Fast Decomposable Submodular Function Minimization using Constrained Total Variation

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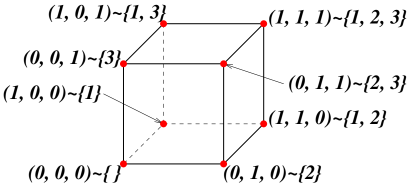
- **Contribution :**
  - Propose a new continuous optimization problem and algorithms to optimize it using $\mathrm{SFM}\mathbf{D}_\mathrm{i}$ oracles.
  - Derive the solution of $\mathrm{SFM}\mathbf{D}$ from the solution of the continuous optimization problem.

Subsets as pseudo boolean functions

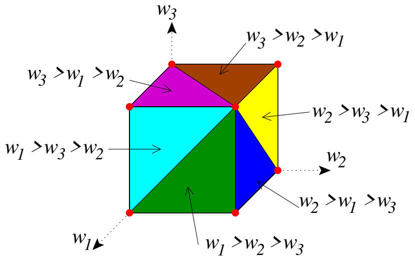
- Subsets may be identified with elements of $\{0, 1\}^p$.



Lovász extension

- Given any set-function F and w such that $w_{j_1} \geqslant \cdots \geqslant w_{j_p}$,

$$f(w) = \sum_{k=1}^{p-1} (w_{j_k} - w_{j_{k+1}}) F(\{j_1, \ldots, j_k\}) + w_{j_p} F(\{j_1, \ldots, j_p\})$$



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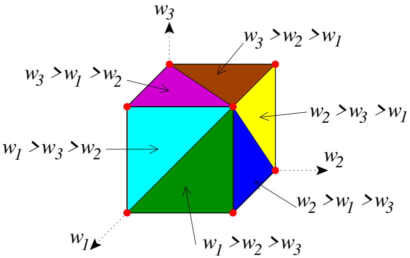
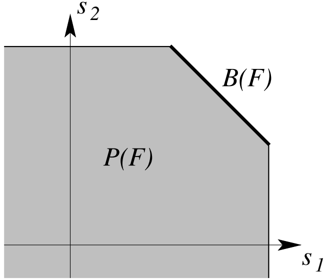
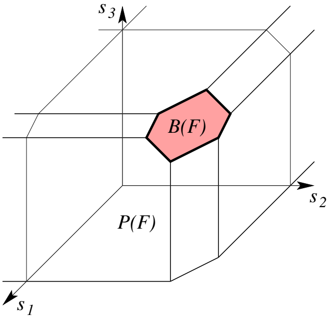


- **Theorem**: F is submodular if and only if f is convex

Submodular Functions and base polyhedra

- Submodular polyhedron:
  $P(F) = \{s \in \mathbb{R}^p,\ \forall A \subset V,\ s(A) \leqslant F(A)\}$
- Base polyhedron: $B(F) = P(F) \cap \{s(V) = F(V)\}$





- **Representation of $f(w)$ as a support function**:

$$\boxed{f(w) = \max_{s \in B(F)} w^\top s}$$

Continuous Optimization

- Goal $(\mathrm{SFM}\mathbf{D})$ $\min_{A \subset V} \sum_{i=1}^{r} F_i(A)$,
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- **Approach.** $\mathrm{SFM}\mathbf{D}_\mathrm{i} \to \mathrm{SFM}\mathbf{C}_\mathrm{i} \to \mathrm{SFM}\mathbf{C} \to \mathrm{SFM}\mathbf{D}$

$\mathrm{SFM}\mathbf{D}_{\mathrm{i}} \rightarrow \mathrm{SFM}\mathbf{C}_{\mathrm{i}} \rightarrow \mathrm{SFM}\mathbf{C} \rightarrow \mathrm{SFM}\mathbf{D}$

1: **Input :** $\mathrm{SFM}\mathbf{D}_i$ for $F_i : 2^V \rightarrow \mathbb{R}$ and $\varepsilon \in \mathbb{R}_+$.
2: **Output :** (w^*, s^*) primal-dual optimal pair of $\mathrm{SFM}\mathbf{C}_i$ for f_i.
3: $A_+ = \mathrm{argmin}_{A \subset V} F_i(A) + \varepsilon|A|$ with a dual certificate $s_+ \in B(F_i)$.
4: $A_- = \mathrm{argmin}_{A \subset V} F_i(A) - \varepsilon|A|$ with a dual certificate $s_- \in B(F_i)$.
5: $w^*(A_+) = -\varepsilon$, $s^*(A_+) = s_+$, $w^*(V \setminus A_-) = \varepsilon$, $s^*(V \setminus A_-) = s_-$
6: $U := A_- \setminus A_+$ and a discrete function $G_i : 2^U$ s.t. $G_i(B) = F_i(A_+ \cup B) - F_i(A_+)$ with Lovász extension $g_i : \mathbb{R}^{|U|} \rightarrow \mathbb{R}$.
7: Solve for optimal solutions of $\min_{w \in \mathbb{R}^{|U|}} g_i(w) + \frac{1}{2}w^2$ and its dual using divide-and-conquer algorithm [Kumar and Bach, 2017] to obtain (w_U^*, s_U^*).
8: $(w^*(U), s^*(U)) = (w_U^*, s_U^*)$

$\mathrm{SFM}\mathbf{D}_{\mathrm{i}} \to \mathrm{SFM}\mathbf{C}_{\mathrm{i}} \to \mathrm{SFM}\mathbf{C} \to \mathrm{SFM}\mathbf{D}$

- $\mathrm{SFM}\mathbf{C}$
  - Primal $\min_{w \in [-\varepsilon,\varepsilon]^n} \sum_{i=1}^r f_i(w) + \frac{1}{2}\|w\|_2^2$.
  - Dual $\max_{(s_1,\ldots,s_r) \in \mathbb{R}^{n \times r}} -\sum_{i=1}^r g_i^*(s_i) - \frac{1}{2}\left\|\sum_{i=1}^r s_i\right\|_2^2$, where

$$g_i^*(s_i) = \inf_{t_i \in B(F_i)} \sup_{w \in [-\varepsilon,\varepsilon]^n} w^\top (s_i - t_i) = \varepsilon \inf_{t_i \in B(F_i)} \|s_i - t_i\|_1.$$

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- Optimization algorithms
  - BCD

$$\forall i\in[r],\; s_i^{\mathrm{new}} = \operatorname*{argmin}_{s_i^{\mathrm{new}}\in\mathbb{R}^n} \; g_i^*(s_i^{\mathrm{new}}) + \frac{1}{2}\Big\|\sum_{j=1}^i s_j^{\mathrm{new}} + \sum_{j=i+1}^r s_j\Big\|_2^2.$$

$\mathrm{SFMD_i} \to \mathrm{SFMC_i} \to \mathrm{SFMC} \to \mathrm{SFMD}$

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  - Acceleration for $r=2$
    $$\begin{aligned} s_2^{\mathrm{new}} &= \operatorname*{argmin}_{s_2^{\mathrm{new}}\in\mathbb{R}^n} g_2^*(s_2^{\mathrm{new}}) + \frac{1}{2}\|t_1 + s_2^{\mathrm{new}}\|^2 \\ s_1^{\mathrm{new}} &= \operatorname*{argmin}_{s_1^{\mathrm{new}}\in\mathbb{R}^n} g_1^*(s_1^{\mathrm{new}}) + \frac{1}{2}\|s_1^{\mathrm{new}} + s_2^{\mathrm{new}}\|_2^2 \\ t_1^{\mathrm{new}} &= s_1^{\mathrm{new}} + \beta(s_1^{\mathrm{new}} - s_1), \end{aligned}$$
    with $\beta = (t-1)/(t+2)$ at iteration t

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$$\boxed{\min_{w \in \mathbb{R}^n} \sum_{i=1}^{r} f_i(w) + \sum_{j=1}^{n} \psi(w_j) \Rightarrow \min_{A \subset V} \sum_{i=1}^{r} F_i(A)}$$

- If w^* is the optimal solution of $\mathrm{SFM}\mathbf{C}$, then $\{w^* \geq 0\}$ is the optimal solution of $\mathrm{SFM}\mathbf{D}$.

Proposition

Given a feasible primal candidate w for $\mathrm{SFM}\mathbf{C}$ with suboptimality η_C, one of the suplevel sets $\{w \geqslant \alpha\}$ of w is an η_D-optimal minimizer of $\mathrm{SFM}\mathbf{D}$, with $\eta_\mathrm{D} = \frac{\eta_\mathrm{C}}{4\varepsilon} + \sqrt{\frac{\eta_\mathrm{C} n}{2}}$.

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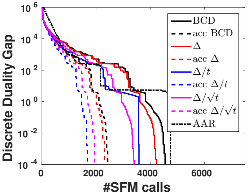
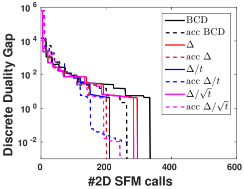
- $\eta_{\mathrm{C}} = \frac{\Delta^2}{t^\alpha}$ where Δ is a notion of diameter of the base polytopes and $\alpha = 2$ for accelerated algorithms and $\alpha = 1$ for plain algorithms.
- $\eta_{\mathrm{D}} = \frac{\Delta\sqrt{n}}{t^{\alpha/2}} + \frac{\Delta^2}{\varepsilon t^\alpha}$.

Experiments - Comparison to state-of-the-art

$3D = 1D + 1D$



$3D = 1D + 2D$



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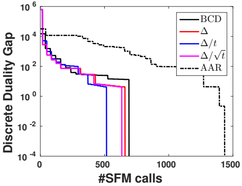
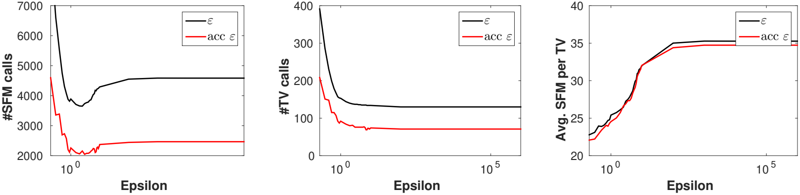


- 2D image of size $n = 2400 \times 2400 = 5.8 \times 10^6$.
- 3D volumetric surface of size $n = 102 \times 100 \times 79 = 8.1 \times 10^5$.

Experiments - Oracle calls for varying ε

Oracle calls for 2D = 1D + 1D for different ε



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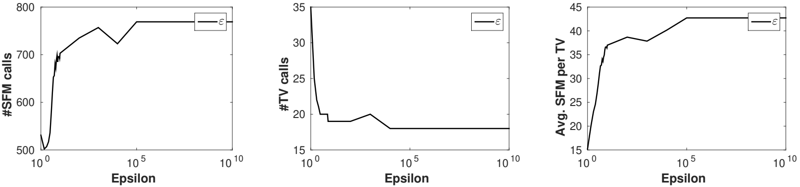

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References I

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Thank you. Questions?