A Construction of Optimal Z-Complementary Code Sets Based on Partially m-shift Orthogonal Complementary Codes

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Abstract

Z-complementary code sets (ZCCSs) are widely used in multi-carrier code-division multiple access (MC-CDMA) and multiple-input multiple-output (MIMO) communication because of their ideal correlation properties within a certain region around the in-phase position named zero correlation zone (ZCZ). In this paper, we introduce the definition of a partially m-shifted orthogonal complementary code, and use it to construct an optimal ZCCS by combining complete complementary codes (CCCs). The resultant optimal ZCCSs have new parameters which have not been reported before.

1 Introduction

In 1951, Golay first proposed the concept of Golay complementary pairs (GCPs) while studying infrared multislit spectroscopy [1]. Ten years later, Golay presented the mathematical definition, properties and constructions of GCPs [2]. GCPs are a pair of sequences that satisfy the sum of aperiodic autocorrelation functions (AACFs) is a Dirac delta function [2]. Inspired by Golay's work, in 1972 Tseng and Liu extended the concept of GCPs to complementary sets (CSs) containing two or more constituent sequences [3]. CSs are a set of sequences that satisfy the sum of aperiodic autocorrelation functions (AACFs) is a Dirac delta function [3]. In addition, any two CSs with zero aperiodic cross-correlation sums (ACCSs) are called mutually orthogonal. Furthermore, a set of CSs that are mutually orthogonal to each other is referred to as a mutually orthogonal complementary sequence sets (MOCSSs). Owing to the ideal correlation properties, MOCSSs have been applied in multi-carrier code division multiple access (MC-CDMA) systems [4], MIMO channel estimation [5] and suppressing the multiple access interference. An (M, N, L)-MOCSS is a family of M CSs, where M denotes the set size (i.e., the number of users), N denotes the flock size (i.e., the number of sub-carriers) and L denotes the sequence length [3]. It is worth noting that the set size of (M, N, L)-MOCSS is upper bounded by the flock size, i.e., $M \leq N$ [6]. When the set size equals the flock size, the MOCSS is called a complete complementary code (CCC) [6]. However, a significant limitation of CCC is that its set size (i.e., the number of users) is upper bounded by the flock size. To support a larger number of users in MC-CDMA systems, Z-complementary code sets (ZCCSs) were proposed by Fan *et al.* [7], which have ideal correlations within a zone around the in-phase position named the zero correlation zone (ZCZ).

In recent years, optimal ZCCSs have attracted a lot of research. For an (M, N, L, Z)-ZCCS, the upper bound of its set size is given by [8], i.e. $M \leq N\lfloor L/Z \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to the real number x. When the equal sign holds, the ZCCS is said to optimal. At present, the systematic construction of optimal ZCCSs can be divided into two types: the direct construction methods based on generalized Boolean functions (GBFs) and the indirect construction methods based on base sequences.

First we review the direct constructions. In 2018, Wu *et al.* [9] first presented a construction of optimal ZCCSs based on generalized Boolean functions (GBFs), and discussed their peak-to-average power ratio (PAPR). In 2019, Sarkar *et al.* [10] constructed optimal ZCCSs from the second-order cosets of the q-ary generalization of the first-order Reed-Muller codes through a graphical representation. Later, in 2020, Sarkar *et al.* [11] proposed a construction of optimal ZCCSs with non-power-of-two lengths based on GBFs. And then in 2021, Sarkar *et al.* [12] also proposed a construction of optimal ZCCSs with non-power-of-two lengths based on Pseudo Boolean functions (PBFs). In addition, based on GBFs, Sarkar *et al.* [13] and Wu *et al.* [14] gave some optimal ZCCSs, respectively. Recently, based on GBFs, Ghosh *et al.* gave a construction of optimal ZCCSs with even lengths [15], and proposed three new classes of optimal binary ZCCSs [16]. Based on extended Boolean functions (EBFs), Shen *et al.* [17] proposed a construction of optimal ZCCSs, and Xiao *et al.* [18] obtained a new class of optimal ZCCSs.

Next, we review the indirect constructions. In [19], Das *et al.* presented a novel construction of optimal ZCCSs described in a z-domain framework by introducing the concept of Z-paraunitary (ZPU) matrices. In 2019, Adhikary *et al.* [20] proposed a construction of optimal ZCCSs based on the Hadamard matrix and Z-complementary pairs (ZCPs), which can obtain optimal ZCCSs of odd and even lengths. The ZCP has zero AACSs within a certain region around the in-phase position named zero correlation zone (ZCZ). A ZCP of length L and ZCZ width Z is abbreviated as (L, Z)-ZCP. Later, based on ZCPs and CCCs, Xie *et al.* [21] also obtained a family of optimal ZCCSs. Recently, combining optimal ZCCSs and CCCs, Yu *et al.* [22] designed two classes of optimal ZCCSs with enlarged parameters in terms of sequence lengths and set size. Based on orthogonal matrices, Cui *et al.* [23] proposed a novel construction of three classes of optimal ZCCSs with flexible lengths.

From the application perspective, modern communication systems require very flexible choices of set sizes, sequence lengths and large ZCZ width without any sacrifice of the desired correlation properties. So, this paper presents a construction of optimal ZCCSs with more flexible choices by introducing the concept of partially m-shift orthogonal complementary code. The partially m-shift orthogonal complementary code has zero autocorrelation and cross-correlation sums for each m time-shift within a certain region around the in-phase position. Based on partially m-shift orthogonal complementary code and CCCs, we get a class of optimal ZCCSs with sequence lengths which have not been reported before.

The remainder of the paper is organized as follows. In Section 2, we will show some basic notations, definitions, and a brief introduction of partially m-shift orthogonal complementary code. In Section 3, we will present the constructions of both ZCCSs and optimal ZCCSs based on partially m-shift orthogonal complementary code and CCCs. In Section 4, we will compare our results with existing ones. Finally, we will conclude our work in Section 5.

2 Preliminaries

In this section, we will present some basic notations, definitions, and a brief introduction of partially m-shift orthogonal complementary code.

- $\mathbf{a}$ and $\mathbf{b}$ denotes two unimodular complex valued sequences of length L, i.e., $\mathbf{a} = \big(a(0), a(1), \cdots, a(L-1)\big)$ and $\mathbf{b} = \big(b(0), b(1), \cdots, b(L-1)\big)$.
- $\mathbf{a}||\mathbf{b} = \big(a(0), \cdots, a(L-1), b(0), \cdots, b(L-1)\big)$ represents the concatenation of $\mathbf{a}$ and $\mathbf{b}$.
- intlv$(\mathbf{a}_0, \mathbf{a}_1, \cdots, \mathbf{a}_{N-1})$= $\big(a_0(0), a_1(0), \cdots, a_{N-1}(0), a_0(1), a_1(1), \cdots, a_{N-1}(1), \cdots, a_0(L-1), a_1(L-1), \cdots, a_{N-1}(L-1)\big)$ denotes the bit-interleaved sequences of $\{\mathbf{a}_0, \mathbf{a}_1, \cdots, \mathbf{a}_{N-1}\}$.
- $e\mathbf{a} = \big(ea(0), ea(1), \cdots, ea(L-1)\big)$, where e is a complex number. When $e = -1$, $e\mathbf{a}$ is written as $-\mathbf{a}$.
- x^* represents the complex conjugate of a complex number x.
- $\lfloor x \rfloor$ denotes the largest integer no more than the real number x. $\lceil x \rceil$ represents the smallest integer greater than or equal to x.
- $\langle k \rangle_N$ is the least non-negative integer of k modulo N, where k and N are two non-negative integers.

Given two complex valued sequences $\mathbf{a}$ and $\mathbf{b}$ of length L, the aperiodic correlation function of $\mathbf{a}$ and $\mathbf{b}$ at time-shift τ is defined as

$$\rho_{\mathbf{a},\mathbf{b}}(\tau) = \begin{cases} \sum\limits_{n=0}^{L-1-\tau} a(n)b^*(n+\tau), & 0 \le \tau \le L-1, \\ 0, & \tau \ge L. \end{cases}$$

When $\mathbf{a} \neq \mathbf{b}$, $\rho_{\mathbf{a},\mathbf{b}}(\tau)$ is called the aperiodic cross-correlation function (ACCF); otherwise, it is called the aperiodic autocorrelation function (AACF) of $\mathbf{a}$. For simplicity, AACF of $\mathbf{a}$ will be denoted by $\rho_{\mathbf{a}}(\tau)$.

A sequence set $A^{(i)}(1 \leq i \leq M)$ contains N constituent sequences of length L, i.e., $A^{(i)} = \{\mathbf{a}_k^i : 0 \leq k < N\}$, where $\mathbf{a}_k^i = \big(a_k^i(0), a_k^i(1), \cdots, a_k^i(L-1)\big)$.The aperiodic cross-correlation function sum of $A^{(i)}$ and $A^{(j)}$ at time-shift τ is defined as

$$\rho_{A^{(i)},A^{(j)}}(\tau) = \sum_{k=0}^{N-1} \rho_{\mathbf{a}_k^i,\mathbf{a}_k^j}(\tau).$$

When $i = j$, $\rho_{A^{(i)},A^{(j)}}(\tau)$ is called the aperiodic autocorrelation function sum (AACFS), denoted by $\rho_{A^{(i)}}(\tau)$ for short.

Definition 1. Let $\mathcal{A} = \left\{A^{(0)}, A^{(1)}, \cdots, A^{(M-1)}\right\}$ be a set containing M sequence sets, where $A^{(i)}$ consists of N constituent sequences of length L. $\mathcal{A}$ is called an ZCCS, if the following equation holds:

$$\rho_{A^{(i)},A^{(j)}}(\tau) = \begin{cases} NL, & \tau = 0, i = j, \\ 0, & 0 < \tau < Z, i = j, \\ 0, & 0 \leq \tau < Z, i \neq j, \end{cases}$$

where Z is the ZCZ width. For simplicity, it is denoted by (M, N, L, Z)-ZCCS.

The following lemma gives a bound on the parameters of an (M, N, L, Z)-ZCCS.

Lemma 2 ([8])**.** *For any (M, N, L, Z)-ZCSS, we have*

$$M \leq N \left\lfloor \frac{L}{Z} \right\rfloor. \tag{1}$$

In this paper, an (M, N, L, Z)-ZCCS is said to be optimal if the equal sign in Eq. (1) holds, i.e., $M = N \left\lfloor \frac{L}{Z} \right\rfloor$. Moreover, when $N = 2$ and $L \neq 2^a 10^b 26^c$, one can have a tighter bound, i.e., $M \leq 2\lceil L/Z - 1 \rceil$ [7]. When the equal sign holds, the ZCCS is called optimal. When $Z = L$, the ZCCS is said an MOCSS, denoted by (M, N, L)-MOCSS; When $Z = L$ and $M = N$, the ZCCS is called a CCC, denoted by (M, M, L)-CCC.

In [24], the authors defined a partially E sequence and used it to construct a ZCP. Below, we will introduce a partially m-shift orthogonal complementary code for constructing ZCCSs.

Definition 3. Let $\mathcal{A} = \left\{A^{(0)}, A^{(1)}, \cdots, A^{(M-1)}\right\}$ be a set containing M sequence sets, where $A^{(i)}$ consists of N constituent sequences of length L. Set the integer m to satisfy $0 < m \leq Z \leq L$, then $\mathcal{A}$ is said to be a partially m-shift orthogonal complementary code if

$$\rho_{A^{(i)},A^{(j)}}(m\tau) = \begin{cases} NL, & \tau = 0, i = j, \\ 0, & 0 < \tau < \frac{Z}{m}, i = j, \\ 0, & 0 \leq \tau < \frac{Z}{m}, i \neq j, \end{cases}$$

where Z is the zone width around the in-phase position. For simplicity, it is denoted by (M, N, L, Z)-partially m-shift orthogonal complementary code.

When $m = 2$ and $M = N = 1$, the partially m-shift orthogonal complementary code is called a partially E sequence [24]; When $m = 2$, $M = 1$ and $Z = L$, the partially m-shift orthogonal complementary code is called an even-shift complementary sequence set (ESCSS) [25]; When $m = 1$, the partially m-shift orthogonal complementary code is called a ZCCS; When $m = 1$, $M = N$ and $Z = L$, the partially m-shift orthogonal complementary code is called a CCC [6]. An (M, N, L, Z)-partially m-shift orthogonal complementary code $\mathcal{A}$ contains M distinct partially m-shift complementary sequence sets, i.e. $\mathcal{A} = \left\{A^{(0)}, A^{(1)}, \cdots, A^{(M-1)}\right\}$, where $A^{(i)}$ satisfies $\rho_{A^{(i)}}(m\tau) = 0, 0 < \tau < \frac{Z}{m}$. In general, for any given (M, N, L, Z)-partially m-shift orthogonal complementary code, the maximum number M of different partially m-shift complementary sequence sets is bounded by

$$M \leq mN \left\lfloor \frac{L}{Z} \right\rfloor. \tag{2}$$

3 Construction of Optimal Z-complementary Codes Sets

In this section, based on partially m-shift orthogonal complementary code and CCC, we will present a construction of ZCCSs with more flexible lengths. Before we begin, let us first define an operator ϕ_m, which is useful for the construction of ZCCSs. Let $A = \{\mathbf{a}_k : 0 \leq k < m\}$ be a set of m constituent sequences of length L_1, where $\mathbf{a}_k = \big(a_k(0), a_k(1), \cdots, a_k(L_1 - 1)\big)$. Let $\mathbf{b} = \big(b(0), b(1), \cdots, b(L_2 - 1)\big)$ be a sequence of length L_2, then $\phi_m(\mathbf{b}, A)$ is a sequence of length L_1L_2 defined as

$$\phi_m(\mathbf{b}, A) = b(0)\mathbf{a}_{\langle 0\rangle_m}||b(1)\mathbf{a}_{\langle 1\rangle_m}|| \cdots ||b(L_2 - 1)\mathbf{a}_{\langle L_2-1\rangle_m}.$$

For better understanding, we will give an example to describe.

Example 4. Let $A = \{\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2\}$ be a set of 3 constituent sequences of length L and $C = \{\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ be a set of 4 constituent sequences of length L. Also let $\mathbf{b} = (1, 1, -1, -1, 1, -1, -1)$ be a sequence of length 7, then

$$\begin{aligned}\phi_3(\mathbf{b}, A) &= \mathbf{a}_0||\mathbf{a}_1|| - \mathbf{a}_2|| - \mathbf{a}_0||\mathbf{a}_1|| - \mathbf{a}_2|| - \mathbf{a}_0;\\ \phi_4(\mathbf{b}, C) &= \mathbf{c}_0||\mathbf{c}_1|| - \mathbf{c}_2|| - \mathbf{c}_3||\mathbf{c}_0|| - \mathbf{c}_1|| - \mathbf{c}_2.\end{aligned}$$

In what follows, we will present the main results of this paper.

Lemma 5. *Let $\mathcal{A} = \{A^{(0)}, A^{(1)}, \cdots, A^{(M_1-1)}\}$ be an (M_1, N_1, L_1, Z_1)-partially m-shift orthogonal complementary code and $\mathcal{B} = \{B^{(0)}, B^{(1)}, \cdots, B^{(M_2-1)}\}$ be an (M_2, M_2, L_2)-CCC, where $A^{(i)} = \{\mathbf{a}^i_0, \mathbf{a}^i_1, \cdots, \mathbf{a}^i_{N_1-1}\}$, $\mathbf{a}^i_l = \big(a^i_l(0), a^i_l(1), \cdots, a^i_l(L_1 - 1)\big)$, $B^{(j)} = \{\mathbf{b}^j_0, \mathbf{b}^j_1, \cdots, \mathbf{b}^j_{M_2-1}\}$, $\mathbf{b}^j_k = \big(b^j_k(0), b^j_k(1), \cdots, b^j_k(L_2 - 1)\big)$, $0 \leq i \leq M_1 - 1$, $0 \leq j, k \leq M_2 - 1$, $0 \leq l \leq N_1 - 1$. Also set integer m to satisfy $m|M_2$, $C^t_k = \{\mathbf{b}^{mt}_k, \mathbf{b}^{mt+1}_k, \cdots, \mathbf{b}^{mt+m-1}_k\}$,* *where* $0 \le t \le M_2/m - 1$. *Define* $\mathcal{S} = \{S^{0,0}, S^{0,1}, \cdots, S^{M_1-1, M_2/m-1}\}$, *where* $S^{i,t}$ *consists of* $N_1 M_2$ *constituent sequences of length* $L_1 L_2$, *i.e.,*

$$\begin{aligned} S^{i,t} = \big\{ & \phi_m(\mathbf{a}_0^i, C_0^t), \phi_m(\mathbf{a}_0^i, C_1^t), \cdots, \phi_m(\mathbf{a}_0^i, C_{M_2-1}^t), \\ & \phi_m(\mathbf{a}_1^i, C_0^t), \phi_m(\mathbf{a}_1^i, C_1^t), \cdots, \phi_m(\mathbf{a}_1^i, C_{M_2-1}^t), \\ & \cdots \\ & \phi_m(\mathbf{a}_{N_1-1}^i, C_0^t), \phi_m(\mathbf{a}_{N_1-1}^i, C_1^t), \cdots, \phi_m(\mathbf{a}_{N_1-1}^i, C_{M_2-1}^t) \big\}. \end{aligned}$$

Then $\mathcal{S}$ *is an* $(M_1 M_2/m, N_1 M_2, L_1 L_2, Z_1 L_2)$*-ZCCS.*

Proof. Let $S^{i,t}$ and $S^{i',t'}$ be two arbitrary sequence sets of $\mathcal{S}$. For any integer $\tau = qL_2 + r$, $n = lM_2 + k$, where $0 \le q \le L_1 - 1$, $0 \le r \le L_2 - 1$, $0 \le l \le N_1 - 1$, $0 \le k \le M_2 - 1$. We proceed with the proof considering the following cases.

- When $r = 0$, we have

$$\begin{aligned} \rho_{S^{i,t}, S^{i',t'}}(\tau) = & \sum_{n=0}^{N_1 M_2 - 1} \rho_{\phi_m(\mathbf{a}^i_{\lfloor \frac{n}{M_2} \rfloor}, C^t_{\langle n \rangle_{M_2}}),\ \phi_m(\mathbf{a}^{i'}_{\lfloor \frac{n}{M_2} \rfloor}, C^{t'}_{\langle n \rangle_{M_2}})}(\tau) \\ = & \sum_{l=0}^{N_1-1} \sum_{k=0}^{M_2-1} \Big[\sum_{h=0}^{L_1-1-q} a_l^i(h) a_l^{i'}(h+q) \cdot \rho_{\mathbf{b}_k^{mt+\langle h \rangle_m}, \mathbf{b}_k^{mt'+\langle h+q \rangle_m}}(0) \Big] \\ = & \sum_{l=0}^{N_1-1} \sum_{h=0}^{L_1-1-q} a_l^i(h) a_l^{i'}(h+q) \cdot \rho_{B^{(mt+\langle h \rangle_m)}, B^{(mt'+\langle h+q \rangle_m)}}(0). \end{aligned} \tag{3}$$

For the case $t \ne t'$, consider $0 \le \tau < Z_1 L_2$ (i.e., $0 \le q \le Z_1 - 1$), since $\mathcal{B}$ is a CCC, $\rho_{B^{(mt+\langle h \rangle_m)}, B^{(mt'+\langle h+q \rangle_m)}}(0) = 0$, we have $\rho_{S^{i,t}, S^{i',t'}}(\tau) = 0$.

For the case $t = t'$, consider $0 \le \tau < Z_1 L_2$ (i.e., $0 \le q \le Z_1 - 1$), when $\langle q \rangle_m \ne 0$, since $\mathcal{B}$ is a CCC, $\rho_{B^{(mt+\langle h \rangle_m)}, B^{(mt'+\langle h+q \rangle_m)}}(0) = 0$, one has $\rho_{S^{i,t}, S^{i',t'}}(\tau) = 0$; when $\langle q \rangle_m = 0$, $\rho_{B^{(mt+\langle h \rangle_m)}, B^{(mt'+\langle h+q \rangle_m)}}(0) = M_2 L_2$, then

$$\begin{aligned} \rho_{S^{i,t}, S^{i',t'}}(\tau) = & \sum_{l=0}^{N_1-1} \sum_{h=0}^{L_1-1-q} a_l^i(h) a_l^{i'}(h+q) \cdot \rho_{B^{(mt+\langle h \rangle_m)}, B^{(mt'+\langle h+q \rangle_m)}}(0) \\ = & M_2 L_2 \cdot \sum_{l=0}^{N_1-1} \sum_{h=0}^{L_1-1-q} a_l^i(h) a_l^{i'}(h+q) \\ = & M_2 L_2 \cdot \rho_{A^{(i)}, A^{(i')}}(m\alpha) \end{aligned}$$

where $0 \le \alpha < \frac{Z_1}{m}$. If $i \ne i'$, since $\mathcal{A}$ is an (M_1, N_1, L_1, Z_1)-partially m-shift orthogonal complementary code, $\rho_{A^{(i)}, A^{(i')}}(m\alpha) = 0$, then $\rho_{S^{i,t}, S^{i',t'}}(\tau) = 0$. If $i = i'$, when $\tau = 0$, $\rho_{S^{i,t}, S^{i',t'}}(\tau) = M_2 L_2 N_1 L_1$; when $1 \le \tau < Z_1 L_2$ (i.e., $1 \le q \le Z_1 - 1$, $1 \le \alpha < \frac{Z_1}{m}$), since $\mathcal{A}$ is an (M_1, N_1, L_1, Z_1)-partially m-shift orthogonal complementary code, we have $\rho_{A^{(i)}, A^{(i')}}(m\alpha) = 0$ and then $\rho_{S^{i,t}, S^{i',t'}}(\tau) = 0$.

- When $r \neq 0$, we have

$$
\begin{aligned}
\rho_{S^{i,t},S^{i',t'}}(\tau) = & \sum_{n=0}^{N_1M_2-1} \rho_{\phi_m(\mathbf{a}^i_{\lfloor \frac{n}{M_2} \rfloor}, C^t_{\langle n \rangle_{M_2}}),\ \phi_m(\mathbf{a}^{i'}_{\lfloor \frac{n}{M_2} \rfloor}, C^{t'}_{\langle n \rangle_{M_2}})}(\tau) \\
= & \sum_{l=0}^{N_1-1} \sum_{k=0}^{M_2-1} \Big[\sum_{h=0}^{L_1-1-q} a^i_l(h) a^{i'}_l(h+q) \cdot \rho_{\mathbf{b}_k^{mt+\langle h \rangle_m}, \mathbf{b}_k^{mt'+\langle h+q \rangle_m}}(r) \\
+ & \sum_{h=0}^{L_1-2-q} a^i_l(h) a^{i'}_l(h+q+1) \cdot \rho_{\mathbf{b}_k^{mt+\langle h \rangle_m}, \mathbf{b}_k^{mt'+\langle h+q+1 \rangle_m}}(r-L_2) \Big] \\
= & \sum_{l=0}^{N_1-1} \Big[\sum_{h=0}^{L_1-1-q} a^i_l(h) a^{i'}_l(h+q) \cdot \rho_{B^{(mt+\langle h \rangle_m)}, B^{(mt'+\langle h+q \rangle_m)}}(r) \\
+ & \sum_{h=0}^{L_1-2-q} a^i_l(h) a^{i'}_l(h+q+1) \cdot \rho_{B^{(mt+\langle h \rangle_m)}, B^{(mt'+\langle h+q+1 \rangle_m)}}(r-L_2) \Big].
\end{aligned}
$$

The case that $r \neq 0$ can be similarly discussed.

According to the discussion above, $\mathcal{S}$ is an $(M_1M_2/m, N_1M_2, L_1L_2, Z_1L_2)$-ZCCS. This completes the proof. □

By Lemma 5, we have the following theorem for the construction of optimal ZCCSs.

Theorem 6. *Let $\mathcal{A} = \{A^{(0)}, \cdots, A^{(M_1-1)}\}$ be an (M_1, N_1, L_1, Z_1)-partially m-shift orthogonal complementary code and satisfy $M_1 = mN_1\lfloor \frac{L_1}{Z_1} \rfloor$. If $\mathcal{B} = \{B^{(0)}, B^{(1)}, \cdots, B^{(M_2-1)}\}$ is an (M_2, M_2, L_2)-CCC, then $\mathcal{S}$ given by Lemma 5 is an optimal $(M_1M_2/m, N_1M_2, L_1L_2, Z_1L_2)$-ZCCS.*

Proof. The proof is analog to that of Lemma 5, so we only need to prove that $\mathcal{S}$ is an optimal $(M_1M_2/m, N_1M_2, L_1L_2, Z_1L_2)$-ZCCS. Since $M_1 = mN_1\lfloor \frac{L_1}{Z_1} \rfloor$, the set size of ZCC-S $\mathcal{S}$ comes up to the theoretical bound in Lemma 2, that is, $M_1M_2/m = N_1M_2\lfloor \frac{L_1}{Z_1} \rfloor = N_1M_2\lfloor \frac{L_1L_2}{Z_1L_2} \rfloor$. Therefore, $\mathcal{S}$ is an optimal $(M_1M_2/m, N_1M_2, L_1L_2, Z_1L_2)$-ZCCS. This completes the proof. □

Below, we will show how to obtain flexible partially m-shifted orthogonal complementary codes for generating optimal ZCCSs with new parameters.

The partially m-shift orthogonal complementary code can be obtained by the computer search, in addition, it can be obtained from the ZCCS by bit-interleaving. For example, $\mathcal{A} = \{A^{(0)}, A^{(1)}\}$ is a $(2, 2, 17, 9)$-ZCCS, i.e.,

$$
\begin{aligned}
A^{(0)} = \{\mathbf{a}^0_0, \mathbf{a}^0_1\} = \{ & (-1,-1,-1,-1,-1,-1,-1,1,-1,1,1,1,-1,-1,-1,1,1), \\
& (-1,-1,1,-1,1,1,-1,-1,1,1,-1,1,-1,1,-1,-1,1)\},
\end{aligned}
$$

$$
\begin{aligned}
A^{(1)} = \{\mathbf{a}^1_0, \mathbf{a}^1_1\} = \{ & (1,-1,-1,1,-1,1,-1,1,1,-1,-1,1,1,-1,1,-1,-1), \\
& (-1,-1,1,1,1,-1,-1,-1,1,-1,1,1,1,1,1,1,1)\}.
\end{aligned}
$$

Then by bit-interleaving, we get a $(2,1,34,18)$-partially 2-shift orthogonal complementary code $\{\mathbf{b}_0^0, \mathbf{b}_0^1\}$, i.e.,

$$\begin{aligned}\mathbf{b}_0^0 = intlv(\mathbf{a}_0^0, \mathbf{a}_1^0) = (&-1,-1,-1,-1,-1,1,-1,-1,-1,1,-1,1,-1,-1,1,-1,-1,\\ &1,1,1,1,-1,1,1,-1,-1,-1,1,-1,-1,1,-1,1,1),\\ \mathbf{b}_0^1 = intlv(\mathbf{a}_0^1, \mathbf{a}_1^1) = (&1,-1,-1,-1,-1,1,1,1,-1,1,1,-1,-1,-1,1,-1,1,\\ &1,-1,-1,-1,1,1,1,1,1,-1,1,1,1,-1,1,-1,1).\end{aligned} \tag{4}$$

And we have $\rho_{\mathbf{b}_0^0}(2) = \rho_{\mathbf{b}_0^0}(4) = \cdots = \rho_{\mathbf{b}_0^0}(16) = 0, \rho_{\mathbf{b}_0^0}(18) = 2$, $\rho_{\mathbf{b}_0^1}(2) = \rho_{\mathbf{b}_0^1}(4) = \cdots = \rho_{\mathbf{b}_0^1}(16) = 0, \rho_{\mathbf{b}_0^1}(18) = 2$ and $\rho_{\mathbf{b}_0^0,\mathbf{b}_0^1}(0) = \rho_{\mathbf{b}_0^0,\mathbf{b}_0^1}(2) = \cdots = \rho_{\mathbf{b}_0^0,\mathbf{b}_0^1}(32) = 0$, $\rho_{\mathbf{b}_0^1,\mathbf{b}_0^0}(0) = \rho_{\mathbf{b}_0^1,\mathbf{b}_0^0}(2) = \cdots = \rho_{\mathbf{b}_0^1,\mathbf{b}_0^0}(32) = 0$. Similarly, based on $(4,4,17,9)$-ZCCS, we can obtain $(4,1,68,36)$-partially 4-shift orthogonal complementary code and $(4,2,34,18)$-partially 2-shift orthogonal complementary code using bit-interleaving, respectively. Thus, a partially m-shift orthogonal complementary code with more flexible parameters can be obtained. Then by Theorem 6, we can offer more flexible choices of optimal ZCCS parameters.

Now, we give some examples of optimal ZCCS to illustrate the result of Theorem 6.

Table 1: Binary $(8,2,32,16)$-partially 2-shift orthogonal complementary code

$\begin{pmatrix}\mathbf{a}_0^0\\ \mathbf{a}_1^0\end{pmatrix} = \begin{pmatrix} +++++--++++++--++++++--+-----++- \\ -----++-+++++--+-----++------++- \end{pmatrix}$
$\begin{pmatrix}\mathbf{a}_0^1\\ \mathbf{a}_1^1\end{pmatrix} = \begin{pmatrix} ++--+-+-++--+-+-++--+-+---++-+-+ \\ --++-+-+++--+-+---++-+-+--++-+-+ \end{pmatrix}$
$\begin{pmatrix}\mathbf{a}_0^2\\ \mathbf{a}_1^2\end{pmatrix} = \begin{pmatrix} +++++--++++++--+-----++-+++++--+ \\ -----++-+++++--++++++--++++++--+ \end{pmatrix}$
$\begin{pmatrix}\mathbf{a}_0^3\\ \mathbf{a}_1^3\end{pmatrix} = \begin{pmatrix} ++--+-+-++--+-+---++-+-+++--+-+- \\ --++-+-+++--+-+-++--+-+-++--+-+- \end{pmatrix}$
$\begin{pmatrix}\mathbf{a}_0^4\\ \mathbf{a}_1^4\end{pmatrix} = \begin{pmatrix} +++++--+-----++-+++++--++++++--+ \\ -----++------++------++-+++++--+ \end{pmatrix}$
$\begin{pmatrix}\mathbf{a}_0^5\\ \mathbf{a}_1^5\end{pmatrix} = \begin{pmatrix} ++--+-+---++-+-+++--+-+-++--+-+- \\ --++-+-+--++-+-+--++-+-+++--+-+- \end{pmatrix}$
$\begin{pmatrix}\mathbf{a}_0^6\\ \mathbf{a}_1^6\end{pmatrix} = \begin{pmatrix} +++++--+-----++------++------++- \\ -----++------++-+++++--+-----++- \end{pmatrix}$
$\begin{pmatrix}\mathbf{a}_0^7\\ \mathbf{a}_1^7\end{pmatrix} = \begin{pmatrix} ++--+-+---++-+-+--++-+-+--++-+-+ \\ --++-+-+--++-+-+++--+-+---++-+-+ \end{pmatrix}$

where 1 and -1 are denoted by $+$ and $-$, respectively.

Example 7. Let $\mathcal{A}$ be a binary $(2,1,34,18)$-partially 2-shift orthogonal complementary code given by (4) and satisfy the equal sign in Eq. (2). Let $\mathcal{B}$ be a binary $(4,4,3)$-CCC from [22, Table III]. Then we can get an optimal binary $(4,4,102,54)$-ZCCS $\mathcal{S}$ from Theorem 6, which is new optimal ZCCSs not presented in previous work.

Example 8. Let $\mathcal{A}$ be a binary $(8,2,32,16)$-partially 2-shift orthogonal complementary code, as shown in Table 1, and satisfy the equal sign in Eq. (2). Let $\mathcal{B}$ be a binary $(4,4,3)$-CCC from [22, Table III]. Then we can get an optimal binary $(16,8,96,48)$-ZCCS $\mathcal{S}$ from Theorem 6.

4 Comparison with the Previous Works

In Table 2, we list the parameters of our proposed optimal ZCCSs with that of the previous works. Compared with the previous construction methods, our proposed construction is different in the following ways:

- In [9, 10, 11, 12, 13, 14, 15, 17, 18, 21], the constructions are based on GBFs, PBFs and EBFs. Hence, for the binary optimal ZCCSs, the parameters are of the form of power of two. Compared to that, we can obtain optimal ZCCSs with non-power-of-two lengths. For example, we get an optimal binary $(4, 4, 102, 54)$-ZCCS, which can not be generated by [9, 10, 11, 12, 13, 14, 15, 17, 18, 21]. In [16], the authors offered optimal binary ZCCS with both power-of-two and non-power-of-two lengths through GBF. However, we can obtain an optimal binary $(4, 4, 102, 54)$-ZCCS, which can not be constructed by [16].

- In [20], Adhikary *et al.* constructed optimal $(2^{n+1}, 2^{n+1}, L, Z)$-ZCCS by using (L, Z)-ZCP and Hadamard matrix, where $Z > \frac{L}{2}$. In addition, in [21], using (L, Z)-ZCP and $(2^{k+1}, 2^{k+1}, 2^m)$-CCC, the authors constructed an optimal $(2^{k+2}, 2^{k+2}, 2^m \cdot L, 2^m \cdot Z)$-ZCCS. Note that, here the flock size and set size are same. However, we can design an optimal ZCCS with a different set size and flock size and a large ZCZ width according to the practical application. For example, we can obtain an optimal binary $(16, 8, 96, 48)$-ZCCS, which can not be constructed by [20, 21].

- In [19], Das *et al.* constructed optimal ZCCSs, based on ZPU matrices. In [22], based on optimal ZCCSs and CCCs, the authors derived optimal ZCCSs with enlarged parameters by using the Kronecker product. Note that, if we represent ZCCSs in [22] as matrices of polynomials as in [19], the same ZCCSs can be obtained by a different approach in [19]. However, based on (M_1, N_1, L_1, Z_1)-partially m-shift orthogonal complementary code and (M_2, M_2, L_2)-CCC, we can generate an optimal $(M_1M_2/m, N_1M_2, L_1L_2, Z_1L_2)$-ZCCS with flexible parameters. In fact, the results of [22] are our special case. For example, when $m = 1$, the parameters of the optimal ZCCS generated by us are the same as the result of [22, Th.1]. In addition, we can obtain an optimal binary $(4, 4, 102, 54)$-ZCCS, which can not be constructed by [19, 22]. In [23], based on $L \times L$ orthogonal matrices and $N \times N$ orthogonal matrices, Cui *et al.* constructed optimal (HL, N, L, Z)-ZCCSs, where $Z|L, N = HZ$. It should be noted that the length of these optimal ZCCSs is limited by the order of the orthogonal matrices. For example, we can get an optimal binary $(4, 4, 102, 54)$-ZCCS, which can not be derived by [23], since the Hadamard matrix of order 102 does not exist.

5 Conclusion

In this paper, we introduced a new concept called partially m-shift orthogonal complementary code. Based on partially m-shift orthogonal complementary code and CCC, we

Table 2: Summary of Existing Optimal ZCCSs

Ref.	Based on	Parameters	Conditions
[9, Th. 2]	GBF	$(M, N, 2^m, 2^m N/M)$	$M = 2^{k+v}, N = 2^k, m \geq 3,$ $v \leq m, k \leq m - v$
[10, Th. 2]	GBF	$(M, N, 2^m, 2^m N/M)$	$M = 2^{k+p+1},$ $N = 2^{k+1}, k + p \leq m$
[11, Th. 2]	GBF	$(M, M, 2^{m-1} + 2,$ $2^{m-2} + 2^{\pi(m-3)} + 1)$	$M = 2^{n+1}, m \geq 3$
[12, Th. 1]	PBF	$(M, N, 2^m M/N, 2^m)$	$M = p \cdot 2^{k+1}, N = 2^{k+1},$ $m \geq 2$, p is prime
[13, Th. 1]	GBF	$(M, N, 2^m, 2^m N/M)$	$M = 2^{n+p}, N = 2^n, p \leq m$
[14, Th. 3]	GBF	$(M, N, 2^m, 2^m N/M)$	$M = 2^{k+v}, N = 2^k,$ $v \leq m, k \leq m - v$
[15, Th. 1]	GBF	$(M, N, 2^m M/N, 2^m)$	$M = k \cdot 2^{n+1}, N = 2^{n+1},$ $k, m, n \in \mathbb{Z}^+$
[16, Th. 1]	GBF	$(M, N, \gamma \cdot M/N, \gamma)$	$M = R \cdot 2^{k+1}, N = 2^{k+1},$ $k \geq 1, m \geq 5,$ $\gamma = 5 \cdot 2^{m-3}$, R is even
[16, Th. 3]	GBF	$(M, M, 3 \cdot \gamma, 2 \cdot \gamma)$	$M = 2^{k+1}, k \geq 1,$ $m \geq 5, \gamma = 5 \cdot 2^{m-3}$
[17, Th. 2]	EBF	$(M, N, q^m, q^m N/M)$	$M = q^{v+1}, N = q,$ $m \geq 2,\ v \leq m$
[18, Th. 4.2]	EBF	$(M, N, q^m, q^m N/M)$	$M = q^{v+d}, N = q^d, q \geq 2,$ $v \leq m, d \leq m - v$
[21, Th. 1]	GBF	$(M, M, 3 \cdot 2^m, 2^{m+1})$	$M = 2^{k+1}, k, m \geq 1$
[21, Th. 3]	ZCP and CCC	$(M, M, L \cdot 2^m, Z \cdot 2^m)$	(L, Z)-ZCP, $M = 2^{k+2}, Z > \frac{L}{2},$ $(2^{k+1}, 2^{k+1}, 2^m)$-CCC
[19, Th. 1]	Butson-type Hadamard Matrices	(M, N, M, N)	$M, N \geq 2$
[19, Th. 2]	Optimal ZPU Matrices and Butson-type Hadamard Matrices	$(M, N, M \cdot N^n, N^{n+1})$	$M, N \geq 2,\ n \geq 0$
[20, Con. 1]	ZCP and Hadamard Matrices	(M, M, L, Z)	(L, Z)-ZCP, $M = 2^{n+1}, Z \geq \lceil \frac{L}{2} \rceil$
[22, Th.1]	Optimal ZCCS and CCC	$(M_1 M_2, N_1 M_2, L_1 L_2, Z_1 L_2)$	(M_2, M_2, L_2)-CCC and Optimal (M_1, N_1, L_1, Z_1)-ZCCS
[22, Th.2]	Optimal ZCCS and CCC	$(M_1, N_1, L_1 L_2 N_1, Z_1 L_2 N_1)$	(N_1, N_1, L_2)-CCC and Optimal (M_1, N_1, L_1, Z_1)-ZCCS
[23, Th.1]	$L \times L$ and $N \times N$ Orthogonal matrices	$(M, N, L, NL/M)$	$M = HL, N = HZ, Z \mid L$
[23, Th.2]	$L \times L$ and $N \times N$ Orthogonal matrices	(M, N, L, Z)	$M = HL, H = \lfloor \frac{N}{Z} \rfloor,$ $\lfloor \frac{L}{Z} \rfloor \cdot (N \mod Z) =$ $(L \mod Z) \lfloor \frac{N}{Z} \rfloor$
Th. 6	CCC and partially m-shift orthogonal complementary code	$(\frac{M_1 M_2}{m}, N_1 M_2, L_1 L_2, Z_1 L_2)$	(M_2, M_2, L_2)-CCC and (M_1, N_1, L_1, Z_1)-partially m-shift orthogonal complementary code satisfying the equal sign in Eq. (2)

presented a construction which can lead to new optimal ZCCSs with sequence lengths. Since the proposed construction depends on the availability of the partially m-shift orthogonal complementary code, the properties of partially m-shift orthogonal complementary code as well as some new constructions can be considered in the future work.

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