MATH 689 - Physics for Mathematicians

Lectures by Igor Zelenko, transcribed by Seth Hoisington

Sepember 5, 2023

Theorem 1 (Darboux)**.** *If σ is a sympectic form on a $2n$-dimensional manifold N, then for all $\lambda \in N$, there exists a neighborhood U of λ and local coordinates $(p_1, \dots, p_n, q^1, \dots, q^n)$ such that*

$$\sigma = dp_1 \wedge dq^1 + \cdots + dp_n \wedge dq^n$$

(See also Arnold, p.229-232)

Such coordinates are called symplectic coordinates and the transition map between symplectic coordinates is called a canonical transformation.

Remark 1.

From linear algebra, for every $\lambda \in N$, we can find a basis in $T_\lambda N$ such that the matrix of the bilinear form σ_λ in this basis is

$$\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

which corresponds to the coordinate basis $\left(\frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n}, \frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^n}\right)$ in (1). In fact, such bases in $T_\lambda N$ are defined modulo the Linear Symplectic group $Sp(T_\lambda N)$. Darboux says that from closure of σ we can choose such a basis at every point of a neighborhood such that the corresponding vector fields commute. Hence, they are coordinate bases w.r.t some coordinates.

1 Lagrangian Submanifolds of a Symplectic Space

1.1 Linear algebra preliminary

Assume that W is a $2n$-dimensional vector space with a non-degenerate skew-symmetric form ω. (e.g. $W = T_\lambda N$, $\omega = \sigma_\lambda$). Given a subspace $L \subseteq W$, let $L^\angle = \{v \in W \mid \omega(v, z) = 0 \forall z \in L\}$. Then since ω is non-degenerate, we have $\dim L^\angle = \dim W - \dim L = 2n - \dim L$. L is called isotropic if $L \subseteq L^\angle$. For example, if $\dim L = 1$, then for all $v, w \in L$, we have that $v = \lambda w$, so, by skew-symmetry,

$$\sigma(v, w) = \lambda\sigma(v, v) = 0.$$

If $L \subseteq W$ is isotropic, then $\dim L \leq \frac{1}{2} \dim W = n$. If L is isotropic and $\dim L = n$, then L is called **Lagrangian**.

Example 1.

In a basis $\left(\frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}, \frac{\partial}{\partial q^1}, \ldots, \frac{\partial}{\partial q^n}\right)$, (in which ω has the matrix $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$), we let $L_0 = \operatorname{span}\left\{\frac{\partial}{\partial q^1}, \ldots, \frac{\partial}{\partial q^n}\right\}$ and let $L_\infty = \operatorname{span}\left\{\frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}\right\}$. Both these subspaces are Lagrangian. More generally, for any symmetric matrix $S = [S_{ij}]_{i,j=1}^n$, we have that the subspace

$$L_S = \operatorname{span}\left\{\frac{\partial}{\partial q^i} + \sum_{j=1}^n S_{ij}\frac{\partial}{\partial p_j}\right\}.$$

is Lagrangian. Indeed, this follows, given that

$$\begin{aligned}
&\omega\left(\frac{\partial}{\partial q^{i_1}} + \sum_{j=1}^n S_{i_1 j}\frac{\partial}{\partial p_j}, \frac{\partial}{\partial q^{i_2}} + \sum_{j=1}^n S_{i_2 j}\frac{\partial}{\partial p_j}\right) \\
&= \sum_i (dp_i \wedge dq^i)\left(\frac{\partial}{\partial q^{i_1}} + \sum_{j=1}^n S_{i_1 j}\frac{\partial}{\partial p_j}, \frac{\partial}{\partial q^{i_2}} + \sum_{j=1}^n S_{i_2 j}\frac{\partial}{\partial p_j}\right) \\
&= S_{i_1 i_2} - S_{i_2 i_1} \\
&= 0
\end{aligned}$$

Moreover L_S is Lagrangian if and only if S is symmetric, and any Lagrangian L transversal to L_∞ is of the form $L = L_S$ for some symmetric matrix S.

Definition 1. *A submanifold L of a symplectic manifold (N, σ) is called Lagrangian if, for every $\lambda \in L$, $T_\lambda L$ is a Lagrangian subspace of $T_\lambda N$. Equivalently, L is Lagrangian if $\sigma|_L = 0$ and $\dim L = \frac{1}{2}\dim N$.*

Example 2. *Let $N = T^*M$ with the canonical symplectic form σ. Then,*

1. *Every fiber of T^*M is Lagrangian.*
2. *Consider M embedded into T^*M are a graph of v-sections, then M is defined by $dp_1 = dp_2 = \cdots = dp_n = 0$, so $\sigma_M = 0$.*

Example 3. *Given a smooth function $f : M \to \mathbb{R}$, the graph of its differential, given by $L_f = \{((df)_q, q) \mid q \in M\} \subseteq T^*M$, is a Lagrangian submanifold. Then (sketching), we can show that for each $\lambda \in T^*M$,*

$$T_\lambda(T^*M) = \operatorname{span}\left\{\frac{\partial}{\partial q^1} + \sum_{j=1}^n \frac{\partial^2 f}{\partial q^i \partial q^j}\frac{\partial}{\partial q^j}\right\}.$$

Since the Hessian matrix $S = \left[\frac{\partial^2 f}{\partial q^i \partial q^j}\right]_{ij}$ is symmetric for any smooth map by Clairaut's theorem, then $L_f = L_S$. Hence, it is Lagrangian.

2 Liouville Integrability

Let (N, σ) be a symplectic manifold. We say that some k Hamiltonians $(F_1, \ldots, F_k) \in C^\infty(N)^k$ are in involution if $\{F_i, F_j\} = 0$ for all $1 \leq i, j \leq k$. If $(F_1, \ldots, F_k)$ are in involution, then for all $\lambda \in N$, $\operatorname{span}\{(\vec{F}_1)_\lambda, \ldots, (\vec{F}_k)_\lambda\}$ is an isotropic subspace of $T_\lambda N$.

We say that the Hamiltonians $(F_1, \ldots, F_k)$ are independent if the vectors $\{(dF_1)_\lambda, \ldots, (dF_k)_\lambda\}$ are linearly independent. By a simple linear transformation, we can conclude that $\{(\vec{F}_1)_\lambda, \ldots, (\vec{F}_k)_\lambda\}$. Therefore, if a set of k Hamiltonians are independent and in involution, then $k \leq n = \frac{\dim N}{2}$.

Assume that $k = n$ and $(F_1, \ldots, F_n)$ are independent in involution. Then, we see that $\operatorname{span}\{\vec{F}_1, \ldots, \vec{F}_n\}$ is an involutive distribution over N since $[\vec{F}_i, \vec{F}_j] = \overrightarrow{\{F_i, F_j\}} = \vec{0} = 0$. Therefore, the distribution is integrable by Frobenius' Theorem, so there exists an n-dimensional integral submanifold, which is in fact Lagrangian ($\sigma(\vec{F}_i, \vec{F}_j) = 0$). Given this, the set $(F_1, \ldots, F_n)$ which is independent in involution is called Liouville Integrable. Let $f \in \mathbb{R}^n$ and let $f = (f_1, \ldots, f_n)$. Then, define

$$N_f = \{\lambda \in N \mid F_i(\lambda) = f_i,\ i = 1, \ldots, n\}.$$

Then if $N_f \neq \emptyset$, then N_f is a Lagrangian submanifold of N. N_f is invariant under the flow generated by $\vec{F}_i$ for every i.

Remark 2. *Note that if $(F_1, \ldots, F_k)$ are integrals of F_1, then the common level set, $N_f = \{F_1 = f_1, \ldots, F_k = f_k\}$ for $f = (f_1, \ldots, f_n) \in \mathbb{R}^n$, if not empty, is a codimension k submanifold in N and it is invariant with respect to the flow of $\vec{F}_1$. So, having k integrals, we can reduce the degrees of freedom by k. The tangent space to N_f at some point $\lambda \in N_f$,*

$$T_\lambda N_f = \ker(dF_1)_\lambda \cap \cdots \cap \ker(dF_k)_\lambda = \left(\operatorname{span}\{\vec{F}_1, \ldots, \vec{F}_k\}\right)^\angle$$

In particular, if $(F_1, \ldots, F_k)$ are in involution, then $\operatorname{span}\left(\vec{F}_1, \ldots, \vec{F}_k\}\right)\big|_\lambda$ *is an isotropic space, so* $\operatorname{span}\left(\vec{F}_1, \ldots, \vec{F}_k\}\right)\big|_\lambda \subseteq T_\lambda N_f$.

Theorem 2 (Arnold-Liouville)**.** *Consider a Hamiltonian system $\dot{\lambda} = \vec{H}(\lambda)$ on N admitting n integrals $(F_1 - H, F_2, \ldots, F_n)$ independent in involution. Let $f \in \mathbb{R}^n$. Then*

1. *If N_f is nonempty, connected, and compact, then N_f is diffeomorphic to an n-dimensional torus, and one can choose global coordinates $\varphi = (\varphi_1, \ldots, \varphi_n) \mod 2\pi$ on T^n such that the flow generated by $\vec{H}$ is conditional periodic, i.e. there exists a vector of frequencies $\omega = \omega(f) \in \mathbb{R}^n$*

such that on N_f, we have that

$$\dot{\varphi} = \omega \Leftrightarrow \begin{cases} \dot{\varphi}_1 = \omega_1 & \varphi_1(t) = \varphi_1(0) + \omega_1 t \mod 2\pi \\ \quad \vdots \qquad \Leftrightarrow & \quad \vdots \\ \dot{\varphi}_n = \omega_n & \varphi_n(t) = \varphi_n(0) + \omega_n t \mod 2\pi \end{cases}$$

2. *Moreover, in the neighborhood $\tilde{N}$ of N_f, there are symplectic coordinates (I, φ) (i.e. $\tilde{N} \cong D \times T^n$) such that $\dot{\lambda} = \vec{H}(\lambda)$ is equivalent to*

$$\begin{cases} \dot{I} = 0 \\ \dot{\varphi} = \omega(I) \end{cases}$$

Note that since I is constant on N_f, then