Infinite matrices of Quantum Mechanics

Initialization code

The Indexed Concatenation Notation can represent infinite matrices

$a, a^{\dagger}$:

Out[•]=

$$\begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \sqrt{4} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \sqrt{5} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{4} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \sqrt{5} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \sqrt{6} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Out[•]=

$$\left\{\underset{k\vDash 1}{\overset{\infty}{\text{€}}}\left[\left\{\overset{k}{\text{€}}[0],\ \sqrt{k},\ \overset{\infty}{\text{€}}[0]\right\}\right]\right\},\ \left\{\underset{k\vDash 1}{\overset{\infty}{\text{€}}}\left[\left\{\overset{k-2}{\text{€}}[0],\ \sqrt{k-1},\ \overset{\infty}{\text{€}}[0]\right\}\right]\right\}$$

Read the rows of a this way:

On row k, start with a sequence of k 0s, followed by $\sqrt{1}$, then an infinite sequence of 0s.

Note: $\left\{\overset{5}{\text{€}}[0]\right\} = \left\{\underset{j\vDash 1}{\overset{5}{\text{€}}}[0]\right\} == \{0, 0, 0, 0, 0\}$, (i.e., a subsequence of 5 copies of 0 inserted into the list (an ordered multiset).

Read the rows of $a^{\dagger}$ this way:

On row k, start with a sequence 0s of length $k-2$, followed by $\sqrt{k-1}$, then an infinite sequence of 0s.

Tricky point: when $k = 1$ or 2, what does it mean to concatenate $k-2$ copies of 0? (!)

Let's break this down.

$\overset{k-2}{\text{€}}[0]$ actually means something like $\underset{j\vDash 1}{\overset{k-2}{\text{€}}}[0]$, so what are the meanings of $\underset{j\vDash 1}{\overset{0}{\text{€}}}[0]$ and $\underset{j\vDash 1}{\overset{-1}{\text{€}}}[0]$? The first clearly means a sequence of 0 copies of 0, so it's an empty subsequence. But we consider $\underset{j\vDash 1}{\overset{-1}{\text{€}}}[0] = 0$, too, since "j starts at 1 and goes to -1" shouldn't even start, we've already passed the termination condition!

These operators lead to the following representation of $\hat{x}$ and $\hat{p}$,

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a)$$
$$\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger - a)\ .$$

Math with I-Cats:

How to add $a^\dagger$ and $\pm a$?

$a^\dagger + a$

$$= \left\{ \underset{k=1}{\overset{\infty}{\text{€}}} \left[\left\{ \overset{k-2}{\text{€}}[0],\ \sqrt{k-1}\,,\ \overset{\infty}{\text{€}}[0] \right\} \right] \right\} + \left\{ \underset{k=1}{\overset{\infty}{\text{€}}} \left[\left\{ \overset{k}{\text{€}}[0],\ \sqrt{k}\,,\ \overset{\infty}{\text{€}}[0] \right\} \right] \right\}$$

$$= \left\{ \underset{k=1}{\overset{\infty}{\text{€}}} \left[\left\{ \overset{k-2}{\text{€}}[0],\ \sqrt{k-1}\,,\ \overset{\infty}{\text{€}}[0] \right\} + \left\{ \overset{k}{\text{€}}[0],\ \sqrt{k}\,,\ \overset{\infty}{\text{€}}[0] \right\} \right] \right\}$$

Treat $k = 1$ separately:

$$= \left\{ \left\{ \overset{1-2}{\text{€}}[0],\ \sqrt{1-1}\,,\ \overset{\infty}{\text{€}}[0] \right\} + \left\{ \overset{1}{\text{€}}[0],\ \sqrt{1}\,,\ \overset{\infty}{\text{€}}[0] \right\},\ \underset{k=2}{\overset{\infty}{\text{€}}} \left[\left\{ \overset{k-2}{\text{€}}[0],\ \sqrt{k-1}\,,\ 0,\ 0,\ \overset{\infty}{\text{€}}[0] \right\} + \left\{ \overset{k-2}{\text{€}}[0],\ 0,\ 0,\ \sqrt{k}\,,\ \overset{\infty}{\text{€}}[0] \right\} \right] \right\}$$

$$= \left\{ \left\{ \sqrt{0}\,,\ \overset{\infty}{\text{€}}[0] \right\} + \left\{ 0,\ \sqrt{1}\,,\ \overset{\infty}{\text{€}}[0] \right\},\ \underset{k=2}{\overset{\infty}{\text{€}}} \left[\left\{ \overset{k-2}{\text{€}}[0],\ \sqrt{k-1}\,,\ 0,\ 0,\ \overset{\infty}{\text{€}}[0] \right\} + \left\{ \overset{k-2}{\text{€}}[0],\ 0,\ 0,\ \sqrt{k}\,,\ \overset{\infty}{\text{€}}[0] \right\} \right] \right\}$$

$$= \left\{ \left\{ 0,\ \sqrt{1}\,,\ \overset{\infty}{\text{€}}[0] \right\},\ \underset{k=2}{\overset{\infty}{\text{€}}} \left[\left\{ \overset{k-2}{\text{€}}[0+0],\ \sqrt{k-1}\,,\ 0,\ \sqrt{k}\,,\ \overset{\infty}{\text{€}}[0+0] \right\} \right] \right\}$$

$$= \left\{ \left\{ 0,\ \sqrt{1}\,,\ \overset{\infty}{\text{€}}[0] \right\},\ \underset{k=2}{\overset{\infty}{\text{€}}} \left[\left\{ \overset{k-2}{\text{€}}[0],\ \sqrt{k-1}\,,\ 0,\ \sqrt{k}\,,\ \overset{\infty}{\text{€}}[0] \right\} \right] \right\}$$

Again, the little catch here is that $\overset{k-2}{\text{€}}[0]$ gives 0 copies for both $k = 1$ and 2, but $k - 1$ copies for $k \geq 2$. Also we didn't really want a square root of anything from the $\sqrt{k-1}$ term when $k = 1$, but we kept it, providing an extra 0 -- before the infinite sequence of more 0s. How to handle this? Always treat such cases separately?

Similarly,

$$a^\dagger - a = \left\{ \left\{ 0,\ -\sqrt{1}\,,\ \overset{\infty}{\text{€}}[0] \right\},\ \underset{k=2}{\overset{\infty}{\text{€}}} \left[\left\{ \overset{k-2}{\text{€}}[0],\ \sqrt{k-1}\,,\ 0,\ -\sqrt{k}\,,\ \overset{\infty}{\text{€}}[0] \right\} \right] \right\}$$

The indexed concatenation notation not only serves to compress the information in a structure that includes (possibly nested) patterns, but we can actually do math on these I-Cat objects in the compressed state!