Optimal few-weight codes from projective spaces

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Abstract

Linear codes with few weights have been extensively developed because of their wide applications in consumer electronics, data storage system, secret sharing, authentication codes, association schemes, and strongly regular graphs. This paper is devoted to two new constructions of linear codes with few weights over the ring $\mathbb{F}_p + u\mathbb{F}_p$ from projective spaces. Moreover, we determine the Lee weight distributions of these codes by investigating the property of the support of the vectors of $\mathbb{F}_p^m$. Via the Gray map, we obtain three classes of linear codes with few weights over $\mathbb{F}_p$. In some cases, these linear codes are proved to be optimal with respect to the Griesmer bound.

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1 Introduction

Let $\mathbb{F}_p$ be the finite field with order p, where p is an odd prime. An $[n,k,d]$ linear code $\mathcal{C}$ over $\mathbb{F}_p$ is a k-dimensional subspace of $\mathbb{F}_p^n$ with minimum Hamming distance d. The *weight enumerator* of $\mathcal{C}$ is the polynomial $1+N_1x+N_2x^2+\cdots+N_nx^n$, where N_i denotes the number of codewords of Hamming weight i in $\mathcal{C}$. The sequence $(1,N_1,N_2,\cdots,N_n)$ is called the *weight distribution* of the code $\mathcal{C}$. The weight distribution contains important information for estimating the probability of error detection and correction. During the past decade, much attention has been paid to determining the weight distribution of a code. Determining the weight distribution of a given code is not an easy task in general. We call $\mathcal{C}$ a t-weight linear code if the number of nonzero N_i in the sequence $(N_1,N_2,\ldots,N_n)$ is equal to t. Linear codes with few weights have been extensively studied because of their significantly important role in consumer electronics, data storage system, secret sharing, authentication codes, association schemes, and strongly regular graphs.

There exists several bounds on the number of codewords in a linear code given the length n and minimum distance d of the code. It is interesting to construct a linear code achieving one bound. For an $[n,k,d]$ linear code over $\mathbb{F}_p$, the Griesmer bound is given by

$$n \geq \sum_{i=0}^{k-1} \lceil \frac{d}{p^i} \rceil,$$

where $\lceil\cdot\rceil$ is the ceiling function. This bound was proved by Griesmer ([3]) for the binary codes and was generalized by Solomon and Stiffler ([15]) for codes over arbitrary finite filed. A linear code C is *optimal* if its parameters n,k and d meet the Griesmer bound [4].

Let $\mathbb{F}_q$ denote the finite field with q elements, where q is a power of p. In [2], Ding and Niederreiter proposed a generic construction of linear codes over $\mathbb{F}_p$. Let $D=\{d_1,d_2,\ldots,d_n\}$ and $\mathrm{Tr}_p^q(\cdot)$ denote the trace function from $\mathbb{F}_q$ to $\mathbb{F}_p$. A linear code of length n over $\mathbb{F}_p$ is defined as:

$$\mathcal{C}_D = \{(\mathrm{Tr}_p^q(ad_1), \mathrm{Tr}_p^q(ad_2), \ldots, \mathrm{Tr}_p^q(ad_n)) : a \in \mathbb{F}_q\}. \quad (1)$$

We call D the defining set of $\mathcal{C}_D$. Let $\mathrm{R}=\mathbb{F}_q+u\mathbb{F}_q$ with $u^2=0$. It is easy to see that R is a local ring with the maximal ideal $\langle u\rangle$. Let $\mathrm{R}_m=\mathbb{F}_{q^m}+u\mathbb{F}_{q^m}$ with $u^2=0$ be an extension ring of R and let R_m^* be the multiplicative group of units of R_m. In [10, 11], the construction defined by (1) was later generalized to codes over finite rings. A linear code over R with a defining set $K=\{d_1,d_2,\ldots,d_n\}\subseteq \mathrm{R}_m^*$ is defined as:

$$\mathcal{C}_K = \{(\mathrm{Tr}(xd_1), \mathrm{Tr}(xd_2), \ldots, \mathrm{Tr}(xd_n)) : x \in \mathrm{R}_m\}, \quad (2)$$

where Tr is the trace function from R_m to R defined by $Tr(a+ub)=\mathrm{Tr}_p^q(a)+u\mathrm{Tr}_p^q(b)$ for $a+ub\in\mathrm{R}_m$. It is easy to see that $\mathcal{C}_K$ defined in (2) is an R-submodule of R^m. Using this construction, some optimal linear codes with few weights over rings have been obtained by selecting the defining sets (see, for instance [9, 7, 12, 13]).

In recent years, several infinite families of optimal or distance-optimal linear codes from simplicial complexes or down sets were constructed (see [1, 5, 6, 14, 16, 17, 18]). In [8], Luo and Ling presented infinite families of optimal or distance-optimal linear codes over $\mathbb{F}_p$ from projective spaces and investigated the locality of these linear codes. Inspired by the work in [8, 16], in this paper, we construct two classes of linear codes with few Lee weights over $\mathrm{R} = \mathbb{F}_p + u\mathbb{F}_p$ with $u^2 = 0$ by employing projective spaces. Let V_A be a subspace of $\mathbb{F}_p^m$ and P_A the corresponding projective space of V_A (see Section 2). Let $K_1 = V_{A_1} + uP_{A_2}$ and $K_2 = P_{A_1} + uV_{A_2}$. Based on the construction defined in (2), two classes of linear codes over $\mathrm{R} = \mathbb{F}_p + u\mathbb{F}_p$ are defined as

$$\mathcal{C}_{K_1} = \{c_{\mathbf{x}} = (\langle \mathbf{x}, \mathbf{d}\rangle_{\mathrm{R}})_{\mathbf{d}\in K_1} | \mathbf{x} \in \mathbb{F}_p^m + u\mathbb{F}_p^m\}$$

and

$$\mathcal{C}_{K_2} = \{c_{\mathbf{x}} = (\langle \mathbf{x}, \mathbf{d}\rangle_{\mathrm{R}})_{\mathbf{d}\in K_2} | \mathbf{x} \in \mathbb{F}_p^m + u\mathbb{F}_p^m\},$$

where $\langle \mathbf{x}, \mathbf{d}\rangle_{\mathrm{R}}$ denotes the inner product of two vectors $\mathbf{x}$ and $\mathbf{d}$ of R^m(see Section 2 for definition).

The rest of this paper is organized as follows. Some preliminaries and notation are given in section 2. In Section 3, we determine the Lee weight distributions of linear codes $\mathcal{C}_{K_1}$ and $\mathcal{C}_{K_2}$ by investigating the property of the support of the vectors of $\mathbb{F}_p^m$. In Section 4, we use the Gray map to obtain some few-weight optimal linear codes over $\mathbb{F}_p$. In Section 5, we make a conclusion.

2 Preliminaries

Let $[m]$ be the set of all integers from 1 to m. Let V_{m+1}^* be the set of all nonzero vectors in vector space $\mathbb{F}_p^{m+1}$. For two vectors $\mathbf{x} = (x_1, x_2, \ldots, x_{m+1})$, $\mathbf{x}' = (x_1', x_2', \ldots, x_{m+1}')$ in V_{m+1}^*, we say that $\mathbf{x}$ and $\mathbf{x}'$ are equivalent if there exists a nonzero $c \in \mathbb{F}_p$ such that $\mathbf{x} = c\mathbf{x}'$. The equivalence class is denoted by $[x_1 : x_2 : \cdots : x_{m+1}]$ and consist of all nonzero scalar multiples of $(x_1, x_2, \ldots, x_{m+1})$. Then the set of equivalence classes is the projective space over F_p with dimension m and is denoted by $PG(m, \mathbb{F}_p)$. The elements of $PG(m, \mathbb{F}_p)$ are called points.

Let A be a nonempty subset of $[m]$. Define an $|A|$-dimensional subspace of $\mathbb{F}_p^m$ as follows:

$$V_A = \{(x_1, x_2, \ldots, x_m) : x_i \in \mathbb{F}_p \text{ if } i \in A \text{ and } x_i = 0 \text{ if } i \notin A\}. \tag{3}$$

Let P_A be the corresponding projective space of V_A. In this paper, we always choose the points of P_A whose first nonzero coordinate position is 1 as a vector representing a equivalence class and express all points of P_A as vectors of length m. It is easy to check that $V_A \backslash \{\mathbf{0}\} = \bigcup_{c\in\mathbb{F}_p^*} cP_A$ and $|P_A| = \frac{p^{|A|}-1}{p-1}$. For example, if $m = 5$ and $A = \{1, 2, 4\}$, then

$$V_{\{1,2,4\}} = \{(x_1, x_2, 0, x_4, 0) : x_1, x_2, x_4 \in \mathbb{F}_p\},$$
$$P_{\{1,2,4\}} = \{(1, x_2, 0, x_4, 0) : x_2, x_4 \in \mathbb{F}_p\} \bigcup \{(0, 1, 0, x_4, 0) : x_4 \in \mathbb{F}_p\} \bigcup \{(0, 0, 0, 1, 0)\}.$$

Below we let $\mathrm{R} = \mathbb{F}_p + u\mathbb{F}_p$ and $\mathrm{R}^m = \mathbb{F}_p^m + u\mathbb{F}_p^m$, where $u^2 = 0$. The inner product of vectors $\mathbf{a} = (a_1, a_2, \ldots, a_m)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_m)$ of R^m is defined by $\langle \mathbf{a}, \mathbf{b} \rangle_R = \sum_{i=1}^m a_i b_i$. Similarly, for two vectors $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{F}_p^m$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_m) \in \mathbb{F}_p^m$, we define the inner product of α and β as $\langle \alpha, \beta \rangle_F = \sum_{i=1}^m \alpha_i \beta_i$.

For any $x + uy \in \mathrm{R}$, $x, y \in \mathbb{F}_p$, define the Gray map ϕ from R to $\mathbb{F}_p^2$ by

$$\phi : \mathrm{R} \to \mathbb{F}_p^2, x + uy \mapsto (y, x + y).$$

For $\mathbf{x} = \alpha + u\beta \in \mathrm{R}^m$, $\alpha \in \mathbb{F}_p^m$ and $\beta \in \mathbb{F}_p^m$, the map ϕ can extend naturally to a map from R^m to $\mathbb{F}_p^{2m}$ as follow:

$$\phi : \mathrm{R}^m \to \mathbb{F}_p^{2m}, \mathbf{x} = \alpha + u\beta \mapsto (\beta, \alpha + \beta).$$

Let $\mathcal{C}$ be a linear code of length m over R. Denote by $w_H(\alpha)$ the Hamming weight of $\alpha \in \mathbb{F}_p^m$. For a codeword $\mathbf{c} = \alpha + u\beta$ of $\mathcal{C}$, the Lee weight of $\mathbf{c}$ is defined to the Hamming weight of its Gray image as follows:

$$w_L(\mathbf{c}) = w_L(\alpha + u\beta) = w_H(\beta) + w_H(\alpha + \beta).$$

The *Lee weight enumerator* of $\mathcal{C}$ of length m is the polynomial $1 + B_1 z + B_2 z^2 + \cdots + B_m z^m$, where B_i denotes the number of codewords of Lee weight i in $\mathcal{C}$. The sequence $(1, B_1, B_2, \cdots, B_m)$ is called the *Lee weight distribution* of the code $\mathcal{C}$.

3 The Lee weight distributions of $\mathcal{C}_{K_1}$ and $\mathcal{C}_{K_2}$

In this section, we present two classes of linear codes over R from projective spaces and determine the Lee weight distributions of these codes.

The support of a vector $\mathbf{a} = (a_1, a_2, \ldots, a_m) \in \mathbb{F}_p^m$, denoted by $\mathrm{Supp}(\mathbf{a})$, is defined by $\mathrm{Supp}(\mathbf{a}) = \{1 \le i \le m : a_i \ne 0\}$. For $A \subseteq [m]$ and $\mathbf{a} \in \mathbb{F}_p^m$, let $\mathbf{a}_A$ be a vector obtained from $\mathbf{a}$ by puncturing on coordinates in $[m] \backslash A$. The following three lemmas are crucial in determining the Lee weight distributions of the codes.

Lemma 1. *[8] Let A_1 and A_2 be two subsets of $[m]$. Then we have the following.*

(i)

$$|S_0 = \{\mathbf{a} \in \mathbb{F}_p^m | \mathrm{Supp}(\mathbf{a}) \cap A_1 = \emptyset\}| = p^{m-|A_1|}$$

and

$$|S_1 = \{\mathbf{a} \in \mathbb{F}_p^m | \mathrm{Supp}(\mathbf{a}) \cap A_1 \ne \emptyset\}| = p^m - p^{m-|A_1|}.$$

(ii)

$$|S_{20} = \{\mathbf{a} \in \mathbb{F}_p^m | \mathrm{Supp}(\mathbf{a}) \cap A_1 = \emptyset, \mathrm{Supp}(\mathbf{a}) \cap A_2 = \emptyset\}| = p^{m-|A_1 \cup A_2|},$$

$$|S_{21} = \{\mathbf{a} \in \mathbb{F}_p^m | \mathrm{Supp}(\mathbf{a}) \cap A_1 = \emptyset, \mathrm{Supp}(\mathbf{a}) \cap A_2 \ne \emptyset\}| = p^{m-|A_1|} - p^{m-|A_1 \cup A_2|},$$

$$|S_{22} = \{\mathbf{a} \in \mathbb{F}_p^m | \mathrm{Supp}(\mathbf{a}) \cap A_1 \ne \emptyset, \mathrm{Supp}(\mathbf{a}) \cap A_2 = \emptyset\}| = p^{m-|A_2|} - p^{m-|A_1 \cup A_2|}$$

and

$$|S_{23} = \{\mathbf{a} \in \mathbb{F}_p^m | \mathrm{Supp}(\mathbf{a}) \cap A_1 \ne \emptyset, \mathrm{Supp}(\mathbf{a}) \cap A_2 \ne \emptyset\}| = p^m - p^{m-|A_1|} - p^{m-|A_2|} + p^{m-|A_1 \cup A_2|}.$$

Lemma 2. *Let A_1 and A_2 be two subsets of $[m]$ and let*

$$T_0 = \{(\mathbf{a}, \mathbf{b}) \in \mathbb{F}_p^m \times \mathbb{F}_p^m | \mathrm{Supp}(\mathbf{a}) \cap A_2 = \emptyset, \mathrm{Supp}(\mathbf{b}) \cap A_1 = \emptyset, \mathrm{Supp}(\mathbf{a}+\mathbf{b}) \cap A_1 = \emptyset\},$$

$$T_1 = \{(\mathbf{a}, \mathbf{b}) \in \mathbb{F}_p^m \times \mathbb{F}_p^m | \mathrm{Supp}(\mathbf{a}) \cap A_2 = \emptyset, \mathrm{Supp}(\mathbf{b}) \cap A_1 = \emptyset, \mathrm{Supp}(\mathbf{a}+\mathbf{b}) \cap A_1 \neq \emptyset\},$$

$$T_2 = \{(\mathbf{a}, \mathbf{b}) \in \mathbb{F}_p^m \times \mathbb{F}_p^m | \mathrm{Supp}(\mathbf{a}) \cap A_2 = \emptyset, \mathrm{Supp}(\mathbf{b}) \cap A_1 \neq \emptyset, \mathrm{Supp}(\mathbf{a}+\mathbf{b}) \cap A_1 = \emptyset\}$$

and

$$T_3 = \{(\mathbf{a}, \mathbf{b}) \in \mathbb{F}_p^m \times \mathbb{F}_p^m | \mathrm{Supp}(\mathbf{a}) \cap A_2 = \emptyset, \mathrm{Supp}(\mathbf{b}) \cap A_1 \neq \emptyset, \mathrm{Supp}(\mathbf{a}+\mathbf{b}) \cap A_1 \neq \emptyset\}.$$

Then

$$|T_0| = p^{2m-|A_1|-|A_1\cup A_2|}, |T_1| = |T_2| = p^{m-|A_1|}(p^{m-|A_2|} - p^{m-|A_1\cup A_2|})$$

and

$$|T_3| = p^{m-|A_2|}(p^m - 2p^{m-|A_1|}) + p^{2m-|A_1|-|A_1\cup A_2|}.$$

Proof. For $\mathbf{a}, \mathbf{b} \in \mathbb{F}_p^m$, let B be a subset of $\mathrm{Supp}(\mathbf{a}) \cap \mathrm{Supp}(\mathbf{b})$ consisting of the coordinates at which $\mathbf{a}+\mathbf{b}$ is nonzero. It can be verified that

$$\begin{aligned} \mathrm{Supp}(\mathbf{a}+\mathbf{b}) &= \Big(\big(\mathrm{Supp}(\mathbf{a}) \cup \mathrm{Supp}(\mathbf{b})\big) \setminus \big(\mathrm{Supp}(\mathbf{a}) \cap \mathrm{Supp}(\mathbf{b})\big) \Big) \cup B \\ &= \big(\mathrm{Supp}(\mathbf{a}) \cap \mathrm{Supp}(\mathbf{b})^c\big) \cup \big(\mathrm{Supp}(\mathbf{b}) \cap \mathrm{Supp}(\mathbf{a})^c\big) \cup B, \end{aligned} \tag{4}$$

where $\mathrm{Supp}(\mathbf{a})^c = [m] \setminus \mathrm{Supp}(\mathbf{a})$ and $\mathrm{Supp}(\mathbf{b})^c = [m] \setminus \mathrm{Supp}(\mathbf{b})$.

Note that $\{(\mathbf{a}, \mathbf{b}) \in \mathbb{F}_p^m \times \mathbb{F}_p^m | \mathrm{Supp}(\mathbf{a}) \cap A_2 = \emptyset, \mathbf{b} \in \mathbb{F}_p^m\} = T_0 \cup T_1 \cup T_2 \cup T_3$. It suffices to determine the size of T_0, T_1, T_2 by Lemma 1 (i) .

Firstly, we determine the size of T_0. By (4), we have

$$\mathrm{Supp}(\mathbf{a}+\mathbf{b}) \cap A_1 = \big(\mathrm{Supp}(\mathbf{a}) \cap \mathrm{Supp}(\mathbf{b})^c \cap A_1\big) \cup \big(\mathrm{Supp}(\mathbf{b}) \cap \mathrm{Supp}(\mathbf{a})^c \cap A_1\big) \cup (B \cap A_1) = \emptyset.$$

This implies that $\mathrm{Supp}(\mathbf{a}) \cap \mathrm{Supp}(\mathbf{b})^c \cap A_1 = \emptyset$. Since $\mathrm{Supp}(\mathbf{b}) \cap A_1 = \emptyset$, we have $\mathrm{Supp}(\mathbf{b})^c \cap A_1 = A_1$. It follows that $\mathrm{Supp}(\mathbf{a}) \cap A_1 = \emptyset$. That is to say, $\mathrm{Supp}(\mathbf{a}+\mathbf{b}) \cap A_1 = \emptyset$ and $\mathrm{Supp}(\mathbf{b}) \cap A_1 = \emptyset$ if and only if $\mathrm{Supp}(\mathbf{a}) \cap A_1 = \emptyset$ and $\mathrm{Supp}(\mathbf{b}) \cap A_1 = \emptyset$. Hence, T_0 can be written as

$$T_0 = \{(\mathbf{a}, \mathbf{b}) \in \mathbb{F}_p^m \times \mathbb{F}_p^m | \mathrm{Supp}(\mathbf{a}) \cap A_2 = \emptyset, \mathrm{Supp}(\mathbf{b}) \cap A_1 = \emptyset, \mathrm{Supp}(\mathbf{a}) \cap A_1 = \emptyset\}.$$

By Lemma 1, we know that $|T_0| = |S_0| \times |S_{20}| = p^{2m-|A_1|-|A_1\cup A_2|}$.

Secondly, we determine the size of T_1. Note that $B \subseteq \mathrm{Supp}(\mathbf{b})$ and $\mathrm{Supp}(\mathbf{b}) \cap A_1 = \emptyset$. Then $B \cap A_1 = \emptyset$ and $\mathrm{Supp}(\mathbf{b})^c \cap A_1 = A_1$. It follows from (4) that

$$\begin{aligned} \mathrm{Supp}(\mathbf{a}+\mathbf{b}) \cap A_1 &= \big(\mathrm{Supp}(\mathbf{a}) \cap \mathrm{Supp}(\mathbf{b})^c \cap A_1\big) \cup \big(\mathrm{Supp}(\mathbf{b}) \cap \mathrm{Supp}(\mathbf{a})^c \cap A_1\big) \cup (B \cap A_1) \\ &= (\mathrm{Supp}(\mathbf{a}) \cap A_1) \cup \emptyset \cup \emptyset = \mathrm{Supp}(\mathbf{a}) \cap A_1 \neq \emptyset. \end{aligned}$$

This implies that $\mathrm{Supp}(\mathbf{a}+\mathbf{b})\cap A_1 \neq \emptyset$ and $\mathrm{Supp}(\mathbf{b})\cap A_1 = \emptyset$ if and only if $\mathrm{Supp}(\mathbf{a})\cap A_1 \neq \emptyset$ and $\mathrm{Supp}(\mathbf{b}) \cap A_1 = \emptyset$. Hence, T_1 can be written as

$$T_1 = \{(\mathbf{a},\mathbf{b}) \in \mathbb{F}_p^m \times \mathbb{F}_p^m | \mathrm{Supp}(\mathbf{a}) \cap A_2 = \emptyset, \mathrm{Supp}(\mathbf{b}) \cap A_1 = \emptyset, \mathrm{Supp}(\mathbf{a}) \cap A_1 \neq \emptyset\}.$$

It then follows from Lemma 1 that $|T_1| = |S_0| \times |S_{22}| = p^{m-|A_1|}(p^{m-|A_2|} - p^{m-|A_1\cup A_2|})$.

Thirdly, we determine the size of T_2. By (4), we know that

$$\mathrm{Supp}(\mathbf{a}+\mathbf{b})\cap A_1 = \big(\mathrm{Supp}(\mathbf{a})\cap\mathrm{Supp}(\mathbf{b})^c\cap A_1\big)\cup\big(\mathrm{Supp}(\mathbf{b})\cap\mathrm{Supp}(\mathbf{a})^c\cap A_1\big)\cup(B\cap A_1) = \emptyset,$$

which implies that $\mathrm{Supp}(\mathbf{a}) \cap \mathrm{Supp}(\mathbf{b})^c \cap A_1 = \emptyset$, $\mathrm{Supp}(\mathbf{b}) \cap \mathrm{Supp}(\mathbf{a})^c \cap A_1 = \emptyset$ and $B \cap A_1 = \emptyset$. It is easy to verify that $\mathrm{Supp}(\mathbf{a}) \cap A_1 \subseteq \mathrm{Supp}(\mathbf{b})$, $\mathrm{Supp}(\mathbf{b}) \cap A_1 \subseteq \mathrm{Supp}(\mathbf{a})$ and $B \cap A_1 = \emptyset$, which implies that $\mathrm{Supp}(\mathbf{a}) \cap A_1 = \mathrm{Supp}(\mathbf{b}) \cap A_1 \neq \emptyset$ from the condition that $\mathrm{Supp}(\mathbf{b}) \cap A_1 \neq \emptyset$. This together $\mathrm{Supp}(\mathbf{b}) \cap A_2 = \emptyset$ implies that $|\mathrm{Supp}(\mathbf{b}) \cap A_1| = |\mathrm{Supp}(\mathbf{a}) \cap A_1| = i$, where $i = 1, 2, \ldots, |A_1\backslash A_2|$. On the other hand, since $\mathrm{Supp}(\mathbf{a} + \mathbf{b}) \cap A_1 = \emptyset$, we have $\mathbf{a}_{A_1} + \mathbf{b}_{A_1} = \mathbf{0}$. Once the coordinates of $\mathbf{b}_{A_1}$ are determined, the corresponding coordinates of $\mathbf{a}_{A_1}$ are also determined. Note that $\mathbf{a}_{A_1\backslash(\mathrm{Supp}(\mathbf{a})\cap A_1)} = \mathbf{0}$ and $\mathbf{b}_{A_1\backslash(\mathrm{Supp}(\mathbf{b})\cap A_1)} = \mathbf{0}$. For each $1 \leq i \leq |A_1\backslash A_2|$, if $|\mathrm{Supp}(\mathbf{b}) \cap A_1| = i$, then there are $(p-1)^i p^{m-|A_1|}\binom{|A_1\backslash A_2|}{i}$ choices for $\mathbf{b}$ and $p^{m-|A_1\cup A_2|}$ choices for $\mathbf{a}$ such that the conditions in T_2 are satisfied. Then

$$|T_2| = \sum_{i=1}^{|A_1\backslash A_2|} \binom{|A_1\backslash A_2|}{i}(p-1)^i p^{m-|A_1|}p^{m-|A_1\cup A_2|} = p^{m-|A_1|}(p^{m-|A_2|} - p^{m-|A_1\cup A_2|}).$$

Note that $\{(\mathbf{a},\mathbf{b}) \in \mathbb{F}_p^m \times \mathbb{F}_p^m | \mathrm{Supp}(\mathbf{a}) \cap A_2 = \emptyset, \mathbf{b} \in \mathbb{F}_p^m\} = T_0 \cup T_1 \cup T_2 \cup T_3$. It follows from Lemma 1 (i) that

$$|T_3| = p^{2m-|A_2|} - |T_0| - |T_1| - |T_2| = p^{m-|A_2|}(p^m - 2p^{m-|A_1|}) + p^{2m-|A_1|-|A_1\cup A_2|}.$$

This completes the proof. □

Lemma 3. *Let A_1 and A_2 be two subsets of $[m]$ and let*

$$R_0 = \{(\mathbf{a},\mathbf{b}) \in \mathbb{F}_p^m \times \mathbb{F}_p^m | \mathrm{Supp}(\mathbf{a}) \cap A_2 \neq \emptyset, \mathrm{Supp}(\mathbf{b}) \cap A_1 = \emptyset, \mathrm{Supp}(\mathbf{a}+\mathbf{b}) \cap A_1 = \emptyset\},$$
$$R_1 = \{(\mathbf{a},\mathbf{b}) \in \mathbb{F}_p^m \times \mathbb{F}_p^m | \mathrm{Supp}(\mathbf{a}) \cap A_2 \neq \emptyset, \mathrm{Supp}(\mathbf{b}) \cap A_1 = \emptyset, \mathrm{Supp}(\mathbf{a}+\mathbf{b}) \cap A_1 \neq \emptyset\},$$
$$R_2 = \{(\mathbf{a},\mathbf{b}) \in \mathbb{F}_p^m \times \mathbb{F}_p^m | \mathrm{Supp}(\mathbf{a}) \cap A_2 \neq \emptyset, \mathrm{Supp}(\mathbf{b}) \cap A_1 \neq \emptyset, \mathrm{Supp}(\mathbf{a}+\mathbf{b}) \cap A_1 = \emptyset\},$$

and

$$R_3 = \{(\mathbf{a},\mathbf{b}) \in \mathbb{F}_p^m \times \mathbb{F}_p^m | \mathrm{Supp}(\mathbf{a}) \cap A_2 \neq \emptyset, \mathrm{Supp}(\mathbf{b}) \cap A_1 \neq \emptyset, \mathrm{Supp}(\mathbf{a}+\mathbf{b}) \cap A_1 \neq \emptyset\}.$$

Then

$$|R_0| = p^{m-|A_1|}(p^{m-|A_1|} - p^{m-|A_1\cup A_2|}),$$
$$|R_1| = |R_2| = p^{m-|A_1|}(p^m - p^{m-|A_2|} - p^{m-|A_1|} + p^{m-|A_1\cup A_2|}),$$

and

$$|R_3| = p^m(p^m - p^{m-|A_2|}) - p^{m-|A_1|}(2p^m - 2p^{m-|A_2|} - p^{m-|A_1|} + p^{m-|A_1\cup A_2|}).$$

Proof. The proof is similar to that of Lemma 2 and is omitted. $\square$

In the following, we always assume that A_1 and A_2 be two nonempty subsets of $[m]$. Let V_{A_1}, V_{A_2} be two subspaces of $\mathbb{F}_p^m$ defined in (3). Let P_{A_2} be the corresponding projective space of V_{A_2}. The following theorem shows that $\mathcal{C}_{K_1}$ has at most four Lee weights, where $K_1 = V_{A_1} + uP_{A_2}$.

Theorem 4. *Let $A_1, A_2 \subseteq [m]$ and $K_1 = V_{A_1} + uP_{A_2} \subseteq \mathrm{R}^m$. Then the code $\mathcal{C}_{K_1}$ is a code with length $\frac{p^{|A_1|}(p^{|A_2|}-1)}{p-1}$ and size $p^{|A_1|+|A_1 \cup A_2|}$. The Lee weight distribution of $\mathcal{C}_{K_1}$ is listed in Table 1, where T_i and R_i are given in Lemmas 2 and 3, $i = 0, 1, 2, 3$.*

Table 1: The Lee weight distribution of $\mathcal{C}_{K_1}$ in Theorem 4

Lee weight i	Frequency B_i
0	$\lvert T_0\rvert$
$p^{\lvert A_1\rvert-1}(p^{\lvert A_2\rvert}-1)$	$\lvert T_1\rvert + \lvert T_2\rvert$
$2p^{\lvert A_1\rvert-1}(p^{\lvert A_2\rvert}-1)$	$\lvert T_3\rvert + \lvert R_3\rvert$
$p^{\lvert A_1\rvert-1}(2p^{\lvert A_2\rvert}-1)$	$\lvert R_1\rvert + \lvert R_2\rvert$
$2p^{\lvert A_1\rvert+\lvert A_2\rvert-1}$	$\lvert R_0\rvert$

Proof. It is clear that $|K_1| = \frac{p^{|A_1|}(p^{|A_2|}-1)}{p-1}$, i.e., the length of $\mathcal{C}_{K_1}$ is $\frac{p^{|A_1|}(p^{|A_2|}-1)}{p-1}$.

For $\mathbf{x} = \alpha + u\beta \in \mathrm{R}^m$, where $\alpha \in \mathbb{F}_p^m, \beta \in \mathbb{F}_p^m$, the Lee weight of the codeword $c_{\mathbf{x}}$ of the code $\mathcal{C}_{K_1}$ is given by

$$\begin{aligned}
w_L(c_{\mathbf{x}}) &= w_L((\langle \mathbf{x}, \mathbf{d}\rangle_{\mathrm{R}})_{\mathbf{d}\in K_1}) = w_L\big((\langle \alpha + u\beta, d_1 + ud_2\rangle_{\mathrm{R}})_{d_1\in V_{A_1}, d_2\in P_{A_2}}\big) \\
&= w_L\big((\langle \alpha, d_1\rangle_F + u(\langle \alpha, d_2\rangle_F + \langle \beta, d_1\rangle_F))_{d_1\in V_{A_1}, d_2\in P_{A_2}}\big) \\
&= w_H((\langle \alpha, d_2\rangle_F + \langle \beta, d_1\rangle_F)_{d_1\in V_{A_1}, d_2\in P_{A_2}}) + w_H((\langle \alpha+\beta, d_1\rangle_F + \langle \alpha, d_2\rangle_F)_{d_1\in V_{A_1}, d_2\in P_{A_2}}) \\
&= |K_1| - \frac{1}{p}\sum_{y\in\mathbb{F}_p}\sum_{d_1\in V_{A_1}}\sum_{d_2\in P_{A_2}} \zeta_p^{y(\langle \alpha, d_2\rangle_F + \langle \beta, d_1\rangle_F)} \\
&\quad + |K_1| - \frac{1}{p}\sum_{y\in\mathbb{F}_p}\sum_{d_1\in V_{A_1}}\sum_{d_2\in P_{A_2}} \zeta_p^{y(\langle \alpha+\beta, d_1\rangle_F + \langle \alpha, d_2\rangle_F)} \\
&= 2p^{|A_1|-1}(p^{|A_2|}-1) - \frac{1}{p}\sum_{y\in\mathbb{F}_p^*}\sum_{d_2\in P_{A_2}} \zeta_p^{y(\langle \alpha, d_2\rangle_F)} \sum_{d_1\in V_{A_1}} \zeta_p^{y(\langle \beta, d_1\rangle_F)} \\
&\quad - \frac{1}{p}\sum_{y\in\mathbb{F}_p^*}\sum_{d_2\in P_{A_2}} \zeta_p^{y(\langle \alpha, d_2\rangle_F)} \sum_{d_1\in V_{A_1}} \zeta_p^{y(\langle \alpha+\beta, d_1\rangle_F)},
\end{aligned} \tag{5}$$

where ζ_p is a primitive p-th root of unity. Next we divide the proof into five cases.

Case 1 ($\mathrm{Supp}(\beta) \cap A_1 = \emptyset, \mathrm{Supp}(\alpha+\beta) \cap A_1 = \emptyset, \mathrm{Supp}(\alpha) \cap A_2 = \emptyset$): Note that $\mathrm{Supp}(\beta) = \mathrm{Supp}(y\beta)$ for any $y \in \mathbb{F}_p^*$. It is easy to check that $\sum_{d_1\in V_{A_1}} \zeta_p^{y(\langle \beta, d_1\rangle_F)} =$

$\sum_{d_1\in V_{A_1}} \zeta_p^{y(\langle \alpha+\beta,d_1\rangle_F)} = p^{|A_1|}$ when $\mathrm{Supp}(\beta)\cap A_1=\emptyset$ and $\mathrm{Supp}(\alpha+\beta)\cap A_1=\emptyset$. In this case,

$$\begin{aligned} w_L(c_{\mathbf{x}}) &= 2p^{|A_1|-1}(p^{|A_2|}-1) - 2p^{|A_1|-1}\sum_{y\in\mathbb{F}_p^*}\sum_{d_2\in P_{A_2}} \zeta_p^{y(\langle \alpha,d_2\rangle_F)} \\ &= 2p^{|A_1|-1}(p^{|A_2|}-1) - 2p^{|A_1|-1}\sum_{d_2'\in V_{A_2}^*} \zeta_p^{\langle \alpha,d_2'\rangle_F} \\ &= 2p^{|A_1|-1}(p^{|A_2|}-1) - 2p^{|A_1|-1}(p^{|A_2|}-1) = 0 \end{aligned}$$

due to $\mathrm{Supp}(\alpha)\cap A_2=\emptyset$ and the fact that $V_{A_2}\backslash\{\mathbf{0}\} = \bigcup_{y\in\mathbb{F}_p^*} cP_{A_2}$. It follows from Lemma 2 that the number of $\mathbf{x}$ with $w_L(c_{\mathbf{x}})=0$ is $|T_0|$.

Case 2 ($\mathrm{Supp}(\beta)\cap A_1=\emptyset, \mathrm{Supp}(\alpha+\beta)\cap A_1=\emptyset, \mathrm{Supp}(\alpha)\cap A_2\neq\emptyset$): Similarly,

$$\sum_{d_1\in V_{A_1}} \zeta_p^{y(\langle \beta,d_1\rangle_F)} = \sum_{d_1\in V_{A_1}} \zeta_p^{y(\langle \alpha+\beta,d_1\rangle_F)} = p^{|A_1|}$$

when $\mathrm{Supp}(\beta)\cap A_1=\emptyset$ and $\mathrm{Supp}(\alpha+\beta)\cap A_1=\emptyset$. In this case, by (5), we have $w_L(c_{\mathbf{x}}) = 2p^{|A_1|+|A_2|-1}$ when $\mathrm{Supp}(\alpha)\cap A_2\neq\emptyset$. It follows from Lemma 3 that the number of $\mathbf{x}$ with $w_L(c_{\mathbf{x}}) = 2p^{|A_1|+|A_2|-1}$ is $|R_0|$.

Case 3 ($\mathrm{Supp}(\beta)\cap A_1=\emptyset, \mathrm{Supp}(\alpha+\beta)\cap A_1\neq\emptyset, \mathrm{Supp}(\alpha)\cap A_2=\emptyset$) or ($\mathrm{Supp}(\beta)\cap A_1\neq\emptyset, \mathrm{Supp}(\alpha+\beta)\cap A_1=\emptyset, \mathrm{Supp}(\alpha)\cap A_2=\emptyset$): Note that

$$\sum_{d_1\in V_{A_1}} \zeta_p^{y(\langle \beta,d_1\rangle_F)} = 0 \qquad \text{or} \qquad \sum_{d_1\in V_{A_1}} \zeta_p^{y(\langle \alpha+\beta,d_1\rangle_F)} = p^{|A_1|}$$

when $\mathrm{Supp}(\beta)\cap A_1\neq\emptyset$ or $\mathrm{Supp}(\alpha+\beta)\cap A_1\neq\emptyset$. In this case, we have $w_L(c_{\mathbf{x}}) = p^{|A_1|-1}(p^{|A_2|}-1)$ when $\mathrm{Supp}(\alpha)\cap A_2=\emptyset$ from (5). It follows from Lemma 2 that the number of $\mathbf{x}$ with $w_L(c_{\mathbf{x}}) = p^{|A_1|-1}(p^{|A_2|}-1)$ is $|T_1|+|T_2|$.

Case 4 ($\mathrm{Supp}(\beta)\cap A_1=\emptyset, \mathrm{Supp}(\alpha+\beta)\cap A_1\neq\emptyset, \mathrm{Supp}(\alpha)\cap A_2\neq\emptyset$) or ($\mathrm{Supp}(\beta)\cap A_1\neq\emptyset, \mathrm{Supp}(\alpha+\beta)\cap A_1=\emptyset, \mathrm{Supp}(\alpha)\cap A_2\neq\emptyset$): By a way similar to the one used in the Case 3, we have $w_L(c_{\mathbf{x}}) = p^{|A_1|-1}(2p^{|A_2|}-1)$ when $\mathrm{Supp}(\alpha)\cap A_2\neq\emptyset$ from (5). By Lemma 3, the number of $\mathbf{x}$ with $w_L(c_{\mathbf{x}}) = p^{|A_1|-1}(2p^{|A_2|}-1)$ is $|R_1|+|R_2|$.

Case 5 ($\mathrm{Supp}(\beta)\cap A_1\neq\emptyset, \mathrm{Supp}(\alpha+\beta)\cap A_1\neq\emptyset, \mathrm{Supp}(\alpha)\cap A_2=\emptyset$) or ($\mathrm{Supp}(\beta)\cap A_1\neq\emptyset, \mathrm{Supp}(\alpha+\beta)\cap A_1\neq\emptyset, \mathrm{Supp}(\alpha)\cap A_2\neq\emptyset$): It is clear that

$$\sum_{d_1\in V_{A_1}} \zeta_p^{y(\langle \beta,d_1\rangle_F)} = \sum_{d_1\in V_{A_1}} \zeta_p^{y(\langle \alpha+\beta,d_1\rangle_F)} = 0$$

when $\mathrm{Supp}(\beta)\cap A_1\neq\emptyset$ and $\mathrm{Supp}(\alpha+\beta)\cap A_1\neq\emptyset$. In this case, $w_L(c_{\mathbf{x}}) = 2p^{|A_1|-1}(p^{|A_2|}-1)$ and the number of $\mathbf{x}$ with $w_L(c_{\mathbf{x}}) = 2p^{|A_1|-1}(p^{|A_2|}-1)$ is $|T_3|+|R_3|$.

To determine the dimension of $\mathcal{C}_{K_1}$,we define a mapping

$$\tau: \mathrm{R}^m \to \mathcal{C}_{K_1}, \mathbf{x}\mapsto \mathbf{c}_{\mathbf{x}}.$$

It is easy to check that τ is an epimorphism from R^m to $\mathcal{C}_{K_1}$. By the homomorphism theorem, $\mathcal{C}_{K_1}$ is isomorphic to $\mathrm{R}^m/\mathrm{Ker}\tau$, where $\mathrm{Ker}\tau = \{\mathbf{x}\in\mathcal{C}_{K_1} | \mathbf{c}_{\mathbf{x}}=\mathbf{0}\}$. Hence, the size of $\mathcal{C}_{K_1}$ is $\frac{p^{2m}}{|T_0|} = p^{|A_1|+|A_1\cup A_2|}$.

This completes the proof. □

If we choose some special subsets A_1,A_2 of $[m]$, then C_{K_1} has three or two Lee weights.

Corollary 5. *Let $A_1, A_2 \subseteq [m]$ with $A_1 \subset A_2$. Then $\mathcal{C}_{K_1}$ is a three-Lee-weight code with length $\frac{p^{|A_1|}(p^{|A_2|}-1)}{p-1}$ and size $p^{|A_1|+|A_2|}$, and the Lee weight distribution of $\mathcal{C}_{K_1}$ is listed in Table 2, where T_i $(i = 0, 3)$ and R_i $(i = 0, 1, 2, 3)$ are given in Lemmas 2 and 3.*

Table 2: The Lee weight distribution of $\mathcal{C}_{K_1}$ in Corollary 5

Lee weight i	Frequency B_i
0	$\lvert T_0\rvert$
$2p^{\lvert A_1\rvert-1}(p^{\lvert A_2\rvert}-1)$	$\lvert T_3\rvert+\lvert R_3\rvert$
$p^{\lvert A_1\rvert-1}(2p^{\lvert A_2\rvert}-1)$	$\lvert R_1\rvert+\lvert R_2\rvert$
$2p^{\lvert A_1\rvert+\lvert A_2\rvert-1}$	$\lvert R_0\rvert$

Proof. It is easy to check that $|T_1| = |T_2| = p^{m-|A_1|}(p^{m-|A_2|} - p^{m-|A_1\cup A_2|}) = 0$ when $A_1 \subset A_2$. Then the conclusion follows from Theorem 4. □

The next corollary follows immediately when $A_1 = A_2$.

Corollary 6. *Let $A_1 = A_2$. Then the code $\mathcal{C}_{K_1}$ is a two-Lee-weight code with length $\frac{p^{|A_1|}(p^{|A_1|}-1)}{p-1}$ and size $p^{2|A_1|}$, and the Lee weight distribution of $\mathcal{C}_{K_1}$ is listed in Table 3, where T_i $(i = 0, 3)$ and R_i $(i = 1, 2, 3)$ are given in Lemmas 2 and 3.*

Table 3: The Lee weight distribution of $\mathcal{C}_{K_1}$ in Corollary 6

Lee weight i	Frequency B_i
0	$\lvert T_0\rvert$
$2p^{\lvert A_1\rvert-1}(p^{\lvert A_2\rvert}-1)$	$\lvert T_3\rvert+\lvert R_3\rvert$
$p^{\lvert A_1\rvert-1}(2p^{\lvert A_2\rvert}-1)$	$\lvert R_1\rvert+\lvert R_2\rvert$

In the following theorem, we determine the Lee weight distributions of the code $\mathcal{C}_{K_2}$, where $K_2 = P_{A_1} + uV_{A_2}$.

Theorem 7. *Let $A_1, A_2 \subseteq [m]$ and $K_2 = P_{A_1} + uV_{A_2} \subseteq \mathrm{R}^m$. Then the code $\mathcal{C}_{K_2}$ is a three-Lee-weight code with length $\frac{p^{|A_2|}(p^{|A_1|}-1)}{p-1}$ and size $p^{|A_1|+|A_1\cup A_2|}$. The Lee weight distribution of $\mathcal{C}_{K_2}$ is listed in Table 4, where T_i and R_i are given in Lemmas 2 and 3, $i = 0, 1, 2, 3$.*

Proof. The proof is similar to that of Theorem 4 and is omitted. □

Corollary 8. *Let $A_1, A_2 \subseteq [m]$ with $A_1 \subseteq A_2$. Then $\mathcal{C}_{K_2}$ is a two-Lee-weight code with length $\frac{p^{|A_2|}(p^{|A_1|}-1)}{p-1}$ and size $p^{|A_1|+|A_2|}$. Moreover, the Lee weight distribution of $\mathcal{C}_{K_2}$ is listed in Table 5, where T_i $(i = 0, 3)$ and R_i $(i = 0, 1, 2, 3)$ are given in Lemmas 2 and 3.*

Table 4: The Lee weight distribution of $\mathcal{C}_{K_2}$ in Theorem 7

Lee weight i	Frequency B_i
0	$\lvert T_0\rvert$
$p^{\lvert A_1\rvert+\lvert A_2\rvert-1}$	$\lvert T_1\rvert+\lvert T_2\rvert$
$2p^{\lvert A_2\rvert-1}(p^{\lvert A_1\rvert}-1)$	$\lvert R_0\rvert+\lvert R_1\rvert+\lvert R_2\rvert+\lvert R_3\rvert$
$2p^{\lvert A_1\rvert+\lvert A_2\rvert-1}$	$\lvert T_3\rvert$

Table 5: The Lee weight distribution of $\mathcal{C}_{K_2}$ in Corollary 8

Lee weight i	Frequency B_i
0	$\lvert T_0\rvert$
$2p^{\lvert A_2\rvert-1}(p^{\lvert A_1\rvert}-1)$	$\lvert R_0\rvert+\lvert R_1\rvert+\lvert R_2\rvert+\lvert R_3\rvert$
$2p^{\lvert A_1\rvert+\lvert A_2\rvert-1}$	$\lvert T_3\rvert$

4 Some optimal linear codes from the image of the codes $\mathcal{C}_{K_1}$ and $\mathcal{C}_{K_2}$ under Gray map

In this section, we show that the image of the codes $\mathcal{C}_{K_1}$ and $\mathcal{C}_{K_2}$ under Gray map have few weights over $\mathbb{F}_p$ and obtain some optimal codes in some case.

Theorem 9. *Let the symbols be the same as those in Corollary 5. Then $\phi(\mathcal{C}_{K_1})$ is a three-weight code over $\mathbb{F}_p$ with parameters $[\frac{2p^{|A_1|}(p^{|A_2|}-1)}{p-1}, |A_1|+|A_2|, 2p^{|A_1|-1}(p^{|A_2|}-1)]$ and the weight enumerator*

$$1+\frac{|T_3|+|R_3|}{|T_0|}z^{2p^{|A_1|-1}(p^{|A_2|}-1)}+\frac{|R_1|+|R_2|}{T_0}z^{p^{|A_1|-1}(2p^{|A_2|}-1)}+\frac{|R_0|}{|T_0|}z^{2p^{|A_1|+|A_2|-1}},$$

where T_i $(i=0,3)$ and R_i $(i=0,1,2,3)$ are given in Lemmas 2 and 3. Furthermore, $\phi(\mathcal{C}_{K_1})$ is optimal with respect to the Griesmer bound.

Proof. By the Gray map, we can get the weight distribution of $\phi(\mathcal{C}_{K_1})$ from the Lee weight distribution of $\mathcal{C}_{K_1}$ in Corollary 5. We now show that $\phi(\mathcal{C}_{K_1})$ is optimal with respect to the Griesmer bound. Note that $\phi(\mathcal{C}_{K_1})$ is a p-ary code with parameters $[\frac{2p^{|A_1|}(p^{|A_2|}-1)}{p-1}, |A_1|+|A_2|, 2p^{|A_1|-1}(p^{|A_2|}-1)]$. Then we have

$$\begin{aligned}
&\sum_{i=0}^{|A_1|+|A_2|-1}\left\lceil\frac{2p^{|A_1|-1}(p^{|A_2|}-1)}{p^i}\right\rceil\\
&=\sum_{i=0}^{|A_1|-1}\left\lceil\frac{2p^{|A_1|-1}(p^{|A_2|}-1)}{p^i}\right\rceil+\sum_{i=|A_1|}^{|A_1|+|A_2|-1}\left\lceil\frac{2p^{|A_1|-1}(p^{|A_2|}-1)}{p^i}\right\rceil\\
&=2p^{|A_1|-1}(p^{|A_2|}-1)+2p^{|A_1|-2}(p^{|A_2|}-1)+\cdots+2(p^{|A_2|}-1)+2p^{|A_2|-1}+2p^{|A_2|-2}+\cdots+2\\
&=2\frac{p^{|A_1|}-1}{p-1}(p^{|A_2|}-1)+2\frac{p^{|A_2|}-1}{p-1}=2p^{|A_1|}(\frac{p^{|A_2|}-1}{p-1}).
\end{aligned}$$

Hence, $\phi(\mathcal{C}_{K_1})$ is optimal with respect to the Griesmer bound. $\square$

Corollary 10. *Let the symbols be the same as those in Corollary 6. Then* $\phi(\mathcal{C}_{K_1})$ *is a two-weight code over* $\mathbb{F}_p$ *with parameters* $[\frac{2p^{|A_1|}(p^{|A_1|}-1)}{p-1}, 2|A_1|, 2p^{|A_1|-1}(p^{|A_1|}-1)]$ *and the weight enumerator*

$$1+\frac{|T_3|+|R_3|}{|T_0|}z^{2p^{|A_1|-1}(p^{|A_1|}-1)}+\frac{|R_1|+|R_2|}{|T_0|}z^{p^{|A_1|-1}(2p^{|A_1|}-1)},$$

where T_i $(i=0,3)$ *and* R_i $(i=1,2,3)$ *are given in Lemmas 2 and 3. Furthermore,* $\phi(\mathcal{C}_{K_1})$ *is optimal with respect to the Griesmer bound.*

By the Gray map, we obtain the following class of two-weight optimal linear codes over $\mathbb{F}_p$.

Theorem 11. *Let the symbols be the same as those in Corollary 8. Then* $\phi(\mathcal{C}_{K_2})$ *is a two-weight linear code over* $\mathbb{F}_p$ *with parameters* $[\frac{2p^{|A_2|}(p^{|A_1|}-1)}{p-1}, |A_1|+|A_2|, 2p^{|A_2|-1}(p^{|A_1|}-1)]$ *and the weight enumerator*

$$1+\frac{|R_0|+|R_1|+|R_2|+|R_3|}{|T_0|}z^{2p^{|A_2|-1}(p^{|A_1|}-1)}+\frac{|T_3|}{|T_0|}z^{2p^{|A_1|+|A_2|-1}},$$

where T_i $(i=0,3)$ *and* R_i $(i=0,1,2,3)$ *are given in Lemmas 2 and 3. Furthermore,* $\phi(\mathcal{C}_{K_2})$ *is optimal with respect to the Griesmer bound.*

Remark 12. (i) It should be noted that these two codes in Corollary 10 and Theorem 11 have different weight distribution. Hence, they are inequivalent to each other even if they have the same parameters when $A_1 = A_2$. (ii) The parameters and weight distributions of our optimal codes are closely related to subsets A_1 and A_2 of $[m]$. It is believed that our construction can produce some optimal codes with flexible and new parameters. Furthermore, we did not find optimal codes equivalent to our optimal linear codes in these references [6, 9, 7, 11, 14, 18].

5 Concluding remarks

In this paper, we presented two classes of linear codes $\mathcal{C}_{K_1}$ and $\mathcal{C}_{K_2}$ over $\mathbb{F}_p + u\mathbb{F}_p$ from projective spaces, where $K_1 = V_{A_1} + uP_{A_2}$ and $K_2 = P_{A_1} + uV_{A_2}$. By investigating the property of the vectors of $\mathbb{F}_p^m$, we determined the Lee weight distribution of these linear codes. We obtained some p-ary optimal linear codes from the Gray image of the codes $\mathcal{C}_{K_1}$ and $\mathcal{C}_{K_2}$.

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