MATH 689 - Physics for Mathematicians

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Definition 1. *A diffeomorphism $\varphi : M \to M$ is called a symmetry of the Lagrangian L if $\varphi^* L = L$, i.e. $L(\varphi(q), \varphi_{*,q}(v)) = L(q, v)$ for all $(q, v) \in TM$.*

1 Proof of Noether's Theorem

Example 1. *Shift/translation in one direction (in local coordinates)*

Let H be described in local coordinates by $(q^1, \ldots, q^n)$. If for some i, L is independent of q^i, then q^i is called cyclic, and $\varphi^s(q^1, \ldots, q^i, \ldots, q^n) = (q^1, \ldots, q^i + s, \ldots, q^n)$. Then φ^s is a symmetry of L.

Theorem 1 (Noether's Theorem)**.** *If an autonomous Lagrangian L admits the one-parameter group of symmetries $\varphi^s : M \to M$, then the mechanical system described by L has a first integral of motion that is written in local coordinates as*

$$I(q, \dot{q}) = \frac{\partial L}{\partial \dot{q}} \frac{d\varphi^s(q)}{ds}\bigg|_{s=0}$$

Proof. Let $X(q) = \frac{d\varphi^s(q)}{ds}\Big|_{s=0}$ (the vector field generating the flow φ^s). Note that by the properties of one-parameter flows, we have that $\frac{d\phi^s}{ds}(q) = X(\varphi^s(q))$. We first note that

$$\frac{d\dot{\varphi}^s(q)}{ds}\bigg|_{s=0} = \frac{\partial}{\partial s}\frac{\partial}{\partial t}\varphi^s(q)|_{s=0} = \dot{X}(q(t))$$

Then we have

$$\begin{aligned}
\frac{d}{dt}I &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}X^i\right) \\
&= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)X^i + \frac{\partial L}{\partial \dot{q}^i}\dot{X}^i \\
&= \frac{\partial L}{\partial q}X^i + \frac{\partial L}{\partial \dot{q}}\dot{X}^i \\
&= \frac{\partial L}{\partial q^s}\bigg|_{s=0}\frac{d\varphi^s(q)}{ds}\bigg|_{s=0} + \frac{\partial L}{\partial \dot{q}^s}\bigg|_{s=0}\frac{d\dot{\varphi}^s(q)}{ds}\bigg|_{s=0} \\
&= \frac{d}{ds}L(\varphi^s(q),\dot{\varphi}^s(q))\big|_{s=0}
\end{aligned}$$

using Euler-Lagrange equation and the chain rule. Since φ^s is a symmetry of L, then $L(\varphi^s(q),\dot{\varphi}^s(q)) = L(q(t),\dot{q}(t))$, so $\frac{d}{ds}L(\varphi^s(q),\dot{\varphi}^s(q))\big|_{s=0} = 0$, giving us that I is a first integral of motion. $\square$

Remark 1. *Coordinate-independence of I in the proof of Noether's Theorem*

The first IoM $I = \frac{\partial L}{\partial \dot{q}}X$ is independent of the choice of coordinates. Fix some $q \in M$ and consider a curve $v^s \in T_qM$ such that $\frac{d}{ds}v^s\big|_{s=0} = X(q)$. Note that we are identifying $X(q) \in T_qM$ as an element of $T_v(T_qM)$. Then, we let $I(q,v) = \frac{\partial}{\partial s}L(q,v^s)\big|_{s=0}$. We see that $\frac{\partial}{\partial s}L(q,v^s)\big|_{s=0} = \frac{\partial L}{\partial \dot{q}}(q,v)\frac{\partial v^s}{\partial s}\big|_{s=0} = \frac{\partial L}{\partial \dot{q}}(q,v)X(q)$

Example 2. *Continuation of Example 1*

Recall the definition of $\varphi^s(q^1,\ldots,q^i,\ldots,q^n) = (q^1,\ldots,q^i+s,\ldots,q^n)$ in local coordinates. We observe that in the canonical coordinates on T_qM, we have that the vector field generated by φ^s is $X(\vec{q}) = \frac{\partial}{\partial q^i}\big|_{\vec{q}}$

Example 3. *Classical momentum is an Integral of Motion*

Consider N particles in $\mathbb{R}^3$. Let $L = T - U$ where T is kinetic energy and U is potential energy. Assume that U depends only on the differences $\vec{x}_a - \vec{x}_b$ for $a,b \in [N]$. In particular, $\frac{\partial U}{\partial \dot{x}^i_\alpha} = 0$. Then for all $\vec{e} \in \mathbb{R}^3$,

$$\varphi^s(\vec{x}_1,\ldots,\vec{x}_N) = (\vec{x}_1 + s\vec{e},\ldots,\vec{x}_N + s\vec{e})$$

is a one-parameter group of symmetries of L. The generator of φ^s is $X =$

$(\vec{e}, \ldots, \vec{e})$, and, as defined in Noether's theorem,

$$\begin{aligned} I(\vec{x}_1, \ldots, \vec{x}_N, \dot{\vec{x}}_1, \ldots, \dot{\vec{x}}_N) &= \frac{\partial L}{\partial \dot{q}} X \\ &= \sum_{\alpha=1}^{N} \sum_{i=1}^{3} \frac{\partial L}{\partial \dot{x}^i_\alpha} e^i \\ &= \sum_{\alpha=1}^{N} \sum_{i=1}^{3} m_\alpha \dot{x}^i_\alpha e^i \\ &= \left\langle \sum_{\alpha=1}^{N} m_\alpha \vec{x}_\alpha, \vec{e} \right\rangle. \end{aligned}$$

Viewing the momentum of the system as the vector $\vec{p} = \sum_{\alpha=1}^{N} \vec{p}_\alpha$, we observe that the above quantity is $\vec{p} \cdot \vec{e}$. Therefore, the total momentum, $\vec{p}$ is constant (since $\langle \vec{p}, \vec{e} \rangle$ for any $\vec{e} \in \mathbb{R}^3$), since we have that $\frac{dI}{dt} = 0$ by Noether's theorem. Therefore, for any trajectory, the momentum of the system is conserved.

Example 4. *Rotations in* $\mathbb{R}^3$

Suppose, as above, that the potential energy U is purely a function of distance between positions: $|\vec{x}_1 - \vec{x}_2|$. Now for any $\vec{w} \in \mathbb{R}^3$, consider the one-parametric group of rotations around this axis with angular velocity $\|\vec{w}\|$. We recall (from Lie group theory) that this group is given by $\{\exp(tA_{\vec{w}}) \in SO_3(\mathbb{R}) \mid t \in \mathbb{R}\}$. Therefore, under the identification, the generator of this flow of rotations is the vector field $Y(\vec{r}) = A_{\vec{w}}\vec{r}$.

We use that $\mathfrak{so}(3) \cong \mathbb{R}^3$ as Lie algebras with multiplication on $\mathbb{R}^3$ given by the cross product. Specifically, for $\vec{w} = (w^1, w^2, w^3)$, we have that the map

$$(w^1, w^2, w^3) \mapsto A_{\vec{w}} = \begin{pmatrix} 0 & -w^3 & w^2 \\ w^3 & 0 & w^1 \\ -w^2 & -w^1 & 0 \end{pmatrix},$$

is a Lie algebra isomorphism, and $A_{\vec{w}}\vec{r} = \vec{w} \times \vec{r}$.

Using Noether, we have that

$$\begin{aligned} I &= \frac{\partial L}{\partial \dot{x}^i_a} Y^i_a \\ &= \sum_i \langle m\dot{\vec{r}}_a, \vec{w} \times \vec{r}_a \rangle \end{aligned}$$

Remark 2. *Note that the quantity* $\frac{\partial L}{\partial \dot{q}} X$ *is well-defined.*

2 Hamiltonian Mechanics

In Lagrangian mechanics, we define a smooth map $L : TM \to \mathbb{R}$. For $(q, v) \in TM (q \in M, v \in T_qM)$, assume that for all $q \in M$, the map $q \mapsto L(q, v)$ is strongly convex, i.e. that

$$d_v^2 L = \sum_{\frac{\partial^2 L}{\partial v^i \partial v^j} dv^i dv^j}$$

is a positive-definite quadratic form. (In Calculus of Variations, this is called the strong Legendre condition).

Let $H(p,q) = \max_{v \in T_q M}(p(v) - L(q,v))$ where $p \in T_q^* M$. Then H is a function on the cotangent bundle TM. Let $h(p,q,v) = p(v) - L(q,v)$ (such that $H(p,q) = \max_{v \in T_q M} h(p,q,v)$). We note that since H is a function on the cotangent bundle, T^*M. Functions on the cotangent bundle are called **Hamiltonians** (anaogous to Lagrangians, which are functions on the tangent bundle).

Example 5. *Hamiltonian for motion in 1 dimension.*

Let $L = \frac{m\dot{x}^2}{2} - U(x)$ for $x \in M = \mathbb{R}$. Then, we observe that $p \in T_x^* M$ is a one-dimensional vector space, meaning that we may identify $p(\cdot) : T_x M \to \mathbb{R}$ as multiplication by a constant p. Therefore, $h(p,x,v) = pv - \frac{mv^2}{2} + U(x)$. Then $\frac{\partial h}{\partial v}\big|_{v=v_0} = (p - mv)|_{v=v_0} = 0$ implies that $p = mv_0$ and $v_0 = \frac{p}{m}$. Therefore,

$$H(p,x) = h(p,x,v_0) = \frac{p^2}{m} - \frac{m\frac{p^2}{m^2}}{2} + U(x) = \frac{p^2}{2m} + U(x).$$

The Hamiltonian corresponding to the Lagrangian L is the total energy of the system. The exact same conclusion holds for motion of particles in $\mathbb{R}^3$.

Note 1. *Cotangent Bundle*

We define $T^*M = \sqcup_{q \in M} T_q^* M$. Let $(q^1, \ldots, q^n)$ be a local coordinate system on M. For some $q \in M$, we have that $\big((dq^1)_q, \ldots, d(q^n)_q\big)$ forms a basis on $T_q^* M$. Any $p \in T_q^* M$ can therefore be written $p = p_1(dq^1)_q + \cdots + p_n(dq^n)_q$ (where $(dq^i)_q\left(\frac{\partial}{\partial q^j}|_q\right) = \delta_j^i$). Therefore, the point $(p,q) \in T^*M$ can be described by the coordinates $(p_1, \ldots, p_n, q^1, \ldots, q^n)$. These coordinates are considered the canonical coordinates on T^*M. Under this coordinate system (and canoncical coordinates on TM), if $v = \sum_{j=1}^n v^j \frac{\partial}{\partial q^j}$ and $p = \sum_{i=1}^n p_i dq^i$, then $p(v) = \sum_{i=1}^n p_i v^i$.

Now, taking v in local coordinates as above, we have that $p(v) = \sum_i p_i v^i$. Then, $h(p,q,v) = \sum_i p_i v^i - L(q,v)$. Then, by strong convexity, the maximum of $v \mapsto h(p,q,v)$ is attained at some $v = v_0$. Therefore, we know that

$\frac{\partial h}{\partial v}\big|_{v=v_0} = 0.$, so in particular, for every i,

$$
\begin{aligned}
0 &= \frac{\partial h}{\partial v^i}\big|_{v=v_0} \\
&= \frac{\partial}{\partial v^i}\left[\sum_i p_i v^i - L(q,v)\right]\big|_{v=v_0} \\
&= \frac{\partial}{\partial v^i}\left[\sum_i p_i v^i\right]\big|_{v=v_0} - \frac{\partial L}{\partial v^i}\big|_{v=v_0} \\
&= p_i - \frac{\partial L}{\partial v^i}\big|_{v=v_0}
\end{aligned}
$$

Thus, $p_i = \frac{\partial L}{\partial v^i}\big|_{v=v_0}$.

Theorem 2 (Equivalence of E-L Equation and Hamiltonian System)**.** *The curve $q(t)$ in M is a solution of E-L, i.e.*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q}(q(t),\dot q(t)) = \frac{\partial L}{\partial q}(q(t),\dot q(t)),$$

if and only if the curve $(q(t),p(t)) \in T^*M$, where $p(t) = \frac{\partial L}{\partial \dot q}(q(t),\dot q(t)) \in T^*M$ is the solution to the following system:

$$
\begin{cases}
\dot q(t) = \dfrac{\partial H}{\partial p}(p(t),q(t)) \\
\dot p(t) = -\dfrac{\partial H}{\partial q}(p(t),q(t))
\end{cases}
$$

where $H(p,q)$ is the Hamiltonian associated with the Lagrangian L.