MATH 689 - Physics for Mathematicians

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Ausust 31, 2023

1 Continued from last class:

Lemma 1. *For any Hamiltonian $H : T^*M \to \mathbb{R}$, there exists a unique vector field, denoted by $\vec{H}$ such that*

$$\iota_{\vec{H}}\sigma = -dH. \tag{1}$$

Moreover, in the canonical coordinates $(p_1, \ldots, p_n, q^1, \ldots, q^n) \in T^*M$,

$$\vec{H} = \sum_i \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}. \tag{2}$$

Thus, a curve $\lambda(t)$ in T^*M satisfies $\lambda(t) = \vec{H}(\lambda(t))$ if and only if

$$\begin{cases} \dot{q}^i(t) = \dfrac{\partial H}{\partial p_i}(p(t), q(t)) \\ \dot{p}^i(t) = -\dfrac{\partial H}{\partial q^i}(p(t), q(t)) \end{cases}$$

Proof. (Continued from previous lecture) We shall now prove (2). Assume $\vec{H} = \sum_i v^i \frac{\partial}{\partial q^i} + w_i \frac{\partial}{\partial p_i}$ and let $z = \sum_i x_i \frac{\partial}{\partial q^i} + y_i \frac{\partial}{\partial p_i}$. Then

$$\begin{aligned} \sigma(\vec{H}, z) &= \sum_i (dp_i \wedge dq^i)(\vec{H}, z) \\ &= \sum_i w_i x^i - v^i y_i. \\ &= \sum_i w_i dq^i(z) - v^i dp_i(z). \end{aligned}$$

Therefore, $\iota_{\vec{H}} = \sum_i w_i dq^i - v^i dp_i$. Since

$$-dH = \sum_i -\frac{\partial H}{\partial q^i}dq^i - \frac{\partial H}{\partial p_i}dp_i,$$

we have that $w_i = -\frac{\partial H}{\partial q^i}$ and $v^i = \frac{\partial H}{\partial p_i}$. $\square$

Definition 1. *$\vec{H}$ is called the* ***Hamiltonian vector field*** *associated to H.*

2 Poisson Brackets

Let $H, G \in C^\infty(T^*M)$. Then the Poisson Bracket $\{H, G\}$ is given by:

$$\{H, G\} := \sigma(\vec{H}, \vec{G}) = dG(\vec{H}) = \vec{H}(G)$$

2.1 Properties of the Poisson Bracket:

- **Skew-symmetry:** $\{H, G\} = -\{G, H\}$. Follows from skew-symmetry of σ.
- **Jacobi Identity** Exercise 3 of Homework 1. Specifically, Since $\overrightarrow{\{H, G\}} = \left[\vec{H}, \vec{G}\right]$, then the Jacobi Identity for the Poisson bracket follows from the Lie bracket (??).
- Thus, the Poisson Bracket is a Lie bracket on $C^\infty(T^*M)$, making the space a (very big) Lie algebra!

2.2 Coordinate Expression for Poisson Brackets in canonical coordinates

We see that $\{H, G\} = \vec{H}(G) = \sum_i \frac{\partial H}{\partial p_i}\frac{\partial G}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial G}{\partial p_i}$, we have that $\{p_i, p_j\} = \{q^i, q^j\} = 0$ and $\{p_i, q^j\} = \delta_{ij}$.

3 First integrals in the Hamiltonian setting

Now, we consider the following Hamiltonian system:

$$\begin{cases} \dot{q}^i = \{H, q^i\} = \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i = \{H, p_i\} = -\dfrac{\partial H}{\partial q^i} \end{cases}$$

Let us consider the first integrals of motion of this system.

Definition 2. *Akin to the Lagrangian case, a* ***trajectory*** *$\lambda(t)$ of the Hamiltonian system defined by H is a curve in T^*M which is a solution of the system $\dot{\lambda} = \vec{H}(\lambda(t))$.*

Definition 3. *A function $G : T^*M \to \mathbb{R}$ is called a first integral of the Hamiltonian system $\dot{\lambda} = \vec{H}\lambda$ if, for any trajectory $\lambda(t)$ of the system, $G(\lambda(t))$ is constant.*

Proposition 1. *G is an integral of the Hamiltonian system defined by H if and only if $\{G, H\} = 0$.*

Proof. First, we see that, in coordinates,

$$\begin{aligned}
\frac{d}{dt}G(\lambda(t)) &= DG_{\lambda(t)}(\dot{\lambda}(t)) \\
&= DG_{\lambda(t)}\left(\vec{H}_{\lambda(t)}\right) \\
&= \sum_i \frac{\partial G}{\partial p_i}\left(\vec{H}_{\lambda(t)}\right)_{p_i} + \frac{\partial G}{\partial q^i}\left(\vec{H}_{\lambda(t)}\right)_{q^i} \\
&= \sum_i \frac{\partial G}{\partial p_i}\left(-\frac{\partial H}{\partial q^i}\right)_{p_i} + \frac{\partial G}{\partial q^i}\left(\frac{\partial H}{\partial p_i}\right) \\
&= \sum_i \frac{\partial H}{\partial p_i}\frac{\partial G}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial G}{\partial p_i} \\
&= \vec{H}(G).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&G(\lambda(t)) \equiv \text{constant} \\
&\Leftrightarrow \frac{d}{dt}G(\lambda(t)) \equiv 0 \\
&\Leftrightarrow \vec{H}(G) = 0 \\
&\Leftrightarrow \{H, G\} = 0
\end{aligned}$$

□

Corollary 1. *H is the integral of $\dot{\lambda} = \vec{H}(\lambda)$*

Proof. $\{H, H\} = 0$ by antisymmetry. □

Theorem 1 (Noether's Theorem in the Hamiltonian setting)**.** *Assume that the mechanical system is defined by a Lagrangian $L : TM \to \mathbb{R}$ and H is the corresponding Hamiltonian (via the Legendre transform). Let X be a vector field on M which generates a 1-parameter group of symmetries of L, φ^s. Then, for $q \in M, p \in T_q^*M$,*

$$H_X(p, q) = p(X(q)),$$

is an integral of the system $\dot{\lambda} = \vec{H}(\lambda)$.

Proof. Recall from the previous lecture that a curve $q(t) \in M$ is a solution to the Lagrangian system if and only if, for $p(t) = \frac{\partial L}{\partial \dot{q}}\big|_{(q(t),\dot{q}(t))}$, then the curve $\lambda(t) = (p(t), q(t))$ is a trajectory for the Hamiltonian system. We also recall that by the original Noether Theorem,

$$I(q, \dot{q}) = \frac{\partial L}{\partial \dot{q}^i}X^i$$

is a first integral for the Lagrangian system. Therefore, letting $\lambda(t) = (p(t), q(t))$ be a trajectory, we have that

$$
\begin{aligned}
H_X(\lambda(t)) &= p(t)\left(X_{q(t)}\right) \\
&= \frac{\partial L}{\partial \dot{q}}\Big|_{(q(t),\dot{q}(t))} X_{q(t)} \\
&= \frac{\partial L}{\partial \dot{q}}\Big|_{(q(t),\dot{q}(t))} \frac{\partial \varphi^s}{\partial s}\Big|_{s=0} \qquad\qquad = I(q(t),\dot{q}(t))
\end{aligned}
$$

This quantity is constant along $(q(t), \dot{q}(t))$, so $H_x(\lambda(t))$ is constant. $\square$

Remark 1. *H_X is called the quasi-impulse of X (by the analogy that $H_{\frac{\partial}{\partial q^i}} = p_i$). H_X is linear on each fiber, and vice-versa if $H \in C^\infty(T^*M)$ is linear on each fiber then there exists $X \in \mathcal{X}(M)$ such that $H = H_X$*

Remark 2. *If $X, Y \in \mathcal{X}(M)$, then $\{H_X, H_Y\} = H_{[X,Y]}$. Therefore, the algebra of infinitesimal symmetries of the Lagrangian (w.r.t Poisson Bracket) is isomorphic to the algebra of integrals of the Hamiltonian system $\dot{\lambda} = \vec{H}(\lambda)$ which is linear on fibers of T^*M.*

Example 1. *Integrals in $\mathbb{R}^3$*

Let $M = \mathbb{R}^3 \sim \vec{F}$. Let

$$L = \frac{m|\dot{\vec{r}}|^2}{2} - U(|\vec{r}|)$$

This is called the central field. Then, the group of symmetries of L is $SO(3)$, so

$$H_i = \langle \vec{r} \times m\dot{\vec{r}}, \vec{e}\rangle = \langle m\dot{\vec{r}}, \vec{e} \times \vec{r}\rangle$$

are integrals of E-L (identifying the vector $\vec{e} \times \vec{r}$ with the generator $Y_{\vec{e}} \in \mathfrak{so}(3)$. Also, $H_i(p, \vec{r}) = p(\vec{e} \times \vec{r})$ are integrals of $\dot{\lambda} = \vec{H}(\lambda)$. (Recall that $H = \frac{\|p\|^2}{2m} + U(\vec{r})$). It can be shown that

$$
\begin{matrix}
(Y_1, Y_2) = Y_3 & & \{H_1, H_2\} = H_3 \\
(Y_2, Y_3) = Y_1 & \Leftrightarrow & \{H_2, H_3\} = H_1 \\
(Y_3, Y_1) = Y_2 & & \{H_3, H_1\} = H_2
\end{matrix}\;.
$$

On the other hand, if $G = H_1^2 + H_2^2 + H_3^2$, then we claim that $\{G, H_i\} = 0$. For example,

$$
\begin{aligned}
\{G, H_3\} &= \{H_1^2 + H_2^2 + H_3^2, H_3\} \\
&= 2H_1\{H_1, H_3\} + 2H_2\{H_2, H_3\} + 2H_3\{H_3, H_3\} \\
&= 2H_1(-H_2) + 2H_2(H_1) + 2H_3(0) \\
&= 0
\end{aligned}
$$

Hence, (H, G, H_i) are integrals which pairwise commute.

Remark 3. *The construction of Hamiltonian vector fields and Poisson brackets can be done exactly the same way for any Symplectic manifold* (N, ω)*.*

All symplectic manifolds of the same dimension are locally equivalent, i.e. if (N_1, ω_1) and (N_2, ω_2) are 2 symplectic manifolds with $\dim N_1 = \dim N_2$, then for all $\lambda_1 \in N_1$ and $\lambda_2 \in N_2$, there exists a neighborhood U_1 and U_2 of each point and a diffeomorphism $\varphi : U_1 \to U_2$ such that $\varphi(\lambda_1) = \lambda_2$ and that $\varphi^* \omega_2 = \omega_1$.

Theorem 2 (Darboux's Theorem)**.** *If* (N, ω) *is a a symplectic manifold of dimension* $2n$*, then around every point* λ*, there exists a coordinate system* $(p_1, \ldots, p_n, q^1, \ldots, q^n)$ *such that* $\omega = \sum_i dp_i \wedge dq^i$*.*

WILL GO THROUGH THE PROOF LATER