Centre for Tropical Livestock Genetics and Health

Analysing sources of phenotypic variation with linear models

Chrissy Rochus

Roslin Institute, University of Edinburgh

THE UNIVERSITY of EDINBURGH

ROSLIN

SRUC

ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Centre for Tropical Livestock Genetics and Health

Sources of phenotypic variation with linear models

Learning objectives:

- Introduction to linear models
- Parts of a model
    - Equation
        - Effects – fixed and random
    - Expectations, assumptions, limitations
- Developing a model
- Matrix format

THE UNIVERSITY of EDINBURGH

ROSLIN

SRUC

ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Centre for Tropical Livestock Genetics and Health

Linear models

$$y_i = \mu + F_i + bx_i + e_i$$

Workhorse of statistical analysis of many kinds of data

Central tool in quantitative genetics

Very versatile & powerful

Note: every model is an assumed simplification of reality

THE UNIVERSITY of EDINBURGH   ROSLIN   SRUC   ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Centre for Tropical Livestock Genetics and Health

Linear models

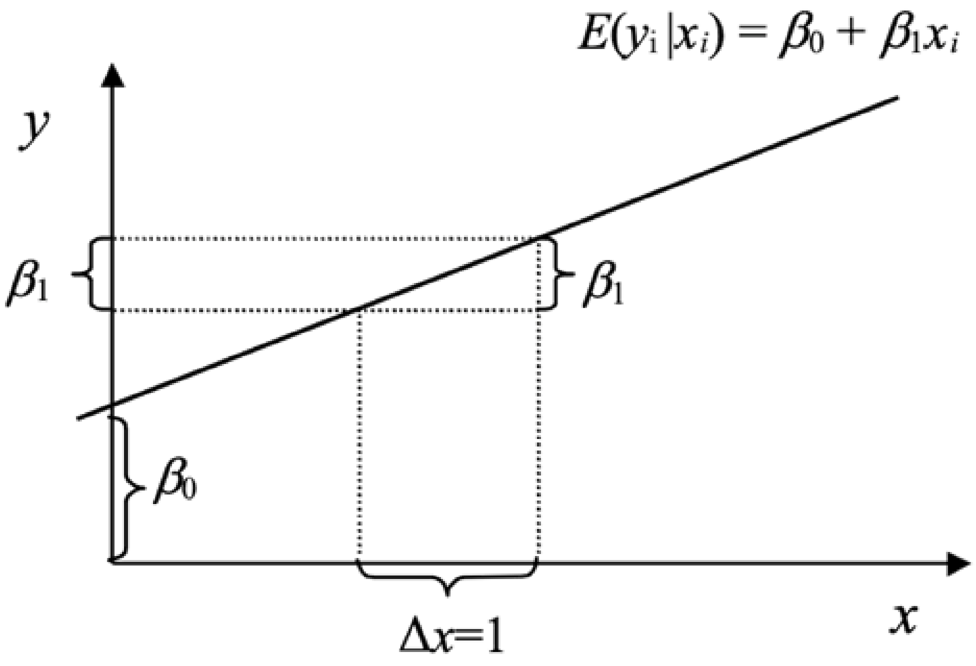


Figure 7.2 Interpretation of parameters for simple linear regression

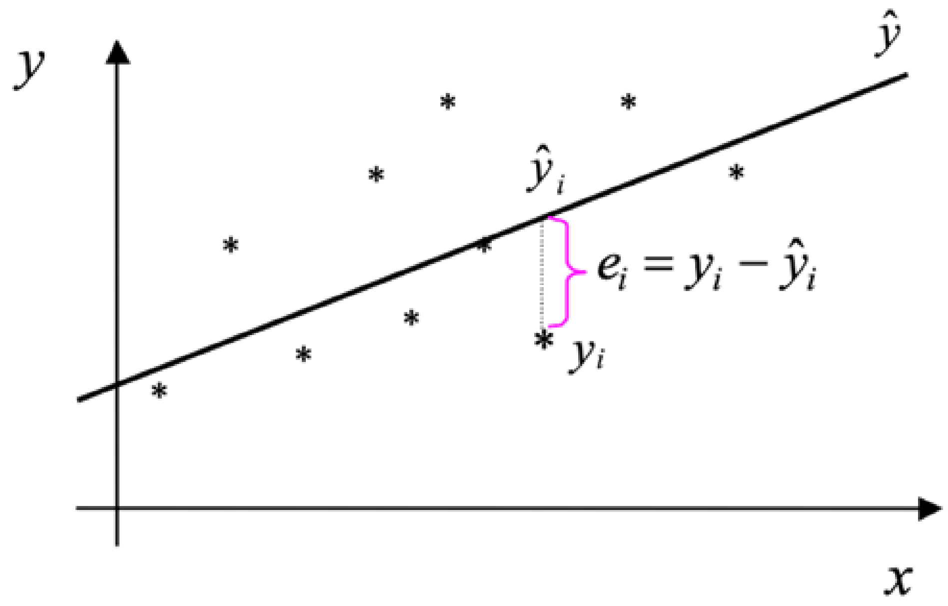


Figure 7.4 Estimated or fitted line of the simple linear regression

Kaps and Lamberson (2004)

Parts of a model

1. The equation
2. Expectations and variance-covariance matrices of random variables
3. Assumptions, restrictions and limitations

A linear model is not complete without all parts

Statistical procedures and strategies are only determined based on a complete model

The equation

$$y_i = \mu + F_i + bx_i + e_i$$

y_i - Dependent variable

μ - Intercept (population reference value)

F_i - Factor with levels

bx_i - Covariate

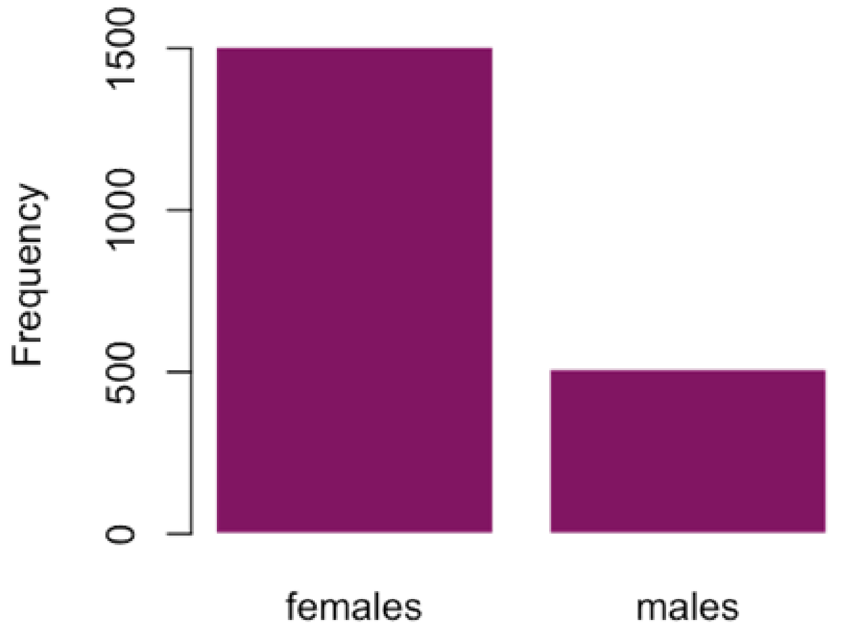
Fixed or random?

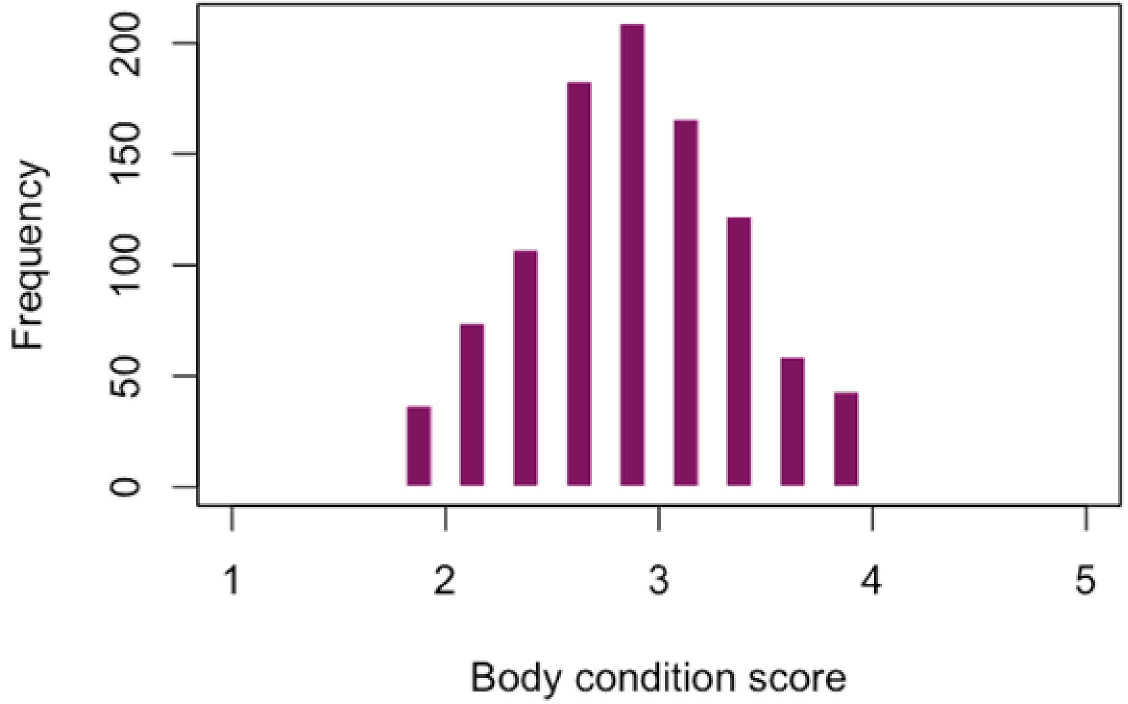
e_i - Residual
(environmental effects/noise, other effects/model misspecification, data errors, …)

Centre for Tropical Livestock Genetics and Health

Variables

Discrete



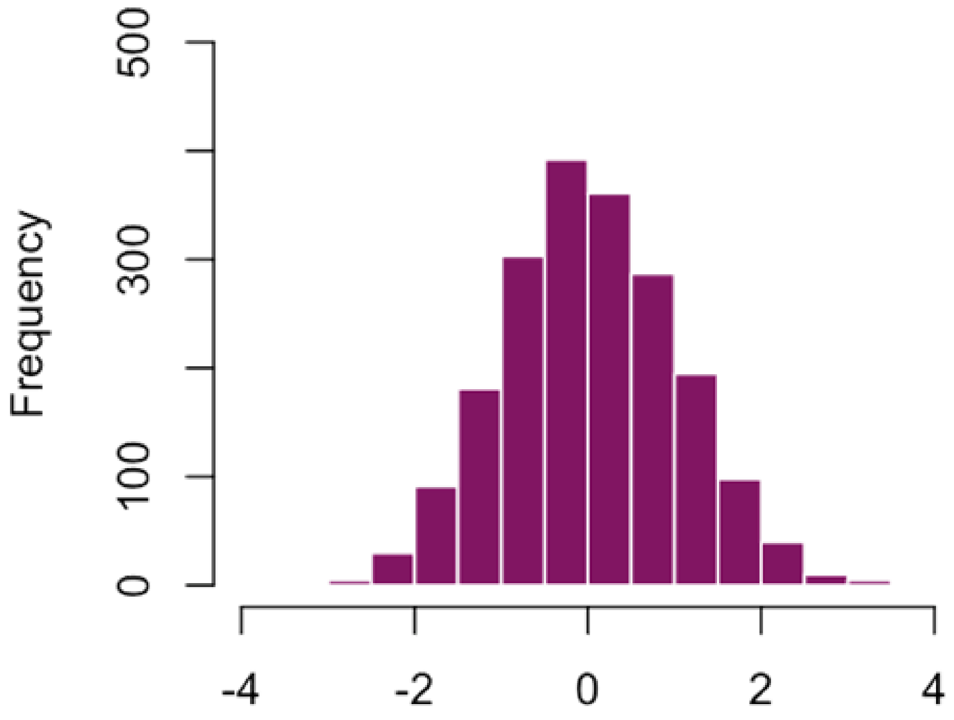


THE UNIVERSITY of EDINBURGH ROSLIN SRUC ILRI

Variables

Centre for Tropical Livestock Genetics and Health

Continuous



THE UNIVERSITY of EDINBURGH ROSLIN SRUC ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Centre for Tropical Livestock Genetics and Health

Fixed vs. random effects

	Fixed	Random
Number of levels	Small or limited to a fixed number	Large
Inferences	Limited to that set of levels	Going to be made to an entire population of conceptual levels
New experiment made	Same factor levels	Completely different levels
Levels of a factor selection	Selected among possible available levels	Sampled from an infinitely large population
Number of records per parameter	Many records	Not many … with many records, random effect behaves practically as a fixed effect
	Regressions of a continuous factor	

THE UNIVERSITY of EDINBURGH ROSLIN SRUC ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Centre for Tropical Livestock Genetics and Health

Fixed vs. random effects

Diet (ex. diets A, B and C)	Animal
Breed (ex. Holstein, Angus, Jersey)	Season (ex. Wet and dry seasons)
Contemporary group	Permanent environment

Centre for Tropical Livestock Genetics and Health

Fixed vs. random effects

Diet (ex. diets A, B and C) - **fixed**	Animal
Breed (ex. Holstein, Angus, Jersey)	Season (ex. Wet and dry seasons)
Contemporary group	Permanent environment

Centre for Tropical Livestock Genetics and Health

Fixed vs. random effects

Diet (ex. diets A, B and C) - **fixed** ✓ Limited to a small, fixed number of levels ✓ Inferences limited to the sets of levels	Animal
Breed (ex. Holstein, Angus, Jersey)	Season (ex. Wet and dry seasons)
Contemporary group	Permanent environment

Centre for Tropical Livestock Genetics and Health

Fixed vs. random effects

Diet (ex. diets A, B and C) - **fixed** ✓ Limited to a small, fixed number of levels ✓ Inferences limited to the sets of levels	Animal - **random** ✓ Inferences going to be made to an entire population of conceptual levels
Breed (ex. Holstein, Angus, Jersey) – **fixed** ✓ Limited to a small, fixed number of levels ✓ Inferences limited to the sets of levels ✓ Levels of the factor selected among possible available levels	Season (ex. Wet and dry seasons) - **fixed** ✓ Limited to a small, fixed number of levels ✓ Inferences limited to the sets of levels ✓ New experiment will have same levels
Contemporary group - **fixed** ✓ Inferences going to be made to an entire population of conceptual levels ✓ New experiment made has completely different levels	Permanent environment - **random** ✓ Number of levels is large

Expectations, assumptions and limitations

Which data was sampled or collected?

Were individuals randomly selected, or met some requirement?

Did data arise from many environments, at random, or chosen?

How to develop a model

Information in literature on what should be included in model

Discuss with others (colleagues, industry, etc. – not everyone will agree what is relevant, which effects are fixed or random)

Test models and change as evidence accumulates

Justify your model

Consider non-linear or other types of models if appropriate

Developing a model – example for beef calf weights

Weights on beef calves taken at 200 days of age are shown in the table below:

Males	Females
198	187
211	194
220	202
	185

Developing a model – example for beef calf weights

Weights on beef calves taken at 200 days of age are shown in the table below:

Males	Females
198	187
211	194
220	202
	185

y_{ij}

y_{ij} 200-day weight

Centre for Tropical Livestock Genetics and Health

Developing a model – example for beef calf weights

Weights on beef calves taken at 200 days of age are shown in the table below:

Males	Females
198	187
211	194
220	202
	185

$$y_{ij} = s_i$$

y_{ij} 200-day weight

s_i effect of sex of calf (fixed effect)

THE UNIVERSITY of EDINBURGH ROSLIN SRUC ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Developing a model – example for beef calf weights

Weights on beef calves taken at 200 days of age are shown in the table below:

Males	Females
198	187
211	194
220	202
	185

$$y_{ij} = s_i + c_j$$

y_{ij} 200-day weight

s_i effect of sex of calf (fixed effect)

c_j effect of calf (random effect)

Developing a model – example for beef calf weights

Weights on beef calves taken at 200 days of age are shown in the table below:

Males	Females
198	187
211	194
220	202
	185

$$y_{ij} = s_i + c_j + e_{ij}$$

y_{ij} 200-day weight

s_i effect of sex of calf (fixed effect)

c_j effect of calf (random effect)

e_{ij} residual, or unexplained, variation (random effect)

Centre for Tropical Livestock Genetics and Health

Developing a model – example for beef calf weights

Expectations

$$E(y_{ij}) = s_i$$
$$E(c_j) = 0$$
$$E(e_{ij}) = 0$$
$$Var(c_j) = \sigma_c^2$$
$$Var(e_{ij}) = \sigma_e^2$$

$$Cov(c_j, c_{j'}) = 0$$
$$Cov(e_{ij}, e_{ij'}) = 0$$
$$Cov(e_j, e_{i'j'}) = 0$$

$$y_{ij} = s_i + c_j + e_{ij}$$

y_{ij}	200-day weight
s_i	effect of sex of calf (fixed effect)
c_j	effect of calf (random effect)
e_{ij}	residual, or unexplained, variation (random effect)

THE UNIVERSITY of EDINBURGH ROSLIN SRUC ILRI

Centre for Tropical Livestock Genetics and Health

Developing a model – example for beef calf weights

Assumptions and limitations

- Calves are the same breed
- Reared in the same environment and time period
- Maternal effects ignored
- Calf effects contain all genetic effects (direct and maternal)
- Weights were accurately recorded at 200 days of age

$$y_{ij} = s_i + c_j + e_{ij}$$

y_{ij}	200-day weight
s_i	effect of sex of calf (fixed effect)
c_j	effect of calf (random effect)
e_{ij}	residual, or unexplained, variation (random effect)

THE UNIVERSITY of EDINBURGH ROSLIN SRUC ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Centre for
Tropical Livestock
Genetics and Health

Matrix algebra

THE UNIVERSITY of EDINBURGH

ROSLIN

SRUC

ILRI

INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Centre for Tropical Livestock Genetics and Health

The equation in matrix format

$$y_{ij} = \mu_i + a_i + e_{ij}$$

$$\downarrow$$

$$\boldsymbol{y} = \boldsymbol{Xb} + \boldsymbol{Za} + \boldsymbol{e}$$

$$y_{11} = \mu_1 + a_1 + e_{11}$$
$$y_{21} = \mu_2 + a_2 + e_{21}$$
$$y_{31} = \mu_3 + a_3 + e_{31}$$

$$\downarrow$$

$$\begin{bmatrix} y_{11} \\ y_{21} \\ y_{31} \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} + \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + \begin{bmatrix} e_{11} \\ e_{21} \\ e_{31} \end{bmatrix}$$

Efficient

- Data storage, and computations run
- A system of linear models can be written in one matrix equation

Centre for Tropical Livestock Genetics and Health

Matrix

A rectangular array of numbers

$$\boldsymbol{M} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

THE UNIVERSITY of EDINBURGH ROSLIN SRUC ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Matrix

A rectangular array of numbers

$$\boldsymbol{M} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

Rows

Columns

Matrix

A rectangular array of numbers

First row

Second column

$$\boldsymbol{M} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

Rows

Columns

Centre for Tropical Livestock Genetics and Health

Matrix

A rectangular array of numbers

First row

Second column

$$\boldsymbol{M} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

Rows

Columns

Column vector

$$\boldsymbol{c} = \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix}$$

Row vector

$$\boldsymbol{r} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \end{bmatrix}$$

Scalar $s = [s_{11}]$

THE UNIVERSITY of EDINBURGH ROSLIN SRUC ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Matrix

A rectangular array of numbers

Column vector

$$\boldsymbol{c} = \begin{bmatrix} 2.5 \\ 5.0 \\ -1.2 \end{bmatrix}$$

$$\boldsymbol{M} = \begin{bmatrix} -20 & 4 & 6 \\ 9 & 60 & -2 \\ 33 & -15 & 8 \end{bmatrix}$$

Row vector

$$\boldsymbol{r} = \begin{bmatrix} 3 & 9 & 11 \end{bmatrix}$$

$$\boldsymbol{E} = \begin{bmatrix} x & y+1 & \sqrt{x-y} \\ (m+n)/n & a^2 & \log d \end{bmatrix}$$

Scalar $s = [13]$

Centre for Tropical Livestock Genetics and Health

Matrix

Naming convention:
Matrix – boldface & uppercase
Vector – boldface & lowercase
Scalar - lowercase

A rectangular array of numbers

$$\boldsymbol{M} = \begin{bmatrix} -20 & 4 & 6 \\ 9 & 60 & -2 \\ 33 & -15 & 8 \end{bmatrix}$$

order

3x3

Column vector

$$\boldsymbol{c} = \begin{bmatrix} 2.5 \\ 5.0 \\ -1.2 \end{bmatrix}$$

3x1

Row vector

$$\boldsymbol{r} = \begin{bmatrix} 3 & 9 & 11 \end{bmatrix}$$

1x3

$$\boldsymbol{E} = \begin{bmatrix} x & y+1 & \sqrt{x-y} \\ (m+n)/n & a^2 & \log d \end{bmatrix}$$

2x3

Scalar $s = [13]$ 1x1

THE UNIVERSITY of EDINBURGH ROSLIN SRUC ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Information on matrices that we need this week

- Special matrices
- Addition
- Multiplication
- Transpose
- Inversion

Information on matrices that we need this week

- Special matrices
- Addition
- Multiplication
- Transpose
- Inversion

Bonus!

Other helpful matrix operations to know for animal breeding

Slides included at the end

Rmarkdown file with how to do matrix operations in R

Special square matrices

Square – equal number of rows and columns

$$\boldsymbol{s} = \begin{bmatrix} 4 & 7 & 0 \\ 0 & -5 & 3 \\ -7 & 3 & 11 \end{bmatrix}$$

Special square matrices

Square – equal number of rows and columns

$$s = \begin{bmatrix} 4 & 7 & 0 \\ 0 & -5 & 3 \\ -7 & 3 & 11 \end{bmatrix}$$

Diagonal – all off-diagonal elements are zero

$$d = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Special square matrices

Square – equal number of rows and columns

$$\boldsymbol{S} = \begin{bmatrix} 4 & 7 & 0 \\ 0 & -5 & 3 \\ -7 & 3 & 11 \end{bmatrix}$$

Diagonal – all off-diagonal elements are zero

$$\boldsymbol{D} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Identity – diagonal matrix, where all diagonal elements are 1

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Centre for Tropical Livestock Genetics and Health

Addition

Two matrices can be added together if they have the same number of rows and columns, ie. They are of the same order and conformable

$$c_{ij} = a_{ij} + b_{ij}$$

$$\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B} = \begin{bmatrix} 9 & 2 \\ -11 & 5 \end{bmatrix} + \begin{bmatrix} 13 & -7 \\ 8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 9 + 13 & 2 + (-7) \\ -11 + 8 & 5 + 4 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -5 \\ -3 & 9 \end{bmatrix}$$

THE UNIVERSITY of EDINBURGH ROSLIN SRUC ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Centre for Tropical Livestock Genetics and Health

Multiplication

Two matrices can be multiplied if the number of columns in the first matrix equals the number of rows in the second

The product matrix has an order of the number of rows of the first matrix by the number of columns of the second

$$c_{ij} = \sum_{j=i}^{m}\sum_{i=1}^{n}\sum_{k=1}^{z} a_{ik}b_{kj}$$

$$\boldsymbol{C} = \boldsymbol{EG} = \begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 4 \\ 4 & -3 & 6 \end{bmatrix}\begin{bmatrix} 4 & -5 \\ 2 & 3 \\ -6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1(4) + 5(2) + 3(-6) & -1(-5) + 5(3) + 3(1) \\ 6(4) + (-2)(2) + 4(-6) & 6(-5) + (-2)(3) + 4(1) \\ 4(4) + (-3)(2) + 6(-6) & 4(-5) + (-3)(3) + 6(1) \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 23 \\ -4 & -32 \\ -26 & -23 \end{bmatrix}$$

Centre for Tropical Livestock Genetics and Health

Multiplication

Two matrices can be multiplied if the number of columns in the first matrix equals the number of rows in the second

The product matrix has an order of the number of rows of the first matrix by the number of columns of the second

$$c_{ij} = \sum_{j=i}^{m} \sum_{i=1}^{n} \sum_{k=1}^{z} a_{ik} b_{kj}$$

$$\boldsymbol{C} = \boldsymbol{EG} = \begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 4 \\ 4 & -3 & 6 \end{bmatrix} \begin{bmatrix} 4 & -5 \\ 2 & 3 \\ -6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1(4) + 5(2) + 3(-6) & -1(-5) + 5(3) + 3(1) \\ 6(4) + (-2)(2) + 4(-6) & 6(-5) + (-2)(3) + 4(1) \\ 4(4) + (-3)(2) + 6(-6) & 4(-5) + (-3)(3) + 6(1) \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 23 \\ -4 & -32 \\ -26 & -23 \end{bmatrix}$$

Multiplication

Two matrices can be multiplied if the number of columns in the first matrix equals the number of rows in the second

The product matrix has an order of the number of rows of the first matrix by the number of columns of the second

$$c_{ij} = \sum_{j=i}^{m}\sum_{i=1}^{n}\sum_{k=1}^{z} a_{ik}b_{kj}$$

$$\boldsymbol{C} = \boldsymbol{EG} = \begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 4 \\ 4 & -3 & 6 \end{bmatrix}\begin{bmatrix} 4 & -5 \\ 2 & 3 \\ -6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1(4) + 5(2) + 3(-6) & -1(-5) + 5(3) + 3(1) \\ 6(4) + (-2)(2) + 4(-6) & 6(-5) + (-2)(3) + 4(1) \\ 4(4) + (-3)(2) + 6(-6) & 4(-5) + (-3)(3) + 6(1) \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 23 \\ -4 & -32 \\ -26 & -23 \end{bmatrix}$$

Multiplication

Two matrices can be multiplied if the number of columns in the first matrix equals the number of rows in the second

The product matrix has an order of the number of rows of the first matrix by the number of columns of the second

$$c_{ij} = \sum_{j=i}^{m}\sum_{i=1}^{n}\sum_{k=1}^{z} a_{ik} b_{kj}$$

$$\boldsymbol{C} = \boldsymbol{EG} = \begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 4 \\ 4 & -3 & 6 \end{bmatrix}\begin{bmatrix} 4 & -5 \\ 2 & 3 \\ -6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1(4) + 5(2) + 3(-6) & -1(-5) + 5(3) + 3(1) \\ 6(4) + (-2)(2) + 4(-6) & 6(-5) + (-2)(3) + 4(1) \\ 4(4) + (-3)(2) + 6(-6) & 4(-5) + (-3)(3) + 6(1) \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 23 \\ -4 & -32 \\ -26 & -23 \end{bmatrix}$$

Centre for Tropical Livestock Genetics and Health

Multiplication

Two matrices can be multiplied if the number of columns in the first matrix equals the number of rows in the second

The product matrix has an order of the number of rows of the first matrix by the number of columns of the second

$$c_{ij} = \sum_{j=i}^{m}\sum_{i=1}^{n}\sum_{k=1}^{z} a_{ik}b_{kj}$$

$$\boldsymbol{C} = \boldsymbol{EG} = \begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 4 \\ 4 & -3 & 6 \end{bmatrix}\begin{bmatrix} 4 & -5 \\ 2 & 3 \\ -6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1(4)+5(2)+3(-6) & -1(-5)+5(3)+3(1) \\ 6(4)+(-2)(2)+4(-6) & 6(-5)+(-2)(3)+4(1) \\ 4(4)+(-3)(2)+6(-6) & 4(-5)+(-3)(3)+6(1) \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 23 \\ -4 & -32 \\ -26 & -23 \end{bmatrix}$$

THE UNIVERSITY of EDINBURGH ROSLIN SRUC ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Multiplication

Centre for Tropical Livestock Genetics and Health

Two matrices can be multiplied if the number of columns in the first matrix equals the number of rows in the second

The product matrix has an order of the number of rows of the first matrix by the number of columns of the second

$$c_{ij} = \sum_{j=i}^{m}\sum_{i=1}^{n}\sum_{k=1}^{z} a_{ik}b_{kj}$$

$$\boldsymbol{C} = \boldsymbol{EG} = \begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 4 \\ 4 & -3 & 6 \end{bmatrix}\begin{bmatrix} 4 & -5 \\ 2 & 3 \\ -6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1(4) + 5(2) + 3(-6) & -1(-5) + 5(3) + 3(1) \\ 6(4) + (-2)(2) + 4(-6) & 6(-5) + (-2)(3) + 4(1) \\ 4(4) + (-3)(2) + 6(-6) & 4(-5) + (-3)(3) + 6(1) \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 23 \\ -4 & -32 \\ -26 & -23 \end{bmatrix}$$

THE UNIVERSITY of EDINBURGH ROSLIN SRUC ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Centre for Tropical Livestock Genetics and Health

Multiplication

Two matrices can be multiplied if the number of columns in the first matrix equals the number of rows in the second

The product matrix has an order of the number of rows of the first matrix by the number of columns of the second

$$c_{ij} = \sum_{j=i}^{m} \sum_{i=1}^{n} \sum_{k=1}^{z} a_{ik} b_{kj}$$

$$\boldsymbol{C} = \boldsymbol{EG} = \begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 4 \\ 4 & -3 & 6 \end{bmatrix} \begin{bmatrix} 4 & -5 \\ 2 & 3 \\ -6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1(4) + 5(2) + 3(-6) & -1(-5) + 5(3) + 3(1) \\ 6(4) + (-2)(2) + 4(-6) & 6(-5) + (-2)(3) + 4(1) \\ 4(4) + (-3)(2) + 6(-6) & 4(-5) + (-3)(3) + 6(1) \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 23 \\ -4 & -32 \\ -26 & -23 \end{bmatrix}$$

THE UNIVERSITY of EDINBURGH ROSLIN SRUC ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Centre for Tropical Livestock Genetics and Health

Multiplication

Two matrices can be multiplied if the number of columns in the first matrix equals the number of rows in the second

The product matrix has an order of the number of rows of the first matrix by the number of columns of the second

$$c_{ij} = \sum_{j=i}^{m} \sum_{i=1}^{n} \sum_{k=1}^{z} a_{ik} b_{kj}$$

$$\boldsymbol{C} = \boldsymbol{EG} = \begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 4 \\ 4 & -3 & 6 \end{bmatrix} \begin{bmatrix} 4 & -5 \\ 2 & 3 \\ -6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1(4) + 5(2) + 3(-6) & -1(-5) + 5(3) + 3(1) \\ 6(4) + (-2)(2) + 4(-6) & 6(-5) + (-2)(3) + 4(1) \\ 4(4) + (-3)(2) + 6(-6) & 4(-5) + (-3)(3) + 6(1) \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 23 \\ -4 & -32 \\ -26 & -23 \end{bmatrix}$$

THE UNIVERSITY of EDINBURGH ROSLIN SRUC ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Multiplication

Two matrices can be multiplied if the number of columns in the first matrix equals the number of rows in the second

The product matrix has an order of the number of rows of the first matrix by the number of columns of the second

$$c_{ij} = \sum_{j=i}^{m}\sum_{i=1}^{n}\sum_{k=1}^{z} a_{ik}b_{kj}$$

$$\boldsymbol{C} = \boldsymbol{EG} = \begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 4 \\ 4 & -3 & 6 \end{bmatrix}\begin{bmatrix} 4 & -5 \\ 2 & 3 \\ -6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1(4) + 5(2) + 3(-6) & -1(-5) + 5(3) + 3(1) \\ 6(4) + (-2)(2) + 4(-6) & 6(-5) + (-2)(3) + 4(1) \\ 4(4) + (-3)(2) + 6(-6) & 4(-5) + (-3)(3) + 6(1) \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 23 \\ -4 & -32 \\ -26 & -23 \end{bmatrix}$$

Centre for Tropical Livestock Genetics and Health

Transpose

For **J**, with order *i* x *k*, the transpose, **J’** has order *i* x *k*

$$\boldsymbol{J} = \begin{bmatrix} 5 & 3 \\ -1 & 4 \\ 6 & -2 \end{bmatrix}$$

$$\boldsymbol{J}' = \boldsymbol{J}^{\boldsymbol{T}} = \begin{bmatrix} 5 & -1 & 6 \\ 3 & 4 & -2 \end{bmatrix}$$

Transpose

For **J**, with order *i* x *k*, the transpose, **J'** has order *i* x *k*

$$\boldsymbol{J} = \begin{bmatrix} 5 & 3 \\ -1 & 4 \\ 6 & -2 \end{bmatrix}$$

$$\boldsymbol{J}' = \boldsymbol{J}^{\boldsymbol{T}} = \begin{bmatrix} 5 & -1 & 6 \\ 3 & 4 & -2 \end{bmatrix}$$

Transpose

For **J**, with order *i* x *k*, the transpose, **J'** has order *i* x *k*

$$\boldsymbol{J} = \begin{bmatrix} 5 & 3 \\ -1 & 4 \\ 6 & -2 \end{bmatrix}$$

$$\boldsymbol{J}' = \boldsymbol{J}^{\boldsymbol{T}} = \begin{bmatrix} 5 & -1 & 6 \\ 3 & 4 & -2 \end{bmatrix}$$

A matrix is not equal to its transpose

Transpose

For **J**, with order *i* x *k*, the transpose, **J'** has order *i* x *k*

A matrix is not equal to its transpose ... Unless it is a symmetric matrix

$$\boldsymbol{J} = \begin{bmatrix} 5 & 3 \\ -1 & 4 \\ 6 & -2 \end{bmatrix}$$

$$\boldsymbol{J}' = \boldsymbol{J}^{\boldsymbol{T}} = \begin{bmatrix} 5 & -1 & 6 \\ 3 & 4 & -2 \end{bmatrix}$$

$$\boldsymbol{W} = \begin{bmatrix} 7 & 2 & -4 \\ 2 & 3 & 0 \\ -4 & 0 & -9 \end{bmatrix}$$

$$\boldsymbol{W}' = \begin{bmatrix} 7 & 2 & -4 \\ 2 & 3 & 0 \\ -4 & 0 & -9 \end{bmatrix}$$

Transpose

For **J**, with order *i* x *k*, the transpose, **J'** has order *i* x *k*

$$\boldsymbol{J} = \begin{bmatrix} 5 & 3 \\ -1 & 4 \\ 6 & -2 \end{bmatrix}$$

$$\boldsymbol{J}' = \boldsymbol{J}^T = \begin{bmatrix} 5 & -1 & 6 \\ 3 & 4 & -2 \end{bmatrix}$$

$$\boldsymbol{W} = \begin{bmatrix} 7 & 2 & -4 \\ 2 & 3 & 0 \\ -4 & 0 & -9 \end{bmatrix}$$

$$\boldsymbol{W}' = \begin{bmatrix} 7 & 2 & -4 \\ 2 & 3 & 0 \\ -4 & 0 & -9 \end{bmatrix}$$

A matrix is not equal to its transpose ... Unless it is a symmetric matrix

Centre for Tropical Livestock Genetics and Health

Inverse

When **W**$^{-1}$ is multiplied by **W**, an identity matrix (**I**) is the product

$$\boldsymbol{W}^{-1}\boldsymbol{W} = \boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Only square matrices can be inverted

For diagonal matrix, **Y**, the inverse is the reciprocal of the diagonal elements

$$\boldsymbol{Y} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\boldsymbol{Y}^{-1} = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & -\frac{1}{10} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Inverse

When $\mathbf{W}^{-1}$ is multiplied by $\mathbf{W}$, an identity matrix ($\mathbf{I}$) is the product

$$\boldsymbol{W}^{-1}\boldsymbol{W} = \boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Only square matrices can be inverted

For diagonal matrix, $\mathbf{Y}$, the inverse is the reciprocal of the diagonal elements

$$\boldsymbol{Y} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\boldsymbol{Y}^{-1} = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & -\frac{1}{10} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Centre for Tropical Livestock Genetics and Health

Inverse

$$\boldsymbol{Z} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$$

THE UNIVERSITY of EDINBURGH ROSLIN SRUC ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Inverse

Calculate the determinant (the difference between the product of the two diagonal elements and the two off-diagonal elements)

$$\boldsymbol{Z} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$$

$$\boldsymbol{Z}^{-1} = \frac{1}{z_{11}z_{22} - z_{12}z_{21}} \begin{bmatrix} z_{22} & -z_{12} \\ -z_{21} & z_{11} \end{bmatrix}$$

Inverse

Calculate the determinant (the difference between the product of the two diagonal elements and the two off-diagonal elements)

$$\boldsymbol{Z} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$$

Reverse diagonal elements, multiplying off-diagonal elements by -1

$$\boldsymbol{Z}^{-\mathbf{1}} = \frac{1}{z_{11}z_{22} - z_{12}z_{21}} \begin{bmatrix} z_{22} & -z_{12} \\ -z_{21} & z_{11} \end{bmatrix}$$

Inverse

Calculate the determinant (the difference between the product of the two diagonal elements and the two off-diagonal elements)

$$\boldsymbol{Z} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$$

Reverse diagonal elements, multiplying off-diagonal elements by -1

$$\boldsymbol{Z}^{-1} = \frac{1}{z_{11}z_{22} - z_{12}z_{21}} \begin{bmatrix} z_{22} & -z_{12} \\ -z_{21} & z_{11} \end{bmatrix}$$

Divide all elements by the determinant

Inversion – 2x2 matrix example

Calculate the determinant (the difference between the product of the two diagonal elements and the two off-diagonal elements)

Reverse diagonal elements, multiplying off-diagonal elements by -1

Divide all elements by the determinant

$$\boldsymbol{Z} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$$

$$\boldsymbol{Z}^{-1} = \frac{1}{z_{11}z_{22} - z_{12}z_{21}} \begin{bmatrix} z_{22} & -z_{12} \\ -z_{21} & z_{11} \end{bmatrix}$$

$$= \frac{1}{3(5) - (-1)(2)} \begin{bmatrix} 5 & -(-1) \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.29 & 0.06 \\ -0.12 & 0.18 \end{bmatrix}$$

The equation

$$y_{ij} = \mu_i + a_i + e_{ij}$$

y_{ij} - record j of the i^{th} animal, dependent/response variable

μ_i - fixed environmental effects of i^{th} animal (year of birth, sex, etc.)

a_i - random additive genetic effect of i^{th} animal

e_{ij} - residual (environmental effects, noise, other effects, model misspecification) effect for record j of the i^{th} animal

The equation in matrix format

$$y_{ij} = \mu_i + a_i + e_{ij}$$

$$\downarrow$$

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{b} + \boldsymbol{Z}\boldsymbol{a} + \boldsymbol{e}$$

$\boldsymbol{y}$	vector of observations
$\boldsymbol{b}$	vector of fixed effects (year of birth, sex, etc.)
$\boldsymbol{a}$	vector of random additive effects
$\boldsymbol{e}$	vector of random residual effects
$\boldsymbol{X}$ and $\boldsymbol{Z}$	incidence matrices relating records to fixed and random effects

The equation in matrix format

$$y_{11} = \mu_1 + a_1 + e_{11}$$
$$y_{21} = \mu_2 + a_2 + e_{21}$$
$$y_{31} = \mu_3 + a_3 + e_{31}$$

$$y_{ij} = \mu_i + a_i + e_{ij}$$

$$\downarrow$$

$$\boldsymbol{y} = \boldsymbol{Xb} + \boldsymbol{Za} + \boldsymbol{e}$$

$$\downarrow$$

$$\begin{bmatrix} y_{11} \\ y_{21} \\ y_{31} \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} + \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + \begin{bmatrix} e_{11} \\ e_{21} \\ e_{31} \end{bmatrix}$$

$\boldsymbol{y}$ vector of observations

$\boldsymbol{b}$ vector of fixed effects (year of birth, sex, etc.)

$\boldsymbol{a}$ vector of random additive genetic effects

$\boldsymbol{e}$ vector of random residual effects

$\boldsymbol{X}$ and $\boldsymbol{Z}$ incidence matrices relating records to fixed and random effects

In matrix format, with fixed effects only

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{b} + \boldsymbol{e}$$

$$[\boldsymbol{X}'\boldsymbol{X}][\widehat{\boldsymbol{b}}] = [\boldsymbol{X}'\boldsymbol{y}]$$

$$[\widehat{\boldsymbol{b}}] = [\boldsymbol{X}'\boldsymbol{X}]^{-1}[\boldsymbol{X}'\boldsymbol{y}]$$

In matrix format, mixed effects model

$$\boldsymbol{y} = \boldsymbol{Xb} + \boldsymbol{Za} + \boldsymbol{e}$$

$$\begin{bmatrix} \boldsymbol{X'X} & \boldsymbol{X'Z} \\ \boldsymbol{Z'X} & \boldsymbol{Z'Z} + \boldsymbol{I}\frac{\sigma_e^2}{\sigma_a^2} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{b}} \\ \boldsymbol{a} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X'y} \\ \boldsymbol{Z'y} \end{bmatrix}$$

$$\begin{bmatrix} \hat{\boldsymbol{b}} \\ \boldsymbol{a} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X'X} & \boldsymbol{X'Z} \\ \boldsymbol{Z'X} & \boldsymbol{Z'Z} + \boldsymbol{I}\frac{\sigma_e^2}{\sigma_a^2} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{X'y} \\ \boldsymbol{Z'y} \end{bmatrix}$$

Mixed effects model example

Ex. Weights on beef calves taken at 200 days of age.

Males	Females
198	187
211	194
220	202
	185

$$y_{ij} = s_i + c_j + e_{ij}$$

y_{ij} 200-day weight

s_i effect of sex of calf (fixed effect)

c_j effect of calf (random effect)

e_{ij} residual, or unexplained, variation (random effect)

Mixed effects model example

Ex. Weights on beef calves taken at 200 days of age.

Males	Females
198	187
211	194
220	202
	185

$$y_{ij} = s_i + c_j + e_{ij}$$

y_{ij} 200-day weight

s_i effect of sex of calf (fixed effect)

c_j effect of calf (random effect)

e_{ij} residual, or unexplained, variation (random effect)

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{s} + \boldsymbol{Z}\boldsymbol{c} + \boldsymbol{e}$$

$\boldsymbol{y}$ vector of weights

$\boldsymbol{s}$ vector of effect of sex

$\boldsymbol{c}$ vector of effect of calf

$\boldsymbol{e}$ vector of random residual effects

$\boldsymbol{X}$ and $\boldsymbol{Z}$ incidence matrices

Mixed effects model example

Ex. Weights on beef calves taken at 200 days of age.

Males	Females
198	187
211	194
220	202
	185

$$y_{ij} = s_i + c_j + e_{ij}$$

y_{ij} 200-day weight

s_i effect of sex of calf (fixed effect)

c_j effect of calf (random effect)

e_{ij} residual, or unexplained, variation (random effect)

$$\boldsymbol{y} = \boldsymbol{Xs} + \boldsymbol{Zc} + \boldsymbol{e}$$

$\boldsymbol{y}$ vector of weights

$\boldsymbol{s}$ vector of effect of sex

$\boldsymbol{c}$ vector of effect of calf

$\boldsymbol{e}$ vector of random residual effects

$\boldsymbol{X}$ and $\boldsymbol{Z}$ incidence matrices

$$\begin{bmatrix} \hat{\boldsymbol{s}} \\ \boldsymbol{c} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X'X} & \boldsymbol{X'Z} \\ \boldsymbol{Z'X} & \boldsymbol{Z'Z} + \boldsymbol{I}\dfrac{\sigma_e^2}{\sigma_c^2} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{X'y} \\ \boldsymbol{Z'y} \end{bmatrix}$$

Centre for Tropical Livestock Genetics and Health

THE UNIVERSITY of EDINBURGH · ROSLIN · SRUC · ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Centre for Tropical Livestock Genetics and Health

Constructing matrices

Males	Females
198	187
211	194
220	202
	185

$$\boldsymbol{y} = \boldsymbol{Xs} + \boldsymbol{Zc} + \boldsymbol{e}$$

$\boldsymbol{y}$	vector of weights
$\boldsymbol{s}$	vector of effect of sex
$\boldsymbol{c}$	vector of effect of calf
$\boldsymbol{e}$	vector of random residual effects
$\boldsymbol{X}$ and $\boldsymbol{Z}$	incidence matrices

$$\begin{bmatrix} \hat{\boldsymbol{s}} \\ \boldsymbol{c} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X'X} & \boldsymbol{X'Z} \\ \boldsymbol{Z'X} & \boldsymbol{Z'Z} + \boldsymbol{I}\dfrac{\sigma_e^2}{\sigma_c^2} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{X'y} \\ \boldsymbol{Z'y} \end{bmatrix}$$

Centre for Tropical Livestock Genetics and Health

Constructing matrices

Males	Females
198	187
211	194
220	202
	185

$$\boldsymbol{y} = \begin{bmatrix} 198 \\ 211 \\ 220 \\ 187 \\ 194 \\ 202 \\ 185 \end{bmatrix}$$

$$\boldsymbol{y} = \boldsymbol{Xs} + \boldsymbol{Zc} + \boldsymbol{e}$$

$\boldsymbol{y}$	vector of weights
$\boldsymbol{s}$	vector of effect of sex
$\boldsymbol{c}$	vector of effect of calf
$\boldsymbol{e}$	vector of random residual effects
$\boldsymbol{X}$ and $\boldsymbol{Z}$	incidence matrices

$$\begin{bmatrix} \hat{\boldsymbol{s}} \\ \boldsymbol{c} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z} \\ \boldsymbol{Z}'\boldsymbol{X} & \boldsymbol{Z}'\boldsymbol{Z} + \boldsymbol{I}\dfrac{\sigma_e^2}{\sigma_c^2} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{X}'\boldsymbol{y} \\ \boldsymbol{Z}'\boldsymbol{y} \end{bmatrix}$$

THE UNIVERSITY of EDINBURGH ROSLIN SRUC ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Constructing matrices

Centre for Tropical Livestock Genetics and Health

Males	Females
198	187
211	194
220	202
	185

$$\boldsymbol{y} = \begin{bmatrix}198\\211\\220\\187\\194\\202\\185\end{bmatrix} \quad \boldsymbol{X} = \begin{bmatrix}1 & 1\\1 & 1\\1 & 1\\1 & 0\\1 & 0\\1 & 0\\1 & 0\end{bmatrix}$$

weights mean sex

$$\boldsymbol{y} = \boldsymbol{Xs} + \boldsymbol{Zc} + \boldsymbol{e}$$

$\boldsymbol{y}$	vector of weights
$\boldsymbol{s}$	vector of effect of sex
$\boldsymbol{c}$	vector of effect of calf
$\boldsymbol{e}$	vector of random residual effects
$\boldsymbol{X}$ and $\boldsymbol{Z}$	incidence matrices

$$\begin{bmatrix}\hat{\boldsymbol{s}}\\\boldsymbol{c}\end{bmatrix} = \begin{bmatrix}\boldsymbol{X'X} & \boldsymbol{X'Z}\\\boldsymbol{Z'X} & \boldsymbol{Z'Z} + \boldsymbol{I}\frac{\sigma_e^2}{\sigma_c^2}\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{X'y}\\\boldsymbol{Z'y}\end{bmatrix}$$

THE UNIVERSITY of EDINBURGH ROSLIN SRUC ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Constructing matrices

Males	Females
198	187
211	194
220	202
	185

$$\boldsymbol{y} = \boldsymbol{Xs} + \boldsymbol{Zc} + \boldsymbol{e}$$

$\boldsymbol{y}$	vector of weights
$\boldsymbol{s}$	vector of effect of sex
$\boldsymbol{c}$	vector of effect of calf
$\boldsymbol{e}$	vector of random residual effects
$\boldsymbol{X}$ and $\boldsymbol{Z}$	incidence matrices

$$\boldsymbol{y} = \begin{bmatrix} 198 \\ 211 \\ 220 \\ 187 \\ 194 \\ 202 \\ 185 \end{bmatrix} \quad \boldsymbol{X} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \quad \boldsymbol{Z} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

weights   mean   sex   calf 1   …   calf 7

$$\begin{bmatrix} \hat{\boldsymbol{s}} \\ \boldsymbol{c} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X'X} & \boldsymbol{X'Z} \\ \boldsymbol{Z'X} & \boldsymbol{Z'Z} + \boldsymbol{I}\dfrac{\sigma_e^2}{\sigma_c^2} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{X'y} \\ \boldsymbol{Z'y} \end{bmatrix}$$

Constructing matrices

Centre for Tropical Livestock Genetics and Health

Males	Females
198	187
211	194
220	202
	185

$$\boldsymbol{y} = \begin{bmatrix}198\\211\\220\\187\\194\\202\\185\end{bmatrix} \quad \boldsymbol{X} = \begin{bmatrix}1&1\\1&1\\1&1\\1&0\\1&0\\1&0\\1&0\end{bmatrix} \quad \boldsymbol{Z} = \begin{bmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix}$$

weights   mean   sex   calf 1 … calf 7

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{s} + \boldsymbol{Z}\boldsymbol{c} + \boldsymbol{e}$$

- $\boldsymbol{y}$ vector of weights
- $\boldsymbol{s}$ vector of effect of sex
- $\boldsymbol{c}$ vector of effect of calf
- $\boldsymbol{e}$ vector of random residual effects
- $\boldsymbol{X}$ and $\boldsymbol{Z}$ incidence matrices

$$\begin{bmatrix}\hat{\boldsymbol{s}}\\ \hat{\boldsymbol{c}}\end{bmatrix} = \begin{bmatrix}\boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z}\\ \boldsymbol{Z}'\boldsymbol{X} & \boldsymbol{Z}'\boldsymbol{Z} + \boldsymbol{I}\frac{\sigma_e^2}{\sigma_c^2}\end{bmatrix}^{-1} \begin{bmatrix}\boldsymbol{X}'\boldsymbol{y}\\ \boldsymbol{Z}'\boldsymbol{y}\end{bmatrix}$$

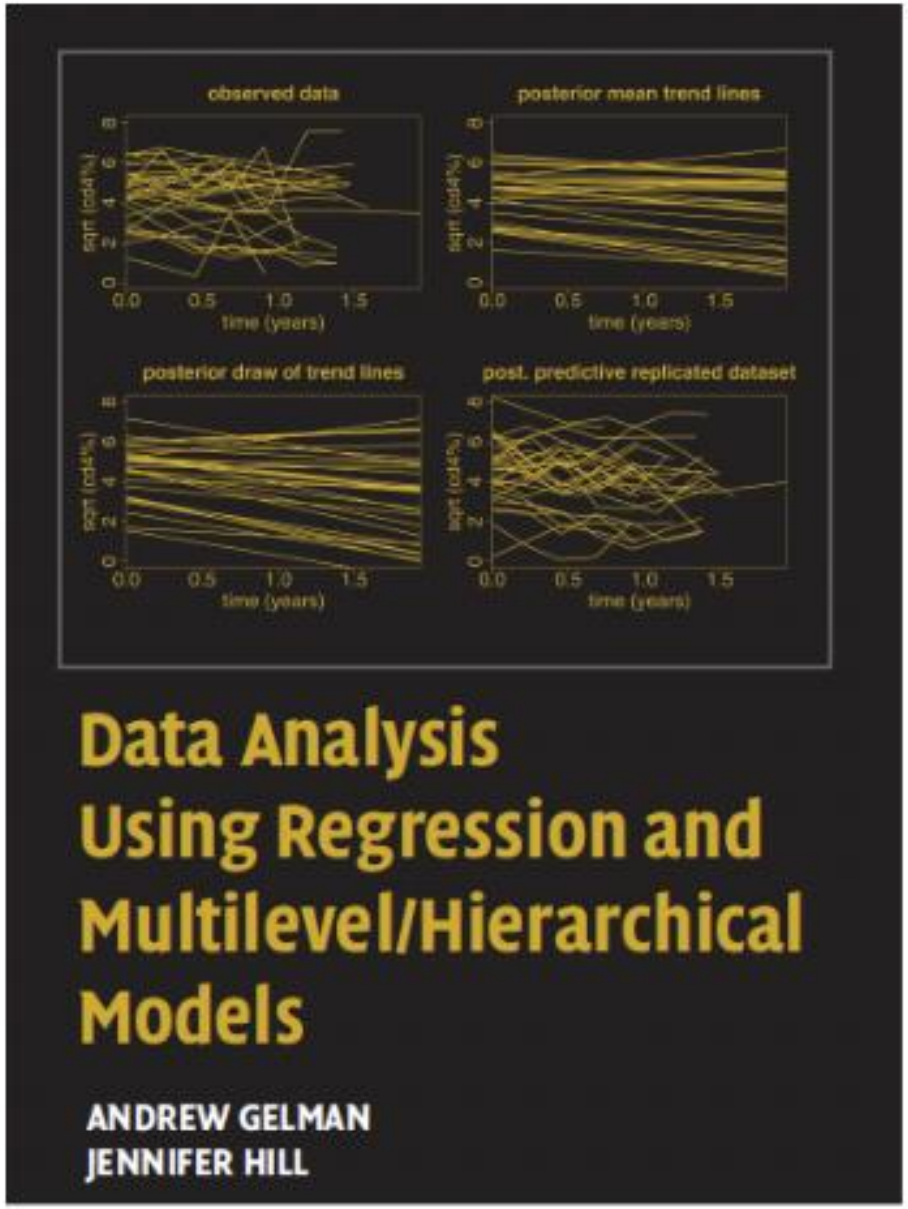
$\boldsymbol{I}$ = 7 by 7 incidence matrix (because random effect has seven levels (calves))

For this lecture, we will assume $\frac{\sigma_e^2}{\sigma_c^2}$ is known

THE UNIVERSITY of EDINBURGH   ROSLIN   SRUC   ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Centre for Tropical Livestock Genetics and Health

Study material



THE UNIVERSITY of EDINBURGH
ROSLIN
SRUC
ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Sources of phenotypic variation with linear models

Learning objectives:

- Introduction to linear models
- Parts of a model
  - Equation
    - Effects – fixed and random
  - Expectations, assumptions, limitations
- Developing a model
- Matrix format

Extra slides

Centre for Tropical Livestock Genetics and Health

Special (square) matrices

Square – equal number of rows and columns

$$\boldsymbol{S} = \begin{bmatrix} 4 & 7 & 0 \\ 0 & -5 & 3 \\ -7 & 3 & 11 \end{bmatrix}$$

Diagonal – all off-diagonal elements are zero

$$\boldsymbol{D} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Identity – diagonal matrix, where all diagonal elements are 1

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Triangular

Lower triangular, all elements above the diagonal are zero

$$\boldsymbol{L} = \begin{bmatrix} 7 & 0 & 0 \\ 2 & 3 & 0 \\ -4 & 15 & 9 \end{bmatrix}$$

Upper triangular, all elements above the diagonal are zero

$$\boldsymbol{U} = \begin{bmatrix} 13 & -2 & 7 \\ 0 & 3 & 11 \\ 0 & 0 & -1 \end{bmatrix}$$

Symmetric – elements above the diagonal equal to the corresponding elements below the diagonal

$$\boldsymbol{B} = \begin{bmatrix} 7 & 2 & -4 \\ 2 & 3 & 0 \\ -4 & 0 & -9 \end{bmatrix}$$

THE UNIVERSITY of EDINBURGH ROSLIN SRUC ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Addition

Two matrices can be added together if they have the same number of rows and columns, ie. They are of the same order and conformable

$$c_{ij} = a_{ij} + b_{ij}$$

$$\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B} = \begin{bmatrix} 9 & 2 \\ -11 & 5 \end{bmatrix} + \begin{bmatrix} 13 & -7 \\ 8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 9+13 & 2+(-7) \\ -11+8 & 5+4 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -5 \\ -3 & 9 \end{bmatrix}$$

Addition

Two matrices can be added together if they have the same number of rows and columns, ie. They are of the same order and conformable

$$c_{ij} = a_{ij} + b_{ij}$$

$$\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B} = \begin{bmatrix} 9 & 2 \\ -11 & 5 \end{bmatrix} + \begin{bmatrix} 13 & -7 \\ 8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 9 + 13 & 2 + (-7) \\ -11 + 8 & 5 + 4 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -5 \\ -3 & 9 \end{bmatrix}$$

Subtraction

A matrix can be subtracted from another if they have the same number of rows and columns, ie. They are of the same order and conformable

$$c_{ij} = a_{ij} + b_{ij}$$

$$\boldsymbol{C} = \boldsymbol{A} - \boldsymbol{B} = \begin{bmatrix} 9 & 2 \\ -11 & 5 \end{bmatrix} + \begin{bmatrix} 13 & -7 \\ 8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 9 - 13 & 2 - (-7) \\ -11 - 8 & 5 - 4 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 9 \\ -19 & 1 \end{bmatrix}$$

Centre for Tropical Livestock Genetics and Health

THE UNIVERSITY of EDINBURGH · ROSLIN · SRUC · ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Multiplication

Two matrices can be multiplied if the number of columns in the first matrix equals the number of rows in the second

The product matrix has an order of the number of rows of the first matrix by the number of columns of the second

$$c_{ij} = \sum_{j=i}^{m} \sum_{i=1}^{n} \sum_{k=1}^{z} e_{ik} g_{kj}$$

$$\boldsymbol{C} = \boldsymbol{EG} = \begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 4 \\ 4 & -3 & 6 \end{bmatrix} \begin{bmatrix} 4 & -5 \\ 2 & 3 \\ -6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1(4) + 5(2) + 3(-6) & -1(-5) + 5(3) + 3(1) \\ 6(4) + (-2)(2) + 4(-6) & 6(-5) + (-2)(3) + 4(1) \\ 4(4) + (-3)(2) + 6(-6) & 4(-5) + (-3)(3) + 6(1) \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 23 \\ -4 & -32 \\ -26 & -23 \end{bmatrix}$$

Multiplication

Two matrices can be multiplied if the number of columns in the first matrix equals the number of rows in the second

The product matrix has an order of the number of rows of the first matrix by the number of columns of the second

$$c_{ij} = \sum_{j=i}^{m}\sum_{i=1}^{n}\sum_{k=1}^{z} a_{ik}b_{kj}$$

$$\boldsymbol{C} = \boldsymbol{EG} = \begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 4 \\ 4 & -3 & 6 \end{bmatrix}\begin{bmatrix} 4 & -5 \\ 2 & 3 \\ -6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1(4)+5(2)+3(-6) & -1(-5)+5(3)+3(1) \\ 6(4)+(-2)(2)+4(-6) & 6(-5)+(-2)(3)+4(1) \\ 4(4)+(-3)(2)+6(-6) & 4(-5)+(-3)(3)+6(1) \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 23 \\ -4 & -32 \\ -26 & -23 \end{bmatrix}$$

Multiplication

Two matrices can be multiplied if the number of columns in the first matrix equals the number of rows in the second

The product matrix has an order of the number of rows of the first matrix by the number of columns of the second

$$c_{ij} = \sum_{j=i}^{m}\sum_{i=1}^{n}\sum_{k=1}^{z} a_{ik}b_{kj}$$

$$\boldsymbol{C} = \boldsymbol{EG} = \begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 4 \\ 4 & -3 & 6 \end{bmatrix}\begin{bmatrix} 4 & -5 \\ 2 & 3 \\ -6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1(4) + 5(2) + 3(-6) & -1(-5) + 5(3) + 3(1) \\ 6(4) + (-2)(2) + 4(-6) & 6(-5) + (-2)(3) + 4(1) \\ 4(4) + (-3)(2) + 6(-6) & 4(-5) + (-3)(3) + 6(1) \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 23 \\ -4 & -32 \\ -26 & -23 \end{bmatrix}$$

Centre for Tropical Livestock Genetics and Health

Multiplication

Two matrices can be multiplied if the number of columns in the first matrix equals the number of rows in the second

The product matrix has an order of the number of rows of the first matrix by the number of columns of the second

$$c_{ij} = \sum_{j=i}^{m} \sum_{i=1}^{n} \sum_{k=1}^{z} a_{ik} b_{kj}$$

$$\boldsymbol{C} = \boldsymbol{EG} = \begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 4 \\ 4 & -3 & 6 \end{bmatrix} \begin{bmatrix} 4 & -5 \\ 2 & 3 \\ -6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1(4) + 5(2) + 3(-6) & -1(-5) + 5(3) + 3(1) \\ 6(4) + (-2)(2) + 4(-6) & 6(-5) + (-2)(3) + 4(1) \\ 4(4) + (-3)(2) + 6(-6) & 4(-5) + (-3)(3) + 6(1) \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 23 \\ -4 & -32 \\ -26 & -23 \end{bmatrix}$$

Multiplication

Two matrices can be multiplied if the number of columns in the first matrix equals the number of rows in the second

The product matrix has an order of the number of rows of the first matrix by the number of columns of the second

$$c_{ij} = \sum_{j=i}^{m}\sum_{i=1}^{n}\sum_{k=1}^{z} a_{ik} b_{kj}$$

$$\boldsymbol{C} = \boldsymbol{EG} = \begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 4 \\ 4 & -3 & 6 \end{bmatrix} \begin{bmatrix} 4 & -5 \\ 2 & 3 \\ -6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1(4) + 5(2) + 3(-6) & -1(-5) + 5(3) + 3(1) \\ 6(4) + (-2)(2) + 4(-6) & 6(-5) + (-2)(3) + 4(1) \\ 4(4) + (-3)(2) + 6(-6) & 4(-5) + (-3)(3) + 6(1) \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 23 \\ -4 & -32 \\ -26 & -23 \end{bmatrix}$$

Multiplication

Two matrices can be multiplied if the number of columns in the first matrix equals the number of rows in the second

The product matrix has an order of the number of rows of the first matrix by the number of columns of the second

$$c_{ij} = \sum_{j=i}^{m} \sum_{i=1}^{n} \sum_{k=1}^{z} a_{ik} b_{kj}$$

$$\boldsymbol{C} = \boldsymbol{EG} = \begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 4 \\ 4 & -3 & 6 \end{bmatrix} \begin{bmatrix} 4 & -5 \\ 2 & 3 \\ -6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1(4) + 5(2) + 3(-6) & -1(-5) + 5(3) + 3(1) \\ 6(4) + (-2)(2) + 4(-6) & 6(-5) + (-2)(3) + 4(1) \\ 4(4) + (-3)(2) + 6(-6) & 4(-5) + (-3)(3) + 6(1) \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 23 \\ -4 & -32 \\ -26 & -23 \end{bmatrix}$$

Multiplication

Two matrices can be multiplied if the number of columns in the first matrix equals the number of rows in the second

The product matrix has an order of the number of rows of the first matrix by the number of columns of the second

$$c_{ij} = \sum_{j=i}^{m} \sum_{i=1}^{n} \sum_{k=1}^{z} a_{ik} b_{kj}$$

$$\boldsymbol{C} = \boldsymbol{EG} = \begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 4 \\ 4 & -3 & 6 \end{bmatrix} \begin{bmatrix} 4 & -5 \\ 2 & 3 \\ -6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1(4) + 5(2) + 3(-6) & -1(-5) + 5(3) + 3(1) \\ 6(4) + (-2)(2) + 4(-6) & 6(-5) + (-2)(3) + 4(1) \\ 4(4) + (-3)(2) + 6(-6) & 4(-5) + (-3)(3) + 6(1) \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 23 \\ -4 & -32 \\ -26 & -23 \end{bmatrix}$$

Centre for Tropical Livestock Genetics and Health

Multiplication

Two matrices can be multiplied if the number of columns in the first matrix equals the number of rows in the second

The product matrix has an order of the number of rows of the first matrix by the number of columns of the second

$$c_{ij} = \sum_{j=i}^{m} \sum_{i=1}^{n} \sum_{k=1}^{z} a_{ik} b_{kj}$$

$$\boldsymbol{C} = \boldsymbol{EG} = \begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 4 \\ 4 & -3 & 6 \end{bmatrix} \begin{bmatrix} 4 & -5 \\ 2 & 3 \\ -6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1(4) + 5(2) + 3(-6) & -1(-5) + 5(3) + 3(1) \\ 6(4) + (-2)(2) + 4(-6) & 6(-5) + (-2)(3) + 4(1) \\ 4(4) + (-3)(2) + 6(-6) & 4(-5) + (-3)(3) + 6(1) \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 23 \\ -4 & -32 \\ -26 & -23 \end{bmatrix}$$

THE UNIVERSITY of EDINBURGH ROSLIN SRUC ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Centre for Tropical Livestock Genetics and Health

Multiplication

Two matrices can be multiplied if the number of columns in the first matrix equals the number of rows in the second

The product matrix has an order of the number of rows of the first matrix by the number of columns of the second

$$c_{ij} = \sum_{j=i}^{m} \sum_{i=1}^{n} \sum_{k=1}^{z} a_{ik} b_{kj}$$

$$\boldsymbol{C} = \boldsymbol{EG} = \begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 4 \\ 4 & -3 & 6 \end{bmatrix} \begin{bmatrix} 4 & -5 \\ 2 & 3 \\ -6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1(4) + 5(2) + 3(-6) & -1(-5) + 5(3) + 3(1) \\ 6(4) + (-2)(2) + 4(-6) & 6(-5) + (-2)(3) + 4(1) \\ 4(4) + (-3)(2) + 6(-6) & 4(-5) + (-3)(3) + 6(1) \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 23 \\ -4 & -32 \\ -26 & -23 \end{bmatrix}$$

Multiplying a scalar with a matrix

Each element of the matrix is multiplied by the scalar

$$\boldsymbol{C} = 4 \times \boldsymbol{E} = \boldsymbol{E} \times 4 = 4 \times \begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 4 \\ 4 & -3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 4(-1) & 4(5) & 4(3) \\ 4(6) & 4(-2) & 4(4) \\ 4(4) & 4(-3) & 4(6) \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 20 & 12 \\ 24 & -8 & 16 \\ 16 & -12 & 24 \end{bmatrix}$$

Centre for Tropical Livestock Genetics and Health

Multiplying a scalar with a matrix

Each element of the matrix is multiplied by the scalar

$$\boldsymbol{C} = 4 \times \boldsymbol{E} = \boldsymbol{E} \times 4 = 4 \times \begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 4 \\ 4 & -3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 4(-1) & 4(5) & 4(3) \\ 4(6) & 4(-2) & 4(4) \\ 4(4) & 4(-3) & 4(6) \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 20 & 12 \\ 24 & -8 & 16 \\ 16 & -12 & 24 \end{bmatrix}$$

THE UNIVERSITY of EDINBURGH ROSLIN SRUC ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Direct product of matrices – Kronecker product

For **H** (order *r* x *s*), and **E** (order *m* x *n*), the direct product, **C**, has an order of *rm* x *sn*

$$\boldsymbol{C} = \boldsymbol{H} \otimes \boldsymbol{E} = \begin{bmatrix} h_{11}\boldsymbol{E} & h_{12}\boldsymbol{E} \\ h_{21}\boldsymbol{E} & h_{22}\boldsymbol{E} \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} 4 & -5 \\ 2 & 3 \end{bmatrix} \otimes \begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 4 \\ 4 & -3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 20 & 12 & 5 & -25 & -15 \\ 24 & -8 & 16 & -30 & 10 & -20 \\ 16 & -12 & 24 & -20 & 15 & -30 \\ -2 & 10 & 6 & -3 & 15 & 9 \\ 12 & -4 & 8 & 18 & -6 & 12 \\ 8 & -6 & 12 & 12 & -9 & 18 \end{bmatrix}$$

Centre for Tropical Livestock Genetics and Health

Direct product of matrices – Kronecker product

For **H** (order *r* x *s*), and **E** (order *m* x *n*), the direct product, **C**, has an order of *rm* x *sn*

$$\boldsymbol{C} = \boldsymbol{H} \otimes \boldsymbol{E} = \begin{bmatrix} h_{11}\boldsymbol{E} & h_{12}\boldsymbol{E} \\ h_{21}\boldsymbol{E} & h_{22}\boldsymbol{E} \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} 4 & -5 \\ 2 & 3 \end{bmatrix} \otimes \begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 4 \\ 4 & -3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 20 & 12 & 5 & -25 & -15 \\ 24 & -8 & 16 & -30 & 10 & -20 \\ 16 & -12 & 24 & -20 & 15 & -30 \\ -2 & 10 & 6 & -3 & 15 & 9 \\ 12 & -4 & 8 & 18 & -6 & 12 \\ 8 & -6 & 12 & 12 & -9 & 18 \end{bmatrix}$$

Direct product of matrices – Kronecker product

Centre for Tropical Livestock Genetics and Health

For **H** (order *r* x *s*), and **E** (order *m* x *n*), the direct product, **C**, has an order of *rm* x *sn*

$$\boldsymbol{C} = \boldsymbol{H} \otimes \boldsymbol{E} = \begin{bmatrix} h_{11}\boldsymbol{E} & h_{12}\boldsymbol{E} \\ h_{21}\boldsymbol{E} & h_{22}\boldsymbol{E} \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} 4 & -5 \\ 2 & 3 \end{bmatrix} \otimes \begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 4 \\ 4 & -3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 20 & 12 & 5 & -25 & -15 \\ 24 & -8 & 16 & -30 & 10 & -20 \\ 16 & -12 & 24 & -20 & 15 & -30 \\ -2 & 10 & 6 & -3 & 15 & 9 \\ 12 & -4 & 8 & 18 & -6 & 12 \\ 8 & -6 & 12 & 12 & -9 & 18 \end{bmatrix}$$

Direct product of matrices – Kronecker product

For **H** (order *r* x *s*), and **E** (order *m* x *n*), the direct product, **C**, has an order of *rm* x *sn*

$$\boldsymbol{C} = \boldsymbol{H} \otimes \boldsymbol{E} = \begin{bmatrix} h_{11}\boldsymbol{E} & h_{12}\boldsymbol{E} \\ h_{21}\boldsymbol{E} & h_{22}\boldsymbol{E} \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} 4 & -5 \\ 2 & 3 \end{bmatrix} \otimes \begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 4 \\ 4 & -3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 20 & 12 & 5 & -25 & -15 \\ 24 & -8 & 16 & -30 & 10 & -20 \\ 16 & -12 & 24 & -20 & 15 & -30 \\ -2 & 10 & 6 & -3 & 15 & 9 \\ 12 & -4 & 8 & 18 & -6 & 12 \\ 8 & -6 & 12 & 12 & -9 & 18 \end{bmatrix}$$

Centre for Tropical Livestock Genetics and Health

Direct product of matrices – Kronecker product

For **H** (order *r* x *s*), and **E** (order *m* x *n*), the direct product, **C**, has an order of *rm* x *sn*

$$\boldsymbol{C} = \boldsymbol{H} \otimes \boldsymbol{E} = \begin{bmatrix} h_{11}\boldsymbol{E} & h_{12}\boldsymbol{E} \\ h_{21}\boldsymbol{E} & h_{22}\boldsymbol{E} \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} 4 & -5 \\ 2 & 3 \end{bmatrix} \otimes \begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 4 \\ 4 & -3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 20 & 12 & 5 & -25 & -15 \\ 24 & -8 & 16 & -30 & 10 & -20 \\ 16 & -12 & 24 & -20 & 15 & -30 \\ -2 & 10 & 6 & -3 & 15 & 9 \\ 12 & -4 & 8 & 18 & -6 & 12 \\ 8 & -6 & 12 & 12 & -9 & 18 \end{bmatrix}$$

Direct product of matrices – Kronecker product

For **H** (order *r* x *s*), and **E** (order *m* x *n*), the direct product, **C**, has an order of *rm* x *sn*

$$\boldsymbol{C} = \boldsymbol{H} \otimes \boldsymbol{E} = \begin{bmatrix} h_{11}\boldsymbol{E} & h_{12}\boldsymbol{E} \\ h_{21}\boldsymbol{E} & h_{22}\boldsymbol{E} \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} 4 & -5 \\ 2 & 3 \end{bmatrix} \otimes \begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 4 \\ 4 & -3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 20 & 12 & 5 & -25 & -15 \\ 24 & -8 & 16 & -30 & 10 & -20 \\ 16 & -12 & 24 & -20 & 15 & -30 \\ -2 & 10 & 6 & -3 & 15 & 9 \\ 12 & -4 & 8 & 18 & -6 & 12 \\ 8 & -6 & 12 & 12 & -9 & 18 \end{bmatrix}$$

Hadamard product

$$\boldsymbol{H} \odot \boldsymbol{X}$$
$$(\boldsymbol{H} \odot \boldsymbol{X})_{\boldsymbol{ij}} = h_{ij}x_{ij}$$

For two matrices of the same order, the product has the same dimensions

$$\boldsymbol{H} \odot \boldsymbol{X} = \boldsymbol{X} \odot \boldsymbol{H}$$

Hadamard product

$$\boldsymbol{H} \odot \boldsymbol{X}$$

$$(\boldsymbol{H} \odot \boldsymbol{X})_{\boldsymbol{ij}} = h_{ij}x_{ij}$$

For two matrices of the same order, the product has the same dimensions

$$\boldsymbol{H} \odot \boldsymbol{X} = \boldsymbol{X} \odot \boldsymbol{H}$$

$$\boldsymbol{E} \odot \boldsymbol{X} = \begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 4 \\ 4 & -3 & 6 \end{bmatrix} \odot \begin{bmatrix} 7 & 2 & -4 \\ 2 & 3 & 0 \\ -4 & 0 & -9 \end{bmatrix}$$

Hadamard product

Centre for Tropical Livestock Genetics and Health

$$\boldsymbol{H} \odot \boldsymbol{X}$$
$$(\boldsymbol{H} \odot \boldsymbol{X})_{\boldsymbol{ij}} = h_{ij}x_{ij}$$

For two matrices of the same order, the product has the same dimensions

$$\boldsymbol{H} \odot \boldsymbol{X} = \boldsymbol{X} \odot \boldsymbol{H}$$

$$\boldsymbol{E} \odot \boldsymbol{X} = \begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 4 \\ 4 & -3 & 6 \end{bmatrix} \odot \begin{bmatrix} 7 & 2 & -4 \\ 2 & 3 & 0 \\ -4 & 0 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} -1(7) & 5(2) & 3(-4) \\ 6(2) & (-2)(3) & 4(0) \\ 4(-4) & -3(0) & 6(-9) \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 10 & -12 \\ 12 & -6 & 0 \\ -16 & 0 & -54 \end{bmatrix}$$

THE UNIVERSITY of EDINBURGH ROSLIN SRUC ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Hadamard product

Centre for Tropical Livestock Genetics and Health

$$\boldsymbol{H} \odot \boldsymbol{X}$$
$$(\boldsymbol{H} \odot \boldsymbol{X})_{\boldsymbol{ij}} = h_{ij} x_{ij}$$

For two matrices of the same order, the product has the same dimensions

$$\boldsymbol{H} \odot \boldsymbol{X} = \boldsymbol{X} \odot \boldsymbol{H}$$

$$\boldsymbol{E} \odot \boldsymbol{X} = \begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 4 \\ 4 & -3 & 6 \end{bmatrix} \odot \begin{bmatrix} 7 & 2 & -4 \\ 2 & 3 & 0 \\ -4 & 0 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} -1(7) & 5(2) & 3(-4) \\ 6(2) & (-2)(3) & 4(0) \\ 4(-4) & -3(0) & 6(-9) \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 10 & -12 \\ 12 & -6 & 0 \\ -16 & 0 & -54 \end{bmatrix}$$

THE UNIVERSITY of EDINBURGH ROSLIN SRUC ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Hadamard product

$$\boldsymbol{H} \odot \boldsymbol{X}$$
$$(\boldsymbol{H} \odot \boldsymbol{X})_{\boldsymbol{ij}} = h_{ij}x_{ij}$$

$$\boldsymbol{H} \odot \boldsymbol{X} = \boldsymbol{X} \odot \boldsymbol{H}$$

For two matrices of the same order, the product has the same dimensions

$$\boldsymbol{E} \odot \boldsymbol{X} = \begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 4 \\ 4 & -3 & 6 \end{bmatrix} \odot \begin{bmatrix} 7 & 2 & -4 \\ 2 & 3 & 0 \\ -4 & 0 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} -1(7) & 5(2) & 3(-4) \\ 6(2) & (-2)(3) & 4(0) \\ 4(-4) & (-3)(0) & 6(-9) \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 10 & -12 \\ 12 & -6 & 0 \\ -16 & 0 & -54 \end{bmatrix}$$

Centre for Tropical Livestock Genetics and Health
THE UNIVERSITY of EDINBURGH
ROSLIN
SRUC
ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Hadamard product

$$\boldsymbol{H} \odot \boldsymbol{X}$$
$$(\boldsymbol{H} \odot \boldsymbol{X})_{\boldsymbol{ij}} = h_{ij}x_{ij}$$

For two matrices of the same order, the product has the same dimensions

$$\boldsymbol{H} \odot \boldsymbol{X} = \boldsymbol{X} \odot \boldsymbol{H}$$

$$\boldsymbol{E} \odot \boldsymbol{X} = \begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 4 \\ 4 & -3 & 6 \end{bmatrix} \odot \begin{bmatrix} 7 & 2 & -4 \\ 2 & 3 & 0 \\ -4 & 0 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} -1(7) & 5(2) & 3(-4) \\ 6(2) & (-2)(3) & 4(0) \\ 4(-4) & (-3)(0) & 6(-9) \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 10 & -12 \\ 12 & -6 & 0 \\ -16 & 0 & -54 \end{bmatrix}$$

Transpose

For **J**, with order *i* x *k*, the transpose, **J'** has order *i* x *k*

$$\boldsymbol{J} = \begin{bmatrix} 5 & 3 \\ -1 & 4 \\ 6 & -2 \end{bmatrix}$$

$$\boldsymbol{J}' = \boldsymbol{J}^{\boldsymbol{T}} = \begin{bmatrix} 5 & -1 & 6 \\ 3 & 4 & -2 \end{bmatrix}$$

Transpose

Centre for Tropical Livestock Genetics and Health

For **J**, with order *i* x *k*, the transpose, **J'** has order *i* x *k*

$$\boldsymbol{J} = \begin{bmatrix} 5 & 3 \\ -1 & 4 \\ 6 & -2 \end{bmatrix}$$

$$\boldsymbol{J}' = \boldsymbol{J}^{\boldsymbol{T}} = \begin{bmatrix} 5 & -1 & 6 \\ 3 & 4 & -2 \end{bmatrix}$$

THE UNIVERSITY of EDINBURGH ROSLIN SRUC ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Transpose

For **J**, with order *i* x *k*, the transpose, **J'** has order *i* x *k*

$$\boldsymbol{J} = \begin{bmatrix} 5 & 3 \\ -1 & 4 \\ 6 & -2 \end{bmatrix}$$

$$\boldsymbol{J}' = \boldsymbol{J}^{\boldsymbol{T}} = \begin{bmatrix} 5 & -1 & 6 \\ 3 & 4 & -2 \end{bmatrix}$$

J is not equal to **J'**

Transpose

For **J**, with order *i* x *k*, the transpose, **J'** has order *i* x *k*

$$\boldsymbol{J} = \begin{bmatrix} 5 & 3 \\ -1 & 4 \\ 6 & -2 \end{bmatrix}$$

$$J' = J^T = \begin{bmatrix} 5 & -1 & 6 \\ 3 & 4 & -2 \end{bmatrix}$$

A matrix is not equal to its transpose

Transpose

For **J**, with order *i* x *k*, the transpose, **J'** has order *i* x *k*

$$\boldsymbol{J} = \begin{bmatrix} 5 & 3 \\ -1 & 4 \\ 6 & -2 \end{bmatrix}$$

$$\boldsymbol{J}' = \boldsymbol{J}^{\boldsymbol{T}} = \begin{bmatrix} 5 & -1 & 6 \\ 3 & 4 & -2 \end{bmatrix}$$

A matrix is not equal to its transpose ... Unless it is a symmetric matrix

$$\boldsymbol{W} = \begin{bmatrix} 7 & 2 & -4 \\ 2 & 3 & 0 \\ -4 & 0 & -9 \end{bmatrix}$$

$$\boldsymbol{W}' = \begin{bmatrix} 7 & 2 & -4 \\ 2 & 3 & 0 \\ -4 & 0 & -9 \end{bmatrix}$$

Transpose

For **J**, with order *i* x *k*, the transpose, **J'** has order *i* x *k*

$$\boldsymbol{J} = \begin{bmatrix} 5 & 3 \\ -1 & 4 \\ 6 & -2 \end{bmatrix}$$

$$\boldsymbol{J}' = \boldsymbol{J}^{\boldsymbol{T}} = \begin{bmatrix} 5 & -1 & 6 \\ 3 & 4 & -2 \end{bmatrix}$$

$$\boldsymbol{W} = \begin{bmatrix} 7 & 2 & -4 \\ 2 & 3 & 0 \\ -4 & 0 & -9 \end{bmatrix}$$

$$\boldsymbol{W}' = \begin{bmatrix} 7 & 2 & -4 \\ 2 & 3 & 0 \\ -4 & 0 & -9 \end{bmatrix}$$

A matrix is not equal to its transpose ... Unless it is a symmetric matrix

Dot product

The sum of the products of two equal length vectors

$$\boldsymbol{a} \cdot \boldsymbol{b} = \sum a_i b_i$$

Dot product

The sum of the products of two equal length vectors

$$\boldsymbol{a} \cdot \boldsymbol{b} = \sum a_i b_i$$

$$\boldsymbol{a} = [7, -2, 1]$$
$$\boldsymbol{b} = [3, 5, -4]$$

$$\boldsymbol{a} \cdot \boldsymbol{b} = 7(3) + (-2)(5) + 1(-4)$$
$$= 7$$

Centre for Tropical Livestock Genetics and Health

Dot product

The sum of the products of two equal length vectors

$$\boldsymbol{a} \cdot \boldsymbol{b} = \sum a_i b_i$$

$$\boldsymbol{a} = [7, -2, 1]$$
$$\boldsymbol{b} = [3, 5, -4]$$

$$\boldsymbol{a} \cdot \boldsymbol{b} = 7(3) + (-2)(5) + 1(-4)$$
$$= 7$$

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a}\boldsymbol{b}'$$

$$\boldsymbol{a} = [7 \quad -2 \quad 1]$$
$$\boldsymbol{b} = [3 \quad 5 \quad -4]$$

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a}\boldsymbol{b}' = [7 \quad -2 \quad 1]\begin{bmatrix} 3 \\ 5 \\ -4 \end{bmatrix}$$

$$= 7(3) + (-2)(5) + 1(-4)$$
$$= 7$$

Centre for Tropical Livestock Genetics and Health

Trace

The sum of the diagonal elements of a matrix

Scalar

$$\boldsymbol{W} = \begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{bmatrix}$$

$$tr(\boldsymbol{W}) = \sum w_{ii}$$

THE UNIVERSITY of EDINBURGH ROSLIN SRUC ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Trace

Centre for Tropical Livestock Genetics and Health

The sum of the diagonal elements of a matrix

Scalar

$$\boldsymbol{W} = \begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{bmatrix}$$

$$tr(\boldsymbol{W}) = \sum w_{ii}$$

$$\boldsymbol{W} = \begin{bmatrix} 7 & 2 & -4 \\ 2 & 3 & 0 \\ -4 & 0 & -9 \end{bmatrix}$$

$$tr(\boldsymbol{W}) = 7 + 3 + (-9) \\ = 1$$

THE UNIVERSITY of EDINBURGH ROSLIN SRUC ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Trace

The sum of the diagonal elements of a matrix

Scalar

$$\boldsymbol{W} = \begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{bmatrix}$$

$$tr(\boldsymbol{W}) = \sum w_{ii}$$

$$\boldsymbol{W} = \begin{bmatrix} 7 & 2 & -4 \\ 2 & 3 & 0 \\ -4 & 0 & -9 \end{bmatrix}$$

$$\begin{aligned} tr(\boldsymbol{W}) &= 7 + 3 + (-9) \\ &= 1 \end{aligned}$$

Centre for Tropical Livestock Genetics and Health

THE UNIVERSITY of EDINBURGH

ROSLIN

SRUC

ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Inversion

When **W**$^{-1}$ is multiplied by **W**, an identity matrix (**I**) is the product

$$\boldsymbol{W}^{-1}\boldsymbol{W} = \boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Only square matrices can be inverted

For diagonal matrix, **Y**, the inverse is the reciprocal of the diagonal elements

$$\boldsymbol{Y} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\boldsymbol{Y}^{-1} = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & -\frac{1}{10} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Inversion

When $\mathbf{W}^{-1}$ is multiplied by $\mathbf{W}$, an identity matrix ($\mathbf{I}$) is the product

Only square matrices can be inverted

For diagonal matrix, $\mathbf{Y}$, the inverse is the reciprocal of the diagonal elements

$$\boldsymbol{W}^{-1}\boldsymbol{W} = \boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{Y} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\boldsymbol{Y}^{-1} = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & -\frac{1}{10} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Inversion – 2x2 matrix example

$$\boldsymbol{Z} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$$

Inversion – 2x2 matrix example

Calculate the determinant (the difference between the product of the two diagonal elements and the two off-diagonal elements)

$$\boldsymbol{Z} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$$

$$\boldsymbol{Z}^{-1} = \frac{1}{z_{11}z_{22} - z_{12}z_{21}} \begin{bmatrix} z_{22} & -z_{12} \\ -z_{21} & z_{11} \end{bmatrix}$$

Inversion – 2x2 matrix example

Centre for Tropical Livestock Genetics and Health

Calculate the determinant (the difference between the product of the two diagonal elements and the two off-diagonal elements)

Reverse diagonal elements, multiplying off-diagonal elements by -1

$$\boldsymbol{Z} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$$

$$\boldsymbol{Z}^{-1} = \frac{1}{z_{11}z_{22} - z_{12}z_{21}} \begin{bmatrix} z_{22} & -z_{12} \\ -z_{21} & z_{11} \end{bmatrix}$$

THE UNIVERSITY of EDINBURGH ROSLIN SRUC ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Centre for Tropical Livestock Genetics and Health

Inversion – 2x2 matrix example

Calculate the determinant (the difference between the product of the two diagonal elements and the two off-diagonal elements)

Reverse diagonal elements, multiplying off-diagonal elements by -1

Divide all elements by the determinant

$$\boldsymbol{Z} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$$

$$\boldsymbol{Z}^{-1} = \frac{1}{z_{11}z_{22} - z_{12}z_{21}} \begin{bmatrix} z_{22} & -z_{12} \\ -z_{21} & z_{11} \end{bmatrix}$$

THE UNIVERSITY of EDINBURGH ROSLIN SRUC ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Centre for Tropical Livestock Genetics and Health

Inversion – 2x2 matrix example

Calculate the determinant (the difference between the product of the two diagonal elements and the two off-diagonal elements)

Reverse diagonal elements, multiplying off-diagonal elements by -1

Divide all elements by the determinant

$$\boldsymbol{Z} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$$

$$\boldsymbol{Z}^{-\mathbf{1}} = \frac{1}{z_{11}z_{22} - z_{12}z_{21}} \begin{bmatrix} z_{22} & -z_{12} \\ -z_{21} & z_{11} \end{bmatrix}$$

$$= \frac{1}{3(5) - (-1)(2)} \begin{bmatrix} 5 & -(-1) \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.29 & 0.06 \\ -0.12 & 0.18 \end{bmatrix}$$

Centre for Tropical Livestock Genetics and Health

Rank

The number of linearly independent rows or columns

A square matrix with rank equal to the number of rows or columns is **full rank**

THE UNIVERSITY of EDINBURGH ROSLIN SRUC ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Rank

$$2x_1 + 5x_2 + 2x_3 = y_1$$
$$5x_1 + 3x_2 + 1x_3 = y_2$$
$$-3x_1 + 2x_2 + 1x_3 = y_3$$

$$\begin{bmatrix} 2 & 5 & 2 \\ 5 & 3 & 1 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\boldsymbol{Nx} = \boldsymbol{y}$$

$$r(\boldsymbol{N}) = 2$$

The number of linearly independent rows or columns

A square matrix with rank equal to the number of rows or columns is **full rank**

In the example, **x'** (vector of solutions) cannot be estimated due to lack of information (because a unique inverse does not exist for **N** because it is not full rank)

If a square matrix is not full rank, the determinant is zero, and a unique inverse does not exist

Generalised inverse

$$NN^{-}N = N$$

A **generalised inverse** can be calculated for a **singular matrix**

Generalised inverses are not unique and can be calculated in different ways

Centre for Tropical Livestock Genetics and Health

Generalised inverse

A **generalised inverse** can be calculated for a **singular matrix**

Generalised inverses are not unique and can be calculated in different ways

$$\boldsymbol{N}\boldsymbol{N}^{-}\boldsymbol{N} = \boldsymbol{N}$$

$$\boldsymbol{N} = \begin{bmatrix} 2 & 5 & 2 \\ 5 & 3 & 1 \\ -3 & 2 & 1 \end{bmatrix}; \ \boldsymbol{A} = \begin{bmatrix} 2 & 5 \\ 5 & 3 \end{bmatrix}$$

THE UNIVERSITY of EDINBURGH ROSLIN SRUC ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Generalised inverse

A **generalised inverse** can be calculated for a **singular matrix**

Generalised inverses are not unique and can be calculated in different ways

$$\boldsymbol{N}\boldsymbol{N}^{-}\boldsymbol{N} = \boldsymbol{N}$$

$$\boldsymbol{N} = \begin{bmatrix} 2 & 5 & 2 \\ 5 & 3 & 1 \\ -3 & 2 & 1 \end{bmatrix}; \ \boldsymbol{A} = \begin{bmatrix} 2 & 5 \\ 5 & 3 \end{bmatrix}$$

$$\boldsymbol{A}^{-1} = \frac{1}{2(3) - 5(5)} \begin{bmatrix} 3 & -5 \\ -5 & 2 \end{bmatrix}$$

$$\boldsymbol{A}^{-1} = \begin{bmatrix} -0.16 & 0.26 \\ 0.26 & -0.11 \end{bmatrix}$$

Centre for Tropical Livestock Genetics and Health

Generalised inverse

A **generalised inverse** can be calculated for a **singular matrix**

Generalised inverses are not unique and can be calculated in different ways

$$\boldsymbol{N}\boldsymbol{N}^{-}\boldsymbol{N} = \boldsymbol{N}$$

$$\boldsymbol{N} = \begin{bmatrix} 2 & 5 & 2 \\ 5 & 3 & 1 \\ -3 & 2 & 1 \end{bmatrix};\ \boldsymbol{A} = \begin{bmatrix} 2 & 5 \\ 5 & 3 \end{bmatrix}$$

$$\boldsymbol{A}^{-1} = \frac{1}{2(3) - 5(5)} \begin{bmatrix} 3 & -5 \\ -5 & 2 \end{bmatrix}$$

$$\boldsymbol{A}^{-1} = \begin{bmatrix} -0.16 & 0.26 \\ 0.26 & -0.11 \end{bmatrix}$$

$$\boldsymbol{N}^{-} = \begin{bmatrix} -0.16 & 0.26 & 0 \\ 0.26 & -0.11 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Centre for Tropical Livestock Genetics and Health

Generalised inverse

A **generalised inverse** can be calculated for a **singular matrix**

Generalised inverses are not unique and can be calculated in different ways

$$\boldsymbol{N}\boldsymbol{N}^{-}\boldsymbol{N} = \boldsymbol{N}$$

$$\boldsymbol{N} = \begin{bmatrix} 2 & 5 & 2 \\ 5 & 3 & 1 \\ -3 & 2 & 1 \end{bmatrix};\ \boldsymbol{A} = \begin{bmatrix} 2 & 5 \\ 5 & 3 \end{bmatrix}$$

$$\boldsymbol{A}^{-1} = \frac{1}{2(3) - 5(5)} \begin{bmatrix} 3 & -5 \\ -5 & 2 \end{bmatrix}$$

$$\boldsymbol{A}^{-1} = \begin{bmatrix} -0.16 & 0.26 \\ 0.26 & -0.11 \end{bmatrix}$$

$$\boldsymbol{N}^{-} = \begin{bmatrix} -0.16 & 0.26 & 0 \\ 0.26 & -0.11 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigenvalues

Eigenvalues are useful in simplifying multivariate evaluations when transforming data

Sum of eigenvalues of a square matrix is equal to its trace (sum of diagonal elements)

Product of eigenvalues is equal to the determinant

For a symmetric matrix, the rank is equal to the number of non-zero eigenvalues

Eigenvalues and eigenvectors

If $\boldsymbol{N}$ is a square symmetric matrix:

$$\boldsymbol{N} = \boldsymbol{UDU}'$$

$\boldsymbol{D}$ is a diagonal matrix, the canonical form of $\boldsymbol{N}$, containing eigenvalues of $\boldsymbol{N}$, and $\boldsymbol{U}$ is an orthogonal matrix of the corresponding eigenvectors

$$|\boldsymbol{N} - d\boldsymbol{I}| = 0$$
$$\boldsymbol{Nu} - d\boldsymbol{u} = 0$$

d is one of the eigenvalues of $\boldsymbol{B}$, and $\boldsymbol{u}$ is the corresponding eigenvector

Cholesky decomposition

Symmetric positive definite matrices can be decomposed into a **product of a lower triangular matrix and its transpose:**

$$\boldsymbol{K} = \boldsymbol{V}\boldsymbol{V}'$$

$$\begin{bmatrix} 9 & 3 & -6 \\ 3 & 5 & 0 \\ -6 & 0 & 21 \end{bmatrix} = \begin{bmatrix} v_{11} & 0 & 0 \\ v_{21} & v_{22} & 0 \\ v_{31} & v_{32} & v_{33} \end{bmatrix} \begin{bmatrix} v_{11} & v_{21} & v_{31} \\ 0 & v_{22} & v_{32} \\ 0 & 0 & v_{33} \end{bmatrix}$$

Centre for Tropical Livestock Genetics and Health

Cholesky decomposition

Symmetric positive definite matrices can be decomposed into a **product of a lower triangular matrix and its transpose:**

$$\boldsymbol{K} = \boldsymbol{V}\boldsymbol{V}'$$

$$\begin{bmatrix} 9 & 3 & -6 \\ 3 & 5 & 0 \\ -6 & 0 & 21 \end{bmatrix} = \begin{bmatrix} v_{11} & 0 & 0 \\ v_{21} & v_{22} & 0 \\ v_{31} & v_{32} & v_{33} \end{bmatrix} \begin{bmatrix} v_{11} & v_{21} & v_{31} \\ 0 & v_{22} & v_{32} \\ 0 & 0 & v_{33} \end{bmatrix}$$

transpose

THE UNIVERSITY of EDINBURGH ROSLIN SRUC ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Centre for Tropical Livestock Genetics and Health

Cholesky decomposition

Symmetric positive definite matrices can be decomposed into a **product of a lower triangular matrix and its transpose:**

$$\boldsymbol{K} = \boldsymbol{V}\boldsymbol{V}'$$

$$\begin{bmatrix} 9 & 3 & -6 \\ 3 & 5 & 0 \\ -6 & 0 & 21 \end{bmatrix} = \begin{bmatrix} v_{11} & 0 & 0 \\ v_{21} & v_{22} & 0 \\ v_{31} & v_{32} & v_{33} \end{bmatrix} \begin{bmatrix} v_{11} & v_{21} & v_{31} \\ 0 & v_{22} & v_{32} \\ 0 & 0 & v_{33} \end{bmatrix}$$

transpose

$$\begin{bmatrix} 9 & 3 & -6 \\ 3 & 5 & 0 \\ -6 & 0 & 21 \end{bmatrix} = \begin{bmatrix} v_{11}(v_{11}) + 0(0) + 0(0) & v_{11}(v_{21}) + 0(v_{22}) + 0(0) & v_{11}(v_{31}) + 0(v_{32}) + 0(v_{33}) \\ v_{21}(v_{11}) + v_{22}(0) + 0(0) & v_{21}(v_{21}) + v_{22}(v_{22}) + 0(0) & v_{21}(v_{31}) + v_{22}(v_{32}) + 0(v_{33}) \\ v_{31}(v_{11}) + v_{32}(0) + v_{33}(0) & v_{31}(v_{21}) + v_{32}(v_{22}) + v_{33}(0) & v_{31}(v_{31}) + v_{32}(v_{32}) + v_{33}(v_{33}) \end{bmatrix}$$

multiplication

THE UNIVERSITY of EDINBURGH ROSLIN SRUC ILRI INTERNATIONAL LIVESTOCK RESEARCH INSTITUTE

Centre for Tropical Livestock Genetics and Health

Cholesky decomposition

Symmetric positive definite matrices can be decomposed into a **product of a lower triangular matrix and its transpose:**

$$\boldsymbol{K} = \boldsymbol{V}\boldsymbol{V}'$$

$$\begin{bmatrix} 9 & 3 & -6 \\ 3 & 5 & 0 \\ -6 & 0 & 21 \end{bmatrix} = \begin{bmatrix} v_{11} & 0 & 0 \\ v_{21} & v_{22} & 0 \\ v_{31} & v_{32} & v_{33} \end{bmatrix} \begin{bmatrix} v_{11} & v_{21} & v_{31} \\ 0 & v_{22} & v_{32} \\ 0 & 0 & v_{33} \end{bmatrix}$$

$$\begin{bmatrix} 9 & 3 & -6 \\ 3 & 5 & 0 \\ -6 & 0 & 21 \end{bmatrix} = \begin{bmatrix} v_{11}(v_{11}) + 0(0) + 0(0) & v_{11}(v_{21}) + 0(v_{22}) + 0(0) & v_{11}(v_{31}) + 0(v_{32}) + 0(v_{33}) \\ v_{21}(v_{11}) + v_{22}(0) + 0(0) & v_{21}(v_{21}) + v_{22}(v_{22}) + 0(0) & v_{21}(v_{31}) + v_{22}(v_{32}) + 0(v_{33}) \\ v_{31}(v_{11}) + v_{32}(0) + v_{33}(0) & v_{31}(v_{21}) + v_{32}(v_{22}) + v_{33}(0) & v_{31}(v_{31}) + v_{32}(v_{32}) + v_{33}(v_{33}) \end{bmatrix}$$

$$\begin{bmatrix} 9 & 3 & -6 \\ 3 & 5 & 0 \\ -6 & 0 & 21 \end{bmatrix} = \begin{bmatrix} v_{11}^2 & v_{11}v_{21} & v_{11}v_{31} \\ v_{11}v_{21} & v_{21}^2 + v_{22}^2 & v_{21}v_{31} + v_{22}v_{32} \\ v_{11}v_{31} & v_{21}v_{31} + v_{22}v_{32} & v_{31}^2 + v_{32}^{22} + v_{33}^2 \end{bmatrix}$$

Cholesky decomposition

Symmetric positive definite matrices can be decomposed into a **product of a lower triangular matrix and its transpose:**

$$\boldsymbol{K} = \boldsymbol{V}\boldsymbol{V}'$$

$$\begin{bmatrix} 9 & 3 & -6 \\ 3 & 5 & 0 \\ -6 & 0 & 21 \end{bmatrix} = \begin{bmatrix} v_{11}^2 & v_{11}v_{21} & v_{11}v_{31} \\ v_{11}v_{21} & v_{21}^2 + v_{22}^2 & v_{21}v_{31} + v_{22}v_{32} \\ v_{11}v_{31} & v_{21}v_{31} + v_{22}v_{32} & v_{31}^2 + v_{32}^{22} + v_{33}^2 \end{bmatrix}$$

$$k_{11} = 9 = v_{11}^2$$
$$v_{11} = \sqrt{9} = 3$$

Cholesky decomposition

Symmetric positive definite matrices can be decomposed into a **product of a lower triangular matrix and its transpose:**

$$\boldsymbol{K} = \boldsymbol{V}\boldsymbol{V}'$$

$$\begin{bmatrix} 9 & 3 & -6 \\ \boxed{3} & 5 & 0 \\ -6 & 0 & 21 \end{bmatrix} = \begin{bmatrix} v_{11}^2 & v_{11}v_{21} & v_{11}v_{31} \\ \boxed{v_{11}v_{21}} & v_{21}^2 + v_{22}^2 & v_{21}v_{31} + v_{22}v_{32} \\ v_{11}v_{31} & v_{21}v_{31} + v_{22}v_{32} & v_{31}^2 + v_{32}^{22} + v_{33}^2 \end{bmatrix}$$

$$k_{11} = 9 = v_{11}^2$$
$$v_{11} = \sqrt{9} = 3$$

$$\boxed{k_{21}} = \boxed{v_{11}v_{21}}$$
$$\boxed{3} = \boxed{3(v_{21})}$$
$$v_{21} = 1$$

Cholesky decomposition

Symmetric positive definite matrices can be decomposed into a **product of a lower triangular matrix and its transpose:**

$$\boldsymbol{K} = \boldsymbol{V}\boldsymbol{V}'$$

$$\begin{bmatrix} 9 & 3 & -6 \\ 3 & 5 & 0 \\ -6 & 0 & 21 \end{bmatrix} = \begin{bmatrix} v_{11}^2 & v_{11}v_{21} & v_{11}v_{31} \\ v_{11}v_{21} & v_{21}^2 + v_{22}^2 & v_{21}v_{31} + v_{22}v_{32} \\ v_{11}v_{31} & v_{21}v_{31} + v_{22}v_{32} & v_{31}^2 + v_{32}^{22} + v_{33}^2 \end{bmatrix}$$

$$k_{11} = 9 = v_{11}^2$$
$$v_{11} = \sqrt{9} = 3$$

$$k_{21} = v_{11}v_{21}$$
$$3 = 3(v_{21})$$
$$v_{21} = 1$$

$$k_{31} = v_{11}v_{31}$$
$$-6 = 3(v_{31})$$
$$v_{31} = -2$$

$$k_{22} = v_{21}^2 + v_{22}^2$$
$$5 = 1 + v_{22}^2$$
$$v_{22} = \sqrt{4} = 2$$

$$k_{32} = v_{21}v_{31} + v_{22}v_{32}$$
$$0 = 1(-2) + 2(v_{32})$$
$$v_{32} = 1$$

$$k_{33} = v_{31}^2 + v_{32}^2 + v_{33}^2$$
$$21 = (-2)^2 + (1)^2 + v_{33}^2$$
$$v_{33} = \sqrt{16} = 4$$

Centre for Tropical Livestock Genetics and Health

Cholesky decomposition

Symmetric positive definite matrices can be decomposed into a **product of a lower triangular matrix and its transpose:**

$$\boldsymbol{K} = \boldsymbol{V}\boldsymbol{V}'$$

$$\begin{bmatrix} 9 & 3 & -6 \\ 3 & 5 & 0 \\ -6 & 0 & 21 \end{bmatrix} = \begin{bmatrix} v_{11}^2 & v_{11}v_{21} & v_{11}v_{31} \\ v_{11}v_{21} & v_{21}^2 + v_{22}^2 & v_{21}v_{31} + v_{22}v_{32} \\ v_{11}v_{31} & v_{21}v_{31} + v_{22}v_{32} & v_{31}^2 + v_{32}^{22} + v_{33}^2 \end{bmatrix}; \qquad V = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ -2 & 1 & 4 \end{bmatrix}$$

$$k_{11} = 9 = v_{11}^2$$
$$v_{11} = \sqrt{9} = 3$$

$$k_{21} = v_{11}v_{21}$$
$$3 = 3(v_{21})$$
$$v_{21} = 1$$

$$k_{31} = v_{11}v_{31}$$
$$-6 = 3(v_{31})$$
$$v_{31} = -2$$

$$k_{22} = v_{21}^2 + v_{22}^2$$
$$5 = 1 + v_{22}^2$$
$$v_{22} = \sqrt{4} = 2$$

$$k_{32} = v_{21}v_{31} + v_{22}v_{32}$$
$$0 = 1(-2) + 2(v_{32})$$
$$v_{32} = 1$$

$$k_{33} = v_{31}^2 + v_{32}^2 + v_{33}^2$$
$$21 = (-2)^2 + (1)^2 + v_{33}^2$$
$$v_{33} = \sqrt{16} = 4$$