# MATH 689 - Physics for Mathematicians

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**Theorem 1** (Equivalence of E-L equation and Hamiltonian system)**.** *The curve $q(t)$ in $M$ is a solution of the E-L equation $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}(q(t), \dot{q}(t)) = \frac{\partial L}{\partial q}(q(t), \dot{q}(t))$ if and only if the curve $(p(t), q(t))$ in $T^*M$, with $p(t) = \frac{\partial L}{\partial \dot{q}}(q(t), \dot{q}(t))$ is the solution to the following system:*

$$\begin{cases} \dot{q}_i(t) = \dfrac{\partial H}{\partial p_i}(p(t), q(t)) \\ \dot{p}_i(t) = -\dfrac{\partial H}{\partial q_i}(p(t), q(t)) \end{cases}$$

*where $(p_1, \dots, p_n, q^1, \dots, q^n)$ are the canonical coordinates on $T^*M$ near $q \in M$, and where $H$ is related to $L$ via the Legendre transform:*

$$H(p, q) = \max_{v \in T_q M} (p(v) - L(q, v))$$

*Proof.* By strong convexity of $v \mapsto L(q, v)$ for all $q$, the maximum of $v \mapsto h(p, q, v)$, if attained is attained at a unique $v := v(p, q)$ and $H(p, q) = h(p, q, v(p, q))$. For $v = v(p, q)$:

$$\begin{aligned} 0 &= \frac{\partial}{\partial v^i} h(p, q, v)\big|_{v=v(p,q)} \\ &= \frac{\partial}{\partial v^i}\left(p_i v^i - L(q, v)\right)\Big|_{v=v(p,q)} \\ &= p_i - \frac{\partial}{\partial v^i} L(q, v)\big|_{v=v(p,q)} \\ \Leftrightarrow p_i &= \frac{\partial}{\partial v^i} L(q, v)\big|_{v=v(p,q)} \end{aligned} \tag{1}$$

Hence by the uniqueness of $v(p, q)$, we find that $v(p(t), q(t)) = \dot{q}(t)$ since

$$\frac{\partial L}{\partial v}\bigg|_{(q,v)=(q(t),\dot{q}(t))} = p(t) = \frac{\partial L}{\partial v}\bigg|_{(q,v)=(q(t),v(p(t),q(t)))}$$

Therefore, $H(p(t), q(t)) = h(p(t), q(t), \dot{q}(t))$, we see that

$$\begin{aligned}
\frac{\partial H}{\partial p_i}(p(t), q(t)) &= \frac{\partial}{\partial p_i} h(p, q, v(p, q))\big|_{(p,q)=(p(t),q(t))} \\
&= \frac{\partial}{\partial p_i} h(p, q, v)\big|_{(p,q,v)=(p(t),q(t),v(p(t),q(t))} \\
&\quad + \frac{\partial}{\partial v} h(p, q, v)\big|_{(p,q,v)=(p(t),q(t),v(p(t),q(t))} \frac{\partial}{\partial p_i} v(p, q)\big|_{(p,q)=(p(t),q(t))}
\end{aligned}$$

Since $\frac{\partial h}{\partial v} = 0$ at $(p, q, v) = (p, q, v(p, q))$, in local coordinates, we have

$$\begin{aligned}
\frac{\partial H}{\partial p_i}(p(t), q(t)) &= \frac{\partial}{\partial p_i} h(p, q, v)\big|_{(p,q,v)=(p(t),q(t),v(p(t),q(t))} \\
&= \frac{\partial}{\partial p_i} \left( \sum_{i=1}^{n} p_i v^i - L(q, v) \right) \big|_{(p,q,v)=(p(t),q(t),v(p(t),q(t))} \\
&= v^i\big|_{(p,q,v)=(p(t),q(t),v(p(t),q(t))} \\
&= \dot{q}^i(t)
\end{aligned}$$

Therefore, we have that, for all $i$,

$$\dot{q}^i(t) = \frac{\partial H}{\partial p^i}(p(t), q(t)).$$

Now, we will verify the second set of equations:

$$\begin{aligned}
\frac{\partial H}{\partial q^i}(p(t), q(t)) &= \frac{\partial}{\partial q^i} h(p, q, v(p, q))\big|_{(p,q)=(p(t),q(t))} \\
&= \frac{\partial}{\partial q^i} h(p, q, v)\big|_{(p,q,v)=(p(t),q(t),v(p(t),q(t))} \\
&\quad + \frac{\partial h}{\partial v}\big|_{(p(t),q(t),v(p(t),q(t))} \frac{\partial}{\partial q^i} v(p, q)\big|_{(p(t),q(t))} \\
&= \frac{\partial}{\partial q^i} h(p, q, v)\big|_{(p,q,v)=(p(t),q(t),v(p(t),q(t))} + 0 \frac{\partial v}{\partial q^i}\big|_{(p(t),q(t))} \\
&= \frac{\partial}{\partial q^i} \left( \sum_i p_i v^i - L(q, v) \right) \big|_{(p,q,v)=(p(t),q(t),\dot{q}(t))} \\
&= -\frac{\partial L}{\partial q^i}\big|_{(p,q,v)=(p(t),q(t),\dot{q}(t))} \\
&= -\frac{\partial L}{\partial q^i}\big|_{(q,v)=(q(t),\dot{q}(t))}
\end{aligned}$$

By Euler-Lagrange, we have that

$$\begin{aligned}\dot{p}(t) &= \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}(q(t), \dot{q}(t)) \\ &= \frac{\partial L}{\partial q}(q(t), \dot{q}(t)) \\ &= -\frac{\partial H}{\partial q^i}(p(t), q(t))\end{aligned}$$

Therefore, $H$ is a solution to the Hamiltonian system. □

# 1 Canonical Symplectic Structure and Coordinate-free Construction of Hamiltonian vector fields in $T^*M$

## 1.1 The tautological (Liouville) 1-form on $T^*M$

Let $\lambda = (p, q) \in T^*M$ where $q \in M, p \in T_q^*M$. Then let $\pi : T^*M \to M$ be the canonical projection: $\pi(p, q) = q$. We want to define a 1-form on $T^*M$ (which is a manifold itself). For some $w \in T_\lambda(T^*M)$. We define

$$S(\lambda)(w) = p(\pi_{*,\lambda}(w))$$

Note that $\pi_{*,\lambda} : T_\lambda(T^*M) \to T_qM$, so $p$ indeed acts upon $\pi_{*,\lambda}(w)$.

**Lemma 1.** *In canonical coordinates* $(p_1, \dots, p_n, q^1, \dots, q^n) \in T^*M$,

$$S = p^1 dq^1 + \cdots + p_n dq^n.$$

*Proof.* In our coordinates, $\pi(p_1, \dots, p_n, q^1, \dots, q^n) = (q^1, \dots, q^n)$. Let $w = v^1 \frac{\partial}{\partial q^1} + \cdots + v^n \frac{\partial}{\partial q^n}$. Then,

$$\begin{aligned}s(w) &= p(d\pi(w)) \\ &= \sum_i p_i dq^i \left(\sum_j v^j \frac{\partial}{\partial q^j}\right) \\ &= \sum_i p_i v^i\end{aligned}$$

On the other hand,

$$\sum_i p_i dq^i(w) = \sum_i p_i dq^i \left(\sum_j v^i \frac{\partial}{\partial q^j} + \sum_j \xi_j \frac{\partial}{\partial p_j}\right) = \sum_i p_i v^i$$

□

Let $\sigma = ds$ (the exterior derivative of the Liouville form).

**Properties of $\sigma$:**

- $\sigma$ is exact (and thus closed)
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  **Lemma 2.** *$\sigma$ is a nondegenerate 2-form. (For every $\lambda \in T^*M$, if for some $w \in T_\lambda(T^*M)$ we have $\iota_w \sigma(\lambda) = 0$, then $w = 0$.)*

  *Proof.* By Lemma 1, in canonical coordinates, $s = p_1 dq^1 + \cdots + p_n dq^n$, so

  $$ds = dp_1 \wedge dq^1 + \cdots + dp_n \wedge dq^n.$$

  In the basis $\left(\frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}, \frac{\partial}{\partial q^1}, \ldots, \frac{\partial}{\partial q^n}\right)$ of $T_\lambda(T^*M)$, the bilinear form corresponds to the matrix

  $$\left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right)$$

  since $\sigma(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}) = 0$, $\sigma(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial q^j}) = \delta_{ij}$, and $\sigma(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}) = 0$. Since this matrix is nonsingular, the form is nondegenerate. □

**Definition 1.** *A closed, non-degenerate 2-form $\omega$ on a manifold $M$ is called a symplectic form (or symplectic structure). If such a form exists on $M$, then the pair $(M, \omega)$ is a symplectic manifold. For any such symplectic manifold, it can be shown that* $\dim(M)$ *is even.*

Therefore, $(T^*M, \sigma)$ is a symplectic manifold for any smooth manifold $M$.

Any smooth function $H : T^*M \to \mathbb{R}$ is called an (autonomous) Hamiltonian.

**Lemma 3.** *For any Hamiltonian $H : T^*M \to \mathbb{R}$, there exists a unique vector field, denoted by $\vec{H}$ such that $\iota_{\vec{H}}\sigma = -dH$. Moreover, in the canonical coordinates $(p_1, \ldots, p_n, q^1, \ldots, q^n) \in T^*M$,*

$$\vec{H} = \frac{\partial H}{\partial p_1}\frac{\partial}{\partial q^1} + \cdots + \frac{\partial H}{\partial p_n}\frac{\partial}{\partial q^n} - \frac{\partial H}{\partial q^1}\frac{\partial}{\partial p_1} + \cdots + \frac{\partial H}{\partial q^n}\frac{\partial}{\partial p_n} = \sum_i \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}.$$

*Thus, a curve $\lambda(t)$ in $T^*M$ satisfies $\lambda(t) = \vec{H}(\lambda(t))$ if and only if*

$$\begin{cases} \dot{q}^i(t) = \dfrac{\partial H}{\partial p_i}(p(t), q(t)) \\ \dot{p}^i(t) = -\dfrac{\partial H}{\partial q^i}(p(t), q(t)) \end{cases}$$

*Proof.* The existence and uniqueness of $\vec{H}$ follows from nondegeneracy. We claim that the interior product, as a map $\iota : T_\lambda(T^*M) \to T^*_\lambda(T^*M)$ given by $Y \mapsto \iota_Y \sigma$, is injective. Indeed, if $\iota_Y \sigma = 0$ for some $Y \in T_\lambda(T^*M)$. However, by nondegeneracy, if $\iota_Y \sigma(X) = \sigma(Y, X) = 0$ for all vectors $X \in T_\lambda(T^*M)$, it follows that $Y = 0$. Surjectivity follows from rank-nullity given that $\dim T_\lambda(T^*M) = \dim T^*_\lambda(T^*M) = 2n$. Therefore, we may let $\vec{H} = \iota^{-1}(-dH)$. (Proof continued in the next lecture) □