Fast Decomposable Submodular Function Minimization using Constrained Total Variation

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Submodular functions

- **Ground Set :** V of n elements
- **Definition**: $F : 2^V \to \mathbb{R}$ is **submodular** if and only if

$$\forall A, B \subseteq V, F(A) + F(B) \geq F(A \cup B) + F(A \cap B)$$

  - Equality for *modular* functions
  - Always assume $F(\varnothing) = 0$.

- **Equivalent definition** :

$$\forall k \in V, A \to F(A \cup \{k\}) - F(A) \text{ is non increasing}$$

  - Diminishing returns property

Decomposable SFM

- **Ground Set :** V of n elements
- **Definition:** $\forall i \in [r]$, $F_i : 2^V \to \mathbb{R}$ is **submodular** if and only if

$$\forall A, B \subseteq V, F_i(A) + F_i(B) \geq F_i(A \cup B) + F_i(A \cap B)$$

  - Equality for *modular* functions
  - Always assume $F_i(\varnothing) = 0$.

- **Goal :** $(\mathrm{SFM}\mathbf{D})$ $\quad \min\limits_{A \subset V} \sum\limits_{i=1}^{r} F_i(A)$

- **Oracles :** $(\mathrm{SFM}\mathbf{D_i})$ $\quad \min\limits_{A \subset V} F_i(A)$

- **Contribution :**
  - Propose a new continuous optimization problem and algorithms to optimize it using $\mathrm{SFM}\mathbf{D_i}$ oracles.
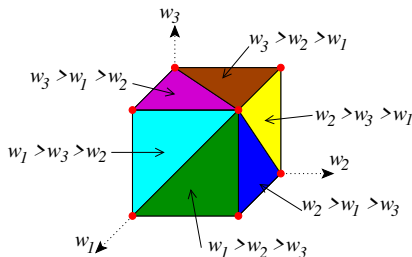 - Derive the solution of $\mathrm{SFM}\mathbf{D}$ from the solution of the continuous optimization problem.

▶ Subsets may be identified with elements of $\{0, 1\}^P$.
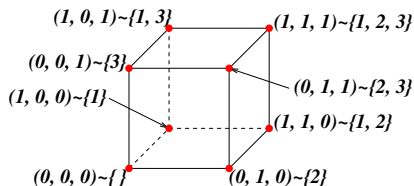
Lovász extension

▶ Given any set-function F and w such that $w_{j_1} \geqslant \cdots \geqslant w_{j_p}$,
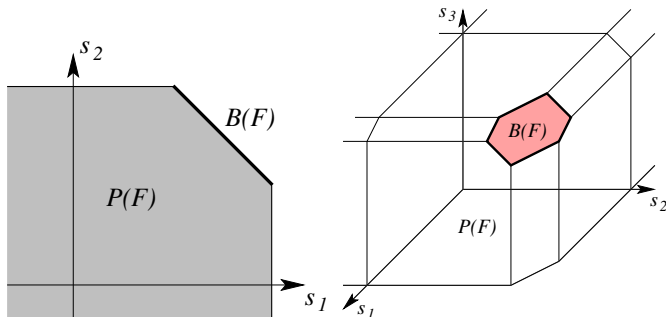
$$f(w) = \sum_{k=1}^{p-1} (w_{j_k} - w_{j_{k+1}}) F(\{j_1, \ldots, j_k\}) + w_{j_p} F(\{j_1, \ldots, j_p\})$$

▶ **Theorem**: F is submodular if and only if f is convex

Submodular Functions and base polyhedra

- Submodular polyhedron:
  $P(F) = \{s \in \mathbb{R}^p, \ \forall A \subset V, \ s(A) \leqslant F(A)\}$
- Base polyhedron: $B(F) = P(F) \cap \{s(V) = F(V)\}$

- **Representation of $f(w)$ as a support function**:

$$f(w) = \max_{s \in B(F)} w^\top s$$

Continuous Optimization

- Goal (SFM**D**) $\min\limits_{A \subset V} \sum\limits_{i=1}^{r} F_i(A)$,

- Oracles (SFM$\mathbf{D}_i$) $\min\limits_{A \subset V} F_i(A)$,

- (SFM**C**) $\min\limits_{w \in \mathbb{R}^n} \sum\limits_{i=1}^{r} f_i(w) + \sum\limits_{j=1}^{n} \psi(w_j)$,

- (SFM$\mathbf{C}_i$) $\min\limits_{w \in \mathbb{R}^n} f_i(w) + \sum\limits_{j=1}^{n} \psi(w_j)$,

- **Related work.** Total variation [Jegelka et al., 2013, Sesh Kumar and Bach, 2017, Ene et al., 2017] considers $\psi(v) = \frac{1}{2}v^2$.

- **Our work.** Let $\varepsilon \in \mathbb{R}_+$ and

$$\psi(w) = \begin{cases} \frac{1}{2}w^2 & \text{if } |w| \leqslant \varepsilon, \\ +\infty & \text{otherwise,} \end{cases}$$

- **Approach.** $\mathrm{SFM}\mathbf{D}_i \rightarrow \mathrm{SFM}\mathbf{C}_i \rightarrow \mathrm{SFM}\mathbf{C} \rightarrow \mathrm{SFM}\mathbf{D}$

SFM$\mathbf{D}_i \to$ SFM$\mathbf{C}_i \to$ SFM$\mathbf{C} \to$ SFM$\mathbf{D}$

1: **Input :** SFM$\mathbf{D}_i$ for $F_i : 2^V \to \mathbb{R}$ and $\varepsilon \in \mathbb{R}_+$.

2: **Output :** (w^*, s^*) primal-dual optimal pair of SFM$\mathbf{C}_i$ for f_i.

3: $A_+ = \operatorname{argmin}_{A \subset V} F_i(A) + \varepsilon|A|$ with a dual certificate $s_+ \in B(F_i)$.

4: $A_- = \operatorname{argmin}_{A \subset V} F_i(A) - \varepsilon|A|$ with a dual certificate $s_- \in B(F_i)$.

5: $w^*(A_+) = -\varepsilon$, $s^*(A_+) = s_+$, $w^*(V \setminus A_-) = \varepsilon$, $s^*(V \setminus A_-) = s_-$

6: $U := A_- \setminus A_+$ and a discrete function $G_i : 2^U$ s.t. $G_i(B) = F_i(A_+ \cup B) - F_i(A_+)$ with Lovász extension $g_i : \mathbb{R}^{|U|} \to \mathbb{R}$.

7: Solve for optimal solutions of $\min_{w \in \mathbb{R}^{|U|}} g_i(w) + \frac{1}{2}w^2$ and its dual using divide-and-conquer algorithm [Sesh Kumar and Bach, 2017] to obtain (w_U^*, s_U^*).

8: $(w^*(U), s^*(U)) = (w_U^*, s_U^*)$

- SFM**C**
  - Primal $\min_{w \in [-\varepsilon, \varepsilon]^n} \sum_{i=1}^{r} f_i(w) + \frac{1}{2} \|w\|_2^2$.
  - Dual $\max_{(s_1, \ldots, s_r) \in \mathbb{R}^{n \times r}} - \sum_{i=1}^{r} g_i^*(s_i) - \frac{1}{2} \left\| \sum_{i=1}^{r} s_i \right\|_2^2$, where

$$g_i^*(s_i) = \inf_{t_i \in B(F_i)} \sup_{w \in [-\varepsilon, \varepsilon]^n} w^\top (s_i - t_i) = \varepsilon \inf_{t_i \in B(F_i)} \|s_i - t_i\|_1.$$

- Optimization algorithms
  - BCD

$$\forall i \in [r], \ s_i^{\text{new}} = \underset{s_i^{\text{new}} \in \mathbb{R}^n}{\operatorname{argmin}} \ g_i^*(s_i^{\text{new}}) + \frac{1}{2} \left\| \sum_{j=1}^{i} s_j^{\text{new}} + \sum_{j=i+1}^{r} s_j \right\|_2^2.$$

  - Acceleration for $r = 2$

$$\begin{aligned}
s_2^{\text{new}} &= \underset{s_2^{\text{new}} \in \mathbb{R}^n}{\operatorname{argmin}} \ g_2^*(s_2^{\text{new}}) + \frac{1}{2} \|t_1 + s_2^{\text{new}}\|^2 \\
s_1^{\text{new}} &= \underset{s_1^{\text{new}} \in \mathbb{R}^n}{\operatorname{argmin}} \ g_1^*(s_1^{\text{new}}) + \frac{1}{2} \|s_1^{\text{new}} + s_2^{\text{new}}\|_2^2 \\
t_1^{\text{new}} &= s_1^{\text{new}} + \beta(s_1^{\text{new}} - s_1),
\end{aligned}$$

with $\beta = (t-1)/(t+2)$ at iteration t

$$\min_{w \in \mathbb{R}^n} \sum_{i=1}^{r} f_i(w) + \sum_{j=1}^{n} \psi(w_j) \Rightarrow \min_{A \subset V} \sum_{i=1}^{r} F_i(A)$$

- If w^* is the optimal solution of $\text{SFM}\mathbf{C}$, then $\{w^* \geq 0\}$ is the optimal solution of $\text{SFM}\mathbf{D}$.

Proposition

Given a feasible primal candidate w for $\text{SFM}\mathbf{C}$ with suboptimality η_{C}, one of the suplevel sets $\{w \geqslant \alpha\}$ of w is an η_{D}-optimal minimizer of $\text{SFM}\mathbf{D}$, with $\eta_{\mathrm{D}} = \frac{\eta_{\mathrm{C}}}{4\varepsilon} + \sqrt{\frac{\eta_{\mathrm{C}} n}{2}}$.

- $\eta_{\mathrm{C}} = \frac{\Delta^2}{t^\alpha}$ where Δ is a notion of diameter of the base polytopes and $\alpha = 2$ for accelerated algorithms and $\alpha = 1$ for plain algorithms.
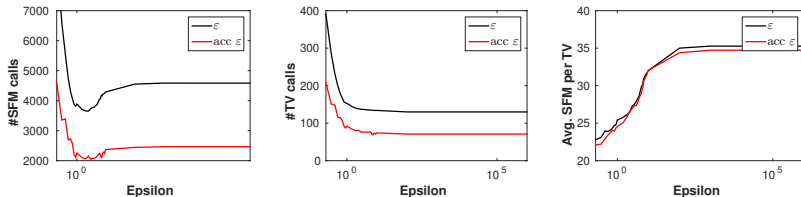- $\eta_{\mathrm{D}} = \frac{\Delta \sqrt{n}}{t^{\alpha/2}} + \frac{\Delta^2}{\varepsilon t^\alpha}$.
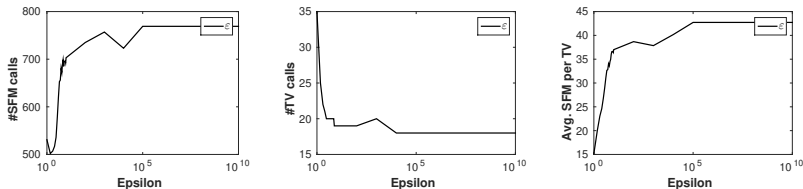
2D = 1D + 1D   3D = 1D + 2D   3D = 1D + 1D + 1D
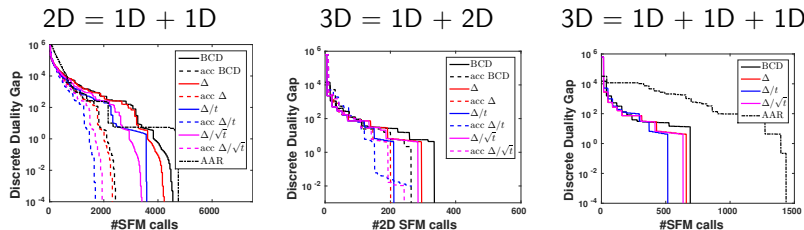
- 2D image of size $n = 2400 \times 2400 = 5.8 \times 10^6$.
- 3D volumetric surface of size $n = 102 \times 100 \times 79 = 8.1 \times 10^5$.

Experiments - Oracle calls for varying ε

Oracle calls for 2D = 1D + 1D for different ε

Oracle calls for 3D = 1D + 1D + 1D for different ε

- 2D image of size $n = 2400 \times 2400 = 5.8 \times 10^6$.
- 3D volumetric surface of size $n = 102 \times 100 \times 79 = 8.1 \times 10^5$.

- Further speed-ups may be achieved by extended the proposed algorithms to [Ene et al., 2017, Li and Milenkovic, 2018]
- Easily parallelizable and it would be interesting to compare to dedicated parallel algorithms for graph cuts [Shekhovtsov and Hlavác, 2011]
- Could be extended to more general submodular optimization problems [Bach, 2016]

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References II

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