Constructing 0.012345679… from 0.111111… without fractions

We start with the infinite decimal $x = 0.111111\ldots$, understood operationally as an infinite stream of 1s, with no reference to fractions or limits.

To compute x^2, we apply ordinary decimal multiplication as an infinite digit algorithm. At the n-th decimal place, the raw (pre-carry) contribution is the sum of all diagonal products: since each digit is 1, this sum equals n.

We then normalize in base 10: the digit written is n mod 10, and the carry to the next place is floor(n/10). Once carries begin (at $n = 10$), they propagate forward in a stable way, producing a repeating cycle of digits.

Explicitly, the output digits settle into the repeating block: 012345679.

Thus the decimal expansion of x^2 is: 0.012345679012345679…

This construction is purely decimal and algorithmic: the repeating pattern emerges from diagonal sums and carry propagation, with no appeal to fractions such as 1/81.