MATH 689 - Physics for Mathematicians

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1 Lagrangian Mechanics

We can describe the position of a system of N particles in $\mathbb{R}^3$ by a set of $3N$ coordinates: $\{(x_\alpha^1, x_\alpha^2, x_\alpha^3)\}_{\alpha=1}^N$.

Given holonomic constraints (constraints on the positions of particles in the system), the system of particles is forced to live on a submanifold M of $\mathbb{R}^{3N}$ (in that the sets of all possible positions is a point on M). M is called the **configuration space** of the system.

Example 1. *Two points joined by a rigid rod of length ℓ in the (x, y)-plane.*

Let $(x_1, y_1), (x_2, y_2)$ be two points joined by a rod of length ℓ. Then, the position of the system can be defined by the location of the first point, and the angle (from 0 to 2π) of the rod. Therefore, the position of the system can be described by the point $(x, y, \theta) \in \mathbb{R}^2 \times S^1 \subseteq \mathbb{R}^4$. Therefore, the configuration space of this system is $\mathbb{R}^2 \times S^1$. Alternatively, the configuration space can be described as the 3-dimensional submanifold of $\mathbb{R}^4$ defined by the zero set of the function $f(x_1, y_1, x_2, y_2) = (x_1 - x_2)^2 + (y_1 - y_2)^2 - \ell$.

Note that knowledge of the position of a system does not determine any information about the future position of the system. To determine the future positions of a system (in classical mechanics), we must know the position of all particles in the configuration space, given by a point q in the configuration space M and a vector $\dot{q} \in T_qM$. Hence, the point $(q, \dot{q})$, which lies in the tangent bundle TM of the configuration space can be used to determine the past and future states of the system, given predetermined equations of motion. In this setting, the tangent bundle TM is called the **Phase space** of the system, and a partipular point in the phase space is called a **state** of the system.

2 The Euler-Lagrange Equation

Example 2. *The equation of motion of a free particle*

Suppose $x \in \mathbb{R}^n$ is a particle with mass m. Then, supposing that no forces are applied to the particle, then by Newton's Second Law, we have that

$$m\frac{d^2x}{dt^2} = 0.$$

Example 3. *The equation of a spring*

Suppose $x \in \mathbb{R}^n$ is a particle with mass m attached to one end of a spring (anchored at the other end) with spring constant k. Then, the spring's motion can be described by the equation

$$m\frac{d^2x}{dt^2} = -kx.$$

Euler-Lagrange gives us an alternative way of describing motion due to conservative (path-independent) forces. For such motions, there exists a smooth function L depending on the state of the system which satisfies the **Euler-Lagrange Equation**:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}.$$

In the first example, the function $L_1(x, \dot{x}) = \frac{m\dot{x}^2}{2}$ satisfies the E-L equation exactly when the equation of motion is satisfied. Similarly, $L_2(x, \dot{x}) = \frac{m\dot{x}^2}{2} - \frac{kx^2}{2}$ satisfies the E-L equation for a spring system (Ex 2.2).

Exercise 1. *Check that the E-L equation is satisfied for the functions L_1, L_2 exactly when a particle satisfies the equations of motion in the examples above.*

Definition 1. *A smooth function $L : TM \to \mathbb{R}$ is called an **autonomous Lagrangian** (time-independent), and a smooth function $L : \mathbb{R} \times TM \to \mathbb{R}$ is called a **non-autonomous Lagrangian** (time-dependent).*

For a given Lagrangian L, we may define a functional A which acts on curves in the configuration space M. For a curve $q : [t_0, t_1] \to M$ such that $q(t_i) = q^i$ for $q_i \in M$ for $i = 0, 1$, we define

$$A(q) = \int_{t_0}^{t_1} L(t, q(t), \dot{q}(t))dt$$

We call this functional an action on the space of curves connecting the points $q^0, q^1 \in M$ at times t_0 and t_1, respectively.

Definition 2. *Let q be a curve in M. Given any one-parameter family of smooth curves $q_s(t)$, for $t \in [t_0, t_1]$ and $s \in (-\varepsilon, \varepsilon)$ such that $q_0 = q$, and $q_s(t_i) = q^i$ for all s, then q is a **critical point** of A if, g*

$$\frac{d}{ds}[A(q_s(t))]\Big|_{s=0} = 0.$$

Theorem 1. *A curve q is a critical point of A if and only if it is a solution to the Euler-Lagrange equation.*

Proof. Fix a one parameter family of curves q_s as in the definition of a critical point of A. Let $h(t) = \frac{\partial}{\partial s}[q_s(t)]\big|_{s=0} \in T_{q(t)}M$. Then $h(t)$ is called a variational vector field. Note that since q_s is constant at the endpoints, then $h(t_0) = h(t_1) = 0$. We have

$$\begin{aligned}\frac{d}{ds}[A(q_s)]\Big|_{s=0} &= \int_{t_0}^{t_1} \frac{\partial}{\partial s} L(t, q_s(t), \dot{q}_s(t))\big|_{s=0} dt \\ &= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q_s}\frac{\partial q_s}{\partial s} + \frac{\partial L}{\partial \dot{q}_s}\frac{\partial \dot{q}}{\partial s} \right)\Big|_{s=0} dt, \\ &= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q_0} h + \frac{\partial L}{\partial \dot{q}_0}\frac{\partial}{\partial s}\frac{\partial q}{\partial t}\Big|_{s=0} \right) dt, \\ &= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q_s}\Big|_{s=0} h + \frac{\partial L}{\partial \dot{q}}\frac{dh}{dt} \right) dt.\end{aligned}$$

Now, we apply integration by parts to the second term:

$$\int_{t_0}^{t_1} \frac{\partial L}{\partial \dot{q}} \dot{h} dt = \frac{\partial L}{\partial \dot{q}} h \Big]_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} h dt.$$

Noting that the boundary term is zero since $h(t_i) = 0$, we find that

$$\begin{aligned}\frac{d}{ds}[A(q_s)]\Big|_{s=0} &= \int_{t_0}^{t_1} \frac{\partial L}{\partial q} h dt - \int_{t_0}^{t_1} \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} h dt \\ &= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \right) h dt.\end{aligned}$$

Define $G(t) := \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$. Note that if L satisfies the E-L equation, we have that $\frac{d}{ds}[A(q_s)]\Big|_{s=0} = \int_{t_0}^{t_1} 0h(t)dt = 0$ for any one-parameter family q_s (as specified), so q is indeed a critical point of A. On the other hand, if we assume that q is a critical point, then $\frac{d}{ds}[A(q_s)]\Big|_{s=0} = 0$, so $\int_{t_0}^{t_1} G(t)h(t)dt = 0$ for all variational vector fields h. We claim that this implies that $G(t) = 0$. Suppose, by way of contradiction that for some t, we have that $G(t) \neq 0$. Then, by continuity, there exists some interval $U = (t-\delta, t+\delta)$ such that $G(s) > 0$ for all $s \in U$. Choose a positive bump function h supported in U (Do we have to show that such an h is a variational vector field??). Then we have that

$$\int_{t_0}^{t_1} G(t)h(t)dt = \int_U G(t) > 0.$$

This contradicts our assumption that $\int_{t_0}^{t_1} G(t)h(t)dt = 0$. ☐

Remark 1. *The above theorem proves the Least-action principle. The minimal trajectory (measured by the functional A) is the path which is determined solely by the equations of motion.*

3 The Least Action Principle

assume we have a system of N particles in $\mathbb{R}^3$ with potential energy U, described by the coordinates $\{(x_\alpha^1, x_\alpha^2, x_\alpha^3)\}_{\alpha=1}^N$. Then, by Newton's Second Law, we have

$$m_\alpha \ddot{x}_\alpha^i = F_\alpha^i = -\frac{\partial U}{\partial x_\alpha^i}.$$

Let $T = \sum_\alpha \sum_i \frac{m_\alpha (\dot{x}_\alpha^i)^2}{2}$ (T represents the kinetic energy of the system). Then, we define the Lagrangian $L = T - U$. Then,

$$\frac{\partial L}{\partial \dot{x}_\alpha^i} = m_\alpha \dot{x}_\alpha^i \text{ and } \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_\alpha^i} = m_\alpha \ddot{x}_\alpha^i.$$

Additionally, $\frac{\partial L}{\partial x_\alpha^i} = -\frac{\partial U}{\partial x_\alpha^i}$, so $\frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x} = F$. Hence, the Lagrangian satisfies the E-L equation exactly when the system satisfies Newton's Laws of Motion.

Theorem 2 (The Least Action Principle)**.** *The motion of the mechanical system under consideration above coincides with an extremal of the action*

$$A(q) = \int_{t_0}^{t_1} (T - U)dt.$$

Remark 2. *By analyzing the second derivative $\frac{d}{ds}A(q_s)$, it can be shown that the extremals in question are indeed local minimizers, justifying the use of the word "least".*

We say that a mechanical system is defined by a Lagrangian L, and its trajectories are solutions to the E-L equation with the given L.

Remark 3. *In local coordinates $(q^1, \dots, q^n)$ on M, the quantity $p_i = \frac{\partial L}{\partial \dot{q}^i}$ is called the generalized momentum conjugated to the coordinate q^i. Thus, the E-L equation can be written in the form $\dot{p}_i = \frac{\partial L}{\partial q^i}$.*

Definition 3. *A function $I : TM \to \mathbb{R}$ is called a (first) integral of motion if, for any trajectory q of a system,*

$$\frac{d}{dt} I(q(t), \dot{q}(t)) \equiv 0,$$

or equivalently, I is constant along the lift of the trajectory to TM.

Remark 4. *First Integrals of Motion can be thought of as conservation laws. Along trajectories determined by the underlying laws of motion, these quantities don't change.*

Example 4. *First integrals of motion of free particle*

Recall that $m\frac{d^2x}{dt^2} = 0$ for a free particle. The integrals of motion are momentum; $p = m\dot{x}$, and kinetic energy: $\frac{m\dot{x}^2}{2}$. It can be shown by taking the time derivative of these quantities that they are indeed integrals of motion.

Example 5. *IoMs of Autonomous Lagrangians*

Assume that the Lagrangian is autonomous ($\frac{dL}{dt} = 0$). Consider the quantity $\sum_i q^i \frac{\partial L}{\partial \dot{q}^i}$ for a trajectory q. Then,

$$\begin{aligned} \frac{d}{dt}\left[\sum_i q^i \frac{\partial L}{\partial \dot{q}^i}\right] &= \sum_i \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} + q^i \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} \\ &= \sum_i \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} + q^i \frac{\partial L}{\partial q^i} \\ &= \frac{d}{dt} L(q(t), \dot{q}(t)) \end{aligned}$$

by Euler-Lagrange and the chain rule. Therefore,

$$\frac{d}{dt}\left[\sum_i q^i \frac{\partial L}{\partial \dot{q}^i} - L(q(t), \dot{q}(t))\right] = 0.$$

Therefore, $H = \sum_i q^i \frac{\partial L}{\partial \dot{q}^i} - L$ is an IoM for the system. Note that when $L = T - U$, then $H = T + U$, which is precisely the conservation of energy (of a closed system).

Example 6. *Lagrangian independent of a certain coordinate*

Suppose L is a Lagrangian such that in a given local coordinate system $(q^1, \ldots, q^n)$, L is independent of q^i for some i. Then $p_i = \frac{\partial L}{\partial \dot{q}^i}$ is an IoM given that, along any trajectory, $\frac{d}{dt} p_i = \frac{d}{dt}\frac{\partial L}{\partial q^i} = \frac{\partial L}{\partial q^i} = 0$ by Euler-Lagrange.

Definition 4. *A diffeomorphism $\varphi : M \to M$ is called a symmetry of the Lagrangian L if $\varphi^* L = L$ (i.e. $L(\varphi(q), \phi_{*,q}(v)) = L(q, v)$ for all $(q, v) \in TM$).*

Example 7. *Lagrangian independent of a coordinate q^i*

Let L be a Lagrangian such that in the local coordinate system $(q^1, \ldots, q^n)$, L is independent of q^i. In this context, the coordinate q^i is called cyclic. We then observe that $\phi^s(q^1, \ldots, q^i, \ldots, q^n) = (q^1, \ldots, q^i + s, \ldots, q^n)$, then ϕ^s is a symmetry of L for all s.