On the Gaussian Mixture Representation of the Laplace Distribution with Location and Scale Parameters and Applications.

Seth Agee

Tennessee Technological University

Objectives

Objectives for today:

- An overview of the Laplace distribution.
- Showing that the Laplace distribution with location and scale parameters has a Gaussian mixture representation.
- An introduction to an asymmetric Laplace distribution.
- Fitting speed data with the Laplace and Asymmetric Laplace distributions.

The Laplace Distribution

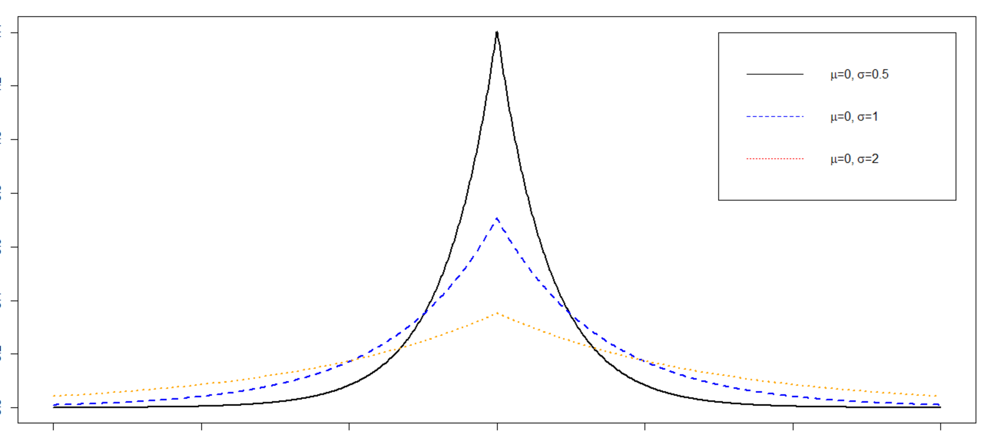

Figure 1:Three Laplace Density Curves with Varying Scale Parameters.

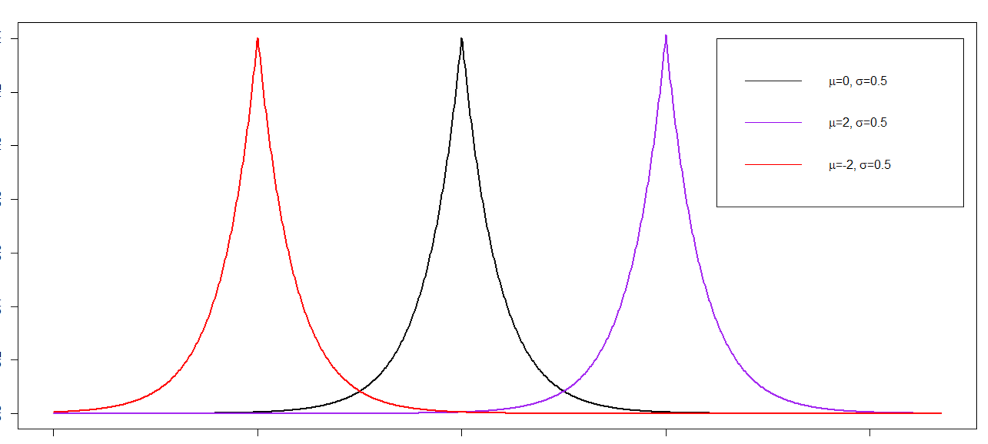

Figure 2:Three Laplace Density Curves with Varying Location Parameters.

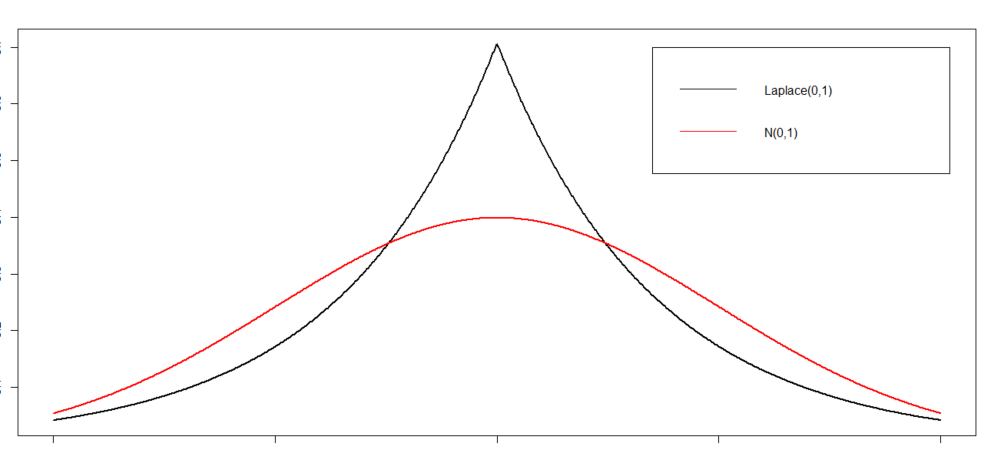

Figure 3:Laplace Distribution and Standard Normal Distribution.

- Very pronounced peak at the center.
- Sometimes called the double Exponential distribution.
- Can be thought of as putting a positive and negative Exponential distribution back-to-back.
- Has cumulative distribution function (CDF) (Kotz, Samuel and Kozubowski, Tomasz and Podgorski, Krzysztof, 2001)

$$F(x) = \begin{cases} \frac{1}{2}e^{(x-\mu)/\sigma} & x \leq \mu \\ 1 - \frac{1}{2}e^{-(x-\mu)/\sigma} & x \geq \mu \end{cases}$$

Notation

From now on we use the following notation:

- Let $\mathcal{E}$ be an Exponential random variable with mean 1.
- Let Z be a standard Normal random variable, N(0,1).
- Let $\mathcal{L}$ be a Laplace random variable.
- Let S be a random sign taking values -1 and +1 with equal probabilities 1/2.

The Representation $\mathcal{L} \sim \mu + \sigma S\mathcal{E}$

First, note that $P(\mathcal{E} \leq x) = 1 - e^{-x}$ when $x \geq 0$ or 0 when x < 0. Now consider

$$\begin{aligned} P(\mu + \sigma S\mathcal{E} \leq x) &= P(S\mathcal{E} \leq \tfrac{x-\mu}{\sigma}) \\ &= P(S=-1)P(\mathcal{E} \geq \tfrac{-x+\mu}{\sigma}) + P(S=1)P(\mathcal{E} \leq \tfrac{x-\mu}{\sigma}). \end{aligned}$$

- If $x \leq \mu$, then $P(\mu + \sigma S\mathcal{E} \leq x) = \frac{1}{2}e^{(x-\mu)/\sigma}$.
- If $x > \mu$, then $P(\mu + \sigma S\mathcal{E} \leq x) = 1 - \frac{1}{2}e^{-(x-\mu)/\sigma}$.
- This gives us exactly the CDF of the Laplace distribution, and so we get $\mathcal{L} \sim \mu + \sigma S\mathcal{E}$.

The Laplace Distribution has a Gaussian Mixture Representation

First, note that moment generating functions (MGFs) of probability distributions are unique. Also note that the MGF of Z is denoted M_Z and the MGF of $\mathcal{E}$ is denoted by $M_{\mathcal{E}}$, and that $M_Z = e^{t^2/2}$ and $M_{\mathcal{E}} = \frac{1}{1-t}$ for t < 1. Let $M_{\mu+\sigma S\mathcal{E}}$ and $M_{\mu+\sigma\sqrt{2\mathcal{E}}Z}$ be the MGFs of $\mu + \sigma S\mathcal{E}$ and $\mu + \sigma\sqrt{2\mathcal{E}}Z$ respectively. Showing these are equal will show that $\mathcal{L} \sim \mu + \sigma\sqrt{2\mathcal{E}}Z$.
For $\mu + \sigma\sqrt{2\mathcal{E}}Z$ we get

$$\begin{aligned} M_{\mu+\sigma\sqrt{2\mathcal{E}}Z} &= E[e^{t(\mu+\sigma\sqrt{2\mathcal{E}}Z)}] = e^{\mu t}E[e^{t\sigma\sqrt{2\mathcal{E}}Z}] = e^{\mu t}E[E[e^{t\sigma\sqrt{2\mathcal{E}}Z}|\mathcal{E}]] \\ &= e^{\mu t}E[M_Z(t\sigma\sqrt{2\mathcal{E}})] = e^{\mu t}E[e^{\sigma^2 t^2\mathcal{E}}] = e^{\mu t}M_{\mathcal{E}}(\sigma^2 t^2) = \frac{e^{\mu t}}{1-\sigma^2 t^2} \end{aligned}$$

For $\mu + \sigma S\mathcal{E}$ we get

$$\begin{aligned} M_{\mu+\sigma S\mathcal{E}} &= E[e^{t(\mu+\sigma S\mathcal{E})}] = e^{\mu t}E[e^{t\sigma S\mathcal{E}}] = e^{\mu t}E[E[e^{t\sigma S\mathcal{E}}|S]] \\ &= e^{\mu t}E[M_{\mathcal{E}}(t\sigma S)] = e^{\mu t}E[\tfrac{1}{1-t\sigma S}] = e^{\mu t}(\tfrac{1}{2}(\tfrac{1}{1+t\sigma}) + \tfrac{1}{2}(\tfrac{1}{1-t\sigma})) = \frac{e^{\mu t}}{1-\sigma^2 t^2} \end{aligned}$$

Thus, $M_{\mu+\sigma S\mathcal{E}} = M_{\mu+\sigma\sqrt{2\mathcal{E}}Z}$, and so $\mathcal{L} \sim \mu + \sigma\sqrt{2\mathcal{E}}Z$ (Ding and Blitzstein, 2018).

Application to Highway Speed Data

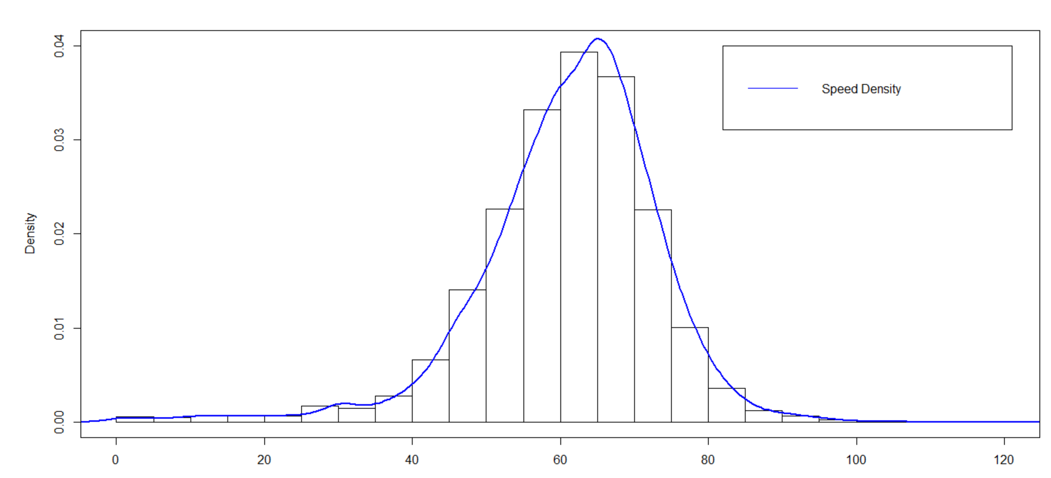

Figure 4:Highway Speed Data with Density Curve.

- Above is highway speed data taken by TDOT censors.
- The speed data has two sources of variation: from the speed and from the measuring device. Then the Laplace distribution may be a better model than the Normal distribution (Geraci and Borja, 2018).
- Peaks relatively sharply, but possible skew is visible.
- This introduces the necessity to consider an asymmetric Laplace distribution in addition to our symmetric one.

An Asymmetric Laplace Distribution

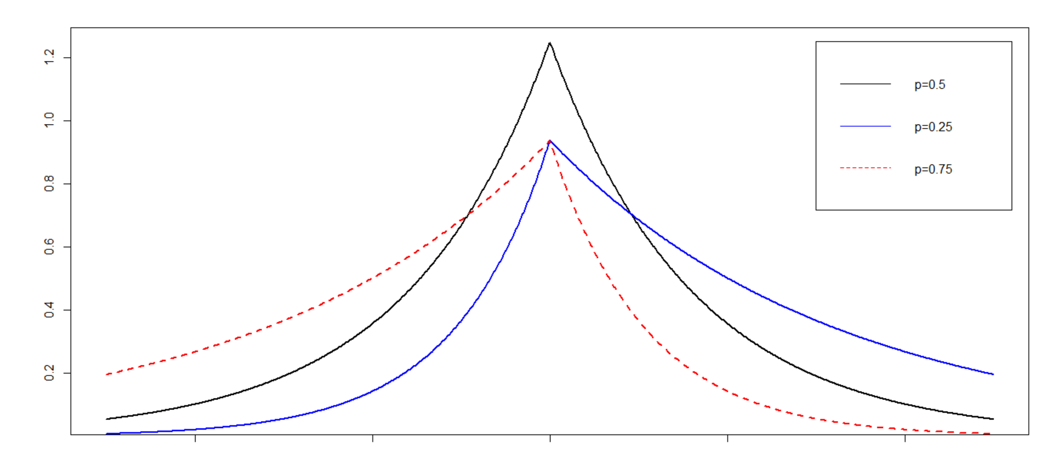

Figure 5:Effects of p on Asymmetric Laplace Distribution.

- Laplace distribution with a built in skewness parameter p which is between 0 and 1.
- Has CDF (Yu and Zhang, 2005)

$$F(x) = \begin{cases} pe^{(1-p)(x-\mu)/\sigma} & x \leq \mu \\ 1 - (1-p)e^{-p(x-\mu)/\sigma} & x > \mu \end{cases}$$

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- Symmetric Laplace distribution when p=0.5.
- Using the R packages `ald`, we can fit our data.

Modelling Results

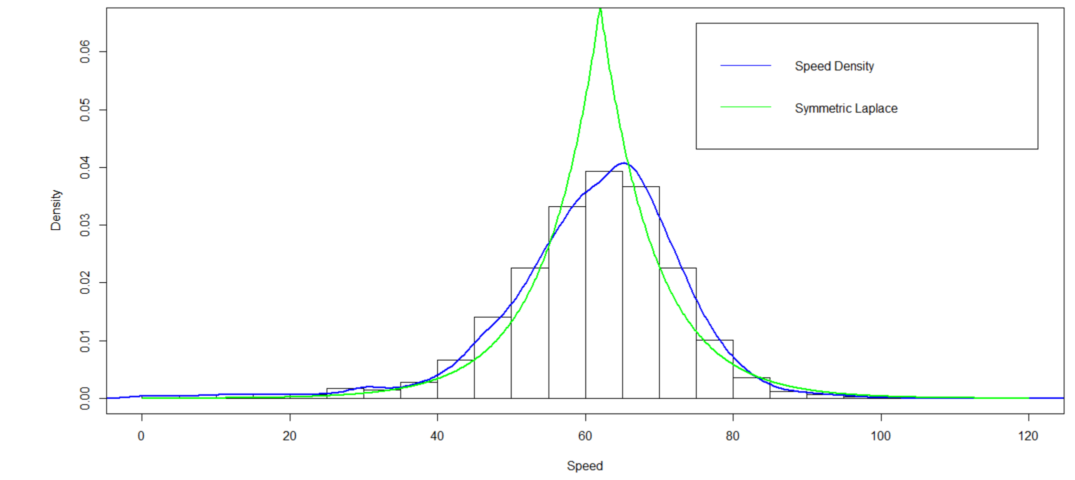

Figure 6:Laplace Distribution on Speed Data.

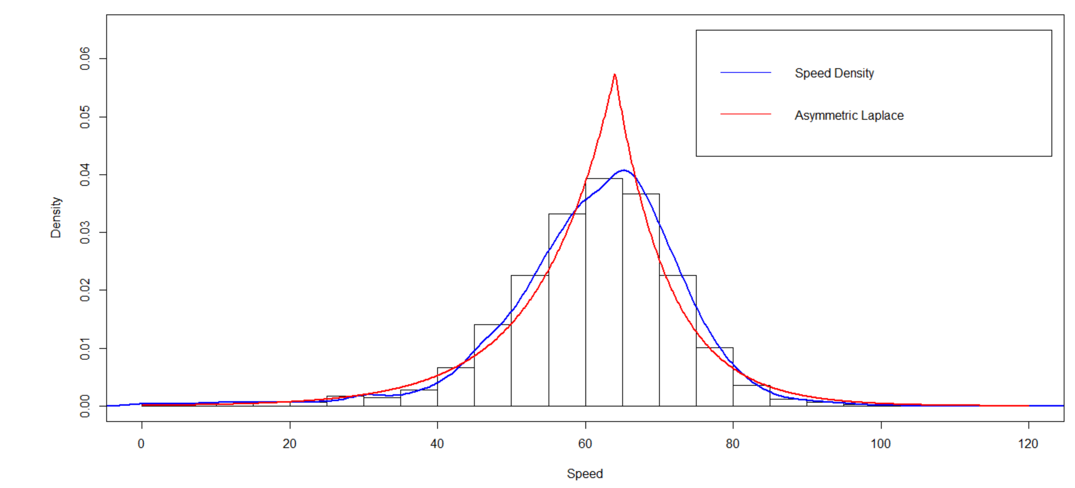

Figure 7:Asymmetric Laplace Distribution on Speed Data.

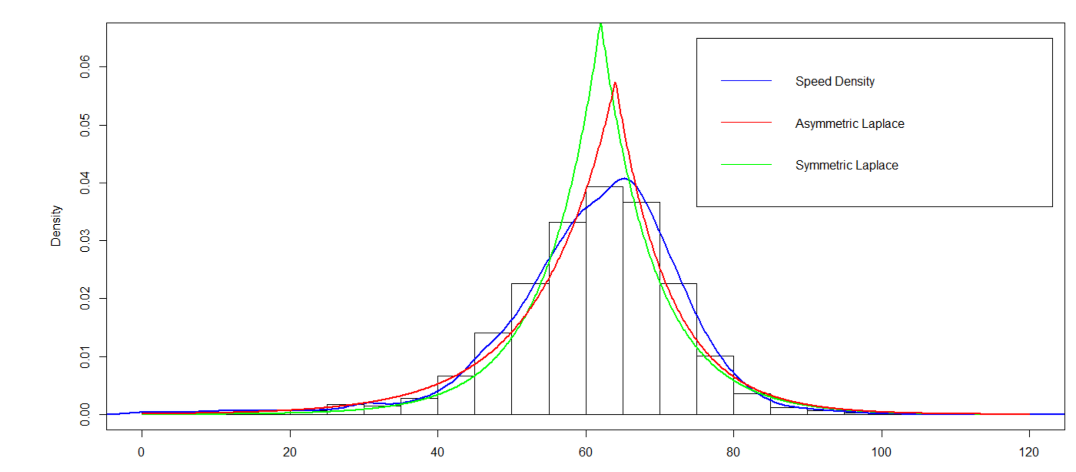

Figure 8:Laplace and Asymmetric Laplace Distributions on Speed Data.

- Using built in MLE functions in `ald`, we first estimate parameters for the symmetric and asymmetric Laplace distributions (Galarza and Lachos, 2018).
- By inspection, we see that the asymmetric Laplace distribution seems to fit better.
- To verify this we use Akaike information criterion:
  - For the symmetric distribution, we get $AIC_1 \approx 6.0 \times 10^6$.
  - For the asymmetric distribution, we get $AIC_2 \approx 5.6 \times 10^6$.
- We get $AIC_1 > AIC_2$, and so the asymmetric model is verified to be a better fit.

Selected References

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- Geraci, M. and Borja, M. C. (2018), Notebook: The Laplace distribution. *Significance*, 15: 10-11. DOI:10.1111/j.1740-9713.2018.01185.x.
- Kotz, Samuel and Kozubowski, Tomasz and Podgorski, Krzysztof (2001), The Laplace Distribution and Generalizations.
- Yu, Keming and Zhang, Jin (2005), A Three-Parameter Asymmetric Laplace Distribution and Its Extension, *Communications in Statistics - Theory and Methods*, 34:9-10, 1867-1879, DOI: 10.1080/03610920500199018.