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Crank Nicolson Method for Solving Parabolic Partial Differential Equations

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Abstract

This paper presents Crank Nicolson method for solving parabolic partial differential equations. Crank Nicolson method is a finite difference method used for solving heat equation and similar partial differential equations. This method is of order two in space, implicit in time, unconditionally stable and has higher order of accuracy.

Key words: Crank Nicolson Method, Finite Difference Method, Exact Solution, Parabolic Equation, Stability

Mathematics Subject Classification: 35A20, 35A35, 35B35, 35K05

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I. INTRODUCTION

The finite difference approximations are one of the simplest and of the oldest methods to solve partial differential equations. It was already known by L. Euler (1707-1783) ca. 1768, in one dimension of space and was probably extended to two dimensions by C. Runge (1856-1927) ca. 1908. The advent of finite difference techniques in numerical applications began in the early 1950s and their development was stimulated by the emergence of computers that offered a convenient framework for dealing with complex problems of science and technology. Theoretical results have been obtained during the last five decades regarding the accuracy, stability and convergence of the finite difference method for partial differential equations.

The mathematical formulation of most problems in science involving rates of change with respect to two or more independent variables, usually representing time, often leads to a partial differential equation.

Problems involving time as one independent variable sometimes lead to parabolic partial differential equations, the simplest of which is the diffusion equation, derived from the theory of heat conduction [11]. The diffusion equation plays an important role in a broad range of practical applications such as fluid mechanics. Only a limited number of special types of parabolic equation have been solved analytically and the usefulness of these solutions is further restricted to problems involving shapes for which the boundary conditions can be satisfied. In such cases numerical methods are some of the very few means of solution.

Crank Nicolson Method for solving parabolic partial differential equations was developed by John Crank and Phyllis Nicolson in the mid-20th century. A practical method for numerical evaluation of partial differential equations of the heat conduction type was considered by [4]. [7] modified the

simple explicit scheme and proved that it is much more stable than the simple explicit case, enabling larger time steps to be used. [8] considered the stability and accuracy of finite difference method for option pricing. However, according to [2], the accuracy of the simple explicit method is barely improved upon.

There are many exhaustive texts on this subject such as ([1], [3], [5], [6], [9], [10], [12], [13]), just to mention few.

In this paper we shall only consider the accuracy of Crank Nicolson method for solving parabolic partial differential equations.

II. CRANK NICOLSON METHOD

This section presents Crank Nicolson method for solving parabolic partial differential equations as follows:

2.1 PARABOLIC PARTIAL DIFFERENTIAL EQUATION

Partial differential equations occur frequently in mathematics, natural sciences and engineering. These are problems involving rates of change of functions of several variables. For examples

- Advection Equation

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = 0$$

- Heat Equation

$$\frac{\partial f}{\partial t} - D \frac{\partial^2 f}{\partial x^2} = 0$$

- Poisson Equation

$$-\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} = u(x, y)$$

- Wave Equation

$$\frac{\partial^2 f}{\partial t^2} - c^2 \frac{\partial^2 f}{\partial y^2} = 0$$

In these equations above x, y are the space coordinates, v, D, c are real positive constants and t, x are often viewed as time and space coordinates respectively.

The general second order linear partial differential equation with two independent variables and one dependent variable is given by

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D = 0 \quad (1)$$

where A, B, C are functions of the independent variables, x, y and D can be a function of

$x, y, u, \frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$. If $B^2 - 4AC = 0$, (1) is called a parabolic partial differential equation. One of

the simple examples of a parabolic PDE is the heat-conduction equation for a metal rod shown in Figure I below;

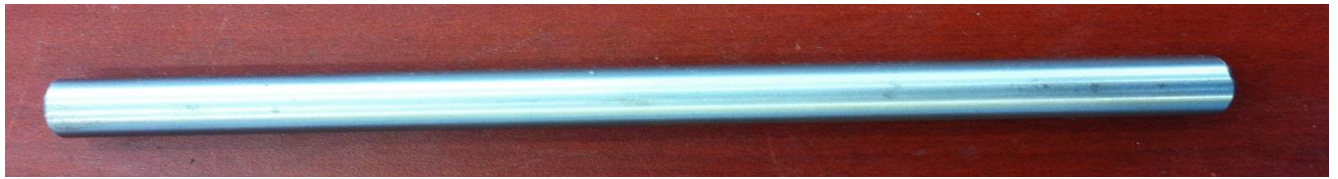


FIGURE I: A METAL ROD

$$\frac{1}{\lambda} \frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} \quad (2)$$

Where f = Temperature as a function of location, x and time, t in which the thermal diffusivity and λ is given by

$$\lambda = \frac{\alpha}{\rho C}$$

where α = Thermal conductivity of rod material, ρ = Density of rod material, C = Specific heat of the rod material

2.2 FINITE DIFFERENCE METHODS

The finite difference methods attempt to solve partial differential equations by approximating the differential equation over the area of integration by a system of algebraic equations. They are a means of obtaining numerical solutions to partial differential equations.

The most common finite difference methods for the solution of partial differential equations are:

- Explicit method
- Implicit method
- Crank Nicolson Method

These methods are closely related but differ in stability, accuracy and execution speed. In the formulation of a partial differential equation problem, there are three components to be considered:

- The partial differential equation.
- The region of space-time on which the partial differential equation is required to be satisfied.
- The ancillary boundary and initial conditions to be met.

Let us consider the diagram below:

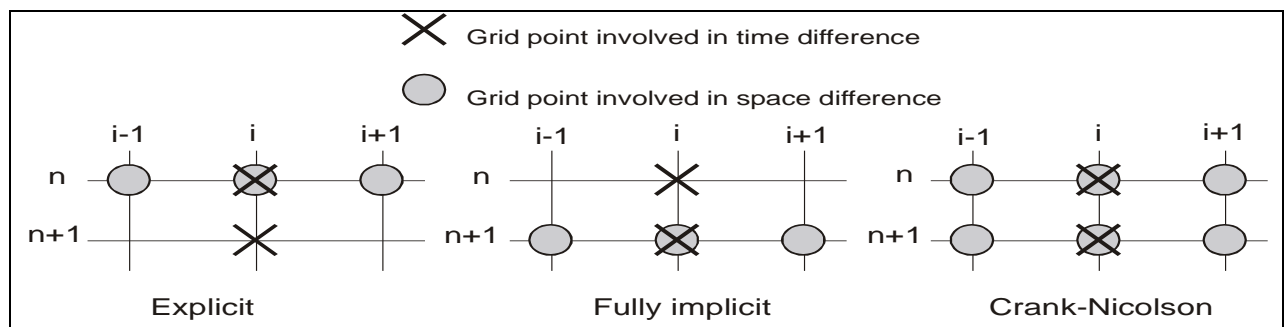


FIGURE II: A COMPUTATIONAL DIAGRAM FOR EXPLICIT AND IMPLICIT METHODS

In finite difference methods we replace the partial derivative occurring in the partial differential equations by approximations based on Taylor series expansions of function near the point or points of interest. The derivative we seek is expressed with many desired order of accuracy.

The finite difference form of the heat equation (2) is now given with the spatial second derivative evaluated from a combination of the derivatives at time steps n and $n + 1$

$$\frac{1}{\lambda} \left(\frac{f_i^{n+1} - f_i^n}{\delta t} \right) = m \left(\frac{f_{i+1}^n - 2f_i^n + f_{i-1}^n}{(\delta x)^2} \right) + (1-m) \left(\frac{f_{i+1}^{n+1} - 2f_i^{n+1} + f_{i-1}^{n+1}}{(\delta x)^2} \right) \quad (3)$$

From the above formula, we will have an explicit scheme when $m=1$ and an implicit scheme when $m=0$ as shown below in equations (4) and (5) respectively:

$$\frac{1}{\lambda} \left(\frac{f_i^{n+1} - f_i^n}{\delta t} \right) = \left(\frac{f_{i+1}^n - 2f_i^n + f_{i-1}^n}{(\delta x)^2} \right) \quad (4)$$

$$\frac{1}{\lambda} \left(\frac{f_i^{n+1} - f_i^n}{\delta t} \right) = \left(\frac{f_{i+1}^{n+1} - 2f_i^{n+1} + f_{i-1}^{n+1}}{(\delta x)^2} \right) \quad (5)$$

In fact m can be any value between 0 and 1, however a common choice for $m=0.5$. (3) becomes;

$$\frac{1}{\lambda} \left(\frac{f_i^{n+1} - f_i^n}{\delta t} \right) = \frac{1}{2} \left(\frac{f_{i+1}^n - 2f_i^n + f_{i-1}^n}{(\delta x)^2} \right) + \frac{1}{2} \left(\frac{f_{i+1}^{n+1} - 2f_i^{n+1} + f_{i-1}^{n+1}}{(\delta x)^2} \right) \quad (6)$$

(6) is called the Crank-Nicolson scheme which is the average of the explicit and implicit schemes.

The above equation (6) can be arranged so that the temperatures at the present time step $(n+1)$ are on the left hand side. By applying these equations to all the nodes, we shall obtain a system with tridiagonal coefficient matrix.

$$\begin{aligned} -\frac{1}{2} \left(\frac{\lambda \delta t}{(\delta x)^2} \right) f_{i-1}^{n+1} + \left(1 + 2 \frac{\lambda \delta t}{(\delta x)^2} \right) f_i^{n+1} - \frac{1}{2} \left(\frac{\lambda \delta t}{(\delta x)^2} \right) f_{i+1}^{n+1} &= \frac{1}{2} \left(\frac{\lambda \delta t}{(\delta x)^2} \right) f_{i-1}^n \\ &+ \left(1 - 2 \frac{\lambda \delta t}{(\delta x)^2} \right) f_i^n - \left(\frac{\lambda \delta t}{(\delta x)^2} \right) f_{i+1}^n \end{aligned} \quad (7)$$

The expression $\frac{\lambda \delta t}{(\delta x)^2}$ is known as the diffusion number and will be denoted by r , i. e. $r = \frac{\lambda \delta t}{(\delta x)^2}$,

then (7) becomes

$$-\frac{r}{2} f_{i-1}^{n+1} + (1+2r) f_i^{n+1} - \frac{r}{2} f_{i+1}^{n+1} = \frac{r}{2} f_{i-1}^n + (1-2r) f_i^n + r f_{i+1}^n \quad (8)$$

Therefore,

$$-r(f_{i-1}^{n+1} + f_{i+1}^{n+1}) + 2(1+r) f_i^{n+1} = r(f_{i-1}^n + f_{i+1}^n) + 2(1-r) f_i^n \quad (9)$$

(9) is called the Crank Nicolson approximation of the problem and its associated boundary and initial conditions are as given.

If we divide the x -interval $0 \leq x \leq 1$ into k equal interval, we have $(k-1)$ internal mesh points per time row. Then for $n=1$ and $i=1, 2, \dots, k-1$, (9) gives a system of $(k-1)$ linear equations for the $(k-1)$ unknown values $f_1^1, f_2^1, f_3^1, \dots, f_{k-1}^1$ in the first time row in terms of the initial values $f_0^0, f_1^0, f_2^0, \dots, f_k^0$ and the boundary values $f_0^1 = 0, f_k^1 = 0$. Similarly for $n=1, n=2$, and so on; that is for each time row we have to solve such a system of $(k-1)$ linear equations resulting from (9). Equation (9) can be written in a matrix equation as

$$Tf = tb \quad (10)$$

where the unknown $f = f^{n+1}$, the known concentrations $b = f^n$ and T, t are tri-diagonal matrices of coefficients defined as

$$\begin{bmatrix} (2+2r) & -r & 0 & \cdots & 0 \\ -r & 2(1+r) & -r & \cdots & 0 \\ 0 & -r & 2(1+r) & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & -r \\ 0 & 0 & 0 & -r & 2(1+r) \end{bmatrix} \begin{bmatrix} f_1^{n+1} \\ f_2^{n+1} \\ f_3^{n+1} \\ \vdots \\ f_{k-1}^{n+1} \end{bmatrix} = \begin{bmatrix} 2(1-r) & r & 0 & \cdots & 0 \\ r & 2(1-r) & r & \cdots & 0 \\ 0 & r & 2(1-r) & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & r \\ 0 & 0 & 0 & r & 2(1-r) \end{bmatrix} \begin{bmatrix} f_1^n \\ f_2^n \\ f_3^n \\ \vdots \\ f_{k-1}^n \end{bmatrix} \quad (11)$$

Crucial to the convergence of this method is the condition (Turner, 1994)

$$r = \frac{\lambda \delta t}{(\delta x)^2} \leq \frac{1}{2}, \text{ this implies that}$$

$$\delta t \leq \frac{(\delta x)^2}{2\lambda} \quad (12)$$

Condition (12) is a drawback in practice. Indeed, in order to attain sufficient accuracy, we have to choose δx small, which makes δt very small by (12). This will make the computation exceptionally lengthy, as more time levels will be required to cover the region. A method that imposes no such

restriction as $r = \frac{\lambda \delta t}{(\delta x)^2}$ was proposed by Crank and Nicolson in [11]

2.3 THE STABILITY OF CRANK NICOLSON METHOD

Consider the equation (9) given below:

$$-r(f_{i-1}^{n+1} + f_{i+1}^{n+1}) + 2(1+r)f_i^{n+1} = r(f_{i-1}^n + f_{i+1}^n) + 2(1-r)f_i^n$$

Worse case solution is given below as:

$$f_i^n = \rho^n (-1)^i \quad (13)$$

Substituting (13) into (9), we have that

$$-r\rho^{n+1}(-1)^{i-1}((-1)^{i-1} + (-1)^{i+1}) + 2(1+r)\rho^{n+1}(-1)^i = r\rho^n(-1)^{i-1}((-1)^{i-1} + (-1)^{i+1}) + 2(1+r)\rho^n(-1)^i$$

$$\rho[-r(-1) - 1 + 2(1+r) - r(-1) + 1] = r(-1) - 1 + 2(1-r) + r(-1) + 1$$

The last equation above can also be written as

$$\rho = \frac{1-2r}{1+2r}, \text{ then}$$

$$\left| \rho = \frac{1-2r}{1+2r} \right| \Rightarrow |\rho| < 1, \quad \forall \quad \rho > 0$$

The Crank Nicolson Method is unconditionally stable and has higher order of accuracy. The price of solving a tridiagonal system at each step is worth paying since this method allows large step sizes. This is the most popular numerical method for heat equation.

III. NUMERICAL EXAMPLES AND RESULTS

This section presents some numerical examples and results as follows:

3.1 NUMERICAL EXAMPLES

EXAMPLE 1

We shall use the Crank-Nicolson method to solve the partial differential equation

$$8 \frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} \quad (14)$$

with the following initial and boundary conditions:

$$\text{Initial conditions: } f(x, 0) = 2000, \quad 0 \leq x \leq 4 \quad (15)$$

$$\text{Boundary conditions: } \frac{\partial f}{\partial x} = \begin{cases} 0.36f - 25.2, & x = 4 \\ 0, & x = 0 \end{cases} \quad (16)$$

SOLUTION:

Using the Crank-Nicolson method, from (9), the heat equation becomes

$$-r(f_{i-1}^{n+1} + f_{i+1}^{n+1}) + 2(1+r)f_i^{n+1} = r(f_{i-1}^n + f_{i+1}^n) + 2(1-r)f_i^n$$

$$\text{Where } r = \frac{\lambda \delta t}{(\delta x)^2}$$

Let $\delta x = 1$, $\delta t = 1$, then we have that

$$r = \frac{\lambda \delta t}{(\delta x)^2} = \frac{0.125}{1} = \frac{1}{8} \leq \frac{1}{2}$$

Substituting r into the above equation (9), we have

$$-\frac{1}{8}(f_{i-1}^{n+1} + f_{i+1}^{n+1}) + 2\left(1 + \frac{1}{8}\right)f_i^{n+1} = \frac{1}{8}(f_{i-1}^n + f_{i+1}^n) + 2\left(1 - \frac{1}{8}\right)f_i^n$$

Therefore,

$$-0.125(f_{i-1}^{n+1} + f_{i+1}^{n+1}) + 2.25f_i^{n+1} = 0.125(f_{i-1}^n + f_{i+1}^n) + 1.75f_i^n \quad (17)$$

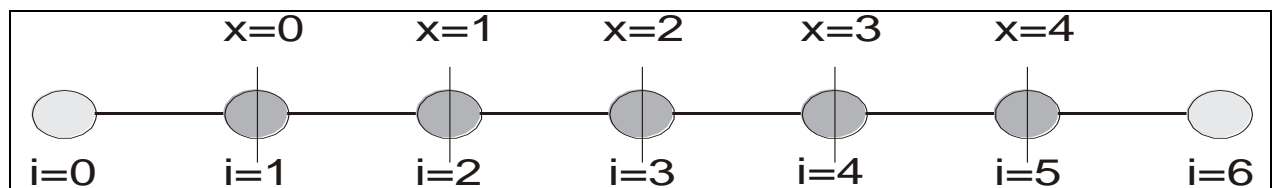


FIGURE III: THE NODES AT ANY GIVEN TIME STEP

There are five unknown temperatures at any given time as shown in Figure 2. The nodes corresponding to $i = 0$ and $i = 5$ are required to accommodate the derivative boundary conditions at the end points.

For node $i = 1$, (17) yields

$$-0.125(f_0^{n+1} + f_2^{n+1}) + 2.25f_1^{n+1} = 0.125(f_0^n + f_2^n) + 1.75f_1^n \quad (18)$$

f_0^{n+1} can be solved in terms of f_1^{n+1} and f_2^{n+1} using the boundary condition

$$\left. \frac{\partial f}{\partial x} \right|_{x=0} = 0.36f - 25.2 \Rightarrow \frac{1}{2\delta x}(f_2^{n+1} - f_0^{n+1}) = 0.36f_1^{n+1} - 25.2$$

Therefore,

$$f_2^{n+1} - f_0^{n+1} = 2(1)(0.36f_1^{n+1} - 25.2)$$

$$f_2^{n+1} - f_0^{n+1} = 0.72f_1^{n+1} - 50.4$$

Then,

$$f_0^{n+1} = f_2^{n+1} - 0.72f_1^{n+1} + 50.4 \quad (19)$$

Similarly, f_5^n can be solved in terms of f_1^n and f_2^n using the same boundary condition

$$f_0^n = f_2^n - 0.72f_1^n + 50.4 \quad (20)$$

Substituting (19) and (20) into (18), we have

$$-0.125(f_2^{n+1} - 0.72f_1^{n+1} + 50.4 + f_2^{n+1}) + 2.25f_1^{n+1} = 0.125(f_2^n - 0.72f_1^n + 50.4 + f_2^n) + 1.75f_1^n$$

Simplifying the last equation further, we obtain

$$2.34f_1^{n+1} - 0.25f_2^{n+1} = 1.66f_1^n + 0.25f_2^n + 12.6 \quad (21)$$

For $2 \leq i \leq 4$, we have (17), which is given by

$$-0.125(f_{i-1}^{n+1} + f_{i+1}^{n+1}) + 2.25f_i^{n+1} = 0.125(f_{i-1}^n + f_{i+1}^n) + 1.75f_i^n$$

At node ($i = 5$),

$$-0.125(f_4^{n+1} + f_6^{n+1}) + 2.25f_5^{n+1} = 0.125(f_4^n + f_6^n) + 1.75f_5^n \quad (22)$$

Using the boundary condition $\left. \frac{\partial f}{\partial x} \right|_{x=4} = 0$, therefore $f_4 = f_6$

$$-0.25f_4^{n+1} + 2.25f_5^{n+1} = 0.125f_4^n + 1.75f_5^n \quad (23)$$

The results generated are shown in the Table 1 below using MATLAB codes.

TABLE I: THE RESULTS GENERATED FROM THE FORMULATION OF THE FINITE DIFFERENCE USING CRANK-NICOLSON METHOD FOR A PARABOLIC EQUATION WITH DERIVATIVE BOUNDARY CONDITIONS

t	f_1 $x = 0.0$	f_2 $x = 1.0$	f_3 $x = 2.0$	f_4 $x = 3.0$	f_5 $x = 4.0$
0.0	2000.00	2000.00	2000.00	2000.00	2000.00
1.0	1850.65	1991.68	1999.54	1999.97	2000.00
2.0	1741.57	1970.71	1997.54	1999.82	1999.97
3.0	1659.03	1943.41	1993.27	1999.34	1999.89
4.0	1594.36	1913.39	1986.68	1998.35	1999.65
5.0	1541.99	1882.70	1978.03	1996.69	1999.18
6.0	1498.33	1852.43	1967.70	1994.27	1998.36
7.0	1460.99	1823.17	1956.04	1991.06	1997.09
8.0	1428.39	1795.18	1943.39	1987.03	1995.30
9.0	1399.43	1768.54	1930.02	1982.23	1992.93
10.0	1373.33	1743.25	1916.17	1976.68	1989.94
11.0	1349.56	1719.25	1902.00	1970.44	1986.30
12.0	1327.69	1696.47	1887.65	1963.56	1982.01
13.0	1307.43	1674.81	1873.23	1956.10	1977.08
14.0	1288.54	1654.18	1858.80	1948.12	1971.53
15.0	1270.83	1634.51	1844.42	1939.65	1965.39
16.0	1254.16	1615.71	1830.14	1930.77	1958.68
17.0	1238.40	1597.70	1815.98	1921.50	1951.45
18.0	1223.45	1580.42	1801.96	1911.89	1943.73
19.0	1209.22	1563.81	1788.09	1901.99	1935.55
20.0	1195.65	1547.82	1774.37	1891.82	1926.96

EXAMPLE 2

Use the Crank-Nicolson method to solve the partial differential equation

$$50 \frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} \quad (24)$$

with the following initial and boundary conditions:

$$\left. \begin{aligned} \text{Initial conditions: } f(x, 0) &= 100x, \quad 0 \leq x \leq 1; \\ f(x, 0) &= 200 - 100x, \quad 0 \leq x \leq 2 \end{aligned} \right\} \quad (25)$$

$$\text{Boundary conditions: } f(0, t) = f(2, t) = 0 \quad (26)$$

SOLUTION

Let $\delta x = 0.2, \delta t = 0.5$

$\Delta x = 0.2, \Delta t = 0.5$, we have

$$r = \frac{\lambda \delta t}{(\delta x)^2} = \frac{(0.02)(0.5)}{(0.2)^2} = \frac{1}{4} \leq \frac{1}{2} \quad (27)$$

Substituting (27) into (9), we obtain

$$-0.25(f_{i-1}^{n+1} + f_{i+1}^{n+1}) + 2(1 + 0.25)f_i^{n+1} = 0.25(f_{i-1}^n + f_{i+1}^n) + 2(1 - 0.25)f_i^n \quad (28)$$

$$-0.25f_{i-1}^{n+1} + 2.5f_i^{n+1} - 0.25f_{i+1}^{n+1} = 0.25f_{i-1}^n + 1.5f_i^n + 0.25f_{i+1}^n \quad (29)$$

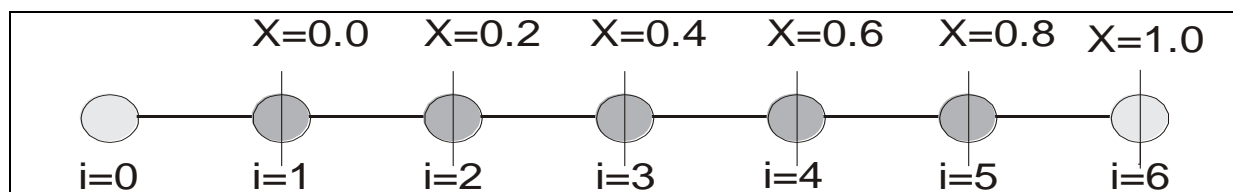


FIGURE IV: THE NODES AT ANY GIVEN TIME STEP

There are six unknown temperatures at any given time as shown in Figure III. The nodes corresponding to $i = 0$ and $i = 6$ are required to accommodate the derivative boundary conditions at the end points.

The initial conditions for the problem are given in the Table II below;

TABLE II

i	1	2	3	4	5	6
$t \setminus x$	0	0.2	0.4	0.6	0.8	1.0
0	0	20	40	60	80	100
0.5	0	V	W	X	Y	Z

From the above Table II, V, W, X, Y, Z are to be calculated.

Due to the symmetry with respect to node ($i = 6$), we have

$$-0.5f_5^{n+1} + 2.5f_6^{n+1} = 0.5f_5^n + 1.5f_6^n$$

The temperatures at the time $t = 0.5$ are the solutions of the following equations in the Table III below;

TABLE III

$i = 2$		$+ 2.5f_2$	$- 0.25f_3$	$=$	40
$i = 3$	$- 0.25f_2$	$+ 2.5f_3$	$- 0.25f_4$	$=$	80
$i = 4$	$- 0.25f_3$	$+ 2.5f_4$	$- 0.25f_5$	$=$	120
$i = 5$	$- 0.25f_4$	$+ 2.5f_5$	$- 0.25f_6$	$=$	160
$i = 6$	$- 0.25f_5$	$+ 2.5f_6$		$=$	190

The results for the above set of equations in Table III above are listed in the Table IV below;

TABLE IV: THE RESULTS GENERATED AT $t = 0, t = 0.5$

i	1	2	3	4	5	6
$t \setminus x$	0	0.2	0.4	0.6	0.8	1.0

0	0	20	40	60	80	100
0.5	0	20	39.99	59.92	79.18	91.84

From the above Table IV, we can see that $V = 20, W = 39.99, X = 59.92, Y = 79.18, Z = 91.84$
 The results obtained for other values of t using MATLAB code are shown in the Table V below

TABLE V: THE RESULTS GENERATED FROM CRANK NICOLSON METHOD FOR THE SOLUTION OF PARTIAL DIFFERENTIAL EQUATION WITH INITIAL AND BOUNDARY CONDITIONS

t	f_1 $x = 0.0$	f_2 $x = 0.2$	f_3 $x = 0.4$	f_4 $x = 0.6$	f_5 $x = 0.8$	f_6 $x = 1.0$
0.0	0.0	20.00	40.00	60.00	80.00	100.00
0.5	0.0	20.00	39.99	59.92	79.18	91.84
1.0	0.0	19.99	39.94	59.59	77.18	86.39
1.5	0.0	19.97	39.82	58.97	75.09	82.31
2.0	0.0	19.92	39.59	58.12	72.89	78.98
2.5	0.0	19.84	39.25	57.12	70.77	76.12
3.0	0.0	19.71	38.82	56.03	68.75	73.58
3.5	0.0	19.54	38.31	54.89	66.83	71.26
4.0	0.0	19.33	37.73	53.72	64.99	69.12
4.5	0.0	19.08	37.11	52.54	63.25	67.12
5.0	0.0	18.80	36.44	51.36	61.57	65.24
5.5	0.0	18.50	35.75	50.19	59.97	63.45
6.0	0.0	18.18	35.04	49.03	58.42	61.75
6.5	0.0	17.84	34.32	47.89	56.93	60.12
7.0	0.0	17.50	33.59	46.77	55.49	58.56
7.5	0.0	17.14	32.86	45.67	54.10	57.05
8.0	0.0	16.79	32.14	44.59	52.75	55.60
8.5	0.0	16.43	31.41	43.53	51.44	54.20
9.0	0.0	16.07	30.70	42.49	50.17	52.84
9.5	0.0	15.71	29.99	41.47	48.94	51.53
10.0	0.0	15.36	29.30	40.48	47.73	50.25

EXAMPLE 3 [11]

Let us consider the following linear equation.

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} \quad 0 \leq x \leq 1, \quad t \geq 0 \quad (30)$$

Subject to the following initial and boundary conditions:

$$f(x, 0) = \sin \pi x; \quad f(0, t) = f(1, t) = 0 \quad (31)$$

The exact solution of the above equation is given by:

$$f(x, t) = e^{-\pi^2 t} \sin \pi x \quad (32)$$

From (9), we have that

$$-r(f_{i-1}^{n+1} + f_{i+1}^{n+1}) + 2(1+r)f_i^{n+1} = r(f_{i-1}^n + f_{i+1}^n) + 2(1-r)f_i^n$$

$$\text{Where } r = \frac{\lambda \delta t}{(\delta x)^2}$$

The Crank-Nicolson approximation (9) of the problem (30) and its associated boundary and initial conditions (31) and (32) respectively are given below as follows:

$$2(1+r)f_i^{n+1} - r(f_{i-1}^{n+1} + f_{i+1}^{n+1}) = 2(1-r)f_i^n + r(f_{i-1}^n + f_{i+1}^n) \quad (33)$$

$$\left. \begin{aligned} f_0^n &= 0, f_1^0 = \sin \pi i \delta x, f_{10}^n = 0 \\ \text{for } i &= 1, 2, 3, 4, 5, n = 0, 1, 2, \dots, 10 \end{aligned} \right\} \quad (34)$$

Taking $\delta x = 0.1$ and $\delta t = 0.001$, we obtain the results to this problem with Matlab codes.

At $x = 0.5$ and $t = 0.005, 0.006, 0.007$, we have the temperature values for the Crank-Nicolson solution (f), as well as their corresponding exact solution values (f^*), and their corresponding errors, (e) in Table VI below;

TABLE VI: CRANK NICOLSON METHOD AND EXACT SOLUTION AT $x = 0.5, \delta t = 0.001$

t	Crank Nicolson Method, (f)	Exact Solution, (f^*)	Error, $e = f^* - f $
0.005	0.9519	0.9519	0.0000
0.006	0.9423	0.9425	0.0002
0.007	0.9328	0.9333	0.0005

EXAMPLE 4

A rod of steel is subjected to a temperature of 100°C on the left end and 25°C on the right end. If the rod is of length 0.05m , use Crank-Nicolson method to find the temperature distribution in the rod from $t = 0$ to $t = 9$ seconds. Use $\delta x = 0.01\text{m}$ and $\delta t = 3\text{s}$.

Given

$$\alpha = 54 \frac{\text{W}}{\text{m} \cdot \text{K}}, \rho = 7800 \frac{\text{kg}}{\text{m}^3}, C = 490 \frac{\text{J}}{\text{kg} \cdot \text{K}}. \quad (35)$$

The initial temperature of the rod is 20°C .

SOLUTION:

$$\lambda = \frac{\alpha}{\rho C} = \frac{54}{7800 \times 490} = 1.4129 \times 10^{-5} \text{ m}^2 / \text{s} \quad (36)$$

Then

$$r = \lambda \frac{\delta t}{(\delta x)^2} = 1.412 \times 10^{-5} \frac{3}{(0.01)^2} = 0.4239 \quad (37)$$

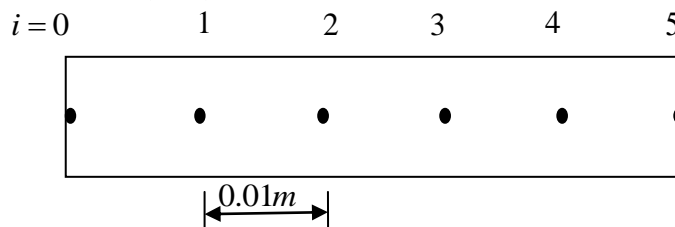


FIGURE V: SCHEMATIC DIAGRAM SHOWING THE NODE REPRESENTATION IN THE MODEL

The boundary conditions are

$$f_0^n = 100^\circ\text{C}, f_5^n = 25^\circ\text{C} \quad \text{for } n = 0, 1, 2, 3 \quad (38)$$

The initial temperature of the rod is 20°C , that is, all the temperatures of the nodes inside the rod are at 20°C , $t = 0$ except for the boundary nodes given by (38). This could be represented as

$$f_i^0 = 20^\circ\text{C}, \text{ for } i = 1, 2, 3, 4. \quad (39)$$

The initial temperature at the nodes inside the rod (when $t = 0$ second) is given by

$$\begin{aligned} f_0^0 &= 100^\circ\text{C} && \text{from (38)} \\ f_1^0 &= 20^\circ\text{C}, f_2^0 = 20^\circ\text{C}, f_3^0 = 20^\circ\text{C}, f_4^0 = 20^\circ\text{C} \} && \text{from (39)} \\ f_5^0 &= 25^\circ\text{C} && \text{from (38)} \end{aligned}$$

The temperature at the nodes inside the rod when $t = 3$ seconds, from (38), we have the boundary condition of the form:

$$f_0^1 = 100^\circ\text{C}, f_5^1 = 25^\circ\text{C}$$

For all the interior nodes, setting $n = 0$ and $i = 1, 2, 3, 4$ in (9) gives the following equations

$$\begin{aligned} i &= 1 \\ -\lambda f_0^1 + 2(1 + \lambda)f_1^1 - \lambda f_2^1 &= f_0^0 + 2(1 - \lambda)f_1^0 + \lambda f_2^0 \\ (-0.4239 \times 100) + 2(1 + 0.4239)f_1^1 - 0.4239f_2^1 &= (0.4239)100 + 2(1 - 0.4239)20 + (0.4239)20 \\ -42.39 + 2.8478f_1^1 - 0.4239f_2^1 &= 42.39 + 23.044 + 8.478 \\ 2.8478f_1^1 - 0.4239f_2^1 &= 116.30 \end{aligned} \quad (40)$$

$$\begin{aligned} i &= 2 \\ -\lambda f_1^1 + 2(1 + \lambda)f_2^1 - \lambda f_3^1 &= \lambda f_1^0 + 2(1 - \lambda)f_2^0 + \lambda f_3^0 \\ -0.4239f_1^1 + 2(1 + 0.4239)f_2^1 - 0.4239f_3^1 &= (0.4239)20 + 2(1 - 0.4239)20 + (0.4239)20 \\ -0.4239f_1^1 + 2.8478f_2^1 - 0.4239f_3^1 &= 40.000 \end{aligned} \quad (41)$$

$$\begin{aligned} i &= 3 \\ -\lambda f_2^1 + 2(1 + \lambda)f_3^1 - \lambda f_4^1 &= \lambda f_2^0 + 2(1 - \lambda)f_3^0 + \lambda f_4^0 \\ -0.4239f_2^1 + 2(1 + 0.4239)f_3^1 - 0.4239f_4^1 &= (0.4239)20 + 2(1 - 0.4239)20 + (0.4239)20 \\ -0.4239f_2^1 + 2.8478f_3^1 - 0.4239f_4^1 &= 40.000 \end{aligned} \quad (42)$$

$$\begin{aligned} i &= 4 \\ -\lambda f_3^1 + 2(1 + \lambda)f_4^1 - \lambda f_5^1 &= \lambda f_3^0 + 2(1 - \lambda)f_4^0 + \lambda f_5^0 \\ -0.4239f_3^1 + 2(1 + 0.4239)f_4^1 - (0.4239)25 &= (0.4239)20 + 2(1 - 0.4239)20 + (0.4239)25 \\ -0.4239f_3^1 + 2.8478f_4^1 - 10.598 &= 8.478 + 23.044 + 10.598 \\ -0.4239f_3^1 + 2.8478f_4^1 &= 52.718 \end{aligned} \quad (43)$$

The coefficient matrix in the above set of equations is tridiagonal. Special algorithms such as Thomas' algorithm are used to solve equation with tridiagonal coefficient matrices

$$\begin{bmatrix} 2.8478 & -0.4239 & 0 & 0 \\ -0.4239 & 2.8478 & -0.4239 & 0 \\ 0 & -0.4239 & 2.8478 & -0.4239 \\ 0 & 0 & -0.4239 & 2.8478 \end{bmatrix} \begin{bmatrix} f_1^1 \\ f_2^1 \\ f_3^1 \\ f_4^1 \end{bmatrix} = \begin{bmatrix} 116.30 \\ 40.000 \\ 40.000 \\ 52.718 \end{bmatrix}$$

The above matrix is tridiagonal. Solving the above matrix we get

$$\begin{bmatrix} f_1^1 \\ f_2^1 \\ f_3^1 \\ f_4^1 \end{bmatrix} = \begin{bmatrix} 44.3720 \\ 23.7460 \\ 20.7970 \\ 21.6070 \end{bmatrix}$$

Hence, the temperature at all the nodes at time, $t = 3$ seconds is

$$\begin{bmatrix} f_0^1 \\ f_1^1 \\ f_2^1 \\ f_3^1 \\ f_4^1 \\ f_5^1 \end{bmatrix} = \begin{bmatrix} 100 \\ 44.3720 \\ 23.7460 \\ 20.7970 \\ 21.6070 \\ 25 \end{bmatrix}$$

The temperature at the nodes inside the rod when $t = 6$ seconds:

$$f_0^2 = 100^\circ\text{C}, f_5^2 = 25^\circ\text{C} \text{ Boundary Condition (38)}$$

For all the interior nodes, putting $n = 1$ and $i = 1, 2, 3, 4$ in (9) gives the following equations

$$i = 1$$

$$\begin{aligned} -\lambda f_0^2 + 2(1 + \lambda)f_1^2 - \lambda f_2^2 &= \lambda f_0^1 + 2(1 - \lambda)f_1^1 + \lambda f_2^1 \\ (-0.4239 \times 100) + 2(1 + 0.4239)f_1^2 - 0.4239f_2^2 &= \\ (0.4239)100 + 2(1 - 0.4239)44.372 + (0.4239)23.746 \\ -42.39 + 2.8478f_1^2 - 0.4239f_2^2 &= 42.39 + 51.125 + 10.066 \\ 2.8478f_1^2 - 0.4239f_2^2 &= 145.971 \end{aligned} \quad (44)$$

$$i = 2$$

$$\begin{aligned} -\lambda f_1^2 + 2(1 + \lambda)f_2^2 - \lambda f_3^2 &= \lambda f_1^1 + 2(1 - \lambda)f_2^1 + \lambda f_3^1 \\ -0.4239f_1^2 + 2(1 + 0.4239)f_2^2 - 0.4239f_3^2 &= \\ (0.4239)44.372 + 2(1 - 0.4239)23.746 + (0.4239)20.797 \\ -0.4239f_1^2 + 2.8478f_2^2 - 0.4239f_3^2 &= 18.809 + 27.360 + 8.8158 \\ -0.4239f_1^2 + 2.8478f_2^2 - 0.4239f_3^2 &= 54.985 \end{aligned} \quad (45)$$

$$i = 3$$

$$\begin{aligned} -\lambda f_2^2 + 2(1 + \lambda)f_3^2 - \lambda f_4^2 &= \lambda f_2^1 + 2(1 - \lambda)f_3^1 + \lambda f_4^1 \\ -0.4239f_2^2 + 2(1 + 0.4239)f_3^2 - 0.4239f_4^2 &= \\ (0.4239)23.746 + 2(1 - 0.4239)20.797 + (0.4239)21.607 \\ -0.4239f_2^2 + 2.8478f_3^2 - 0.4239f_4^2 &= 10.066 + 23.962 + 9.1592 \\ -0.4239f_2^2 + 2.8478f_3^2 - 0.4239f_4^2 &= 43.187 \end{aligned} \quad (46)$$

$$i = 4$$

$$\begin{aligned}
 -\lambda f_3^2 + 2(1+\lambda)f_4^2 - \lambda f_5^2 &= \lambda f_3^1 + 2(1-\lambda)f_4^1 + \lambda f_5^1 \\
 -0.4239f_3^2 + 2(1+0.4239)f_4^2 - (0.4239)25 &= \\
 (0.4239)20.797 + 2(1-0.4239)21.607 + (0.4239)25 \\
 -0.4239f_3^2 + 2.8478f_4^2 - 10.598 &= 8.8158 + 24.896 + 10.598 \\
 -0.4239f_3^2 + 2.8478f_4^2 &= 54.908
 \end{aligned} \tag{47}$$

The simultaneous linear equations (44) – (47) can be written in matrix form as

$$\begin{bmatrix} 2.8478 & -0.4239 & 0 & 0 \\ -0.4239 & 2.8478 & -0.4239 & 0 \\ 0 & -0.4239 & 2.8478 & -0.4239 \\ 0 & 0 & -0.4239 & 2.8478 \end{bmatrix} \begin{bmatrix} f_1^2 \\ f_2^2 \\ f_3^2 \\ f_4^2 \end{bmatrix} = \begin{bmatrix} 145.971 \\ 54.985 \\ 43.187 \\ 54.908 \end{bmatrix}$$

Solving the above set of equations, we get

$$\begin{bmatrix} f_1^2 \\ f_2^2 \\ f_3^2 \\ f_4^2 \end{bmatrix} = \begin{bmatrix} 55.8830 \\ 31.0750 \\ 23.1740 \\ 22.7300 \end{bmatrix}$$

Hence, the temperature at all the nodes at time, $t = 6$ seconds is

$$\begin{bmatrix} f_0^2 \\ f_1^2 \\ f_2^2 \\ f_3^2 \\ f_4^2 \\ f_5^2 \end{bmatrix} = \begin{bmatrix} 100 \\ 55.8830 \\ 31.0750 \\ 23.1740 \\ 22.7300 \\ 25 \end{bmatrix}$$

The temperature at the nodes inside the rod when $t = 9$ seconds:

$$f_0^3 = 100^\circ\text{C}, f_5^3 = 25^\circ\text{C} \quad \text{Boundary Condition (38)}$$

For all the interior nodes, setting $n = 2$ and $i = 1, 2, 3, 4$ in (9) gives the following equations

$i = 1$

$$\begin{aligned}
 -\lambda f_0^3 + 2(1+\lambda)f_1^3 - \lambda f_2^3 &= \lambda f_0^2 + 2(1-\lambda)f_1^2 + \lambda f_2^2 \\
 (-0.4239 \times 100) + 2(1+0.4239)f_1^3 - 0.4239f_2^3 &= \\
 (0.4239)100 + 2(1-0.4239)55.883 + (0.4239)31.075 \\
 -42.39 + 2.8478f_1^3 - 0.4239f_2^3 &= 42.39 + 64.388 + 13.173 \\
 2.8478f_1^3 - 0.4239f_2^3 &= 162.34
 \end{aligned} \tag{48}$$

$i = 2$

$$\begin{aligned}
 -\lambda f_1^3 + 2(1+\lambda)f_2^3 - \lambda f_3^3 &= \lambda f_1^2 + 2(1-\lambda)f_2^2 + \lambda f_3^2 \\
 -0.4239f_1^3 + 2(1+0.4239)f_2^3 - 0.4239f_3^3 &= \\
 (0.4239)55.883 + 2(1-0.4239)31.075 + (0.4239)23.174 \\
 -0.4239f_1^3 + 2.8478f_2^3 - 0.4239f_3^3 &= 23.689 + 35.805 + 9.8235 \\
 -0.4239f_1^3 + 2.8478f_2^3 - 0.4239f_3^3 &= 69.318
 \end{aligned} \tag{49}$$

$i=3$

$$\begin{aligned}
 -\lambda f_2^3 + 2(1+\lambda)f_3^3 - \lambda f_4^3 &= \lambda f_2^2 + 2(1-\lambda)f_3^2 + \lambda f_4^2 \\
 -0.4239f_2^3 + 2(1+0.4239)f_3^3 - 0.4239f_4^3 &= \\
 (0.4239)31.075 + 2(1-0.4239)23.174 + (0.4239)22.730 \\
 -0.4239f_2^3 + 2.8478f_3^3 - 0.4239f_4^3 &= 13.173 + 26.701 + 9.635 \\
 -0.4239f_2^3 + 2.8478f_3^3 - 0.4239f_4^3 &= 49.509
 \end{aligned} \tag{50}$$

$i=4$

$$\begin{aligned}
 -\lambda f_3^3 + 2(1+\lambda)f_4^3 - \lambda f_5^3 &= \lambda f_3^2 + 2(1-\lambda)f_4^2 + \lambda f_5^2 \\
 -0.4239f_3^3 + 2(1+0.4239)f_4^3 - (0.4239)25 &= \\
 (0.4239)23.174 + 2(1-0.4239)22.730 + (0.4239)25 \\
 -0.4239f_3^3 + 2.8478f_4^3 - 10.598 &= 9.8235 + 26.190 + 10.598 \\
 -0.4239f_3^3 + 2.8478f_4^3 &= 57.210
 \end{aligned} \tag{51}$$

The simultaneous linear equations (48) – (51) can be written in matrix form as

$$\begin{bmatrix} 2.8478 & -0.4239 & 0 & 0 \\ -0.4239 & 2.8478 & -0.4239 & 0 \\ 0 & -0.4239 & 2.8478 & -0.4239 \\ 0 & 0 & -0.4239 & 2.8478 \end{bmatrix} \begin{bmatrix} f_1^3 \\ f_2^3 \\ f_3^3 \\ f_4^3 \end{bmatrix} = \begin{bmatrix} 162.34 \\ 69.318 \\ 49.509 \\ 57.210 \end{bmatrix}$$

Solving the above set of equations, we get

$$\begin{bmatrix} f_1^3 \\ f_2^3 \\ f_3^3 \\ f_4^3 \end{bmatrix} = \begin{bmatrix} 62.6040 \\ 37.6130 \\ 26.5620 \\ 24.0420 \end{bmatrix}$$

Hence, the temperature at all the nodes at time, $t = 9$ seconds is

$$\begin{bmatrix} f_0^3 \\ f_1^3 \\ f_2^3 \\ f_3^3 \\ f_4^3 \\ f_5^3 \end{bmatrix} = \begin{bmatrix} 100 \\ 62.6040 \\ 37.6130 \\ 26.5620 \\ 24.0420 \\ 25 \end{bmatrix}$$

The Exact solution to the above problem in example 4 is given by:

$$f(x, t) = 100 - 1500x + \sum_{m=1}^{\infty} \left\{ \frac{10(-1)^m - 160}{m\pi} \right\} \exp\left((-1.4129 \times 10^{-5})(20m\pi)^2 t\right) \sin(20m\pi x) \tag{52}$$

We shall present in the Table VII below, the results obtained at interior nodes $n = 2$ and $i = 1, 2, 3, 4$ only in the context of exact solution by substituting the values of x and t gives the temperature inside the rod at a particular location and time.

**TABLE VII: COMPARISON OF TEMPERATURE OBTAINED AT INTERIOR NODES
 $n = 2$ AND $i = 1, 2, 3, 4$ USING CRANK NICOLSON METHOD**

Temperature at Nodes	Crank Nicolson Method ($^{\circ}\text{C}$)	Exact Solution ($^{\circ}\text{C}$)	Error Generated ($^{\circ}\text{C}$)
f_1^3	62.6040	62.5100	0.0940
f_2^3	37.6130	37.0840	0.5290
f_3^3	26.5620	25.8440	0.2820
f_4^3	24.0420	23.6100	0.4320

The temperature distribution along the length of the rod at different times is plotted in Figure VI.

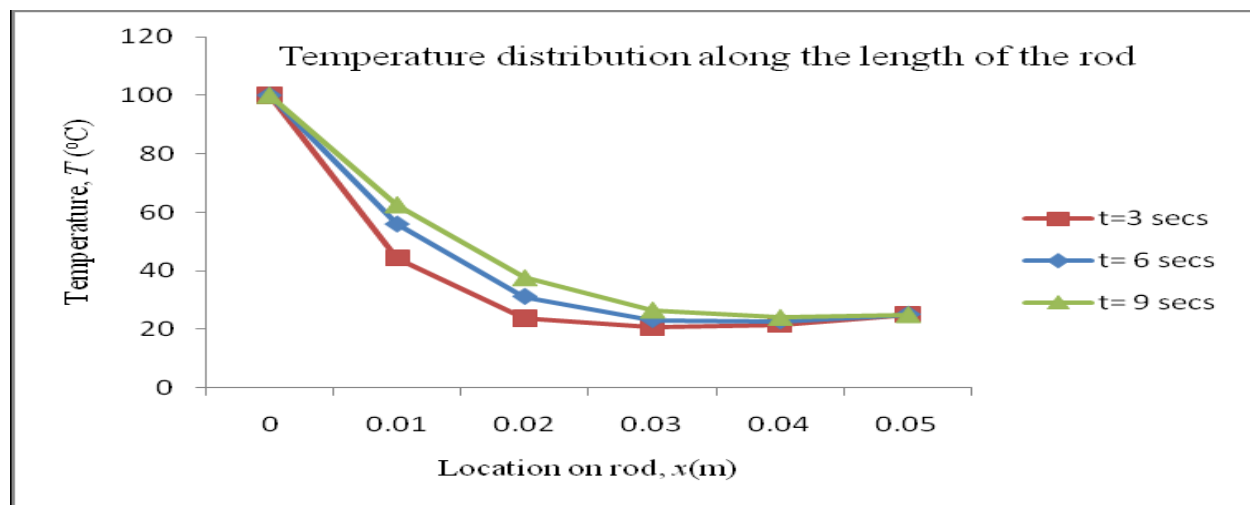


FIGURE VI: TEMPERATURE DISTRIBUTION IN ROD FROM CRANK-NICOLSON METHOD

IV. DISCUSSION OF RESULTS

From Tables 1 and 5, we can see that Crank Nicolson method is good for solving parabolic heat equation. Tables 6 and 7 demonstrate that the Crank Nicolson method performs well, is mutually consistent and agree with the exact solution. However, Crank Nicolson method provides better accuracy and it only requires the solution of a very simple system of linear equations (namely, a tridiagonal system) at every time level.

V. CONCLUSION

Crank Nicolson method has become a very popular finite difference method for solving parabolic partial differential equations. This method has its own advantages and disadvantages. Crank Nicolson method is robust, unconditionally stable, has higher order of accuracy up to $O((\Delta x)^2, (\Delta t)^2)$ and converges faster than other two finite difference methods but the main problem of this method is that it starts to oscillate when the coefficient of the second derivative (the *diffusion term*) is very small or when the coefficient of the first derivative (the *convection term*) is large (or both).

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