DD2437 – Artificial Neural Networks and Deep Architectures (annda)

Lecture 5: **Radial basis function NN and introduction to competitive learning**

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Lecture overview

- Interpolation problem and radial-basis functions (RBFs)
- RBF networks – hybrid learning
- Weight interpretation
- Competitive mechanisms for unsupervised learning

The separability of patterns

Cover’s theorem

“A complex pattern classification problem, projected nonlinearly to a high-dimensional space, is more likely to be linearly separable than in low dimensional space, especially if it is not populated too densely.”

Cover, 1965

The separability of patterns

Cover’s theorem

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Cover, 1965

So, we need to have:

1) Many nonlinear mappings $\varphi_i(\mathbf{x} \in \mathrm{R}^M)$: $\mathrm{R}^M \rightarrow \mathrm{R}^1$, where $i = 1,.., N$ (large)
2) Linear function for separability in high N-dimensional space

The radial-basis-function (RBF) technique

Nonlinear mapping with the use of *radial-basis-functions* (RBFs):

$$\varphi_i\left(\left\|\mathbf{x}-\mathbf{x}_i\right\|\right)$$

$\mathbf{x}_i$ – RBF centre

$\|\cdot\|$ – vector norm, often Euclidean

$\varphi_i(r)$ – kernel function, often Gaussian

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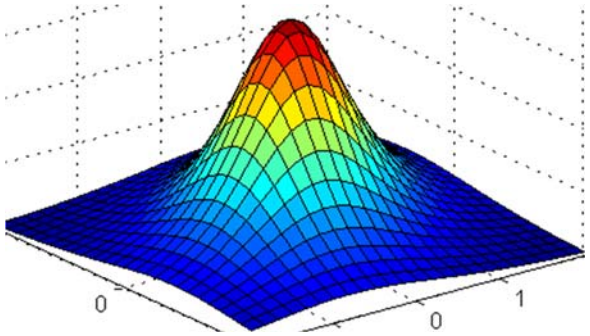
$$\varphi_i\left(r=\left\|\mathbf{x}-\mathbf{x}_i\right\|\right)=\exp\left(-\frac{r^2}{2\sigma^2}\right)$$



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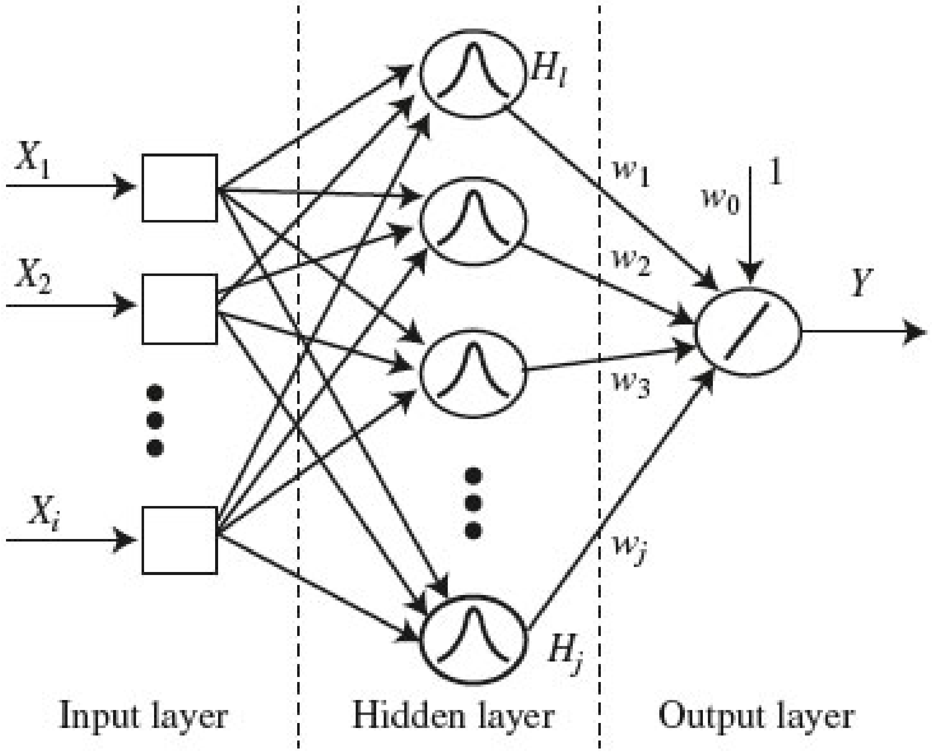
Linear operation in N-dimensional space:

$$F(\mathbf{x})=\sum_{i=1}^{N} w_i\varphi_i\left(\left\|\mathbf{x}-\mathbf{x}_i\right\|\right)$$

The RBF neural network concept



In the *exact interpolation*, the size of the hidden layer, N, is equal to the number of samples n

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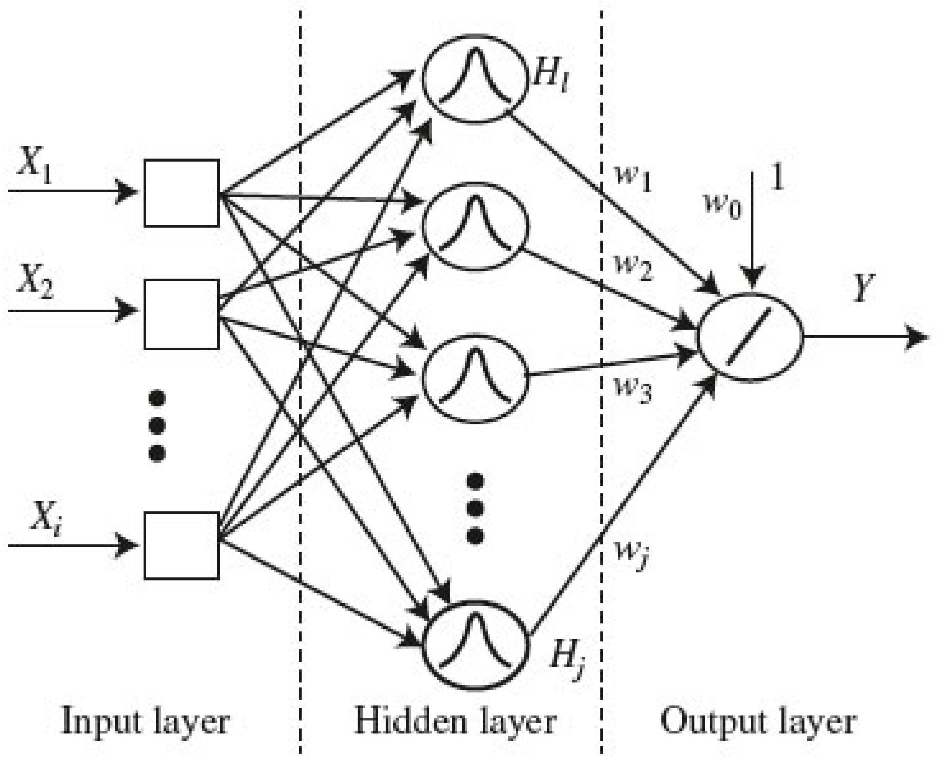

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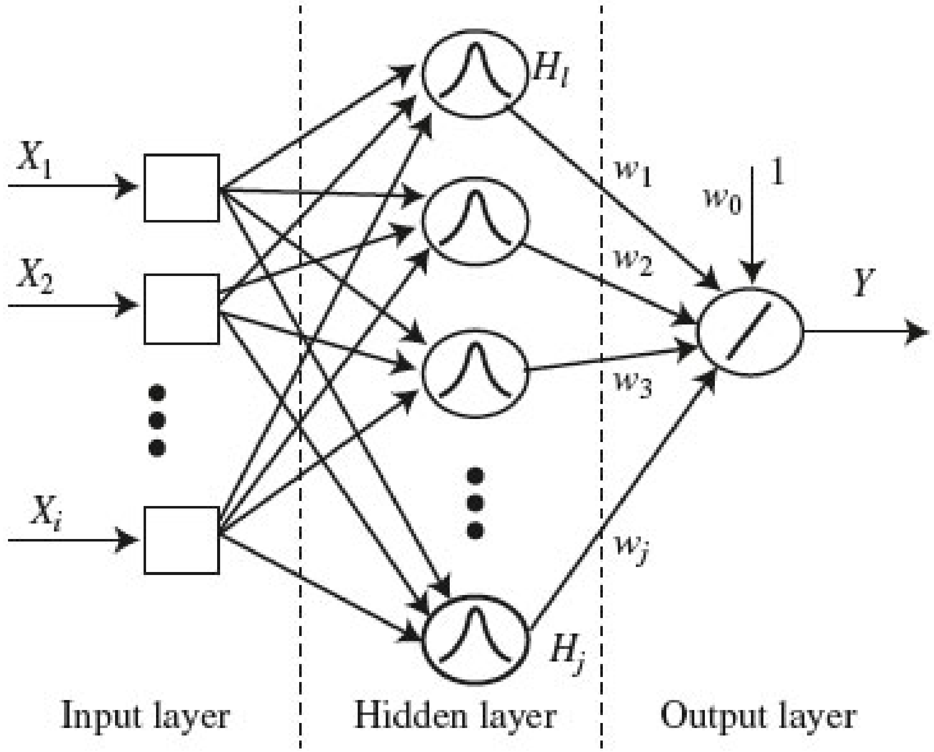
BUT this is not robust especially if a lot of samples are corrupted with noise!

The RBF neural network concept



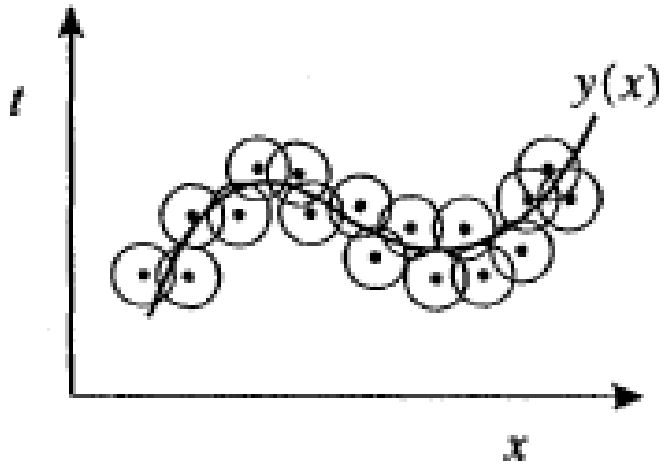
In the *exact interpolation*, the size of the hidden layer, N, is equal to the number of samples n

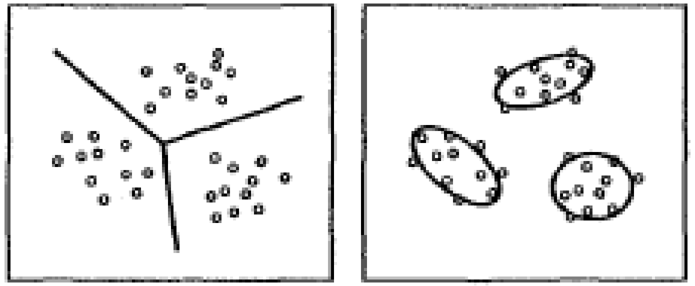
BUT this is not robust especially if a lot of samples are corrupted with noise!

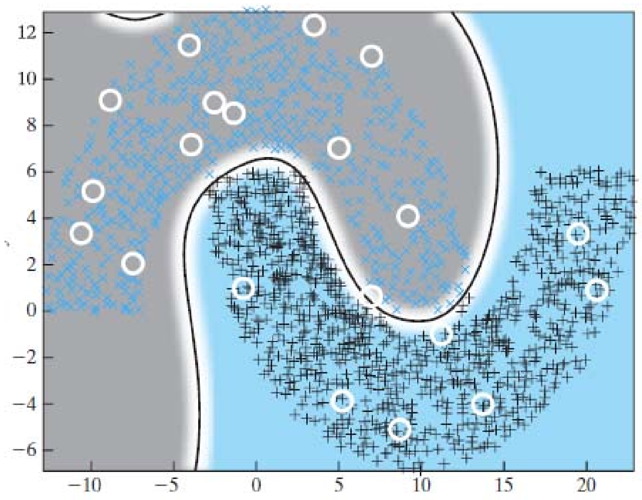
We need modifications:

1) $N < n$
2) centres $\mathbf{x}_i$ different from samples
3) widths, σ, also differ across RBF nodes
4) it is possible to include biases

Examples

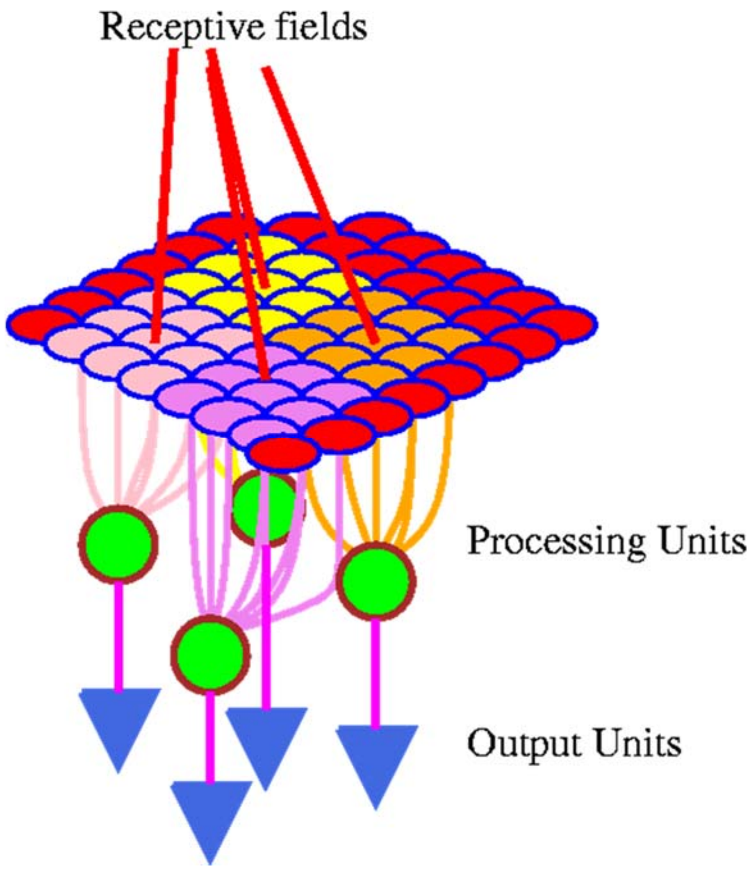






See *Bishop (1995)* for Bayesian interpretation of classification with RBF networks

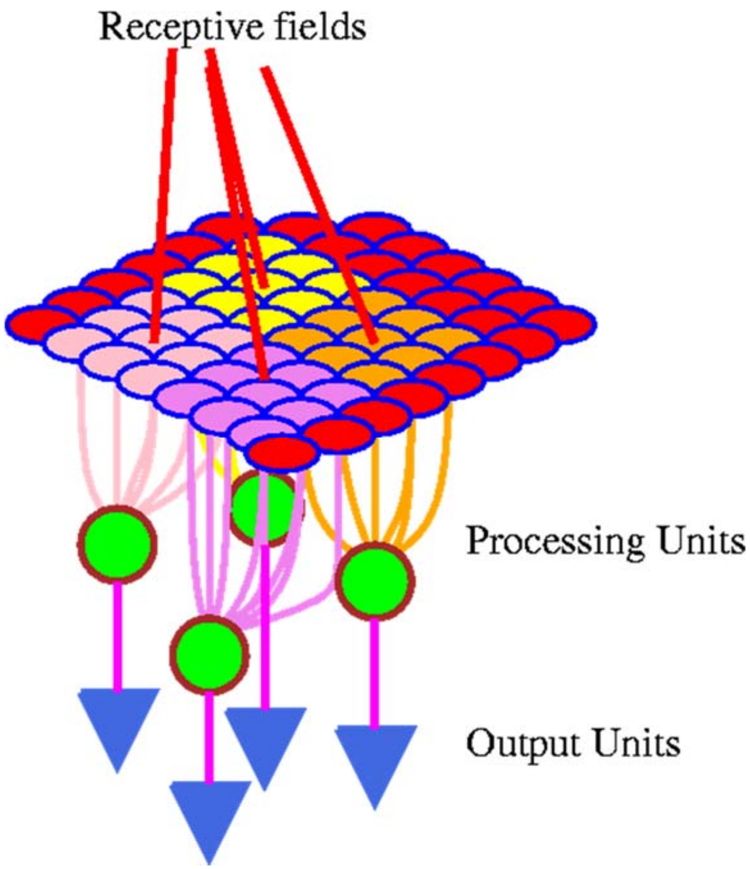
Analogy with receptive fields



The **receptive field** of a unit is the region of the input space from which a stimulus pattern evokes a response.

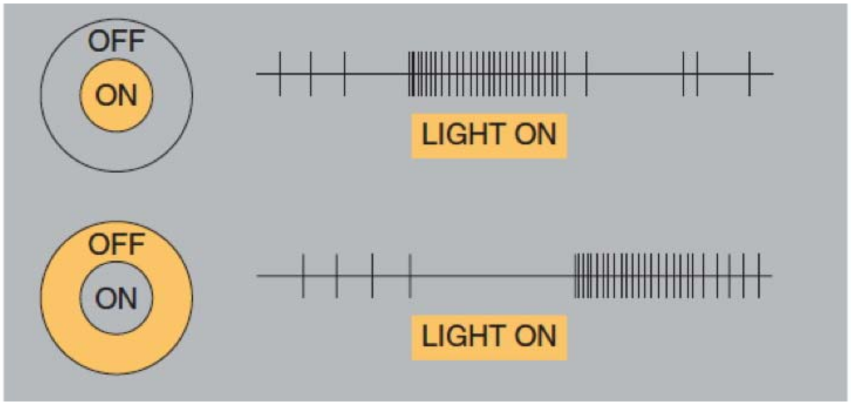
Haykin, 2009

Analogy with receptive fields

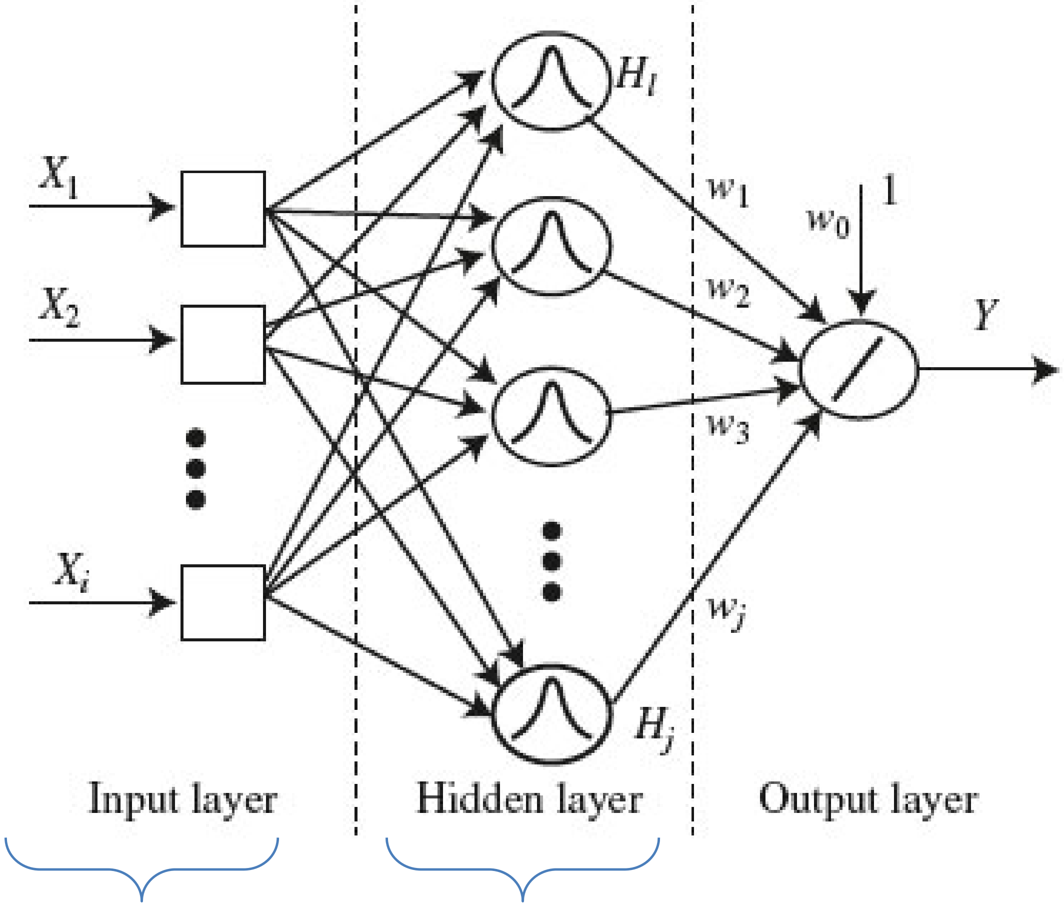


Example:

Retinal ON-centre and OFF-centre cells



Hybrid learning algorithm

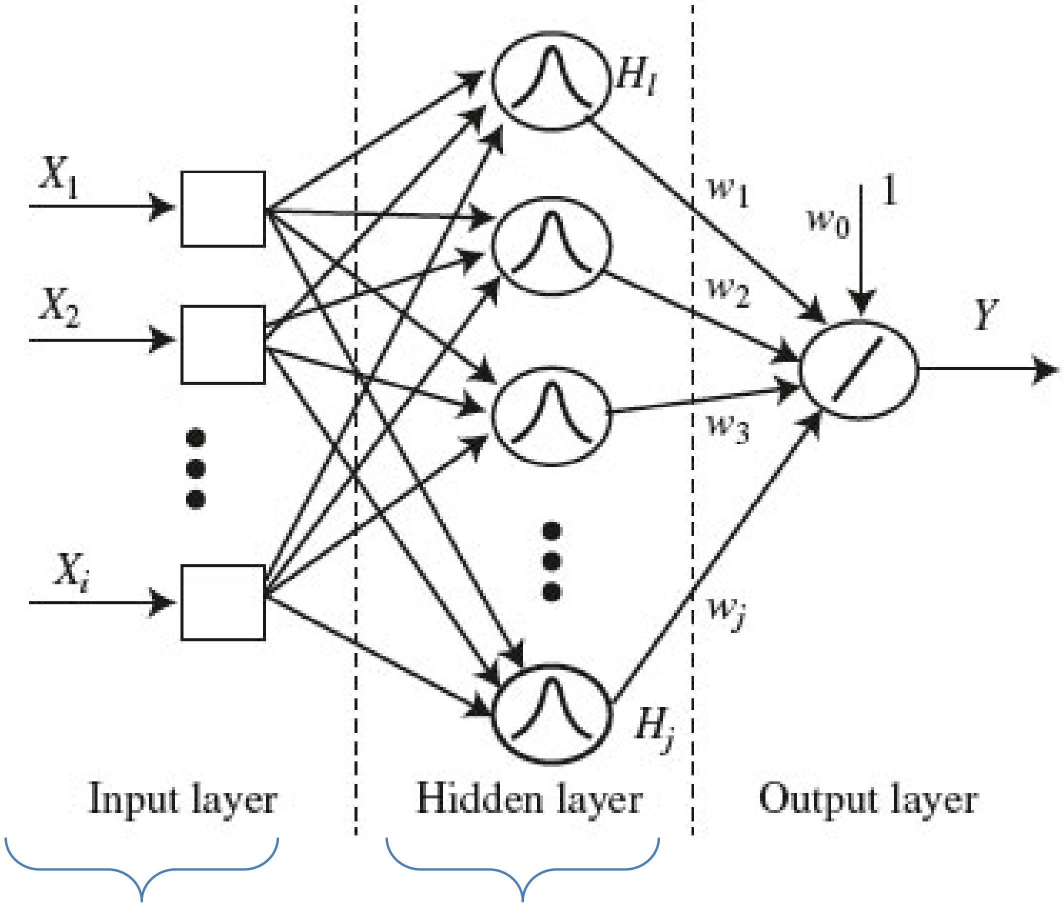


M **input** units N **hidden** units

n **training** samples

mapping: $\mathbb{R}^M \to \mathbb{R}^N \to \mathbb{R}^1$

Hybrid learning algorithm



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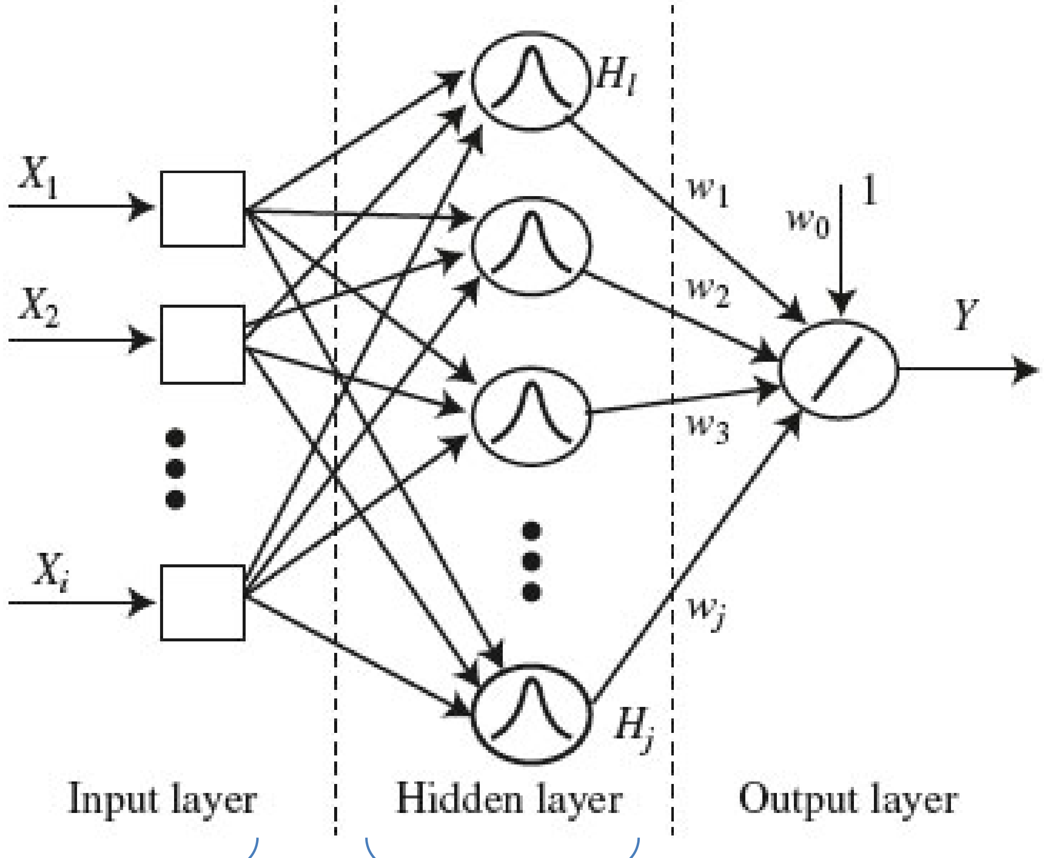

M **input** units

N **hidden** units

Size of the **input layer** determined by the dimensionality of the input.

Hidden layer

- N has to be decided
- centres and widths of hidden units have to be identified

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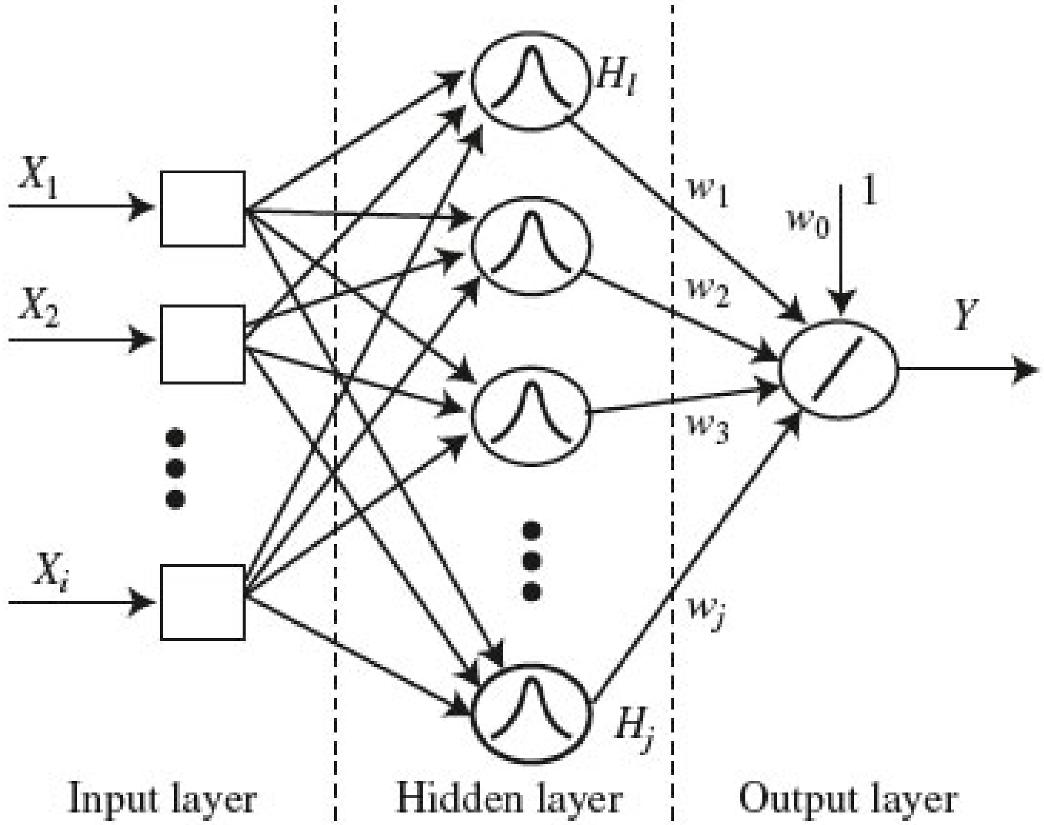
Hybrid learning algorithm



M **input units** N **hidden units**

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Size of the **input layer** determined by the dimensionality of the input.

Hidden layer

- N has to be decided
- centres and widths of hidden units have to be identified

Output layer performs a linear mapping – e.g., training with least square methods

$$\begin{bmatrix} \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1N} \\ \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2N} \\ . & . & . & . \\ \varphi_{n1} & \varphi_{n2} & \cdots & \varphi_{nN} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ . \\ w_N \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ . \\ d_n \end{bmatrix}$$

Hybrid learning – output layer

- Least-squares fitting – batch approach

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- Recursive least-squares estimation (see *Haykin*, RLS algorithm)

delta rule:

$$\begin{aligned} \mathbf{\Delta w} &= -\eta \nabla_{\mathbf{w}} \hat{\xi} \\ &= -\eta \frac{1}{2} \nabla_{\mathbf{w}} (f(x_k) - \mathbf{\Phi}(x_k)^\top \mathbf{w})^2 \\ &= \eta (f(x_k) - \mathbf{\Phi}(x_k)^\top \mathbf{w}) \mathbf{\Phi}(x_k) \\ &= \eta e \mathbf{\Phi}(x_k) \end{aligned}$$

$$\mathbf{\Phi}(x_k) = \begin{pmatrix} \phi_1(x_k) \\ \phi_2(x_k) \\ \vdots \\ \phi_n(x_k) \end{pmatrix}$$

RBF NN vs MLP

- Hidden units in MLP rely on weighted linear summations of inputs (a matter of interpretation: *it could be seen as the correlation or scalar prod*)
  - RBFs rely on distance to prototype vectors

RBF NN vs MLP

- Hidden units in MLP rely on weighted linear summations of inputs
  - RBFs rely on distance to prototype vectors
- In MLPs, function approximation is defined as by a nested sum of weighted summations
  - In RBFs, the approximation is defined by a single weighted sum

RBF NN vs MLP

- Hidden units in MLP rely on weighted linear summations of inputs
  - RBFs rely
- In MLPs, fun   es weighted su
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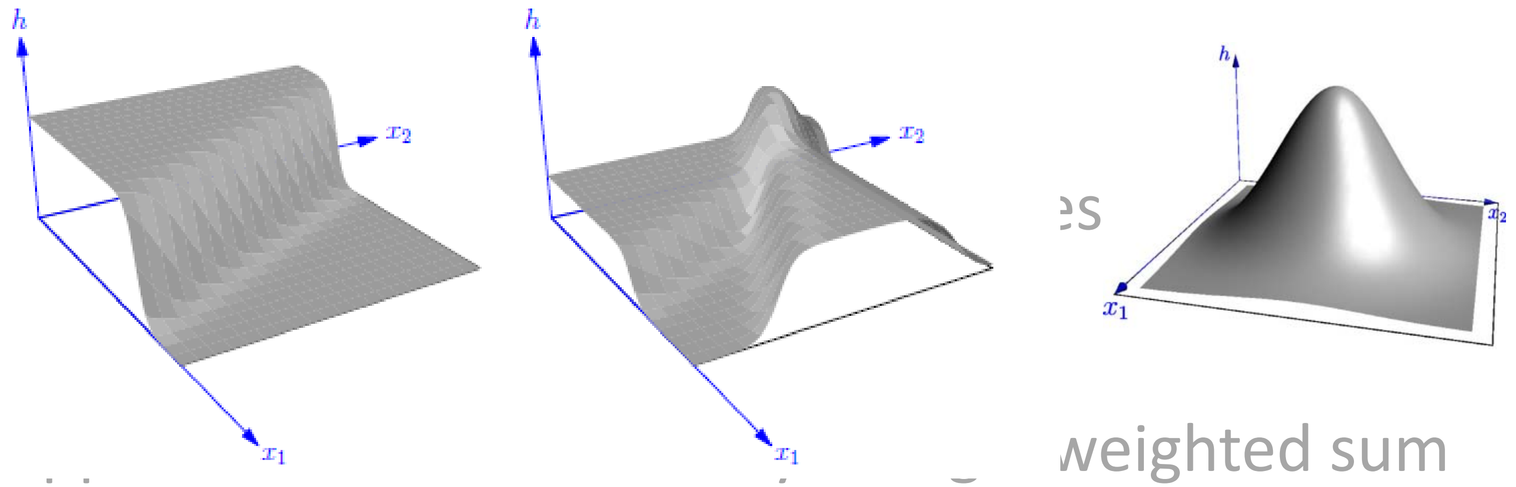
Weighted sum of Base Functions

- MLPs form distributed activations (many hidden units contribute to the output for a given input, which partly leads to local minima etc.)
  - In RBFs, very few local basis functions (wrt. input) are activated for a given input

RBF NN vs MLP

- In MLPs, usually all parameters/weights are trained at the same time
    - ➢ In RBFs, hybrid two-stage training is used

RBF NN vs MLP

- In MLPs, usually all parameters/weights are trained at the same time
  - ➢ In RBFs, hybrid two-stage training is used
- MLPs rely on complex multi-layer architecture
  - ➢ RBF NNs have simple one-hidden-layer architecture

Hybrid learning – RBF layer

- Clustering algorithms
  - k-means clustering
  - clustering with Kohonen feature maps (SOMs), vector quantization (VQ)
  - estimate of the cluster width (variance problem)
- Gaussian mixture models: expectation-maximization (EM) alg.

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- Supervised tuning of the RBF parameters with gradient descent

$$\frac{\partial E}{\partial \mu_{ji}} = \sum_n \sum_k \{y_k(\mathbf{x}^n) - t_k^n\} w_{kj} \exp\left(-\frac{\|\mathbf{x}^n - \boldsymbol{\mu}_j\|^2}{2\sigma_j^2}\right) \frac{(x_i^n - \mu_{ji})}{\sigma_j^2}$$

$$\frac{\partial E}{\partial \sigma_j} = \sum_n \sum_k \{y_k(\mathbf{x}^n) - t_k^n\} w_{kj} \exp\left(-\frac{\|\mathbf{x}^n - \boldsymbol{\mu}_j\|^2}{2\sigma_j^2}\right) \frac{\|\mathbf{x}^n - \boldsymbol{\mu}_j\|^2}{\sigma_j^3}$$

Bishop, 1995

Hybrid learning – RBF layer

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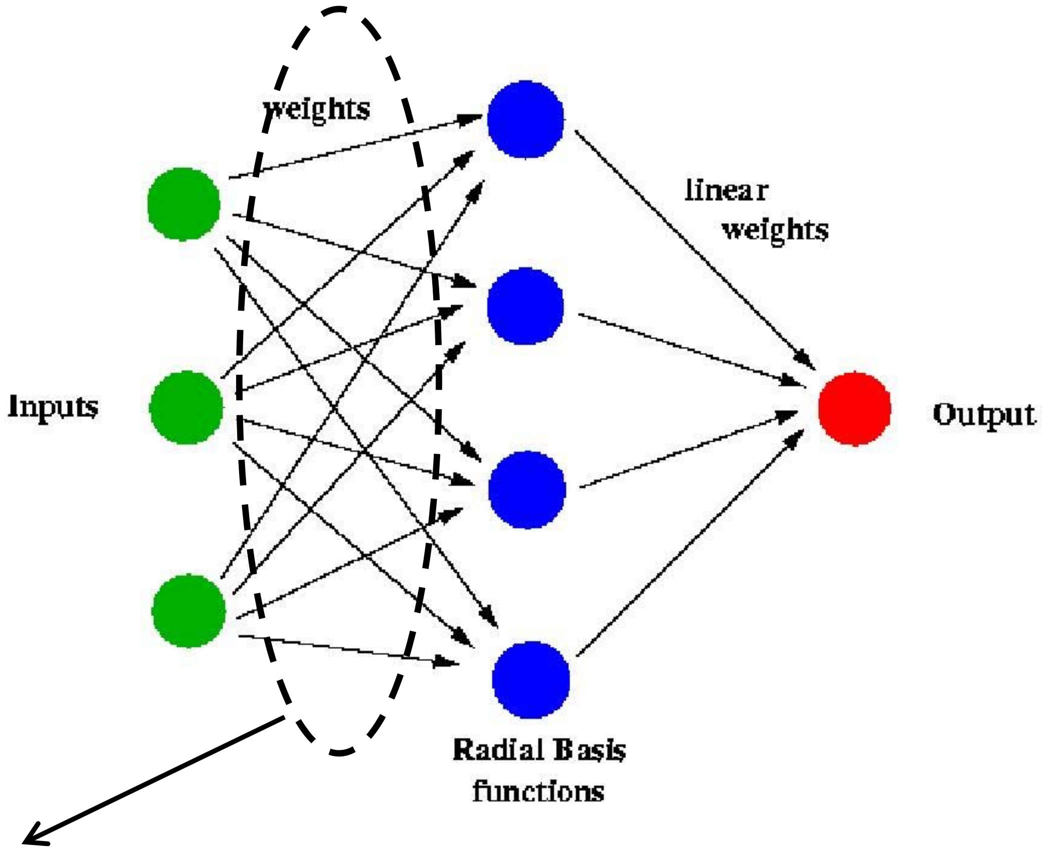
Bishop, 1995

- Subset selection with orthogonal least squares

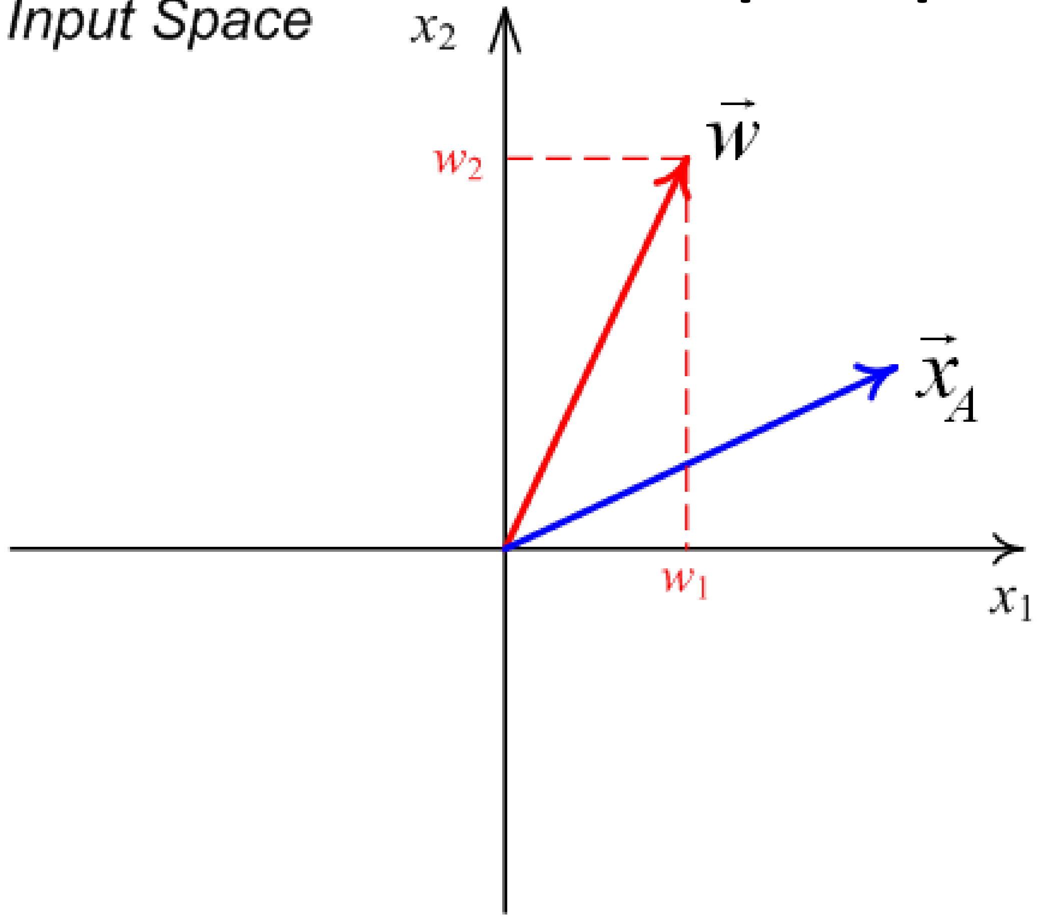
Interpretation of the weights to hidden layer

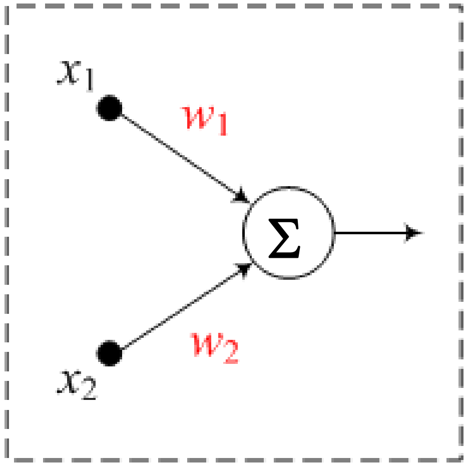


Do these connections have weights?

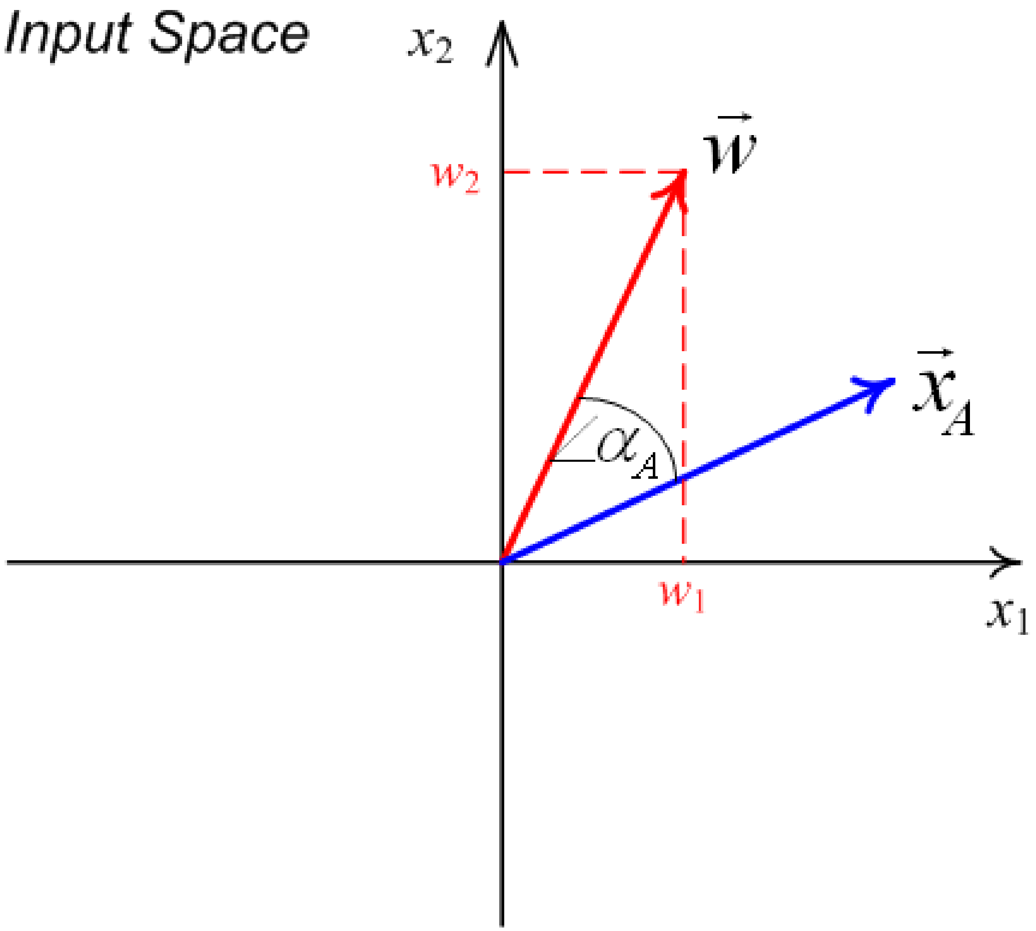
Weights in the input space

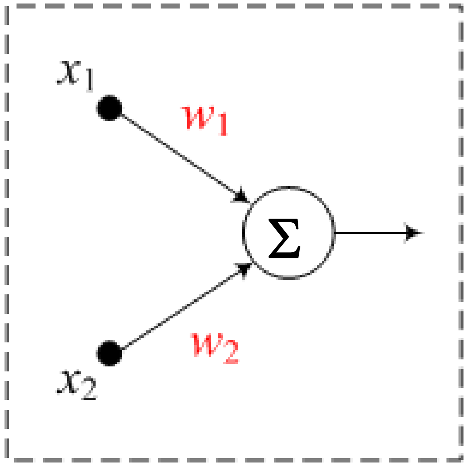
Recall perceptron





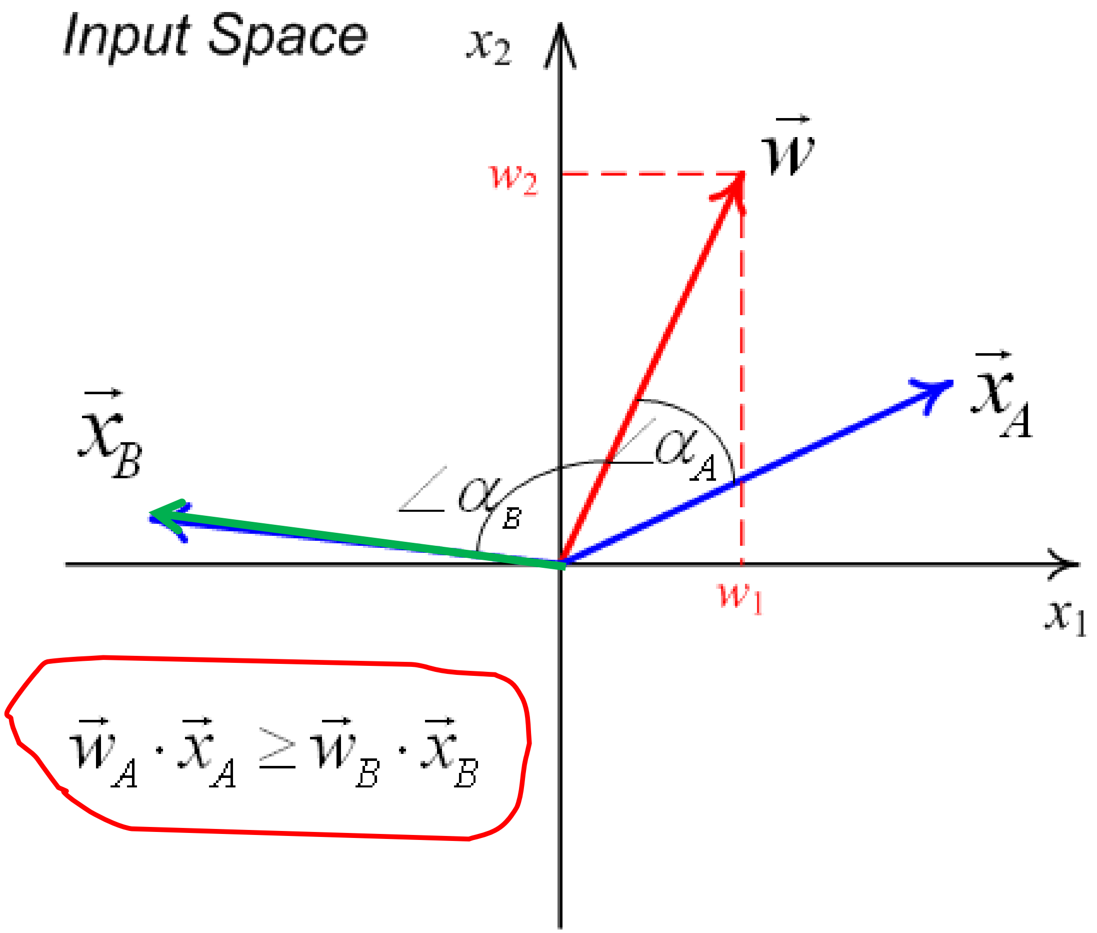
Weights in the input space – scalar product



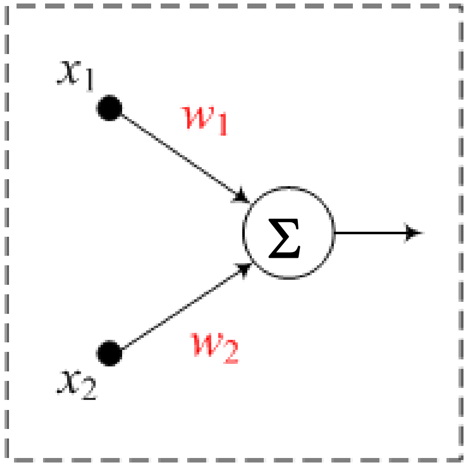


$$\vec{w}\cdot\vec{x} = \sum_{i=1}^{2} w_i x_i = w_1 x_1 + w_2 x_2 = \|\vec{w}\|\|\vec{x}\|\cos(\angle\alpha)$$

Weights in the input space – scalar product



$$\vec{w}_A \cdot \vec{x}_A \geq \vec{w}_B \cdot \vec{x}_B$$



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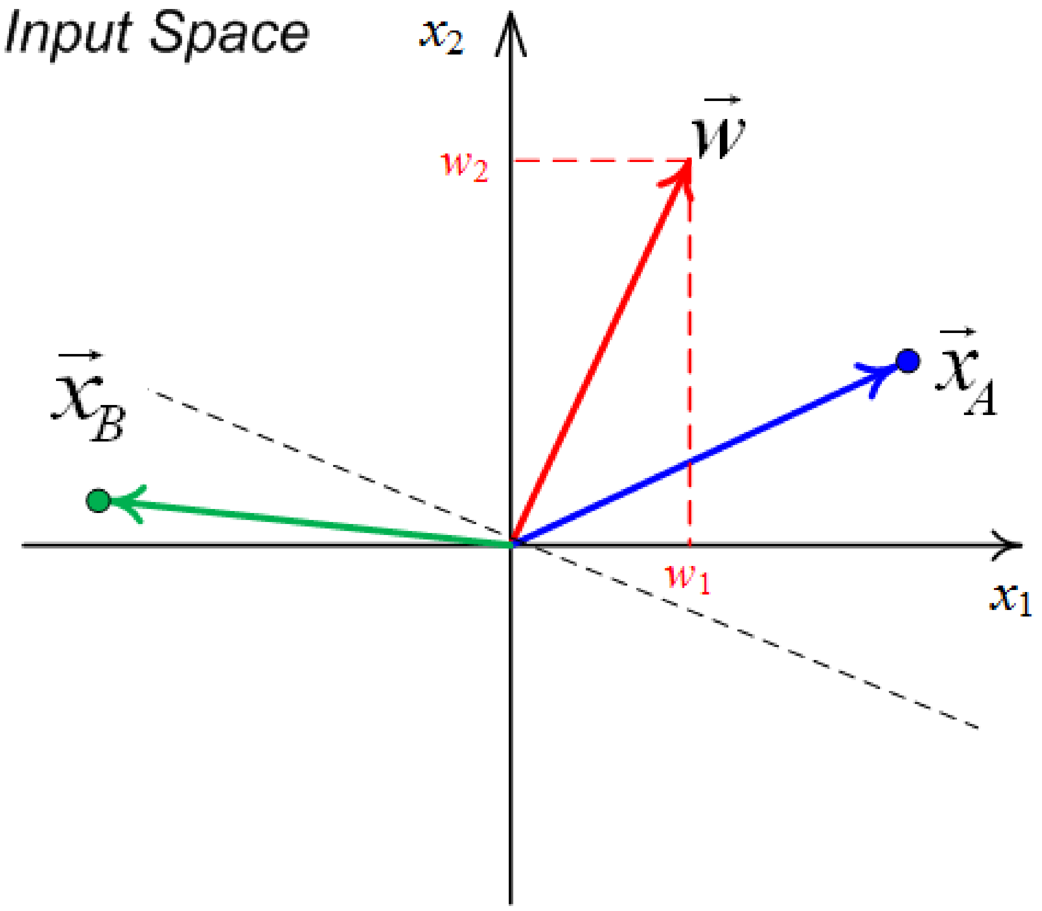
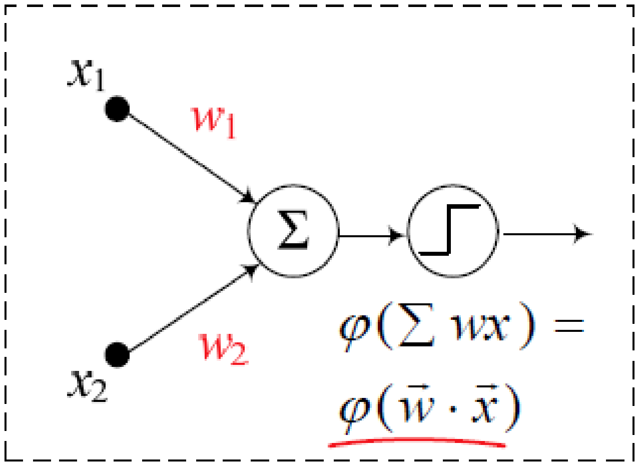
Weights in the input space – perceptron classifier





$$\varphi(\sum wx) = \underline{\varphi(\vec{w} \cdot \vec{x})}$$

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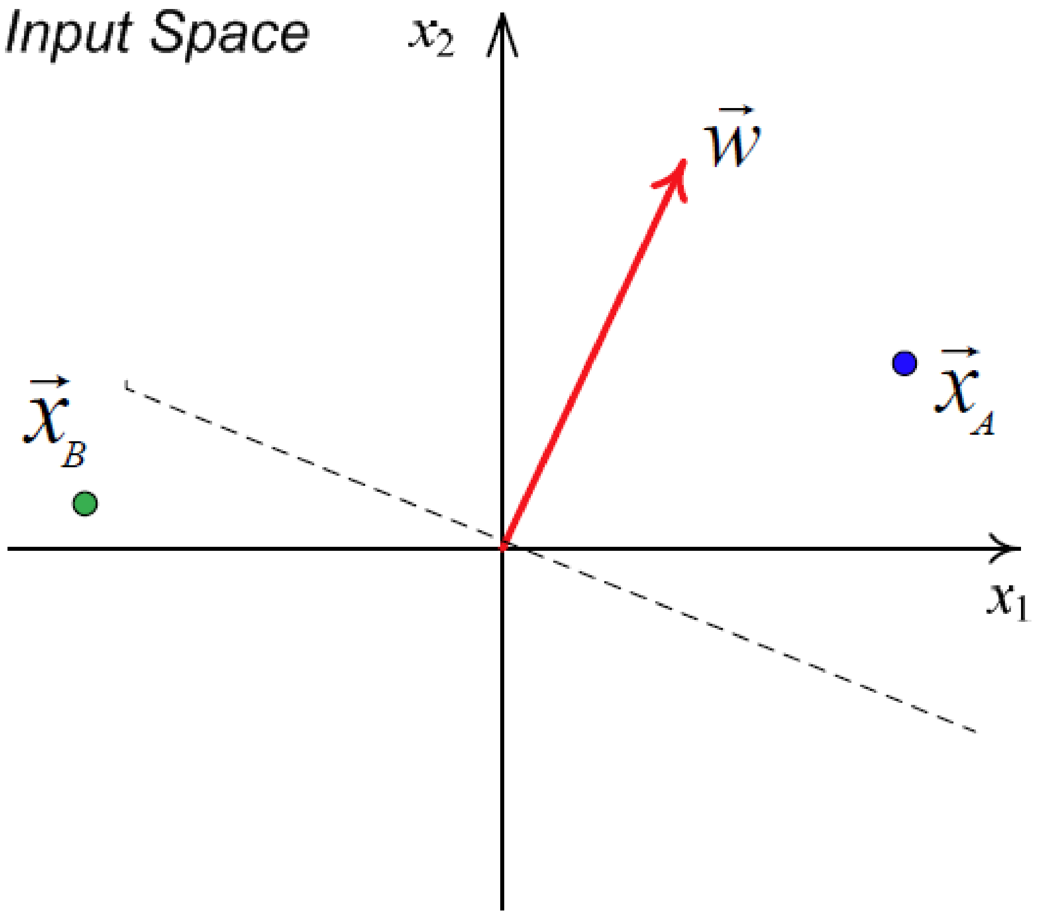
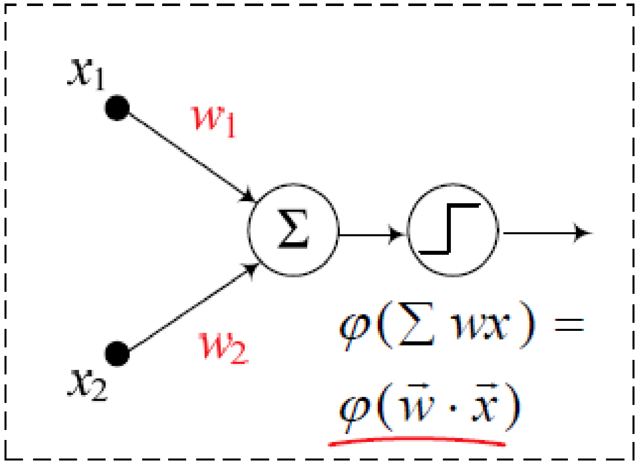
Weights in the input space – perceptron classifier





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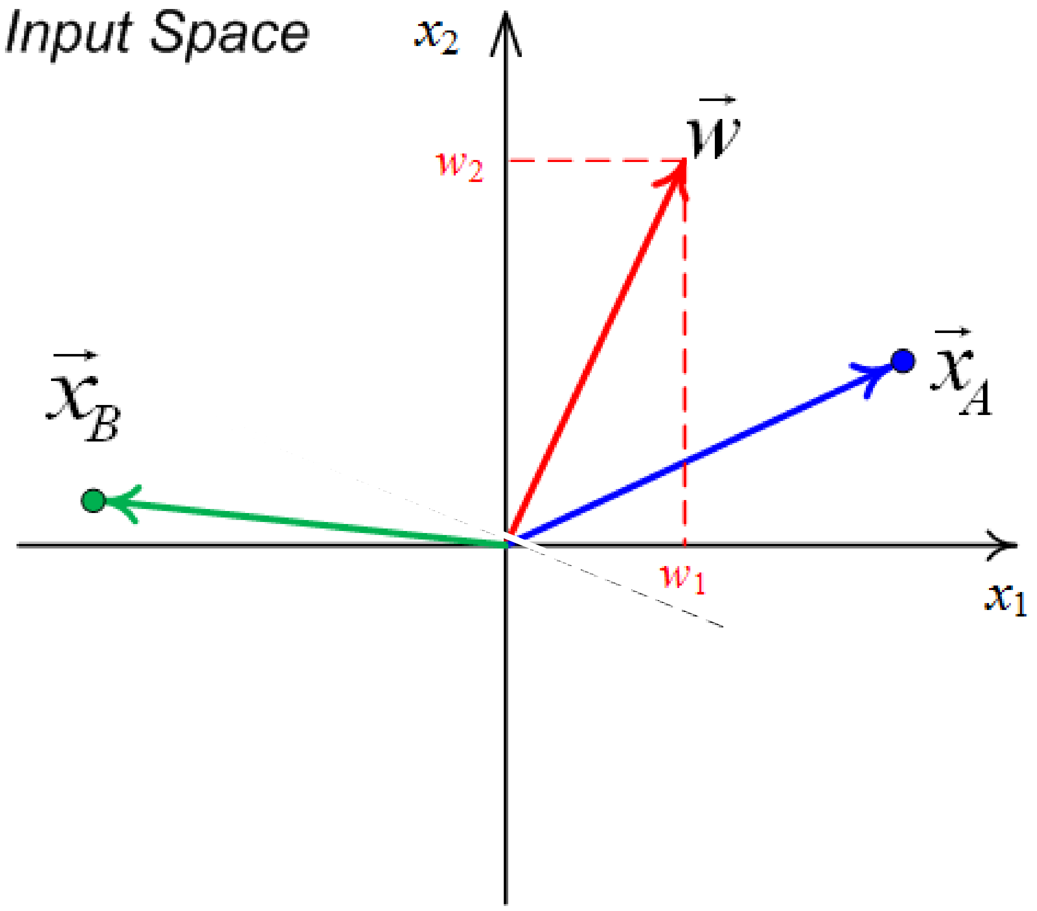
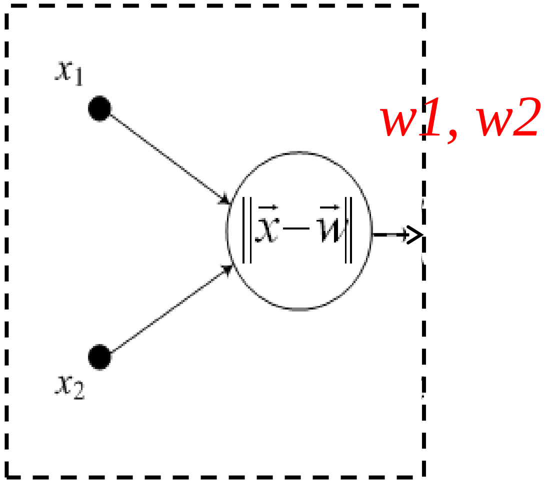
based on scalar product

$$\vec{w}\cdot\vec{x} = \sum_{i=1}^{2} w_i x_i = w_1 x_1 + w_2 x_2 = \|\vec{w}\|\|\vec{x}\|\cos(\angle\alpha)$$

Weights in the input space – Euclidean distance

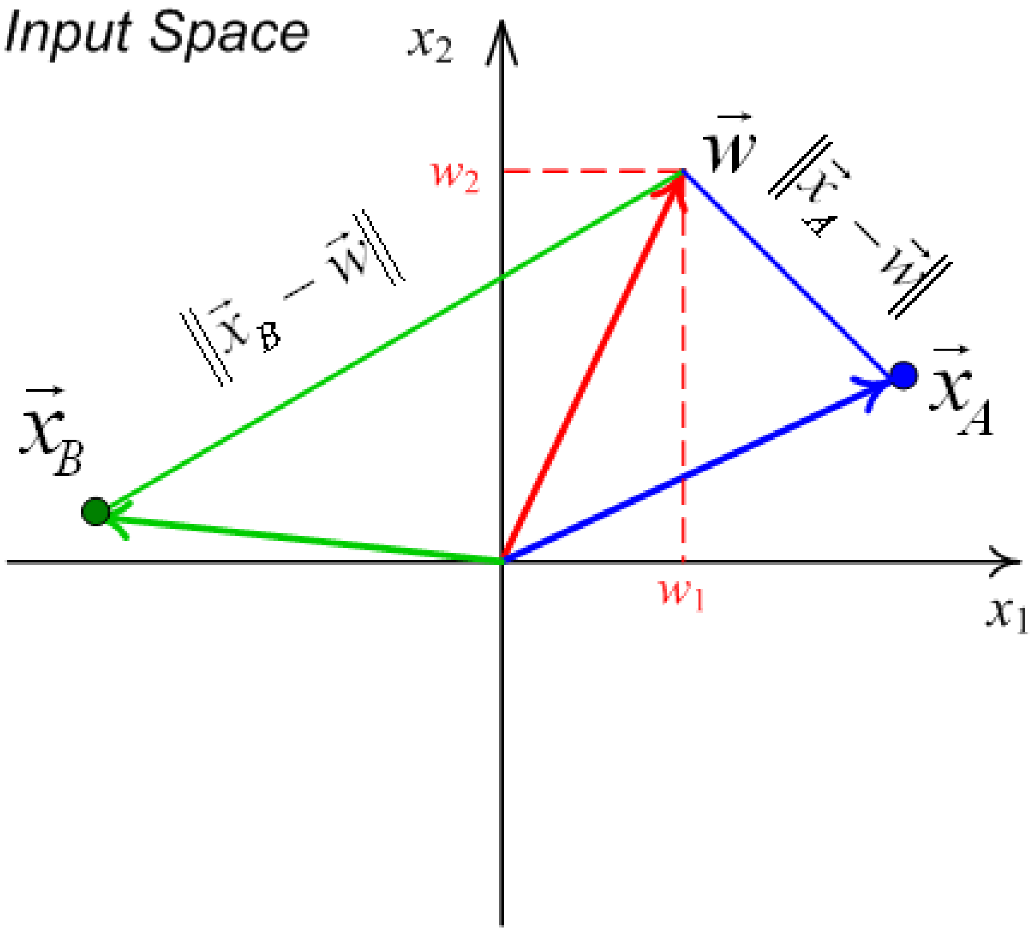


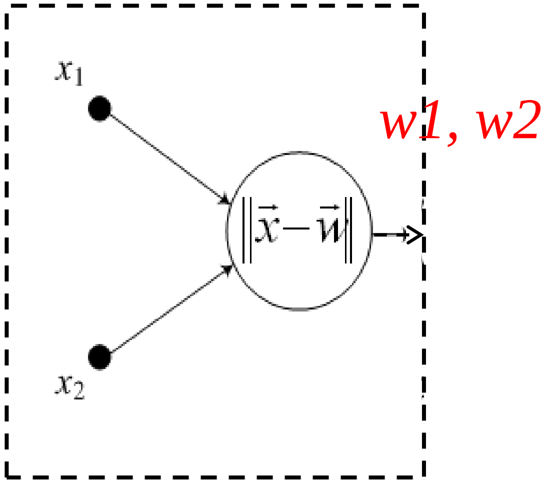


Euclidean distance measure: $\|\vec{x}-\vec{w}\|$

$$\|\vec{x}-\vec{w}\| = \sqrt{\sum_{i=1}^{2}(x_i - w_i)^2}$$

Weights in the input space – Euclidean distance





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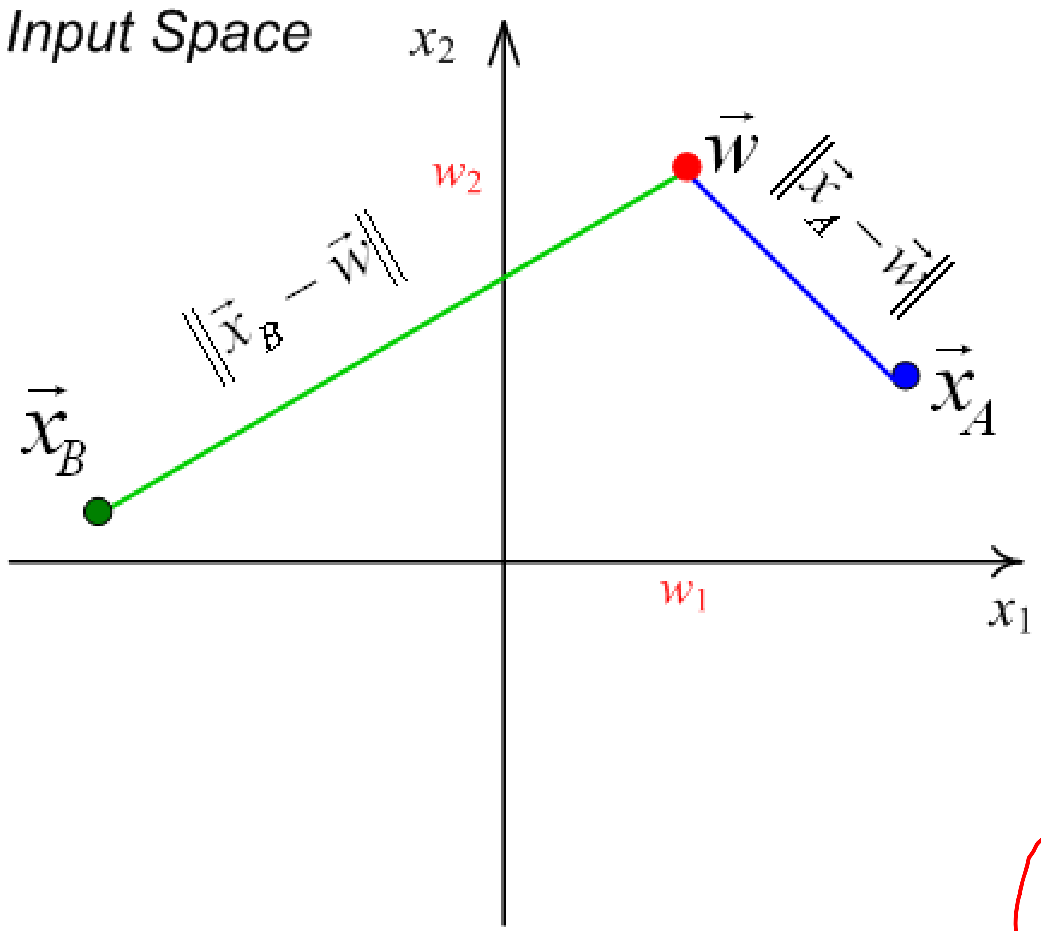
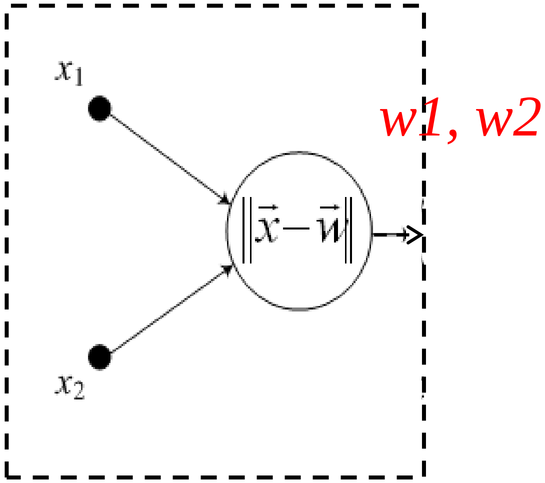
Weights in the input space – Euclidean distance



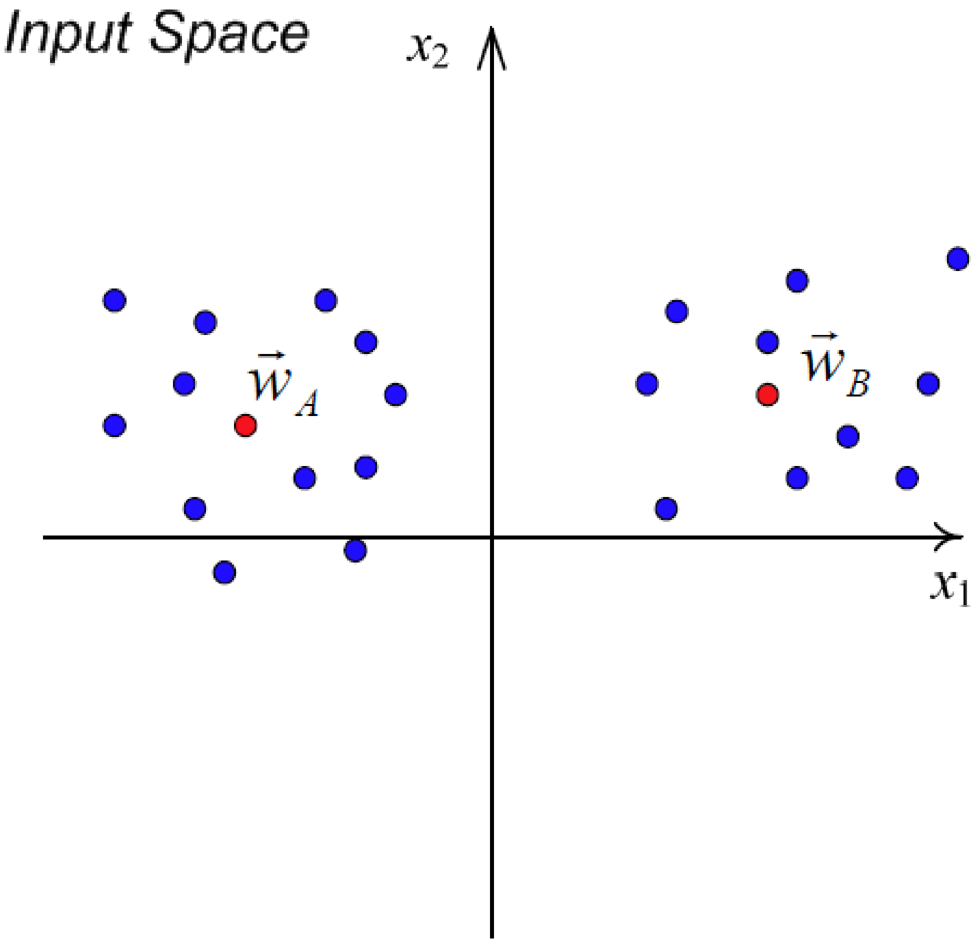


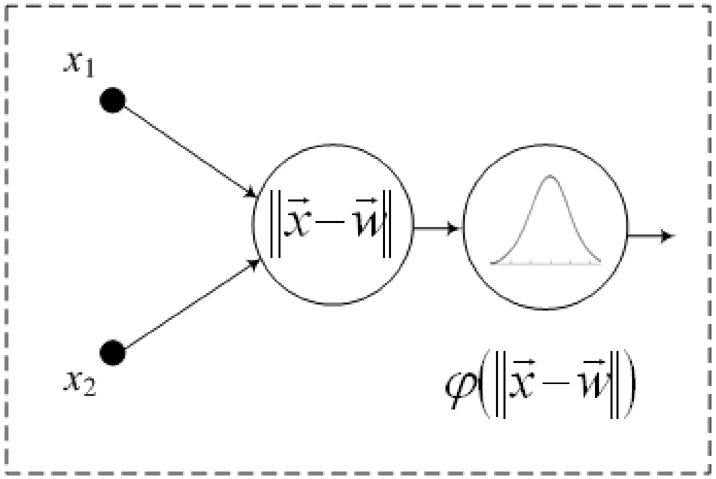
Euclidean distance measure: $\|\vec{x}-\vec{w}\|$

$$\|\vec{x}_A-\vec{w}\| < \|\vec{x}_B-\vec{w}\|$$

$$\|\vec{x}-\vec{w}\| = \sqrt{\sum_{i=1}^{2}(x_i-w_i)^2}$$

Weights in the input space – proximity measure

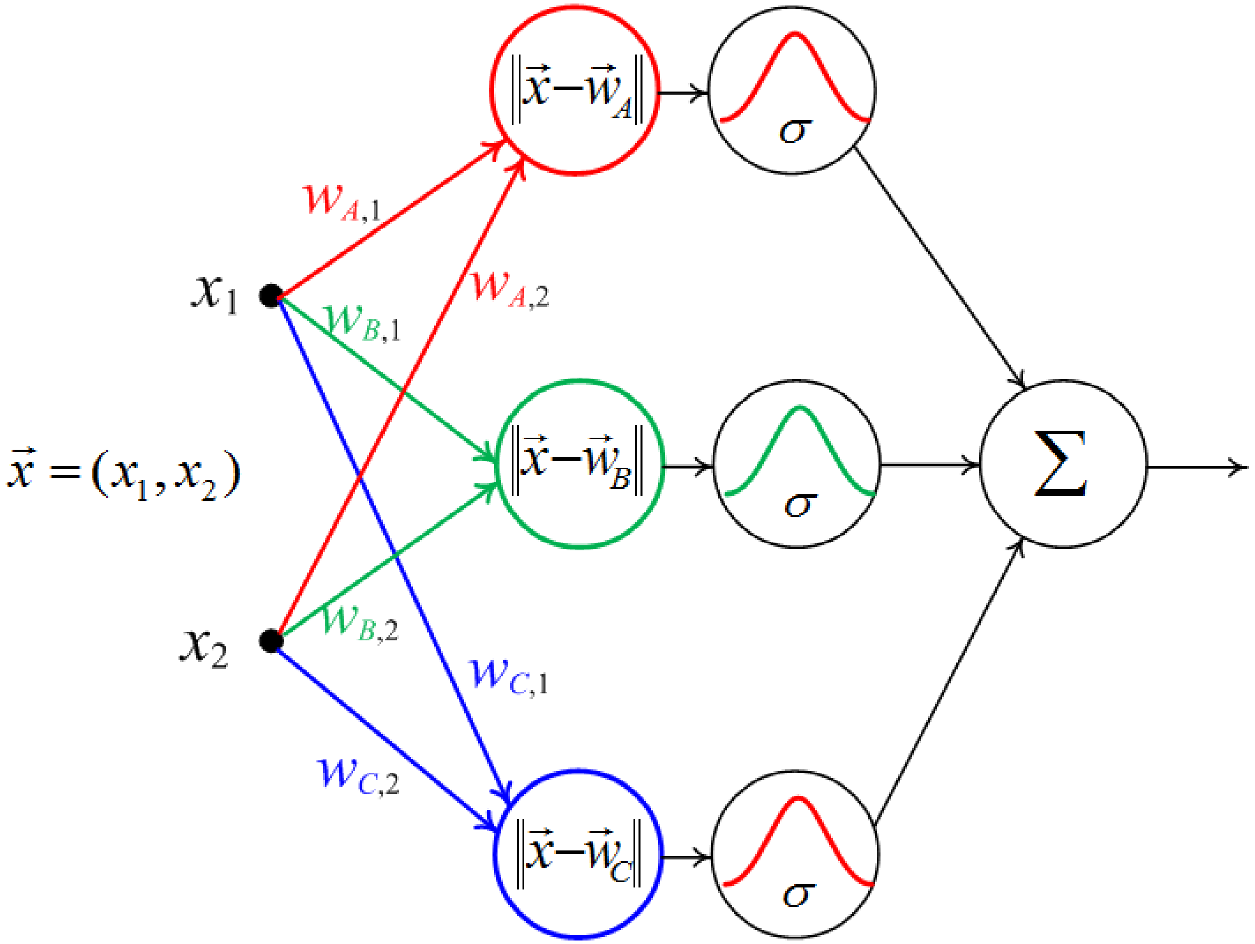




based on Euclidean distance

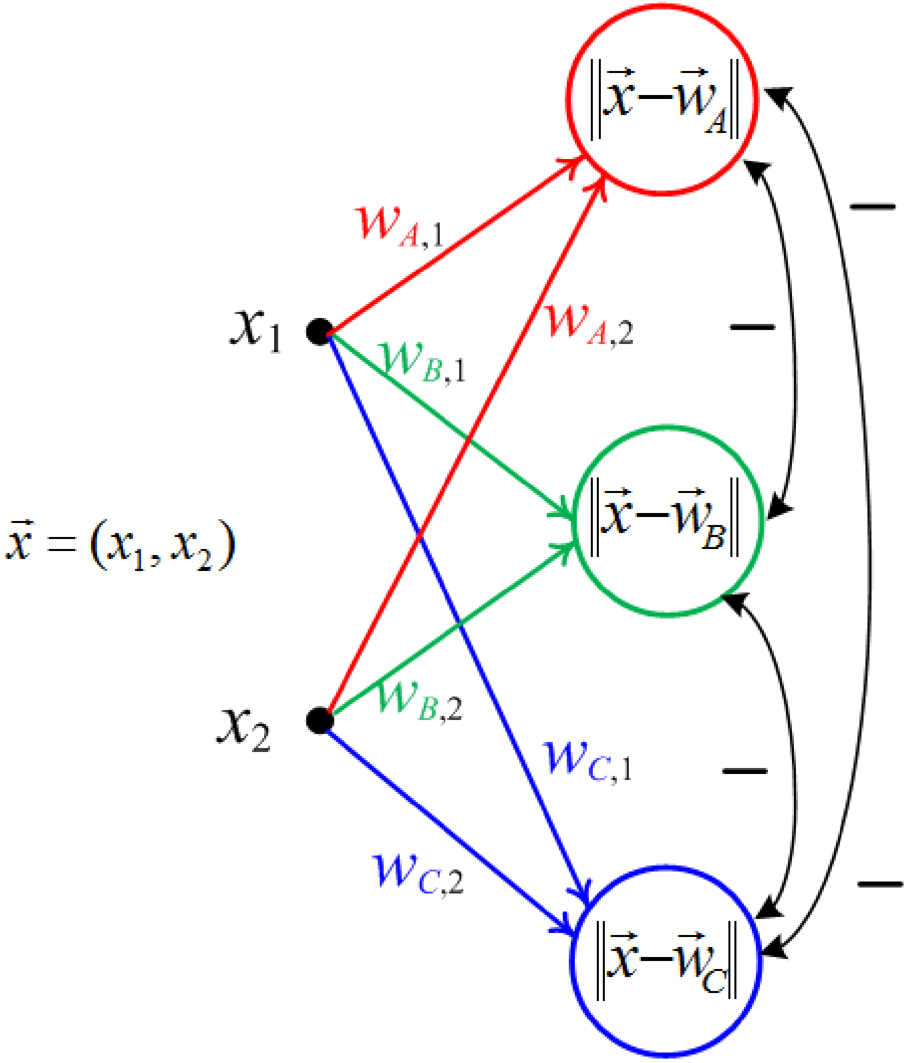
$$\varphi(\|\vec{x}-\vec{w}\|) = \exp\left(-\frac{\|\vec{x}-\vec{w}\|}{\sigma^2}\right)$$

RBF network – interpretation of the input weights



$$\varphi(\vec{x}, \vec{w}) = \exp\left(\frac{-\|\vec{x} - \vec{w}\|}{\sigma^2}\right)$$

Competition and winner-take-all mechanism

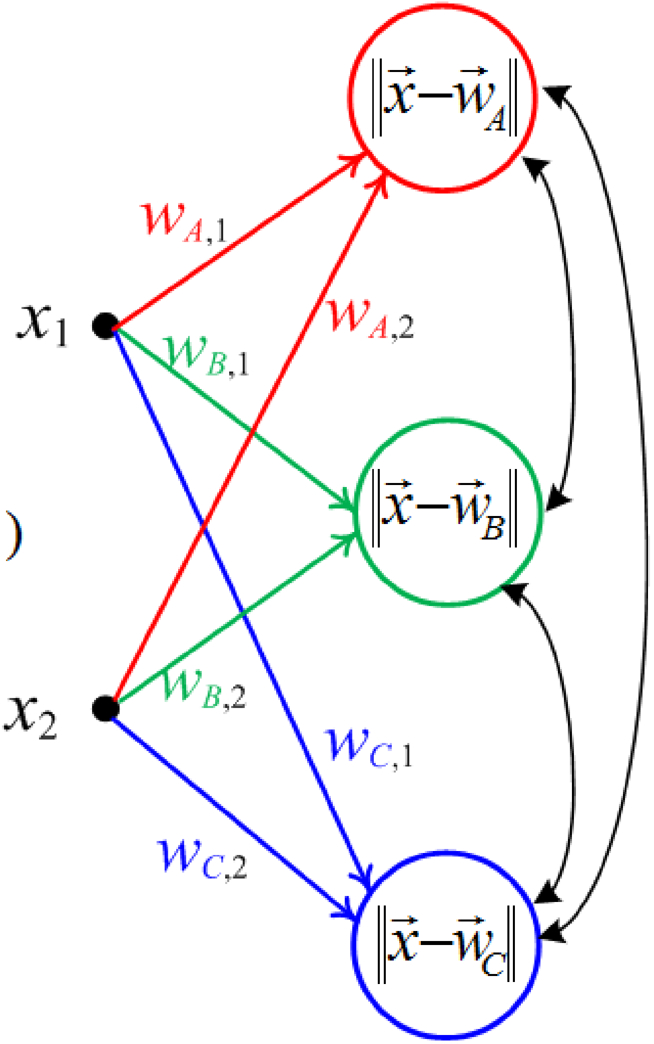


Competition between nodes – **Winner Takes All** (WTA)

$$\vec{w}_A=(w_{A,1},w_{A,2})$$
$$\vec{w}_B=(w_{B,1},w_{B,2})$$
$$\vec{w}_C=(w_{C,1},w_{C,2})$$

Competitive mechanisms

$\vec{x} = (x_1, x_2)$



If the **red** node w_A wins, then:

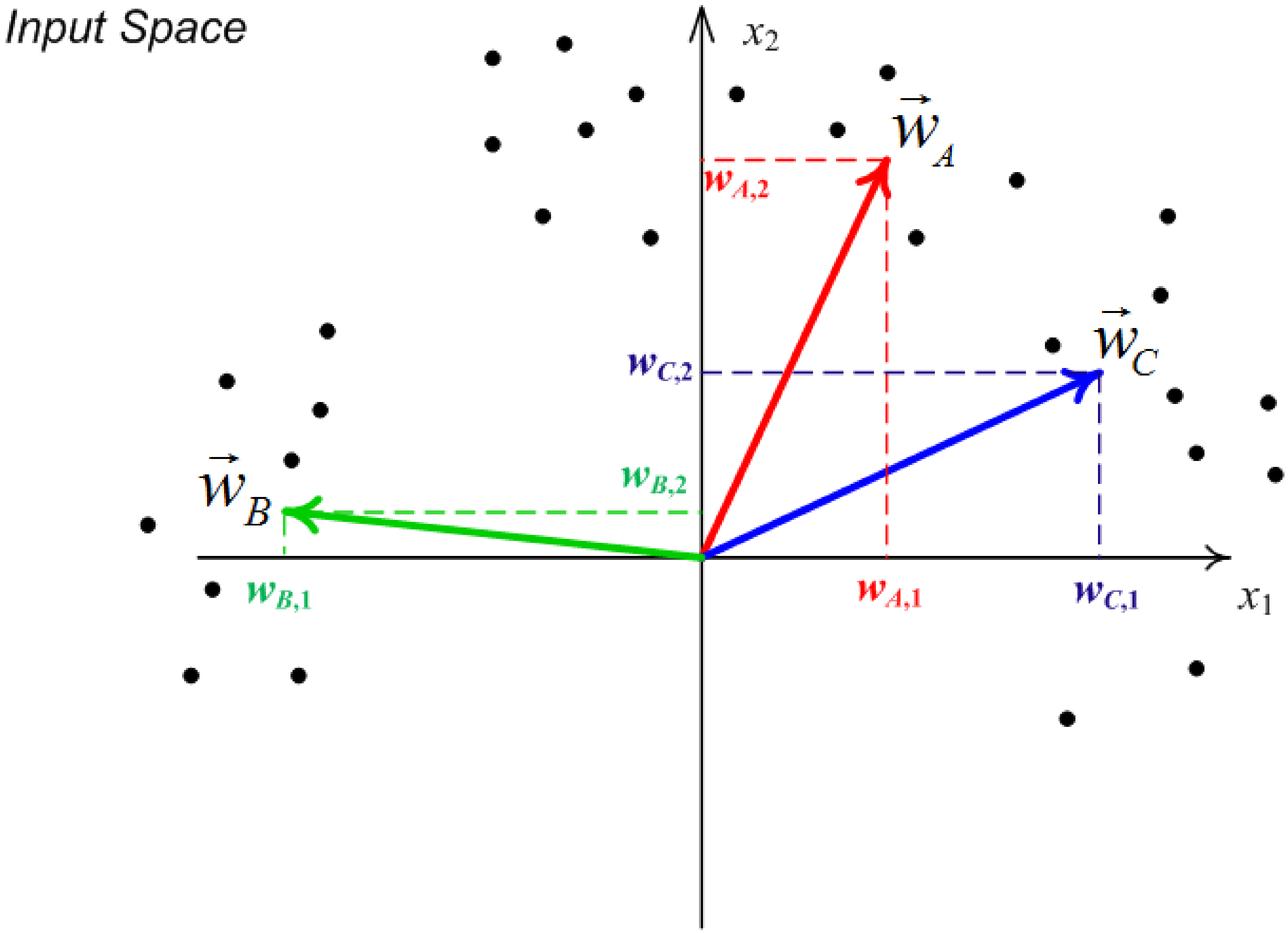
$$\Delta\vec{w}_A = \eta\vec{x}$$

OR

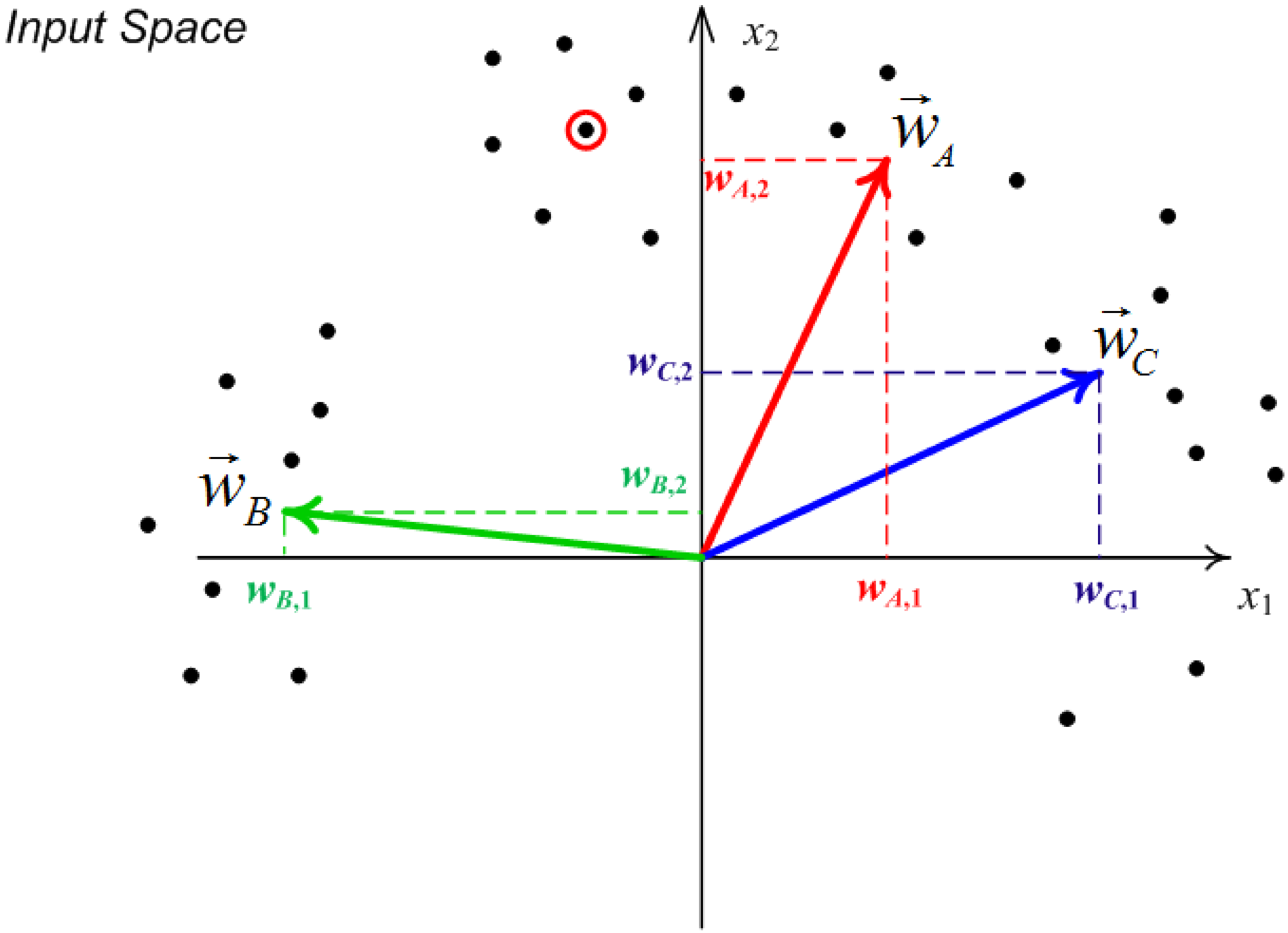
$$\Delta\vec{w}_A = \eta(\vec{x} - \vec{w}_A)$$

$$\vec{w}_A = (w_{A,1}, w_{A,2})$$
$$\vec{w}_B = (w_{B,1}, w_{B,2})$$
$$\vec{w}_C = (w_{C,1}, w_{C,2})$$

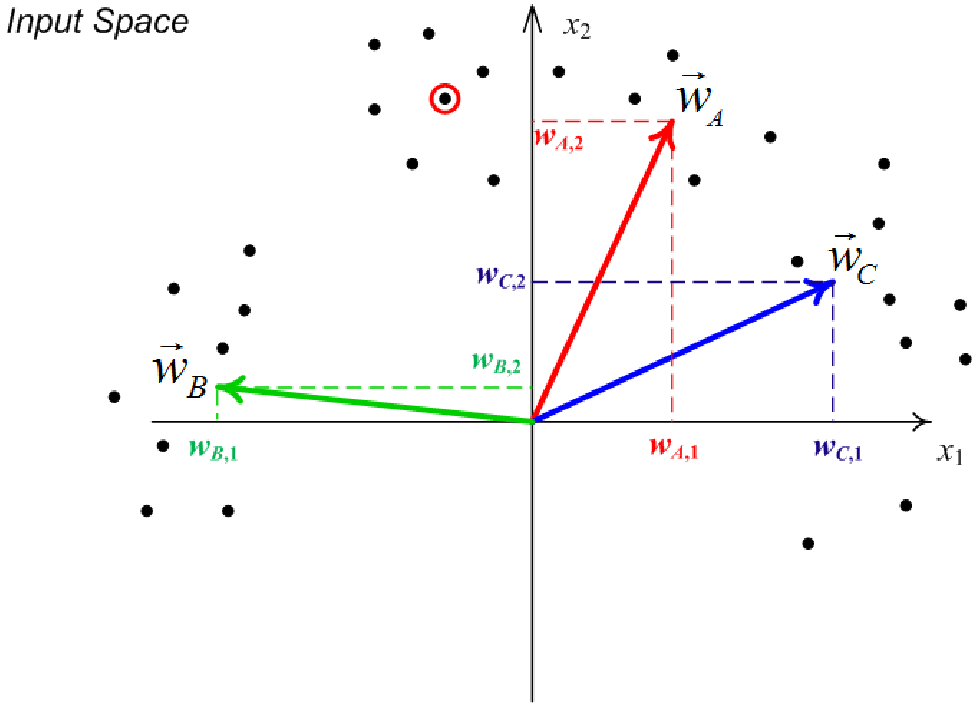
Competition – update of the input weights, example



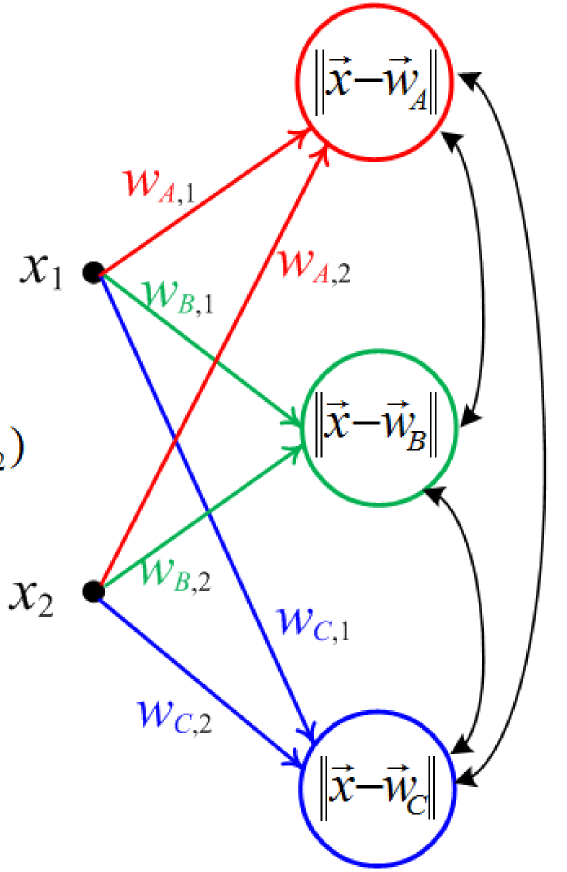
Competition – update of the input weights



Competition – update of the input weights



$$\vec{x} = (x_1, x_2)$$



$$\vec{w}_A = (w_{A,1}, w_{A,2})$$
$$\vec{w}_B = (w_{B,1}, w_{B,2})$$
$$\vec{w}_C = (w_{C,1}, w_{C,2})$$

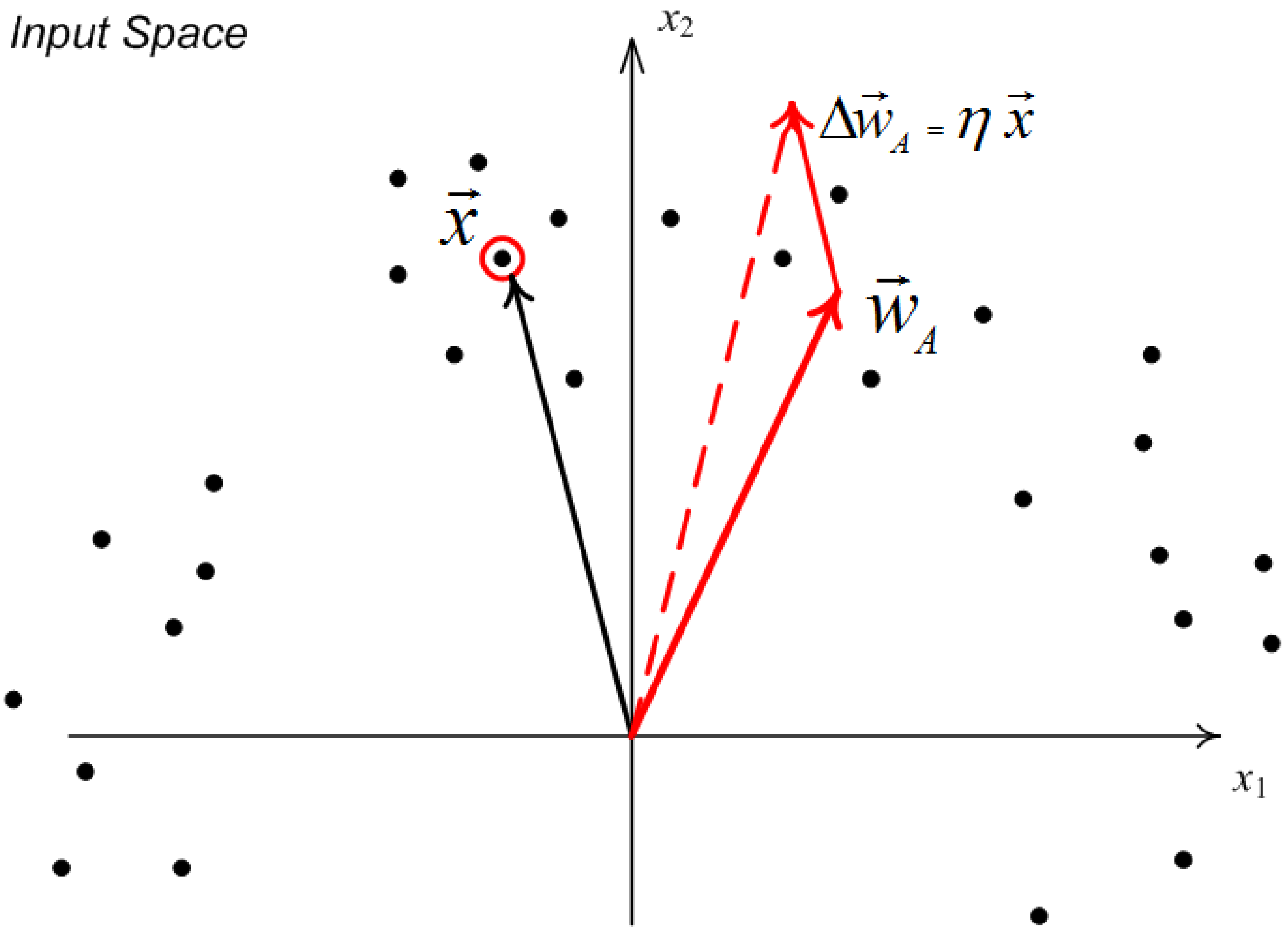
If the **red** node w_A wins, then:

$$\Delta\vec{w}_A = \eta\vec{x}$$

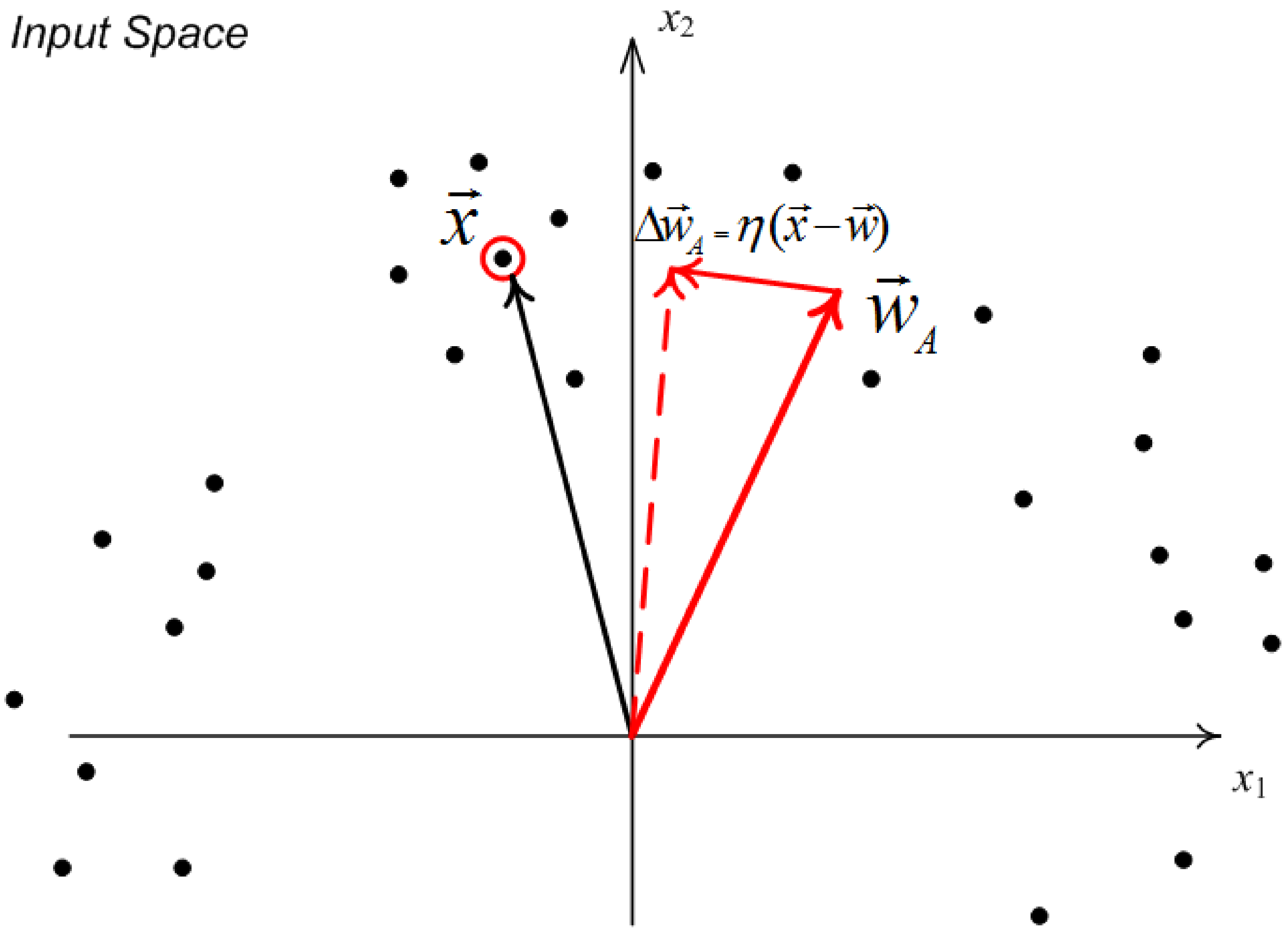
OR

$$\Delta\vec{w}_A = \eta(\vec{x} - \vec{w}_A)$$

Competition – update of the input weights, ver. 1



Competition – update of the input weights, ver. 2



Input Space
x_2
$\vec{x}$
$\Delta\vec{w}_A = \eta(\vec{x}-\vec{w})$
$\vec{w}_A$
x_1