Convex Optimization for Parallel Energy Minimization

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Overview

Submodularity and Examples

Notations and Definitions

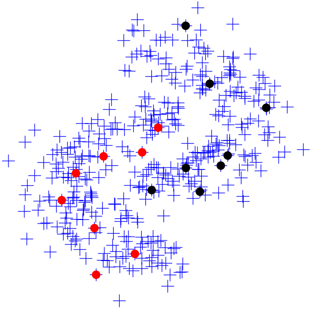
SFM and the corresponding smooth problems

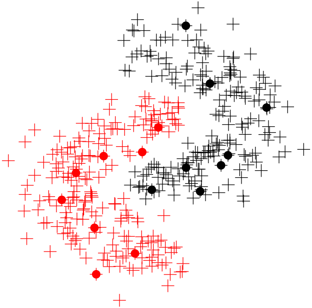
Active set methods for submodular minimization

Active set methods for parallel submodular minimization

Results

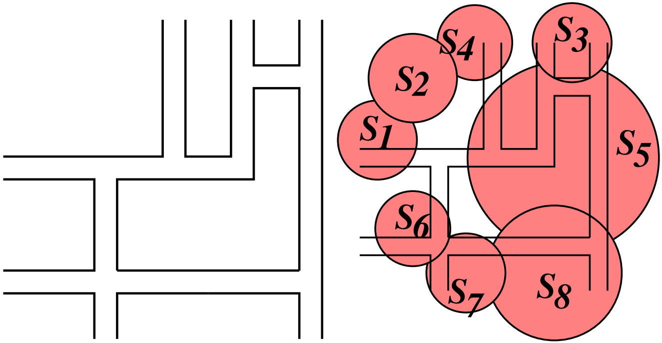
Semi - supervised clustering





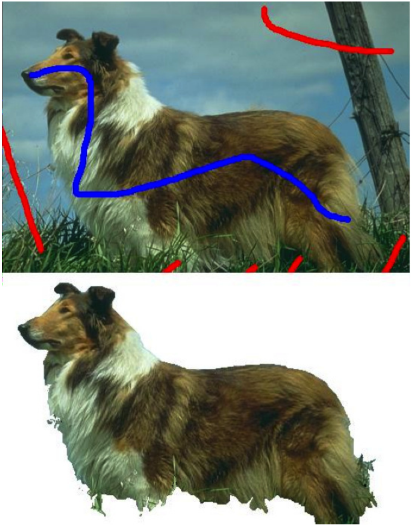
- Submodular function minimization

Sensor placement



- Krause and Guestrin, 2005.
- Submodular function maximization

Energy Minimization in Computer Vision



- Graph cuts and image segmentation

Total Variation Denoising

- Chambolle, 2005.

Submodular functions

- **Definition**: $F : 2^V \to \mathbb{R}$ is **submodular** if and only if

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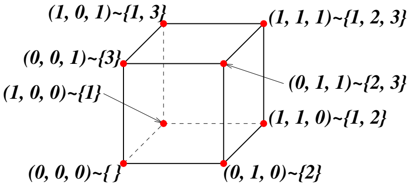
  - Diminishing returns property

Subsets as pseudo boolean functions

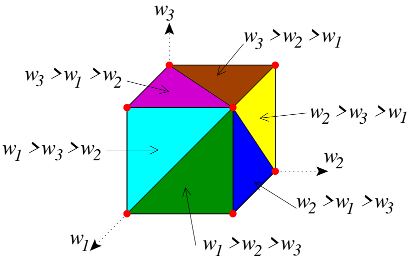
- Subsets may be identified with elements of $\{0,1\}^p$.



Lovász extension

- Given any set-function F and w such that $w_{j_1} \geqslant \cdots \geqslant w_{j_p}$,

$$f(w) = \sum_{k=1}^{p-1} (w_{j_k} - w_{j_{k+1}}) F(\{j_1, \ldots, j_k\}) + w_{j_p} F(\{j_1, \ldots, j_p\})$$



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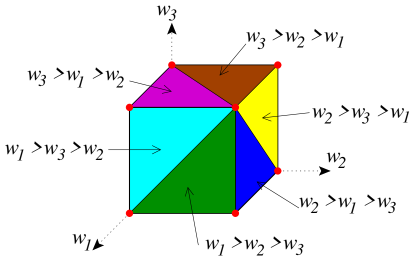

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  - Extension from $\{0,1\}^p$ to $\mathbb{R}^p$

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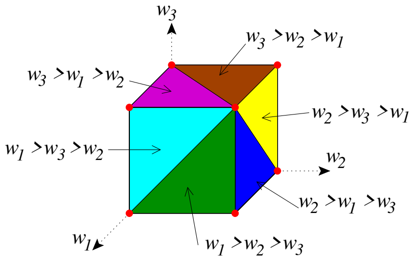

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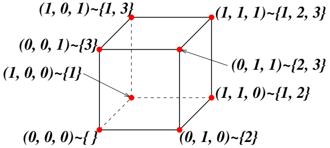
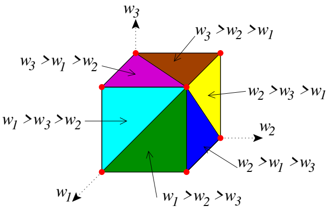
- **Theorem**: F is submodular if and only if f is convex

Cut functions and Lovász extension





- **Cut function.** $F : 2^V \to \mathbb{R}$ such that $F(A) = \sum_{i \in V} \sum_{j \in \mathcal{N}(i)} a_{ij} |1_{i \in A} - 1_{j \in A}|$.

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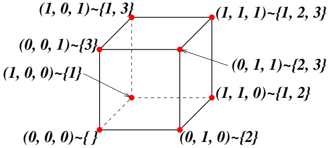

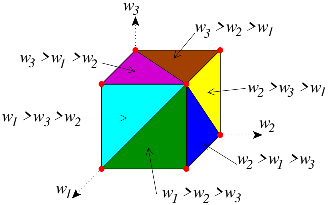

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- **Lovász extension.** $f : [0,1]^p \to \mathbb{R}$ such that $f(w) = \sum_{i \in V} \sum_{j \in \mathcal{N}(i)} a_{ij} |w_i - w_j|$.

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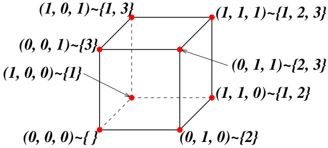

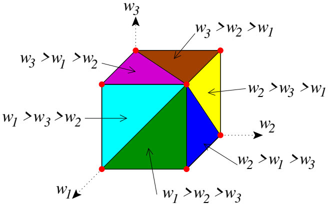

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- If F is a cut function then f is the corresponding total variation problem.

From submodular minimization to smooth problems

- **Related optimization problems**

(D) Discrete $\min_{A \subset V} F(A) - u(A)^1 = \min_{w \in \{0,1\}^p} f(w) - u^\top w$

(C) Continuous $\min_{w \in [0,1]^p} f(w) - u^\top w$

(S) Smooth $\min_{w \in \mathbb{R}^p} f(w) - u^\top w + \frac{1}{2}\|w\|_2^2$

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$$\min_{A \subset V} F(A) - u(A) + \lambda |A|, \forall \lambda \in \mathbb{R}$$

  - $\arg\min_{A \subseteq V} F(A) - u(A) + \lambda|A| = \{k, w_k \geqslant \lambda\}$
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- If **(D)** is graph cut $\Rightarrow$ **(S)** is parametric maxflow.

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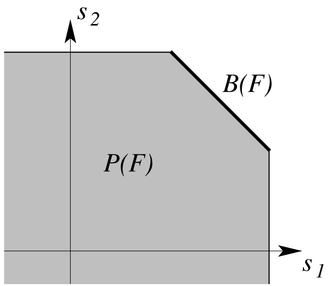
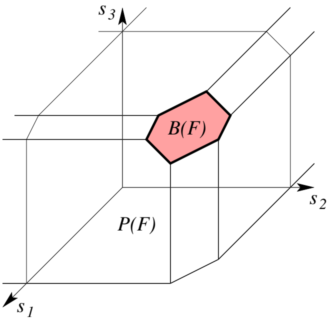
Submodular Functions and base polyhedra

- Submodular polyhedron:
  $P(F) = \{s \in \mathbb{R}^p,\ \forall A \subset V,\ s(A) \leqslant F(A)\}$
- Base polyhedron: $B(F) = P(F) \cap \{s(V) = F(V)\}$
- Many facets (up to 2^p), many extreme points (up to $p!$)





Lovász extension and base polyhedra

- **Fundamental property**: If F is submodular, maximizing linear functions may be done by a "greedy algorithm"
  - Let $w \in \mathbb{R}_+^p$ such that $w_{j_1} \geqslant \cdots \geqslant w_{j_p}$
  - Let $s_{j_k} = F(\{j_1, \ldots, j_k\}) - F(\{j_1, \ldots, j_{k-1}\})$ for $k \in \{1, \ldots, p\}$
  - Then $f(w) = \max_{s \in P(F)} w^\top s = \max_{s \in B(F)} w^\top s$

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- **Primal - Dual**
  - **Primal** : $\min_{w \in \mathbb{R}^p} f(w) - u^\top w + \frac{1}{2}\|w\|_2^2$
  - **Dual** : $\max_{s \in B(F)} -\frac{1}{2}\|u - s\|_2^2$
  - At optimal, $w^* = u - s^*$

Faces of the Base Polyhedon and Ordered Partitions

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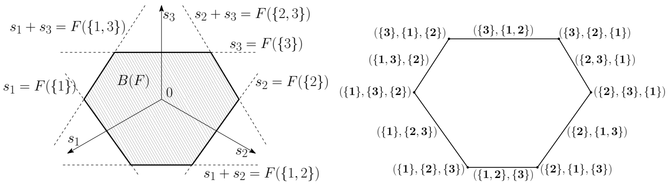

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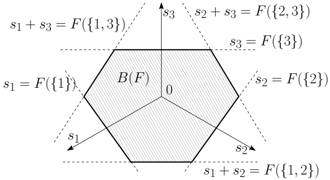

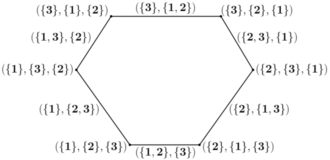

- Given an Ordered Partition $\mathcal{A} = (A_1, \ldots, A_m)$
- Let $B_i = A_1 \cup \ldots \cup A_i, \forall i = 1, \ldots, m$. $B_m = V$

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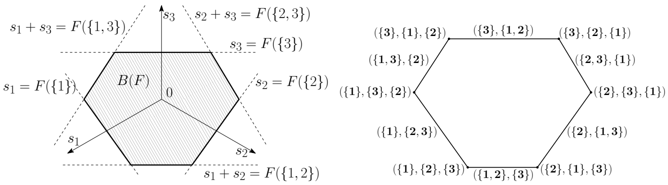

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- Let $B_i = A_1 \cup \ldots \cup A_i, \forall i = 1, \ldots, m$. $B_m = V$
- Outer approximation of $B(F)$ by the ordered partition $\mathcal{A}$

$$\widehat{B}^{\mathcal{A}}(F) = \{s \in \mathbb{R}^n, s(V) = F(V), \forall i = \{1, \ldots, m{-}1\}, s(B_i) \leq F(B_i)\}$$

Outer Approximation of Base Polytope and Optimization

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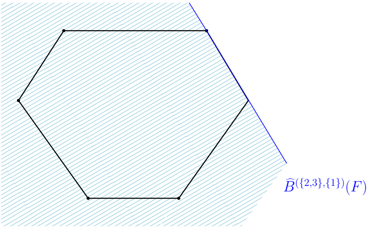

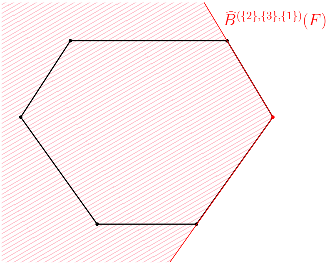

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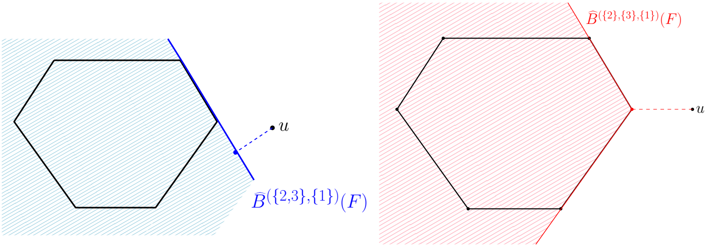

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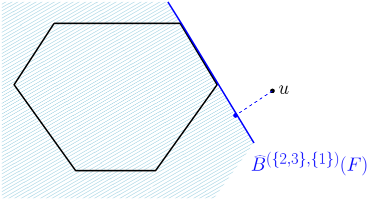

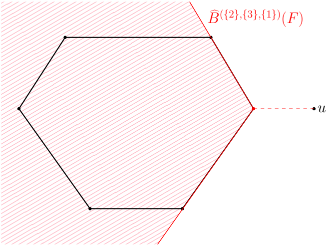

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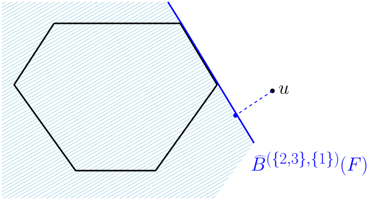

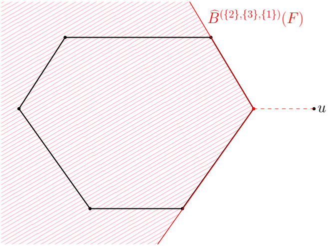

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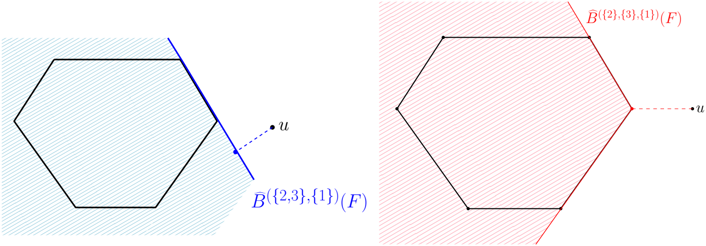

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  - $f(w) = \sum_{i=1}^m v_i [F(B_i) - F(B_{i-1})]$
- $\min_{v \in \mathbb{R}^m} \sum_{i=1}^m v_i [F(B_i) - F(B_i - 1) - u(A_i)] + \frac{1}{2} \sum_{i=1}^m |A_i| v_i^2$ such that $v_1 \ge \ldots \ge v_m$

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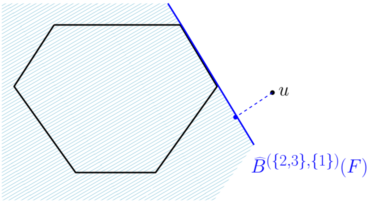

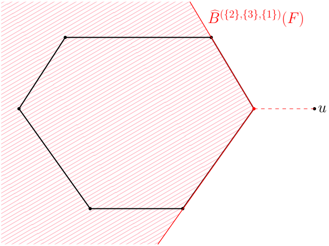

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- Isotonic regression solved using wpav, $s = u - w$
- Refines $\mathcal{A}$ by merging partitions and ensuring $v_1 > \ldots > v_{m'}$

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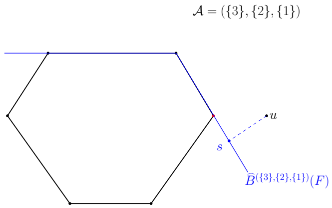
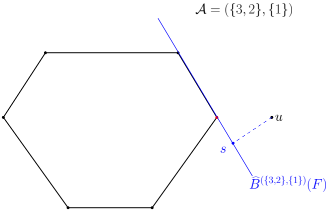
- If $F(C) - s(C) < 0$ then split all the ordered partitions of $\mathcal{A}$ that intersect with C

Active Set Algorithm



- Input : Submodular function F with SFM oracle, $u \in \mathbb{R}^p$, ordered partition $\mathcal{A}$
- Algorithm : iterate until convergence
  (a) Project u onto outer-approximation $\widehat{B}^{\mathcal{A}}(F)$ (using isotonic regression)

Active Set Algorithm



- Input : Submodular function F with SFM oracle, $u \in \mathbb{R}^p$, ordered partition $\mathcal{A}$
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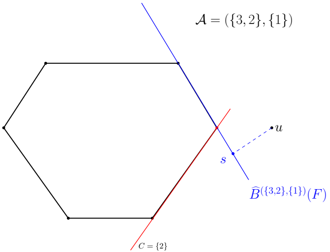
  (b) Refine partitions by merging partitions with equal values

Active Set Algorithm



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- Algorithm : iterate until convergence
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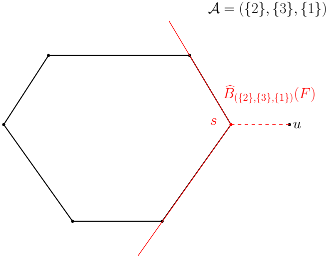
  (b) Refine partitions by merging partitions with equal values
  (c) Check for optimality of s by SFM oracle to solve $C = \min_{A \in V} F(C) - s(C)$

Active Set Algorithm



- Input : Submodular function F with SFM oracle, $u \in \mathbb{R}^p$, ordered partition $\mathcal{A}$
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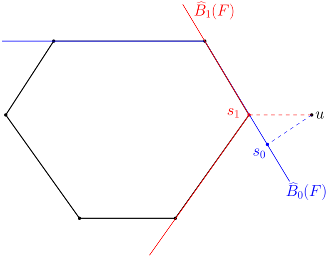
  (c) Check for optimality of s by SFM oracle to solve $C = \min_{A \in V} F(C) - s(C)$
  (d) If not optimal then split the ordered partitions of $\mathcal{A}$ that intersect with C and goto (a).

Active Set Algorithm



- Input : Submodular function F with SFM oracle, $u \in \mathbb{R}^p$, ordered partition $\mathcal{A}$
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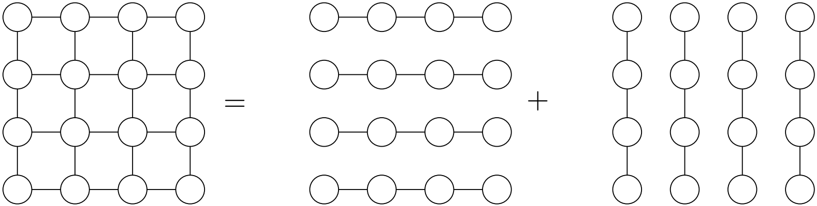
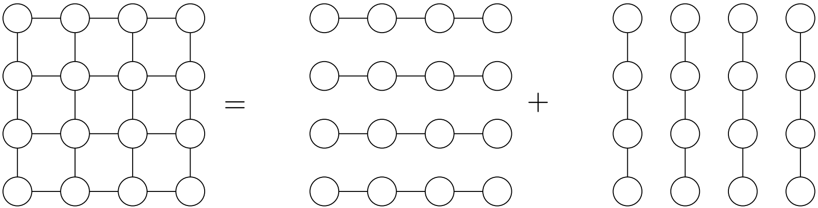
Decomposable functions



- **Goal:** Use simpler SFM oracles to solve (S) and as a consequence solve (D)

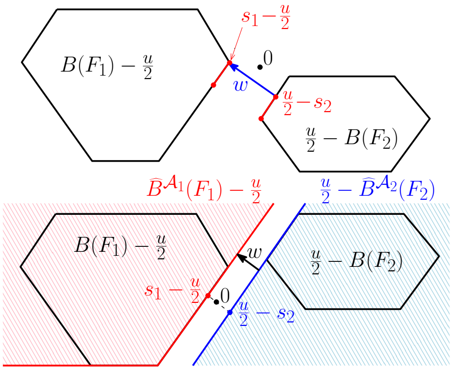
Decomposable functions



- **Goal:** Use simpler SFM oracles to solve (S) and as a consequence solve (D)
- **(D)** : $\min_{A \subset V} F_1(A) + F_2(A) - u(A)$
- **(S)** : $\min_{w \in \mathbb{R}^p} f_1(w) + f_2(w) - u^\top w + \frac{1}{2}\|w\|_2^2$
- **Dual** : $\min_{s_1 \in B(F_1) - u/2, -s_2 \in u/2 - B(F_2)} \|s_1 - (-s_2)\|_2$

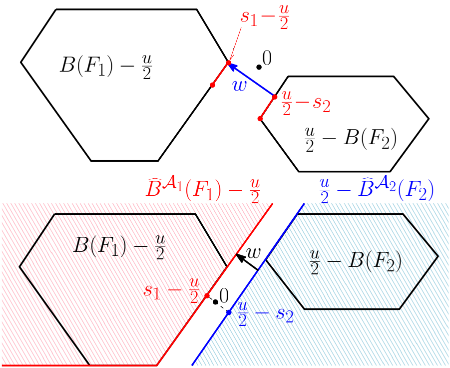
Closest points between two polytopes

Dual : $\min_{s_1 \in B(F_1)-u/2,\, s_2 \in u/2-B(F_2)} \|s_1 - s_2\|_2$



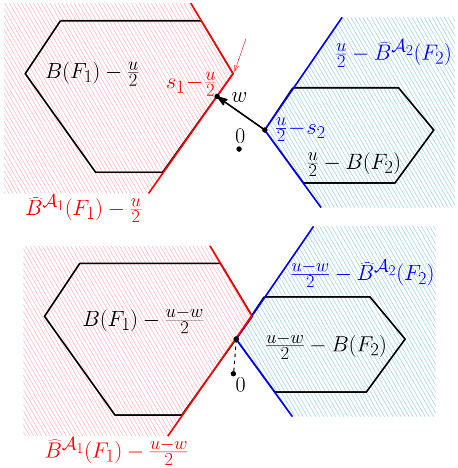
Closest points between two polytopes

Dual : $\min_{s_1 \in B(F_1)-u/2, s_2 \in u/2-B(F_2)} \|s_1 - s_2\|_2$



- Reflection methods for user-friendly submodular optimization. S.Jegelka, F. Bach and S. Sra, NIPS-2013.
- Uses expensive TV oracles.

Translated intersecting polytopes



Active set method for decomposable functions

- **Input**: Submodular function F_1 and F_2 with SFM oracles, $u \in \mathbb{R}^n$, ordered partitions $\mathcal{A}_1, \mathcal{A}_2$
- **Algorithm**: iterate until convergence (i.e., $\varepsilon_1 + \varepsilon_2$ small enough)

Active set method for decomposable functions

- **Input**: Submodular function F_1 and F_2 with SFM oracles, $u \in \mathbb{R}^n$, ordered partitions $\mathcal{A}_1, \mathcal{A}_2$
- **Algorithm**: iterate until convergence (i.e., $\varepsilon_1 + \varepsilon_2$ small enough)
  (a) Find $\mathcal{A} = \mathrm{coalesce}(\mathcal{A}_1, \mathcal{A}_2)$ and run isotonic regression to minimize $f(w) - u^\top w + \frac{1}{2}\|w\|_2^2$ such that w is compatible with $\mathcal{A}$.

Active set method for decomposable functions

- **Input**: Submodular function F_1 and F_2 with SFM oracles, $u \in \mathbb{R}^n$, ordered partitions $\mathcal{A}_1, \mathcal{A}_2$
- **Algorithm**: iterate until convergence (i.e., $\varepsilon_1 + \varepsilon_2$ small enough)
  (a) Find $\mathcal{A} = \mathrm{coalesce}(\mathcal{A}_1, \mathcal{A}_2)$ and run isotonic regression to minimize $f(w) - u^\top w + \frac{1}{2}\|w\|_2^2$ such that w is compatible with $\mathcal{A}$.
  (b) Run accelerated Dykstra's algorithm to find the projection of 0 onto the intersection of $\widehat{B}^{\mathcal{A}_1}(F_1) - u/2 + w/2$ and $u/2 - w/2 - \widehat{B}^{\mathcal{A}_2}(F_2)$.

Active set method for decomposable functions

- **Input**: Submodular function F_1 and F_2 with SFM oracles, $u \in \mathbb{R}^n$, ordered partitions $\mathcal{A}_1, \mathcal{A}_2$
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  (c) Merge the sets in $\mathcal{A}_j$ which are tight for s_j, $j \in \{1, 2\}$.

Active set method for decomposable functions

- **Input**: Submodular function F_1 and F_2 with SFM oracles, $u \in \mathbb{R}^n$, ordered partitions $\mathcal{A}_1, \mathcal{A}_2$
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  (c) Merge the sets in $\mathcal{A}_j$ which are tight for s_j, $j \in \{1, 2\}$.
  (d) Check optimality by solving $\min_{C_{j,i_j} \subseteq A_{j,i_j}} F_j(B_{j,i_j-1} \cup C_{j,i_j}) - F_j(B_{j,i+j-1}) - s_j(C_{j,i_j})$ for $i_j \in \{1, \ldots, m_j\}$, Monitor ε_1 and ε_2 such that $F_j(C_j) - s_j(C_j) \geqslant -\varepsilon_j$, $j = 1, 2$.

Active set method for decomposable functions

- **Input**: Submodular function F_1 and F_2 with SFM oracles, $u \in \mathbb{R}^n$, ordered partitions $\mathcal{A}_1, \mathcal{A}_2$
- **Algorithm**: iterate until convergence (i.e., $\varepsilon_1 + \varepsilon_2$ small enough)
  (a) Find $\mathcal{A} = \mathrm{coalesce}(\mathcal{A}_1, \mathcal{A}_2)$ and run isotonic regression to minimize $f(w) - u^\top w + \frac{1}{2}\|w\|_2^2$ such that w is compatible with $\mathcal{A}$.
  (b) Run accelerated Dykstra's algorithm to find the projection of 0 onto the intersection of $\widehat{B}^{\mathcal{A}_1}(F_1) - u/2 + w/2$ and $u/2 - w/2 - \widehat{B}^{\mathcal{A}_2}(F_2)$.
  (c) Merge the sets in $\mathcal{A}_j$ which are tight for s_j, $j \in \{1, 2\}$.
  (d) Check optimality by solving $\min_{C_{j,i_j} \subseteq A_{j,i_j}} F_j(B_{j,i_j-1} \cup C_{j,i_j}) - F_j(B_{j,i+j-1}) - s_j(C_{j,i_j})$ for $i_j \in \{1, \ldots, m_j\}$, Monitor ε_1 and ε_2 such that $F_j(C_j) - s_j(C_j) \geqslant -\varepsilon_j$, $j = 1, 2$.
  (e) If both s_1 and s_2 not optimal, for all C_{j,i_j} which are different from $\varnothing$ and A_{j,i_j}, split partitions.

Active set method for decomposable functions

- **Input**: Submodular function F_1 and F_2 with SFM oracles, $u \in \mathbb{R}^n$, ordered partitions $\mathcal{A}_1, \mathcal{A}_2$
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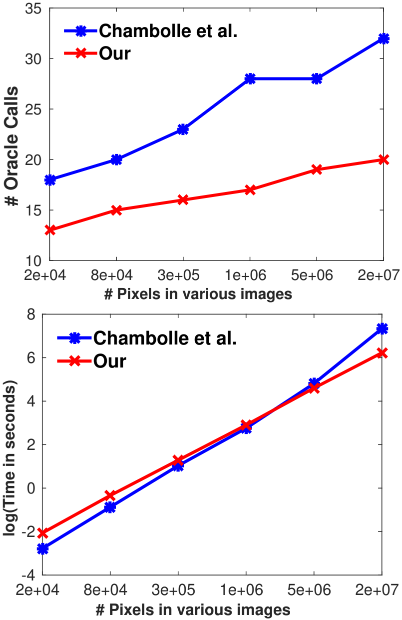
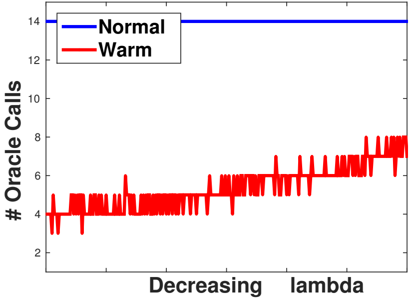
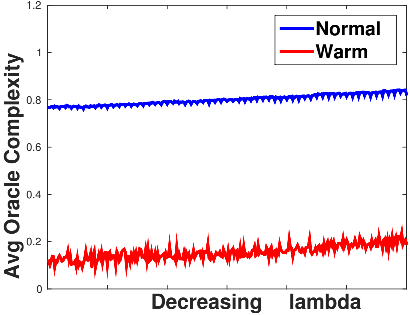
  (e) If both s_1 and s_2 not optimal, for all C_{j,i_j} which are different from $\varnothing$ and A_{j,i_j}, split partitions.
- **Output**: $w \in \mathbb{R}^n$ and $s_1 \in B(F_1)$, $s_2 \in B(F_2)$.

Results for 2D TV without decomposition

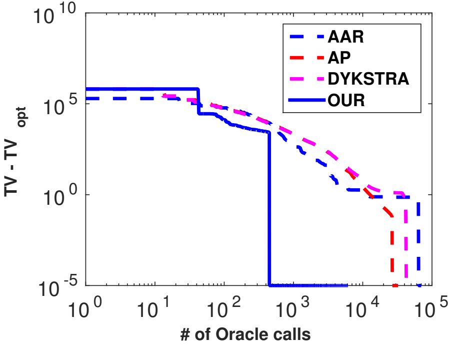


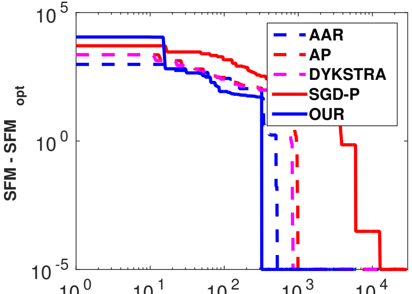
Results for 2D TV without decomposition (Warm Start)





Results for 2D TV with decomposition into 1D





Thank You. Questions?