Martingale representation, Girsanov's theorem, and a generalization of the Black-Scholes model

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Section 1

Introduction to options

- Suppose a corporation wishes to buy 100 bushels of corn in one year's time, and due to budget constraints they will not be willing to pay any more than \$5 per bushel.
- They may then go to a farmer and sign an agreement, ensuring that they will be able to buy their corn for at most \$5 per bushel, regardless of the current market price of corn.
  - The corporation wants to buy a call option at the \$5 strike price for corn.
- Of course, the farmer does not wish to lose any money on this deal, and so he takes some amount of money, which we call premium, initially.

Introduction to options continued

- The scenario we have laid out gives rise to two interesting questions:
  - Given an initial amount of money, is there a strategy that the farmer can use to make sure he has the exact amount needed to cover his losses at the end of the contract period?
  - If such a strategy exists, how much do we need to charge for the premium?
- The first question is about the existence of a "replicating strategy," and the second is about "option pricing."

Portfolios

- A *portfolio* is a pair of processes φ_t and ψ_t which describe respectively the number of units of stock and of bond which we hold at time t.
- We say a portfolio (φ_t, ψ_t) with stock price X_t, bond price B_t, and value at time t, $V_t = \varphi_t X_t + \psi_t B_t$, is *self-financing* if and only if

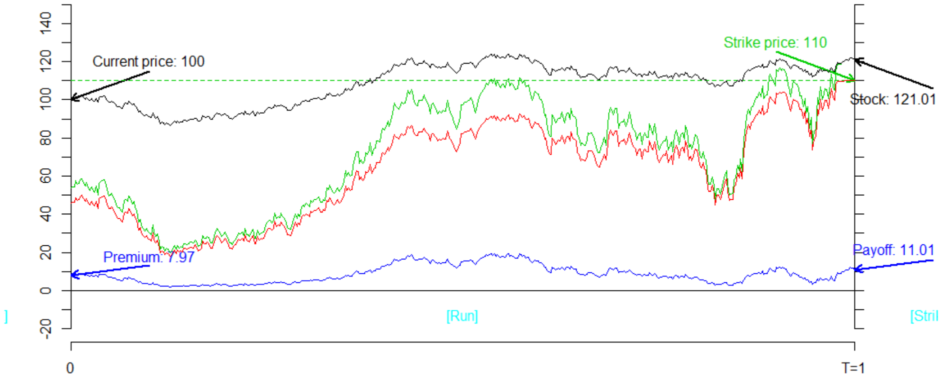
$$dV_t = \varphi_t dX_t + \psi_t dB_t.$$

- A self-financing portfolio requires no additional funds after the formation of the portfolio, and all trades are financed by buying or selling assets within the portfolio.

Replicating strategy

- Given a riskless bond B, a risky stock S with volatility σ, and a claim C on events up to time T, a *replicating strategy* for C is a self-financing portfolio (φ_t, ψ_t) such that $\int_0^T \sigma_t^2 \varphi_t^2 dt < \infty$ almost surely and
$$V_T = \varphi_T S_T + \psi_T B_T = C.$$
- If C gives the amount an option contract seller needs to pay off at the time T, the existence of a replicating strategy implies that the price of C at time t is given by V_t. Specifically, it allows us to determine the theoretical price of an option contract premium by calculating V_0.
  - The self-financing property ensures the portfolio achieves the exact pay off amount using only the money paid by the buyer.

Visualization of replicating strategy



In order to find such a strategy, we first need a model for the stock price.

Formulating our question

- Let k be the strike price of an option contract which can be exercised at some time T. If the seller wishes to give the buyer the exact value of a stock price X at time T, the value of his portfolio should be

$$V(T) = [X(T) - k]_+ = \begin{cases} X(T) - k & \text{if } X(T) - k \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

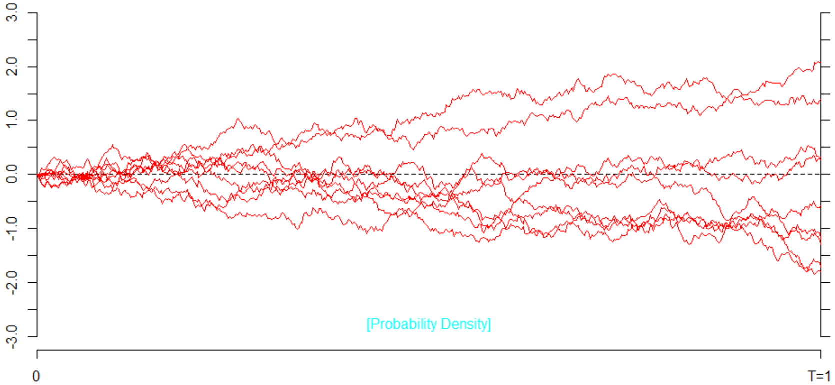
- If there exists a replicating strategy for $[X(T) - k]_+$, we know exactly what the premium should be.

Section 2

Brownian motion

- Brownian motion is the behaviour of a particle suspended in liquid, originally studied by scientists
- Brownian motion was observed to
  - Have highly irregular paths, having no derivative at any point;
  - The paths appeared to be independent of one another.
- Due to these observations, we say that a stochastic process $(W_t)_{t=0}^{\infty}$ is a Brownian motion on the space $(\Omega, \mathcal{F}, P)$ if
  - The process starts at 0 (almost surely);
  - The process has independent increments;
  - For $0 \leq s < t$ the increment $W_t - W_s$ is normally distributed with mean 0 and variance $t - s$.
- Brownian motion will be the source of randomness in our model for the stock price.

Brownian motion visual



Stochastic differential equations

- We can define the integral of certain processes f with respect to Brownian motion
$$\int f(s,\omega)dW_s(\omega),$$
which we call the *Itô integral*.
- Using this, we can also define a stochastic differential equation or *Itô process* by
$$X_t = X_0 + \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)dW_s$$
for some certain functions u and v.
- We can write this in the differential form
$$dX_t = u_t dt + v_t dW_t.$$
- These Itô processes give us a way to describe the evolution of a system over time subject to randomness (such as a stock price).

A general stock price model

- Generally, it is assumed that stock prices and bond prices grow exponentially over time.
- Then for bond price B and stock price X, we may consider the differential equations given by

$$\begin{aligned} dB_t &= r(t)B_t dt \\ dX_t &= \mu(t,\omega)X_t dt + \sigma(t,\omega)X_t dU_t, \end{aligned}$$

  where $(U_t)_{t\geq 0}$ is a Brownian motion.
- Solutions of the above would be exponential.
- What kind of restrictions must be put on the drift term μ and volatility term σ?

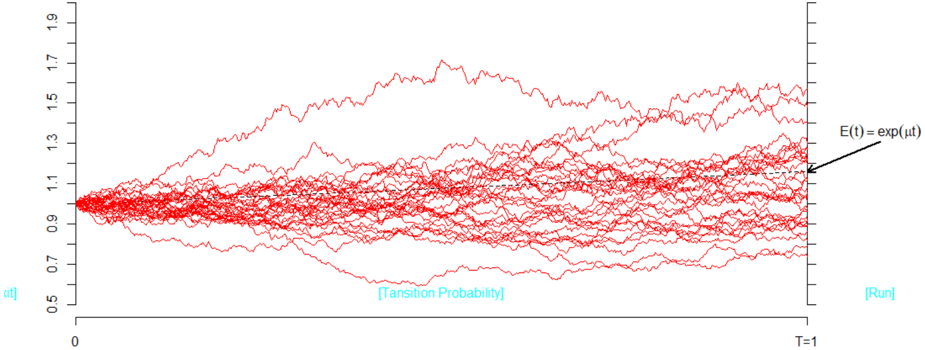
Black-Scholes model

- We first assume the same model as Black and Scholes originally considered.
- This model assumes a constant drift and volatility:

$$dB_t = rB_t dt; \quad B_0 = 1,$$
$$dX_t = \mu X_t dt + \sigma X_t dU_t; \quad X_0 = P_0 > 0,$$

where $(U_t)_{t \geq 0}$ is a Brownian motion on $(\Omega, \mathcal{F}, P)$, r is the fixed interest rate of the bond, μ is the constant drift term, and σ is the constant nonzero volatility term.

Visual of the model



- It appears to be a Brownian motion, but with an additional drift term.

Generalized Black-Scholes model

- Now, we allow the drift and volatility to be deterministic functions of time.
- Consider the following differential equations for a bond price B and stock price X:

$$dB_t = r(t)B_t dt; \quad B(0) = 1,$$
$$dX_t = \mu(t,\omega)X_t dt + \sigma(t)X_t dU_t; \quad X(0) = P_0 > 0,$$

  where U is a Brownian motion on $(\Omega, \mathcal{F}, P)$, μ, σ, and r satisfy those conditions necessary to make the above equations Itô processes and $\frac{\mu(t)-r(t)}{\sigma(t)}$ satisfies Novikov's condition.
- These requirements are technical, but ensure that the above is well defined.
- We will talk more about Novikov's condition later.

Diffusions

- A *diffusion* $(X_t)_{t\geq 0}$ is a stochastic process which satisfies

$$dX_t = \mu(X_t)dt + \sigma(X_t)dU_t; \quad t \geq s, X(s) = x,$$

where U is a Brownian motion, and μ, σ satisfy

$$|\mu(x) - \mu(y)| + |\sigma(x) - \sigma(y)| \leq D|x - y|$$

for all $x, y \in \mathbb{R}$ and some constant D.

- Diffusions are very similar to Itô processes, but μ and σ rely only upon the value X_t, and are not functions of t.

Diffusion model

- In the final model we will consider, we assume that the stock price is a diffusion process.
- For bond price B and stock price X:

$$dB_t = r(B_t)dt; \quad B(0) = 1$$
$$dX_t = \mu(X_t)dt + \sigma(X_t)dU_t; \quad X(0) = x$$

  where $r(x) = r \cdot x$ for constant r, the processes μ and σ satisfy the conditions in the definition of Itô diffusion, and $\frac{\mu(X_t) - r(X_t)}{\sigma(X_t)}$ satisfies Novikov's condition
- These requirements are again technical, but ensure that the above is well defined.

Section 3

Itô's formula

Theorem

Let X_t be an Itô process given by

$$dX_t = u_t dt + v_t dW_t.$$

Let $g(t,x) \in C^2([0,\infty) \times \mathbb{R})$. Then

$$Y_t = g(t, X_t)$$

is again an Itô process and

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t) \cdot (dX_t)^2$$

where $(dX_t)^2 = dX_t \cdot dX_t$ is computed according to the rules

$$dt \cdot dt = dt \cdot dW_t = dW_t \cdot dt = 0, \ dW_t \cdot dW_t = dt.$$

Itô's product rule

Corollary

Let X_t, Y_t be processes given by

$$dX_t = \sigma_t dW_t + \mu_t dt$$

$$dY_t = \rho_t dW_t + \nu_t dt$$

for Brownian motion W. Then

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + \sigma_t \rho_t dt = X_t dY_t + Y_t dX_t + dX_t dY_t.$$

- We now have the ability to more easily find solutions of stochastic differential equations.

Section 4

Martingales

Definition

Let $\mathcal{M}_t$ be "all information available at the time t" (more formally, the filtration). Then a stochastic process M_t is a *martingale* if

1. M_t is $\mathcal{M}_t$-measurable for all $t \geq 0$.
2. $E(M_{t+s} \mid \mathcal{M}_t) = M_t$ for all $s \geq 0$.

- Martingales can be thought of as a sequence of random variables that represent winnings in a fair game of chance. The player does not expect to win or lose at any stage of the game since they are equally likely to happen, so the expected amount at a future stage is just the current amount.
- Brownian motion is a martingale, but Itô processes are only semi-martingales. This is due to the drift term present in Itô processes.

Martingale representation theorem

Theorem

Suppose M_t is an $\mathcal{F}_t$-martingale (with respect to P), $(U_t)_{t\geq 0}$ is a Brownian motion, and that $M_t \in L^2(\mathcal{F}_t, P)$ for all $t \geq 0$. Then there exists a unique stochastic process $g(s,\omega)$ (satisfying some measurability and integrability conditions) for all $t \geq 0$ and

$$M_t(\omega) = E[M_0] + \int_0^t g(s,\omega)dU_s$$

almost surely for all $t \geq 0$.

- It is true that the Itô integral $\int_0^t g(s,\omega)dU_s$ is an $\mathcal{F}_t$-martingale for appropriate g. The above theorem tells us that the converse is also true: given any martingale, we can represent it in terms of an Itô integral.

Section 5

Girsanov's theorem

- Let $(Y_t)_{t\geq 0}$ be an Itô process of the form $dY_t = a(t,\omega)dt + dU_t$ where $t \leq T$, $Y_0 = 0$, and $T \leq \infty$ is a given constant and $(U_t)_{t\geq 0}$ is a Brownian motion on $(\Omega, \mathcal{F}_T, P)$.
- Let $M_t = \exp\left(-\int_0^t a(s,\omega)dU_s - \frac{1}{2}\int_0^t a^2(s,\omega)ds\right)$ where $t \leq T$.

Theorem

Assume that $a(s,\omega)$ satisfies Novikov's condition

$$E\left[\exp\left(\frac{1}{2}\int_0^T a^2(s,\omega)ds\right)\right] < \infty$$

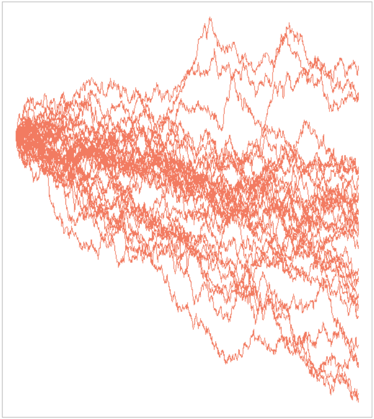
where $E = E_P$ is the expectation with respect to P. Define the measure Q on $(\Omega, \mathcal{F}_T)$ by

$$dQ(\omega) = M_T(\omega)dP(\omega).$$

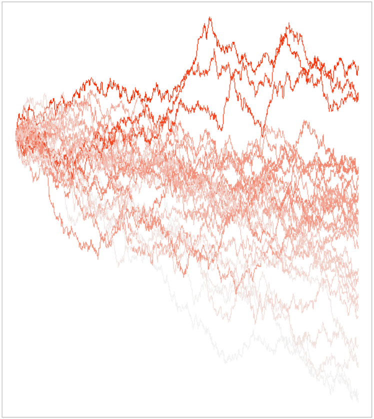
Then Y_t is a Brownian motion with respect to Q for $t \leq T$.

Changing the measure

Girsanov's theorem tells us that there is a way to get rid of the drift term, making a Brownian motion with drift into a standard Brownian motion, and thus a martingale.



30 paths of a Brownian motion with negative drift



The same paths reweighted according to the Girsanov formula

Section 6

Return to replicating strategies

- Recall the Black-Scholes model for stock price X and bond price B:

$$
\begin{aligned}
dB_t &= r(t) B_t dt \\
dX_t &= \mu(t,\omega) X_t dt + \sigma(t,\omega) X_t dU_t,
\end{aligned}
$$

  where $(U_t)_{t \geq 0}$ is a Brownian motion.
- The proof showing that the replicating strategy exists in this case is similar to the generalized case, so we only show the simpler version.

Black-Scholes replicating strategy

Step 1

The process $Y_t = B_t^{-1} X_t$ is a martingale under the measure Q, as was given in Girsanov's theorem.

- Define $dW_t = \frac{\mu - r}{\sigma} dt + dU_t$. Then by Girsanov's theorem, W_t becomes Brownian motion under the measure Q.
- Apply Itô's product rule to find that $dY_t = \sigma Y_t dW_t$. We can see that this is "driftless" and is therefore a martingale.
- This representation of Y in terms of W will be important in step 2.

Black-Scholes replicating strategy contd.

Step 2

Under the filtration $\mathcal{G}_t$ of W_t, the best estimate of $[X_T - k]_+$ at the time t is given by $Z_t = e^{-rT} E_{Q_T}([X_T - k]_+ \mid \mathcal{G}_t)$. There is a unique process φ_t such that $dZ_t = \varphi_t dY_t$.

- Z_t is the best estimate of the payoff due to the fact that the conditional expectation is in fact a projection.
- We can also show that Z_t is a martingale using properties of the conditional expectation.
- Now, using the martingale representation theorem, there exists a unique process g such that $Z_t = E[Z_0] + \int_0^t g_s dW_s$.
- Now, use part 1 to define $\varphi_t = \frac{g_t}{\sigma Y_t}$ and so we have $Z_t = E[Z_0] + \int_0^t \varphi_s dY_s$.

Black-Scholes replicating strategy contd.

Step 3

Let $\psi_t = Z_t - \varphi_t Y_t$ and $V_t = B_t Z_t = e^{-r(T-t)} E_{Q_T}([X_T - k]_+ \mid \mathcal{G}_t)$. Then (φ_t, ψ_t) is a replicating strategy for $[X_T - k]_+$.

- We have $V_t = \varphi_t X_t + \psi_t B_t$.
- We can show that V_t satisfies the self-financing property $dV_t = \varphi_t dX_t + \psi_t dB_t$ using Itô's product rule.
- Since $V_T = [X_T - k]_+$, we have that (φ_t, ψ_t) is a replicating strategy.

Diffusion model replicating strategy

Theorem

There exists a replicating strategy (φ, ψ) *for the claim* $[X_T - k]_+$ *under the diffusion model. The processes* φ *and* ψ *are given by* $\varphi(t, X_t) = g_x(t, X_t)B_t$ *and* $\psi(t, X_t) = Z_t - g_x(t, X_t)X_t$ *where*

$$g_x(t, X_t) = \frac{\partial}{\partial x}(E_{Q_T}(B_T^{-1}[X_{T-t} - k]_+ \mid X_0 = x))$$

and

$$Z_t = E_{Q_T}(B_T^{-1}[X_{T-t} - k]_+ \mid X_0 = x).$$

- This result relies upon the Kolmogorov backward equation.
- The processes in the replicating strategy are given explicitly, which we were not able to find in the previous result. This is due to the fact that we no longer rely upon the martingale representation theorem.

Section 7

The Black-Scholes formula

Theorem

Under the Black-Scholes model, we can find that the option premium price is given by

$$V_0 = P_0 \Phi\left(\frac{(r + \frac{\sigma^2}{2})T + \ln\frac{P_0}{k}}{\sigma\sqrt{T}}\right) - ke^{-rT}\Phi\left(\frac{(r - \frac{\sigma^2}{2})T + \ln\frac{P_0}{k}}{\sigma\sqrt{T}}\right).$$

- Φ is the normal cumulative distribution function. The reason this appears is due to the fact that Brownian motion has normally distributed increments.
- This formula can be easily computed and is therefore well suited to many applications.

The generalized Black-Scholes formula

Theorem

Under the generalized Black-Scholes model, the option premium price is given by

$$e^{s^2/2+m+\xi}\Phi\left(\frac{m+s^2-\ln k}{s}\right)-e^{\xi}k\Phi\left(\frac{m-\ln k}{s}\right)$$

where $\xi=-\int_0^T r_s ds$, $m=\int_0^T\left(r_s-\frac{\sigma_s^2}{2}\right)ds+\ln P_0$, *and* $s^2=\int_0^T\sigma_r^2 dr$.

- We gain generality in the model, but lose the ability to easily compute the premium price.
- In many cases where the functions in the given model are complex, this price may not be possible to compute.