CHAPTER 13

BLACK-SCHOLES OPTION PRICING MODEL

FIN2325 with Dr. Velthuis

SUMMARY

- Stock Return Distribution
- The Assumptions of the BS Model
- The Importance of Volatility
- The Black-Scholes Model
  - Received the Nobel Prize in 1997
- The Implied Volatility
- Extensions

Why do *I* need to understand Option Pricing?

- Options are everywhere:
  - Performance bonuses
  - Government bailouts
  - Patent value
  - Mining raw resources
  - United Airlines FareLock

  - *Real* options (applied to real assets):
    - Land value (option to develop it)
    - Mergers and acquisitions
    - Investments or building property, with various options wrt managerial *flexibility*
      - Option to expand
      - Option to delay
      - Option to abandon
  - Capital structure of a company (esp. if valuing a distressed company)
    - Debt: bond + short put option on assets
    - Equity: call option on assets of the firm
  - Decision making in general
    - Tradeoffs, "should I bring an umbrella?"
- Everything else equal, *the more uncertainty there is, the more valuable the option!*

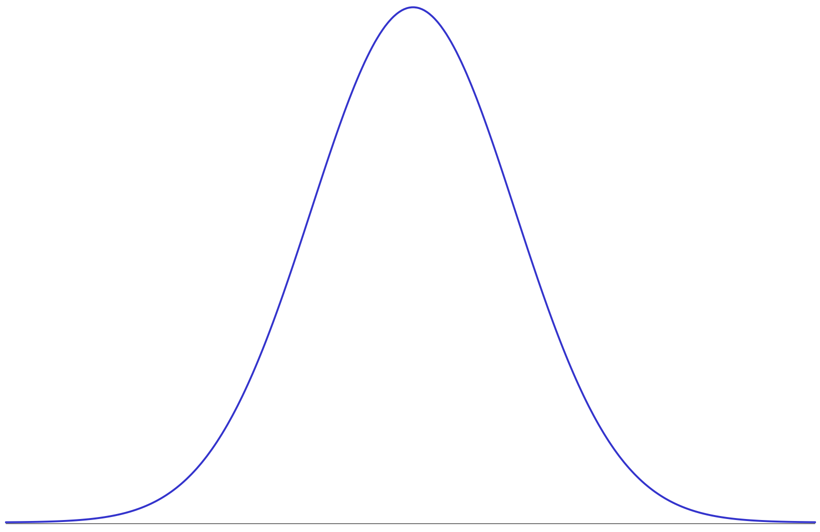
Stock Return Distribution

- Stock prices follow a random walk
- How to characterize a random walk?

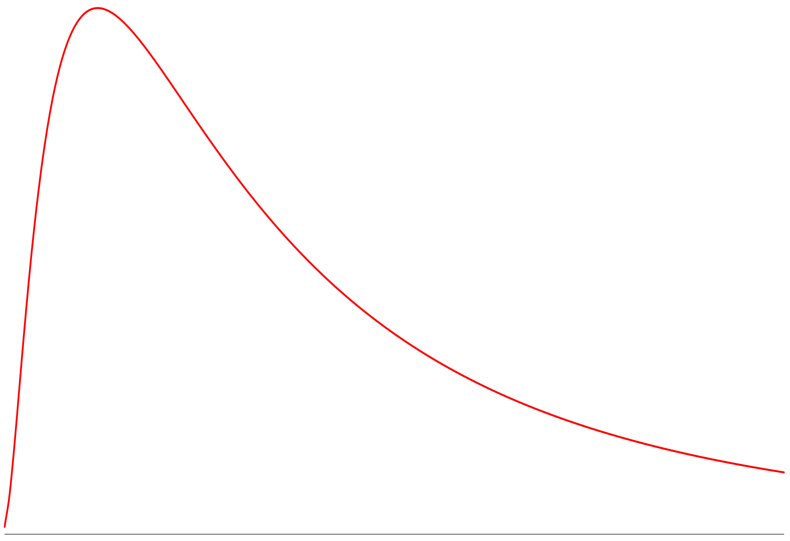
Stock Return Distribution

- Proportional changes (returns) in the stock price during a short time period are normally distributed
- Stock price follows a log-normal distribution
  - This means that the logarithm of the stock price is normally distributed
- A normal distribution is symmetric, and a log-normal distribution is skewed

Stock Return Distribution



normal



lognormal

Stock Return Distribution

- Two parameters characterize the stock price distribution
  - The expected return (mean, μ)
  - The volatility (standard deviation, σ)

Stock Return Distribution

- The logarithmic price, $ln\ S_T$, follows a normal distribution

$$\ln S_T \sim N\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right]$$

- The logarithmic return, $\ln \frac{S_T}{S_0}$, follows a normal distribution

$$\ln \frac{S_T}{S_0} \sim N\left[\left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right]$$

  - where μ is the annualized mean return and σ is the annualized return volatility
  - The expected return on the stock with continuous compounding is $\mu - \sigma^2/2$
  - This reflects the difference between arithmetic and geometric means

How to Estimate Mean Return and Stock Return Volatility Using Historical Data

- Mean return: average return
  - For pricing options, the mean return is irrelevant (risk-neutral valuation)

- Volatility: sample standard deviation of stock return

How to Rescale Volatility

- In pricing options, we use 252 trading days to convert daily volatility to annualized volatility, and ignore non-trading days, i.e.,

$$\text{Annualized volatility} = \text{volatility per day} \times \sqrt{252}$$

Assumptions Underlying the BS Model

- Stock price follows a log-normal distribution
- The market is perfect: no transaction costs, and no taxes
- There are no dividend distributions on the stock
- There is no riskless arbitrage opportunity
- Trading is continuous
- One can borrow and lend at the risk-free rate
- The interest rate is constant
- The volatility of stock returns is constant

The Derivation of the BS Model

- It is similar to the no-arbitrage analysis used in deriving the one-step binomial model
- Set up a riskless portfolio using the option and the stock
- The return on the portfolio is the same as the risk-free rate in the absence of riskless arbitrage
- Since the underlying source of uncertainty affects both stock and option, the construction of the riskless portfolio is possible

Inputs to the BS Model

- Current stock price, S
- Strike price, K
- Maturity of the option, T
- Interest rates, r
- Volatility of stock return, σ
  - A measure of uncertainty about stock return
  - Typically in the range between 10-50% per year

The Black-Scholes Model

- The European call price (or put price) is given by

$$c = SN(d_1) - Ke^{-rT}N(d_2)$$

$$p = Ke^{-rT}N(-d_2) - SN(-d_1)$$

where

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

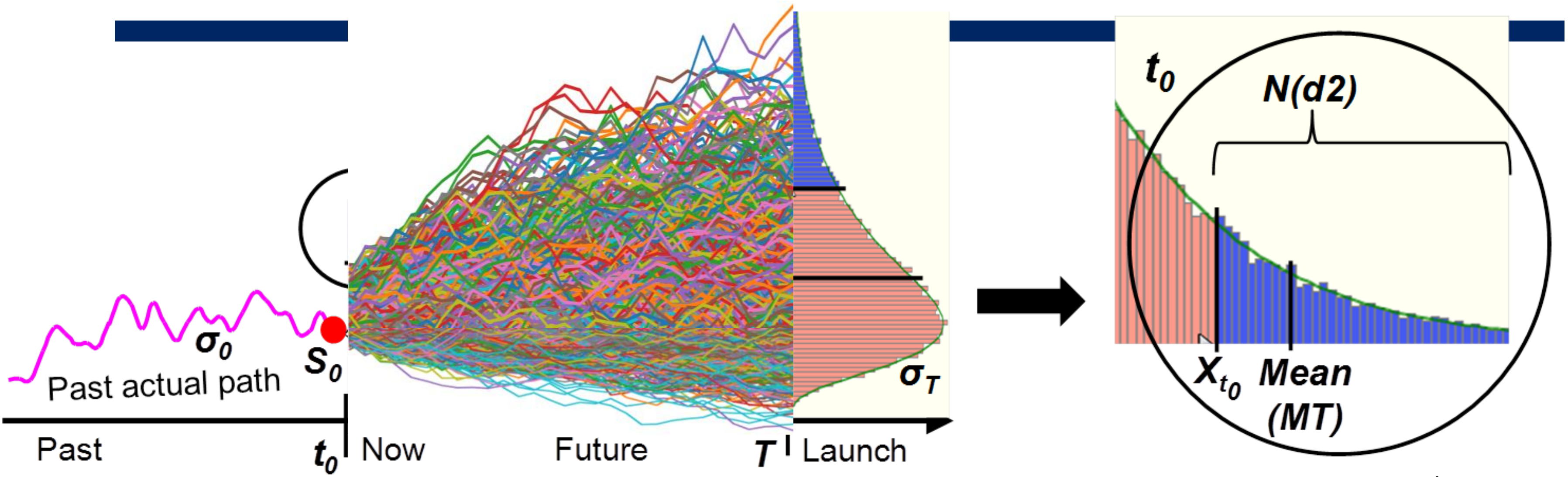
and $N(d_1)$ and $N(d_2)$ are cumulative probability functions for a standard normal distribution

Properties of Black-Scholes Formula

- As S_0 becomes very large:
  - d_1 and d_2 become large
  - $N(d_1)$ and $N(d_2)$ tend to 1
  - $N(-d_1)$ and $N(-d_2)$ tend to 0
  - c tends to $S_0 - Ke^{-rT}$
  - p tends to zero

- As S_0 becomes very small
  - c tends to zero
  - p tends to $Ke^{-rT} - S_0$

Graphical Representation



By: ScottMathews

- $N(d_2)$ the risk-neutral probability that the call option will be exercised
  - Probability of the stock price being in the tail of the distribution
- $N(d_1)$ is the value of the option payoff relative to that of the stock
  - $SN(d_1)$ is the present value of the expected stock price at expiration in a risk-neutral world, given that the option is being exercised

Risk-Neutral Valuation

- The variable μ does not appear in the Black-Scholes equation
- The equation is independent of all variables affected by risk preference
- This is consistent with the risk-neutral valuation principle

The Black-Scholes Model

- Use the table provided in the textbook to get $N(d_1)$ and $N(d_2)$
  - Examples:

  $d_1 = 0.85$, $N(d_1) = 0.8023$

  $d_2 = -0.12$, $N(d_2) = 0.4522$

  Let's verify these results

$N(-d1) = 1 - N(d1)$

Example:

$N(0.85) = .8023$

$N(-0.12) = .4522$

What is

$N(0) = ?$

$N(.75) = ?$

$N(-1.23) = ?$

Table values show area under the normal curve from σ to left

	Second Decimal									
σ	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

THE BLACK-SCHOLES MODEL: EXAMPLE

- Consider a 6-month call option, the current stock price is $42, the strike price is $40, the risk-free rate is 10% per year, and the volatility is 20% per year

THE BLACK-SCHOLES MODEL: EXAMPLE

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$= \frac{\ln(42/40) + (0.1 + 0.2^2/2)0.5}{0.2\sqrt{0.5}}$$

$$= 0.7693$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.769 - 0.2\sqrt{0.5} = 0.6278$$

THE BLACK-SCHOLES MODEL: EXAMPLE

- Check the Normal distribution table

$$N(d_1) = N(0.769) = 0.7794$$

$$N(d_2) = N(0.627) = 0.7357$$

- The call price is

$$\begin{aligned} c &= SN(d_1) - Ke^{-rT}N(d_2) \\ &= 42 \times 0.7794 - 40 \times e^{-0.10*0.5} \times 0.7349 \\ &= 4.76 \end{aligned}$$

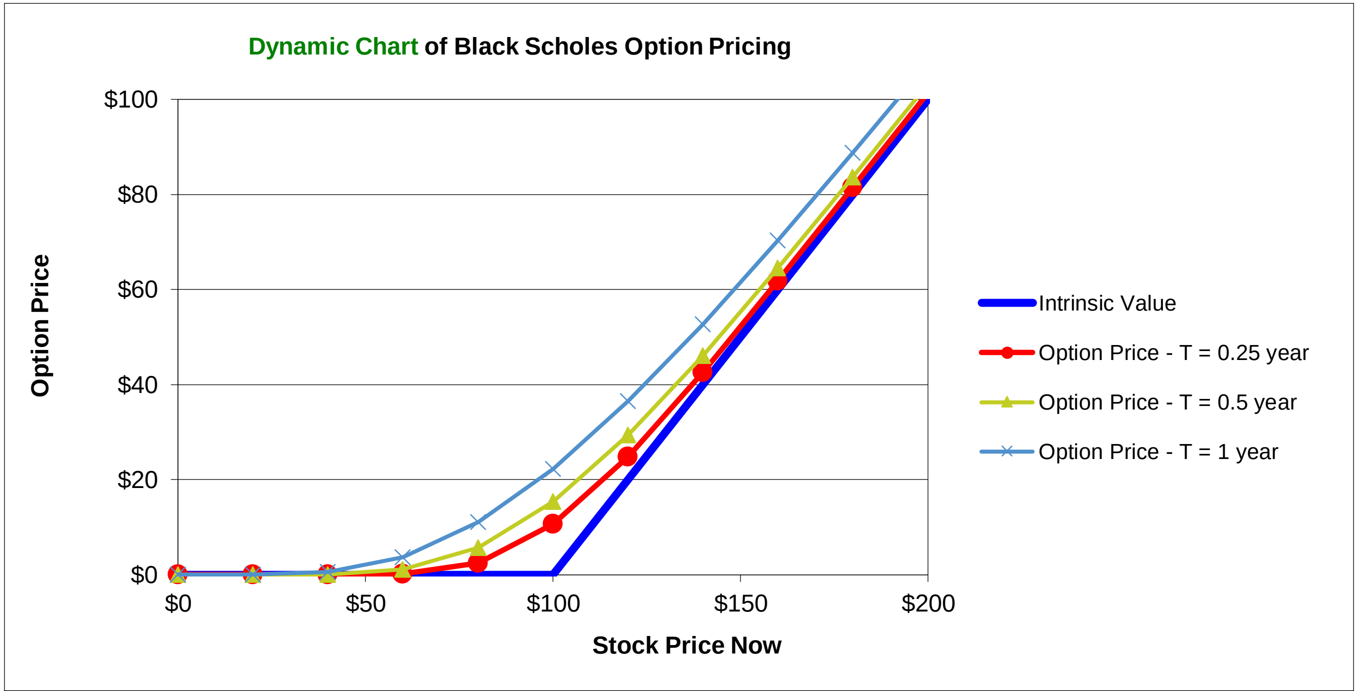
BS Options Dynamics Example

- The Black-Scholes formula for a call option gives a price that tends to max(S-K,0) as T→0

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} = \frac{\ln(S/K)}{\sigma\sqrt{T}} + \frac{(r + \sigma^2/2)}{\sigma}T$$

$\ln(S/K) > 0$ if S>K

$\ln(S/K) < 0$ if S<K



The Implied Volatility

- The volatility is unobservable
- One can use historical stock return data to calculate stock return volatility--*sample standard deviation*
- One can also use the observed option price and the BS model to back out the volatility--*implied volatility (IV)*
- IV is the volatility implied by the option price observed in the market and is forward-looking

The Implied Volatility

- How to find the IV for a given option price
  - We can not write the IV as a function of the option price
  - Thus, no formula is available to calculate the IV directly
  - Use interactive search (trial-and-error) procedure to find the IV

The Implied Volatility

- Intuition: the option price is an increasing function of volatility
- If volatility is too high, the calculated option price would be larger than the observed option price, then try a smaller volatility
- If volatility is too low, the calculated option price would be smaller than the observed option price, then try a larger volatility

The Volatility Smile

- A volatility smile shows the variation of the implied volatility with the strike price
- The volatility smile is the same whether calculated from European call options or European put options. (This follows from put-call parity.)

The Volatility Smile

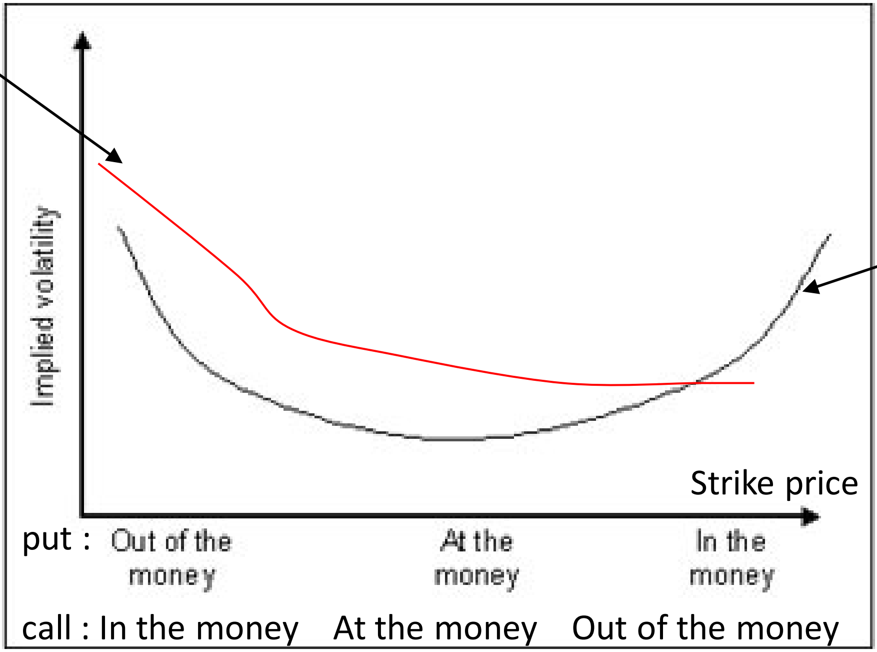
- The BS model assumes that the volatility is constant
- If this is true, the IV for options with different maturity or different strikes would be the same
- In practice, implied volatilities of in, at, and out-of-the money options are generally different.

The Volatility Smile

- Empirically, IVs from out-of-money and in-the-money options are higher than those from at-the-money options
- The volatility "smile" or "smirk"
- The smile or smirk may reflect traders beliefs that the distribution of stock returns have fatter tails than the normal curve implies

VOLATILITY SMILES AND SMIRKS

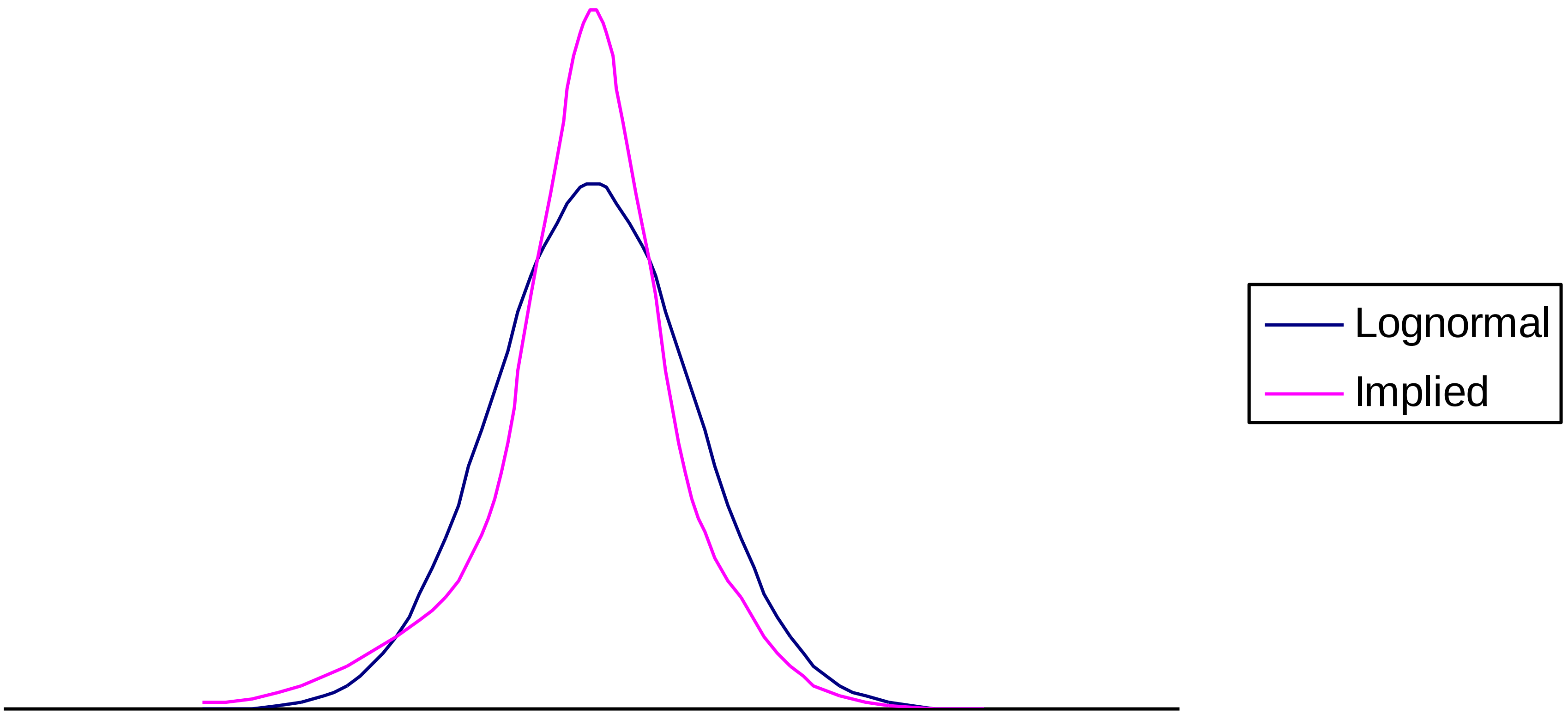
Smirk for equity options:



Smile for foreign currency options

(fatter left and right tails)

Implied Distribution for Equity Options



Properties of Implied Distribution for Equity Options

- The left tail is heavier and the right tail is less heavy than the lognormal distribution
- It is also “more peaked” than the lognormal distribution

Reasons for Smile in Equity Options

- Leverage
  - As a company's equity declines in value, the company's leverage increases. This means that the equity becomes more risky and its volatility increases
- Crashophobia
  - Prior to the stock market crash of 1987, implied volatilities were much less dependent on strike prices
  - Afterwards, traders became concerned about the possibility of another crash and started pricing options accordingly

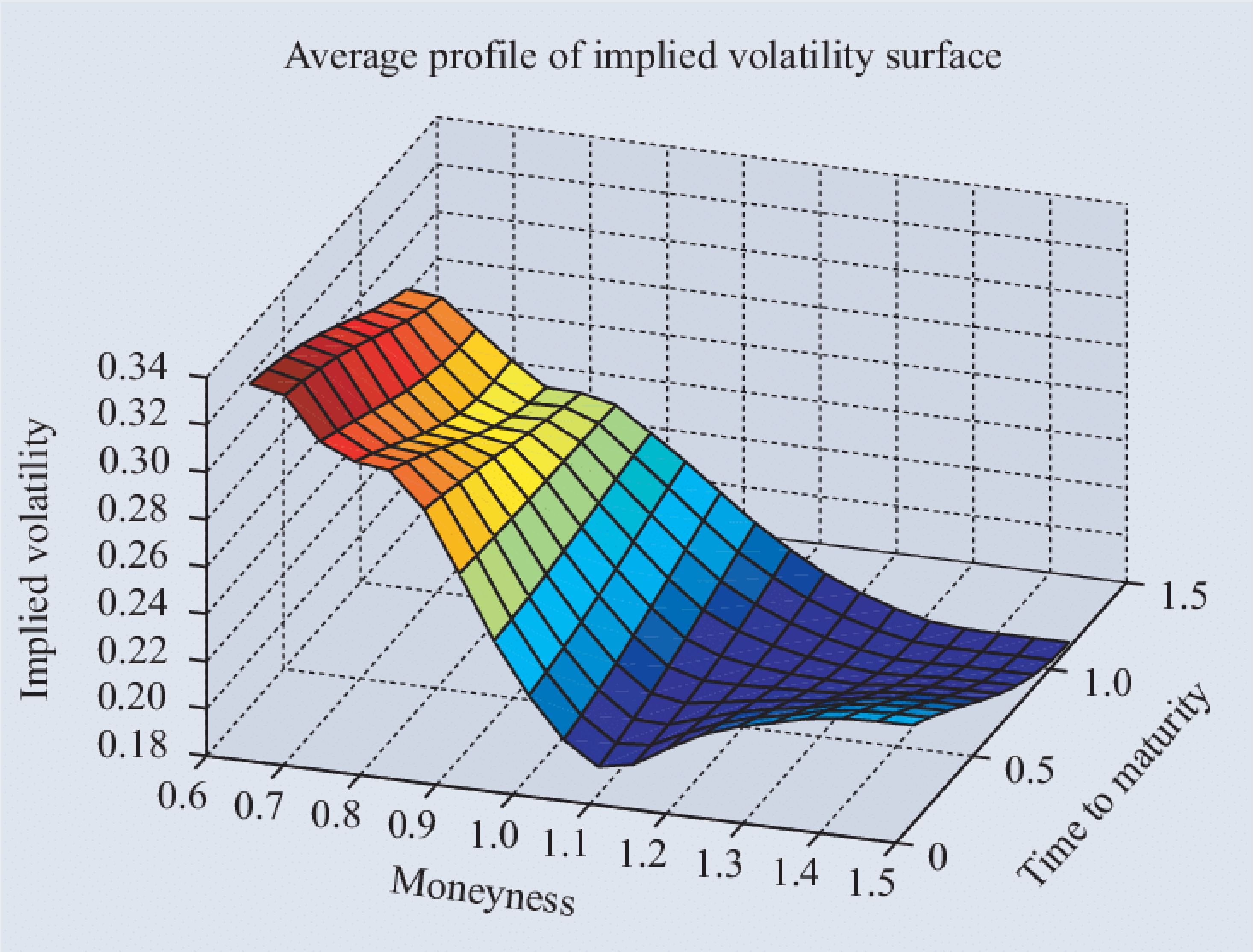
Volatility Term Structure

- In addition to calculating a volatility smile, traders also calculate a volatility term structure
- This shows the variation of implied volatility with the time to maturity of the option

Volatility Term Structure

- The volatility term structure tends to be downward sloping when volatility is high and upward sloping when it is low

EXAMPLE OF A VOLATILITY SURFACE



By: Jose Da Fonseca

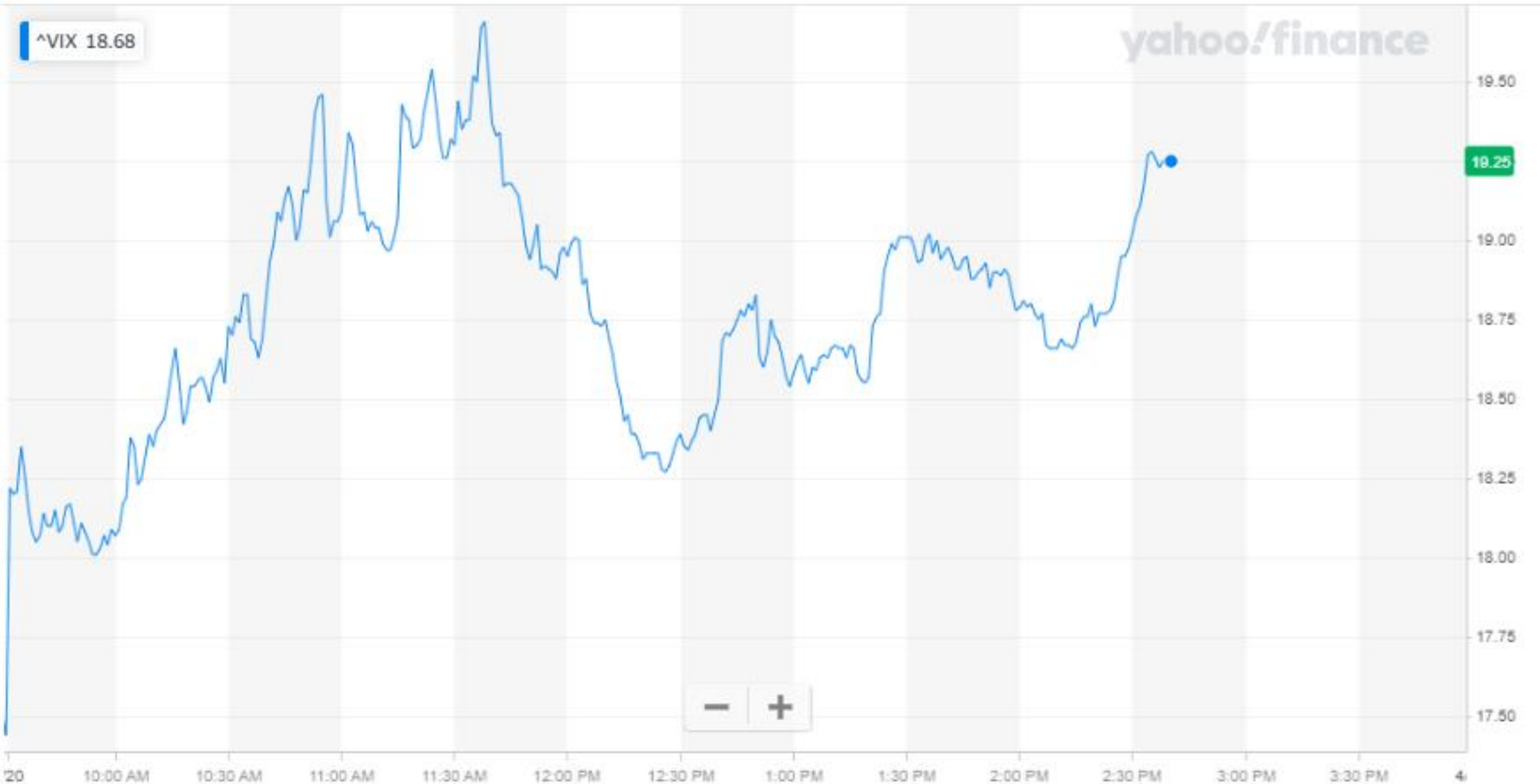
Volatility Index: VIX

- Introduced by Professor Bob Whaley at Duke University in 1993
- It provides investors with market estimates of expected volatility
- It is calculated by interpolating the IV of 30-day SPX calls/puts (S&P 500 index options) based on options with 23-37 days to expiration
  - Original Cboe Volatility Index (VXO) was based on eight S&P 100® Index ("OEX® Index") call/put options

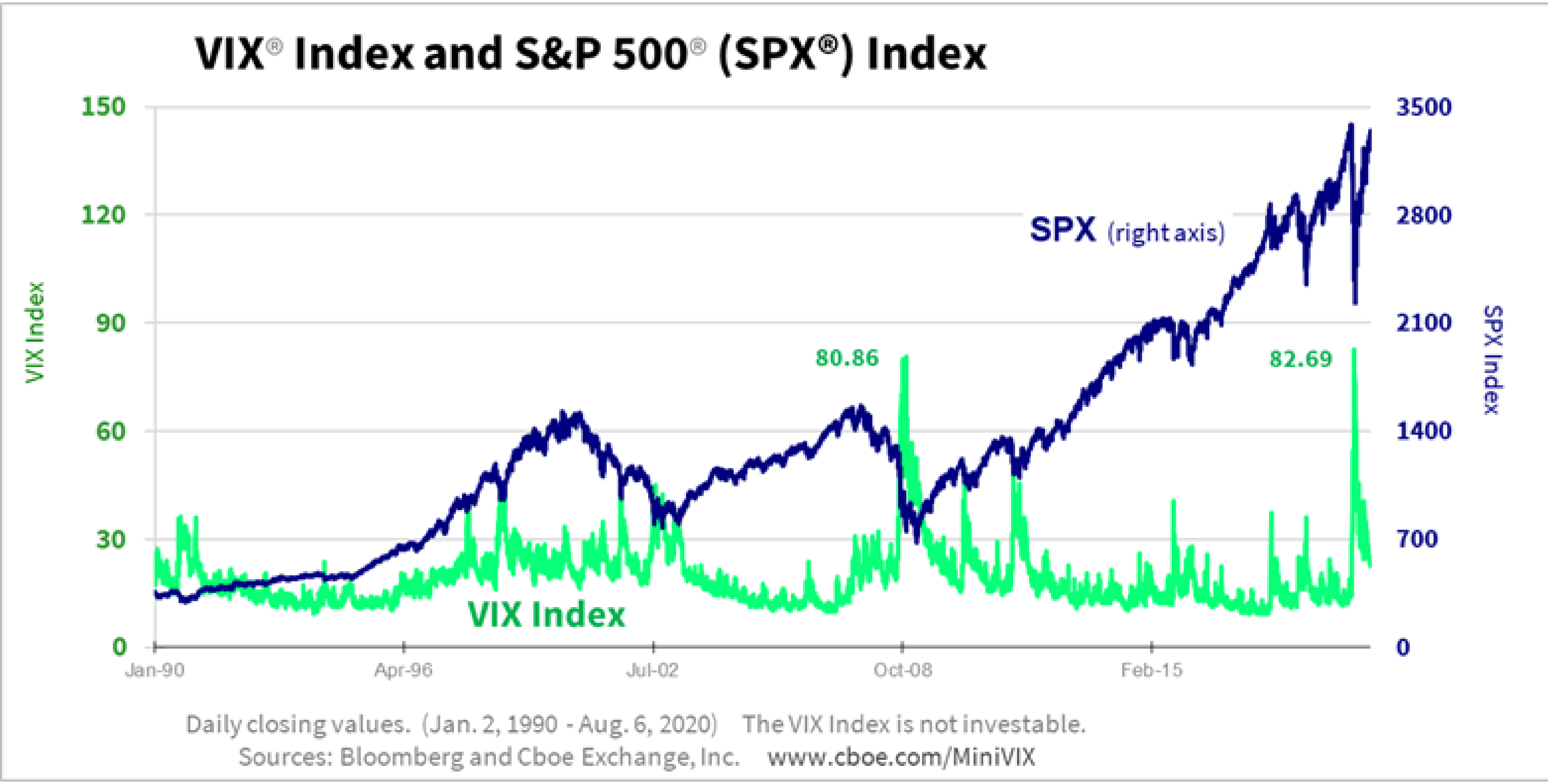
Volatility Index--VIX

- Options are 30-day at-the-money index options
- VIX is calculated and disseminated every 15 seconds (8:30 a.m. to 3:15 p.m. CT, and nowadays also between 2:15 a.m. CT and 8:15 a.m. CT)
- Between 1990 and April 2021, the average value of VIX was 19.5%
- Trading strategies based on the VIX index
  - VXX
  - XIV (inverse)
- Nasdaq Volatility Index: VXN

VIX – Intraday, 4/20/2021



VIX – History



VIX® Index and S&P 500® (SPX®) Index
150
120
90
60
30
0
3500
2800
2100
1400
700
0
VIX Index
SPX Index
SPX (right axis)
80.86
82.69
VIX Index
Jan-90
Apr-96
Jul-02
Oct-08
Feb-15
Daily closing values. (Jan. 2, 1990 - Aug. 6, 2020) The VIX Index is not investable.
Sources: Bloomberg and Cboe Exchange, Inc. www.cboe.com/MiniVIX

What Causes Volatility?

- New information
- Trading frictions: bid-ask spread, minimum tick size, transaction costs
- Trading itself: It was found that the volatility is much larger during trading hours than non-trading hours
- Opening volatility is 20% larger than closing volatility

Extensions of the BS model

- Dividends
- American options

Dividends

- European options on dividend-paying stocks can be valued by substituting the stock price less the present value of dividends into the Black-Scholes formula

- S → S – PV(dividends)

Dividends

- Only dividends with ex-dividend dates *during the life of the option* should be included
- The "dividend" should be the expected reduction in the stock price on the ex-dividend date
  - Taking into account tax impact

American Calls

- An American call on a non-dividend-paying stock
  - should never be exercised early
- An American call on a dividend-paying stock
  - It is sometimes optimal to exercise immediately prior to an ex-dividend date

Black's Approximation for Dealing with Dividends in American Call Options

- Set the American price equal to the maximum of the two European prices
  - The 1st European price is for an option maturing at the same time as the American option
  - The 2nd European price is for an option maturing just before the final ex-dividend date

Black's Approximation: Example

- Consider a 5-month call option, the current stock price is \$50, the strike price is \$50, the risk-free rate is 10% per year, and the volatility is 40% per year. Dividends of \$2 are paid after 2 months and 4 months.
  - $S=50$
  - $K=50$
  - $T-t=5/12$
  - $\sigma=40\%$
  - $r=10\%$
  - $D_1 = D_2 = 2$, after 2/12 and 4/12 years, respectively

Black's Approximation: Example

- Use DerivaGem to find:
  1. European price for an option maturing at the same time as the American option:
     - c = 3.9567
  2. European price for an option maturing just before the final ex-dividend date:
     - T-t = 4/12
     - D_1 = 2, no D_2
     - c = 4.2687
- Black's approximation suggests an option price of 4.27
- American call option price using Binomial model with 500 tree steps: C = 4.4240

SUMMARY

- **Stock prices follow a log-normal distribution**
- Create a tree considering various up/down moves
- *Early exercise* can be accommodated
- **Black-Scholes model provides a closed-form solution for pricing European options**
- $c = SN(d_1) - Ke^{-rT}N(d_2)$
- $p = Ke^{-rT}N(-d_2) - SN(-d_1)$
- **Implied volatility is a forward-looking measure of volatility, derived from observed option prices**
- Implied vol can vary with the strike price, leading to the so-called *"volatility smile" (or volatility smirk)*

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Homework 15

Chapter 13