CHAPTER 13
Valuing Stock Options: The Black-Scholes-Merton Model

Practice Questions

Problem 13.8.
A stock price is currently $40. Assume that the expected return from the stock is 15% and its volatility is 25%. What is the probability distribution for the rate of return (with continuous compounding) earned over a one-year period?

In this case $\mu = 0.15$ and $\sigma = 0.25$. From equation (13.4) the probability distribution for the rate of return over a one-year period with continuous compounding is:

$$\phi\left(0.15 - \frac{0.25^2}{2},\ 0.25^2\right)$$

i.e.,

$$\phi(0.11875,\ 0.25^2)$$

The expected value of the return is 11.875% per annum and the standard deviation is 25.0% per annum.

Problem 13.9.
A stock price has an expected return of 16% and a volatility of 35%. The current price is $38.

a) *What is the probability that a European call option on the stock with an exercise price of $40 and a maturity date in six months will be exercised?*
b) *What is the probability that a European put option on the stock with the same exercise price and maturity will be exercised?*

a) The required probability is the probability of the stock price being above $40 in six months time. Suppose that the stock price in six months is S_T. The probability distribution of ln S_T is

$$\phi\left\{\ln 38 + \left(0.16 - \frac{0.35^2}{2}\right)0.5,\ 0.35^2 \times 0.5\right\}$$

i.e.,

$$\phi(3.687,\ 0.247^2)$$

Since $\ln 40 = 3.689$, the required probability is

$$1 - N\left(\frac{3.689 - 3.687}{0.247}\right) = 1 - N(0.008)$$

From normal distribution tables $N(0.008) = 0.5032$ so that the required probability is 0.4968.

b) In this case the required probability is the probability of the stock price being less than $40 in six months. It is

$$1 - 0.4968 = 0.5032$$

Problem 13.10.

Prove that, with the notation in the chapter, a 95% confidence interval for S_T is between

$$S_0 e^{(\mu-\sigma^2/2)T-1.96\sigma\sqrt{T}} \qquad \text{and} \qquad S_0 e^{(\mu-\sigma^2/2)T+1.96\sigma\sqrt{T}}$$

From equation (13.2), ln S_T has the distribution

$$\phi\left\{\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right\}$$

95% confidence intervals for $\ln S_T$ are therefore

$$\ln S_0 + (\mu - \frac{\sigma^2}{2})T - 1.96\sigma\sqrt{T}$$

and

$$\ln S_0 + (\mu - \frac{\sigma^2}{2})T + 1.96\sigma\sqrt{T}$$

95% confidence intervals for S_T are therefore

$$e^{\ln S_0 + (\mu-\sigma^2/2)T-1.96\sigma\sqrt{T}} \qquad \text{and} \qquad e^{\ln S_0 + (\mu-\sigma^2/2)T+1.96\sigma\sqrt{T}}$$

i.e.

$$S_0 e^{(\mu-\sigma^2/2)T-1.96\sigma\sqrt{T}} \qquad \text{and} \qquad S_0 e^{(\mu-\sigma^2/2)T+1.96\sigma\sqrt{T}}$$

Problem 13.11.

A portfolio manager announces that the average of the returns realized in each of the last 10 years is 20% per annum. In what respect is this statement misleading?

This problem relates to the material in Section 13.2 and Business Snapshot 13.1. The statement is misleading in that a certain sum of money, say $1,000, when invested for 10 years in the fund would have realized a return (with annual compounding) of less than 20% per annum.

The average of the returns realized in each year is always greater than the return per annum (with annual compounding) realized over 10 years. The first is an arithmetic average of the returns in each year; the second is a geometric average of these returns.

Problem 13.12.

Assume that a non-dividend-paying stock has an expected return of μ and a volatility of σ. An innovative financial institution has just announced that it will trade a derivative that pays off a dollar amount equal to

$$\frac{1}{T}\ln\left(\frac{S_T}{S_0}\right)$$

at time T. The variables S_0 and S_T denote the values of the stock price at time zero and time T.

a) *Describe the payoff from this derivative.*
b) *Use risk-neutral valuation to calculate the price of the derivative at time zero.*

a) The derivative will pay off a dollar amount equal to the continuously compounded return on the security between times 0 and T.

b) The expected value of $\ln(S_T / S_0)$ is, from equation (13.4), $(\mu - \sigma^2 / 2)T$. The expected payoff from the derivative is therefore $\mu - \sigma^2 / 2$. In a risk-neutral world this becomes $r - \sigma^2 / 2$. The value of the derivative at time zero is therefore:

$$\left(r - \frac{\sigma^2}{2} \right) e^{-rT}$$

Problem 13.13.

What is the price of a European call option on a non-dividend-paying stock when the stock price is \$52, the strike price is \$50, the risk-free interest rate is 12% per annum, the volatility is 30% per annum, and the time to maturity is three months?

In this case, $S_0 = 52$, $K = 50$, $r = 0.12$, $\sigma = 0.30$, and $T = 0.25$.

$$d_1 = \frac{\ln(52/50) + (0.12 + 0.3^2 / 2)0.25}{0.30\sqrt{0.25}} = 0.5365$$

$$d_2 = d_1 - 0.30\sqrt{0.25} = 0.3865$$

The price of the European call is

$$52N(0.5365) - 50e^{-0.12\times0.25}N(0.3865)$$

$$= 52 \times 0.7042 - 50e^{-0.03} \times 0.6504$$

$$= 5.06$$

or \$5.06.

Problem 13.14.

What is the price of a European put option on a non-dividend-paying stock when the stock price is \$69, the strike price is \$70, the risk-free interest rate is 5% per annum, the volatility is 35% per annum, and the time to maturity is six months?

In this case, $S_0 = 69$, $K = 70$, $r = 0.05$, $\sigma = 0.35$, and $T = 0.5$.

$$d_1 = \frac{\ln(69/70) + (0.05 + 0.35^2 / 2) \times 0.5}{0.35\sqrt{0.5}} = 0.1666$$

$$d_2 = d_1 - 0.35\sqrt{0.5} = -0.0809$$

The price of the European put is

$$70e^{-0.05\times0.5}N(0.0809) - 69N(-0.1666)$$

$$= 70e^{-0.025} \times 0.5323 - 69 \times 0.4338$$

$$= 6.40$$

or \$6.40.

Problem 13.15.

A call option on a non-dividend-paying stock has a market price of $\$2.50$. *The stock price is \$15, the exercise price is \$13, the time to maturity is three months, and the risk-free interest rate is 5% per annum. What is the implied volatility?*

In the case $c = 2.5$, $S_0 = 15$, $K = 13$, $T = 0.25$, $r = 0.05$. The implied volatility must be calculated using an iterative procedure.
A volatility of 0.2 (or 20% per annum) gives $c = 2.20$. A volatility of 0.3 gives $c = 2.32$. A volatility of 0.4 gives $c = 2.507$. A volatility of 0.39 gives $c = 2.487$. By interpolation the implied volatility is about 0.396 or 39.6% per annum.
The implied volatility can also be calculated using DerivaGem. Select equity as the Underlying Type in the first worksheet of DG400f.xls. Select Black-Scholes European as the Option Type. Input stock price as 15, the risk-free rate as 5%, time to exercise as 0.25, and exercise price as 13. Leave the dividend table blank because we are assuming no dividends. Select the button corresponding to call. Select the implied volatility button. Input the Price as 2.5 in the second half of the option data table. Hit the *Enter* key and click on calculate. DerivaGem will show the volatility of the option as 39.64%.

Problem 13.16.
Show that the Black–Scholes–Merton formula for a call option gives a price that tends to $\max(S_0 - K, 0)$ *as* $T \to 0$.

$$\begin{aligned} d_1 &= \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \\ &= \frac{\ln(S_0/K)}{\sigma\sqrt{T}} + \frac{r + \sigma^2/2}{\sigma}\sqrt{T} \end{aligned}$$

As $T \to 0$, the second term on the right hand side tends to zero. The first term tends to $+\infty$ if $\ln(S_0/K) > 0$ and to $-\infty$ if $\ln(S_0/K) < 0$. Since $\ln(S_0/K) > 0$ when $S_0 > K$ and $\ln(S_0/K) < 0$ when $S_0 < K$, it follows that

$d_1 \to \infty$ as $T \to 0$ when $S_0 > K$
$d_1 \to -\infty$ as $T \to 0$ when $S_0 < K$

Similarly

$d_2 \to \infty$ as $T \to 0$ when $S_0 > K$
$d_2 \to -\infty$ as $T \to 0$ when $S_0 < K$

Under the Black-Scholes-Merton formula the call price, c is given by:

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

From the above results, when $S_0 > K$, $N(d_1) \to 1.0$ and $N(d_2) \to 1.0$ as $T \to 0$ so that $c \to S_0 - K$. Also, when $S_0 < K$, $N(d_1) \to 0$ and $N(d_2) \to 0$ as $T \to 0$ so that $c \to 0$. These results show that $c \to \max(S_0 - K, 0)$ as $T \to 0$.

Problem 13.17.
Explain carefully why Black's approach to evaluating an American call option on a dividend-paying stock may give an approximate answer even when only one dividend is anticipated. Does the answer given by Black's approach understate or overstate the true option value? Explain your answer.

Black's approach in effect assumes that the holder of option must decide at time zero whether

it is a European option maturing at time t_n (the final ex-dividend date) or a European option maturing at time T. In fact, the holder of the option has more flexibility than this. The holder can choose to exercise at time t_n if the stock price at that time is above some level but not otherwise. Furthermore, if the option is not exercised at time t_n, it can still be exercised at time *T*.
It appears that Black's approach should understate the true option value. This is because the holder of the option has more alternative strategies for deciding when to exercise the option than the two strategies implicitly assumed by the approach. These alternative strategies add value to the option.
However, this is not the whole story! The standard approach to valuing either an American or a European option on a stock paying a single dividend applies the volatility to the stock price less the present value of the dividend. (The procedure for valuing an American option is explained in Chapter 18.) Black's approach when considering exercise just prior to the dividend date applies the volatility to the stock price itself. Black's approach therefore assumes more stock price variability than the standard approach in some of its calculations. In some circumstances it can give a higher price than the standard approach.

Problem 13.18.
Consider an American call option on a stock. The stock price is $70, the time to maturity is eight months, the risk-free rate of interest is 10% per annum, the exercise price is $65, and the volatility is 32%. A dividend of $1 is expected after three months and again after six months. Use the results in the appendix to show that it can never be optimal to exercise the option on either of the two dividend dates. Use DerivaGem to calculate the price of the option.

With the notation in the text

$$D_1 = D_2 = 1, \quad t_1 = 0.25, \quad t_2 = 0.50, \quad T = 0.6667, \quad r = 0.1 \quad \text{and} \quad K = 65$$

$$K(1 - e^{-r(T-t_2)}) = 65(1 - e^{-0.1\times 0.1667}) = 1.07$$

Hence

$$D_2 < K(1 - e^{-r(T-t_2)})$$

Also:

$$K(1 - e^{-r(t_2-t_1)}) = 65(1 - e^{-0.1\times 0.25}) = 1.60$$

Hence:

$$D_1 < K(1 - e^{-r(t_2-t_1)})$$

It follows from the conditions established in the Appendix to Chapter 13 that the option should never be exercised early. The option can therefore be value as a European option.
The present value of the dividends is

$$e^{-0.25\times 0.1} + e^{-0.50\times 0.1} = 1.9265$$

Also:

$$S_0 = 68.0735, \quad K = 65, \quad \sigma = 0.32, \quad r = 0.1, \quad T = 0.6667$$

$$d_1 = \frac{\ln(68.0735/65) + (0.1 + 0.32^2/2)0.6667}{0.32\sqrt{0.6667}} = 0.5626$$

$$d_2 = d_1 - 0.32\sqrt{0.6667} = 0.3013$$

$$N(d_1) = 0.7131, \quad N(d_2) = 0.6184$$

and the call price is

$$68.0735 \times 0.7131 - 65e^{-0.1\times 0.6667} \times 0.6184 = 10.94$$

or \$10.94.

DerivaGem can be used to calculate the price of this option. Select equity as the Underlying Type in the first worksheet of DG400f.xls. Select Black-Scholes European as the Option Type. Input stock price as 70, the volatility as 32%, the risk-free rate as 10%, time to exercise as =8/12, and exercise price as 65. In the dividend table, enter the times of dividends as 0.25 and 0.50, and the amounts of the dividends in each case as 1. Select the button corresponding to call. Hit the *Enter* key and click on calculate. DerivaGem will show the value of the option as \$10.942.

Problem 13.19.

A stock price is currently \$50 and the risk-free interest rate is 5%. Use the DerivaGem software to translate the following table of European call options on the stock into a table of implied volatilities, assuming no dividends. Are the option prices consistent with the assumptions underlying Black–Scholes–Merton?

Stock Price	*Maturity = 3 months*	*Maturity = 6 months*	*Maturity = 12 months*
54	7.00	8.30	10.50
50	3.50	5.20	7.50
55	1.60	2.90	5.10

Using DerivaGem we obtain the following table of implied volatilities

Stock Price	*Maturity = 3 months*	*Maturity = 6 months*	*Maturity = 12 months*
54	37.78	34.99	34.02
50	32.12	32.78	32.03
55	31.98	30.77	30.45

To calculate first number, select equity as the Underlying Type in the first worksheet of DG400f.xls. Select Black-Scholes European as the Option Type. Input stock price as 50, the risk-free rate as 5%, time to exercise as 0.25, and exercise price as 45. Leave the dividend table blank because we are assuming no dividends. Select the button corresponding to call. Select the implied volatility button. Input the Price as 7.0 in the second half of the option data table. Hit the *Enter* key and click on calculate. DerivaGem will show the volatility of the option as 37.78%. Change the strike price and time to exercise and recompute to calculate the rest of the numbers in the table.

The option prices are not exactly consistent with Black–Scholes–Merton. If they were, the implied volatilities would be all the same. We usually find in practice that low strike price options on a stock have significantly higher implied volatilities than high strike price options on the same stock. This phenomenon is discussed in Chapter 19.

Problem 13.20.

Show that the Black–Scholes–Merton formulas for call and put options satisfy put–call parity.

The Black–Scholes–Merton formula for a European call option is

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

so that

$$c + Ke^{-rT} = S_0 N(d_1) - Ke^{-rT} N(d_2) + Ke^{-rT}$$

or

$$c + Ke^{-rT} = S_0 N(d_1) + Ke^{-rT}[1 - N(d_2)]$$

or

$$c + Ke^{-rT} = S_0 N(d_1) + Ke^{-rT} N(-d_2)$$

The Black–Scholes–Merton formula for a European put option is

$$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$$

so that

$$p + S_0 = Ke^{-rT} N(-d_2) - S_0 N(-d_1) + S_0$$

or

$$p + S_0 = Ke^{-rT} N(-d_2) + S_0[1 - N(-d_1)]$$

or

$$p + S_0 = Ke^{-rT} N(-d_2) + S_0 N(d_1)$$

This shows that the put–call parity result

$$c + Ke^{-rT} = p + S_0$$

holds.

Problem 13.21.

Show that the probability that a European call option will be exercised in a risk-neutral world is, with the notation introduced in this chapter, $N(d_2)$*. What is an expression for the value of a derivative that pays off \$100 if the price of a stock at time T is greater than* K *?*

The probability that the call option will be exercised is the probability that $S_T > K$ where S_T is the stock price at time T. In a risk neutral world the probability distribution of ln S_T is

$$\phi\left\{\ln S_0 + (r - \sigma^2/2)T,\ \sigma^2 T\right\}$$

The probability that $S_T > K$ is the same as the probability that $\ln S_T > \ln K$. This is

$$1 - N\left[\frac{\ln K - \ln S_0 - (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right]$$

$$= N\left[\frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right]$$

$$= N(d_2)$$

The expected value at time T in a risk neutral world of a derivative security which pays off \$100 when $S_T > K$ is therefore

$$100N(d_2)$$

From risk neutral valuation the value of the security is

$$100e^{-rT} N(d_2)$$

Further Questions

Problem 13.22.
If the volatility of a stock is 18% per annum, estimate the standard deviation of the percentage price change in (a) one day, (b) one week, and (c) one month.

(a) $18/\sqrt{252} = 1.13\%$
(b) $18/\sqrt{52} = 2.50\%$
(c) $18/\sqrt{12} = 5.20\%$

Problem 13.23.
A stock price is currently $50. Assume that the expected return from the stock is 18% per annum and its volatility is 30% per annum. What is the probability distribution for the stock price in two years? Calculate the mean and standard deviation of the distribution. Determine the 95% confidence interval.

In this case $S_0 = 50$, $\mu = 0.18$ and $\sigma = 0.30$. The probability distribution of the stock price in two years, S_T, is lognormal and is, from equation (13.2), given by:

$$\ln S_T \sim \phi\left[\ln 50 + \left(0.18 - \frac{0.09}{2}\right)2,\ 0.3^2 \times 2\right]$$

i.e.,

$$\ln S_T \sim \phi(4.18,\ 0.42^2)$$

The mean stock price is from equation (13.3)

$$50e^{0.18\times 2} = 50e^{0.36} = 71.67$$

and the standard deviation is

$$50e^{0.18\times 2}\sqrt{e^{0.09\times 2} - 1} = 31.83$$

95% confidence intervals for $\ln S_T$ are

$$4.18 - 1.96 \times 0.42 \quad \text{and} \quad 4.18 + 1.96 \times 0.42$$

i.e.,

$$3.35 \quad \text{and} \quad 5.01$$

These correspond to 95% confidence limits for S_T of

$$e^{3.35} \quad \text{and} \quad e^{5.01}$$

i.e.,

$$28.52 \quad \text{and} \quad 150.44$$

Problem 13.24. (Excel file)
Suppose that observations on a stock price (in dollars) at the end of each of 15 consecutive weeks are as follows:
30.2, 32.0, 31.1, 30.1, 30.2, 30.3, 30.6, 33.0,
32.9, 33.0, 33.5, 33.5, 33.7, 33.5, 33.2
Estimate the stock price volatility. What is the standard error of your estimate?

The calculations are shown in the table below

$$\sum u_i = 0.09471 \quad \sum u_i^2 = 0.01145$$

and an estimate of standard deviation of weekly returns is:

$$\sqrt{\frac{0.01145}{13} - \frac{0.09471^2}{14 \times 13}} = 0.02884$$

The volatility per annum is therefore $0.02884\sqrt{52} = 0.2079$ or 20.79%. The standard error of this estimate is

$$\frac{0.2079}{\sqrt{2 \times 14}} = 0.0393$$

or 3.9% per annum.

Week	Closing Stock Price ($)	Price Relative $= S_i / S_{i-1}$	Weekly Return $u_i = \ln(S_i / S_{i-1})$
1	30.2		
2	32.0	1.05960	0.05789
3	31.1	0.97188	–0.02853
4	30.1	0.96785	–0.03268
5	30.2	1.00332	0.00332
6	30.3	1.00331	0.00331
7	30.6	1.00990	0.00985
8	33.0	1.07843	0.07551
9	32.9	0.99697	–0.00303
10	33.0	1.00304	0.00303
11	33.5	1.01515	0.01504
12	33.5	1.00000	0.00000
13	33.7	1.00597	0.00595
14	33.5	0.99407	–0.00595
15	33.2	0.99104	–0.00900

Problem 13.25.

A financial institution plans to offer a derivative that pays off a dollar amount equal to S_T^2 at time T where S_T is the stock price at time T. Assume no dividends. Defining other variables as necessary use risk-neutral valuation to calculate the price of the derivative at time zero. (Hint: The expected value of S_T^2 can be calculated from the mean and variance of S_T given in Section 13.1.)

From Section 13.1, if E denotes expected value,

$$E(S_T) = S_0 e^{\mu T}$$

$$\text{var}(S_T) = S_0^2 e^{2\mu T}(e^{\sigma^2 T} - 1)$$

Because $\text{var}(S_T) = E[(S_T)^2] - [E(S_T)]^2$, it follows that $E[(S_T)^2] = \text{var}(S_T) + [E(S_T)]^2$ so that

$$E[(S_T)^2] = S_0^2 e^{2\mu T}(e^{\sigma^2 T} - 1) + S_0^2 e^{2\mu T}$$

$$= S_0^2 e^{(2\mu + \sigma^2)T}$$

In a risk-neutral world $\mu = r$ so that

$$E[(S_T)^2] = S_0^2 e^{(2r+\sigma^2)T}$$

Using risk-neutral valuation, the value of the derivative security at time zero is

$$e^{-rT} E[(S_T)^2]$$

$$= S_0^2 e^{(2r+\sigma^2)T} e^{-rT}$$

$$= S_0^2 e^{(r+\sigma^2)T}$$

Problem 13.26.

Consider an option on a non-dividend-paying stock when the stock price is $30, the exercise price is $29, the risk-free interest rate is 5% per annum, the volatility is 25% per annum, and the time to maturity is four months.

a. *What is the price of the option if it is a European call?*
b. *What is the price of the option if it is an American call?*
c. *What is the price of the option if it is a European put?*
d. *Verify that put–call parity holds.*

In this case $S_0 = 30$, $K = 29$, $r = 0.05$, $\sigma = 0.25$ and $T = 4/12$

$$d_1 = \frac{\ln(30/29) + (0.05 + 0.25^2/2) \times 4/12}{0.25\sqrt{0.3333}} = 0.4225$$

$$d_2 = \frac{\ln(30/29) + (0.05 - 0.25^2/2) \times 4/12}{0.25\sqrt{0.3333}} = 0.2782$$

$$N(0.4225) = 0.6637, \quad N(0.2782) = 0.6096$$

$$N(-0.4225) = 0.3363, \quad N(-0.2782) = 0.3904$$

a. The European call price is

$$30 \times 0.6637 - 29e^{-0.05\times 4/12} \times 0.6096 = 2.52$$

or $2.52.

b. The American call price is the same as the European call price. It is $2.52.

c. The European put price is

$$29e^{-0.05\times 4/12} \times 0.3904 - 30 \times 0.3363 = 1.05$$

or $1.05.

d. Put-call parity states that:

$$p + S = c + Ke^{-rT}$$

In this case $c = 2.52$, $S_0 = 30$, $K = 29$, $p = 1.05$ and $e^{-rT} = 0.9835$ and it is easy to verify that the relationship is satisfied,

Problem 13.27.

Assume that the stock in Problem 13.26 is due to go ex-dividend in 1.5 months. The expected dividend is 50 cents.

a. *What is the price of the option if it is a European call?*
b. *What is the price of the option if it is a European put?*
c. *Use the results in the Appendix to this chapter to determine whether there are any circumstances under which the option is exercised early.*

a. The present value of the dividend must be subtracted from the stock price. This gives a new stock price of:

$$30-0.5e^{-0.125\times 0.05}=29.5031$$

and

$$d_1=\frac{\ln(29.5031/29)+(0.05+0.25^2/2)\times 0.3333}{0.25\sqrt{0.3333}}=0.3068$$

$$d_2=\frac{\ln(29.5031/29)+(0.05-0.25^2/2)\times 0.3333}{0.25\sqrt{0.3333}}=0.1625$$

$$N(d_1)=0.6205;\quad N(d_2)=0.5645$$

The price of the option is therefore

$$29.5031\times 0.6205-29e^{-0.05\times 4/12}\times 0.5645=2.21$$

or $2.21.

b. Because

$$N(-d_1)=0.3795,\quad N(-d_2)=0.4355$$

the value of the option when it is a European put is

$$29e^{-0.05\times 4/12}\times 0.4355-29.5031\times 0.3795=1.22$$

or $1.22.

c. If t_1 denotes the time when the dividend is paid:

$$K(1-e^{-r(T-t_1)})=29(1-e^{-0.05\times 0.2083})=0.3005$$

This is less than the dividend. Hence the option should be exercised immediately before the ex-dividend date for a sufficiently high value of the stock price.

Problem 13.28.

Consider an American call option when the stock price is $18, the exercise price is $20, the time to maturity is six months, the volatility is 30% per annum, and the risk-free interest rate is 10% per annum. Two equal dividends of 40 cents are expected during the life of the option, with ex-dividend dates at the end of two months and five months. Use Black's approximation and the DerivaGem software to value the option. Suppose now that the dividend is D on each ex-dividend date. Use the results in the Appendix to determine how high D can be without the American option being exercised early.

We first value the option assuming that it is not exercised early, we set the time to maturity equal to 0.5. There is a dividend of 0.4 in 2 months and 5 months. Other parameters are $S_0=18$, $K=20$, $r=10\%$, $\sigma=30\%$. DerivaGem gives the price as 0.7947. We next value the option assuming that it is exercised at the five-month point just before the final dividend. DerivaGem gives the price as 0.7668. The price given by Black's approximation is therefore 0.7947. (DerivaGem also shows that the American option price calculated using the binomial model with 100 time steps is 0.8243.)

It is never optimal to exercise the option immediately before the first ex-dividend date when

$$D_1\le K[1-e^{-r(t_2-t_1)}]$$

where D_1 is the size of the first dividend, and t_1 and t_2 are the times of the first and second dividend respectively. Hence we must have:

$$D_1 \le 20[1-e^{-(0.1\times 0.25)}]$$

that is,

$$D_1 \le 0.494$$

It is never optimal to exercise the option immediately before the second ex-dividend date when:

$$D_2 \le K(1-e^{-r(T-t_2)})$$

where D_2 is the size of the second dividend. Hence we must have:

$$D_2 \le 20(1-e^{-0.1\times 0.0833})$$

that is,

$$D_2 \le 0.166$$

It follows that the dividend can be as high as 16.6 cents per share without the American option being worth more than the corresponding European option.