Wilczek, Phys. Rev. D9 (1974) 980.
[47] H.D. Politzer, Phys. Rev. D9 (1974) 2174.
[48] E.D. Bloom, in: Proc. Intern. Symposium on Electron and Photon Interactions, Bonn, August 1973, to be published.
[49] A. De Rújula et al., Possible Non-Regge Behavior of Electroproduction Structure Functions, Princeton preprint, March 1974.
[50] D. Bailin, A. Love and D. Nanopoulos, Lett. Nuovo Cimento 9 (1974) 501.
[51] S. Adler, Phys. Rev. 143 (1966) 1144.
[52] C. Llewellyn Smith, Phys. Reports 3 (1972) 261.
[53] F. Wilczek, private communication;
M. Calvo, to be published.
[54] G. Parisi, Phys. Létters 43B (1973) 207.
[55] A. De Rújula, Phys. Rev. Letters 32 (1974) 1143.
[56] D. Gross and S. Treiman, Phys. Rev. Letters 32 (1974) 1145.
[57] D. Gross, Phys. Rev. Letters 32 (1974) 1071.
[58] G. Parisi, Detailed Predictions for the P-N Structure Functions ..., Columbia preprint, March 1974.
[59] A. De Rujula, H. Georgi and H.D. Politzer, The Breakdown of Scaling in Neutrino and Electron Scattering, Harvard preprint, June 1974.
[60] E.D. Bloom and F. Gilman, Phys. Rev. D4 (1971) 2901.
[61] K. Strauch, in: Proc. Intern. Symposium on Electron and Photon Interactions, Bonn, August 1973, to be published.
[62] T. Appelquist and H. Georgi, Phys. Rev. D8 (1973) 4000.
[63] A. Zee, Phys. Rev. D8 (1973) 4038.
[64] J.S. Kang, Scalar Contribution to the e^+e^- Annihilation in Asymptotically Free Theories, Brandeis preprint, February 1974.
[65] J. Kogut, Electron Position Annihilation and Speculations on the Renormalization Group of Strong Interactions, Cornell preprint, January 1974.
[66] H. Georgi and S. Glashow, Phys. Rev. Letters 32 (1974) 438.
[67] H. Georgi, H. Quinn and S. Weinberg, Hierarchy of Interactions in Unified Gauge Theories, Harvard preprint, May 1974.
[68] F. Reines and M.F. Crouch, Phys. Rev. Letters 32 (1974) 493.
[69] T. Appelquidst and J. Carazzone, to be published.
[70] E. Poggio and T. Appelquist, Bound States and Asymptotic Scaling Laws in Field Theory, Harvard preprint, June 1974.
[71] L. Landau, A. Abrikosov and I. Khalatnikov, Dokl. Akad. Nauk USSR 95 (1954) 1177.