CHAPTER 12

BINOMIAL OPTION PRICING MODEL

FIN2325 with Dr. Velthuis

Learning Goals

- Option pricing models
- One-step binomial models for European options
- Risk-neutral valuation
- Two-step binomial models
- European vs. American options
- Hedge ratio
- Binomial trees in practices

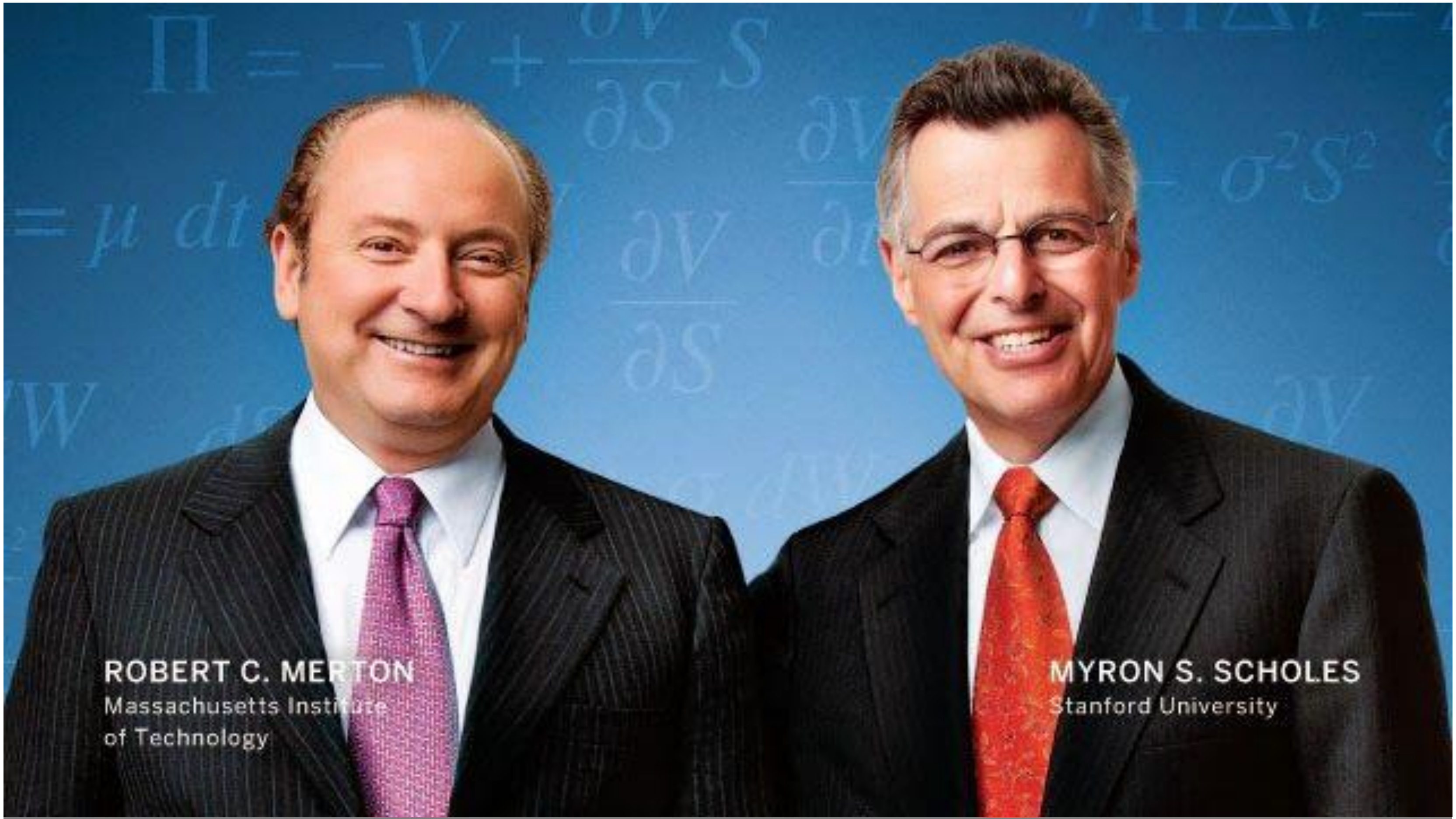
OPTION PRICING MODELS

- The Black-Scholes model (1973)
  - Developed by Black, Scholes and Merton to price European options
    - 1997 Nobel Prize Recipients
  - Use stochastic, continuous time calculus (rocket science!)
  - Merton and Scholes were principals of the infamous Long-term Capital Management (LTCM)

Option Pricing Models

- The Binomial model (1978, Cox-Ross-Rubinstein)
  - Discrete time version of B-S more suitable to options with optimal early exercise
  - Make simplified assumptions that stock prices are binomially distributed
- Monte Carlo Simulation
  - Used for exotic options or with non-normal distributions

Nobel Prize in Economics



Fischer Black passed away in 1995 before Nobel Prize was awarded to his colleagues in 1997

A One-Step Binomial Model

A One-Step Binomial Model (European Options)

1. Use no-arbitrage approach
   – Set up a risk-less portfolio (stocks & option)
   – Find the value of the portfolio on T (maturity)
   – Find the value of the portfolio today (t=0)
   – Back out the value of an option

A One-Step Binomial Model (European Options)

2. Use risk-neutral valuation approach
   - Compute risk-neutral probability of price movement (up or down)
   - Compute PV of an option using the option's payoff on the expiration day T

A Simple Binomial Model

- A stock price is currently \$20
- In three months it will be either \$22 or \$18

Stock Price = \$22

Stock price = \$20

Stock Price = \$18

Pricing a Call Option

- A 3-month call option on the stock has a strike price of 21

Stock Price = $22

Call Price = $1

Stock Price = $20

Call Price = ?

Stock Price = $18

Call Price = $0

Setting Up a Riskless Portfolio

- Consider the Portfolio: Δ: long Δ shares

  -1: short 1 call option

$$
\begin{array}{ccc}
 & & 22\Delta - 1 \\
\Delta S - c & \begin{array}{c}\nearrow \\ \searrow\end{array} & \\
 & & 18\Delta
\end{array}
$$

- The portfolio is riskless when

  $22\Delta - 1 = 18\Delta$ or $\Delta = 0.25$ (hedge ratio)

Setting Up a Riskless Portfolio

- The riskless portfolio = 0.25*S - c
- The stock price and call price move in the same direction, which makes it possible to set up a riskless portfolio if we long stocks and short call options
- The hedge ratio for call options should be positive, so we set up the riskless portfolio as
  $\Delta S - c$

Setting Up a Riskless Portfolio

- An example of hedge ratio for calls
  - AAPL
  - A call option
  - Strike price K=120
  - 2021/4/6, S_1=\$126.21, c_1=\$7.85
  - 2021/4/7, S_2=\$127.90, c_2=\$9.15
  - Hedge ratio

  $\Delta = (9.15-7.85)/(127.90 - 126.21) = 0.77 > 0$

Setting Up a Riskless Portfolio

- An example of hedge ratio for puts
  - AAPL
  - A put option
  - Strike price K=135
  - 2021/4/6, S_1=\$126.21, p_1=\$9.70
  - 2021/4/7, S_2=\$127.90, p_2=\$8.70
  - Hedge ratio on 2021/4/7:

  $\Delta = (8.70-9.70)/(127.90 - 126.21) = -0.59 < 0$

Valuing the Portfolio (Risk-Free Rate is 12%, T=3 months)

- The riskless portfolio is:
  - Long 0.25 shares
  - Short 1 call option
- The value of the portfolio in 3 months is
  - $0.25 \times 22 - 1 = 4.50$
- The value of the portfolio today is
  - $4.5e^{-0.12 \times 3/12} = 4.3670$

Valuing the Call Option

- Recall

  value of the portfolio today = ΔS - c_{today}

- c_{today} = ΔS - portfolio value

  = 0.25 x 20 - 4.367

  = 0.633

Pricing a Put Option

- A 3-month put option on the stock has a strike price of 21

Stock Price = \$22

Put Price = \$ 0

Stock Price = \$20

Put Price = ?

Stock Price = \$18

Put Price = \$ 3

Setting Up a Riskless Portfolio

- Consider the Portfolio: - Δ: shares
  1: put option

$$-\Delta S + P \quad \begin{cases} \nearrow \; -22\Delta \\ \searrow \; -18\Delta + 3 \end{cases}$$

- The portfolio is riskless when

  $-22\Delta = -18\Delta + 3$ or $\Delta = -0.75$

Setting Up a Riskless Portfolio

- Riskless portfolio = (-1) (-0.75) S + p

  = 0.75 S + p
- The stock price and put price move in opposite directions, which makes it possible to set up a riskless portfolio if we long stocks and long put options
- The hedge ratio for put options should be negative, so we set up the riskless portfolio as
  $-\Delta S + p$

Valuing the Portfolio (Risk-Free Rate is 12%)

The value of the portfolio in 3 months is

- $-\Delta 22 = -(-0.75)\text{x}22 = 16.5$

- The value of the portfolio today is
  - $16.5e^{-0.12 \times 3/12} = 16.01$

Valuing the Put Option

- Recall

  value of the portfolio today = $-\Delta S + p_{today}$

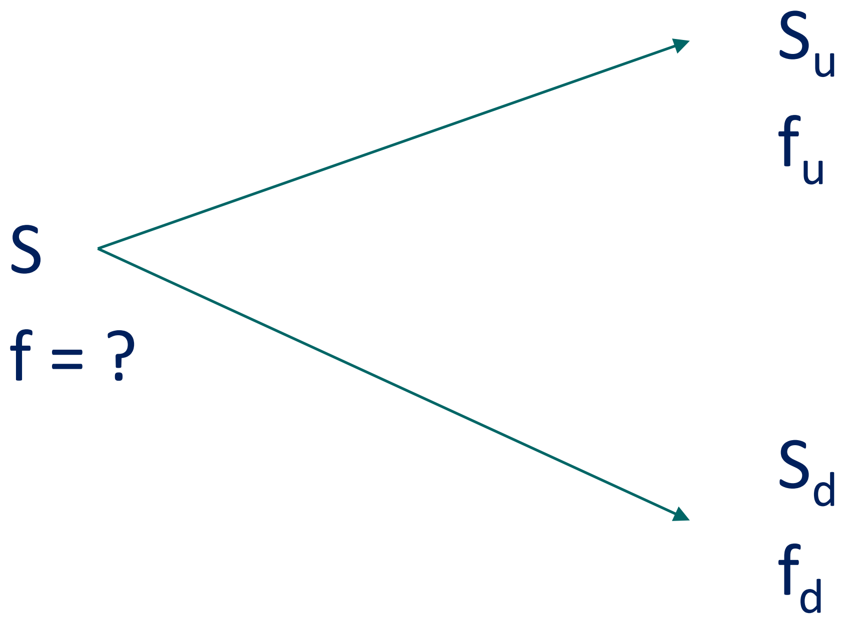
- p_{today} = portfolio value $+\Delta S$

  = 16.01+(-0.75)x20

  = 1.01

Generalization

- A derivative lasts for time T and is dependent on a stock



GENERALIZATION

- Consider the portfolio that is long Δ shares and short 1 derivative

$$\Delta S - f \quad \begin{cases} \nearrow \; \Delta S_u - f_u \\ \searrow \; \Delta S_d - f_d \end{cases}$$

- The portfolio is riskless when

$$\Delta S_u - f_u = \Delta S_d - f_d$$

$$\Delta = \frac{f_u - f_d}{S_u - S_d}$$

GENERALIZATION

- Value of the portfolio at time T is $\Delta S_u - f_u$
- Value of the portfolio today is $(\Delta S_u - f_u)e^{-rT}$
- Another expression for the portfolio value today is $\Delta S - f$
- Hence, $\Delta S - f = (\Delta S_u - f_u)e^{-rT}$

- $f = \Delta S - (\Delta S_u - f_u)e^{-rT}$

Generalization

- Substituting for Δ we obtain

$$f = [p\, f_u + (1 - p) f_d] e^{-rT}$$

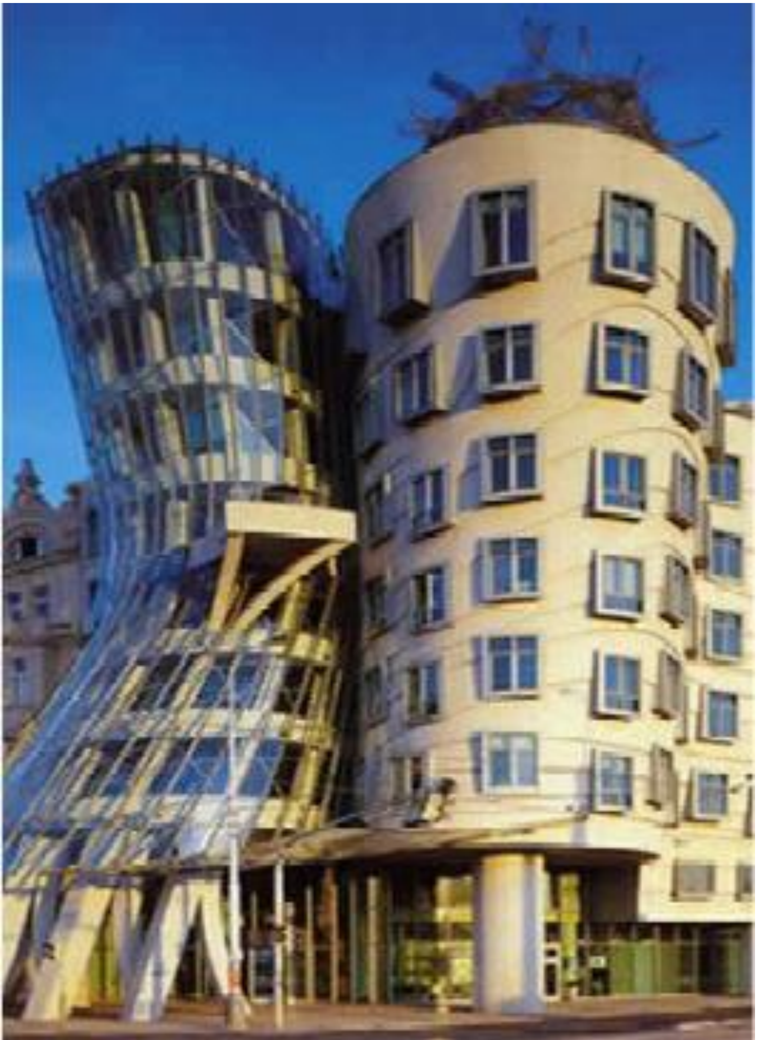
where

$$u = \frac{S_u}{S} \qquad d = \frac{S_d}{S}$$

p: risk neutral probability that the stock price will move up

$$p = \frac{e^{rT} - d}{u - d}$$

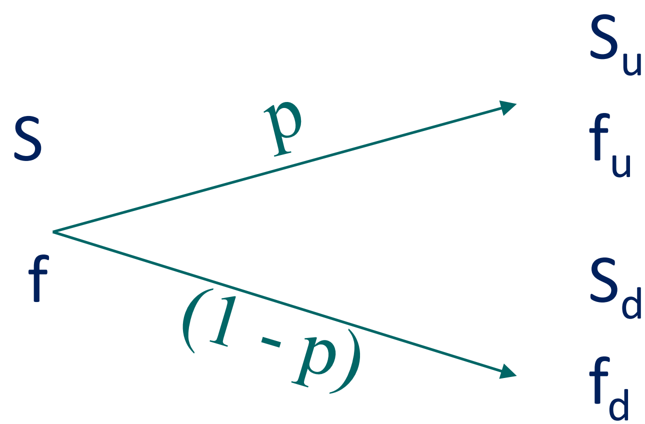
Risk-Neutral Valuation: Intuition



- The effect of medicine:
  Pain ➔ Joy
- Risky asset ➔ Risk-less asset
- Unknown discount rate ➔
  Risk-free rate

Risk-Neutral Valuation

- $f = [p\, f_u + (1 - p) f_d\,]e^{-rT}$
- The variables p and (1 - p) can be interpreted as the risk-neutral probabilities of up and down movements
- The value of a derivative is its expected payoff in a risk-neutral world discounted at the risk-free rate



Irrelevance of Stock's Expected Return

- When we value an option, the true probabilities of stock price moving up and down (i.e. expected return) are irrelevant
  - Key reason is that we calculate the option value in terms of the price of the underlying stock
  - The true probabilities of future up and down movements are already incorporated into the stock price.
  - As investors become more risk averse, stock prices decline

Irrelevance of Stock's Expected Return

- When we are valuing an option in terms of the underlying stock
  - The expected return on the stock is irrelevant
  - The expected return on the option is irrelevant

Real World vs. Risk-Neutral World

- Real world:
  - Stock: $S = [q\, Su + (1 - q)S_d]e^{-r_s T}$
  - Option: $f = [q\, f_u + (1 - q)f_d]e^{-r_o T}$
  - q: probability of an up-move in the real world
  - r_S is discount rate of stock
  - r_O is discount rate of option
- Risk-neutral world:
  - Stock: $S = [p\, S_u + (1 - p)S_d]e^{-rT}$
  - Option: $f = [p\, f_u + (1 - p)f_d]e^{-rT}$
  - p: probability of an up-move in the risk-neutral world
  - r is risk-free rate

Real World vs. Risk-Neutral World

- Assume: S = 20, Su = 22, Sd = 18, r = 12%, T = 3/12,

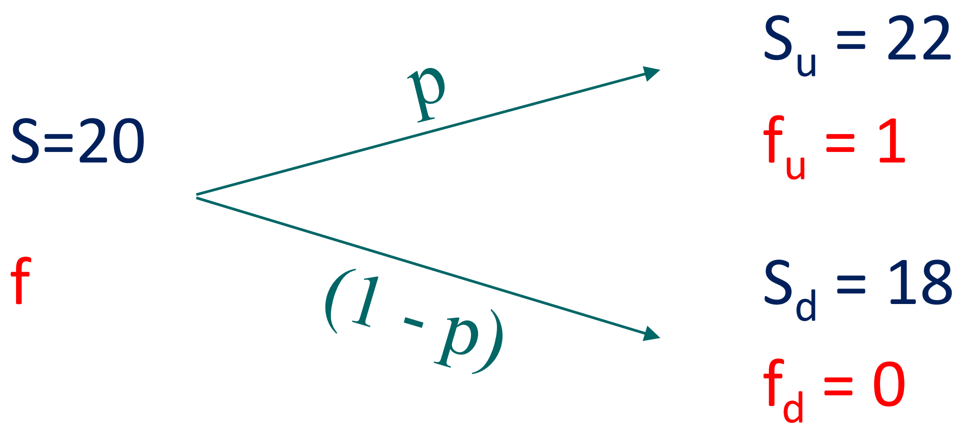
  r_S = 16%, and K = 21
- Then, $S = [q\,Su + (1-q)S_d\,]e^{-r_sT}$

  $20 = [q\,22 + (1-q)18\,]e^{-0.16*3/12}$

  → q = 0.7041
- The expected payoff from the option in the real world is then: $q\,f_u + (1-q)f_d$

  $= 0.7041 * 1 + (1 - 0.7041) * 0$
- Now we are stuck! We don’t know the discount rate to discount this payoff and we can’t value the option…
  - Because we know the correct value of the option is 0.633, we can deduce that the correct discount rate is 42.58%. This is because 0.633 = 0.7041 exp(-0.4258*3/12)

Original Example Revisited (Call Option, r=0.12, T=3 months)



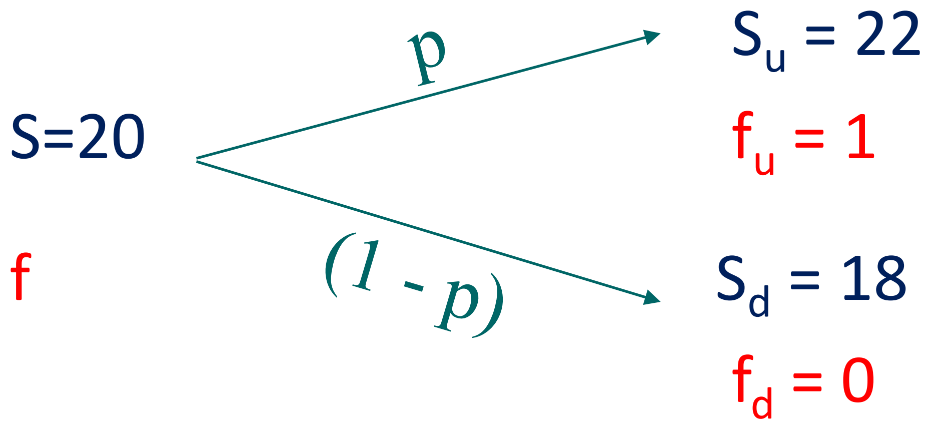
- P: risk-neutral probability (The expected return on the stock is the risk-free rate)

$20e^{0.12 \times 0.25} = p*22 + (1 - p)*18;$ $\quad p= 0.6523$

- Alternatively, we can use the formula to compute p

$$p = \frac{e^{rT} - d}{u - d}$$

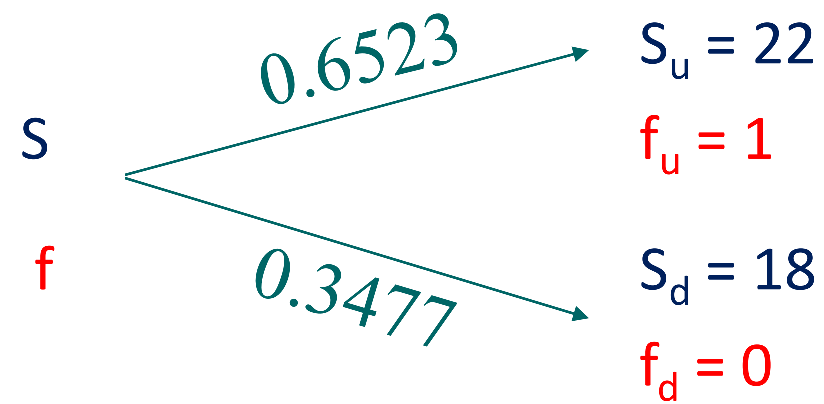
Original Example Revisited (Call Option)



- u=22/20=1.1, d=18/20=0.9

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.12 \times 0.25} - 0.9}{1.1 - 0.9} = 0.6523$$

Valuing the Call Option



- The value of the call is $[pf_u + (1-p)f_d]e^{-rT}$

  $[0.6523 \times 1 + 0.3477 \times 0]\ e^{-0.12 \times .25} = ?$

ORIGINAL EXAMPLE REVISITED (PUT OPTION, STRIKE PRICE= 21)

$$
\begin{array}{lcl}
 & & S_u = 22 \\
 & \overset{p}{\nearrow} & f_u = 0 \\
S=20 & & \\
f & \underset{(1-p)}{\searrow} & S_d = 18 \\
 & & f_d = 3
\end{array}
$$

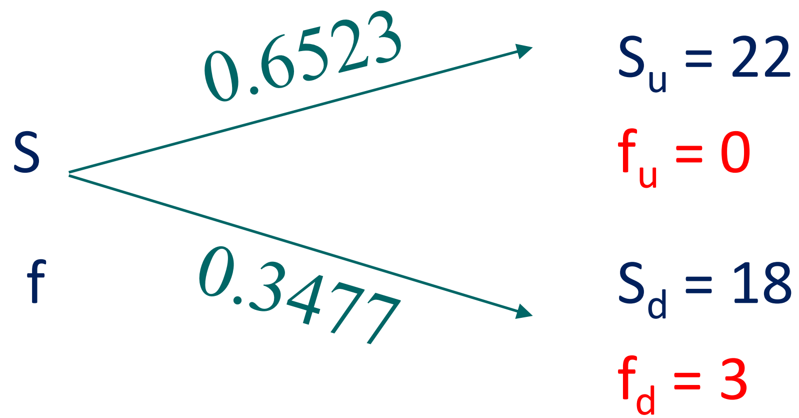
$u = \dfrac{S_u}{S}$

$d = \dfrac{S_d}{S}$

- We use the formula to compute p

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.12 \times 0.25} - 0.9}{1.1 - 0.9} = 0.6523$$

VALUING THE PUT OPTION

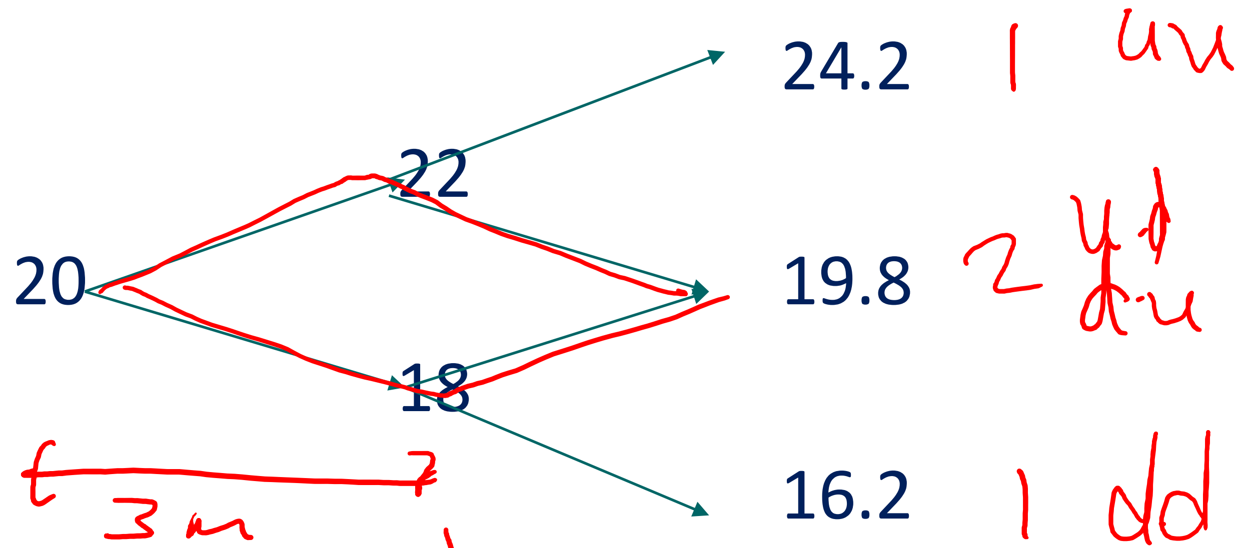


- The value of the put is $[pf_u + (1-p)f_d]e^{-rT}$

$[0.6523 \times 0 + 0.3477 \times 3]\ e^{-0.12 \times .25} = ?$

A Two-Step Binomial Model

A Two-Step Binomial Model

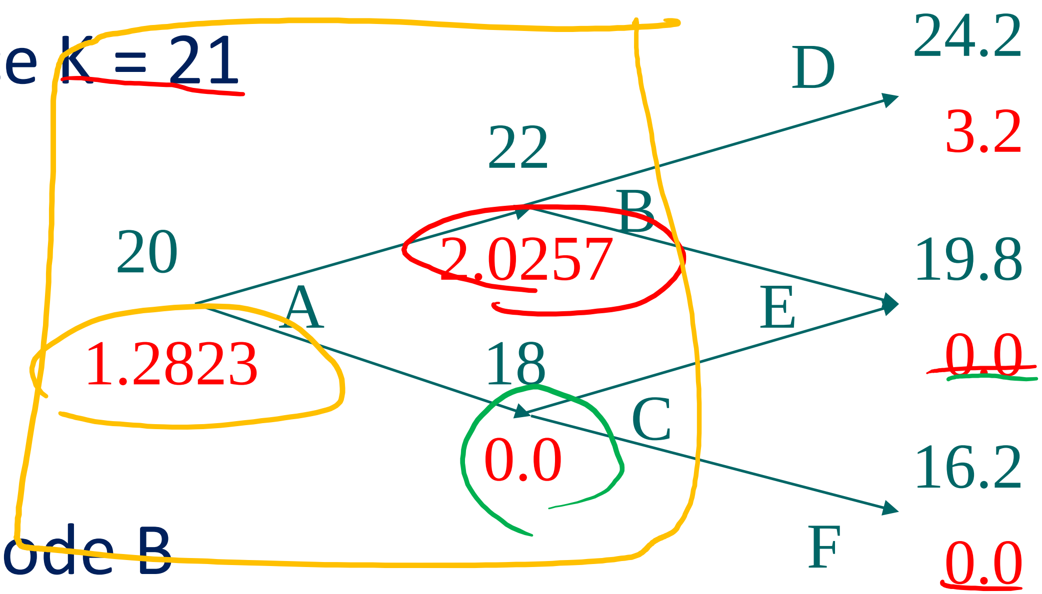


- Each time step is 3 months
- Risk-free rate r=0.12
- Risk neutral probability p=0.6523

Valuing a Call Option

$\max(S_T - K, 0)$

- Strike price K = 21



$\max(24.2 - 21, 0)$

- Value at node B

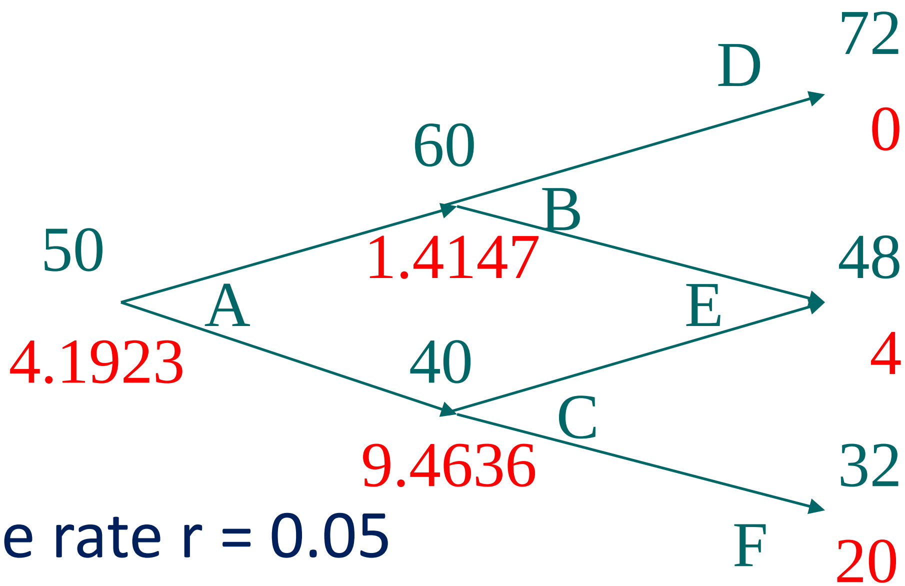
  $= e^{-0.12 \times 0.25}(0.6523 \times 3.2 + 0.3477 \times 0) = 2.0257$

- Value at node A

  $= e^{-0.12 \times 0.25}(0.6523 \times 2.0257 + 0.3477 \times 0)= 1.2823$

Valuing a Put Option: Exercise

- The strike price K = 52



- Risk-free rate r = 0.05
- Each time step is one year, T = 1 year
- Compute u, d, and risk-neutral probability p

Valuing a Put Option: Exercise

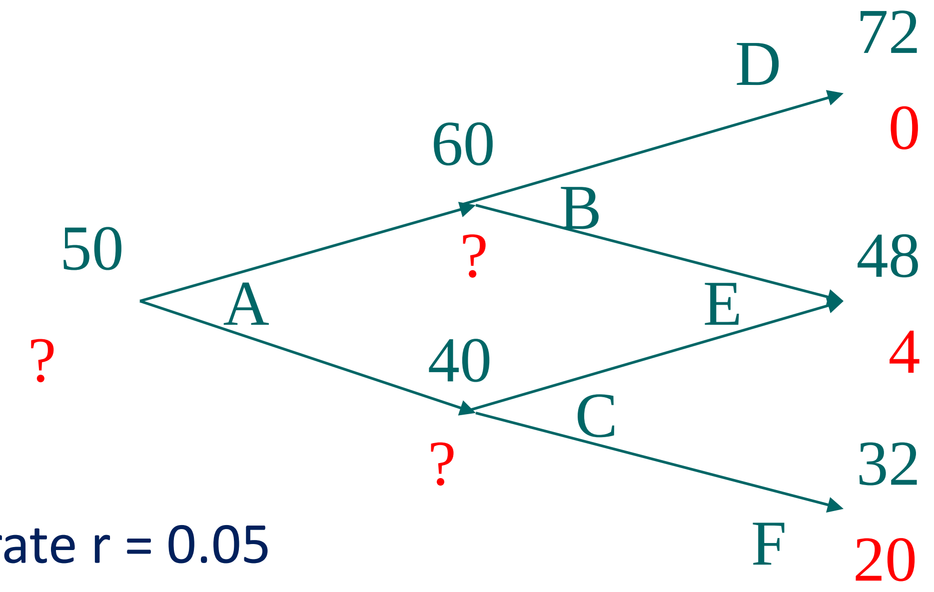
- $u = 60/50 = 1.2$, $d = 40/50 = 0.8$
- $p = [e^{(0.05*1)} - 0.8]/(1.2-0.8) = 0.6282$
- B: option value $= [p*0+(1-p)*4] * e^{(-0.05*1)} = 1.4147$
- C: option value $= [p*4+(1-p)*20]*e^{(-0.05*1)} = 9.4636$
- A: option value

  $= [p*1.4147 + (1-p)*9.4636]*e^{(-0.05*1)} = 4.19$

AMERICAN OPTIONS (EARLY EXERCISE)

What Happens When an Option is American? Value an American Put Option

- The strike price X = 52
-



- Risk-free rate r = 0.05
- Each time step is one year, T = 1 year
- Need to decide whether we exercise the option early

What Happens When an Option is American

- At node B
  - If exercise put option, value=0
  - If not exercise, value=1.4147
    - $1.4147 is the price of an European option that has the same terms
  - Thus, do not exercise

What Happens When an Option is American

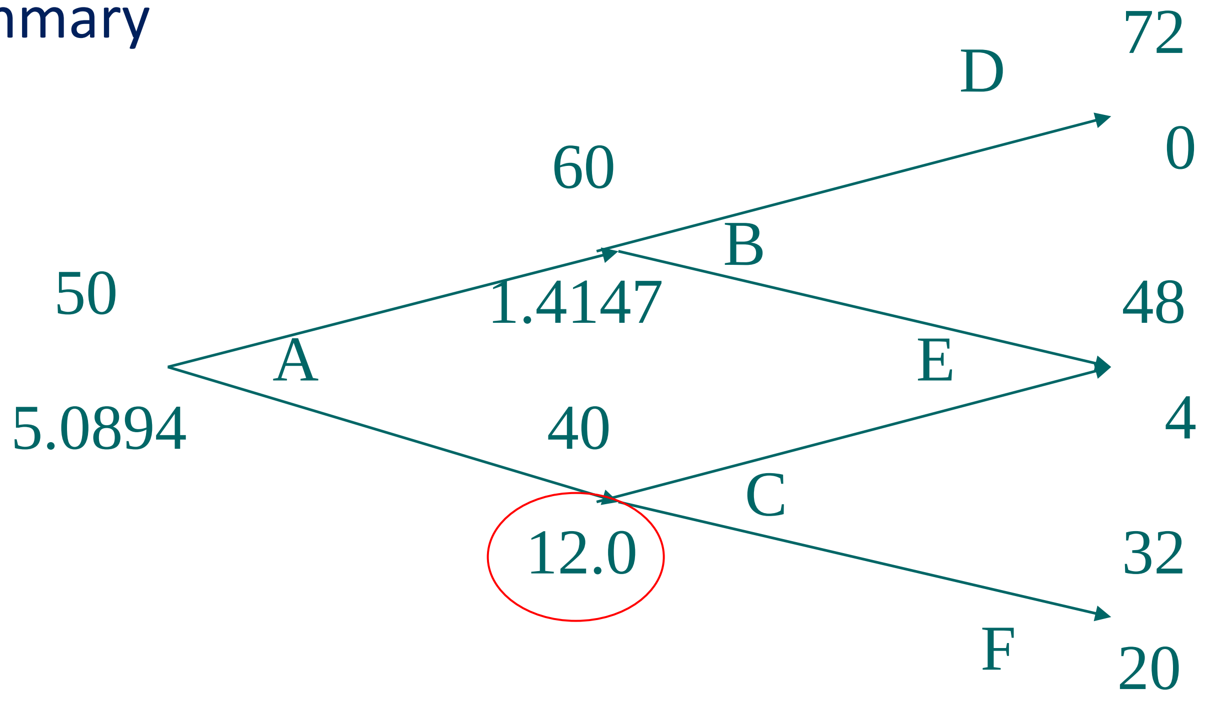
- At node C:
  - If exercise put option, value=52-40=12
  - If not exercise, value=9.4636
    - The price of an European option that has the same terms
  - Thus, ***exercise*** the put option

What Happens When an Option is American

- At node A:
  - If exercise put option, value=52-50=2
  - If not exercise, value=5.0894
    - The price of an European option that has the same terms
  - Thus, do not exercise

WHAT HAPPENS WHEN AN OPTION IS AMERICAN

- Summary



DELTA

- Delta (Δ) is the ratio of the change in the price of a stock option to the change in the price of the underlying stock
- The value of Δ varies from node to node
- Delta—hedge ratio

Binomial Trees in Practice

- We design the tree to represent the behavior of a stock price in a risk-neutral world
- We choose the tree parameters p , u , and d so that the tree gives correct values for the mean and standard deviation of the stock price changes during Δt

CHOOSING *U* AND *D*

- One way of matching the volatility is to set

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = 1/u = e^{-\sigma\sqrt{\Delta t}}$$

  where σ is the volatility and Δt is the length of the time step.

- This is the approach used by Cox, Ross, and Rubinstein

Tree Parameters for a Non-dividend Paying Stock

When Δt is small, a solution is

$$u = e^{\sigma\sqrt{\Delta t}}$$

(up movement)

$$d = e^{-\sigma\sqrt{\Delta t}}$$

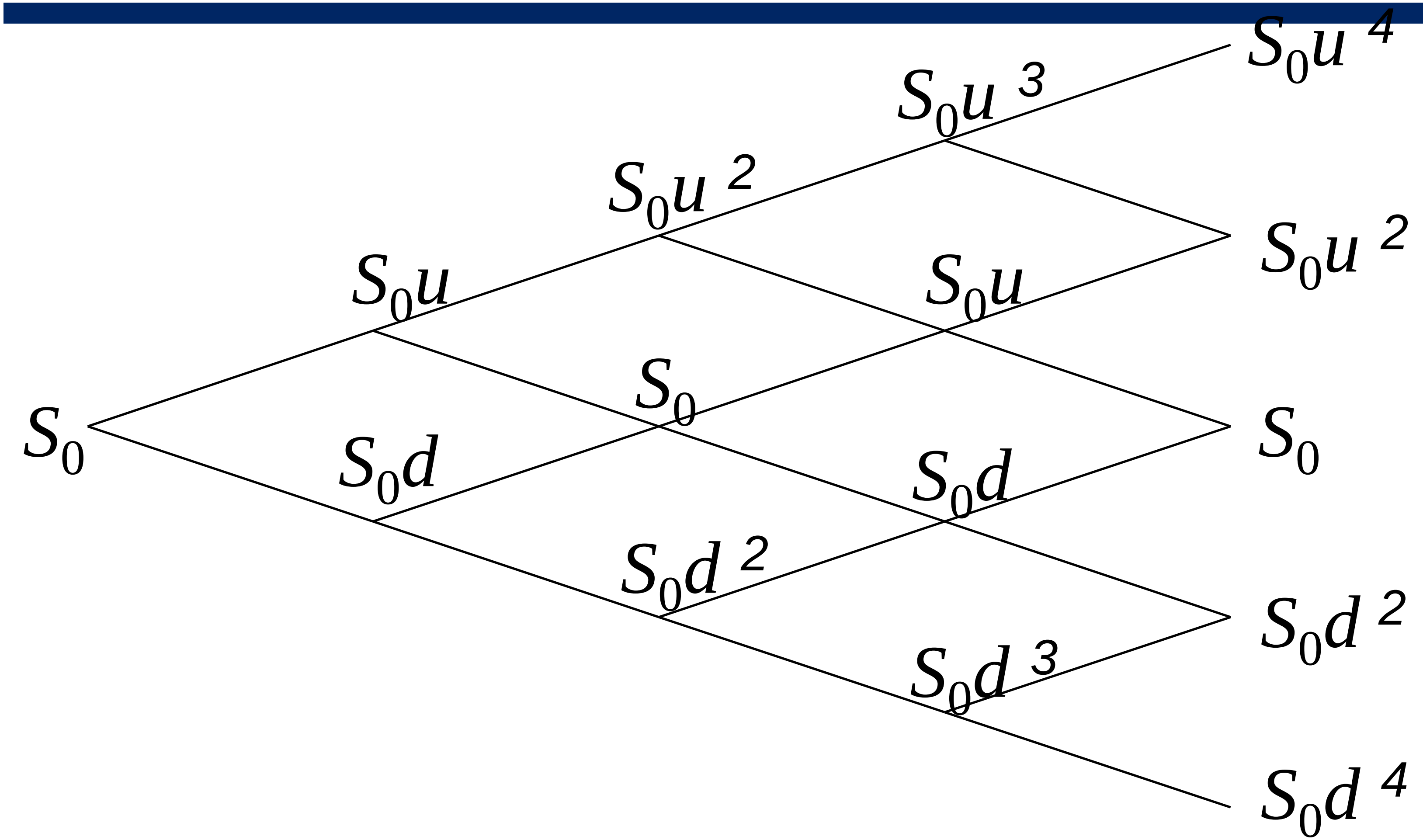
(down movement)

$$p = \frac{a-d}{u-d}$$

$$a = e^{r\,\Delta t}$$

(growth factor)

The Complete Tree



Backwards Induction

- We know the value of an option at the final nodes
  - Using the stock price and the strike price
- We work back through the tree using risk-neutral valuation to calculate the value of the option at each node, testing for early exercise when appropriate

Example: Put Option

$S_0 = 50$; $K = 50$; $r = 10\%$; $\sigma = 40\%$;

$T = 5$ months $= 0.4167$;

$\Delta t = 1$ month $= 0.0833$

The parameters imply

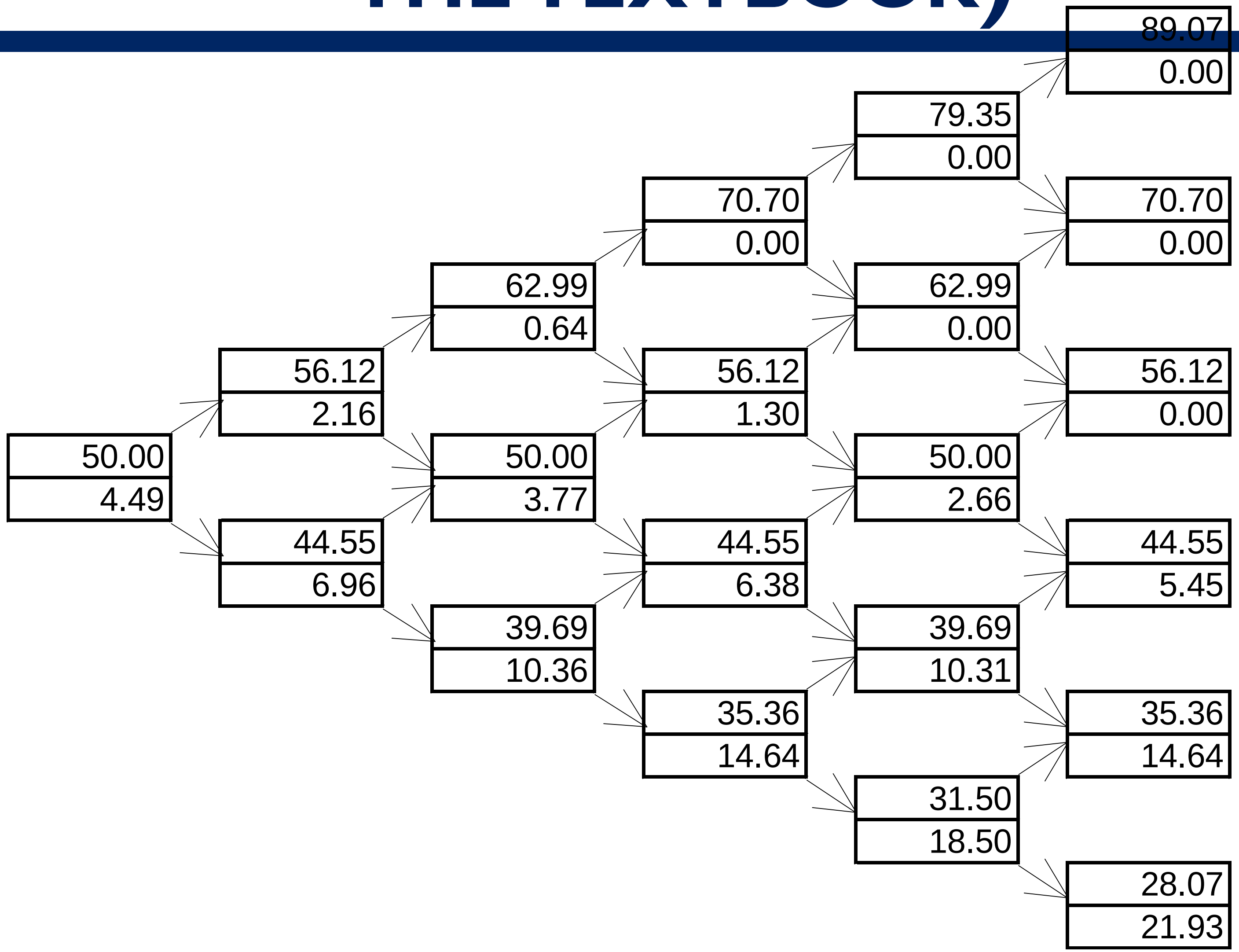
$u = e^{\sigma\sqrt{\Delta t}}$

$d = 1/u = e^{-\sigma\sqrt{\Delta t}}$

$u = 1.1224$; $d = 0.8909$;

$a = 1.0084$; $p = 0.5073$

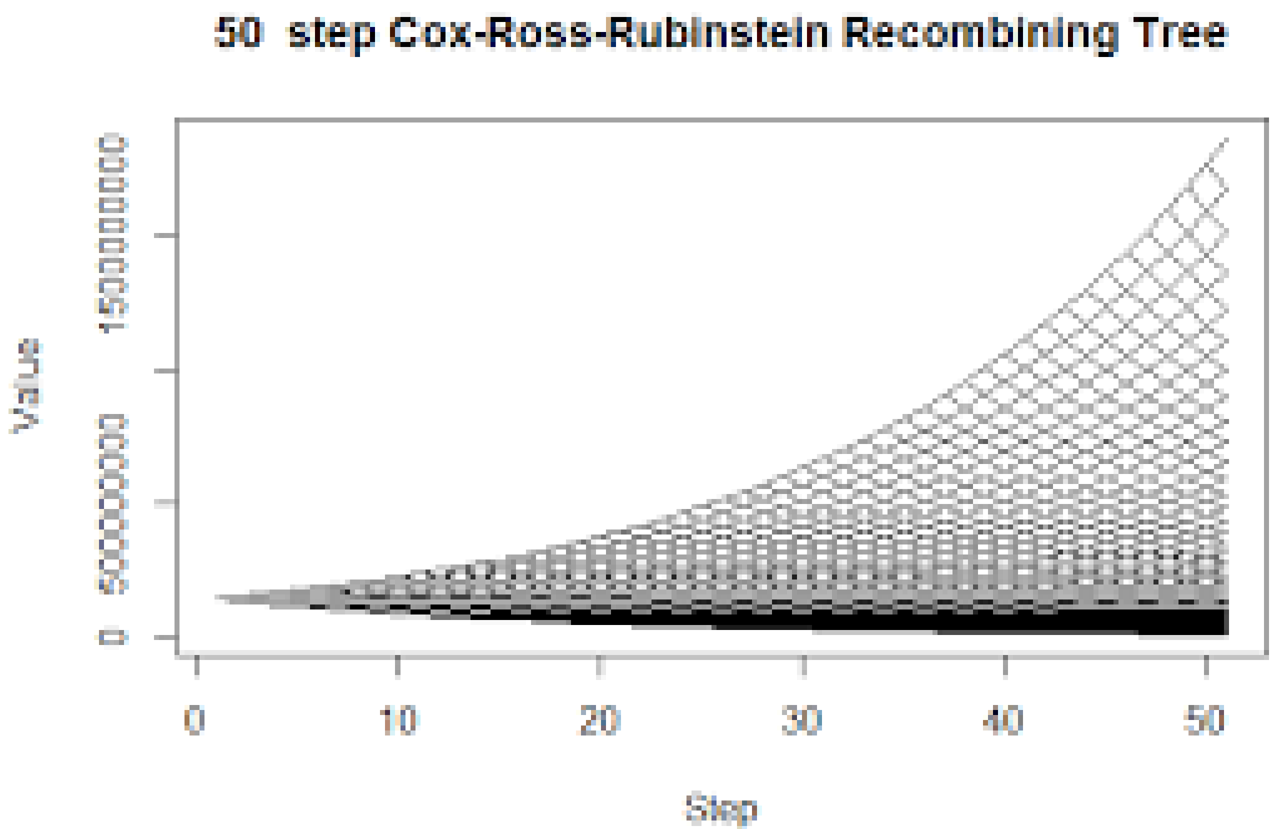
EXAMPLE (USE DERIVAGEM FROM THE TEXTBOOK)



Increasing the Number of Steps

- The Binomial-Tree price of an European option converges to its Black-Scholes price
- In practice at least 30 time steps are necessary to give good option values
- Excel example

Binomial Trees In Practice…



Effect of Dividends on Tree

- When dealing with dollar dividends, the tree does not recombine after a dividend payout.
- Alternatively, considering dividend yields can be more easily accommodated.

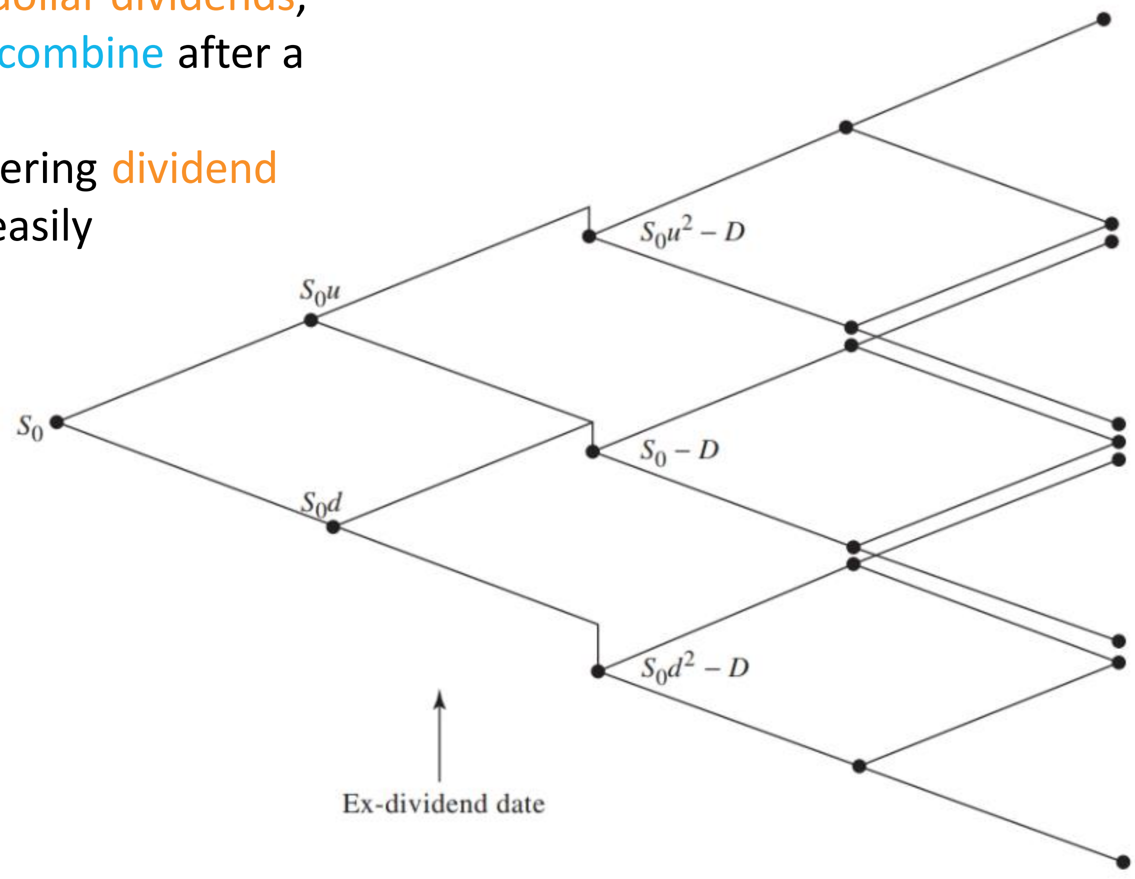


Figure 18.6 Tree when dollar amount of dividend is assumed known and volatility is assumed constant

Options on Other Assets

- When a stock price pays continuous dividends at rate q we construct the tree in the same way but set $a = e^{(r-q)\Delta t}$
  - For options on stock indices, q equals the dividend yield on the index
  - For options on a foreign currency, q equals the foreign risk-free rate
  - For options on futures contracts $q = r$

OPTIONS ON OTHER ASSETS

- The probability of an up move

$$p = \frac{a - d}{u - d}$$

$a = e^{r\Delta t}$ for a non-dividend paying stock

$a = e^{(r-q)\Delta t}$ for a stock index where q is the dividend yield on the index

$a = e^{(r-r_f)\Delta t}$ for a currency where r_f is the foreign risk-free rate

$a = 1$ for a futures contract

Summary

- **Binomial model can be used to price European and American options**
- Create a tree considering various up/down moves
- *Early exercise* can be accommodated
- **No arbitrage relations can be used to derive option price**
- Use hedge ratio Delta to create a riskless portfolio
- **Risk neutral valuation is a powerful method to find derivative prices.**
- Compute risk-neutral probabilities
- Discount payoffs at risk-free rate

Finance 2325

Homework 17

Chapter 12. An introduction to binomial trees

Questions 1, 2, 3, 4, 9, 10, 11

Homework 18

Chapter 12. An introduction to binomial trees

Questions 5, 6, 7, 8, 16

Chapter 18. Binomial trees in practice

Questions 8, 9