Multiple Spectrally Null Constrained Complete Complementary Codes of Various Lengths Over Small Alphabet

Rajen Kumar*, Palash Sarkar†, Prashant Kumar Srivastava‡, Sudhan Majhi §

Abstract

Complete complementary codes (CCCs) are highly valuable in the fields of information security, radar and communication. The spectrally null constrained (SNC) problem arises in radar and modern communication systems due to the reservation or prohibition of specific spectrums from transmission. The literature on SNC-CCCs is somewhat limited in comparison to the literature on traditional CCCs. The main objective of this paper is to discover several configurations of SNC-CCCs that possess more flexibility in their parameters. The proposed construction utilised the existing CCCs and mutually orthogonal sequences. The proposed construction can cover almost all lengths with the smallest alphabets $\{-1, 0, 1\}$. Further, the idea of SNC-CCC is extended to multiple SNC-CCCs with an inter-set zero cross-correlation zone (ZCCZ). Through the propose construction, we could control the cross-correlation magnitude outside the ZCCZ. Consequently, the resulting codes possess both aperiodic and periodic inter-set ZCCZ and feature a low magnitude of cross-correlation value outside the ZCCZ.

A Golay complementary pair (GCP) indicates a pair of sequences whose sum of aperiodic auto-correlation functions (AACFs) results in zero at nonzero time shifts. Golay uncovered a sequence pair that can be used during the research of multislit spectroscopy [1, 2]. GCPs are comprehensively utilised in engineering applications, particularly in radar systems and communication systems. These applications include channel estimation [3, 4], design of the physical uplink control channel [5], non-orthogonal multiple access [6], radar

*Rajen Kumar is with the Department of Mathematics, Indian Institute of Technology Patna, India, email:rajen_2021ma04@iitp.ac.in. The work of Rajen Kumar was supported in part by the CSIR, India, under award letter 09/1023(0034)/2019-EMR-I.

†Palash Sarkar is with the Department of Informatics, Selmer Center, University of Bergen, Norway, email:palash.sarkar@uib.no. The work of Palash Sarkar was supported by the Research Council of Norway under Grant 311646/O70.

‡Prashant Kumar Srivastava is with the Department of Mathematics, Indian Institute of Technology Patna, India, email:pksri@iitp.ac.in.

§Sudhan Majhi is with the Department of Electrical Communication and Engineering, Indian Institute of Science Bangalore, India, email:smajhi@iisc.ac.in. The work of Sudhan Majhi was supported by the SERB, Govt. of India, under grant no. CRG/2022/000529 and EEQ/2022/001018.

waveform design [7], control of peak-to-average power ratio for multi-carrier communication systems [8], and more. Tseng and Liu presented the idea of a Golay complementary set (GCS) comprising more than two sequences. The sum of the AACFs for all sequences is zero except at zero time shift [9]. Due to their similar characteristics to GCPs, GCSs are also utilised in several communication and radar systems [10, 11]. Furthermore, GCS has the added benefit of a greater code rate compared to GCP, in addition to its variable length advantage [12–14].

A mutually orthogonal Golay complementary set (MOGCS) is a collection of K GCSs. Each GCS in the MOGCS has M sequences, each of length L. Additionally, the cross-correlation function between distinct GCSs is zero. A MOGCS is referred to as a complete complementary code (CCC) when K is equal to M [15]. For implementing multi-antenna or multi-user systems, it is important to consider the cross-correlation characteristics across sets of sequences. This is particularly relevant for systems such as CCC based code division multiple access (CDMA) and multi-input multi-output (MIMO) radar [16–19]. The idea of CCC extended to multiple CCC with an inter-set zero cross-correlation zone (ZCCZ) [20, 21], which is similar to the Z complementary code set (ZCCS). The idea of CCC also extended to multiple CCC with inter-set low cross-correlation, which is similar to the Quasi complementary code set (QCCS).

In systems that use orthogonal frequency division multiplexing (OFDM), some sub-carriers are designated as reserved and are not allowed to transmit signals [22]. For instance, the direct current sub-carrier is specifically allocated, known as spectrally null constrained (SNC), to prevent any discrepancies in the D/A and A/D converters during radio frequency transmission [23]. The increasing need for OFDM or multi-carrier CDMA sequences with spectrum null constraints, also known as non-contiguous sequences, is primarily motivated by their potential applications in cognitive radio (CR) communications [24]. Transmission on sub-carriers not used by primary users constrains secondary users in OFDM-based CR transmissions. The Third-Generation Partnership Project Long-Term Evolution enhanced licenced-assisted access and the New Radio in Unlicenced (NR-U) implemented interlaced transmission, with the null locations of the SNC sequences being regularly distributed (although the nulls in NR-U are unevenly spaced). It is also important to think about the spectral null constraint when using the CCC as omnidirectional precoding for a rectangular array that is not all the same size. In the IEEE P802.15.4z standard, the average power permitted in ultra-wide-band is very low. Therefore, the sequence design will consider the inclusion of null to decrease the average power. To summarise, several situations in sequence design require the use of null constraints.

Only a few of the conventional GCSs and CCCs take into account this limitation, which has been addressed in [5, 23, 25–29]. Sahin and Yang extended the conventional GCPs to address the SNC problem, as described in [5] and [26]. In [23], Zhou *et al.* sequentially built the SNC-MOGCSs/SNC-GCSs using an iterative approach. They used two sequences extracted from a GCP as the initial seed sequence and then introduced a certain amount of zeros into these two sequences. As a result, new sequences were obtained with a zero correlation zone. Hence, a challenging issue arises regarding the methodology for constructing SNC-CCC. Shen *et al.* proposed a method for constructing SNC-CCC using extended Boolean functions and graphs [28]. However, the parameters are only in the power of p when elements of code are considered from the qth root of unity and zero, for $p \mid q$ and $p \geq 2$. In machine-type communication, alphabet size plays a major role and must be minimum [30]. However, there are gaps in the SNC-CCC proposed in [28] in terms of lengths and alphabet sizes. For example, when the alphabets are $-1, 1, 0$, set size, code size and length are restricted to in the form of the power-of-two. We are strongly motivated to include a greater range of parameters for SNC-CCC in comparison to existing literature. The proposed construction not only provides SNC-CCCs with new parameters but also provides flexibility in the alphabet and the length of the constituent sequences. It may be noted that codes are referred to as CCC, when it is a traditional CCC, and codes with nulls are referred to as SNC-CCC.

In the proposed construction, we use existing CCCs and mutually orthogonal sequences (MOSs) as seeds. By performing the concatenation operation in a specific way, as described in Section 2, we obtain multiple SNC-CCCs over a small alphabet. It may be noted that our smallest alphabet is $\{-1, 0, 1\}$, on which the proposed construction is capable of generating almost all possible lengths. The proposed multiple SNC-CCCs also have a ZCCZ property with respect to both periodic and aperiodic correlation. With these properties, the obtained code set is useful for multi-cell MC-CDMA systems, where the users inside a cell enjoy interference-free communication due to the ideal correlation property of a SNC-CCC and the users from two different cells also enjoy interference-free communication within the ZCCZ. Our study also revealed that we can control non-zero inter-set cross-correlation magnitude values outside the ZCCZ. We consider this an opportunity to minimise the upper bound for inter-set cross-correlation magnitude values of the proposed multiple SNC-CCCs.

We structure the subsequent sections of the paper as follows: Section 1 establishes appropriate notations and definitions. Section 2 introduces new constructions for SNC-CCC and multiple SNC-CCC and provides an example to illustrate this. Further, we explain the ZCCZ width of the multiple SNC-CCC and conclude with the low inter-set cross-correlation value. In Section 3, a comparison has been given with existing literature. Based on the proposed work, we have highlighted three problems that we may consider as our future work in Section 4. The paper is concluded in Section 5.

1 Preliminaries

Before anything starts, let us specify the notation and definitions that will be utilised consistently throughout this paper.

Definition 1. Let $\mathbf{a} = (a_1, a_2, \ldots, a_L)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_L)$ be two complex-valued sequences of length L and τ be an integer. Define

$$\mathcal{C}(\mathbf{a}, \mathbf{b})(\tau) = \begin{cases} \sum_{i=1}^{L-\tau} a_{i+\tau} b_i^*, & 0 \leq \tau < L, \\ \sum_{i=1}^{L+\tau} a_i b_{i-\tau}^*, & -L < \tau < 0, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

is called ACCF of $\mathbf{a}$ and $\mathbf{b}$ at time shift τ, where $(\cdot)^*$ represents complex conjugation. When $\mathbf{a} = \mathbf{b}$, $\mathcal{C}(\mathbf{a}, \mathbf{b})(\tau)$ is called AACF of $\mathbf{a}$ and is denoted by $\mathcal{C}(\mathbf{a})(\tau)$. Further, periodic cross-correlation function (PCCF) of $\mathbf{a}$ and $\mathbf{b}$ at time shift τ is defined as

$$\Theta(\mathbf{a},\mathbf{b})(\tau)=\mathcal{C}(\mathbf{a},\mathbf{b})(\tau)+\mathcal{C}(\mathbf{a},\mathbf{b})(\tau-L). \tag{2}$$

Definition 2. Let $\mathbf{C}=\{C_k:1\leq k\leq M\}$ be a set of M matrices (codes), each having order $M\times L$. And C_k is defined as

$$C_k=\begin{bmatrix}\mathbf{c}_1^k\\ \mathbf{c}_2^k\\ \vdots\\ \mathbf{c}_M^k\end{bmatrix}_{M\times L}, \tag{3}$$

where $\mathbf{c}_j^k(1\leq j\leq M,1\leq k\leq K)$ is the j-th row sequence of C_k.Then ACCF between C_{k_1} and C_{k_2} is defined by

$$\mathcal{C}\left(C_{k_1},C_{k_2}\right)(\tau)=\sum_{\nu=1}^{M}\mathcal{C}\left(\mathbf{c}_\nu^{k_1},\mathbf{c}_\nu^{k_2}\right)(\tau). \tag{4}$$

When $C_{k_1}=C_{k_2}$, $\mathcal{C}(C_{k_1},C_{k_2})(\tau)$ is called AACF of C_{k_1} and is denoted by $\mathcal{C}(C_{k_1})(\tau)$. Similarly, the PCCF of between C_{k_1} and C_{k_2} is defined by

$$\Theta\left(C_{k_1},C_{k_2}\right)(\tau)=\sum_{\nu=1}^{M}\Theta\left(\mathbf{c}_\nu^{k_1},\mathbf{c}_\nu^{k_2}\right)(\tau). \tag{5}$$

Definition 3. Let $\mathbf{a}=(a_1,a_2,\ldots,a_L)$ be any complex-valued sequence and $N=\{x\in\mathbb{N}:a_x=0\}$ is non-empty set, $\mathbf{a}$ is called a SNC sequence. A CCC is called an SNC-CCC if there is at least one SNC sequence in the CCC [28].

Definition 4. Let $\mathbf{C}=\{C_k:1\leq k\leq M\}$ be a set of M codes of order $M\times L$ and it follows

$$\mathcal{C}(C_{k_1},C_{k_2})(\tau)=\begin{cases}ML-\epsilon & k_1=k_2,\tau=0\\ 0 & \text{otherwise},\end{cases} \tag{6}$$

where, ϵ is the number of zeros in a code. When $\epsilon=0$, it is referred to as traditional aperiodic CCC and for $\epsilon\geq 1$, we refer to it as aperiodic SNC-CCC. It is trivial that aperiodic CCC also satisfies the ideal periodic correlation properties. Therefore, an aperiodic CCC can also be called a periodic CCC. To avoid possible confusion between the terms aperiodic and periodic CCC, we will exclusively use the term CCC throughout this paper.

Definition 5. Let $\mathfrak{C}=\{\mathbf{C}^j:1\leq j\leq P\}$ be a collection of P many (M,L)-CCCs, i.e., $\mathbf{C}^j=\left\{C_k^j:1\leq k\leq M\right\}$, where $1\leq j\leq P,P\geq 2$. If any two codes from different CCCs $\mathbf{C}^{j_1}$ and $\mathbf{C}^{j_2}$ with $1\leq j_1\neq j_2\leq P$ follows

$$\begin{aligned}&\mathcal{C}(C_{k_1}^{j_1},C_{k_2}^{j_2})(\tau)=0,\ |\tau|<Z_A,\\ &\qquad\delta_A=\max\left\{|\mathcal{C}(C_{k_1}^{j_1},C_{k_2}^{j_2})(\tau)|:j_1\neq j_2,1\leq k_1,k_2\leq M,Z\leq|\tau|\leq L-1\right\},\end{aligned} \tag{7}$$

where $1 \leq k_1, k_2 \leq M$, then we denote $\mathfrak{C}$ as aperiodic (P, M, L, Z_A, δ_A)-CCCs. Similarly,

$$\begin{aligned}\Theta(C_{k_1}^{j_1}, C_{k_2}^{j_2})(\tau) = 0,\ |\tau| < Z_P,\\ \delta_P = \max\left\{|\Theta(C_{k_1}^{j_1}, C_{k_2}^{j_1})(\tau)| : j_1 \neq j_2, 1 \leq k_1, k_2 \leq M, Z \leq |\tau| \leq L-1\right\},\end{aligned} \tag{8}$$

where $1 \leq k_1, k_2 \leq M$, then we denote $\mathfrak{C}$ as periodic (P, M, L, Z_P, δ_P)-CCCs.

Let $\mathbf{a}$ and $\mathbf{b}$ be two complex sequences of identical length and said to be orthogonal if the dot product $\langle \mathbf{a}, \mathbf{b} \rangle$ is equal to 0. We refer to the set of sequences as MOSs when the number of sequences exceeds two and the dot product of any two sequences is zero. A construction of P many MOSs with length P is suggested in [31].

2 Proposed Construction

In this section, we describe our main method of construction. First, we provide a new method, which involves the concatenation of zeros and matrices with some scalar multiplications. Scalars must be selected meticulously to ensure they do not impact the elements of matrices following multiplication.

Construction 6. *Let $C_1, C_2, \ldots, C_P$ be a set of $M \times L$ matrices and $\mathbf{b} = (b_1, b_2, \ldots, b_P)$ be a sequence of length P. For $K > P$, then we define*

$$\mathcal{R}^{\mathcal{P}(n)}(C_1, C_2, \ldots, C_P; \mathbf{b}) = \left[\mathbf{0}^{n_1} \parallel b_1 C_1 \parallel \mathbf{0}^{n_2} \parallel b_2 C_2 \parallel \mathbf{0}^{n_3} \parallel \cdots \parallel \mathbf{0}^{n_P} \parallel b_P C_P \parallel \mathbf{0}^{n_P+1}\right], \tag{9}$$

where, $\mathcal{P}(n) = (n_1, n_2, \ldots, n_{P+1})$, partition of n with $P+1$ non-negative integers, $n = n_1 + n_2 + \cdots + n_{P+1}$, $\mathbf{0}^{n_1}$ represents a zero matrix of size $M \times n_1$ and $\parallel$ represents concatenation of two matrices.

First we consider a (M, L)-CCC, $P \leq M$, such that $P \mid M$ and MOSs of length P. Now, for any positive integers n, we take a partition with $P+1$ non-negative integers. The partition of n decides the position and numbers of nulls in the proposed codes.

Theorem 7. *Let $\mathbf{C}$ be a (M, L)-CCC, $P \mid M$, $\mathbf{b}^1, \mathbf{b}^2, \ldots, \mathbf{b}^P$ be MOSs of length P. Now, define*

$$B_{\nu P+\mu} = \mathcal{R}^{\mathcal{P}(n)}(C_{\nu P+1}, C_{\nu P+2}, \ldots, C_{(\nu+1)P}; \mathbf{b}^{\mu}), \tag{10}$$

for $0 \leq \nu < \frac{M}{P}$, $1 \leq \mu \leq P$. Then $\mathbf{B} = \{B_1, B_2, \ldots, B_M\}$ is a $(M, PL+n)$ SNC-CCC.

Since the alphabets of CCC and MOSs are identical, the resulting SNC-CCCs likewise possess the same alphabets, with an additional zero. The position of nulls can be determined by the partition of n, which is employed in the construction.

Example 8. Let

$$C_1 = \begin{bmatrix} + + + \\ + + - \\ - + - \\ - + - \end{bmatrix}, C_2 = \begin{bmatrix} + - + \\ + + - \\ - - + \\ + + + \end{bmatrix}, C_3 = \begin{bmatrix} + - - \\ + + + \\ - + - \\ + - - \end{bmatrix} \text{ and } C_4 = \begin{bmatrix} + - - \\ + - + \\ + + + \\ - + + \end{bmatrix},$$

be a $(4,3)$-CCC from [32], where $+$ and $-$ represent 1 and -1, respectively. Now, assume $P = 2$, $\mathbf{b}^1 = (1,1)$, $\mathbf{b}^2 = (1,-1)$, $n_1 = 0, n_2 = 3$ and $n_3 = 0$. Then from **Theorem** 7,

$$\begin{aligned} B_1 &= \begin{bmatrix} ++{+}000+-+ \\ ++{-}000++- \\ ++{-}000--+ \\ -+{-}000+++ \end{bmatrix}, \qquad & B_2 &= \begin{bmatrix} ++{+}000-+- \\ ++{-}000--+ \\ ++{-}000++- \\ -+{-}000--- \end{bmatrix}, \\ B_3 &= \begin{bmatrix} +-{-}000+-- \\ ++{+}000+-+ \\ -+{-}000+++ \\ +-{-}000-++ \end{bmatrix}, \text{ and } & B_4 &= \begin{bmatrix} +-{-}000-++ \\ ++{+}000-+- \\ -+{-}000--- \\ +-{-}000+-- \end{bmatrix}, \end{aligned} \tag{11}$$

is a $(4,9)$ SNC-CCC.

Remark 9. In the **Example** 8, there is the freedom to decide on various values n_1, n_2 and n_3.

Let $\pi_1, \pi_2, \ldots$ be permutations of $\{1, 2, \ldots, M\}$ such that

$$\pi_{j_1}(i_1 P + \mu) \neq \pi_{j_2}(i_2 P + \mu), \tag{12}$$

for $j_1 \neq j_2$, $0 \leq i_1, i_2 < \frac{M}{P}$ and $1 \leq \mu < P$, we have P many such permutations.

Theorem 10. *Let* $\mathbf{C}$ *be a* (M, L)*-CCC,* $P \mid M$*,* $\mathbf{b}^1, \mathbf{b}^2, \ldots, \mathbf{b}^P$ *be MOSs of length* P *and* $\pi_0, \pi_1, \ldots, \pi_{P-1}$ *be permutations as defined in* (12)*. Now, define*

$$B_{jM+\nu P+\mu} = \mathcal{R}^{\mathcal{P}(n)}(C_{\pi_j(\nu P+1)}, C_{\pi_j(\nu P+2)}, \ldots, C_{\pi_j((\nu+1)P)}; \mathbf{b}^\mu), \tag{13}$$

for $0 \leq \nu < \frac{M}{P}$*,* $1 \leq \mu \leq P$ *and* $0 \leq j \leq P-1$*. Then, each* $\mathbf{B}^j = \{B_{jM+1}, B_{jM+2}, \ldots, B_{(j+1)M}\}$ *is a* $(M, PL+n)$ *SNC-CCC and by combining all SNC-CCCs it becomes a multiple CCCs with inter-set ZCCZ.*

Remark 11. The elements in the partition n can be used to obtain the ZCCZ width of multiple SNC-CCCs. $\lambda + L$ will be the periodic and aperiodic ZCCZ, where

$$\lambda = \min_{2 \leq i \leq P} \{n_i\}. \tag{14}$$

Now, we present a method to decrease the magnitude of the cross-correlation value outside the ZCCZ.

Corollary 12. *Let* $\mathcal{P}(n)$ *be partition of* n *i.e.,* $n = n_1 + n_2 + \ldots + n_{P+2}$ *such that* $n_{i_1} \neq n_{i_2}$ *for* $2 \leq i_1, i_2 \leq P$ *in* ***Theorem*** *10 then* δ_A *become* LM*. Therefore, we have a aperiodic* $(P, M, PL+n, L+\lambda, LM)$*-CCCs. Further, if* $n_{i_1} \neq n_{i_2} \mod L$ *for* $2 \leq i_1, i_2 \leq P$ *in* ***Theorem*** *10 then* δ_P *become* LM*. This type of* $\mathcal{P}(n)$ *is possible for* $n \geq \frac{P(P-1)}{2}$*. Therefore, we have a periodic* $(P, M, PL+n, L, LM)$*-CCCs.*

Let $\pi_1, \pi_2, \ldots, \pi_P$ be permutations of $\{1, 2, \ldots, M\}$ as define in (12), further for any two permutation π_{j_1} and π_{j_2}, when

$$\begin{aligned} \pi_{j_1}(i_1P + \mu_1) =& \pi_{j_2}(i_2P + \mu_2), \text{then} \\ \pi_{j_1}(i_1P + \mu_1 + \alpha) \neq& \pi_{j_2}(i_2P + \mu_2 + \alpha), \end{aligned} \tag{15}$$

for $1 \leq \mu_1 + \alpha, \mu_2 + \alpha \leq P$.

Corollary 13. ***Theorem*** *10 provides multiple SNC-CCCs such that cross-correlation is non-zero at only one time shift and the value is equal to LM when permutations satisfy* (15)*. Therefore, we have both aperiodic $(P, M, PL + n, L + \lambda, LM)$-CCCs and periodic $(P, M, PL + n, L, LM)$-CCCs.*

We are providing one more example to illustrate the multiple SNC-CCCs with ZCCZ.

Example 14. In **Example** 8, we consider as identity permutation ($\pi_1 = (1, 2, 3, 4)$) and construct a SNC-CCC. As $P = 2$, we have second permutation $\pi_2 = (4, 1, 2, 3)$ such that π_1 and π_2 satisfies (15). With the same $\mathbf{C}$, $\mathbf{b}^1$ and $\mathbf{b}^2$, we have one CCC as given in **Example** 8 and the other CCC is as given below

$$\begin{aligned} B_5 &= \begin{bmatrix} + - -000 + ++ \\ + - +000 + +- \\ + + +000 - +- \\ - + +000 - +- \end{bmatrix}, & B_6 &= \begin{bmatrix} + - -000 - -- \\ + - +000 - -+ \\ + + +000 + -+ \\ - + +000 + -+ \end{bmatrix}, \\ B_7 &= \begin{bmatrix} + - +000 + -- \\ + + -000 + ++ \\ - - +000 - +- \\ + + +000 + -- \end{bmatrix}, \text{ and} & B_8 &= \begin{bmatrix} + - +000 - ++ \\ + + -000 - -- \\ - - +000 + -+ \\ + + +000 - ++ \end{bmatrix}. \end{aligned} \tag{16}$$

Combining $B_1, B_2, \ldots, B_8$ is a multiple SNC-CCCs with aperiodic ZCCZ width is 6.

Choosing a set of permutations plays a role in a low correlation magnitude value. To ensure such permutation exists, we are providing an example. Let $M = 4$, $P = 4$ then $\pi_1 = (1, 2, 3, 4)$, $\pi_2 = (2, 3, 4, 1)$, $\pi_3 = (3, 4, 1, 2)$ and $\pi_4 = (4, 1, 2, 3)$, satisfy (12) but not (15). But $\pi_1 = (1, 2, 3, 4)$, $\pi_2 = (4, 3, 2, 1)$, $\pi_3 = (3, 1, 4, 2)$ and $\pi_4 = (2, 4, 1, 3)$ satisfies (15).

3 Comparison

3.1 Comparison with [26] and [33]

[26] and [33] generated SNC-GCS via generalised Boolean functions and generated parameters closely multiple of 2. Furthermore, their findings are restricted to SNC-GCSs and do not extend to SNC-CCCs. However, in the proposed construction, every code is SNC-GCSs, which are not restricted to a multiple of 2 only.

3.2 Comparision with [23]

The method in [23] implements an iterative strategy using an existing CCC to generate SNC-MOGCSs. Meanwhile, the proposed construction provides SNC-CCCs.

3.3 Comparision with [28]

SNC-CCCs has been constructed using an extended Boolean function in [28] with diversified parameters. Notably, these parameters consistently involve powers of p for $p \geq 2$, where the elements are obtained from the qth root of unity and zero for $(p \mid q)$. The given example demonstrates that the proposed construction offers more flexibility.

4 Future directions

Based on our contribution in this paper, we would like to introduce the following future works:

1. In **Corollary** 13, the set permutation used satisfying (15). However, constructing a set of permutations satisfying (15) is not straightforward. We consider it to be our future research problem.

2. In the current literature, we do not have sufficient information on the optimal collection of multiple SNC-CCCs in relation to their maximum magnitude of inter-set cross-correlation value. This limitation leads to a future direction on deriving a lower correlation bound for multiple collections of SNC-CCCs.

3. Besides, as we can see in our proposed construction of multiple SNC-CCCs, we also have a ZCCZ, which leads us to the natural question, "What will be the relationship between the ZCCZ width and the other parameters of our multiple collection of SNC-CCCs?"

5 Conclusion

In this paper, with the help of MOSs, we developed a method to construct SNC-CCCs, with flexible parameters. The proposed construction can cover almost all the possible lengths over the alphabet $\{-1, 0, 1\}$. Further, we have extended the construction to produce multiple SNC-CCCs with inter-set ZCCZ. The proposed construction includes a wider range of parameters in relation to length and alphabet size. Furthermore, we have shown that restriction can be made on the highest cross-correlation magnitude value outside the ZCCZ width, assuring that the multiple SNC-CCCs possess not only inter-set ZCCZ width but also exhibit a low cross-correlation magnitude value outside the ZCCZ width.

6 Proof of Theorems

In this section we complete the proof of Theorem 7 and Theorem 10.

Proof of Theorem 7. We complete the proof in two parts, one is for AACF and second is for ACCF values.

Case 1. Let $1 \leq t_1 = \nu_1 P + \mu_1 \leq M$ and $\tau \neq 0$, then $\mathcal{C}(B_{t_1})(\tau)$ is written as linear sum of AACF of a code at non-zero time shift and ACCF between two different codes from $\{C_{\nu_1+1}, C_{\nu_1+2}, \ldots, C_{\nu_1+P}\}$, which results $\mathcal{C}(B_{t_1})(\tau) = 0$ for any τ except at $\tau = 0$.

Case 2. Consider two distinct integer $n_1 = \nu_1 P + \mu_1$ and $n_2 = \nu_2 P + \mu_2$ such that $1 \leq \nu_1 P + \mu_1 \neq \nu_2 P + \mu_2 \leq M$, for $0 \leq \nu < \frac{M}{P}$, $1 \leq \mu \leq P$. Again, we consider two subcases on the basis of ν_1, ν_2 to complete the proof.

Subcase (i): $\nu_1 = \nu_2$. Since $\mathbf{b}^{\nu_1}$ is orthogonal to $\mathbf{b}^{\nu_2}$, it implies that $\langle \mathbf{b}^{\nu_1}, \mathbf{b}^{\nu_2} \rangle = 0$. Therefore,

$$\mathcal{C}(B_{t_1}, B_{t_2})(0) = \mathbf{b}^{\nu_1} \cdot \mathbf{b}^{\nu_2 *} \sum_{i=1}^{P} \mathcal{C}(C_{\nu_1+i})(0) = 0. \tag{17}$$

Now, assume $\tau \neq 0$, then $\mathcal{C}(B_{t_1}, B_{t_2})(\tau)$ is a linear sum of ACCF between two different codes from $\{C_{\nu_1+1}, C_{\nu_1+2}, \ldots, C_{\nu_1+P}\}$, which results $\mathcal{C}(B_{t_1}, B_{t_2})(\tau) = 0$.

Subcase (ii): $\nu_1 \neq \nu_2$, then $\mathcal{C}(B_{t_1}, B_{t_2})(\tau)$ is written as linear sum of AACF of a code at non-zero time shift and ACCF between two different codes from $\{C_{\nu_1+1}, C_{\nu_1+2}, \ldots, C_{\nu_1+P}, C_{\nu_2+1}, C_{\nu_2+2}, \ldots, C_{\nu_2+P}\}$, which results $\mathcal{C}(B_{t_1}, B_{t_2})(\tau) = 0$ for any τ.

From the above cases the proof is complete. □

Proof of Theorem 10. Each $\mathbf{B}^j$ is a $(M, PL+n)$ SNC-CCC, for $1 \leq j \leq P-1$. Now, consider $B_{t_1}^{j_1} \in \mathbf{B}^{j_1}$ and $B_{t_2}^{j_2} \in \mathbf{B}^{j_2}$. The value of $\mathcal{C}(B_{t_1}^{j_1}, B_{t_2}^{j_2})(\tau)$ become non-zero when it include AACF of any code from $\{C_1, C_2, \ldots, C_M\}$, that happens only for $\tau = L + n_i$ for some $2 \leq i \leq P$. There for $\mathcal{C}(B_{t_1}^{j_1}, B_{t_2}^{j_2})(\tau) = 0$ for $|\tau| < L + \lambda$, where $\lambda = \min\{n_i : 2 \leq i \leq P\}$. This completes the proof. □

References

[1] Marcel J. E. Golay. Static multislit spectrometry and its application to the panoramic display of infrared spectra∗. *J. Opt. Soc. Am.*, 41(7):468–472, Jul 1951.

[2] M. Golay. Complementary series. *IRE Transactions on Information Theory*, 7(2):82–87, 1961.

[3] P. Spasojevic and C.N. Georghiades. Complementary sequences for ISI channel estimation. *IEEE Trans. Inf. Theory*, 47(3):1145–1152, 2001.

[4] H.M. Wang, X.Q. Gao, B. Jiang, X.H. You, and W. Hong. Efficient MIMO channel estimation using complementary sequences. *IET Communications*, 1:962–969(7), October 2007.

[5] Alphan Şahin and Rui Yang. An uplink control channel design with complementary sequences for unlicensed bands. *IEEE Trans. Wireless Commun.*, 19(10):6858–6870, 2020.

[6] Nam Yul Yu. Binary Golay spreading sequences and Reed-Muller codes for uplink grant-free NOMA. *IEEE Trans. Commun.*, 69(1):276–290, 2021.

[7] Ali Pezeshki, A. Robert Calderbank, William Moran, and Stephen D. Howard. Doppler resilient Golay complementary waveforms. *IEEE Trans. Inf. Theory*, 54(9):4254–4266, 2008.

[8] J.A. Davis and J. Jedwab. Peak-to-mean power control in OFDM, Golay complementary sequences, and Reed-Muller codes. *IEEE Trans. Inf. Theory*, 45(7):2397–2417, 1999.

[9] Chin-Chong Tseng and C. Liu. Complementary sets of sequences. *IEEE Trans. Inf. Theory*, 18(5):644–652, 1972.

[10] H. D. Nguyêñ and G. E. Coxson. Doppler tolerance, complementary code sets, and generalised thue–morse sequences. *IET Radar, Sonar & Amp; Navigation*, 10:1603–1610, 2016.

[11] JoaquíN Aparicio and Takuya Shimura. Asynchronous detection and identification of multiple users by multi-carrier modulated complementary set of sequences. *IEEE Access*, 6:22054–22069, 2018.

[12] K.G. Paterson. Generalized Reed-Muller codes and power control in OFDM modulation. *IEEE Trans. Inf. Theory*, 46(1):104–120, 2000.

[13] Kai-Uwe Schmidt. Complementary sets, generalized Reed–Muller codes, and power control for OFDM. *IEEE Trans. Inf. Theory*, 53(2):808–814, 2007.

[14] Palash Sarkar, Sudhan Majhi, and Zilong Liu. A direct and generalized construction of polyphase complementary sets with low PMEPR and high code-rate for OFDM system. *IEEE Trans. Commun.*, 68(10):6245–6262, 2020.

[15] N. Suehiro and M. Hatori. N-shift cross-orthogonal sequences. *IEEE Trans. Inf. Theory*, 34(1):143–146, 1988.

[16] Xiqing Liu, Yvonne Huang, Chih-Yu Chang, and Hsiao-Hwa Chen. Generalized complementary coded scrambling multiple access for MIMO communications. *IEEE Trans. Veh. Technol.*, 70(12):13047–13061, 2021.

[17] Si-Yue Sun, Hsiao-Hwa Chen, and Wei-Xiao Meng. A survey on complementary-coded MIMO CDMA wireless communications. *IEEE Communications Surveys & Tutorials*, 17(1):52–69, 2015.

[18] Jun Tang, Ning Zhang, Zhikun Ma, and Bo Tang. Construction of doppler resilient complete complementary code in MIMO radar. *IEEE Trans. Signal Process.*, 62(18):4704–4712, 2014.

[19] Si-Yue Sun, Hsiao-Hwa Chen, and Wei-Xiao Meng. A framework to construct three-dimensional complementary codes for multiuser MIMO systems. *IEEE Trans. Veh. Technol.*, 64(7):2861–2874, 2015.

[20] Xinyu Men and Yubo Li. New construction of multiple complete complementary codes with inter-set zero cross-correlation zone. *IEEE Signal Process. Lett.*, 29:1958–1962, 2022.

[21] Liying Tian, Yubo Li, and Chengqian Xu. Multiple complete complementary codes with inter-set zero cross-correlation zone. *IEEE Trans. Commun.*, 68(3):1925–1936, 2020.

[22] IEEE standard for wireless lan medium access control (MAC) and physical layer (PHY) specifications. *IEEE Std 802.11-1997*, pages 1–445, 1997.

[23] Yajing Zhou, Yang Yang, Zhengchun Zhou, Kushal Anand, Su Hu, and Yong Liang Guan. New complementary sets with low PAPR property under spectral null constraints. *IEEE Trans. Inf. Theory*, 66(11):7022–7032, 2020.

[24] Benjamin R. Hamilton, Xiaoli Ma, John E. Kleider, and Robert J. Baxley. OFDM pilot design for channel estimation with null edge subcarriers. *IEEE Trans. Wireless Commun.*, 10(10):3145–3150, 2011.

[25] A. Gavish and A. Lempel. On ternary complementary sequences. *IEEE Trans. Inf. Theory*, 40(2):522–526, 1994.

[26] Alphan Şahin and Rui Yang. A generic complementary sequence construction and associated encoder/decoder design. *IEEE Trans. Commun.*, 69(11):7691–7705, 2021.

[27] Roman N. Ipanov, Alexandr I. Baskakov, Nikolay Olyunin, and Min-Ho Ka. Radar signals with ZACZ based on pairs of D-code sequences and their compression algorithm. *IEEE Signal Process. Lett.*, 25(10):1560–1564, 2018.

[28] Bingsheng Shen, Yang Yang, Zhengchun Zhou, and Sihem Mesnager. Constructions of spectrally null constrained complete complementary codes via the graph of extended boolean functions. *IEEE Trans. Inf. Theory*, 69(9):6028–6039, 2023.

[29] Nishant Kumar, Palash Sarkar, and Sudhan Majhi. Construction of spectrally-null-constrained zero-correlation zone sequences with flexible support. *Cryptography and Communications*, May 2024.

[30] Palash Sarkar, Chunlei Li, Sudhan Majhi, and Zilong Liu. New correlation bound and construction of Quasi-complementary sequence sets. *IEEE Trans. Inf. Theory*, 70(3):2201–2223, 2024.

[31] Rajen Kumar, Prashant Kumar Srivastava, and Sudhan Majhi. A direct construction of type-II Z complementary code set with arbitrarily large codes, 2023.

[32] Yu Tao, Adhikari Avik, Ranjan, Wang Yanyan, and Yang Yang. New class of optimal Z-complementary code sets. *IEEE Signal Process. Lett.*, pages 1–1, 2022.

[33] Bingsheng Shen, Yang Yang, Pingzhi Fan, and Zhengchun Zhou. Constructions of non-contiguous complementary sequence sets and their applications. *IEEE Trans. Wireless Commun.*, 21(7):4871–4882, 2022.