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ON FINITE-DIFFERENCE SOLUTIONS OF THE HEAT EQUATION IN SPHERICAL COORDINATES

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The finite-difference solution for the temperature distribution within a sphere exposed to a nonuniform surface heat flux involves special difficulties because of the presence of mathematical singularities. For this reason, the adequacy of some finite-difference representations of the heat diffusion equation is examined. In particular, neglecting the contribution from the term causing the singularity is shown as an accurate and efficient method of treating a singularity in spherical coordinates. A method based on the superposition principle is investigated and found quite suitable for this kind of problem in spherical coordinates.

INTRODUCTION

The solution of the transient heat diffusion equation in spherical coordinates has recently attracted attention. Werley and Gilligan [1] solved for the transient temperature distribution within a sphere when it was exposed to a directed uniform heat flux. This problem is important in the area of magnetic thermonuclear reactor engineering, where it is proposed to use a rain of high-speed liquid spheres to remove heat and collect simultaneously large fluxes of particles [2, 3]. The solution of Werley and Gilligan is adequate for short durations when convective and radiative heat transfers are negligible. Duffy [4] has extended the analysis of Werley and Gilligan by including radiative heat transfer. However, his analytical solution is limited to small heat flux densities. For more complex problems, it is necessary to use numerical methods to obtain the solution. To solve the time evolution of the temperature field within a sphere submitted to a nonuniform surface heat flux, the numerical scheme must handle singularities at the poles and center.

Because few papers have dealt with how to resolve these singularities, we intend to examine the adequacy of finite-difference representations for the solution of the heat diffusion equation in the vicinity of a singularity. First the singularities encountered in spherical coordinates are discussed mathematically, followed by a description of algorithms used to resolve these singularities. Finally, results are presented that compare the various algorithms proposed for a particular solution of a two-dimensional problem where the analytical solution is known.

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NOMENCLATURE

C_p	thermal capacity (J/kg · K)	θ	longitudinal position in sphere (radian)
f	fraction of mesh size [between 0 and 1]	λ	thermal conductivity (W/m · K)
I	number of mesh points in r direction	μ	roots of transcendental Eq. (A.3)
J	number of mesh points in ϕ direction	ρ	density (kg/m^3)
K	number of mesh points in θ direction	τ	dimensionless time [$\alpha t/R^2$]
P_n	Legendre polynomial of order n	ϕ	latitudinal position in sphere (radian)
(q/A)	surface heat flux density (W/m^2)		
r	radial position in sphere (m)		
R	radius of sphere (m)	**Subscripts**	
t	time (s)		
T	temperature (K)	a	analytical
U	intermediate temperature in radial direction (K)	avg	average
		i	mesh point in r direction
V	intermediate temperature in latitudinal direction (K)	j	mesh point in ϕ direction
		k	mesh point in θ direction
W	intermediate temperature in longitudinal direction (K)	m	summation indices
		n	summation indices
α	thermal diffusivity (m^2/s)[$\alpha = \lambda/\rho C_p$]	r	radial direction
β	roots of transcendental Eq. (A.2)	t	at time t
γ	roots of transcendental Eq. (A.4)	0	at time zero
ϵ	average temperature error (K)	ϕ	latitudinal direction

SINGULARITIES IN SPHERICAL COORDINATES

The transient heat conduction equation in spherical coordinates, assuming constant physical properties and no heat generation within the sphere, is given by

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial T}{\partial \phi}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 T}{\partial \theta^2} \tag{1}$$

When Eq. (1) is solved numerically, each term of the right-hand side of the equation introduces certain difficulties:

1. The term in the radial direction is singular at $r = 0$. However, if symmetry conditions prevail, the singularity is easily reduced using L'Hôpital's rule.

$$\lim_{r\to 0}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = \lim_{r\to 0}\left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}\right) = \frac{\partial^2 T}{\partial r^2} + 2\frac{\partial^2 T}{\partial r^2} = 3\frac{\partial^2 T}{\partial r^2} \tag{2}$$

In the case where the surface heat flux density is not uniform over the surface of the sphere, the symmetry condition at the geometric center is usually not satisfied. Indeed, the symmetry condition is satisfied only when there are no latitudinal and longitudinal variations, that is, for a one-dimensional problem. However, in practice, in finite-difference solutions, it is usually necessary to assume that symmetry exists at the center to simplify the solution of the problem.

2. The second term (latitudinal direction) is singular at $r = 0$, $\phi = 0$, and ϕ

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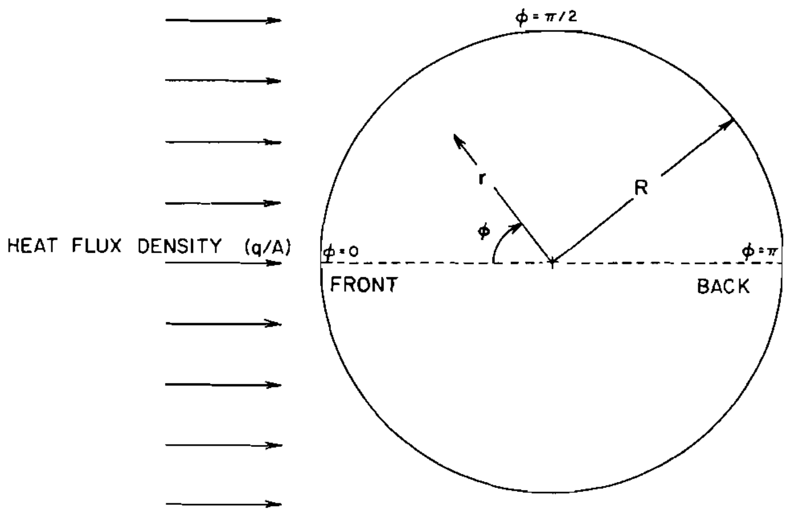

Fig. 1 Geometry of the problem.

$= \pi$. However, if symmetry prevails at $\phi = 0$ and $\phi = \pi$, the singularity at both $\phi = 0$ and $\phi = \pi$ can be resolved by applying L'Hôpital's rule:

$$\begin{aligned}\lim_{\phi \to 0,\pi} \frac{1}{r^2 \sin\phi} \frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial T}{\partial \phi}\right) &= \lim_{\phi \to 0,\pi}\left(\frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\cos\phi}{r^2 \sin\phi}\frac{\partial T}{\partial \phi}\right) \\ &= \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} = \frac{2}{r^2}\frac{\partial^2 T}{\partial \phi^2}\end{aligned} \tag{3}$$

Although singularities at $\phi = 0$ and $\phi = \pi$ are resolved for a symmetrical problem, the singularity at $r = 0$ cannot be eliminated.

3. For the third term (longitudinal direction), the singularities at $r = 0$, $\phi = 0$, and $\phi = \pi$ cannot be eliminated using L'Hôpital's rule.

Therefore the finite-difference solution of a three-dimensional problem in spherical coordinates involves several singularities.

FORMULATION OF THE PROBLEM

The evaluation of various numerical algorithms in the neighborhood of singularities is best done with a known analytical solution. The analytical solution with the greater degree of complexity is the one presented by Werley and Gilligan [1] for a sphere exposed to a directed heat flux as shown in Fig. 1. It is assumed the thermal and physical properties of the sphere are constant and the sphere does not loose heat by radiation and/or convection. In this problem, there is no longitudinal variation (i.e., independent of θ), reducing the problem to a two-dimensional problem. A description of the analytical equation is presented in the Appendix.

The problem is then to determine the transient temperature distribution within a sphere. In the sample problem, the sphere is made of copper ($\lambda = 366.8$ W/m·k,

C_p = 385 J/kg·K, ρ = 8890 kg/m^3, and α = 1.072 cm^2/s). Initially the sphere is at thermal equilibrium with its surrounding at a temperature T_0. Subsequently, the temperature distribution in the sphere undergoes a transient state by allowing heat to flow at its boundaries. To obtain the temperature distribution within the sphere, Eq. (1) will be solved numerically with the following initial conditions:

$$\begin{aligned} T = T_0 \text{ for } 0 &\leqq r \leqq R \\ 0 &\leqq \phi \leqq 2\pi \\ 0 &\leqq \theta \leqq 2\pi \end{aligned} \tag{4}$$

In order to examine the adequacy of finite-difference representations for all singularities in spherical coordinates, the term in the longitudinal direction is kept and the problem is solved numerically as if it were a three-dimensional problem. Since the results obtained with the various algorithms are compared with a known analytical solution, it is assumed that symmetry conditions prevail. The following boundary conditions can be written:

$$\left.\frac{\partial T}{\partial r}\right|_{r=0} = 0 \tag{5}$$

$$-\lambda \left.\frac{\partial T}{\partial r}\right|_{r=R} = \frac{q}{A} \cos\phi \qquad 0 \leq \phi \leq \frac{\pi}{2} \tag{6}$$

$$-\lambda \left.\frac{\partial T}{\partial r}\right|_{r=R} = 0 \qquad \frac{\pi}{2} \leq \phi \leq \pi \tag{7}$$

$$\left.\frac{\partial T}{\partial \phi}\right|_{\phi=0} = 0 \tag{8}$$

$$\left.\frac{\partial T}{\partial \phi}\right|_{\phi=\pi} = 0 \tag{9}$$

$$\left.\frac{\partial T}{\partial \theta}\right|_{\theta=0} = 0 \tag{10}$$

$$\left.\frac{\partial T}{\partial \theta}\right|_{\theta=\pi} = 0 \tag{11}$$

In other words, it is assumed that the sphere is insulated at the back. Equation (5) is strictly true for an angle $\phi = \pi/2$. Because of the assumed symmetry, it is only necessary to consider the upper hemisphere.

DESCRIPTION OF THE PROPOSED ALGORITHMS

The algorithms developed for this study were programmed for a three-dimensional problem even though the analytical solution available is limited to a two-

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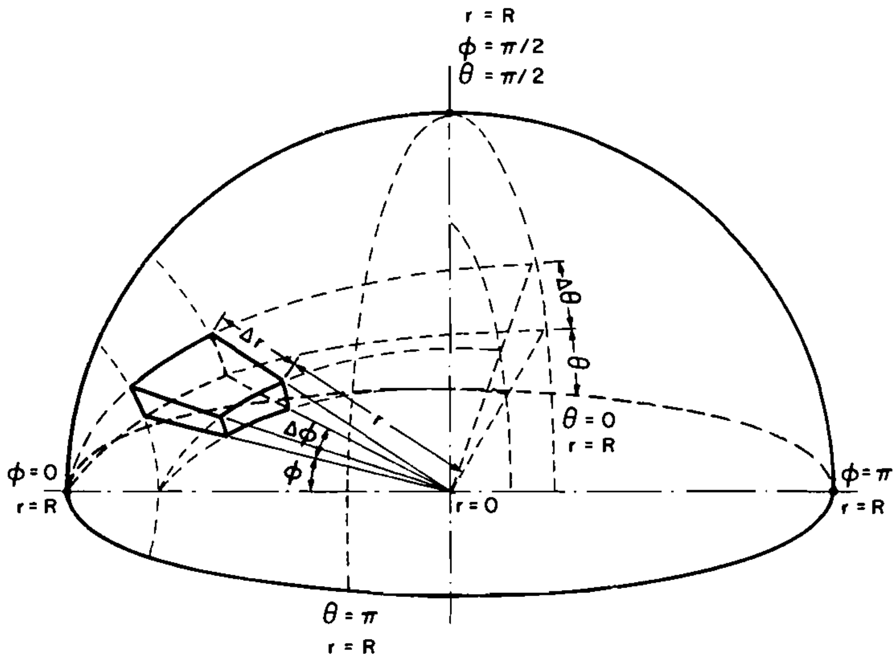

Fig. 2 Finite difference notation.

dimensional case, because we wish to also examine the three singularities introduced by the contribution of the term in the longitudinal direction.

Two different numerical methods were compared in this study: the method based on the superposition principle and the method of Brian. These will be described in turn. Figure 2 gives the notations used to discretize the portion of the sphere for which the temperature field is sought. The r axis is divided into $I - 1$ spherical shells of thickness $\Delta r = R/(I - 1)$, the ϕ axis is divided into $J - 1$ angular sections of angle $\Delta\phi = \pi/(J - 1)$, and the θ axis is divided into $K - 1$ angular sections of angle $\Delta\theta = \pi/(K - 1)$. With this notation, $[IJK]$ mesh points are defined and correspond to (i, j, k), where i varies from 1 to I, j from 1 to J, and k from 1 to K.

Method Based on the Superposition Principle

This method is a version of the alternating-direction implicit (ADI) method, where a three-dimensional problem is treated as three one-dimensional problems [5]. The temperature field at the end of each time step is obtained by summing the three one-dimensional temperature differences calculated over the time step:

$$T_{i,j,k,t+\Delta t} = U_{i,j,k} + V_{i,j,k} + W_{i,j,k} - 2T_{i,j,k,t} \tag{12}$$

where $U_{i,j,k}$, $V_{i,j,k}$, and $W_{i,j,k}$ are the three intermediate temperatures in the radial, the latitudinal, and the longitudinal directions, respectively, obtained by solving the three one-dimensional problems.

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The formulation of the algorithm based on the superposition principle is given by:

$$\frac{U_{i,j,k} - T_{i,j,k,t}}{\alpha\,\Delta t} = \frac{U_{i+1,j,k} - 2U_{i,j,k} + U_{i-1,j,k}}{\Delta r^2} + \frac{U_{i+1,j,k} - U_{i-1,j,k}}{(i-1)\,\Delta r^2} \tag{13}$$

$$\frac{V_{i,j,k} - T_{i,j,k,t}}{\alpha\,\Delta t} = \frac{V_{i,j+1,k} - 2V_{i,j,k} + V_{i,j-1,k}}{[(i-1)\,\Delta r]^2\,\Delta\phi^2} + \frac{\cos\phi(V_{i,j+1,k} - V_{i,j-1,k})}{2[(i-1)\,\Delta r]^2 \sin\phi\,\Delta\phi} \tag{14}$$

$$\frac{W_{i,j,k} - T_{i,j,k,t}}{\alpha\,\Delta t} = \frac{W_{i,j,k+1} - 2W_{i,j,k} + W_{i,j,k-1}}{[(i-1)\,\Delta r]^2 \sin^2\phi\Delta\theta^2} \tag{15}$$

respectively, for the radial, the latitudinal, and the longitudinal directions. Equations (13), (14), and (15) apply only for interior mesh points ($2 \le i \le I - 1$; $2 \le j \le J - 1$, and $2 \le k \le K - 1$).

For mesh points located on a boundary, special consideration is necessary. For the radial direction at $r = 0$ ($i = 1$), Eq. (2) applies and the finite-difference equation becomes

$$\frac{U_{1,j,k} - T_{1,j,k,t}}{\alpha\,\Delta t} = 6\left[\frac{U_{2,j,k} - U_{1,j,k}}{(\Delta r)^2}\right] \tag{16}$$

On the surface of the sphere $r = R$ ($i = I$), there is a heat flux into the surface given by

$$-\lambda\frac{\partial U}{\partial r} = \left(\frac{q}{A}\right)_\phi \tag{17}$$

In finite-difference form, Eq. (17) is best represented by the central difference form

$$-\lambda\frac{U_{I+1,j,k} - U_{I-1,j,k}}{2\,\Delta r} = \left(\frac{q}{A}\right)_j \tag{18}$$

where $(I + 1)$ is an imaginary grid point located at a distance Δr beyond the boundary. The imaginary point is eliminated by combining Eqs. (13) and (18) to yield

$$\frac{U_{I,j,k} - T_{I,j,k,t}}{\alpha\,\Delta t} = \frac{2(U_{I-1,j,k} - U_{I,j,k})}{(\Delta r)^2} + \frac{2(q/A)_j}{\lambda\,\Delta r}\left(1 + \frac{1}{I-1}\right) \tag{19}$$

The surface heat flux density for a three-dimensional problem would be function of both ϕ and θ. For the directed heat flux of the present problem, the surface heat-flux density is only a function of ϕ:

$$\begin{aligned} \left(\frac{q}{A}\right)_j &= \left(\frac{q}{A}\right)\cos[(j-1)\Delta\phi] && 0 \le j \le \text{integer}\left(\frac{J}{2} + 1\right) \\ &= 0 && \text{integer}\left(\frac{J}{2} + 1\right) \le j \le J \end{aligned} \tag{20}$$

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The number of mesh points J in the latitudinal direction is usually chosen as an odd number in order to have a mesh point located at $\pi/2$.

The formulation of Eq. (20) is mathematically correct. However, because of the geometry of the sphere, energy is not conserved when this formulation is used. To circumvent this problem it is necessary to specify the heat flux density at each mesh point as the average heat flux density over the length of the arc extending halfway to the two neighboring mesh points. This procedure allows the energy to be conserved to within 0.1%.

At $\phi = 0$ ($j = 1$), Eq. (3) applies and the symmetry condition requires that

$$V_{i,0,k} = V_{i,2,k} \tag{21}$$

so that the finite-difference equation becomes

$$\frac{V_{i,1,k} - T_{i,1,k,t}}{\alpha\,\Delta t} = 4\,\frac{V_{i,2,k} - V_{i,1,k}}{[(i-1)\,\Delta r]^2\,\Delta\phi^2} \tag{22}$$

Similarly, at $\phi = \pi$ ($j = J$), the finite-difference equation becomes

$$\frac{V_{i,J,k} - T_{i,J,k,t}}{\alpha\,\Delta t} = 4\,\frac{V_{i,J-1,k} - V_{i,J,k}}{[(i-1)\,\Delta r]^2\,\Delta\phi^2} \tag{23}$$

Similarly, assuming symmetry at $\theta = 0$ ($k = 1$) and $\theta = \pi$ ($k = K$), the finite-difference equations are given respectively by Eqs. (24) and (25).

$$\frac{W_{i,j,1} - T_{i,j,1,t}}{\alpha\,\Delta t} = 2\,\frac{W_{i,j,2} - W_{i,j,1}}{[(i-1)\,\Delta r]^2\sin^2\phi\,\Delta\theta^2} \tag{24}$$

$$\frac{W_{i,j,K} - T_{i,j,K,t}}{\alpha\,\Delta t} = 2\,\frac{W_{i,j,K-1} - W_{i,j,K}}{[(i-1)\,\Delta r]^2\sin^2\phi\,\Delta\theta^2} \tag{25}$$

As mentioned before, Eqs. (14), (15), and (22) through (25) are singular at the mesh point located at the geometric center. Three methods can be used to resolve this singularity problem.

The first algorithm is simply not to include the latitudinal and the longitudinal contributions of the equation at the geometric center. This proposed method is motivated by the fact that at the geometric center, there are no variations in the latitudinal and the longitudinal variables since the angular variations are made on a single point. This implies that Eqs. (14), (15), and (22) through (25) are solved only for $i = 2$ to $i = I$.

The second method is to exclude the point $r = 0$ by introducing a small, but finite, interior surface. This method can be termed as the hollow-sphere approximation. When $r = 0$ ($i = 1$), the term $[(i - 1)\,\Delta r]$ is replaced by

$$\delta r = f_r\,\Delta r \tag{26}$$

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where f_r is a number between 0 and 1 so that the null radius is replaced by a small nonzero value.

The third method consists of approximating the region of singularity in a Cartesian formulation. However, this method adds to the complexity of the solution since it is necessary to consider and match the two regions defined respectively in the spherical and Cartesian coordinates. In this investigation only the first two approaches will be considered.

The three sets of Eqs. (13), (16), and (19); (14), (22), and (23); (15), (24), and (25), form tridiagonal matrix systems. The temperature distribution at the end of the time step is computed from Eq. (12).

It is important to note that J times K values of the temperature at the geometric center are calculated since the temperature distribution is calculated in the radial direction at each angle of the latitudinal and longitudinal directions. The temperature at the geometric center at the end of each time step is calculated as the average value of all these temperatures calculated for the same mesh point. This averaging procedure compensates for the assumption made that symmetry condition prevails at the geometric center.

Equations (15), (24), and (25) are also singular at latitudinal angles $\phi = 0$ and $\phi = \pi$ due to the sin ϕ function appearing in the denominator. The two methods proposed for the singularity at the geometric center can also be used for the present singularity. For the first algorithm, Eqs. (15), (24), and (25) are solved only for $j = 2$ to $j = J - 1$, since there are no variations in the longitudinal direction at $\phi = 0$ and $\phi = \pi$. For the second algorithm, a hollow-shaft approximation is used whereby mesh points at $\phi = 0$ and $\phi = \pi$ are eliminiated by introducing a small, but finite, interior surface. At these two latitudinal angles, ϕ is replaced by

$$\begin{aligned} \phi &= f_\phi \, \Delta\phi && \text{for } j = 1 \\ \phi &= \pi - f_\phi \, \Delta_\phi && \text{for } j = J \end{aligned} \tag{27}$$

where f_ϕ is a number between 0 and 1 so that sin ϕ will not become zero at these two angles.

Method of Brian

The method proposed by Brian [6] is an alternating-direction implicit (ADI) method where the heat diffusion equation is solved implicitly, alternating in the radial, latitudinal, and longitudinal directions. The algorithm is given by the following equations:

$$\begin{aligned} \frac{U_{i,j,k} - T_{i,j,k,t}}{\alpha(\Delta t/2)} &= \frac{U_{i+1,j,k} - 2U_{i,j,k} + U_{i-1,j,k}}{\Delta r^2} + \frac{U_{i+1,j,k} - U_{i-1,j,k}}{(i-1)\,\Delta r^2} \\ &+ \frac{T_{i,j+1,k,t} - 2T_{i,j,k,t} + T_{i,j-1,k,t}}{[(i-1)\,\Delta r]^2\,\Delta\phi^2} + \frac{\cos\phi(T_{i,j+1,k,t} - T_{i,j-1,k,t})}{2[(i-1)\,\Delta r]^2 \sin\phi\,\Delta\phi} \\ &+ \frac{T_{i,j,k+1,t} - 2T_{i,j,k,t} + T_{i,j,k-1,t}}{[(i-1)\,\Delta r]^2 \sin^2\phi\,\Delta\theta^2} \end{aligned} \tag{28}$$

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$$\frac{V_{i,j,k} - T_{i,j,k,t}}{\alpha(\Delta t/2)} = \frac{U_{i+1,j,k} - 2U_{i,j,k} + U_{i-1,j,k}}{\Delta r^2} + \frac{U_{i+1,j,k} - U_{i-1,j,k}}{(i-1)\,\Delta r^2} + \frac{V_{i,j+1,k} - 2V_{i,j,k} + V_{i,j-1,k}}{[(i-1)\,\Delta r]^2\,\Delta\phi^2} + \frac{\cos\phi\,(V_{i,j+1,k} - V_{i,j-1,k})}{2[(i-1)\,\Delta r]^2 \sin\phi\,\Delta\phi} + \frac{T_{i,j,k+1,t} - 2T_{i,j,k,t} + T_{i,j,k-1,t}}{[(i-1)\,\Delta r]^2 \sin^2\phi\,\Delta\theta^2} \tag{29}$$

$$\frac{T_{i,j,k,t+\Delta t} - V_{i,j,k}}{\alpha(\Delta t/2)} = \frac{U_{i+1,j,k} - 2U_{i,j,k} + U_{i-1,j,k}}{\Delta r^2} + \frac{U_{i+1,j,k} - U_{i-1,j,k}}{(i-1)\,\Delta r^2} + \frac{V_{i,j+1,k} - 2V_{i,j,k} + V_{i,j-1,k}}{[(i-1)\,\Delta r]^2\,\Delta\phi^2} + \frac{\cos\phi(V_{i,j+1,k} - V_{i,j-1,k})}{2[(i-1)\,\Delta r]^2 \sin\phi\,\Delta\phi} + \frac{T_{i,j,k+1,t+\Delta t} - 2T_{i,j,k,t+\Delta t} + T_{i,j,k-1,t+\Delta t}}{[(i-1)\,\Delta r]^2 \sin^2\phi\,\Delta\theta^2} \tag{30}$$

This very popular method has the net advantage of being unconditionally stable, whereas the method based on the superposition principle is not [7, 8].

The same boundary conditions discussed earlier apply as well for the method of Brian. For this method, to resolve the singularity problem, the algorithms of the hollow-sphere and the hollow-shaft approximations were used.

RESULTS AND DISCUSSION

All the results presented in this section were obtained for a sphere, made of copper and having a diameter of 10 cm. The sphere is exposed to a directed heat flux of 100 W/cm^2. To evaluate the accuracy of the various methods compared in this paper, an average temperature error is used. It is defined as the average of the square root of the sum of the square of the error between the numerically calculated temperature and the analytical temperature. It is given by

$$\epsilon = \sqrt{\frac{\sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} (T_{i,j,k} - T_a)^2}{IJK}} \tag{31}$$

where T_a is the analytical temperature computed at each point from Eq. (A1). In each case the summation for the calculation of the analytical temperature was assumed to have converged when the variation of the temperature was less than 10^{-5}°C. The convergence was faster for large time level and small ratio of (r/R). The mean-square-error criterion used in the paper gives smaller values than the actual mean temperature error. This criterion was chosen to weight larger temperature differences more heavily.

Figure 3 shows the results obtained with the two numerical methods compared in this investigation. The method of Brian employs the hollow-sphere and the hollow-

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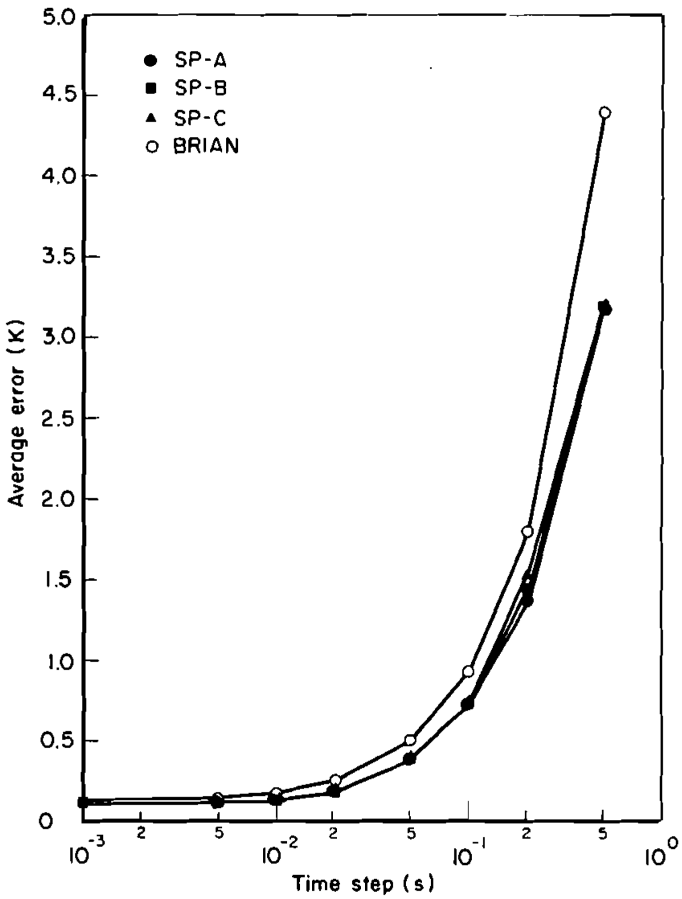

Fig. 3 Curves of the average temperature error as a function of the time increment for various algorithms.

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shaft approximations, whereas the method based on the superposition principle uses three different algorithms. The first algorithm (SP-A) assumes that symmetry prevails at the center for the solution in the radial direction but the contribution of the center point is not included for the calculation in the latitudinal and the longitudinal directions in order to remove the singularity at $r = 0$. As explained before, there are in fact no variations of the temperature in the latitudinal and the longitudinal directions at that point. For the same reason, the longitudinal contribution is not included for mesh points located at $\phi = 0$ and $\phi = \pi$. The second algorithm based on the superposition principle (SP-B) uses the hollow-sphere and the hollow-shaft approximations where the null radius at the geometric center is replaced by a small fraction f_r of the radical mesh size and the null sin ϕ is eliminated by respectively adding and subtracting a small fraction f_ϕ of the latitudinal mesh size at $\phi = 0$ and $\phi = \pi$. In this case, for the method of Brian and the method based on the superposition principle, fractions f_r and f_ϕ were 0.1. For the third algorithm based on the superposition principle (SP-C), instead of assuming symmetry at the geometric cen-

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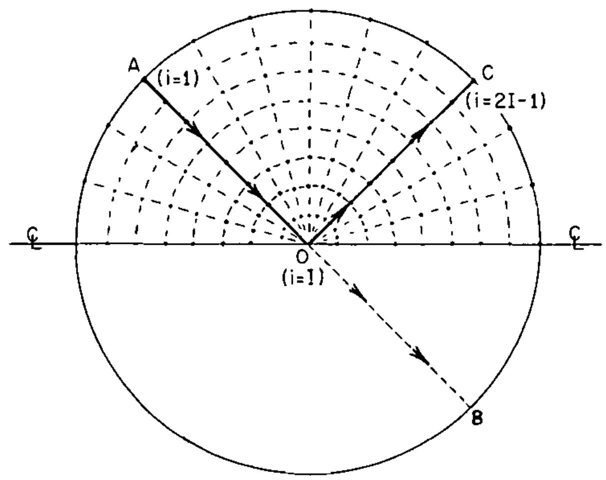

Fig. 4 Methodology of solution for SP-C algorithm.

ter, the problem is solved starting with a surface mesh point, going through the geometric-center mesh point, and ending at the antipode mesh point as shown by line *AOB* in Fig. 4. The central finite-difference equation for the center point is formed with the center mesh point and the two adjacent mesh points located on the same axis. In practice, because of the symmetry condition at $\phi = 0$ and $\phi = \pi$, the mesh point in the other half of the sphere is replaced by its mirror-image mesh point so that the solution was obtained along the trajectory *AOC* in Fig. 4.

The results of Fig. 3 as well as all the results presented in this paper were obtained following a heating period of 10 s. The temperature distribution at the end

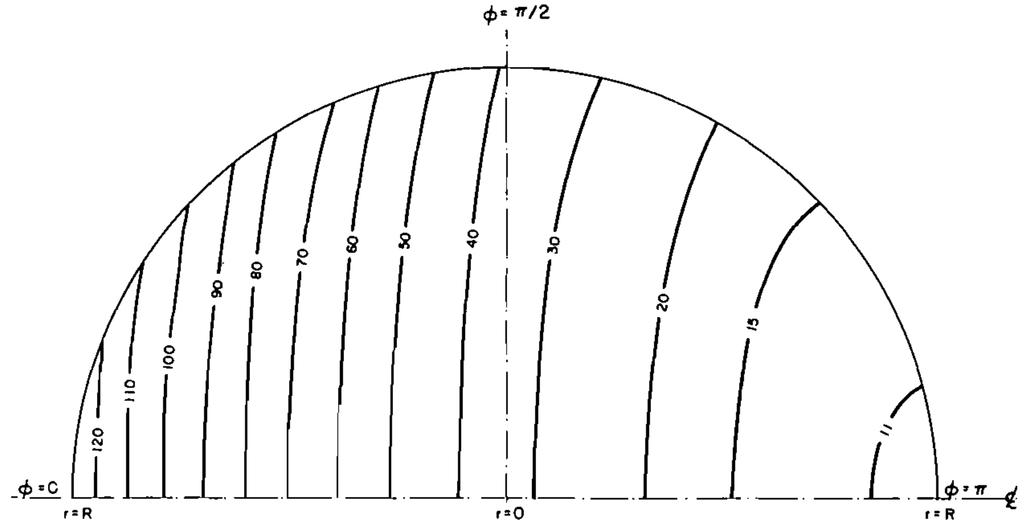

Fig. 5 Temperature distribution at a heating period of 10 s.

of this heating period is presented in Fig. 5. Even under severe heating conditions and sharp temperature gradients, the two methods are very accurate.

The three curves of Fig. 3 for the three algorithms of the superposition principle are identical. The results obtained with the method of Brian and the methods based on the superposition principle are identical for small time increments. As the time increment is increased, the method of Brian becomes less accurate than the methods based on the superposition principle. For the method of Brian, it is necessary to use the hollow-sphere and the hollow-shaft approximations in each of the three directions, which certainly contributes to increasing the inaccuracy of this method.

One additional factor that must be taken into account in the selection of a numerical method is the computation time necessary to obtain the solution of the problem. When compared with the pure method based on the superposition principle (SP-A), the two other algorithms based on the superposition principle (SP-B) and (SP-C) and the method of Brian took, respectively, 1.24, 1.10, and 1.96 more computation time on a VAX-780 computer.

The similarity in the results obtained for the three forms of the superposition principle suggests that the calculation of the temperature field within the sphere is not sensitive to the assumption made for the geometric center. This is reasonable because if the number of radial mesh points for uniform mesh spacings is sufficiently large, the influence of the center point becomes negligible since the volume of each slab is proportional to the square power of the radius. Moreover, the averaging procedure performed at the geometric center at each time increment pushes the solution toward that found with the symmetry assumption.

The results of Fig. 3 for the method based on the superposition principle show that ignoring completely the contribution of the center point when solving in the latitudinal and longitudinal directions amounts to replacing the null radius by a small nonzero radius. Then, one can ask the following question: Is there an optimum value of the fraction of mesh size that would produce the best representation of the temperature field? Figure 6 shows the results for the average temperature errors as a function of the fraction of the radial mesh size, f_r, used to replace the null radius for the two numerical methods and various time increments. For the method of Brian, the same fraction ($f_r = f_\phi$) was also used to replace the null ($\sin\phi$) in the longitudinal direction. The results show that the method of Brian is very sensitive to the fraction of mesh size used to replace the singularity in the algorithm. The method based on the superposition principle is insensitive to the fraction of mesh size even for fractions as low as 10^{-6}. It is interesting to note that a fraction of 1 is very accurate, suggesting that the center point can be ignored completely when solving for the radial direction. It is as efficient as any other methods and also has the advantage of being more simple. For identical time increments, the method based on the superposition principle is more accurate than the method of Brian.

Figure 7 gives the results for the average temperature error as a function of the number of mesh points in the radial and latitudinal directions for the method of superposition and the method of Brian, respectively. These results show that the accuracy of numerical methods in spherical conditions is strongly dependent on the number of mesh points in the latitudinal direction and varies weakly with the number of mesh points in the radial direction. This strong dependence in the latitudinal direction is caused by the rapid variation of the heat flux density at the surface of the

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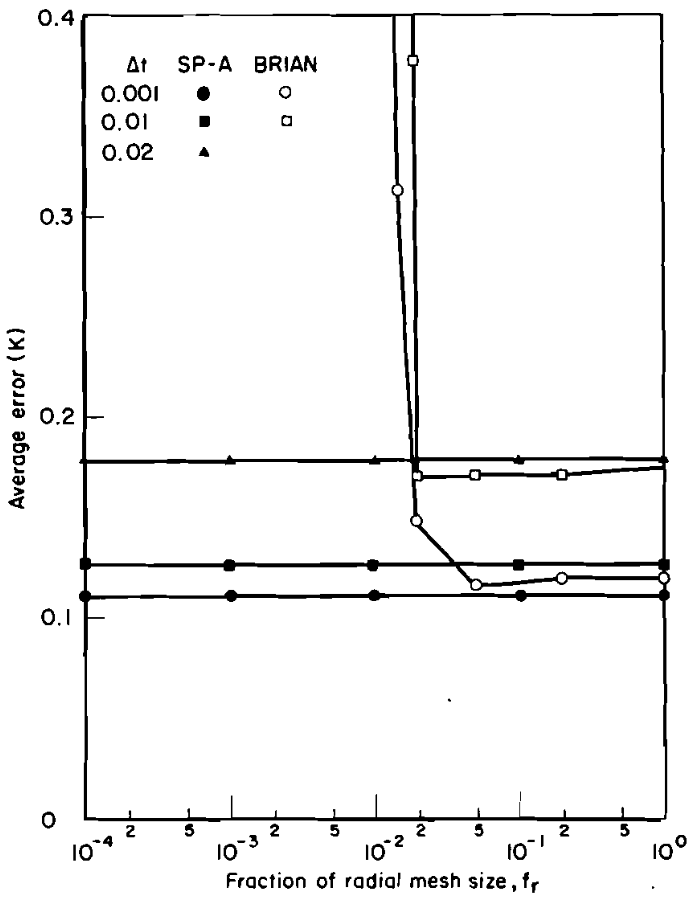

Fig. 6 Curves of the average temperature error as a function of the fraction of the radial mesh size, f_r, and time increment for the two numerical methods.

sphere. Essentially, eight mesh points in the radial direction are sufficient for an adequate representation of the temperature field. For better accuracy, it is preferable to increase the number of mesh points in the latitudinal direction. The two numerical methods behave similarly in their accuracy with respect to the number of mesh points. However, the method based on the superposition principle fares slightly better than the method of Brian.

Since the method of Brian is very popular, a parametric study on the fractions f_r and f_ϕ has been done, with the results presented in Fig. 8. The results show that there is an additive effect of f_r and f_ϕ on the accuracy of the hollow-sphere and hollow-shaft approximations when the method of Brian is used and that for an accurate representation of the temperature field, the product of f_r and f_ϕ must be smaller than 0.001. Here again, large fractions f_r and f_ϕ are adequate to solve singularity problems in spherical coordinates. The longitudinal contribution is the most sensitive term because of the presence of $r^2 \sin^2 \phi$ in the denominator. Choosing small fractions f_r and f_ϕ leads to ill-conditioned matrices that considerably affect the results.

The method of Brian is affected the most, since the longitudinal contribution appears in the solution of the three alternating directions.

All the results presented in this study were obtained with only three mesh points ($K = 3$) in the longitudinal direction, since the surface heat flux density was not a function of Θ. Increasing the number of mesh points in the longitudinal direction gives identical results to those presented earlier, provided that fractions f_r and f_ϕ are chosen large enough as shown in Fig. 9. An increase in the number of mesh points results in a smaller value of $\Delta\Theta^2$, which further contributes to the ill-conditioning of matrices. Therefore choosing large fractions f_r and f_ϕ—say, larger than 0.1—produces accurate results.

Unfortunately, no analytical solutions exist for the three-dimensional problem in spherical coordinates. Consequently, it is impossible to test which algorithm would be adequate for coping with singularities associated with a truly three-dimensional case. However, with the results obtained for the directed heat flux problem considered in this investigation, it is safe to state that not including the contribution of the

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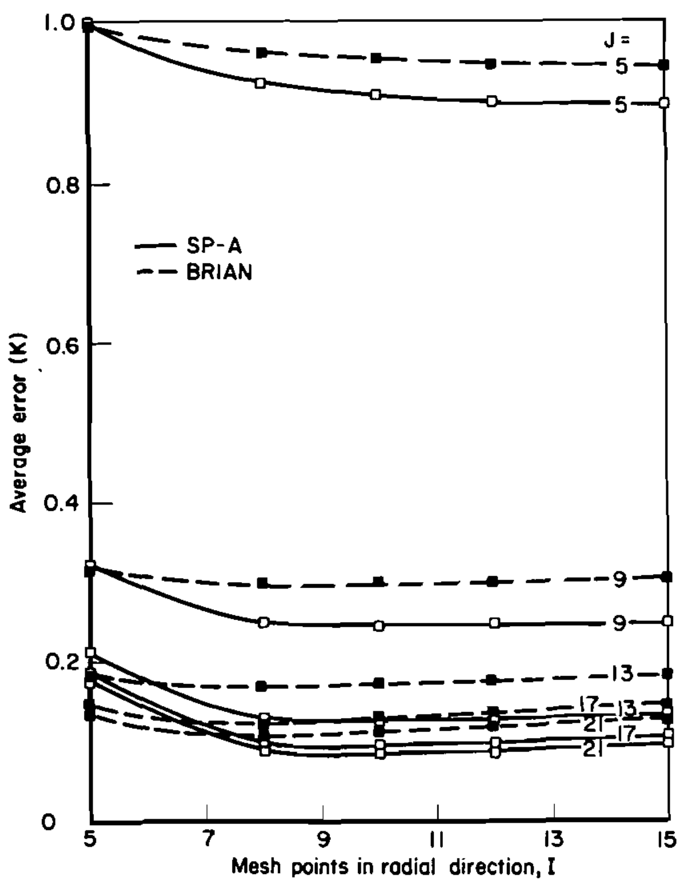

Fig. 7 Curves of the average temperature error as a function of the number of mesh points in both the radial and latitudinal directions for the two numerical methods.

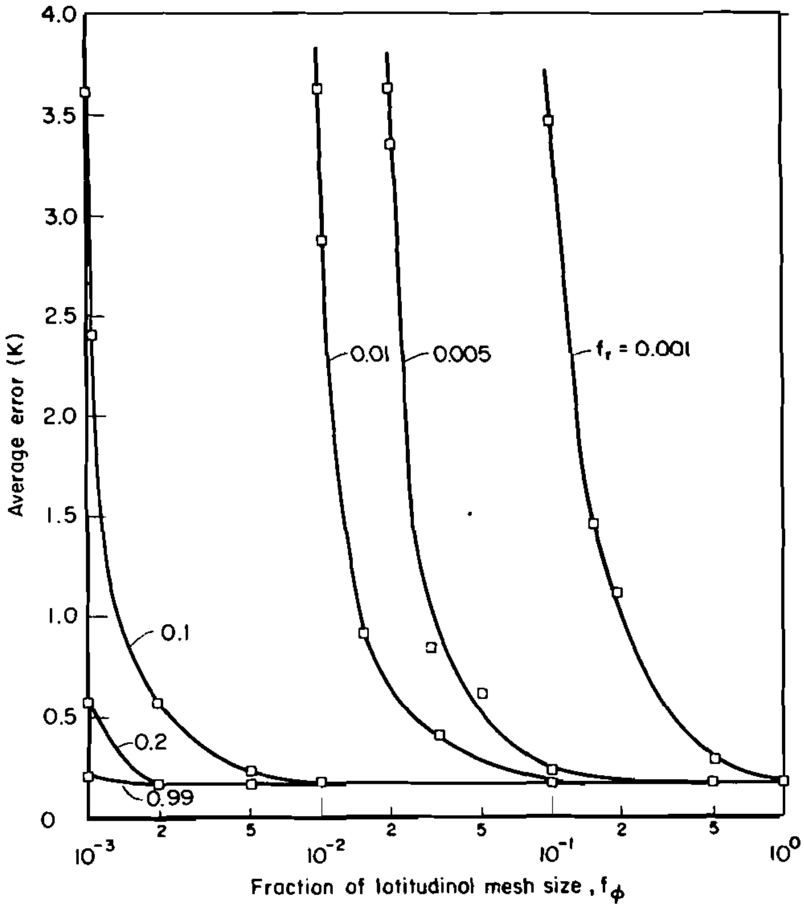

Fig. 8 Curves of the average temperature error as a function of the fractions f_r and f_ϕ for the method of Brian.

terms causing singularities or using the hollow-sphere and the hollow-shaft approximations with large fractions of the mesh size seem to lead to an adequate representation of the temperature field within the sphere, provided that the number of mesh points is sufficiently large.

CONCLUSION

This paper considered the adequacy of finite-difference representations for problems that have singularities. To test the validity of the various algorithms, the heat equation in spherical coordinates was solved numerically in three dimensions and compared with a known two-dimensional analytical solution.

The two numerical methods tested, the method of Brian and the method based on the superposition principle, provide adequate representation of the temperature

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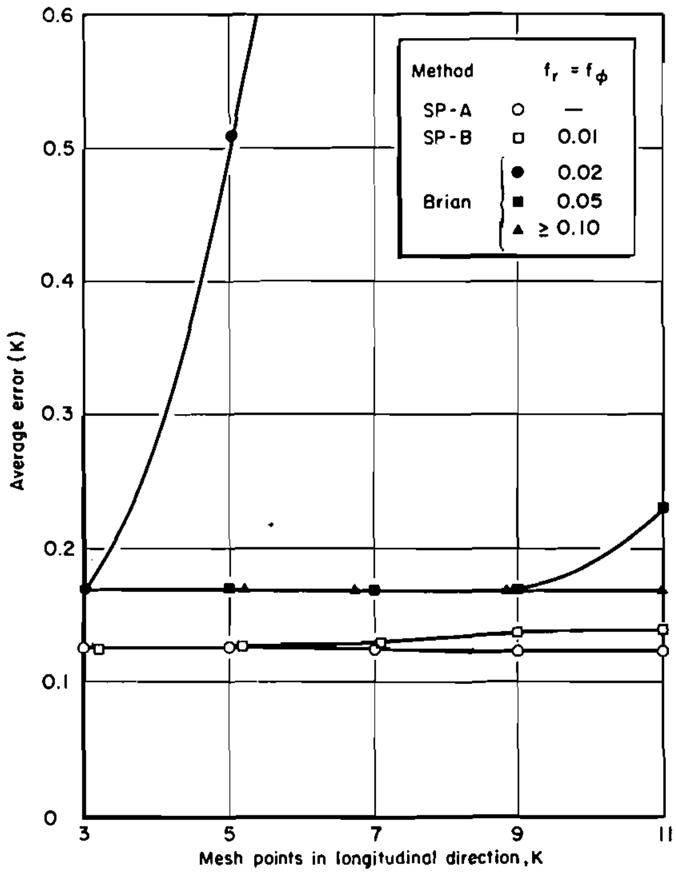

Fig. 9 Curves of the average temperature error as a function of the number of mesh points in the longitudinal direction for the two numerical methods (I = 10, J = 13, t = 10 s).

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field within the sphere. In all cases, the method based on the superposition problem gave slightly better results than the popular method of Brian, with the additional benefit of requiring less computation time. Excluding the contribution of the terms in the finite-difference equations containing the singularity is the simplest and most efficient manner to solve singularity problems. The accuracy of this algorithm compares well with the hollow-sphere and the hollow-shaft approximations algorithm, provided that the fraction of the mesh size replacing the singularity is sufficiently large.

APPENDIX

The analytical solution of the transient heat conduction equation, with a uniform initial temperature, located in a uniform field of directed heat flux as derived by Werley and Gilligan [1], is given by the following equation:

$$\frac{\lambda T(r,\phi,\tau)}{R(q/A)} = \frac{\lambda T_0}{R(q/A)} + \frac{3\tau}{4} + \frac{5(r/R)^2 - 3}{40} - \frac{1}{2(r/R)} \sum_{m=1}^{\infty} \frac{\sin(\beta_m r/R)\, e^{-\beta_m^2 \tau}}{\beta_m^2 \sin(\beta_m)}$$
$$+ \cos\phi \left[\frac{(r/R)}{2} - \left(\frac{r}{R}\right)^{-1/2} \sum_{m=1}^{\infty} \frac{J_{3/2}(\mu_m r/R)\, e^{-\mu_m^2 \tau}}{J_{3/2}(\mu_m)(\mu_m^2 - 2)} \right]$$
$$- \sum_{n=1}^{\infty} \frac{2n + 1/2}{2n^2 + n - 1} P_{2n}(0) P_{2n}(\cos\phi) \left\{ \frac{(r/R)^{2n}}{4} - \left(\frac{r}{R}\right)^{-1/2} \right.$$
$$\left. - \left(\frac{r}{R}\right)^{-1/2} \sum_{m=1}^{\infty} \frac{J_{2n+1/2}(\gamma_m r/R)\, e^{-\gamma_m^2 \tau}}{J_{2n+1/2}(\lambda_m)[\gamma_m^2 - 2n(2n+1)]} \right\} \tag{A1}$$

where β_m, μ_m, and γ_m are the roots of the following transcendental equations:

$$\tan \beta_m = \beta_m \tag{A2}$$

$$\frac{d}{d\mu}\left[\frac{J_{3/2}(\mu_m)}{\sqrt{\mu_m}}\right] = 0 \tag{A3}$$

$$\frac{d}{d\gamma}\left[\frac{J_{2n+1/2}(\gamma_m)}{\sqrt{\gamma_m}}\right] = 0 \tag{A4}$$

with β_0, μ_0, $\gamma_0 = 0$.

For certain values of (r/R) and small τ, a large number of terms is required before convergence is achieved with an accuracy of less than 0.01%. For example, the closer the ratio (r/R) is to unity, the larger the number of terms is required.

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