Balanced Binary Sequences with Favourable Autocorrelation from Cyclic Relative Difference Sets

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Abstract

In this paper, we propose a balanced binary sequence of even period $2u$ for some even values of u with 5-level autocorrelation from a cyclic relative difference set with parameters $(u, 2, u-1, \frac{u}{2}-1)$. We further identify its half-period as those having an optimal odd autocorrelation. Various relations of these with some previous constructions are discussed.

1 Introduction

Binary sequences with good autocorrelation properties are advantageous for synchronization in various communication systems [6, 7]. There have been a lot of results on the constructions of sequences (binary, almost binary, ternary, non-binary, polyphase, almost polyphase, etc) for the last half century or more for improved performance of various communications systems. Most of the sequences in this paper are over the binary alphabet $\mathbb{F}_2 = \{0, 1\}$ but the correlation is computed over $\mathbb{C}$ with the correspondence

$$x \in \mathbb{F}_2 = \{0, 1\} \leftrightarrow (-1)^x \in \mathbb{C}.$$

When we are given a binary sequence $\mathbf{s} = \{s(i) \in \mathbb{F}_2 | i = 0, 1, ..., L-1\}$ of length L, we may consider its (usual) periodic expansion for computing its periodic autocorrelation. In that sense, we will use the term 'length' and 'period' of a binary sequence interchangeably. Then, the periodic autocorrelation of $\mathbf{s}$ at shift τ, denoted by $C_{\mathbf{s}}(\tau)$, is given by

$$C_{\mathbf{s}}(\tau) = \sum_{i=0}^{L-1} (-1)^{s(i)+s(i+\tau)}, \tag{1}$$

where $i + \tau$ is computed mod L. There is an alternative way of expanding the sequence $\mathbf{s}$ of length L periodically. Let $\mathbf{s}'$ be a complement of $\mathbf{s}$ defined by

$$s'(i) = s(i) + 1, \quad i = 0, 1, ..., L-1.$$

Then, the alternative periodic expansion, called odd-periodic expansion, is to repeat $\mathbf{s}$ in concatenation with $\mathbf{s}'$ of total length $2L$. The autocorrelation of $\mathbf{s}$ with this type of expansion is called the odd autocorrelation of $\mathbf{s}$. The odd autocorrelation at shift τ with $0 \le \tau < L$, denoted by $C_{\mathbf{s}}^{odd}(\tau)$, is given by

$$C_{\mathbf{s}}^{odd}(\tau) = \sum_{i=0}^{L-\tau-1} (-1)^{s(i)+s(i+\tau)} + \sum_{i=L-\tau}^{L-1} (-1)^{s(i)+s(i+\tau)+1}, \tag{2}$$

where $i+\tau$ is computed mod L. In fact, $C_{\mathbf{s}}(\tau)$ can be said to be an even autocorrelation.

The binary sequence $\mathbf{s}$ of even period L is said to have optimal autocorrelation [7,9] if

$$C_{\mathbf{s}}(\tau) = \begin{cases} 0 \text{ or } -4 & \text{if } L \equiv 0 \ (mod\ 4) \\ 2 \text{ or } -2 & \text{if } L \equiv 2 \ (mod\ 4). \end{cases}$$

for any $\tau \neq 0$. A lot of binary sequences of even period L above with (non-perfect) optimal autocorrelation have been constructed [4,5,15,19,21,26], which would be best possible in terms of its periodic autocorrelation, since the perfect binary sequence is known only for $L = 4$ [7].

Instead of suppressing all the out-of-phase autocorrelation magnitudes, one started to consider having all-zero out-of-phase autocorrelations except for one non-zero value at some $\tau \neq 0$. It is called almost perfect sequences [24] and investigated immediately by many others [10,16–18] and further generalized into some non-binary zero-correlation zone sequences [20,22,23]. We would like to mention that [17,18] established some fundamental relation between cyclic relative difference sets and almost perfect binary sequences, which is very much similar to the relation between cyclic difference sets and binary sequences with two-level autocorrelation. For example, binary NTU sequences [16] are closely related with binary sequences from a cyclic relative difference set [10], which will be mentioned at the very last Remark of this paper. In fact, the main result of this paper is a full generalization of [10].

In search of sequences with better autocorrelation property, on the other hand, almost binary sequences or ternary sequences have been studied a lot [12,13,17]. Here, an almost binary sequence is a ternary sequence over $\{0, +1, -1\}$ but the term 0 occurs only once or a few times. Such sequences with 'perfect' autocorrelation have been found, for example, in [13].

Some reviews on the binary and almost binary sequences with good odd autocorrelation follows now. In [14], especially in Section IV. A. 4 there, a binary sequence of even period is said to have an odd optimal autocorrelation if the magnitude of out of phase odd autocorrelation is no larger than 2. The binary sequences with low or optimal odd periodic autocorrelation have also been proposed a lot [12–14,16,25].

In this paper, we propose (Theorem 3) a balanced binary sequence of even length $2u$ for some integer u with 5-level autocorrelation from a cyclic relative difference set. The out-of-phase autocorrelation magnitudes are all zero except for three indices at which the value is either $2u$ ($\tau = u$, once) or 4 (at some $\tau \neq 0, u$ twice). We observe the half period of this sequence and found that it has optimal odd autocorrelation (Theorem 4). Furthermore, we explain some of the known constructions for sequences with good (even or odd) autocorrelation are closely related with two main results of this paper using an relative difference set (RDS).

This paper is organized as follows, Section 2 introduces some preliminaries. Section 3 discusses two main results of this paper. Section 4 explains the relation between our constructions and other known constructions, especially in [12, 16]. Section 5 concludes this paper with a conjecture on the binary sequences of even period with optimal odd autocorrelation.

2 Preliminaries

2.1 Notation

We will fix the following notation throughout the paper.

- $\mathbb{Z}$ is the set of integers and $\mathbb{Z}_L$ is the integers mod L.
- $\mathbb{C}$ is the set of complex numbers and $\mathbb{F}_q$ is the finite field of size q.
- Given a binary sequence $\mathbf{s} = \{s(i) \in \mathbb{F}_2 | i = 0, 1, ..., L-1\}$ of length L, the periodic autocorrelation of $\mathbf{s}$ at shift τ, denoted by $C_{\mathbf{s}}(\tau)$, is given by (1) and the odd autocorrelation $C_{\mathbf{s}}^{odd}(\tau)$ is given by (2), both in the beginning of Introduction.
- For a subset X of $\mathbb{Z}_L$ and an element $\tau \in \mathbb{Z}_L$, we define

  $$\Delta_X(\tau) \triangleq |(\tau + X) \cap X|,$$

  where $\tau + X = \{\tau + x | x \in X\}$. Note that

  $$\Delta_X(\tau) = \Delta_X(-\tau)$$

  for any subset X and any τ.

2.2 Relative Difference Set

Definition 1 (Relative Difference Sets [2, 8, 17])**.** Let u,v,k, λ be positive integers. A (u, v, k, λ) relative difference set (RDS) D in the (additive) cyclic group $\mathbb{Z}_{uv}$ relative to its subgroup $(u) = u\mathbb{Z}_{uv}$ is a k-subset $\{d_1, d_2, \ldots, d_k\} \subset \mathbb{Z}_{uv}$, satisfying the following condition:

$$\Delta_D(d) = \begin{cases} \lambda, & d \in \mathbb{Z}_{uv} \backslash u\mathbb{Z}_{uv} \\ k, & d = 0 \\ 0, & \text{otherwise,} \end{cases} \tag{3}$$

for any $d \in \mathbb{Z}_{uv}$. Throughout this paper, we call this a (u, v, k, λ) RDS without refering to the cyclic group $\mathbb{Z}_{uv}$ and its subgroup $(u) = u\mathbb{Z}_{uv}$.

It is well-known that a (u, k, λ)-cyclic difference set (CDS) in $\mathbb{Z}_u$ is a $(u, v = 1, k, \lambda)$-RDS in $\mathbb{Z}_u$ (relative to its trivial subgroup $\{0\}$). We are mostly interested in the case where

$v = 2$ [2] and $k = u-1$ so that the parameters become $(u, v = 2, k = u-1, \lambda = \frac{u}{2}-1)$, since the existence of a cyclic (u, v, k, λ)-RDS implies the relation

$$k(k-1) = \lambda v(u-1).$$

This set of parameters further implies that u itself must be even. The following provides an equal-size partition of $\mathbb{Z}_{2u}$ so that a binary sequence can be constructed from such RDS D.

Proposition 2. *Let D be a $(u, 2, u-1, \frac{u}{2}-1)$-RDS. Then, $\mathbb{Z}_{2u}$ can be decomposed into the following disjoint union:*

$$\mathbb{Z}_{2u} = D \cup (u+D) \cup \{z\} \cup \{z+u\},$$

for some z.

Proof. By (3), $\Delta_D(u) = 0$. Therefore,

$$D \cap (u+D) = \emptyset.$$

Therefore,

$$|\mathbb{Z}_{2u} \backslash (D \cup (u+D))| = 2u - 2k = 2.$$

Therefore, $\mathbb{Z}_{2u} \backslash (D \cup (u+D))$ is non-empty. Let z be a member. If $z+u \in D$, then $z = z+u+u \in u+D$. If $z+u \in u+D$, then $z \in D$. Therefore, we also have $z+u \in \mathbb{Z}_{2u} \backslash (D \cup (u+D))$. □

3 Binary Sequences with Favourable Autocorrelation from RDS

In this section, we propose a balanced binary sequence $\mathbf{s} = \{s(i) | i \in \mathbb{Z}_{2u}\}$ of length $2u$ with 5-level autocorrelation from a $(u, 2, u-1, \frac{u}{2}-1)$-RDS and discuss its two variations with (somewhat) better correlation property: the first is its half period portion of length u which is still balanced with 4-level optimal odd autocorrelation; the second is its one-bit-changed version so that the result is almost balanced but with 3-level autocorrelation so that it is almost perfect.

Theorem 3 (Main Construction)**.** *Let D be a $(u, v = 2, k = u-1, \lambda = \frac{u}{2}-1)$-RDS. Let $z \in \mathbb{Z}_{2u}$ so that $\mathbb{Z}_{2u}$ is partitioned as in Proposition 2:*

$$\mathbb{Z}_{2u} = D \cup (u+D) \cup \{z\} \cup \{z+u\}. \tag{4}$$

Define a binary sequence $\mathbf{s} = \{s(i) \,|\, i = 0, 1, ..., 2u-1\}$ as follows:

$$s(i) = \begin{cases} 0, & i \in D \cup \{z\} \\ 1, & i \in (u+D) \cup \{u+z\}. \end{cases} \tag{5}$$

Then, the periodic (even) autocorrelation of $\mathbf{s}$ becomes:

$$C_{\mathbf{s}}(\tau) = \begin{cases} 2u, & \tau = 0 \\ -2u, & \tau = u \\ 4, & \tau, -\tau \in -z+u+D \\ -4, & \tau, -\tau \in -z+D \\ 0, & \textit{otherwise}. \end{cases} \tag{6}$$

Proof. From the relation (4) and definition of the sequence $\mathbf{s}$ in (5), the autocorrelation of $\mathbf{s}$ at shift τ is calculated as follows:

$$\begin{aligned} C_{\mathbf{s}}(\tau) &= \sum_{i\in\mathbb{Z}_{2u}} (-1)^{s(i)+s(i+\tau)} \\ &= \sum_{i\in D}(-1)^{s(i)+s(i+\tau)} + \sum_{i\in u+D}(-1)^{s(i)+s(i+\tau)} + (-1)^{s(z)+s(z+\tau)} + (-1)^{s(z+u)+s(z+u+\tau)}. \end{aligned} \tag{7}$$

The first sum in (7) can be split into the following three cases: (a) $i \in D$ and $i+\tau \in D$ so that $s(i)+s(i+\tau) = 0$, (b) $i \in D$ and $i+\tau \in u+D$ so that $s(i)+s(i+\tau) = 1$ and (c) $i \in D$ and $i+\tau \in \{z, u+z\}$ so that $s(i)+s(i+\tau) = s(i+\tau)$ which is 1 if $i+\tau = z$ and 0 if $i+\tau = u+z$. Then the case (a) becomes

$$\sum_{\substack{i\,\in\,D\\ i\,+\,\tau\,\in\,D}} (+1) = |D \cap (-\tau + D)| = |\tau + D \cap D| = \Delta_D(\tau).$$

Similarly, the case (b) becomes

$$\sum_{\substack{i\,\in\,D\\ i\,+\,\tau\,\in\,u\,+\,D}} (-1) = -|D \cap (-\tau + u + D)| = -\Delta_D(u-\tau).$$

Similarly, the second sum can be split into the following three cases: (a) $i \in u+D$ and $i+\tau \in D$ so that $s(i)+s(i+\tau) = 1$, (b) $i \in u+D$ and $i+\tau \in u+D$ $s(i)+s(i+\tau) = 0$, and (b) $i \in u+D$ and $i+\tau \in \{z, u+z\}$. Then, similar to the first two cases of the first sum, the cases (a) and (b) become:

$$\sum_{\substack{i\,\in\,u\,+\,D\\ i\,+\,\tau\,\in\,D}} (-1) = -|(u+D) \cap (-\tau + D)| = -\Delta_D(u+\tau)$$

and

$$\sum_{\substack{i\,\in\,u\,+\,D\\ i\,+\,\tau\,\in\,u\,+\,D}} (+1) = |u + D \cap (u - \tau + D)| = \Delta_{u+D}(\tau) = \Delta_D(\tau).$$

Therefore, the autocorrelation of $\mathbf{s}$ at shift τ becomes:

$$\begin{aligned} C_{\mathbf{s}}(\tau) = {} & 2\Delta_D(\tau) - \Delta_D(u+\tau) - \Delta_D(u-\tau) \\ & + \sum_{\substack{i\,\in\,D\,\cup\,(u\,+\,D)\\ i\,+\,\tau\,=\,z,\,z\,+\,u}} (-1)^{s(i)+s(i+\tau)} + (-1)^{s(z+\tau)} + (-1)^{s(z+u+\tau)+1} \end{aligned} \tag{8}$$

For the cases of either $\tau = 0$ or $\tau = u$, recall that

$$(D \cup (u+D)) \cap \{z, z+u\} = \emptyset.$$

Therefore, the middle sum in (8) vanishes in this case. When $\tau = 0$, we have

$$C_{\mathbf{s}}(0) = 2\Delta_D(0) + 2(-1)^0 = 2k + 2 = 2u,$$

which is the length of $\mathbf{s}$. Similarly, when $\tau = u$,

$$C_{\mathbf{s}}(u) = -2\Delta_D(0) + 2(-1)^1 = -2k - 2 = -2u.$$

Now, consider the case when $\tau \neq 0, u$. Then, the first line of $C_{\mathbf{s}}(u)$ in (8) becomes

$$2\Delta_D(\tau) - \Delta_D(u+\tau) - \Delta_D(u-\tau) = 0,$$

since $\Delta_D(\tau) = \Delta_D(u \pm \tau) = \lambda = \frac{u}{2} - 1$. Now, the middle sum in (8) becomes the sum of only two terms

$$\begin{aligned} \sum_{\substack{i \,\in\, D \cup (u+D) \\ i+\tau \,=\, z,\, z+u}} (-1)^{s(i)+s(i+\tau)} &= (-1)^{s(z-\tau)+s(z)} + (-1)^{s(z+u-\tau)+s(z+u)} \\ &= (-1)^{s(z-\tau)} + (-1)^{s(z+u-\tau)+1}, \end{aligned}$$

since, in this case,

$$\{z-\tau, z+u-\tau\} \subset (D \cup (u+D)),$$

and hence, there are only two terms corresponding to $i = z - \tau$ and $i = z + u - \tau$. Therefore, (8) finally becomes

$$C_{\mathbf{s}}(\tau) = (-1)^{s(z-\tau)} + (-1)^{s(z+u-\tau)+1} + (-1)^{s(z+\tau)} + (-1)^{s(z+u+\tau)+1}.$$

Therefore, finally, when $\tau \neq 0, u$,

$$C_{\mathbf{s}}(\tau) = \begin{cases} -4, & z-\tau, z+\tau \in D \\ 4, & z-\tau, z+\tau \in u+D \\ 0, & \text{otherwise.} \end{cases}$$

This proves the theorem. □

The binary sequence $\mathbf{s} = \{s(i) \,|\, i = 0, 1, ..., 2u-1\}$ constructed from above theorem is balanced since

$$|D \cup \{z\}| = |(u+D) \cup \{u+z\}|.$$

Note that $(u+D) \cup \{u+z\}$ can be represented also as $u+(D \cup \{z\})$. This explains its some special periodic property. When it is (cyclically) shifted by half the period, then the result is a complement of the original sequence. Therefore, its half period portion of length u is expanded odd-periodically, the result is the same as the (even) periodic expansion of the original sequence $\mathbf{s}$ of length $2u$. The proof of the following is straightforward from Theorem 3 and the discussions so far.

Theorem 4. *Let* $\mathbf{s}$ *be the binary sequence of period* $2u$ *constructed from Theorem 3 with some* $(u, 2, u-1, \frac{u}{2}-1)$*-RDS and an integer* z *satisfying the relation* (4)*. Define the binary sequence* $\mathbf{t}$ *of period* u *as follows, for* $i = 0, 1, ..., u-1$,

$$t(i) = s(i).$$

Then the odd autocorrelation of $\mathbf{t}$ *at shift* τ *with* $0 \le \tau < u$ *becomes:*

$$C_{\mathbf{t}}^{odd}(\tau) = \begin{cases} u, & \tau = 0 \\ 2, & \tau, -\tau \in -z + u + D \\ -2, & \tau, -\tau \in -z + D \\ 0, & \textit{otherwise.} \end{cases}$$

Remark 5. The binary sequence $\mathbf{t}$ constructed from Theorem 4 is optimal in the sense of minimizing the maximum magnitude of out of phase odd autocorrelation described in Section IV. A. 4 of [14], as mentioned in Introduction.

4 The relation between our construction and other known construction

In this section, we discuss the relation between our construction in Theorems 3 and 4 from an RDS of parameters $(u, 2, u-1, \frac{u}{2}-1)$ and other previous constructions, for example, those in [12, 16] which were given without mentioning any RDS structure. In fact, an example of an RDS can be constructed by using some finite field structures [8], and the construction of a binary sequence can be stated without mentioning any RDS structures and by simply using a subset D of the integers mod uv. The sequences in [12, 14, 16] are in fact constructed in this way without mentioning any RDS structures.

First, we will give a brief explanation of those from [12] and [14]. We use the following additional notations in this section.

- q is an odd prime power.
- $\mathbb{F}_{q^2}$ is the finite field with q^2 elements.
- α is a primitive element of $\mathbb{F}_{q^2}$.
- $\beta \triangleq \alpha^{q+1}$ is a primitive element of $\mathbb{F}_q$.
- For any non-zero $b \in \mathbb{F}_q$, we use, for $0 \le j < q-1$,

$$\log_\beta(b) = j \quad \text{if and only if} \quad b = \beta^j.$$

- $\mathrm{Tr}(a) \in \mathbb{F}_q$ is the trace of $a \in \mathbb{F}_{q^2}$ defined by

$$\mathrm{Tr}(a) = a + a^q.$$

A construction of some binary sequences with optimal odd autocorrelation is given in [14], where the binary sequence is obtained from a ternary $\{0, \pm 1\}$ odd perfect sequence by replacing 0 with one of $\{+1, -1\}$. Here, an odd perfect sequence is defined as those having all the out-of-phase odd autocorrelation values equal to zero. The resulting binary sequence is not odd perfect, but has optimal odd autocorrelation as those from Theorem 4. In fact, it is also given in [12] as follows:

Definition 6 (Modified Krengel Sequences [12, 14])**.** The binary sequence $\mathbf{x} = \{x(i) \mid i = 0, 1, ..., q\}$ of length $(q+1)$ is defined as

$$x(i) = \begin{cases} 1, & \log_\beta(\mathrm{Tr}(\alpha^i)) \text{ is odd} \\ 0, & \mathrm{Tr}(\alpha^i) = 0 \text{ or else } \log_\beta(\mathrm{Tr}(\alpha^i)) \text{ is even.} \end{cases}$$

We just note the values of i above so that $x(i) = 0$ except for the case $\mathrm{Tr}(\alpha^i) = 0$. It is not difficult to observe that the set of these values of i forms an RDS of parameters $(q+1, 2, q, \frac{q-1}{2})$ [8] in the structure of $\mathbb{F}_q$ and its extension $\mathbb{F}_{q^2}$. That is, claim that

$$D \triangleq \{i \in \mathbb{Z}_{2(q+1)} \mid \log_\beta(\mathrm{Tr}(\alpha^i)) \text{ is even}\}. \tag{9}$$

is a $(q+1, 2, q, \frac{q-1}{2})$-RDS in $\mathbb{Z}_{2(q+1)}$ relative to its subgroup $(q+1)\mathbb{Z}_{2(q+1)}$. For the proof, see Cor. 5.1.1 in [8] or Sec. 2.2 in [17].

Remark 7. The modified Krengel sequence $\mathbf{x}$ of length $q+1$ is the same as those constructed from Theorem 4 with the RDS D in (9).

Nogami, Tada and Uehara [16] proposed some binary sequences as in the following definition for some specific parameters.

Definition 8 (Binary NTU Sequences [16])**.** The binary sequence $\mathbf{y} = \{y(i) \mid i = 0, 1, ..., 2(q+1) - 1\}$ of length $2(q+1)$ is defined in [16] as

$$y(i) = \begin{cases} 1, & \log_\beta(\mathrm{Tr}(\alpha^i)) \text{ is odd} \\ 0, & \mathrm{Tr}(\alpha^i) = 0 \text{ or else } \log_\beta(\mathrm{Tr}(\alpha^i)) \text{ is even.} \end{cases}$$

We call this sequence the binary NTU sequence.

It is interesting that the only difference between this sequence and the modified Krengel sequence is the range of i which defines the sequence. It can be also seen that only one term is changed from the construction in Theorem 3 at index $u+z$ so that the result is no longer balanced (we may call this almost balanced) but with better autocorrelation property which is only 3-level. In fact, this binary sequence has been defined to be the almost perfect sequences [18, 24] and Pott and Bradley proved [18] that they are equivalent to some $(u, 2, u-1, \lambda)$-RDS in $\mathbb{Z}_{2u}$ relative to its subgroup $u\mathbb{Z}_{2u}$.

Remark 9. These are all equivalent to an almost perfect binary sequence of period $2(q+1)$:

1. A cyclic relative difference set with parameter $(q+1, 2, q, (q-1)/2)$ in $\mathbb{Z}_{2u}$ relative to its subgroup $u\mathbb{Z}_{2u}$.
2. Binary NTU sequence of length $2(q+1)$.
3. Modified binary sequence of length $2u$ obtained by chaning one term at index $z+u$ from those constructed in Theorem 3 with $u = q+1$.

5 Concluding Remarks

We propose a construction of a balanced binary sequence of even period with 5-level autocorrelation in Theorem 3 from an RDS of parameters $(u, 2, u-1, \frac{u}{2}-1)$, which is slightly different from the almost perfect binary sequences from this RDS as mentioned in Remark 9. We further identify its half-period in Theorem 4 as those having optimal odd autocorrelation. All of the sequences of our constructions are derived from any $(u, v=2, k=u-1, \lambda=\frac{u}{2}-1)$-RDS when $u=q+1$ for an odd prime power q [8,17].

We find out that the binary sequence of period with optimal odd autocorrelation derived from [12,14] can be constructed from Theorem 4 with the $(u=q+1, 2, q, \frac{q-1}{2})$-RDS constructed from [8,17].

All the known parameters of a $(u, v=2, k=u-1, \lambda)$-RDS are $(u=q+1, v=2, k=q, \lambda=\frac{q-1}{2})$ for some odd prime power q. There exist several non-equivalence classes of $(u=q+1, v=2, k=q, \lambda=\frac{q-1}{2})$-RDS [1,3,11,17]. Indeed, our construction in Theorem 4 give some binary sequences of period $q+1$ with optimal odd autocorrelation. We conjecture that it is the only way of getting a binary sequence of period $q+1$ with optimal odd autocorrelation for some odd prime power q.

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