MATH 689 - Physics for Mathematicians

Lectures by Igor Zelenko, transcribed by Seth Hoisington

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1 Recall: The Lorentz boost along x_1-axis

$$\begin{cases} c\tilde{t} = \dfrac{\frac{y}{c}x_1 + ct}{\sqrt{1-\frac{u^2}{c^2}}} \\ \tilde{x}_1 = \dfrac{x_1 + \frac{u}{c}(ct)}{\sqrt{1-\frac{u^2}{c^2}}} \\ \tilde{x}_2 = x_2 \\ \tilde{x}_3 = x_3 \end{cases} \Rightarrow \begin{cases} \dfrac{\tilde{E}}{c} = \dfrac{\frac{u}{c}p_1 + \frac{E}{c}}{\sqrt{1-\frac{u^2}{c^2}}} \\ \tilde{p}_1 = \dfrac{p_1 + \frac{u}{c}\frac{E}{c}}{\sqrt{1-\frac{u^2}{c^2}}} \\ \tilde{p}_2 = p_2 \\ \tilde{p}_3 = p_3 \end{cases}$$

2 More about 4-vectors and 4-tensors

We work in $\mathbb{R}^4$ with coordinates $(x^0 = ct, x^1, x^2, x^3)$. This space is called Minkowski space-time. The Lorentzian is defined by $\Delta\ell^2 = (\nabla x^0)^2 - (\nabla x^1)^2 - (\nabla x^2)^2 - (\nabla x^3)^2$. Let

$$(g_{\mu v}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Then, $(\Delta\ell)^2 = \sum_{\mu,\nu} g_{\mu\nu}\Delta x^\mu \Delta x^\nu =$. The indefinite inner product

$$a \cdot \hat{a} = a^0\hat{a}^0 - a^1\hat{a}^1 - a^2\hat{a}^2 - a^3\hat{a}^3$$

The linear transformation which preserves the Lorentzian inner product are called Lorentzian transformations, and are all of the form $x^\mu \mapsto \tilde{x}^\mu = \sum_{\nu=0}^3 a^\mu_\nu x^\nu$. H is Lorentzian if and only if $a_\rho(g_{\mu\nu})a_\sigma = \sum_{\mu,\nu} g_{\mu\nu} a^\mu_\rho a^\nu_\sigma = g_{\rho\sigma}$. It follows from this fact that if a_μ is a column vector of the matrix (a^ν_μ), then

$$a_0 \cdot a_0 = 1, a_\rho \cdot a_\rho = -1 \quad \text{for } \mu = 1, 2, 3, \text{ and for } \rho \neq \sigma \quad a_\rho \cdot a_\sigma = 0.$$

The set of all such Lorentz transformations forms a Lie group, called the Lorentz group, $O(1,3)$. In the last lecture, we saw that coordinate transformations between two inertial frames K and K' where the later is moving along the3 x^1-axis of the former with velocity u are described by Lorentz transformations when they preserve the origin (i.e. $(0,0) \mapsto (0,0)$). Therefore, the most general space-time coordinate transforms are affine transforms such that the linear part is Lorentzian. Transformations of this more general type are called **Poincare transformations** and the form a Lie group called the Poincare group.

Definition 1. *A **4-vector** is any 4-component quantity* (A_0, A_1, A_2, A_3) *depending on an inertial frame such that when passing to another inertial frame* $\tilde{K}$ *via a Poincaré transform, it transforms as* $A^\mu \to \tilde{A}^\mu = \sum_{\nu=0}^3 a_\nu^\mu A^\nu$.

It can be asked why it is appropriate to use the above definition as opposed to the standard, more general 4-vector as defined linear-algebraically. In this definition, care is taken to allow for the quantity labelled the 4-vector to be viewed in any inertial frame. This definition is useful since special relativity requires us to view all inertial frames equally, so all relevant four-component quantities in this context should be 4-vectors under this definition. A prime example of a 4-vector is given by the relativistic 4-momentum/energy-momentum vector $\left(\frac{E}{c}, p\right) = \left(\frac{E}{c}, p_1, p_2, p_3\right)$ as discussed in the previous lecture. Similarly, one can define a 4-covector and more generally the notion of 4-tensors of type (k, ℓ). To do so, we use the lowering-index convention: $(g^{\lambda\rho})_{\lambda,\rho=0}^3 = (g_{\mu\nu})^{-1}$.

Definition 2. *A **4-covector** is any 4-component quantity* (A_0, A_1, A_2, A_3) *such that* $A_\mu \mapsto \tilde{A}_\mu = \sum_{\nu=0}^3 a_\mu^\nu A_\nu$ *for a Lorentz transformation* (a_μ^ν)

Remark 1. *If* $(A^\mu)_{\mu=0}^3$ *is a 4-vector, then* $A_\mu = \sum_{\nu=0}^3 g_{\mu\nu} A^\nu$ *is a 4-covector. To see this, we observe that* $A^\nu = \sum_{\rho=0}^3 g^{\nu\rho} A_\rho$. *Thus,*

$$\tilde{A}_\mu = \sum_{\nu=0}^3 g_{\mu\nu} \tilde{A}^\nu = \sum_{\nu,\lambda=0}^3 g_{\mu\nu} a_\lambda^\nu A^\lambda = \sum_{\nu,\lambda,\rho=0}^3 g_{\mu\nu} g^{\lambda\rho} a_\lambda^\nu A_\rho.$$

Noting that $(a_\rho^\lambda)^{-1} = (g_{\mu\nu})(a_{\lambda\rho})(g^{\mu\nu})$, *we have that* $\tilde{A}_\mu = \sum_{\nu=0}^3 ((a_\tau^\sigma)^{-1})_\mu^\rho A_\rho$. *Furthermore, from the form of* g, *we have that* $(A_0, A_1, A_2, A_3) = (A^0, -A^1, A^2, -A^3)$.

Geometrically, a covector is an element of the dual space of the Minkowski space, and the Lorentzian inner product identifies a vector space with its dual via the lowering index operation: $A_\mu = \sum_{\nu=0}^3 g_{\mu\nu} A^\nu$. More generally, given nonnegative integers k and ℓ, a tuple

$$\left(A_{\nu_1,\ldots,\nu_\ell}^{\mu_1,\ldots,\mu_k}\right), \quad \begin{array}{l} 0 \leq \mu_i \leq 3, \quad 1 \leq i \leq k \\ 0 \leq \nu_i \leq 3, \quad 1 \leq j \leq \ell \end{array}$$

is called a (k, ℓ)-tensor if under the transformation $x_\mu \mapsto \sum_{\nu=0}^3 a_\nu^\mu x_\nu + b^\mu$ between two inertial frames, it transforms as follows:

$$\left(\tilde{A}^{\mu_1,\ldots,\mu_k}{}_{\nu_1,\ldots,\nu_\ell}\right) = \sum_{0 \leq \sigma_1,\ldots,\sigma_k,\rho_1\ldots,\rho_\ell \leq 3} (a_{\sigma_1}^{\mu_1} \cdots a_{\sigma_k}^{\mu_k} \left((a_\beta^\alpha)^{-1}\right)_{\nu_1}^{\rho_1} \cdots \left((a_\beta^\alpha)^{-1}\right)_{\nu_\ell}^{\rho_\ell} A_{\nu_1,\ldots,\nu_\ell}^{\mu_1,\ldots,\mu_k},$$

This generalizes the previous notion, as 4-vectors are given by $k = 1$ and $\ell = 0$, and likewise, 4-covectors are given by $k = 0$ and $\ell = 1$.

Example 1. *Relativistic Angular Momentum*

The Lagrangian for relativistic a free particle is $L = -mc^2\sqrt{1-\frac{v^2}{c^2}}$ is Lorentz-invariant. So in the context of Noether's theorem, the Lorentz group can be a source of symmetries for our Lagrangian. More concretely, we can produce six independent first integrals given that we can define six linearly independent one-parameter subgroups which act as symmetries for four-vectors (as $6 = \dim SO^+(1,3)$). The Lagrangian is autonomous, but we can still define symmetries of space-time per Problem 2 on Homework 1. Considering rotations in the three spatial coordinates, we may define an analogous angular-momentum quantity. Recall that for a position vector $\vec{x}$ and a (classical) momentum $\vec{p}$, we have $\vec{L} = \vec{x} \times \vec{p}$. Replacing the classical momentum with the relativistic momentum $\vec{p} = M(v)\vec{v} = \frac{m}{\sqrt{1-\frac{|v|^2}{c^2}}}\vec{v}$ gives us three integrals of motion since the relativistic momentum is conserved. Additionally, since the 4-vector $\left(\frac{E}{c}, \vec{p}\right)$ is conserved under the Lorentz boost, we may define three integrals of motion as the three components of the vector $\vec{N} = \vec{p}t - \frac{E}{c^2}\vec{x}$ (**Exercise**). Multiplying those components by c, we may write all six integrals of motion as the following bivector ($(2,0)$-tensor):

$$L = (ct, \vec{x}) \wedge \left(\frac{E}{c}, \vec{p}\right) = \left(\begin{array}{c|ccc} 0 & -N^1c & -N^2c & -N^3c \\ \hline -N^1c & 0 & L^{12} & -L^{31} \\ -N^2c & -L^{12} & 0 & L^{23} \\ -N^3c & L^{31} & -L^{23} & 0 \end{array}\right),$$

where $\vec{L} = (L^{23}, L^{31}, L^{12})$ is the space-only relativistic momentum and $\vec{N} = (N^1, N^2, N^3)$ is the integrals from the Lorentz boost. Thinking of the left 4-vector as the space-time analogue of the position vector, and given the right 4-vector's status as the space-time analogue of momentum, (and that $\wedge = \times$ in three dimensions) it is natural to define this quantity as the relativistic angular momentum.

Note: I have moved the introductory discussion on classical field theory to Lecture 10 for the sake of organization.