An effective approach to enumerate universal cycles for *k*-permutations

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Abstract

A universal cycle for k-permutations is a cyclic arrangement in which each k-permutation appears exactly once as k consecutive elements. In this paper, we study the enumeration problem of universal cycles for k-permutations (Problem 477[10]) and obtain exact formulae for $k = 3, 4$.

1 Introduction

Universal cycles were introduced by Chung, Diaconis, and Graham [4] as generalizations of de Bruijn cycles [2], which are cyclic binary sequence of length 2^n that contain every binary n-tuple. Universal cycles are connected with Gray codes deeply [12, 15]. In this paper we consider the universal cycles for k-permutations. Given a positive integer n, let $[n] = \{1, 2, \ldots, n\}$. A *k-permutation* is an ordered arrangement of k distinct elements in $[n]$, $1 \le k \le n$. Let $P_{n,k}$ be the set of all k-permutations of the n-set $[n]$. Obviously, $|P_{n,k}| = n!/(n-k)!$. Let $C = (c_1, c_2, \ldots, c_{|P_{n,k}|})$ be a cyclic arrangement (or periodic sequence), where each $c_i \in [n]$ for $1 \le i \le |P_{n,k}|$. If in C each k-permutation appears exactly once as k consecutive elements, then we say that C is a *universal cycle* for $P_{n,k}$. For example, if $n = 4$ and $k = 2$, then (123413242143) is a universal cycle for $P_{4,2}$

It is obvious that there is no universal cycle for k-permutations when $k = n$. Jackson [9] showed that the universal cycle for k-permutations always exists when $k < n$. There are lots of results about the construction of universal cycles for k-permutations, mainly for the case that $k = n-1$ named *shorthand permutations* [6, 7, 11, 13]. Wong [17] introduced the relaxed shorthand notation to encode permutations. Recently, Sawada and Williams [14] consider the universal cycle for strings with fixed-content. An interesting problem is to enumerate distinct universal cycles for k-permutations. This problem was formally presented in [10].

Problem 1. (Problem 477 [10]) How many different universal cycles for $P_{n,k}$ exist?

As far as we know, this enumeration problem is still open. When $k = 1$, the number of universal cycles for $P_{n,1}$ is obviously equal to $(n-1)!$. However, for $k \geq 2$, it is not easy to enumerate universal cycles. The number of universal cycles for $P_{n,2}$ and $P_{n,3}$ were obtained in [3], using Eulerian tours on certain digraphs and their adjacency matrices, their powers, and corresponding eigenvalues. The key is to find an algebraic relation between the adjacency matrix and the all one matrix, and thus to determine these eigenvalues. However, for $k \geq 4$, it is complicated to determine the enumeration formula by using the method in [3]. In this paper, we propose a new method to find eigenvalues of the adjacency matrix and thus count the number of universal cycles. Based on this method, we obtain the exact formula for $k = 4$, and this also gives a new proof of the exact formula for $k = 3$.

Let us recall some definitions and concepts for digraphs. For a vertex v in a digraph, its out-degree is the number of arcs with initial vertex v, and its in-degree is the number of arcs with final vertex v. A digraph is balanced if each vertex has the same in-degree and out-degree. Obviously, a digraph contains an Eulerian tour if and only if the digraph is connected and balanced (see, for example, [1, Theorem 1.7.2]).

Given a digraph D, its *adjacency matrix* is the (0,1)-matrix $A = (a_{i,j})$ where $a_{i,j} = 1$ if $v_i v_j$ is an arc of D, and $a_{i,j} = 0$ otherwise. Let Γ be the diagonal matrix of the vertex out-degrees. The *Laplacian matrix* of D is defined as $L = \Gamma - A$. The eigenvalues of L are called the Laplacian eigenvalues of D.

Now we introduce the definition of the transition digraph. Let D be a digraph with vertex set $P_{n,k-1}$. The arcs of D satisfy the following rule: for any two vertices $i_1 i_2 \cdots i_{k-1}$ and $j_1 j_2 \cdots j_{k-1}$, there is an arc from $i_1 i_2 \cdots i_{k-1}$ to $j_1 j_2 \cdots j_{k-1}$ if and only if $i_s = j_{s-1}$ for $2 \leq s \leq k-1$, and $i_1 \neq j_{k-1}$. Such a digraph is called the transition digraph of $P_{n,k}$. Let uv be an arc in D with initial vertex u and final vertex v. If $u = i_1 i_2 \cdots i_{k-1}$, then $v = i_2 i_3 \cdots i_{k-1} i_k$, where $i_k \in [n] \backslash \{i_1, i_2, \ldots, i_{k-1}\}$, and so the arc uv may be regarded as the k-permutation $i_1 i_2 \cdots i_{k-1} i_k$. On the other hand, any k-permutation $i_1 i_2 \cdots i_{k-1} i_k$ in $P_{n,k}$ is represented by an arc with initial vertex $i_1 i_2 \cdots i_{k-1}$ and final vertex $i_2 i_3 \cdots i_{k-1} i_k$. Jackson [9] showed that such transition digraph is balanced and connected. One can see that any Eulerian tour in this transition digraph corresponds to a universal cycle for $P_{n,k}$, which leads to the following proposition directly.

Proposition 2. *The number of distinct universal cycles for $P_{n,k}$ is equal to the number of Eulerian tours of its transition digraph.*

This proposition implies that it is sufficient to consider the number of Eulerian tours in the transition digraph of $P_{n,k}$. Let D be a connected balanced digraph, and let $\epsilon(D)$ denote the number of Eulerian tours of D. We use $d^+(v)$ to denote the out-degree of a vertex v. There is a surprising connection between the number of Eulerian tours and Laplacian eigenvalues, given by the next lemma.

Lemma 3. ([16]) *Let D be a connected balanced digraph with vertex set V. If the Laplacian* *eigenvalues of D are $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{|V|-1} > \mu_{|V|} = 0$, then*

$$\epsilon(D) = \frac{1}{|V|}\mu_1\mu_2\cdots\mu_{|V|-1}\prod_{v\in V}(d^+(v)-1)!. \tag{1}$$

According to Lemma 3, to count the number of universal cycles for $P_{n,k}$, it is enough to compute the corresponding Laplacian eigenvalues. Let D be the transition digraph of $P_{n,k}$ with adjacency matrix A and Laplacian matrix L. Since D is regular, the eigenvalues of L can be determined by the eigenvalues of A. Hence the eigenvalues of A are the key to count the number of universal cycles. However, the eigenvalues of A are usually more difficult to compute directly, since the order of A is $n!/(n-k+1)!$. In order to determine the eigenvalues of A, we introduce the representation matrix T of the transition digraph D (see Definition 9 for details). The first main result of the paper establishes an algebraic relation of the representation matrix T and adjacency matrix A.

Lemma 4. *The minimal polynomial of A divides the characteristic polynomial of T.*

From [8, Corollary 3.3.4] and Lemma 4, the eigenvalues of A are contained in the set of eigenvalues of T, without counting multiplicities. Note that the order of T is $\sum_{i=0}^{k-1}\frac{(k-1!)^2}{(i!)^2(k-1-i)!}$. Clearly, if n is sufficiently large for k, then the order of T is generally much smaller than the order of A. Therefore, Lemma 4 provides an efficient approach to find all possible vaules of eigenvalues of A. Once the values of eigenvalues are determined, we use the standard techniques from spectral graph theory to determine their multiplicities and thus the number of universal cycles can be obtained from Lemma 3. In particular, we obtain the exact formulae for $P_{n,3}$ and $P_{n,4}$ in the following results.

Theorem 5. ([3]) *The number of universal cycles for $P_{n,3}$ is equal to $n^{n-2}(n-1)^{\frac{n(n-3)}{2}-1}(n-2)^{n-1}(n-3)^{\frac{(n-1)(n-2)}{2}}((n-3)!)^{n(n-1)}$.*

Theorem 6. *The number of universal cycles for $P_{n,4}$ is equal to $n^{n-2}(n-1)^{\frac{n(n-3)}{2}-1}(n-2)^{\frac{n(n-2)(n-4)}{3}-2}(n-3)^{\frac{(n-1)(3n-2)}{2}-1}(n-4)^{(n-1)(n-2)-1}(n^2-7n+13)^{\frac{n(n-2)(n-4)}{3}}((n-4)!)^{n(n-1)(n-2)}$.*

The rest of this paper is organized as follows. In Section 2 we introduce some definitions and properties of the adjacency matrix and the so-called representation matrix of transition digraph with vertex set $P_{n,k}$. The detailed proofs of Theorems 5 and 6 are provided in Section 3 and Section 4, respectively.

2 Adjacency matrix and representation matrix

For the sake of convenience, in this section, we consider the transition digraph D with vertex set $P_{n,m}$, $1 \leq m \leq n-2$. At this time the Eulerian tours in D correspond to universal cycles for $P_{n,m+1}$. Let A be the adjacency matrix of D. We use $\tau_l(u,v)$ to denote the number of walks from u to v in D with length $l \geq 0$. Thus the (u,v)-entry of A^l is equal to $\tau_l(u,v)$.

For a square matrix M, let $\mathrm{tr}(M)$ denote the trace of M. We note that the diagonal entry of A^l is the number of closed walks of length l. Therefore, the transition digraph D has $\mathrm{tr}(A^l)$ closed walks of length l.

Lemma 7. *Let D be the transition digraph with vertex set $P_{n,m}$. Then its adjacency matrix A satisfies the following properties:*
(1) $\mathrm{tr}(A^0) = n!/(n-m)!$*;*
(2) if $1 \le l \le m$*, then* $\mathrm{tr}(A^l) = 0$*;*
(3) if $n \ge m+1$*, then* $\mathrm{tr}(A^{m+1}) = n!/(n-m-1)!$.

Proof. Since A^0 is an identity matrix, $\mathrm{tr}(A^0) = |P_{n,m}| = n!/(n-m)!$. Note that a closed l-walk in D is equivalent to a periodic sequence $(c_1, c_2, \ldots, c_l)$ which satisfies the following conditions: any m consecutive elements form an m-permutation in $P_{n,m}$, and any $m+1$ consecutive elements form an $(m+1)$-permutation in $P_{n,m+1}$. This means that, in D, there are no closed walks of length less than $m+1$. Hence $\mathrm{tr}(A^l) = 0$ for any $1 \le l \le m$, and $\mathrm{tr}(A^{m+1}) = |P_{n,m+1}| = n!/(n-m-1)!$. $\square$

About $\tau_l(u,v)$, since D is vertex-transitive, there is an automorphism θ such that $\theta(u) = 12\cdots m$ and $\theta(v) = \beta$ where $\beta \in P_{n,m}$. Then $\tau_l(u,v) = \tau_l(12\cdots m, \beta)$. Thus, in order to discuss the number of l-walks in D, it suffices to consider the number of l-walks from the vertex $12\cdots m$ to any other vertex in D.

A *multiset* is a collection in which elements may occur more than once. The number of times an element occurs in a multiset is called its multiplicity. The cardinality of a multiset is the sum of the multiplicities of its elements. Let $S = \{1, 2, \ldots, m, n^{[n-m]}\}$ be a multiset, where the multiplicity of the element n is $n-m$. Let $P^*_{n,m}$ be the set of all arrangements of m elements in the n-multiset S. Obviously,

$$|P^*_{n,m}| = \sum_{i=0}^{m} \frac{m!}{i!}\binom{m}{i} = \sum_{i=0}^{m} \frac{(m!)^2}{(i!)^2(m-i)!}.$$

Let δ be an integer-valued function on $[n]$ with

$$\delta(i) = \begin{cases} i, & \text{if } i \le m, \\ n, & \text{if } i > m, \end{cases}$$

for any $1 \le i \le n$. We shall define a map $\phi : P_{n,m} \mapsto P^*_{n,m}$ such that for any permutation $b_1 b_2 \cdots b_m \in P_{n,m}$,

$$\phi(b_1 b_2 \cdots b_m) = \delta(b_1)\delta(b_2)\cdots\delta(b_m).$$

Clearly, ϕ is a surjection. Moreover, for any $\alpha \in P^*_{n,m}$, the preimage $\phi^{-1}(\alpha)$ is the set of all permutations of $P_{n,m}$ that map to α under ϕ. For convenience, we may assume that all the arrangements in $P^*_{n,m} = \{\alpha^{(1)}, \ldots, \alpha^{(|P^*_{n,m}|)}\}$ are listed by lexicographical order, that is,

$$\alpha^{(1)} \preccurlyeq \alpha^{(2)} \preccurlyeq \cdots \preccurlyeq \alpha^{(|P^*_{n,m}|)},$$

where $\alpha^{(i)} = a_1^{(i)} a_2^{(i)} \cdots a_m^{(i)}$. In particular, $\alpha^{(1)} = 12\cdots m$ and $\alpha^{(|P^*_{n,m}|)} = nn\cdots n$.

Based on the fact that D is vertex-transitive, we have the following result.

Lemma 8. *Suppose* $\alpha^{(i)} \in P^*_{n,m}$*. Then* $\tau_l(\alpha^{(1)}, b) = \tau_l(\alpha^{(1)}, b')$ *for any* $b, b' \in \phi^{-1}(\alpha^{(i)})$*.*

We use $\tau_l^*(\alpha^{(1)}, \alpha^{(i)})$ to denote the number of l-walks from $12\cdots m$ to any permutation in $\phi^{-1}(\alpha^{(i)})$, that is, $\tau_l^*(\alpha^{(1)}, \alpha^{(i)}) = \tau_l(\alpha^{(1)}, b)$ where $b \in \phi^{-1}(\alpha^{(i)})$.

For $l \geq 0$, we let $X^l = (\tau_l^*(\alpha^{(1)}, \alpha^{(1)}), \tau_l^*(\alpha^{(1)}, \alpha^{(2)}), \ldots, \tau_l^*(\alpha^{(1)}, \alpha^{(|P^*_{n,m}|)}))$. Clearly, $X^0 = (1, 0, \ldots, 0)$. To study the property of X^l, we need some auxiliary tools.

Let $\varrho : P^*_{n,m} \times P^*_{n,m} \mapsto \{0, 1\}$ and $\sigma : P^*_{n,m} \mapsto \{0, 1, \ldots, m\}$ be two functions defined as follows:

$$\varrho(\alpha^{(i)}, \alpha^{(j)}) = \begin{cases} 1, & \text{if } a_2^{(i)} \cdots a_m^{(i)} = a_1^{(j)} \cdots a_{m-1}^{(j)} \text{ and } a_1^{(i)} \neq a_m^{(j)}, \\ 1, & \text{if } a_2^{(i)} \cdots a_m^{(i)} = a_1^{(j)} \cdots a_{m-1}^{(j)} \text{ and } a_1^{(i)} = a_m^{(j)} = n, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\sigma(\alpha^{(i)}) = \#\{a_t^{(i)} : 1 \leq t \leq m, a_t^{(i)} = n\}.$$

Now we introduce the definition of the representation matrix for a transition digraph.

Definition 9. Let D be a transition digraph with vertex set $P_{n,m}$. The representation matrix, denoted by T, of D is defined as follows:

- T is a matrix of order $|P^*_{n,m}|$;
- for any two arrangements $\alpha^{(i)}$ and $\alpha^{(j)}$ in $P^*_{n,m}$, the (i,j)-entry of T is

$$T(i,j) = \begin{cases} n - m - \sigma(\alpha^{(j)}), & \text{if } \varrho(\alpha^{(i)}, \alpha^{(j)}) = 1 \text{ and } a_1^{(i)} = n, \\ 1, & \text{if } \varrho(\alpha^{(i)}, \alpha^{(j)}) = 1 \text{ and } a_1^{(i)} \neq n, \\ 0, & \text{otherwise}. \end{cases}$$

We remark that the matrix T can also be viewed as a quotient-like matrix for the transition digraph D (see, e.g., [5]).

The following lemma describes a relation between the representation matrix T and the vector X^l.

Lemma 10. *Let $X^l = (\tau_l^*(\alpha^{(1)}, \alpha^{(1)}), \tau_l^*(\alpha^{(1)}, \alpha^{(2)}), \ldots, \tau_l^*(\alpha^{(1)}, \alpha^{(|P^*_{n,m}|)}))$ and T defined as above. Then $X^l T = X^{l+1}$.*

Proof. Set $q = |P^*_{n,m}|$. Let $X^l = (\tau_l^*(\alpha^{(1)}, \alpha^{(1)}), \tau_l^*(\alpha^{(1)}, \alpha^{(2)}), \ldots, \tau_l^*(\alpha^{(1)}, \alpha^{(q)}))$. It suffices to prove that, for $1 \leq s \leq q$,

$$\tau_{l+1}^*(\alpha^{(1)}, \alpha^{(s)}) = \sum_{i=1}^{q} \tau_l^*(\alpha^{(1)}, \alpha^{(i)}) T(i, s).$$

Suppose that $\alpha^{(s)} = a_1 a_2 \cdots a_m$. Let $b = b_1 b_2 \cdots b_m$ be a permutation in $P_{n,m}$ such that $b_1 b_2 \cdots b_m \in \phi^{-1}(\alpha^{(s)})$. Then $\delta(b_i) = a_i$ for $1 \leq i \leq m$. We define three subsets of $[n]$:

$R = \{z : z \neq b_m, z b_1 b_2 \cdots b_{m-1} \in P_{n,m}\}$, $R_1 = \{z \in R : z \leq m\}$ and $R_2 = \{z \in R : z > m\}$.

Obviously, $R = R_1 \cup R_2$ and $R_1 \cap R_2 = \emptyset$. For any $z \in R$, there is an arc from $zb_1 \cdots b_{m-1}$ to $b_1 b_2 \cdots b_m$ in the transition graph. If $z \in R_1$, then $\phi(zb_1 \cdots b_{m-1}) = za_1 a_2 \cdots a_{m-1}$. If $z \in R_2$, then $\phi(zb_1 \cdots b_{m-1}) = na_1 a_2 \cdots a_{m-1}$. It follows that

$$\begin{aligned}
\tau^*_{l+1}(\alpha^{(1)}, \alpha^{(s)}) &= \tau_{l+1}(\alpha^{(1)}, b) \\
&= \sum_{z \in R} \tau_l(\alpha^{(1)}, zb_1 b_2 \cdots b_{m-1}) \\
&= \sum_{z \in R_1} \tau_l(\alpha^{(1)}, zb_1 b_2 \cdots b_{m-1}) + \sum_{z \in R_2} \tau_l(\alpha^{(1)}, zb_1 b_2 \cdots b_{m-1}) \\
&= \sum_{z \in R_1} \tau^*_l(\alpha^{(1)}, za_1 a_2 \cdots a_{m-1}) + \sum_{z \in R_2} \tau^*_l(\alpha^{(1)}, na_1 a_2 \cdots a_{m-1}).
\end{aligned}$$

Since $\phi(b_1 b_2 \cdots b_m) = \alpha^{(s)}$, the permutation $b_1 b_2 \cdots b_m$ contains $\sigma(\alpha^{(s)})$ integers greater than m. It is easy to see that $|R_2| = n - m - \sigma(\alpha^{(s)})$. Set $\beta = a_1 a_2 \cdots a_{m-1}$. Then we have

$$\tau^*_{l+1}(\alpha^{(1)}, \alpha^{(s)}) = \sum_{z \in R_1} \tau^*_l(\alpha^{(1)}, z\beta) + (n - m - \sigma(\alpha^{(s)}))\tau^*_l(\alpha^{(1)}, n\beta). \tag{2}$$

On the other hand, by the definition of T, one can see that

$$\sum_{i=1}^{m} \tau^*_l(\alpha^{(1)}, \alpha^{(i)})T(i, s) = \sum_{z \in R_3} \tau^*_l(\alpha^{(1)}, z\beta) + (n - k - \sigma(\alpha^{(s)}))\tau^*_l(\alpha^{(1)}, n\beta), \tag{3}$$

where $R_3 = \{z \leq m : z \neq a_m, z\beta \in P^*_{n,m}\}$. Clearly, $R_3 = R_1$. Combining (2) and (3), the required equality follows. $\square$

Lemma 11. *Let T and A be the representation matrix and adjacency matrix of a transition digraph D respectively. If $f(\lambda)$ is a polynomial such that $f(T) = \mathbf{0}$, then $f(A) = \mathbf{0}$.*

Proof. Let $f(\lambda) = \sum_{i=0}^{s} c_i \lambda^i$ such that $f(T) = \sum_{i=0}^{s} c_i T^i = \mathbf{0}$. It suffices to show that

$$f(A) = \sum_{i=0}^{s} c_i A^i = \mathbf{0}. \tag{4}$$

Let u and v be vertices in D ($u = v$ is allowed). Thus, to show (4), it suffices to prove that

$$\sum_{i=0}^{s} c_i \tau_i(u, v) = 0. \tag{5}$$

According to Lemma 8, it is enough to prove that for any $\alpha^{(j)} \in P^*_{n,m}$,

$$\sum_{i=0}^{s} c_i \tau^*_i(\alpha^{(1)}, \alpha^{(j)}) = 0. \tag{6}$$

By the definition of X^l and Lemma 10, Equation (6) is equivalent to

$$\sum_{i=0}^{s} c_i X^i = \sum_{i=0}^{s} c_i X^0 T^i = X^0 (\sum_{i=0}^{s} c_i T^i) = X^0 f(T) = \mathbf{0}.$$

Hence the proof is complete. □

Proof of Lemma 4. Suppose that $p(\lambda)$ is the characteristic polynomial of T. The Cayley-Hamilton Theorem implies that $p(T) = \mathbf{0}$. By Lemma 11, one can see that $p(\lambda)$ is a monic polynomial that annihilates A. This leads to that the minimal polynomial of A divides $p(\lambda)$, which completes the proof. □

3 Enumeration formula for $k = 3$

In this section we re-derive the exact formula for $P_{n,3}$, which was first obtained in [3] using a different method. In this case, we recall that the transition graph D is defined on the vertex set $P_{n,2}$ and $P^*_{n,2}$ is the set of all arrangements of 2 elements of $\{1, 2, n^{[n-2]}\}$. We may list the arrangements in $P^*_{n,2}$ by lexicographical order, as follows:

$$12 \preccurlyeq 1n \preccurlyeq 21 \preccurlyeq 2n \preccurlyeq n1 \preccurlyeq n2 \preccurlyeq nn.$$

Thus, the representation matrix T can be written as

$$T = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ n-2 & n-3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & n-2 & n-3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & n-3 & n-3 & n-4 \end{bmatrix}.$$

A simple calculation shows that the characteristic polynomial of T is

$$p_T(\lambda) = (\lambda - n + 2)(\lambda - 1)(\lambda + 1)(\lambda^2 + \lambda + n - 2)^2.$$

According to Lemma 4, we obtain that the characteristic polynomial of A is

$$p_A(\lambda) = (\lambda - n + 2)(\lambda - 1)^{t_1}(\lambda + 1)^{t_2}(\lambda^2 + \lambda + n - 2)^{t_3},$$

where t_1, t_2, t_3 are nonnegative integers. Let p and q be two roots of $\lambda^2 + \lambda + n - 2 = 0$. It follows that the eigenvalues of A are listed as $n-2, 1^{[t_1]}, (-1)^{[t_2]}, p^{[t_3]}, q^{[t_3]}$ with their multiplicites. Using the relationship between eigenvalues and trace of a matrix, it follows from Lemma 7 that

$$\begin{cases} \operatorname{tr}(A^0) = t_1 + t_2 + 2t_3 + 1 = n(n-1), \\ \operatorname{tr}(A) = n - 2 + t_1 - t_2 + t_3(p+q) = 0, \\ \operatorname{tr}(A^2) = (n-2)^2 + t_1 + t_2 + t_3(p^2 + q^2) = 0. \end{cases} \quad (7)$$

Since p and q are roots of $\lambda^2+\lambda+n-2=0$, one can see that

$$\begin{cases} p+q=-1, \\ p^2+q^2=-2n+5. \end{cases} \tag{8}$$

Combining (7) and (8), we obtain that $t_1=(n-1)(n-2)/2$, $t_2=n(n-3)/2$ and $t_3=n-1$. Then the next result follows directly.

Lemma 12. *Let A be the adjacency matrix of the transition digraph defined on $P_{n,2}$. Then the eigenvalues of A are $n-2, 1^{[(n-1)(n-2)/2]}, (-1)^{[n(n-3)/2]}, p^{[n-1]}, q^{[n-1]}$, where p and q are roots of $\lambda^2+\lambda+n-2=0$.*

Since the out-degree of any vertex in the transition digraph is $(n-2)$, the Laplacian matrix L and the adjacency matrix A satisfy the equation $L=(n-2)I-A$. Hence the eigenvalues of L can be obtained by the above lemma.

Corollary 13. *Let L be the Laplacian matrix of the transition digraph defined on $P_{n,2}$. Then the eigenvalues of L are $0, (n-3)^{[(n-1)(n-2)/2]}, (n-1)^{[n(n-3)/2]}, (n-2-p)^{[n-1]}, (n-2-q)^{[n-1]}$, where p and q are roots of $\lambda^2+\lambda+n-2=0$.*

We are now in a position to prove Theorem 5.

Proof of Theorem 5. Proposition 2 shows that the number of distinct universal cycles for $P_{n,3}$ is equal to the number of Eulerian tours in D. Combining Lemma 3 and Corollary 13, one can see that the number of Eulerian tours in D is

$$\epsilon(D)=\frac{1}{n(n-1)}(n-3)^{\frac{(n-1)(n-2)}{2}}(n-1)^{\frac{n(n-3)}{2}}(n-2-p)^{n-1}(n-2-q)^{n-1}\prod_{v\in V(D)}(n-3)!. \tag{9}$$

Since p and q are roots of $\lambda^2+\lambda+n-2=0$, it follows that

$$(n-2-p)(n-2-q)=n(n-2).$$

Therefore, we obtain that

$$\epsilon(D)=n^{n-2}(n-1)^{\frac{n(n-3)}{2}-1}(n-2)^{n-1}(n-3)^{\frac{(n-1)(n-2)}{2}}((n-3)!)^{n(n-1)},$$

and the result follows. □

4 Enumerating formula for $k=4$

We now derive the enumeration formula for $P_{n,4}$ using this method. Let D be the corresponding transition digraph. Clearly, D is defined on $P_{n,3}$. All arrangements in $P^*_{n,3}$ can be listed by lexicographical order, as follows:

$$\begin{aligned} 123 &\preccurlyeq 12n \preccurlyeq 132 \preccurlyeq 13n \preccurlyeq 1n2 \preccurlyeq 1n3 \preccurlyeq 1nn \preccurlyeq 213 \preccurlyeq 21n \preccurlyeq 231 \preccurlyeq 23n \preccurlyeq 2n1 \\ &\preccurlyeq 2n3 \preccurlyeq 2nn \preccurlyeq 312 \preccurlyeq 31n \preccurlyeq 321 \preccurlyeq 32n \preccurlyeq 3n1 \preccurlyeq 3n2 \preccurlyeq 3nn \preccurlyeq n12 \preccurlyeq n13 \\ &\preccurlyeq n1n \preccurlyeq n21 \preccurlyeq n23 \preccurlyeq n2n \preccurlyeq n31 \preccurlyeq n32 \preccurlyeq n3n \preccurlyeq nn1 \preccurlyeq nn2 \preccurlyeq nn3 \preccurlyeq nnn \end{aligned}$$

Table 1: Power sums of the roots

sum	value	sum	value	sum	value	sum	value
a_1+a_2	1	$\sum_{i=1}^3 b_i$	1	$\sum_{i=1}^3 c_i$	-1	$\sum_{i=1}^3 d_i$	-1
$a_1^2+a_2^2$	-1	$\sum_{i=1}^3 b_i^2$	-1	$\sum_{i=1}^3 c_i^2$	3	$\sum_{i=1}^3 d_i^2$	$-2n+7$
$a_1^3+a_2^3$	-2	$\sum_{i=1}^3 b_i^3$	$3n-11$	$\sum_{i=1}^3 c_i^3$	$3n-13$	$\sum_{i=1}^3 d_i^3$	$3(n-3)(4-n)-1$
$a_1^4+a_2^4$	-1	$\sum_{i=1}^3 b_i^4$	$4n-13$	$\sum_{i=1}^3 c_i^4$	$-4n+19$	$\sum_{i=1}^3 d_i^4$	$2(n-3)(3n-11)+1$

According to Definition 9, one can determine the representation matrix T of D, which is exhibited in the Appendix. Using MAPLE, we obtain that the characteristic polynomial of T is

$$p_T(\lambda)=(\lambda-n+3)(\lambda+1)^2(\lambda^2-\lambda+1)^2(\lambda^3-\lambda^2+\lambda-n+3)^3(\lambda^3+\lambda^2-\lambda-n+3)^3 \\ (\lambda^3+\lambda^2+(n-3)\lambda+(n-3)^2)^3.$$

Then by Lemma 4, it follows that the characteristic polynomial of A is

$$p_A(\lambda)=(\lambda-n+3)(\lambda+1)^{t_1}(\lambda^2-\lambda+1)^{t_2}(\lambda^3-\lambda^2+\lambda-n+3)^{t_3}(\lambda^3+\lambda^2-\lambda-n+3)^{t_4} \\ (\lambda^3+\lambda^2+(n-3)\lambda+(n-3)^2)^{t_5},$$

where t_1,t_2,t_3,t_4,t_5 are indetermined nonnegative integers. We may assume that the roots of the irreducible factors of $p_A(\lambda)$ are as follows:

- the roots of $\lambda^2-\lambda+1$ are denoted by a_1 and a_2;
- the roots of $\lambda^3-\lambda^2+\lambda-n+3$ are denoted by b_1,b_2 and b_3;
- the roots of $\lambda^3+\lambda^2-\lambda-n+3$ are denoted by c_1,c_2 and c_3;
- the roots of $\lambda^3+\lambda^2+(n-3)\lambda+(n-3)^2$ are denoted by d_1,d_2 and d_3.

The power sums of the above roots are presented in Table 1. Note that the eigenvalues of A are

$$n-3,(-1)^{[t_1]},a_1^{[t_2]},a_2^{[t_2]},b_1^{[t_3]},b_2^{[t_3]},b_3^{[t_3]},c_1^{[t_4]},c_2^{[t_4]},c_3^{[t_5]},d_1^{[t_5]},d_2^{[t_5]},d_3^{[t_5]}.$$

According to Lemma 7, it follows that

$$\begin{cases} \mathrm{tr}(A^0)=1+t_1+2t_2+3t_3+3t_4+3t_5=n(n-1)(n-2) \\ \mathrm{tr}(A)=n-3-t_1+t_2(a_1+a_2)+t_3\sum_{i=1}^3 b_i+t_4\sum_{i=1}^3 c_i+t_5\sum_{i=1}^3 d_i=0 \\ \mathrm{tr}(A^2)=(n-3)^2+t_1+t_2(a_1^2+a_2^2)+t_3\sum_{i=1}^3 b_i^2+t_4\sum_{i=1}^3 c_i^2+t_5\sum_{i=1}^3 d_i^2=0 \\ \mathrm{tr}(A^3)=(n-3)^3-t_1+t_2(a_1^3+a_2^3)+t_3\sum_{i=1}^3 b_i^3+t_4\sum_{i=1}^3 c_i^3+t_5\sum_{i=1}^3 d_i^3=0 \\ \mathrm{tr}(A^4)=(n-3)^4+t_1+t_2(a_1^4+a_2^4)+t_3\sum_{i=1}^3 b_i^4+t_4\sum_{i=1}^3 c_i^4+t_5\sum_{i=1}^3 d_i^4=n!/(n-4)! \end{cases}$$

Combining with the power sums in Table 1, it follows from the calculation that

$$\begin{cases} t_1=n(n-2)(n-4)/3-1, \\ t_2=n(n-2)(n-4)/3, \\ t_3=(n-1)(n-2)/2, \\ t_4=(n-1)(n-2)/2-1, \\ t_5=n-1. \end{cases} \tag{10}$$

Let L be the Laplacian matrix of D. Since the out-degree of any vertex in D is $n-3$, we have $L=(n-3)I-A$. Thus, the eigenvalues of L are

$$0,(n-2)^{[t_1]},(n-3-a_1)^{[t_2]},(n-3-a_2)^{[t_2]},(n-3-b_1)^{[t_3]},(n-3-b_2)^{[t_3]},(n-3-b_3)^{[t_3]},\\ (n-3-c_1)^{[t_4]},(n-3-c_2)^{[t_4]},(n-3-c_3)^{[t_5]},(n-3-d_1)^{[t_5]},(n-3-d_2)^{[t_5]},(n-3-d_3)^{[t_5]}.$$

Since a_1 and a_2 are roots of $\lambda^2-\lambda+1=0$, we obtain that $a_1+a_2=1$ and $a_1a_2=1$. It follows that $(n-3-a_1)(n-3-a_2)=n^2-7n+13$. Similarly, one can see that

$$\begin{cases} (n-3-b_1)(n-3-b_2)(n-3-b_3)=(n-3)^2(n-4), \\ (n-3-c_1)(n-3-c_2)(n-3-c_3)=(n-1)(n-3)(n-4), \\ (n-3-d_1)(n-3-d_2)(n-3-d_3)=n(n-3)^2. \end{cases}$$

Thus the product of the nonzero eigenvalues of L, denoted by z, is

$$z=(n-2)^{t_1}(n^2-7n+13)^{t_2}((n-3)^2(n-4))^{t_3}((n-1)(n-3)(n-4))^{t_4}(n(n-3)^2)^{t_5},$$

where the values of t_i, $i=1,2,\ldots,5$, are from (10). By Lemma 3, the number of Eulerian tours of D is

$$\begin{aligned} \epsilon(D) &= z\frac{(n-3)!}{n!}\prod_{v\in V(D)}(d^+(v)-1)! \\ &= \frac{z}{n(n-1)(n-2)}((n-4)!)^{n(n-1)(n-2)} \\ &= n^{n-2}(n-1)^{\frac{n(n-3)}{2}-1}(n-2)^{\frac{n(n-2)(n-4)}{3}-2}(n-3)^{\frac{(n-1)(3n-2)}{2}-1}(n-4)^{(n-1)(n-2)-1} \\ &\quad (n^2-7n+13)^{\frac{n(n-2)(n-4)}{3}}((n-4)!)^{n(n-1)(n-2)}, \end{aligned}$$

which yields the exact formula in Theorem 6.

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Appendix: transpose of the representation matrix $\boldsymbol{T}$

$$
\left[\begin{array}{cccccccccccccccccccccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n-3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & n-4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n-3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n-4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n-5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n-4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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\end{array}\right]
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