CHAPTER 18
Binomial Trees in Practice

Practice Questions

Problem 18.8.
Consider an option that pays off the amount by which the final stock price exceeds the average stock price achieved during the life of the option. Can this be valued from a binomial tree using backwards induction?

No! This is an example of a *path-dependent option*. The payoff depends on the path followed by the stock price as well as on its final value. The option cannot be valued by starting at the end of the tree and working backward, because the payoff at a final branch depends on the path used to reach it. European options for which the payoff depends on the average stock price can be valued using Monte Carlo simulation, as described in Section 18.6.

Problem 18.9.
A nine-month American put option on a non-dividend-paying stock has a strike price of $49. The stock price is $50, the risk-free rate is 5% per annum, and the volatility is 30% per annum. Use a three-step binomial tree to calculate the option price.

In this case, $S_0 = 50$, $K = 49$, $r = 0.05$, $\sigma = 0.30$, $T = 0.75$, and $\Delta t = 0.25$. Also

$$u = e^{\sigma\sqrt{\Delta t}} = e^{0.30\sqrt{0.25}} = 1.1618$$

$$d = \frac{1}{u} = 0.8607$$

$$a = e^{r\Delta t} = e^{0.05\times 0.25} = 1.0126$$

$$p = \frac{a-d}{u-d} = 0.5043$$

$$1 - p = 0.4957$$

The output from DerivaGem for this example is shown in the Figure S18.1. The calculated price of the option is $4.29. Using 100 steps the price obtained is $3.91.

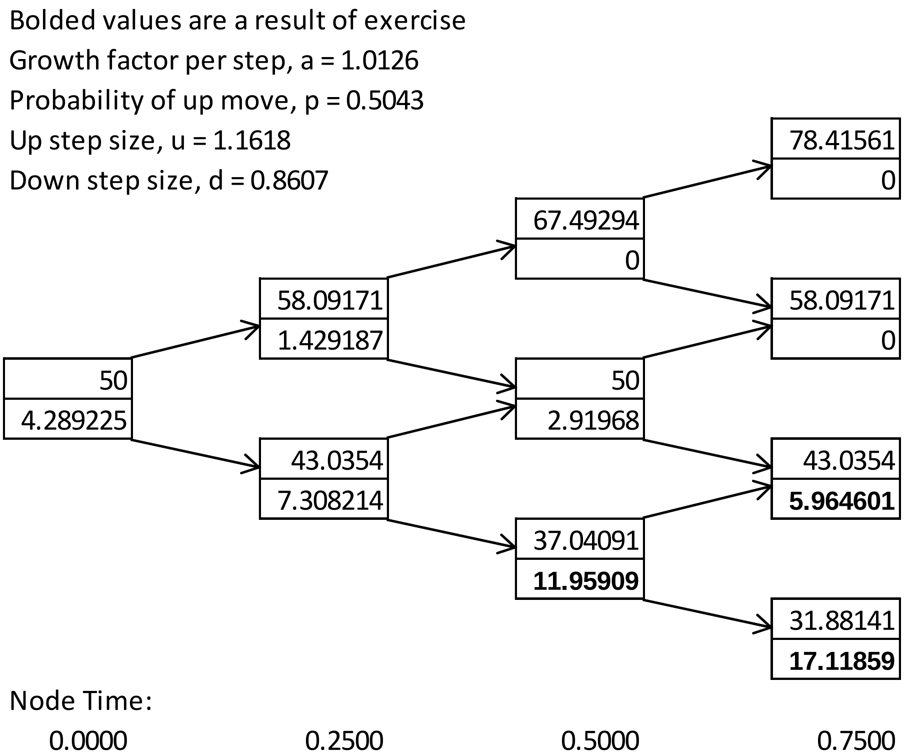


Figure S18.1 Tree for Problem 18.9

Problem 18.10.

Use a three-time-step tree to value a nine-month American call option on wheat futures. The current futures price is 400 cents, the strike price is 420 cents, the risk-free rate is 6%, and the volatility is 35% per annum. Estimate the delta of the option from your tree.

In this case $F_0 = 400$, $K = 420$, $r = 0.06$, $\sigma = 0.35$, $T = 0.75$, and $\Delta t = 0.25$. Also

$$u = e^{0.35\sqrt{0.25}} = 1.1912$$

$$d = \frac{1}{u} = 0.8395$$

$$a = 1$$

$$p = \frac{a-d}{u-d} = 0.4564$$

$$1-p = 0.5436$$

The output from DerivaGem for this example is shown in the Figure S18.2. The calculated price of the option is 42.07 cents. Using 100 time steps the price obtained is 38.64. The option's delta is calculated from the tree is

$$(79.971 - 11.419)/(476.498 - 335.783) = 0.487$$

When 100 steps are used the estimate of the option's delta is 0.483.

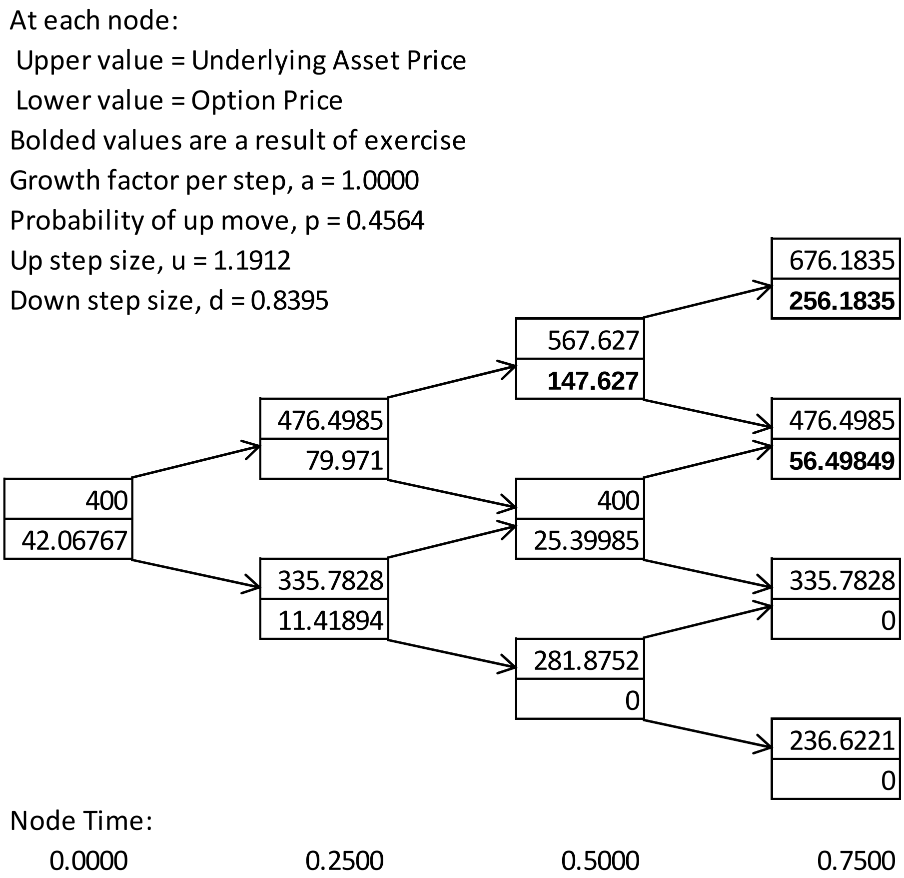


Figure S18.2 Tree for Problem 18.10

Problem 18.11.

A three-month American call option on a stock has a strike price of $20. The stock price is $20, the risk-free rate is 3% per annum, and the volatility is 25% per annum. A dividend of $2 is expected in 1.5 months. Use a three-step binomial tree to calculate the option price.

In this case the present value of the dividend is $2e^{-0.03\times0.125}=1.9925$. We first build a tree for $S_0=20-1.9925=18.0075$, $K=20$, $r=0.03$, $\sigma=0.25$, and $T=0.25$ with $\Delta t=0.08333$. This gives Figure S18.3. For nodes between times0 and 1.5 months we then add the present value of the dividend to the stock price. The result is the tree in Figure S18.4. The price of the option calculated from the tree is 0.674. When 100 steps are used the price obtained is 0.690.

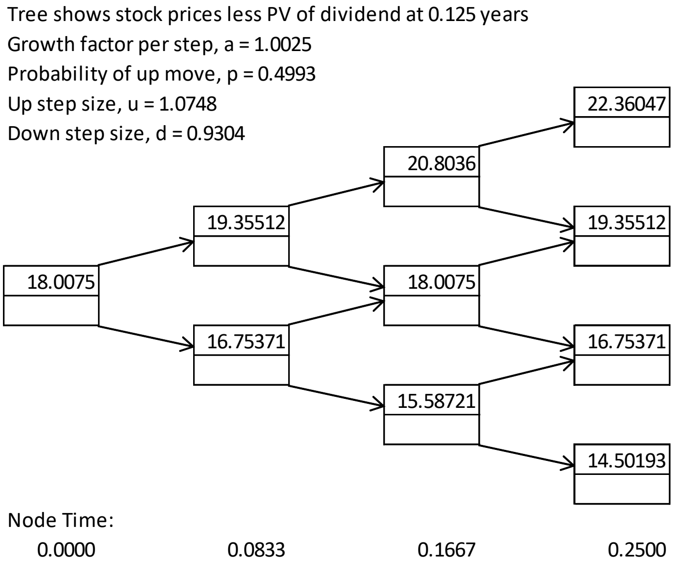


Figure S18.3 First tree for Problem 18.11

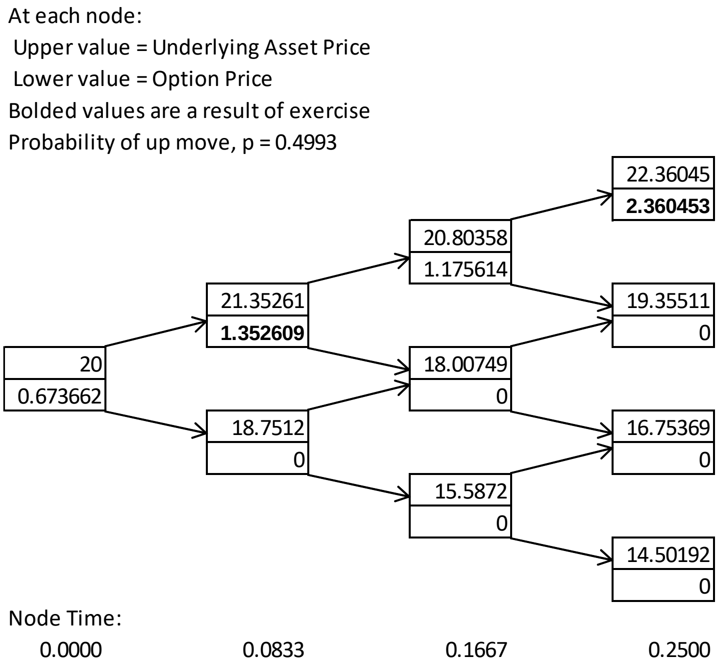


Figure S18.4 Final Tree for Problem 18.11

Problem 18.12.

A one-year American put option on a non-dividend-paying stock has an exercise price of $18. The current stock price is $20, the risk-free interest rate is 15% per annum, and the volatility *of the stock is 40% per annum. Use the DerivaGem software with four three-month time steps to estimate the value of the option. Display the tree and verify that the option prices at the final and penultimate nodes are correct. Use DerivaGem to value the European version of the option. Use the control variate technique to improve your estimate of the price of the American option.*

In this case $S_0 = 20$, $K = 18$, $r = 0.15$, $\sigma = 0.40$, $T = 1$, and $\Delta t = 0.25$. The parameters for the tree are

$$u = e^{\sigma\sqrt{\Delta t}} = e^{0.4\sqrt{0.25}} = 1.2214$$
$$d = 1/u = 0.8187$$
$$a = e^{r\Delta t} = 1.0382$$
$$p = \frac{a-d}{u-d} = \frac{1.0382-0.8187}{1.2214-0.8187} = 0.545$$

The tree produced by DerivaGem for the American option is shown in Figure S18.5. The estimated value of the American option is $1.29.

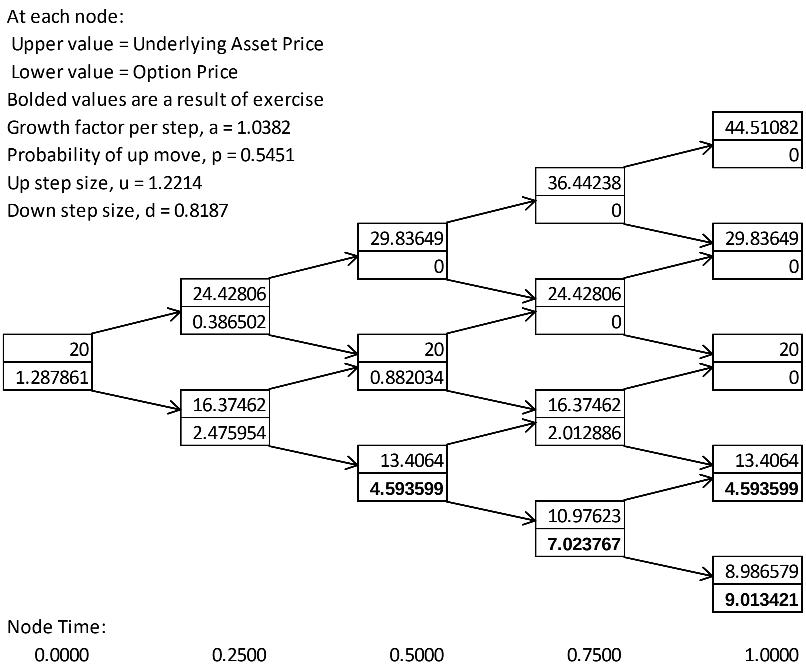


Figure S18.5 Tree to evaluate American option for Problem 18.12

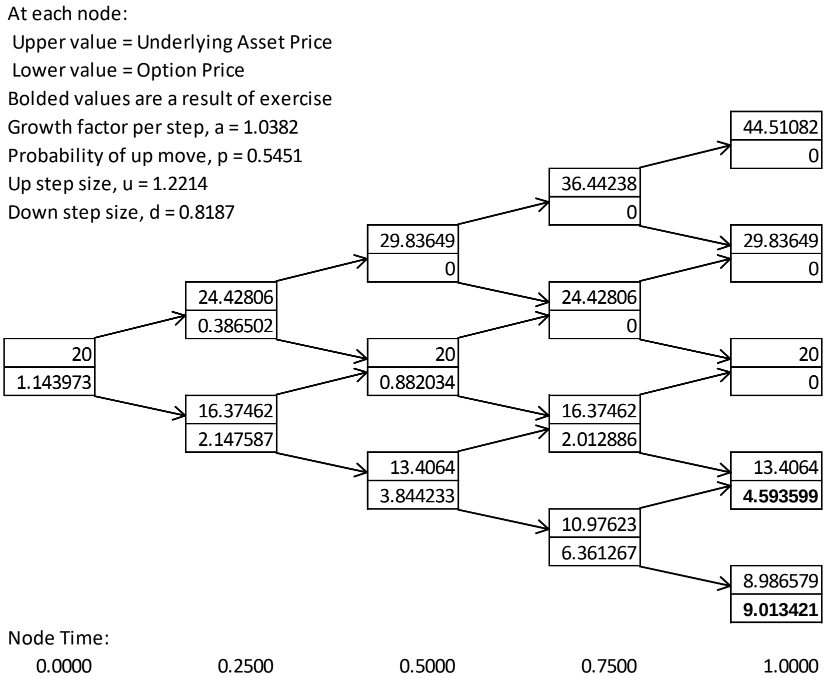


Figure S18.6 Tree to evaluate European option in Problem 18.12

As shown in Figure S18.6, the same tree can be used to value a European put option with the same parameters. The estimated value of the European option is \$1.14. The option parameters are $S_0 = 20$, $K = 18$, $r = 0.15$, $\sigma = 0.40$ and $T = 1$

$$d_1 = \frac{\ln(20/18) + 0.15 + 0.40^2/2}{0.40} = 0.8384$$
$$d_2 = d_1 - 0.40 = 0.4384$$

$$N(-d_1) = 0.2009; \quad N(-d_2) = 0.3306$$

The true European put price is therefore

$$18e^{-0.15} \times 0.3306 - 20 \times 0.2009 = 1.10$$

This can also be obtained from DerivaGem. The control variate estimate of the American put price is therefore 1.29 + 1.10 −1.14 = \$1.25.

Problem 18.13.

A two-month American put option on a stock index has an exercise price of 480. The current level of the index is 484, the risk-free interest rate is 10% per annum, the dividend yield on the index is 3% per annum, and the volatility of the index is 25% per annum. Divide the life of the option into four half-month periods and use the binomial tree approach to estimate the value of the option.

In this case $S_0 = 484$, $K = 480$, $r = 0.10$, $\sigma = 0.25$ $q = 0.03$, $T = 0.1667$, and $\Delta t = 0.04167$

$$u = e^{\sigma\sqrt{\Delta t}} = e^{0.25\sqrt{0.04167}} = 1.0524$$

$$d = \frac{1}{u} = 0.9502$$

$$a = e^{(r-q)\Delta t} = 1.00292$$

$$p = \frac{a-d}{u-d} = \frac{1.0029-0.9502}{1.0524-0.9502} = 0.516$$

The tree produced by DerivaGem is shown in the Figure S18.7. The estimated price of the option is $14.93.

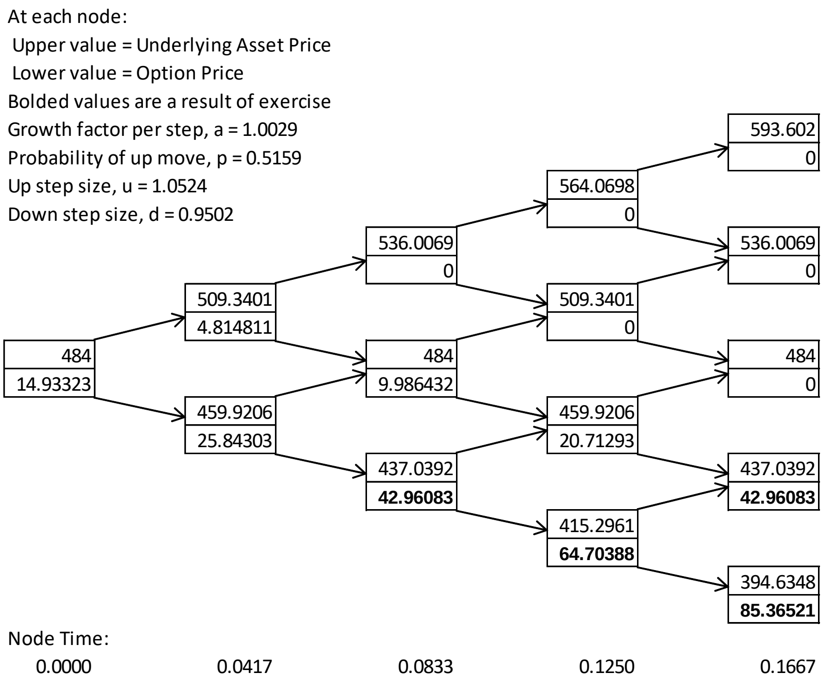


Figure S18.7 Tree to evaluate option in Problem 18.13

Problem 18.14.

How would you use the control variate approach to improve the estimate of the delta of an American option when the binomial tree approach is used?

First the delta of the American option is estimated in the usual way from the tree. Denote this by Δ_A^*. Then the delta of a European option which has the same parameters as the American option is calculated in the same way using the same tree. Denote this by Δ_B^*. Finally the true European delta, Δ_B, is calculated using the formulas in Chapter 17. The control variate estimate of delta is then:

$$\Delta_A^* - \Delta_B^* + \Delta_B$$

Problem 18.15.
How would you use the binomial tree approach to value an American option on a stock index when the dividend yield on the index is a function of time?

When the dividend yield is constant

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = \frac{1}{u}$$

$$p = \frac{a-d}{u-d}$$

$$a = e^{(r-q)\Delta t}$$

Making the dividend yield, q, a function of time makes a, and therefore p, a function of time. However, it does not affect u or d. It follows that if q is a function of time we can use the same tree by making the probabilities a function of time. The interest rate r can also be a function of time as described in Section 18.4.

Further Questions

Problem 18.16.
An American put option to sell a Swiss franc for dollars has a strike price of $0.80 and a time to maturity of one year. The volatility of the Swiss franc is 10%, the dollar interest rate is 6%, the Swiss franc interest rate is 3%, and the current exchange rate is 0.81. Use a three-time-step tree to value the option. Estimate the delta of the option from your tree.

The tree is shown in Figure S18.8. The value of the option is estimated as 0.0207. and its delta is estimated as

$$\frac{0.006221-0.041153}{0.858142-0.764559} = -0.373$$

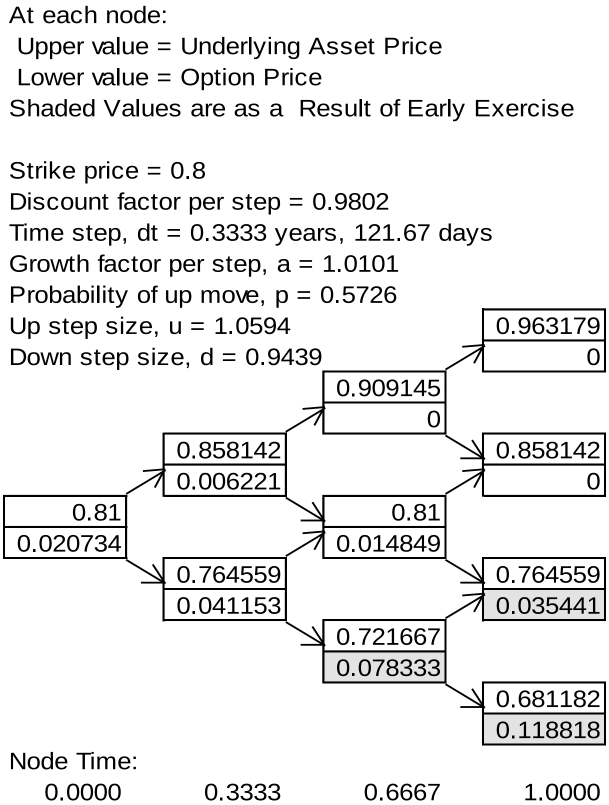


Figure S18.8 Tree for Problem 18.16

Problem 18.17.

A one-year American call option on silver futures has an exercise price of $9.00. The current futures price is $8.50, the risk-free rate of interest is 12% per annum, and the volatility of the futures price is 25% per annum. Use the DerivaGem software with four three-month time steps to estimate the value of the option. Display the tree and verify that the option prices at the final and penultimate nodes are correct. Use DerivaGem to value the European version of the option. Use the control variate technique to improve your estimate of the price of the American option.

In this case $F_0 = 8.5$, $K = 9$, $r = 0.12$, $T = 1$, $\sigma = 0.25$, and $\Delta t = 0.25$. The parameters for the tree are

$$u = e^{\sigma\sqrt{\Delta t}} = e^{0.25\sqrt{0.25}} = 1.1331$$

$$d = \frac{1}{u} = 0.8825$$

$$a = 1$$

$$p = \frac{a-d}{u-d} = \frac{1-0.8825}{1.1331-0.8825} = 0.469$$

The tree output by DerivaGem for the American option is shown in Figure S18.9. The estimated value of the option is $0.596. The tree produced by DerivaGem for the European version of the option is shown in Figure S18.10. The estimated value of the option is $0.586. The Black–Scholes price of the option is $0.570. The control variate estimate of the price of the option is therefore

$$0.596 + 0.570 - 0.586 = 0.580$$

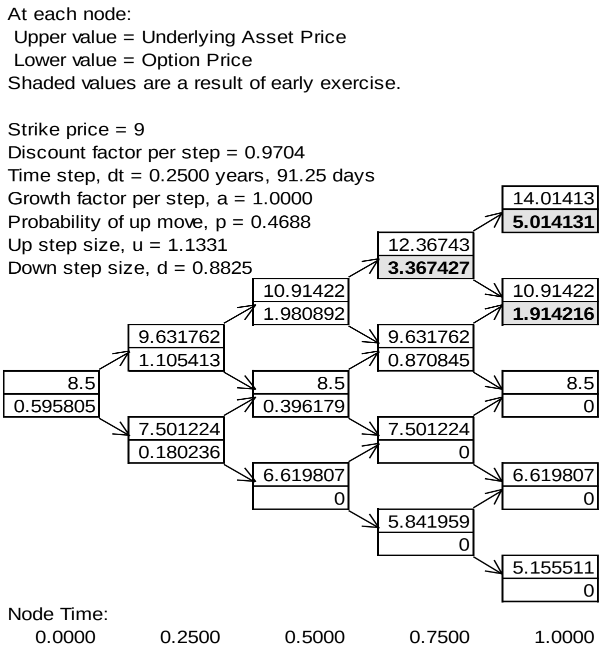


Figure S18.9 Tree for American option in Problem 18.17

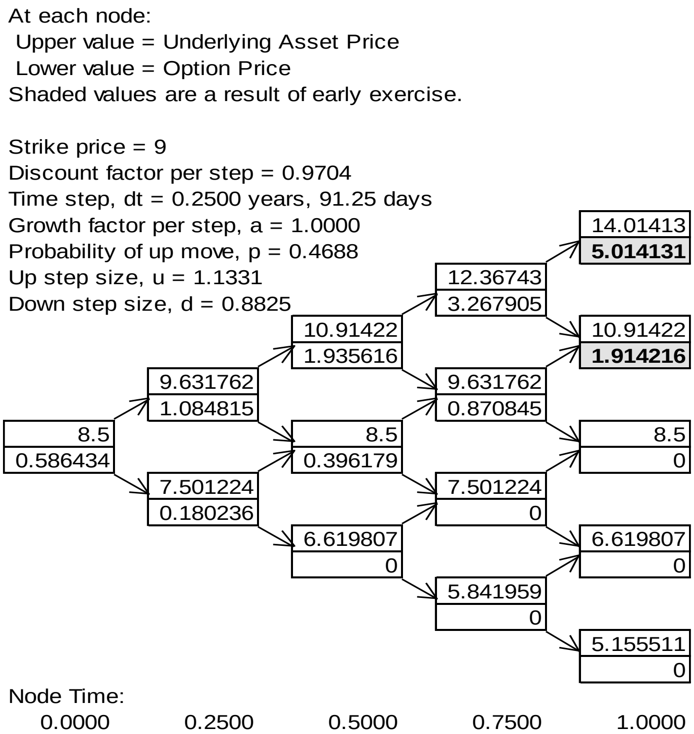


Figure S18.10 Tree for European option in Problem 18.17

Problem 18.18.

A six-month American call option on a stock is expected to pay dividends of $1 per share at the end of the second month and the fifth month. The current stock price is $30, the exercise price is $34, the risk-free interest rate is 10% per annum, and the volatility of the part of the stock price that will not be used to pay the dividends is 30% per annum. Use the DerivaGem software with the life of the option divided into 100 time steps to estimate the value of the option. Compare your answer with that given by Black's approximation (see Section 13.10.)

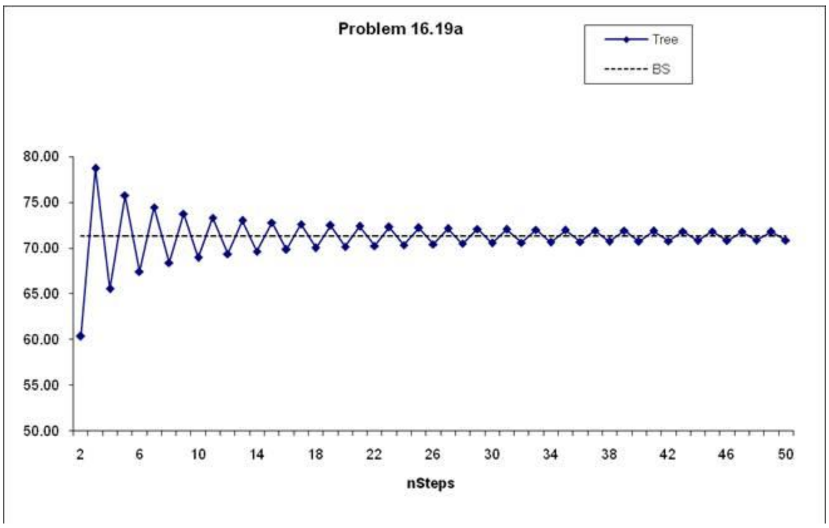
DerivaGem gives the value of the option as 1.0349. Black's approximation sets the price of the American call option equal to the maximum of two European options. The first lasts the full six months. The second expires just before the final ex-dividend date. In this case the software shows that the first European option is worth 0.957 and the second is worth 0.997. Black's model therefore estimates the value of the American option as 0.997.

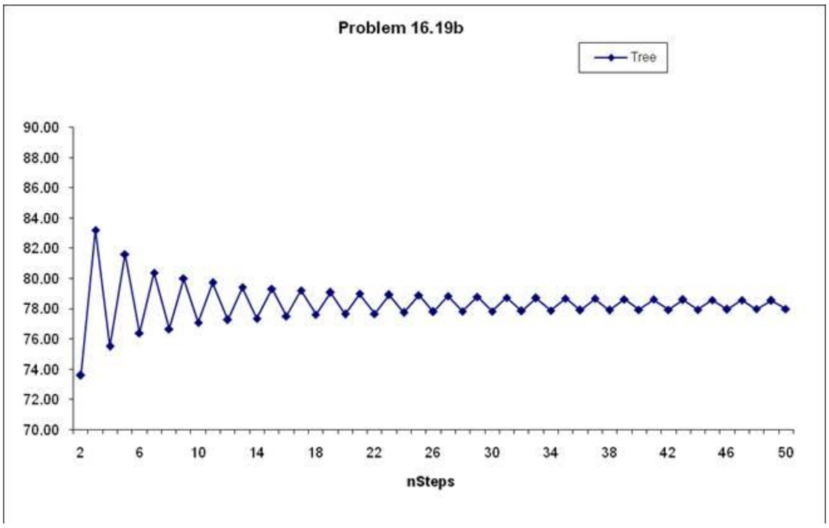
Problem 18.19. (Excel file)

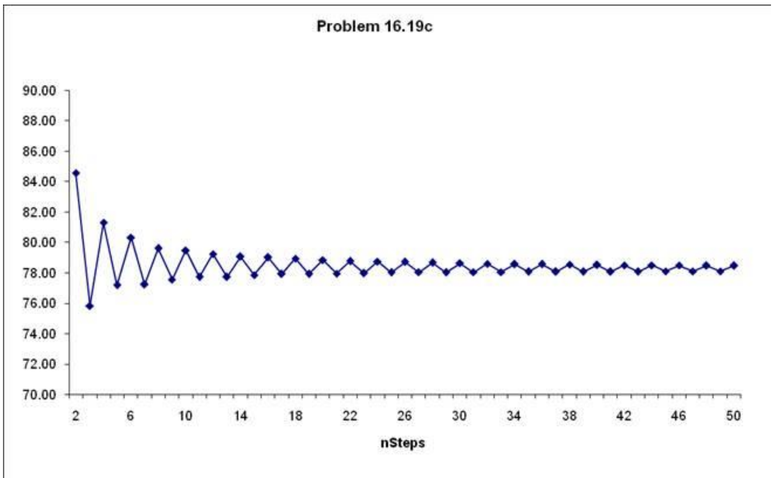
The DerivaGem Application Builder functions enable you to investigate how the prices of options calculated from a binomial tree converge to the correct value as the number of time steps increases. (See Figure 18.4 and Sample Application A in DerivaGem.) Consider a put option on a stock index where the index level is 900, the strike price is 900, the risk-free rate is 5%, the dividend yield is 2%, and the time to maturity is 2 years

a. *Produce results similar to Sample Application A on convergence for the situation where the option is European and the volatility of the index is 20%.*
b. *Produce results similar to Sample Application A on convergence for the situation where the option is American and the volatility of the index is 20%.*
c. *Produce a chart showing the pricing of the American option when the volatility is 20% as a function of the number of time steps when the control variate technique is used.*
d. *Suppose that the price of the American option in the market is 85.0. Produce a chart showing the implied volatility estimate as a function of the number of time steps.*

The results, produced by making modifications to Sample Application A, are shown in Figure S18.11.







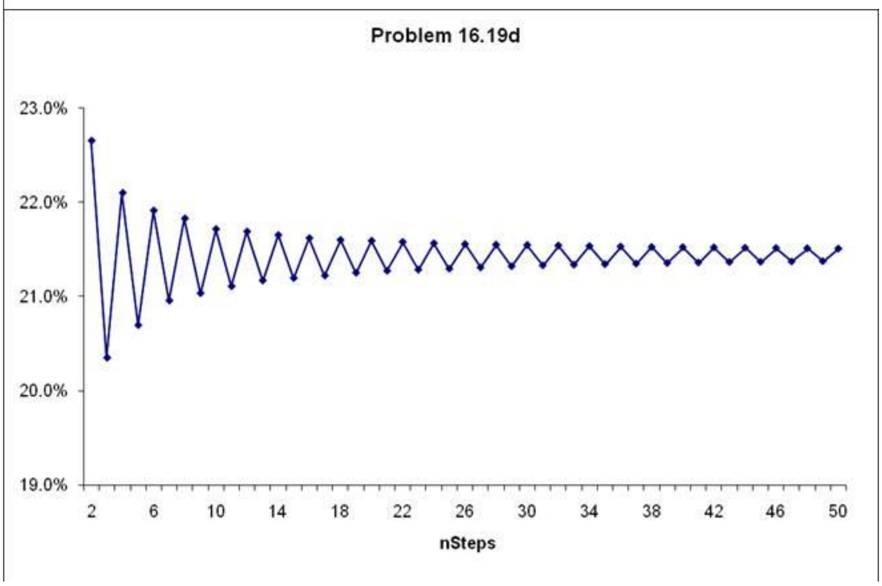


Figure S18.11: Convergence Charts for Problem 18.19

Problem 18.20

Estimate delta, gamma, and theta from the tree in Example 18.1. Explain how each can be interpreted.

Delta is (33.64−6.13)/(327.14−275.11) = 0.5288. This is the rate of change of the option price with respect to the futures price. Gamma is

$$\frac{(56.73-12.90)/(356.73-300)-(12.90-0)/(300-252.29)}{0.5\times(356.73-252.29)}=0.009$$

This is the rate of change of delta with respect to the futures price. Theta is (12.9−19.16)/0.16667=−37.59 per year or −0.1029 per calendar day.

Problem 18.21

How much is gained from exercising early at the lowest node at the nine-month point in Example 18.2?

Without early exercise the option is worth 0.2535 at the lowest node at the 9 month point. With early exercise it is worth 0.2552. The gain from early exercise is therefore 0.0017.

Problem 18.22

A four-step Cox–Ross–Rubinstein binomial tree is used to price a one-year American put option on an index when the index level is 500, the strike price is 500, the dividend yield is 2%, the risk-free rate is 5%, and the volatility is 25% per annum. What is the option price, delta, gamma, and theta? Explain how you would calculate vega and rho.

The tree is shown in Figure S18.12. the option price is 41.27. Delta, gamma and theta are −0.436, 0.0042, and −0.067. To calculate vega, the volatility could be increased to 26% and the tree reconstructed to get a new option value (43.09). The increase in the option value (1.82) is the vega per 1% change in volatility. To calculate rho, the interest rate could be increased from 5% to 6% and the tree reconstructed to get a new option value (39.67). The change in the option value (−1.60) is the rho per 1% change in the interest rate.

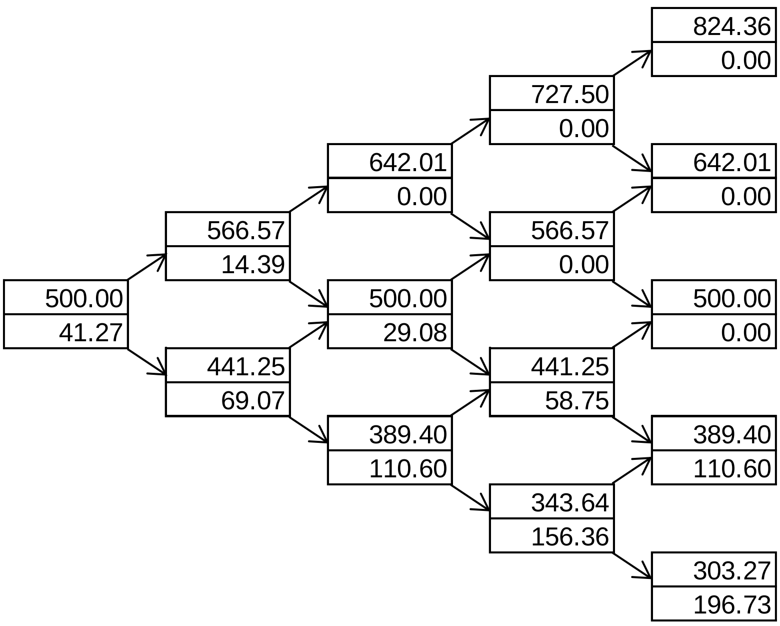


Figure S18.12: Tree for Problem 18.22