New Constructions of Two-Dimensional Binary Z-Complementary Array Pairs

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Abstract

The Z-complementary array pair (ZCAP) is a two-dimensional extension of the Z-complementary sequence pair, applied to radar waveform design, spatial synchronization, and two-dimensional multi-carrier CDMA systems. We suggest optimal binary Type-I and Type-II ZCAPs using concatenation, interleaving, and iteration techniques to further ZCAP research. The zero correlation zone (ZCZ) ratio of the constructed ZCAPs approaches 1. Our constructions provide new design methods for ZCAPs and improve the flexibility and adaptability of ZCAP parameters beyond the current literature.

Key words:Z-complementary array pair, Z-complementary sequence pair, zero correlation zone.

1 Introduction

In 1951, while investigating an optical problem in multislit spectroscopy, M.J.E. Golay introduced the concept of complementary sequence pairs, i.e., pairs of sequences with zero aperiodic autocorrelation and zero in each nonzero time shift, known as Golay complementary pairs (GCPs) [1]. However, available lengths for GCPs are very limited. For instance, the length of a binary GCP exists exclusively in the form$2^\alpha 10^\beta 26^\gamma$[2], where α, β, and γ are nonnegative integers. To obtain flexible sequence length, Fan et al. introduced the concept of "zero correlation zone (ZCZ)" in 2007 [3], which defines an aperiodic Z-complementary pair (ZCP) by specifying that the aperiodic autocorrelation sum of sequence pairs is zero only in a segment of intervals. The Golay complementary array pair (GCAP) is a two-dimensional extended form of GCP [4], with row length and column

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length of $2^\alpha 10^\beta 26^\gamma$. The Z-complementary array pair (ZCAP) as the extension of GCAP can provide flexible row and column lengths [5], which can be applied to two-dimensional radars [6], two-dimensional synchronization [7], and two-dimensional multi-carrier CDMA systems [8]. ZCAPs possess the two-dimensional ZCZ property where in the sum of the aperiodic autocorrelation functions of consistent arrays is zero in a specific area. Similar to ZCPs, ZCAPs have an extremely wide parameter range and are divided into Type-I ZCAPs and Type-II ZCAPs according to different positions of ZCZ in the array [9].

In contrast to the extensively studied ZCPs over many years [10-21], there are comparatively fewer research methods and results available for the ZCAPs. In general, ZCAP construction approaches include the direct methods and the indirect methods. Two-dimensional Boolean functions are an effective tool for directly generating ZCAPs. By designing different kinds of two-dimensional Boolean functions, Pai obtained ZCAPs with array size $2^n \times \sum\limits_{\alpha=t+1}^{m-1} d_\alpha 2^{\alpha-1} + 2^v$[22], Abhishek Roy constructed ZCAPs with array size $(2^{m_1-1} + 2^{n+1}) \times (2^{m_2} + 4)$[23], and Zhang constructed ZCAPs with array size $14 \cdot 2^{n-4} \times 2^m$[24]. Accordingly, the direct methods yield limited results in terms of array size and ZCZ size. To enlarge the parameter range, the indirect methods play a key role. Based on the existing ZCPs and ZCAPs, Pai et al. utilized the Kronecker product, concatenation operation, and interleaving operation to obtain Type-I ZCAPs with varying array sizes. Besides, as Type-II ZCAP can suppress asynchronous interference in MC-CDMA system, Pai et al. also proposed a design for Type-II ZCAPs, but the parameters remained constrained[9]. Consequently, in this paper, we generate four class of two-dimensional ZCAPs with new parameters, including optimal Type-I ZCAPs and Type-II ZCAPs by employing concatenation, interleaving, and interation techniques.

The rest of the paper is organized as follows. In Section 2, some basic notations and definitions are provided. In Section 3, several construction methods of Type-I ZCAPs, including optimal ZCAPs, are presented using the concatenation, Kronecker product, and interleaving methods, and the construction results of various parameters are obtained. In Section 4, Type-II ZCAPs are designed based on the iteration method. In Section 5, the parameters of the constructed ZCAPs are analyzed and compared to the literature. Section 6 concludes this paper.

2 Preliminaries

Let $\boldsymbol{C}$ be a binary array of size $L_1 \times L_2$, $\boldsymbol{C} = (c_{i,j}, 0 \le i < L_1, 0 \le j < L_2)$, where $(c_{i,j}) \in \{+1, -1\}$. For convenience, let " + " and " − " denote "1" and "−1", respectively.

- x^* stands for the conjugate of the complex number x.
- $(\cdot)^T$ denotes the transposition of $(\cdot)$, where $(\cdot)$ can represent a sequence or an array.
- Let $\boldsymbol{D} = (d_{i,j}, 0 \le i < L_1, 0 \le j < L_2)$ be also an array of size $L_1 \times L_2$. c_j and d_j denote the respective column vectors of arrays $\boldsymbol{C}$ and $\boldsymbol{D}$. The concatenation operation of arrays $\boldsymbol{C}$ and $\boldsymbol{D}$ is expressed as

$$\boldsymbol{C} \parallel \boldsymbol{D} = (c_0, c_1, \ldots, c_{L_2-1}, d_0, d_1, \ldots, d_{L_2-1}), \tag{1}$$

where "$\|$" stands for concatenation operator.

• $\odot$ represents the interleaving operation, and the interleaving of arrays $\boldsymbol{C}$ and $\boldsymbol{D}$ is expressed as follows

$$\boldsymbol{C} \odot \boldsymbol{D} = (c_0, d_0, c_1, d_1 \ldots, c_{L_2-1}, d_{L_2-1}), \tag{2}$$

where c_j and d_j denote the column vectors of arrays $\boldsymbol{C}$ and $\boldsymbol{D}$, respectively.

Definition 1. The two-dimensional aperiodic cross-correlation function (ACCF) of arrays $\boldsymbol{C}$ and $\boldsymbol{D}$ at the shift (u_1, u_2) is defined as

$$\rho(\boldsymbol{C}, \boldsymbol{D}; u_1, u_2) = \sum_{i=0}^{L_1-1-u_1} \sum_{i=0}^{L_2-1-u_2} c_{i,j} d_{i+u_1, j+u_2}, \tag{3}$$

where $0 \leq u_1 < L_1, 0 \leq u_2 < L_2$. When $\boldsymbol{C} = \boldsymbol{D}$, $\rho(\boldsymbol{C}, \boldsymbol{C}; u_1, u_2)$ is called the aperiodic autocorrelation function (AACF), denoted by $\rho(\boldsymbol{C}; u_1, u_2)$. If $L_1 = 1$, the array $\boldsymbol{C}$ becomes a sequence $\boldsymbol{C} = (c_j, j = 0, 1, \ldots, L_2 - 1)$, then the AACF of the sequence $\boldsymbol{C}$ can be expressed as

$$\rho(\boldsymbol{C}; u) = \sum_{j=0}^{L_2-1-u} c_j c_{j+u}, \tag{4}$$

Definition 2. [3] If a pair of sequences $\boldsymbol{a}$ and $\boldsymbol{b}$ of length L satisfies

$$\rho(\boldsymbol{a}; u) + \rho(\boldsymbol{b}; u) = \begin{cases} 2L, u = 0 \\ 0, -Z < u < Z, u \neq 0 \end{cases} \tag{5}$$

where Z represents the ZCZ width, then the sequence pair $(\boldsymbol{a}, \boldsymbol{b})$ is called a Z-complementary pairs, denoted as $(L, Z) - \text{ZCP}$.

Definition 3. [3] If two $(L, Z) - \text{ZCPs}$ $(\boldsymbol{a}, \boldsymbol{b})$ and $(\boldsymbol{c}, \boldsymbol{d})$ satisfy

$$\rho(a, c; u) + \rho(b, d; u) = 0, \ \textit{where } 0 \leq u < Z \tag{6}$$

then $(\boldsymbol{a}, \boldsymbol{b})$ and $(\boldsymbol{c}, \boldsymbol{d})$ are said to be mates to each other.

Definition 4. [4] If the AACF sum of arrays $\boldsymbol{C}$ and $\boldsymbol{D}$ of size $L_1 \times L_2$ satisfies

$$\rho(\boldsymbol{C}; u_1, u_2) + \rho(\boldsymbol{D}; u_1, u_2) = \begin{cases} 2L_1L_2, (u_1, u_2) = (0, 0) \\ 0, 0 \leq |u_1| < Z_1, 0 \leq |u_2| < Z_2, (u_1, u_2) \neq (0, 0) \end{cases} \tag{7}$$

then $(\boldsymbol{C}, \boldsymbol{D})$ is called a Type-I ZCZ complementary array pair (ZCAP), denoted as Type-I $((L_1, L_2), (Z_1, Z_2)) - \text{ZCAP}$.

In particular, the proof of ZCAPs in the following theorems of this paper is given only for $u_1 \geq 0$ and $u_2 \geq 0$. The proof process for $u_1 \leq 0$ and $u_2 \leq 0$ is similar, so it is omitted.

Definition 5. [5] If the AACF sum of arrays $\boldsymbol{C}$ and $\boldsymbol{D}$ of size $L_1 \times L_2$ satisfies

$$\rho(\boldsymbol{C}; u_1, u_2) + \rho(\boldsymbol{D}; u_1, u_2) = 0, L_1 - Z_1 < |u_1| < L_1 \text{ or } L_2 - Z_2 < |u_2| < L_2 \tag{8}$$

then $(\boldsymbol{C}, \boldsymbol{D})$ is referred to as a Type-II $((L_1, L_2)(Z_1, Z_2)) - \text{ZCAP}$.

Definition 6. [22] The ZCZ ratio of $((L_1, L_2)(Z_1, Z_2)) - \text{ZCAP}$ is defined as

$$ZCZ_{ratio} = \frac{Z_1 Z_2}{L_1 L_2} \tag{9}$$

Definition 7. [10] A binary odd-length $(L, Z) - \text{ZCP}$ $(\boldsymbol{a}, \boldsymbol{b})$ is said to be Z-optimal if the parameters satisfy $Z = \frac{L+1}{2}$, and a binary even-length $(L, Z) - \text{ZCP}$ is said to be Z-optimal if the parameters satisfy $Z = L - 2$.

Definition 8. [9] Let $(\boldsymbol{A}, \boldsymbol{B})$ be a binary $((L_1, L_2)(Z_1, Z_2)) - \text{ZCAP}$, where L_1 and L_2 are odd. If $Z_1 Z_2 = \left(\frac{L_1+1}{2}\right)\left(\frac{L_2+1}{2}\right)$, then $(\boldsymbol{A}, \boldsymbol{B})$ is said to be Z-optimal. For an binary $((L_3, L_4)(Z_3, Z_4)) - \text{ZCAP}$, if one of its dimensions is odd and the other is even, then it is called an optimal ZCAP when $Z_1 Z_2 \leq \max\left((L_3 - 1) L_4, L_3 (L_4 - 1)\right)$.

3 Construction of Type-I ZCAPs

In this section, two classes of ZCAPs are constructed by the concatenation operation and interleaving methods based ZCPs and ZCAPs. By utilizing the optimal odd-length ZCP as the base sequence in Theorem 1, we can optimize the resulting two-dimensional ZCAP.

Theorem 1. *Let* $(\boldsymbol{a}, \boldsymbol{b})$ *be an* $(L, Z) - \text{ZCP}$ *and* $(\boldsymbol{c}, \boldsymbol{d}) = \left(\overleftarrow{\boldsymbol{b}}, -\overleftarrow{\boldsymbol{a}}\right)$ *be a mate of* $(\boldsymbol{a}, \boldsymbol{b})$*, where* $\overleftarrow{\boldsymbol{a}}$ *and* $\overleftarrow{\boldsymbol{b}}$ *represents the reverse of* $\boldsymbol{a}$ *and* $\boldsymbol{b}$*, respectively. The array pairs* $(\boldsymbol{M}, \boldsymbol{N})$ *are constructed by the following operations.*

I. $\boldsymbol{M} = (\boldsymbol{a}^T \parallel \boldsymbol{c}^T \parallel \boldsymbol{a}^T), N = (\boldsymbol{b}^T \parallel \boldsymbol{d}^T \parallel \boldsymbol{b}^T)$;

II. $\boldsymbol{M} = \left(\boldsymbol{a}^T \parallel \boldsymbol{a}^T \parallel -\boldsymbol{a}^T \parallel \boldsymbol{c}^T \parallel -\boldsymbol{a}^T\right), \boldsymbol{N} = \left(\boldsymbol{b}^T \parallel \boldsymbol{b}^T \parallel -\boldsymbol{b}^T \parallel \boldsymbol{d}^T \parallel -\boldsymbol{b}^T\right);$

III. $\boldsymbol{M} = \left(\boldsymbol{a}^T \parallel \boldsymbol{a}^T \parallel \boldsymbol{c}^T \parallel -\boldsymbol{c}^T \parallel \boldsymbol{a}^T \parallel -\boldsymbol{c}^T \parallel -\boldsymbol{a}^T\right),$
$\boldsymbol{N} = \left(\boldsymbol{b}^T \parallel \boldsymbol{b}^T \parallel \boldsymbol{d}^T \parallel -\boldsymbol{d}^T \parallel \boldsymbol{b}^T \parallel -\boldsymbol{d}^T \parallel -\boldsymbol{b}^T\right);$

IV. $\boldsymbol{M} = \left(\boldsymbol{a}^T \parallel \boldsymbol{a}^T \parallel \boldsymbol{c}^T \parallel \boldsymbol{a}^T \parallel -\boldsymbol{a}^T \parallel -\boldsymbol{c}^T \parallel \boldsymbol{a}^T \parallel -\boldsymbol{c}^T \parallel -\boldsymbol{a}^T\right),$
$\boldsymbol{N} = \left(\boldsymbol{b}^T \parallel \boldsymbol{b}^T \parallel \boldsymbol{d}^T \parallel \boldsymbol{b}^T \parallel -\boldsymbol{b}^T \parallel -\boldsymbol{d}^T \parallel \boldsymbol{b}^T \parallel -\boldsymbol{d}^T \parallel -\boldsymbol{b}^T\right);$

V. $\boldsymbol{M} = \left(\boldsymbol{a}^T \parallel \boldsymbol{a}^T \parallel \boldsymbol{c}^T \parallel \boldsymbol{a}^T \parallel \boldsymbol{c}^T \parallel -\boldsymbol{c}^T \parallel \boldsymbol{a}^T \parallel -\boldsymbol{c}^T \parallel -\boldsymbol{a}^T \parallel \boldsymbol{c}^T \parallel \boldsymbol{a}^T\right),$
$\boldsymbol{N} = \left(\boldsymbol{b}^T \parallel \boldsymbol{b}^T \parallel \boldsymbol{d}^T \parallel \boldsymbol{b}^T \parallel \boldsymbol{d}^T \parallel -\boldsymbol{d}^T \parallel \boldsymbol{b}^T \parallel -\boldsymbol{d}^T \parallel -\boldsymbol{b}^T \parallel \boldsymbol{d}^T \parallel \boldsymbol{b}^T\right);$

VI. $\boldsymbol{M} = \left(\boldsymbol{a}^T \parallel \boldsymbol{a}^T \parallel \boldsymbol{a}^T \parallel \boldsymbol{a}^T \parallel -\boldsymbol{a}^T \parallel \boldsymbol{a}^T \parallel -\boldsymbol{a}^T \parallel \boldsymbol{c}^T \parallel \boldsymbol{c}^T \parallel -\boldsymbol{c}^T \parallel -\boldsymbol{c}^T \parallel \boldsymbol{c}^T \parallel -\boldsymbol{a}^T\right),$
$\boldsymbol{N} = \left(\boldsymbol{b}^T \parallel \boldsymbol{b}^T \parallel \boldsymbol{b}^T \parallel \boldsymbol{b}^T \parallel -\boldsymbol{b}^T \parallel -\boldsymbol{b}^T \parallel -\boldsymbol{b}^T \parallel \boldsymbol{d}^T \parallel \boldsymbol{d}^T \parallel -\boldsymbol{d}^T \parallel \boldsymbol{d}^T \parallel \boldsymbol{d}^T \parallel -\boldsymbol{b}^T\right).$

The array pairs are $((L, 3), (Z, 2)) - \text{ZCAP}$*,* $((L, 5), (Z, 3)) - \text{ZCAP}$*,* $((L, 7), (Z, 4)) - \text{ZCAP}$*,* $((L, 9), (Z, 5)) - \text{ZCAP}$*,* $((L, 11), (Z, 6)) - \text{ZCAP}$*,* $((L, 13), (Z, 7)) - \text{ZCAP}$ *respectively.*

Proof. Initially, we illustrate the conclusion with Construction I as an example. For $0 \leq u_1 < Z$, consider the three cases below.

1) For $u_2 = 0$,

$$\rho(\boldsymbol{M}; u_1, 0) = 2\rho(\boldsymbol{a}; u_1) + \rho(\boldsymbol{c}; u_1), \tag{10}$$

$$\rho(\boldsymbol{N}; u_1, 0) = 2\rho(\boldsymbol{b}; u_1) + \rho(\boldsymbol{d}; u_1), \tag{11}$$

we have

$$\rho(\boldsymbol{M}; u_1, 0) + \rho(\boldsymbol{N}; u_1, 0) = 2\left(\rho(\boldsymbol{a}; u_1) + \rho(\boldsymbol{b}; u_1)\right) + \rho(\boldsymbol{c}; u_1) + \rho(\boldsymbol{d}; u_1) = 0 \tag{12}$$

2) For $u_2 = 1$,

$$\rho(\boldsymbol{M}; u_1, 1) = \rho(\boldsymbol{a}, \boldsymbol{c}; u_1) + \rho(\boldsymbol{c}, \boldsymbol{a}; u_1), \tag{13}$$

$$\rho(\boldsymbol{N}; u_1, 1) = \rho(\boldsymbol{b}, \boldsymbol{d}; u_1) + \rho(\boldsymbol{d}, \boldsymbol{b}; u_1), \tag{14}$$

Since $(\boldsymbol{a}, \boldsymbol{b})$ and $(\boldsymbol{c}, \boldsymbol{d})$ are mates to each other, we have

$$\rho(\boldsymbol{M}; u_1, 1) + \rho(\boldsymbol{N}; u_1, 1) = 0 \tag{15}$$

3) For $u_2 = 2$,

$$\rho(\boldsymbol{M}; u_1, 2) = \rho(\boldsymbol{a}; u_1), \tag{16}$$

$$\rho(\boldsymbol{N}; u_1, 2) = \rho(\boldsymbol{c}; u_1), \tag{17}$$

When $u_1 = 0$, we have

$$\rho(\boldsymbol{M}; 0, 2) + \rho(\boldsymbol{N}; 0, 2) = 2L. \tag{18}$$

From the above, it is clear that $(\boldsymbol{M}, \boldsymbol{N})$ is an $((L,3),(Z,2)) - \mathrm{ZCAP}$. The proof processes of other constructions are comparable to Construction I, so we omitted them. □

Remark 1. If $(\boldsymbol{a}, \boldsymbol{b})$ is an odd-length Z-optimal ZCP, then by Definition 7 $Z = \frac{L+1}{2}$ holds. To verify the optimality of the ZCAPs, we incorporate the ZCAP parameters into the equation $Z_1 Z_2 = \left(\frac{L_1+1}{2}\right)\left(\frac{L_2+1}{2}\right)$ in Definition 8 to calculate them independently, and it can be observed that the ZCAPs produced by Theorem 1 are Z-optimal, these optimal ZCAPs are not mentioned in the previous literature. In recent years, researchers have investigated and produced abundant results on odd-length Z-optimal ZCPs [10], which facilitates the generation of odd-dimension Z-optimal ZCAPs using our methods.

Example 1. Take a Z-optimal $(3,2) - \mathrm{ZCP}\,(\boldsymbol{a}, \boldsymbol{b}) = (+++, +-+)$, then the mate of $(\boldsymbol{a}, \boldsymbol{b})$ is $(\boldsymbol{c}, \boldsymbol{d}) = (+-+, ---)$. By Construction I, we can obtain that $\boldsymbol{M} = \left(\boldsymbol{a}^T \parallel \boldsymbol{c}^T \parallel \boldsymbol{a}^T\right) = \begin{pmatrix} +++ \\ +-+ \\ +++ \end{pmatrix}$ and $\boldsymbol{N} = \left(\boldsymbol{b}^T \parallel \boldsymbol{d}^T \parallel \boldsymbol{b}^T\right) = \begin{pmatrix} +-+ \\ --- \\ +-+ \end{pmatrix}$. The AACF sum of $\boldsymbol{M}$ and $\boldsymbol{N}$ is $\begin{bmatrix} 2 & 0 & 6 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 18 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 6 & 0 & 2 \end{bmatrix}$, so $(\boldsymbol{M}, \boldsymbol{N})$ is a $((3,3),(2,2)) - \mathrm{ZCAP}$. Since $Z_1 Z_2 = \left(\frac{L_1+1}{2}\right)\left(\frac{L_2+1}{2}\right) = 4$, according to Definition 8, then $(\boldsymbol{M}, \boldsymbol{N})$ is an optimal ZCAP.

Theorem 2. *Let $(\boldsymbol{A}, \boldsymbol{B})$ be an $((L_1, L_2),(Z_1, Z_2)) - \mathrm{ZCAP}$, then the two operations on $\boldsymbol{A}$ and $\boldsymbol{B}$ are performed as below.*

Construction I (Interleaving):

$$\boldsymbol{M}^1 = (\boldsymbol{A} \odot \boldsymbol{B}) = \left(U_{\boldsymbol{A},1} U_{\boldsymbol{B},1} U_{\boldsymbol{A},2} U_{\boldsymbol{B},2} \cdots U_{\boldsymbol{A},L_2} U_{\boldsymbol{B},L_2}\right), \tag{19}$$

$$\boldsymbol{N}^1 = (\boldsymbol{A} \odot (-\boldsymbol{B})) = \left(U_{\boldsymbol{A},1}\left(-U_{\boldsymbol{B},1}\right) U_{\boldsymbol{A},2}\left(-U_{\boldsymbol{B},2}\right) \cdots U_{\boldsymbol{A},L_2}\left(-U_{\boldsymbol{B},L_2}\right)\right), \tag{20}$$

or

$$\boldsymbol{M}^1=\begin{pmatrix}\boldsymbol{A}\\ \odot\\ \boldsymbol{B}\end{pmatrix}=\begin{pmatrix}V_{\boldsymbol{A},1}\\ V_{\boldsymbol{B},1}\\ V_{\boldsymbol{A},2}\\ V_{\boldsymbol{B},2}\\ \vdots\\ V_{\boldsymbol{A},L_1}\\ V_{\boldsymbol{B},L_1}\end{pmatrix},\boldsymbol{N}^1=\begin{pmatrix}\boldsymbol{A}\\ \odot\\ -\boldsymbol{B}\end{pmatrix}=\begin{pmatrix}V_{\boldsymbol{A},1}\\ -V_{\boldsymbol{B},1}\\ V_{\boldsymbol{A},2}\\ -V_{\boldsymbol{B},2}\\ \vdots\\ V_{\boldsymbol{A},L_1}\\ -V_{\boldsymbol{B},L_1}\end{pmatrix} \tag{21}$$

where $U_{\boldsymbol{A},r}$ and $U_{\boldsymbol{B},r}$ denote the rth column of the arrays $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively, and $V_{\boldsymbol{A},r}$ and $V_{\boldsymbol{B},r}$ denote the rth row of the arrays $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively, then (M^1,N^1) is the $(L_1,2L_2),(Z_1,2Z_2)-ZCAP$ or $(2L_1,L_2),(2Z_1,Z_2)-ZCAP$.

Construction II (Iteration):

Let $\left(\boldsymbol{M}^1,\boldsymbol{N}^1\right)=((\boldsymbol{A}\odot\boldsymbol{B}),(\boldsymbol{A}\odot-\boldsymbol{B}))$ and $\left(\overline{\boldsymbol{M}^1},\overline{\boldsymbol{N}^1}\right)=\left(\begin{pmatrix}\boldsymbol{A}\\ \odot\\ \boldsymbol{B}\end{pmatrix},\begin{pmatrix}\boldsymbol{A}\\ \odot\\ -\boldsymbol{B}\end{pmatrix}\right)$. Take $\left(\boldsymbol{M}^1,\boldsymbol{N}^1\right)$ and $\left(\overline{\boldsymbol{M}^1},\overline{\boldsymbol{N}^1}\right)$ as seeds for subsequent interleaving iteration operations, the interleaving iteration expressions are as follows,

$$\boldsymbol{M}^k=\boldsymbol{M}^{k-1}\odot\boldsymbol{N}^{k-1},\boldsymbol{N}^k=\boldsymbol{M}^{k-1}\odot\left(-\boldsymbol{N}^{k-1}\right). \tag{22}$$

$$\overline{\boldsymbol{M}^k}=\left(\overline{\boldsymbol{M}^{k-1}}\odot\overline{\boldsymbol{N}^{k-1}}\right),\overline{\boldsymbol{N}^k}=\left(\overline{\boldsymbol{M}^{k-1}}\odot-\overline{\boldsymbol{N}^{k-1}}\right). \tag{23}$$

Then, $\left(\boldsymbol{M}^k,\boldsymbol{N}^k\right)$ obtained after k interleaving iterations is a $\left(\left(L_1,2^kL_2\right),\left(Z_1,2^kZ_2\right)\right)-ZCAP$, and $\left(\overline{\boldsymbol{M}^k},\overline{\boldsymbol{N}^k}\right)$ is a $\left(\left(2^kL_1,L_2\right),\left(2^kZ_1,Z_2\right)\right)-ZCAP$.

Proof. For Construction I in Theorem 2, take the column interleaving method as an example, then $\boldsymbol{M}^1=(\boldsymbol{A}\odot\boldsymbol{B}),\boldsymbol{N}^1=(\boldsymbol{A}\odot(-\boldsymbol{B}))$, when $0\le u_1<L_1,0\le u_2<L_2$, we can obtain the following equations.

$$\rho\left(\boldsymbol{M}^1;u_1,2u_2\right)=\rho\left(\boldsymbol{A};u_1,u_2\right)+\rho\left(\boldsymbol{B};u_1,u_2\right), \tag{24}$$

$$\rho\left(\boldsymbol{N}^1;u_1,2u_2\right)=\rho\left(\boldsymbol{A};u_1,u_2\right)+\rho\left(\boldsymbol{B};u_1,u_2\right), \tag{25}$$

$$\rho\left(\boldsymbol{M}^1;u_1,2u_2+1\right)=\rho\left(\boldsymbol{A},\boldsymbol{B};u_1,u_2\right)+\rho\left(\boldsymbol{B},\boldsymbol{A};u_1,u_2+1\right), \tag{26}$$

$$\rho\left(\boldsymbol{N}^1;u_1,2u_2+1\right)=-\rho\left(\boldsymbol{A},\boldsymbol{B};u_1,u_2\right)-\rho\left(\boldsymbol{B},\boldsymbol{A};u_1,u_2+1\right). \tag{27}$$

According to Definition 4, we can get

$$\rho\left(\boldsymbol{M}^1;u_1,2u_2\right)+\rho\left(\boldsymbol{N}^1;u_1,2u_2\right)=2\left(\rho\left(\boldsymbol{A};u_1,u_2\right)+\rho\left(\boldsymbol{B};u_1,u_2\right)\right)=0, \tag{28}$$

$$\rho\left(\boldsymbol{M}^1;u_1,2u_2+1\right)+\rho\left(\boldsymbol{N}^1;u_1,2u_2+1\right)=0, \tag{29}$$

Let $0<t_2<2Z_2$, when $0<u_1<Z_1,0<t_2<2Z_2$,

$$\rho\left(\boldsymbol{M}^1;u_1,t_2\right)+\rho\left(\boldsymbol{N}^1;u_1,t_2\right)=0. \tag{30}$$

Therefore $\left(\boldsymbol{M}^1,\boldsymbol{N}^1\right)$ is $((L_1,2L_2),(Z_1,2Z_2))-$ZCAP. The proof of the row interleaving is similar to that of the column interleaving, so it is omitted. Moreover, Construction II's proof procedure of is analogous to that of Construction I, so it is omitted. □

Figure 1: The ACCF sum of $(\boldsymbol{M}^1, \boldsymbol{N}^1)$

Figure 2: The ACCF sum of $(\boldsymbol{M}^2, \boldsymbol{N}^2)$

Example 2. $(\boldsymbol{A}, \boldsymbol{B}) = \left(\left(\begin{smallmatrix}+++\\+-+\\+++\end{smallmatrix}\right), \left(\begin{smallmatrix}+-+\\---\\+-+\end{smallmatrix}\right)\right)$ is a $((3,3),(2,2)) - \mathrm{ZCAP}$, then by Construction I, we can get $\boldsymbol{M}^1 = (\boldsymbol{A} \odot \boldsymbol{B}) = \left(\begin{smallmatrix}+++-++\\+---+-\\+++-++\end{smallmatrix}\right)$, $\boldsymbol{N}^1 = (\boldsymbol{A} \odot -\boldsymbol{B}) = \left(\begin{smallmatrix}+-+++-\\++-+++\\+-+++-\end{smallmatrix}\right)$. Take $\boldsymbol{M}^2 = \begin{pmatrix}\boldsymbol{M}^1\\ \odot \\ \boldsymbol{N}^1\end{pmatrix} = \left(\begin{smallmatrix}+++-++\\+-+++-\\+---+-\\++-+++\\+++-++\\+-+++-\end{smallmatrix}\right)$, $\boldsymbol{N}^2 = \begin{pmatrix}\boldsymbol{M}^1\\ \odot \\ -\boldsymbol{N}^1\end{pmatrix} = \left(\begin{smallmatrix}+++-++\\-+---+\\+---+-\\--+---\\+++-++\\-+---+\end{smallmatrix}\right)$, the AACF sum of $(\boldsymbol{M}^1, \boldsymbol{N}^1)$ is shown in Figure 1 and the AACF sum of $(\boldsymbol{M}^2, \boldsymbol{N}^2)$ is shown in Figure 2, so $(\boldsymbol{M}^1, \boldsymbol{N}^1)$ is a $((3,6)(2,4)) - \mathrm{ZCAP}$, $(\boldsymbol{M}^2, \boldsymbol{N}^2)$ is a $((6,6)(4,4)) - \mathrm{ZCAP}$.

4 Constructions of Type-II ZCAPs

There are very few Type-II ZCAP constructions in the existing literature. Type-II ZCAPs were achieved by using one-dimensional Type-II ZCP for the Kronecker product and concatenation operation in [9]. However, how to obtain Type-II ZCAP with flexible size is still an open question. In this section, we first use two arbitrary arrays with the same row number to construct a class of Type-II ZCAPs by concatenation operation. Subsequently, we extend the ZCAP parameters by iteration operation.

Theorem 3. *Let A be an array of size $L \times N_1$ and B be an array of size $L \times N_2$. Perform the following concatenation operation on A and B,*

$$C^0 = A \parallel B, D^0 = A \parallel -B, \tag{31}$$

then the array pair (C^0, D^0) is a Type-II $((L, N_1 + N_2)(L, \min(N_1, N_2) + 1)) - \mathrm{ZCAP}$.

Proof. Let us discuss the following two cases for $0 < u_1 < L$. Case 1: When $N_1 \le N_2$, the following three intervals are discussed: For $0 < u_2 \le N_1$, we have

$$\rho(C^0; u_1, u_2) = \rho(A; u_1, u_2) + \rho(B; u_1, u_2) + \rho(A, B; u_1, N_1 - u_2), \tag{32}$$

$$\rho(D^0; u_1, u_2) = \rho(A; u_1, u_2) + \rho(B; u_1, u_2) - \rho(A, B; u_1, N_1 - u_2), \tag{33}$$

thus,

$$\rho(C^0; u_1, u_2) + \rho(D^0; u_1, u_2) = 2\left(\rho(A; u_1, u_2) + \rho(B; u_1, u_2)\right) \tag{34}$$

For $N_1 < u_2 \le N_2$,

$$\rho(C^0; u_1, u_2) = \rho(B; u_1, u_2) + \rho(A, B; u_1, u_2 - N_1), \tag{35}$$

$$\rho(D^0; u_1, u_2) = \rho(B; u_1, u_2) - \rho(A, B; u_1, u_2 - N_1) \tag{36}$$

then

$$\rho(C^0; u_1, u_2) + \rho(D^0; u_1, u_2) = 2\rho(B; u_1, u_2). \tag{37}$$

Figure 3: The ACCF sum of ($\boldsymbol{C}^0$, $\boldsymbol{D}^0$)

For $N_2 < u_2 \leq N_1 + N_2$,

$$\rho(C^0; u_1, u_2) + \rho(D^0; u_1, u_2) = \rho(A, B; u_1, u_2 - N_2) - \rho(A, B; u_1, u_2 - N_2) = 0 \tag{38}$$

To sum up, when $N_1 \leq N_2$, we have

$$\rho(C^0; u_1, u_2) + \rho(D^0; u_1, u_2) = \begin{cases} 2\left(\rho(A; u_1, u_2) + \rho(B; u_1, u_2)\right), 0 < u_2 \leq N_1, \\ 2\rho(B; u_1, u_2), N_1 < u_2 \leq N_2, \\ 0, N_2 < u_2 \leq N_1 + N_2. \end{cases} \tag{39}$$

Therefore, (C^0, D^0) is a Type- II $((L, N_1 + N_2)\,(L, N_1 + 1)) - \text{ZCAP}$.

Case 2: When $N_1 > N_2$, for $0 < u_2 \leq N_1$, we have

$$\rho(C^0; u_1, u_2) = \rho(A; u_1, u_2) + \rho(B; u_1, u_2) + \rho(A, B; u_1, N_1 - u_2) \tag{40}$$

$$\rho(D^0; u_1, u_2) = \rho(A; u_1, u_2) + \rho(B; u_1, u_2) - \rho(A, B; u_1, N_1 - u_2) \tag{41}$$

For $N_1 < u_2 \leq N_1 + N_2$,

$$\rho(C^0; u_1, u_2) + \rho(D^0; u_1, u_2) = \rho(A, B; u_1, u_2 - N_1) - \rho(A, B; u_1, u_2 - N_1) = 0. \tag{42}$$

Consequently, it is clear that when $N_1 > N_2$,

$$\rho(C^0; u_1, u_2) + \rho(D^0; u_1, u_2) = \begin{cases} 2\left(\rho(A; u_1, u_2) + \rho(B; u_1, u_2)\right), 0 < u_2 \leq N_1, \\ \\ 0, N_1 < u_2 \leq N_1 + N_2. \end{cases} \tag{43}$$

then (C^0, D^0) is an $((L, N_1 + N_2)\,(L, N_2 + 1)) - \text{ZCAP}$. From the above, it is clear that (C^0, D^0) is a Type-II $((L, N_1 + N_2)\,(L, \min(N_1, N_2) + 1)) - ZCAP$. $\square$

Example 3. Let $A = \begin{pmatrix} +-++ \\ ++-+ \\ +++- \end{pmatrix}$ and $B = \begin{pmatrix} +- \\ +- \\ ++ \end{pmatrix}$. According to Theorem 3, we can obtain $C^0 = A \parallel B = \begin{pmatrix} +-+++- \\ ++-++- \\ +++-++ \end{pmatrix}$ and $D^0 = A \parallel -B = \begin{pmatrix} +--+-+ \\ ++-+-+ \\ +++--- \end{pmatrix}$. The AACF sum of C^0 and D^0 is shown in Figure 3, then it can be seen that (C^0, D^0) is a Type-II $((2, 5)\,(2, 3)) - \text{ZCAP}$.

Remark 2 : Since the order of the basis matrix selected by Theorem 3 is arbitrary, the parameters of the ZCAP constructed by Theorem 3 are very flexible. In fact, Theorem 3 can yield optimal Type-II odd-dimensional ZCAP. For example, when $L = 7, N_1 = 3, N_2 = 10$, an odd-dimensional Type-II $((7, 13)\,(7, 4)) - -\text{ZCAP}$ is obtained. According to Definition 9, $((7, 13)\,(7, 4)) - \text{ZCAP}$ is an optimal Type-II ZCAP. Besides, In addition, the optimal Type-II ZCAP with parameters such as Type-II $((7, 13)\,(7, 4)) - \text{ZCAP}$ and Type-II $((9, 17)\,(9, 5)) - \text{ZCAP}$ can be generated.

Theorem 4. *Using* (C^0, D^0) *obtained from Theorem 3 as the initial array, after k iterations, we obtain* $\left(C^k, D^k\right)$*, where*

$$C^k = C^{k-1} \parallel D^{k-1}, D^k = C^{k-1} \parallel -D^{k-1} \tag{44}$$

then $\left(C^k, D^k\right)$ *is a Type-II* $\left(L, 2^k\left(N_1+N_2\right), \left(L, 2^k\left(N_1+N_2\right)-\min\left(N_1, N_2\right)+1\right)\right)-\mathrm{ZCAP}$.

Proof. This proof is akin to Theorem 3 and can be derived through mathematical induction, hence it is omitted. □

Remark 3 : The ZCZ ratio of the Type-II ZCAPs can be calculated according to Definition 7. In Theorem 4, the ZCZ ratio of the constructed ZCAP is that

$$ZCZ_{ratio} = \frac{Z_1 Z_2}{L_1 L_2} = \frac{2^k L\left(N_1+N_2\right)}{2^k L\left(N_1+N_2\right)-\min\left(N_1, N_2\right)+1} = 1 - \frac{\min\left(N_1, N_2\right)-1}{2^k\left(N_1+N_2\right)} \tag{45}$$

According to Eq. (45), when the number of iteration k is large enough, the ZCZ ratio of the Type-II ZCAPs is close 1.

5 Comparison of ZCAP parameters

Until now, the construction methods and outcomes of ZCAPs are relatively inadequate. We compare the parameters of the ZCAPs constructed in this paper with those in existing literature in Table 1. Type- I ZCAPs were directly constructed by Boolean functions in [22]-[24]. [5] and [9] provided more Type-I and Type-II ZCAPs by indirect methods. Although ZCAPs in [9] can exist all array size, many ZCZ size cannot be obtained. Table 1 indicates that the parameters of proposed ZCAPs cannot be produced by the previous constructions, such as optimal Type-I $((3,9),(2,5))-\mathrm{ZCAP}$, Type-I $((6,6),(4,4))-ZCAP$, Type-II $((2,24)(2,21))-\mathrm{ZCAP}$, optimal Type-II $((7,13)(7,4))-\mathrm{ZCAP}$.

6 Conclusions

In this paper, Type-I and Type-II ZCAPs with new parameters are proposed by indirect construction methods. For Type-I ZCAPs, Theorem 1 involves concatenating ZCPs to construct Type-I ZCAPs with multiple parameters, including optimal binary ZCAPs. In Theorem 2, interleaving and iteration procedures are performed on binary ZCAPs with small sizes to obtain binary ZCAPs with large sizes. For Type-II ZCAPs, we employ arbitrary two arrays and apply concatenation and iteration operations to achieve optimal Type-II ZCAPs. The outstanding advantage of the method is the absence of initial array restrictions, making the parameters of Type-II ZCAPs flexible, with the ZCZ ratio approaching 1 as the number of iterations increases.

Table 1: Comparison of ZCAPs

Ref.	ZCAP Parameters	Types of ZCAPs	ZCZ_{ratio}	Optimality (Y/N)	Methods
[22]	$\left(\begin{array}{c}\left(2^n, 2^{m-1}+\sum\limits_{\alpha=t+1}^{m-1} d_\alpha 2^{\alpha-1}+2^v\right), \\ \left(2^n, 2^{t-1}+2^v\right)\end{array}\right)$	Type-I	$\frac{2^{t-1}+2^v}{2^{m-1}+\sum\limits_{\alpha=t+1}^{m-1} d_\alpha 2^{\alpha-1}+2^v}$	N	Boolean function
[23]	$\left(\begin{array}{c}\left(2^{m_1-1}+2^{n+1}, 2^{m_2}+4\right), \\ \left(2^{\pi(n+1)}+2^{n+1}, 2^{m_2-2}+2^{\varphi(m_2-3)}+1\right)\end{array}\right)$	Type-I	$\frac{\left(2^{\pi(n+1)}+2^{n+1}\right)\left(2^{m_2-2}+2^{\varphi(m_2-3)}+1\right)}{\left(2^{m_1-1}+2^{n+1}\right)\left(2^{m_2}+4\right)}$	N	Boolean function
[24]	$\left(\left(14 \cdot 2^{n-4}, 2^m\right), \left(12 \cdot 2^{n-4}, 2^m\right)\right)$	Type-I	$6/7$	N	Boolean function
	$\left(\left(14, 2^m\right), \left(12, 2^m\right)\right)$		$6/7$	N	
[5]	$((4, L), (4, Z))$	Type-I	Z/L	N	Concatenation
	$((2^m, 2^n L), (2^m, Z))$		$Z/2^n L$	N	
	$((2L_1, L_2), (L_1, Z))$		$Z/2L_2$	N	Kronecker product
[9]	$((L_1, 2L_2), (Z_1, Z_2))$	Type-I	$\frac{Z_1 Z_2}{2L_1 L_2}$	N	Kronecker product and concatenation
	$((L_1, 2L_2), (Z_1, 2L_2))$		Z_1/L_1	N	
	$((L_1, 2L_2+1), (Z_1, Z_2))$		$\frac{Z_1 Z_2}{L_1(2L_2+1)}$	N	
	$((L_1, 2L_2+1), (Z_1, L_2+1))$		$\frac{Z_1(L_2+1)}{L_1(2L_2+1)}$	Y	
	$((L_1, 2L_2+2), (Z_1, L_2+1))$		$Z_1/2L_1$	N	Concatenation
	$((L_1, 2L_2), (Z_1, 2Z_2))$		$\frac{Z_1 Z_2}{L_1 L_2}$	N	Kronecker product and interleaving
	$((L_1 L_3, L_2 L_4), (Z_1, Z_2))$		$\frac{Z_1 Z_2}{L_1 L_2 L_3 L_4}$	N	Kronecker product
	$((L_1, 2L_2), (Z_1, L_2+1))$	Type-II	$\frac{Z_1(L_2+1)}{2L_1 L_2}$	N	Kronecker product and concatenation
	$((L_1, 2L_2), (Z_1, 2L_2))$		Z_1/L_1	N	
	$((L_1, 2L_2+1), (Z_1, L_2+1))$		$\frac{Z_1(L_2+1)}{L_1(2L_2+1)}$	N	
Thm. 1	$((L, 3), (Z, 2))$	Type-I	$2Z/3L$	Y	Concatenation
	$((L, 5), (Z, 3))$		$3Z/5L$		
	$((L, 7), (Z, 4))$		$4Z/7L$		
	$((L, 9), (Z, 5))$		$5Z/9L$		
	$((L, 11), (Z, 6))$		$6Z/11L$		
	$((L, 13), (Z, 7))$		$7Z/13L$		
Thm. 2	$\left(\left(L_1, 2^k L_2\right), \left(Z_1, 2^k Z_2\right)\right)$	Type-I	$\frac{Z_1 Z_2}{L_1 L_2}$	N	Interleaving and iteration
	$\left(\left(2^k L_1, L_2\right), \left(2^k Z_1, Z_2\right)\right)$		$\frac{Z_1 Z_2}{L_1 L_2}$		
Thm. 3	$((L, N_1+N_2), (L, \min(N_1, N_2)+1))$	Type-II	$\frac{\min(N_1,N_2)+1}{N_1+N_2}$	Y	Concatenation
Thm. 4	$\left(\begin{array}{c}\left(L, 2^k\left(N_1+N_2\right)\right), \\ \left(L, 2^k\left(N_1+N_2\right)-\min\left(N_1, N_2\right)+1\right)\end{array}\right)$		$1-\frac{\min(N_1,N_2)-1}{2^k(N_1+N_2)}$	N	Iteration

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