CHAPTER 18 & 22

MONTE CARLO SIMULATION & EXOTIC OPTIONS

FIN2325 with Dr. Velthuis

Today

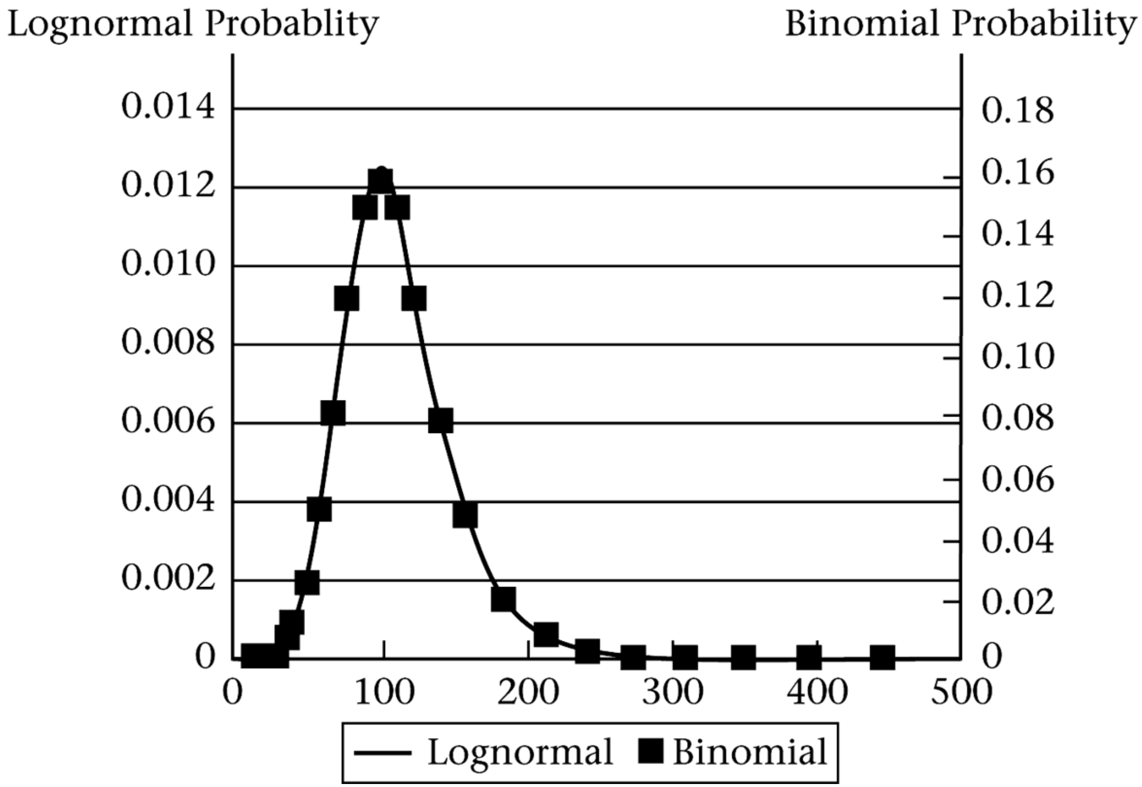
- Binomial distribution of stock prices
  - Rolling one die
  - Rolling two dice
  - Rolling three dice
  - Monte Carlo simulation
- Lognormal distribution of stock prices
  - Black-Scholes model
  - Monte Carlo simulation

Stock Return Distribution

- The binomial model assumes that continuously compounded returns follow a random walk
- Black-Scholes assume that the stock price follows a log-normal distribution
  - Stock prices are positive
  - The distribution is skewed to the right
  - Continuously compounded returns on the stock are normally distributed
- The binomial tree approximates the log-normal distribution

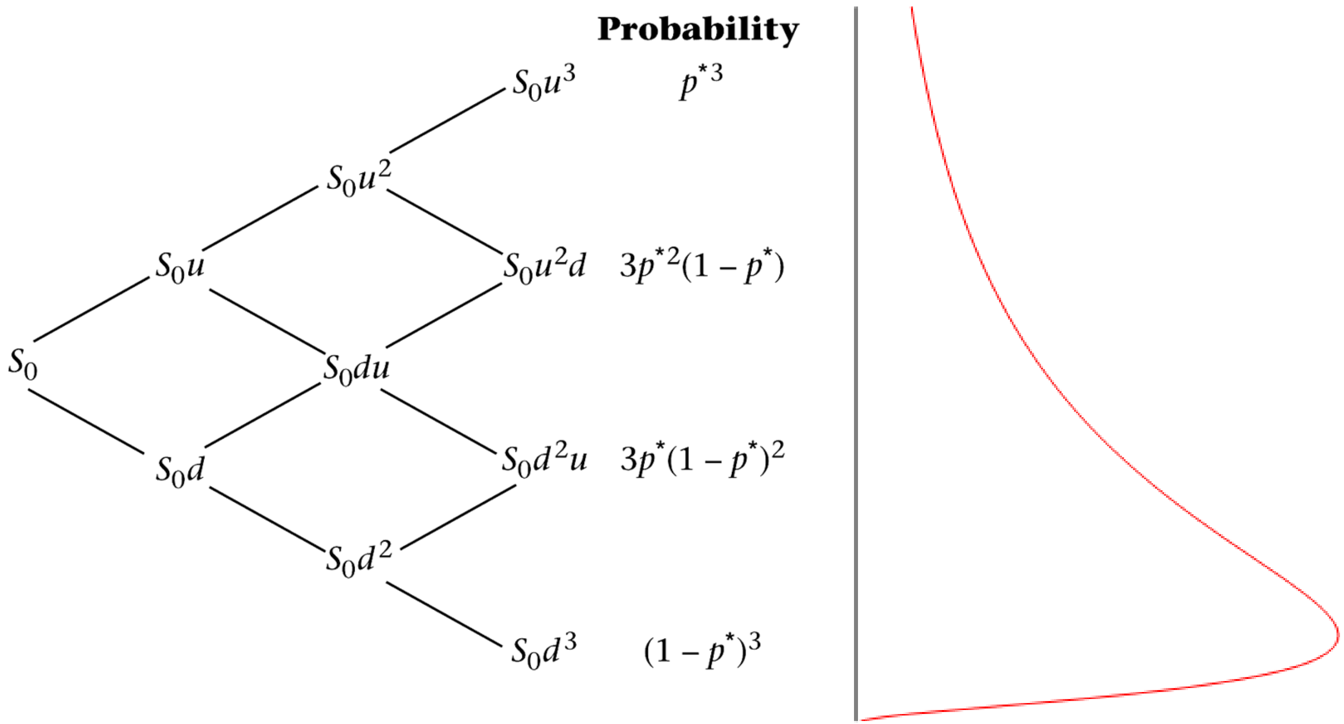
Stock Return Distribution

- A tree with 25 time steps is closely related to the log-normal distribution

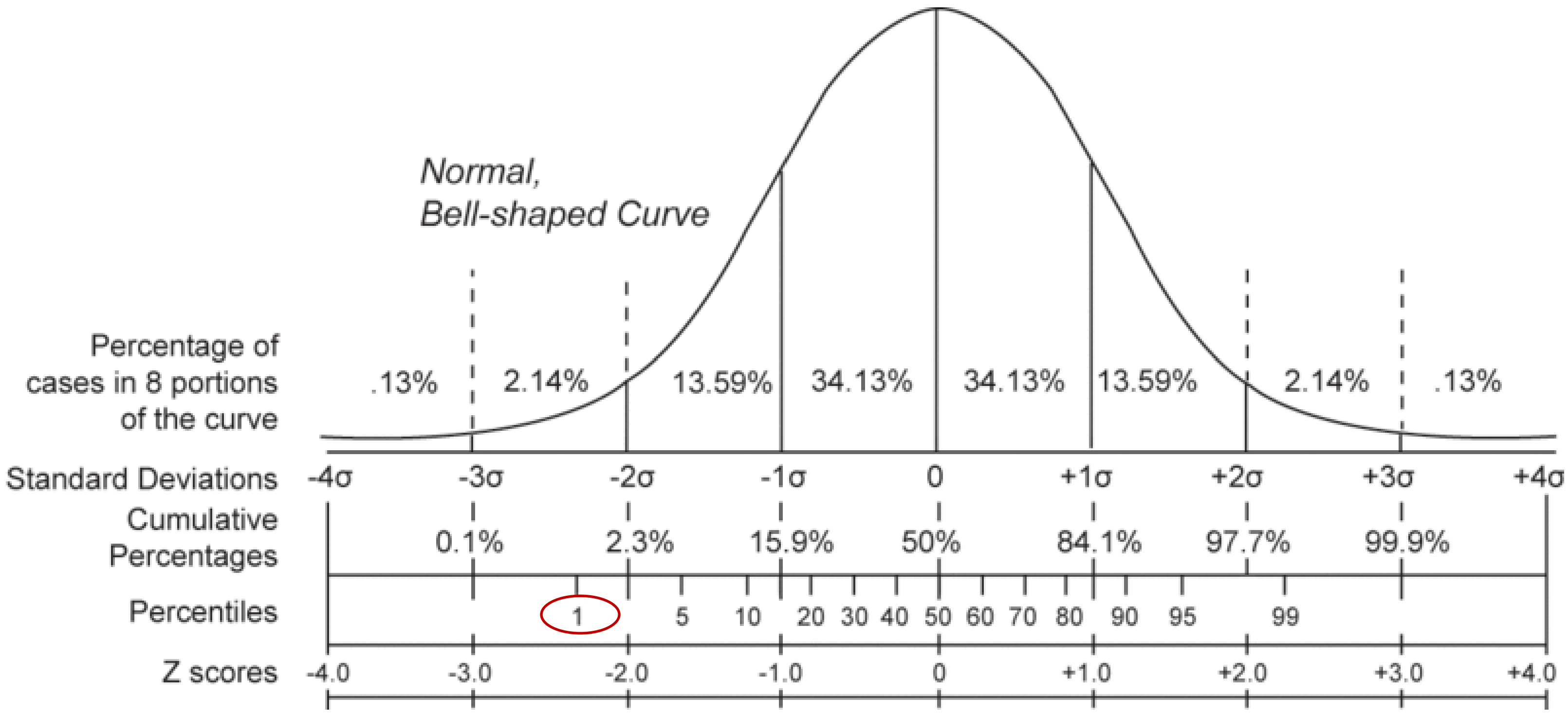


Stock Return Distribution

- The binomial tree assigns probabilities to different possible stock prices at maturity of the option



STANDARD NORMAL DISTRIBUTION



- The likelihood of drawing a number smaller than -2.33 standard deviations from the mean is 1%

Valuing Simple Options I

- We define a simple European call on the outcome of a roll of a die at date T (S_T)
- Suppose r=0.10, T=1 year, K = 3.5
- The outcomes are as follows:

Probability	p	0.166667	0.166667	0.166667	0.166667	0.166667	0.166667
Outcome roll die	ST	1	2	3	4	5	6
Option value	Max(0,S_T-K)	0	0	0	0.5	1.5	2.5
Expected Payoff:		0.75					
PV(Exp. Payoff)		0.68					

VALUING SIMPLE OPTIONS I

- We define a simple European call on the outcome of a roll of a die at date T (S_T)
- Suppose r=0.10, T=1 year, K = 3.5
- The value of the call today is:

$$c = e^{-0.1*1}\left[\frac{1}{6}\max(0,6-3.5)+\frac{1}{6}\max(0,5-3.5) + \frac{1}{6}\max(0,4-3.5)+\frac{1}{6}\max(0,3-3.5) + \frac{1}{6}\max(0,2-3.5)+\frac{1}{6}\max(0,1-3.5)\right]$$

$$= \frac{1}{6} * [(6-3.5)+(5-3.5)+(4-3.5)]\mathrm{e}^{-0.1*1} = 0.68$$

Valuing Simple Options I

- Rewriting gives:

$$c = \frac{1}{6}[6+5+4]\mathrm{e}^{-0.1} - (3.5)\left[\frac{1}{6}+\frac{1}{6}+\frac{1}{6}\right]\mathrm{e}^{-0.1}$$

$$= E(S_T)\mathrm{e}^{-0.1} \times \frac{1}{6}[6+5+4]\frac{1}{E(S_T)} - (3.5)\mathrm{e}^{-0.1} \times \frac{3}{6}$$

$$= S \cdot (Pr_1) - Ke^{-rT}(Pr_2)$$

  - Since S $= E(S_T)\mathrm{e}^{-0.1}$ from risk-neutral valuation
  - $Pr_1 = \frac{1}{6}\frac{[6+5+4]}{E(S_T)} = \frac{1}{2}\frac{\frac{[6+5+4]}{3}}{E(S_T)}$
  - $Pr_2 = \frac{3}{6} = \frac{1}{2}$

- This is analogous to the Black-Scholes formula:

$$c = SN(d_1) - Ke^{-rT}N(d_2)$$

VALUING SIMPLE OPTIONS I

- Interpretations:
  - $S \cdot (Pr_1)$: expected value of the payoff from exercising the call when it is in-the-money
    - Pr_1: "adjusted probability": the probability of the call being in the money, adjusting for how much in the money it is, in relative terms to $E(S_T)$
  - $Ke^{-rT}(Pr_2)$: present value of the expected cost of exercising the call when it is in-the-money
    - Pr_2: the probability of the call being in the money

Valuing Options Through Monte Carlo Simulation

- The binomial distribution is an approximation to the lognormal distribution

- The Black-Scholes Model assumes that stock prices follow a log-normal distribution and provides a closed-form solution for the option price that can be evaluated directly

- Monte Carlo simulation results approach Black-Scholes option prices in the limit

Monte Carlo Simulation

- The basic concept is that games of chance can be played to approximate solutions to real world problems

Advantages of Monte Carlo

- Verify results of Binomial and Black-Scholes Models

- Can be used to value exotic options with "path-dependent" payoffs

Monte Carlo Simulations

1. Create a model of how underlying asset prices change
   - We can let the computer roll the dice for us, or,
   - Generate stock prices based on a given distribution
2. Simulate many possible paths for the stock price
3. Compute the payoff of the option for each trial
   - The simulation results constitute the distribution of the option payoff
   - The mean payoff is used to derive the option price

I. SIMULATING STOCK PRICES

- Continuously compounded stock returns are assumed to be normally distributed and prices are log-normally distributed

- Adapting this assumption to simulate stock prices, we get

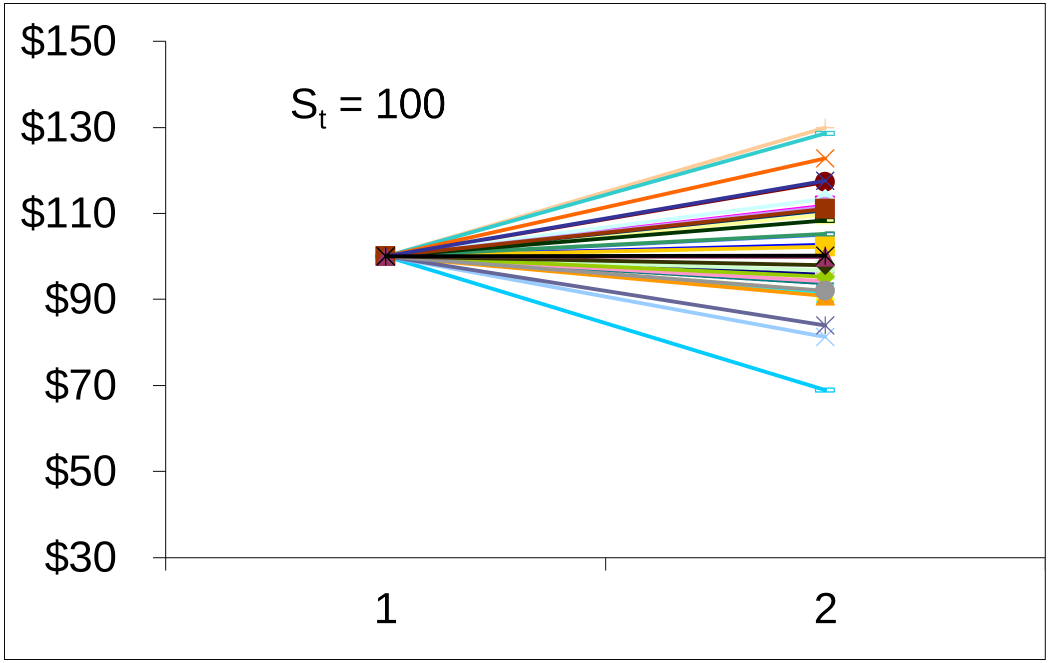
$$S_{t+\Delta t} = S_t e^{\left(\mu - 0.5\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}\cdot Z}$$

  where μ=mean return, σ=volatility, Z=random variable (standard normal)

- ***When simulating stock prices for option valuation, set μ=r, this is an example of "risk-neutral" pricing***

2. Simulate Many Price Paths

- To simulate multiple future price paths, draw a set of standard normal Z's and substitute the results into the stock price equation



3. Calculate PV of Average Payoffs Across Paths

- For each stock price path *i*, calculate the payoff of the option, $Payoff_i$
- The Monte Carlo price is

$$\boxed{f = e^{-rT} \cdot \frac{1}{n} \sum_{i=1}^{n} Payoff_i}$$

Examples of Monte Carlo Valuation – European Option

- Example: Value a 12-month European call where S=\$100, K=\$100, r=10%, and σ=20%

  - Generate random Z~N(0,1)

$$S_T = Se^{\left(0.10-0.5*0.20^2\right)*1+0.20\sqrt{1}\cdot Z}$$

  - For each stock price, compute

$$Payoff_i = \max(0, S_T - \$100)$$

10,000x

- Average over the payoffs and compute option price

$$c = e^{-0.1*1}E(Payoff) = \quad ?????$$

versus \$13.27 Black-Scholes price

Pricing Exotic Options Using Monte Carlo Simulation

- We can use Monte Carlo simulation to generate a stock price path by considering small steps through time Δt:

$$S_{t+\Delta t} = S_t e^{\left(r-0.5\sigma^2\right)\Delta t+\sigma\sqrt{\Delta t}\cdot Z}$$

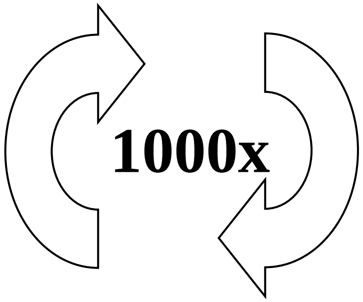
- Using this price path, a path dependent, exotic option can be valued

Exotic Options - Example

- Asian options: payoff based on average price over some period (not just terminal price, S_T)
  - e.g.
  - Call Payoff = Max(0, $S_{avg} - K$)
  - Put Payoff = Max(0, $K - S_{avg}$)

Examples of Monte Carlo Valuation –Asian Option

- Compute stock prices based on thirty random numbers Z(1), Z(2), …, Z(30).
  - $S_1 = S_0 e^{(r-0.5\sigma^2)1/365+\sigma\sqrt{\frac{1}{365}}\cdot Z(1)}$
  - $S_2 = S_1 e^{(r-0.5\sigma^2)1/365+\sigma\sqrt{\frac{1}{365}}\cdot Z(2)}$
  - $S_3 = S_2 e^{(r-0.5\sigma^2)1/365+\sigma\sqrt{\frac{1}{365}}\cdot Z(3)}$
  - …
  - $S_{30} = S_{29} e^{(r-0.5\sigma^2)1/365+\sigma\sqrt{\frac{1}{365}}\cdot Z(30)}$



  - The payoff of the call for each path *i:*

$$c_{asian}^{i} = e^{-r*30/365} * \max\left[\frac{S_1 + S_2 + \cdots + S_{30}}{30} - K, 0\right]$$

- The value of the call is

$$c_{asian} = \frac{1}{1000}\sum_{i=1}^{1000} c_{asian}^{i}$$

Exotic Options

- Traded in the over the counter market
- Less actively traded
- These options are more profitable for a derivatives dealer (e.g. investment bank)

Why Exotic Options

- Meet a specific hedging need
- Tax, accounting, legal, or regulatory reasons
- Represent a specific view on potential future movements in market variables
- Occasionally a derivatives dealer tries to benefit at the expense of unwary fund managers

BARRIER OPTIONS

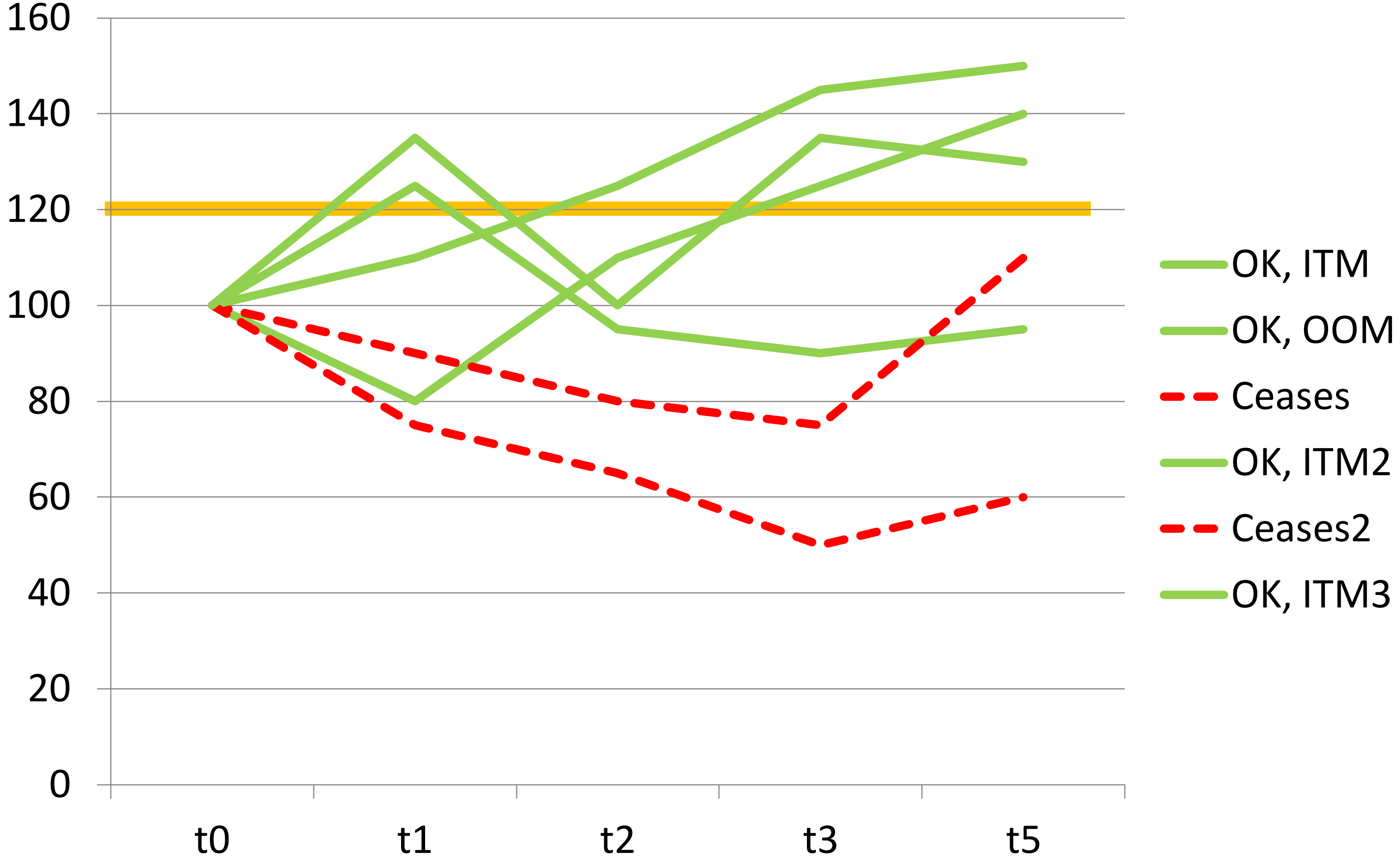
- Knock-in options: come into existence only if asset price hits barrier before option maturity
- Knock-out options: ceases to exist if asset price hits barrier before option maturity
- Less expensive than regular options

Barrier Options

- Up options: asset price must hit barrier from below (S < Barrier)
- Down options: asset price must hit barrier from above (S > Barrier)
- Option may be a put or a call
- Eight possible combinations

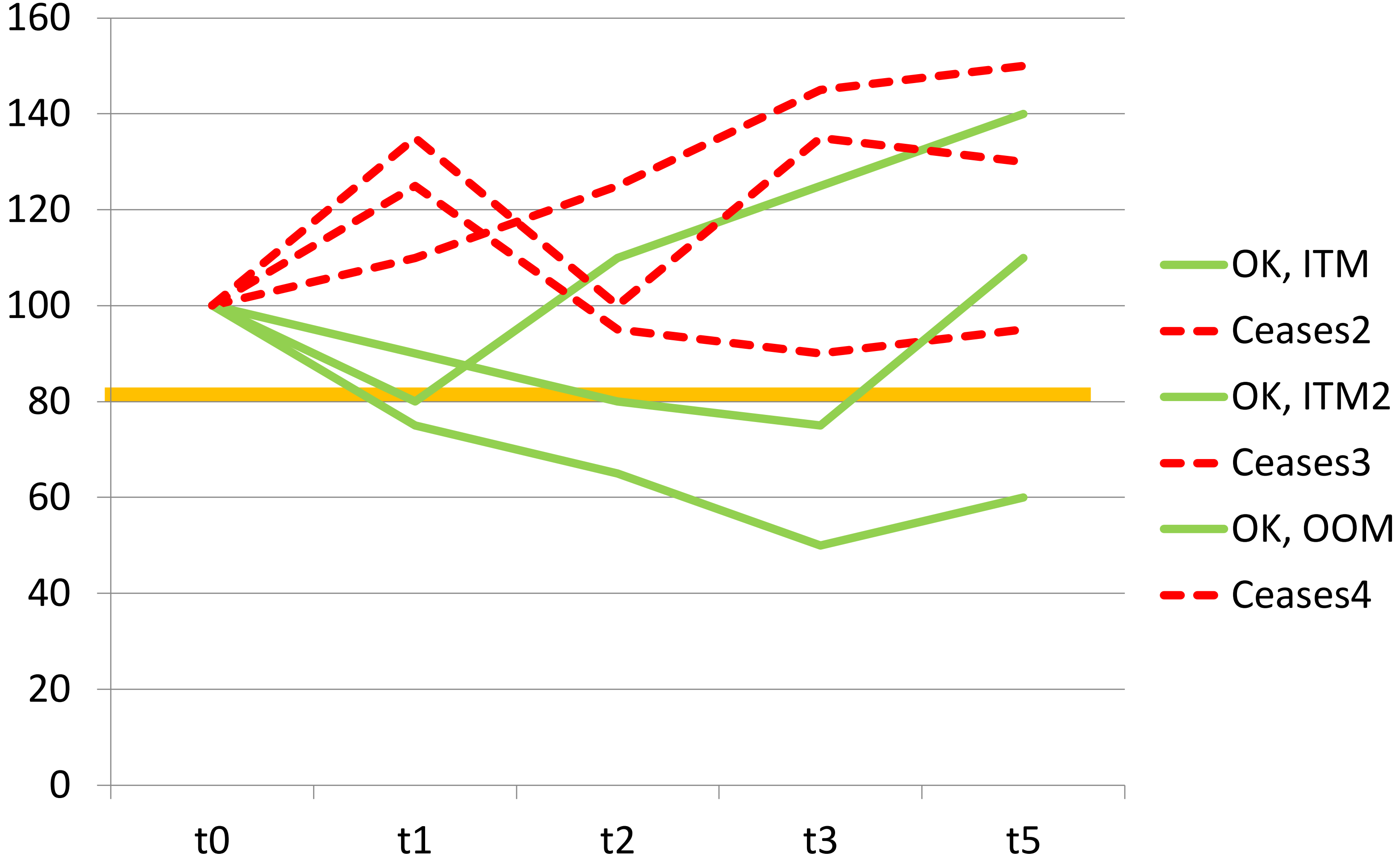
UP-AND-OUT CALL OPTION
BARRIER = 120, K=100

UP-AND-IN CALL OPTION
BARRIER = 120, K = 100



Down-and-out call option
barrier = 80, K=100

DOWN-AND-IN CALL OPTION
BARRIER = 80, K = 100



Parity Relations

$$c = c_{\mathrm{ui}} + c_{\mathrm{uo}}$$

$$c = c_{\mathrm{di}} + c_{\mathrm{do}}$$

$$p = p_{\mathrm{ui}} + p_{\mathrm{uo}}$$

$$p = p_{\mathrm{di}} + p_{\mathrm{do}}$$

Lookback Options

- Floating lookback call pays $S_T - S_{\min}$ at time T
  - Allows buyer to buy stock at lowest observed price in some interval of time
  - Payoff = max($S_T - S_{\min}$,0)
- Floating lookback put pays $S_{\max} - S_T$ at time T
  - Allows buyer to sell stock at highest observed price in some interval of time
  - Payoff = max($S_{\max} - S_T$,0)

Lookback Options

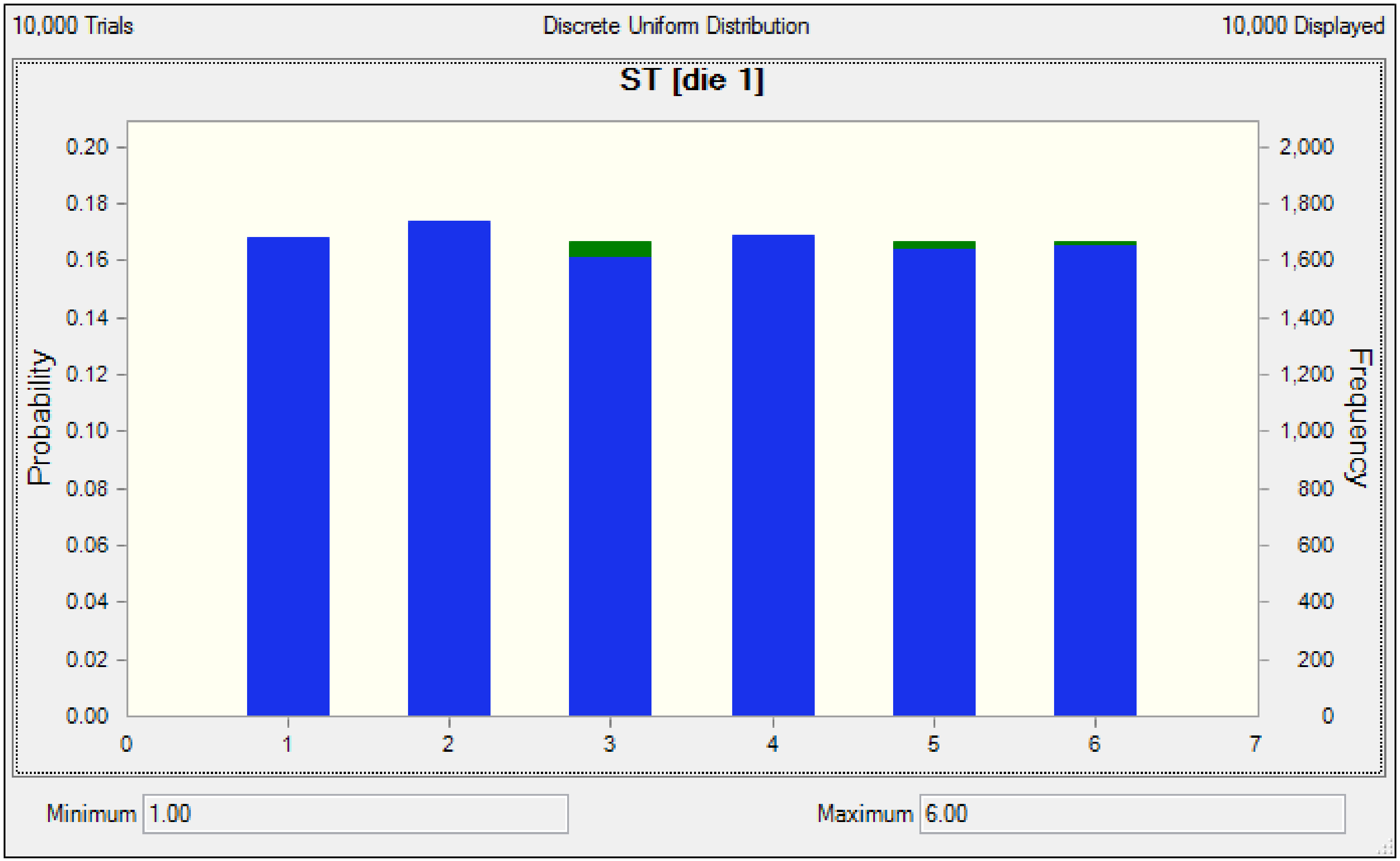
- Fixed lookback call pays off the maximum asset price minus a strike price
  - Payoff = $\max(S_{max} - K, 0)$
- Fixed lookback put pays off the strike price minus the minimum asset price
  - Payoff = $\max(K - S_{min}, 0)$

Asian Options

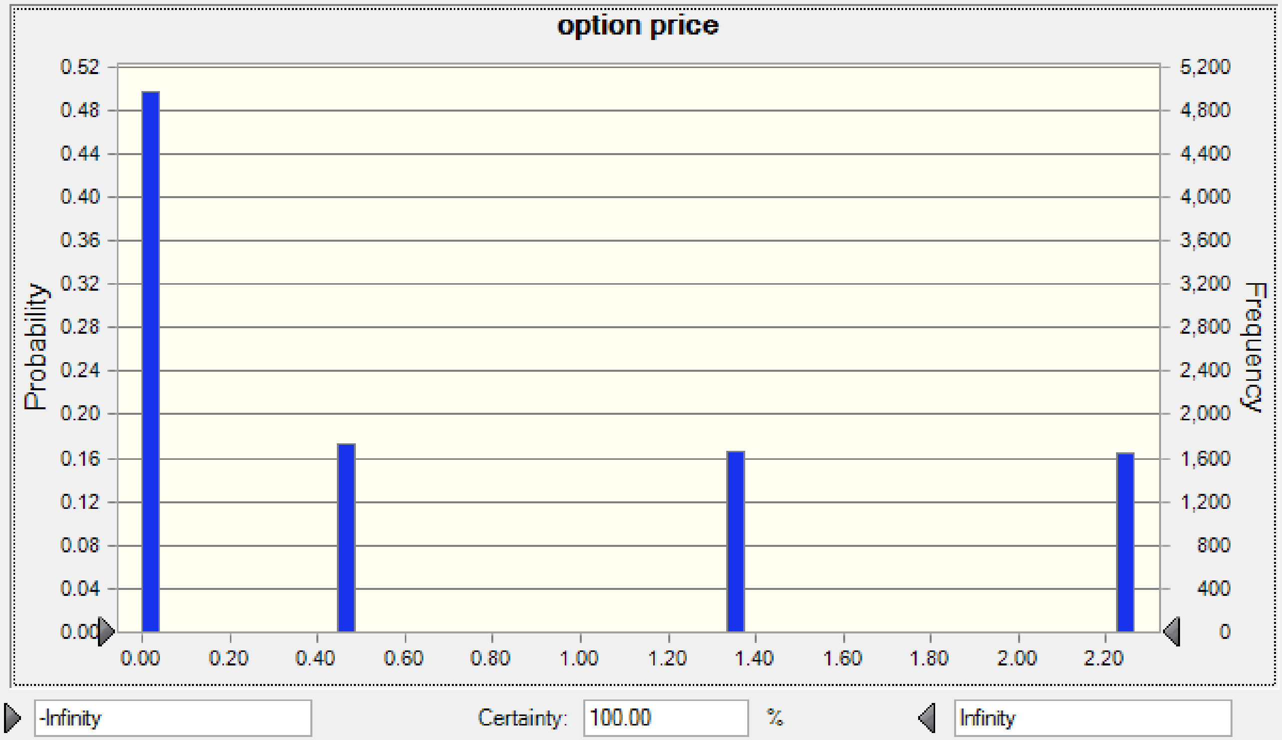
- Payoff related to average stock price
- Average Price options pay:
  - $\max(S_{\text{ave}} - \text{K}, 0)$ (call), or
  - $\max(K - S_{\text{ave}}, 0)$ (put)
    - Can guarantee that the average price received for an asset in frequent trading over a period of time is above some level
- Average Strike options pay:
  - $\max(S_T - S_{\text{ave}}, 0)$ (call), or
  - $\max(S_{\text{ave}} - S_T, 0)$ (put)
    - Can guarantee that the average price paid for an asset in frequent trading over a period of time is not greater than the final price

MONTE CARLO SIMULATION RESULTS

DISTRIBUTION OF THE OUTCOME OF A ROLL OF A DIE



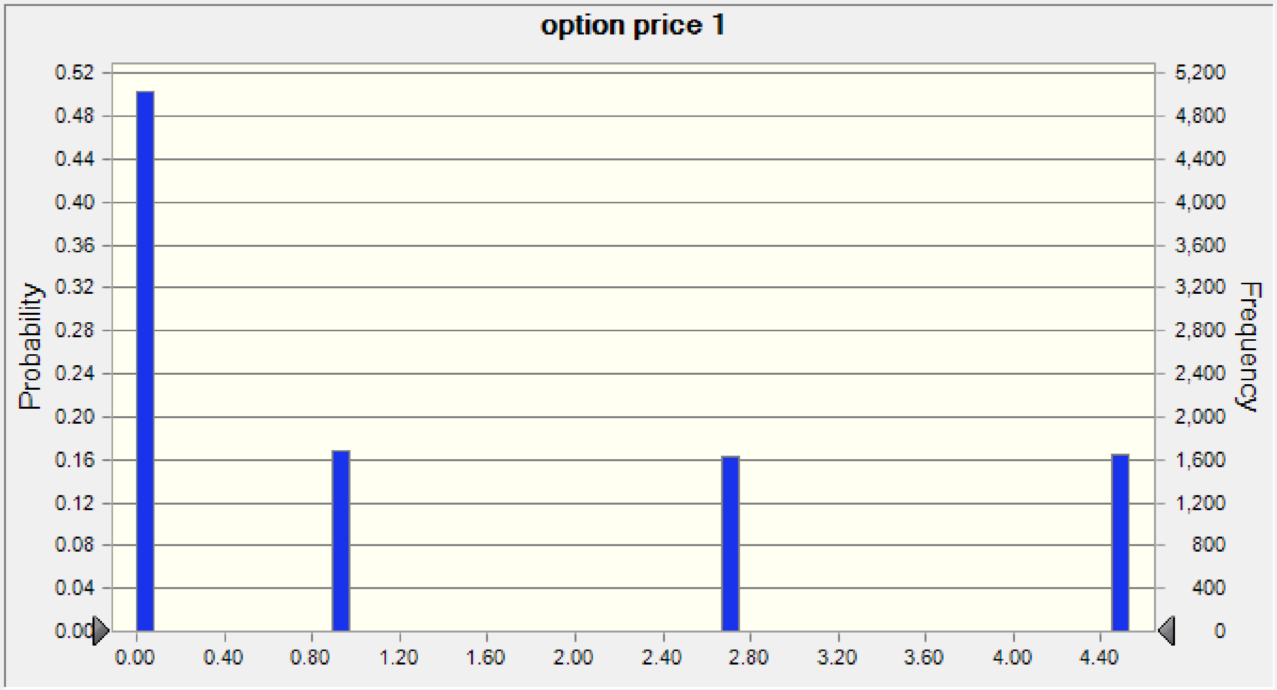
EUROPEAN CALL BASED ON THE OUTCOME OF A ROLL OF A DIE AT DATE T



Statistic	Forecast values
Trials	10,000
Base Case	0.00
Mean	0.68
Median	0.45
Mode	0.00
Standard Deviation	0.85
Variance	0.73
Skewness	0.8924
Kurtosis	2.23
Coeff. of Variation	1.26
Minimum	0.00
Maximum	2.26
Mean Std. Error	0.01

EUROPEAN CALL BASED ON 2 TIMES OUTCOME OF A ROLL OF A DIE

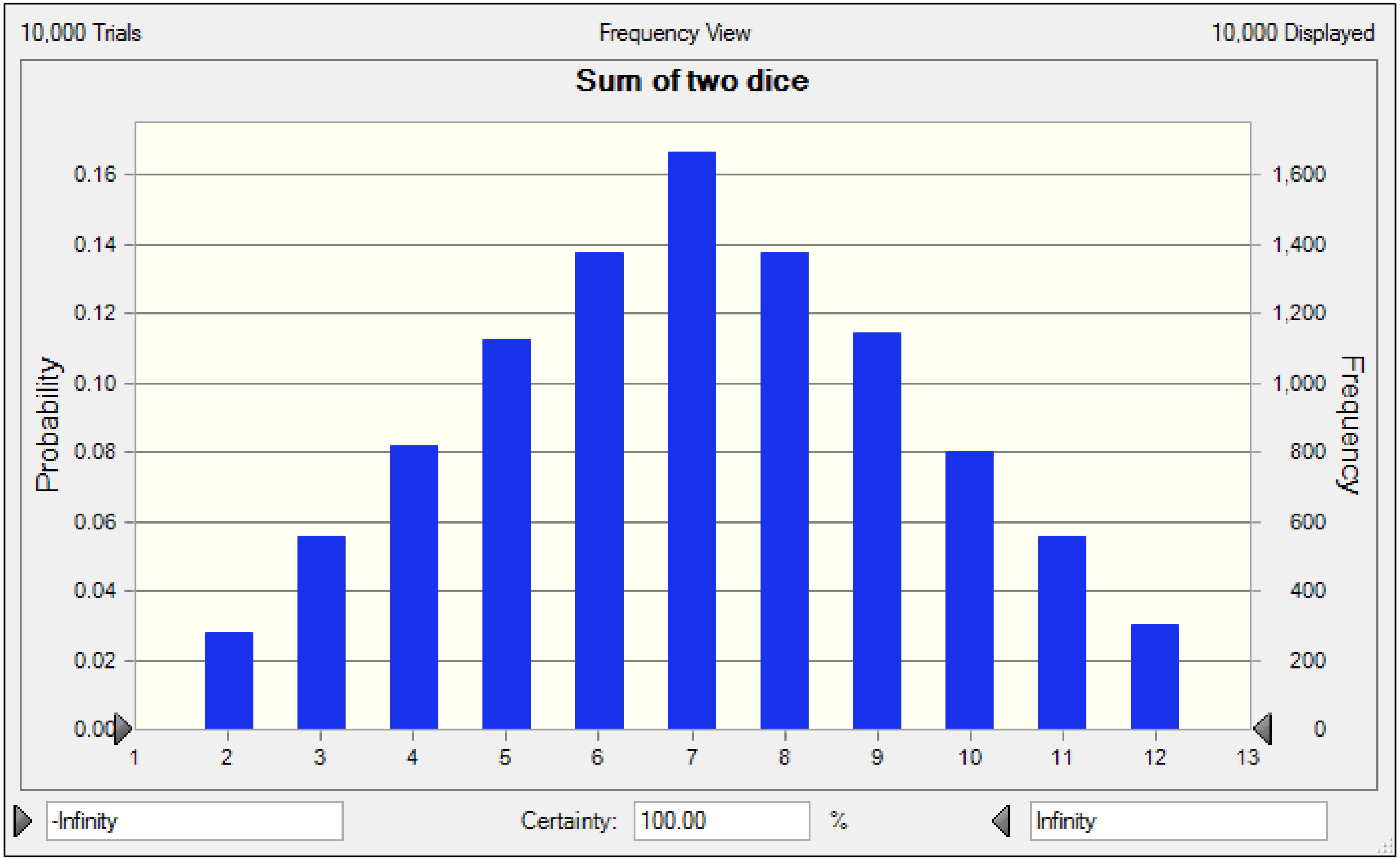
10,000 Trials Split View 10,000 Displayed



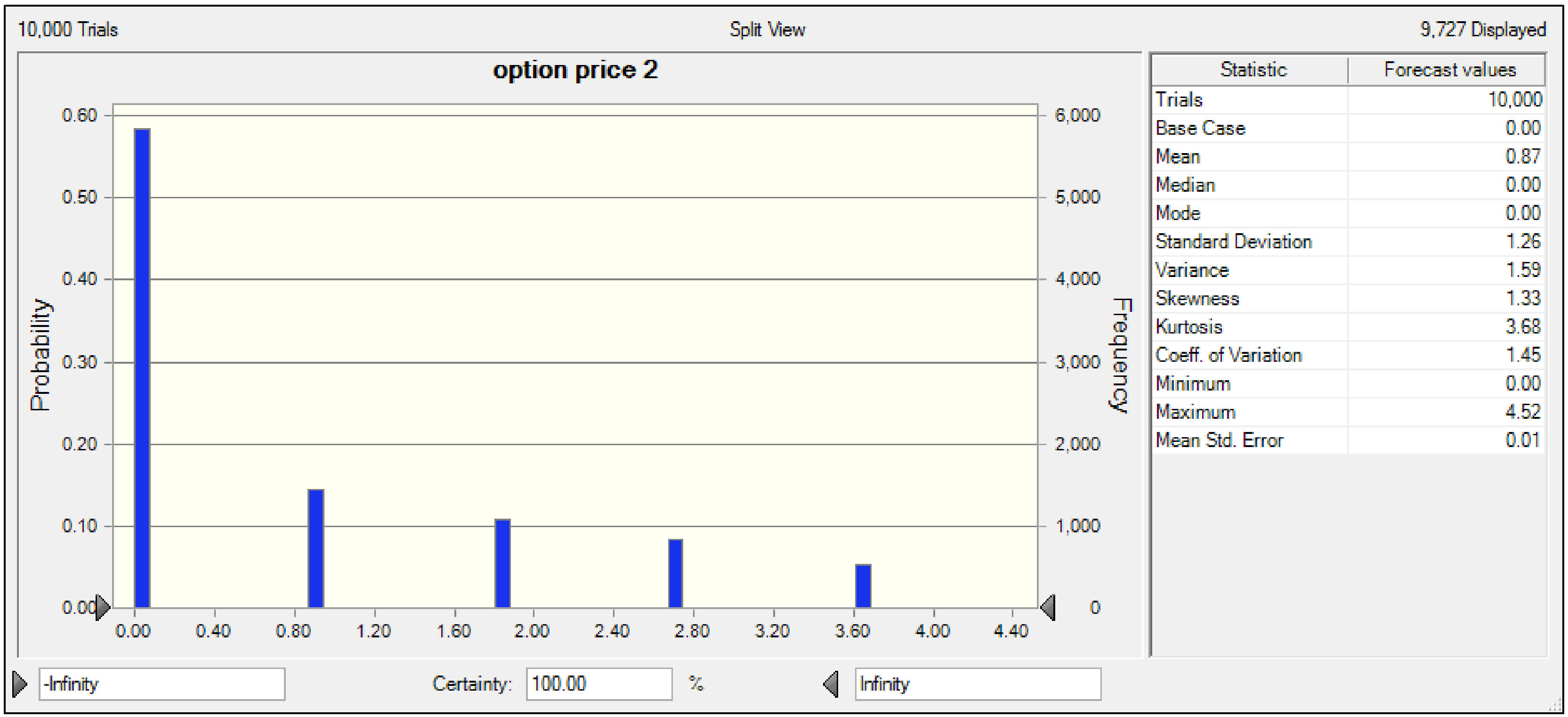
Statistic	Forecast values
Trials	10,000
Base Case	0.00
Mean	1.35
Median	0.00
Mode	0.00
Standard Deviation	1.71
Variance	2.92
Skewness	0.9013
Kurtosis	2.24
Coeff. of Variation	1.27
Minimum	0.00
Maximum	4.52
Mean Std. Error	0.02

-Infinity Certainty: 100.00 % Infinity

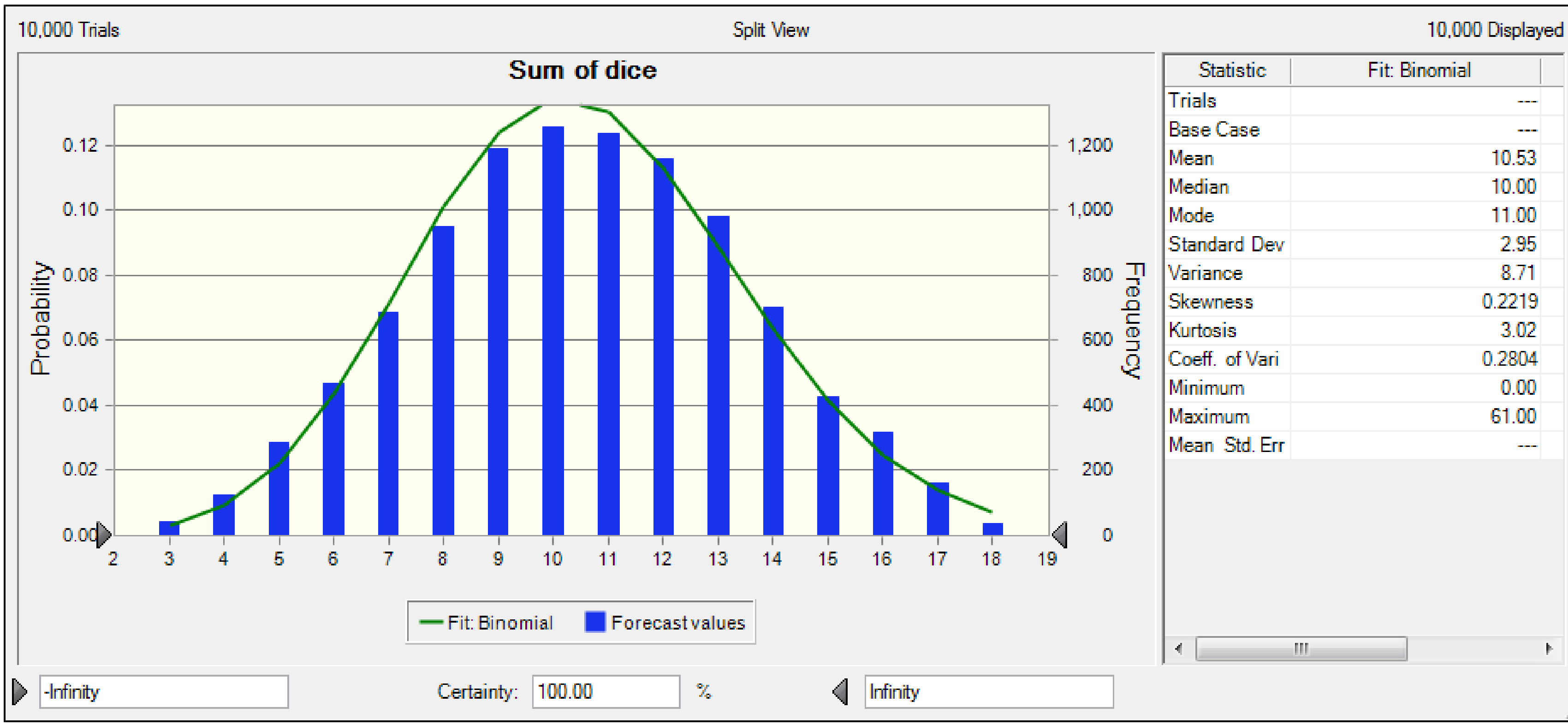
DISTRIBUTION OF THE SUM OF THE OUTCOMES OF THE ROLL OF TWO DICE



EUROPEAN CALL BASED ON THE SUM OF THE OUTCOMES OF THE ROLL OF TWO DICE AT DATE T

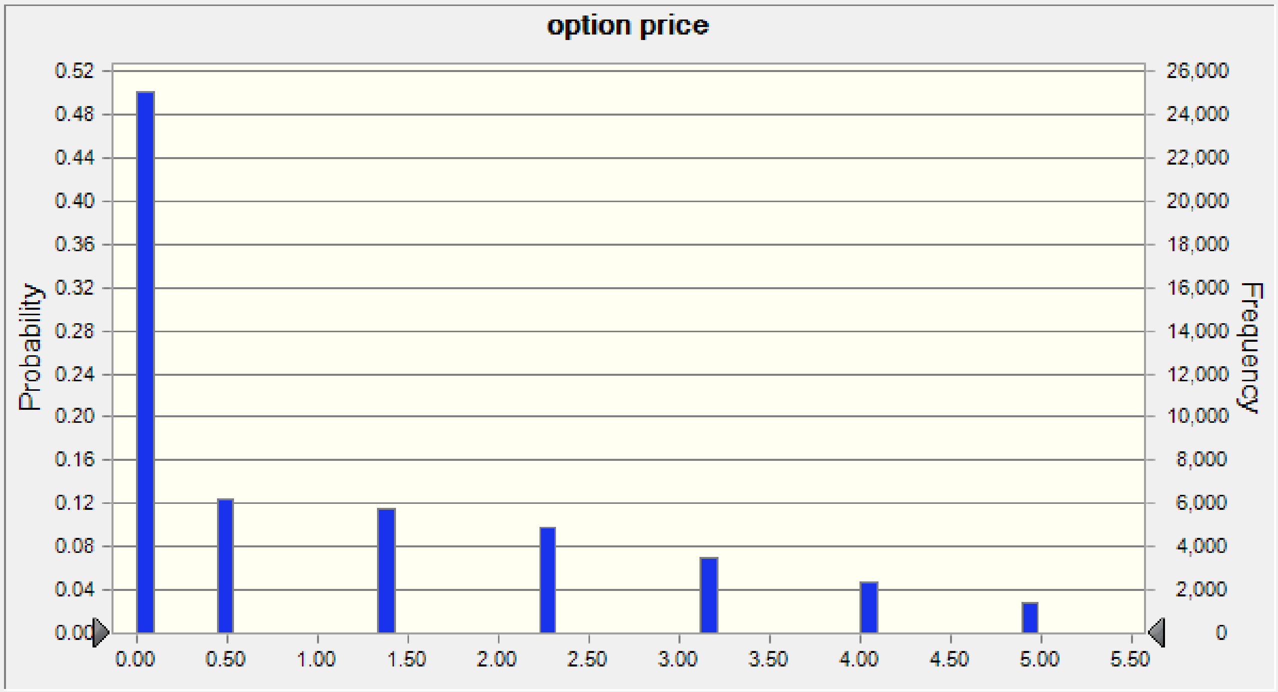


DISTRIBUTION OF THE SUM OF THE OUTCOMES OF THE ROLL OF THREE DICE



EUROPEAN CALL BASED ON THE SUM OF THE OUTCOMES OF THE ROLL OF THREE DICE

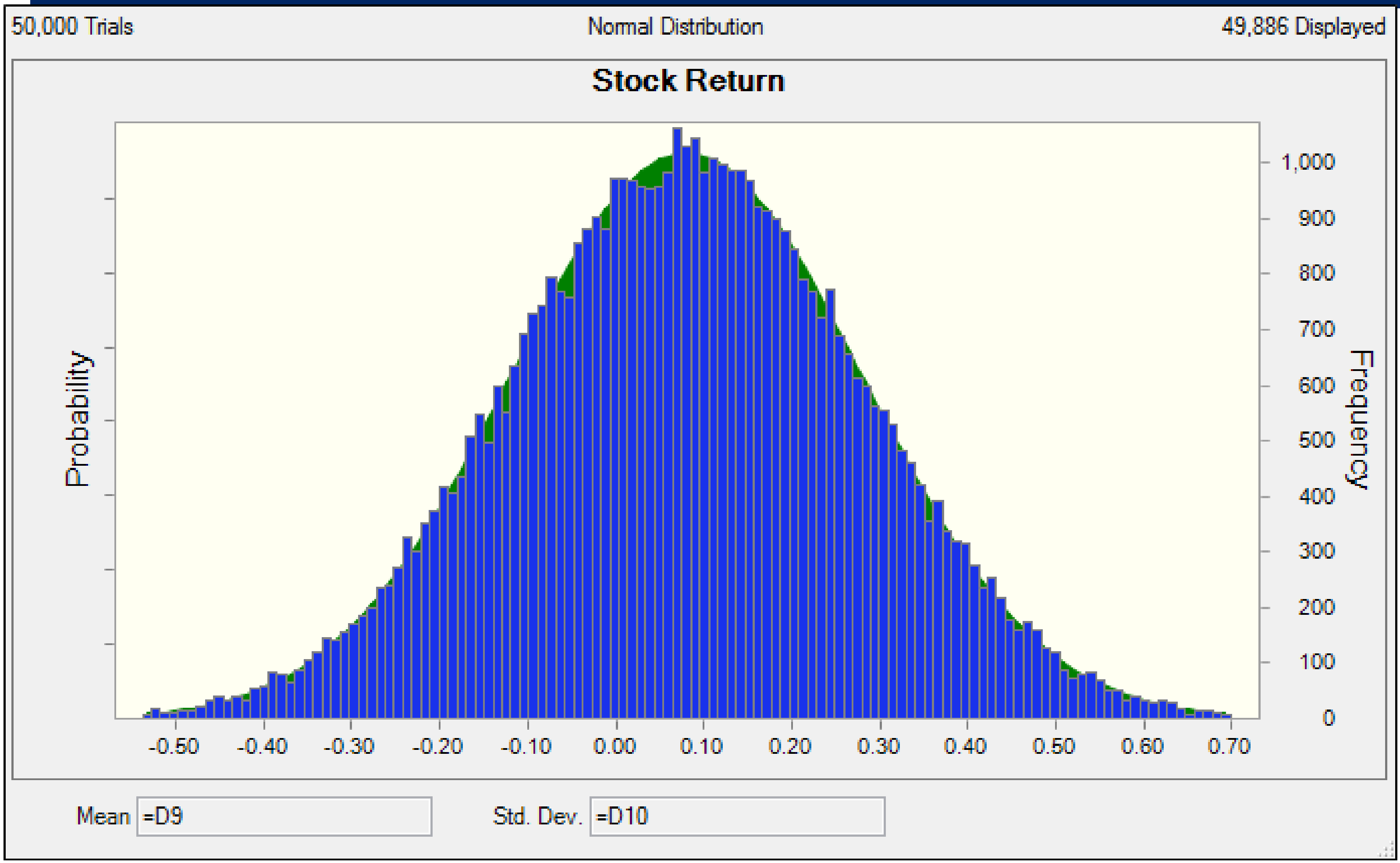
50,000 Trials   Split View   49,072 Displayed



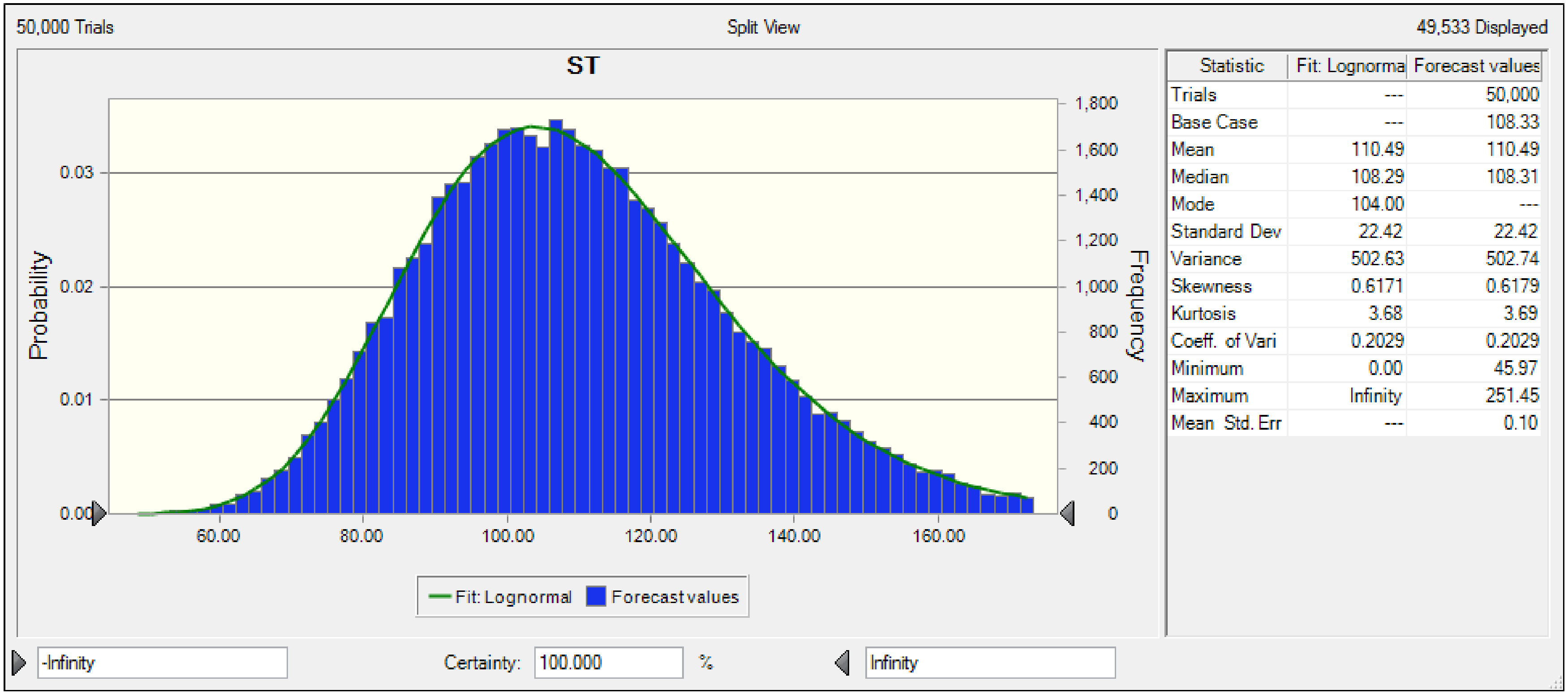
Statistic	Forecast values
Trials	50,000
Base Case	0.00
Mean	1.10
Median	0.00
Mode	0.00
Standard Deviation	1.55
Variance	2.40
Skewness	1.44
Kurtosis	4.25
Coeff. of Variation	1.41
Minimum	0.00
Maximum	6.79
Mean Std. Error	0.01

-Infinity   Certainty: 100.000 %   Infinity

NORMAL DISTRIBUTION OF THE STOCK RETURNS

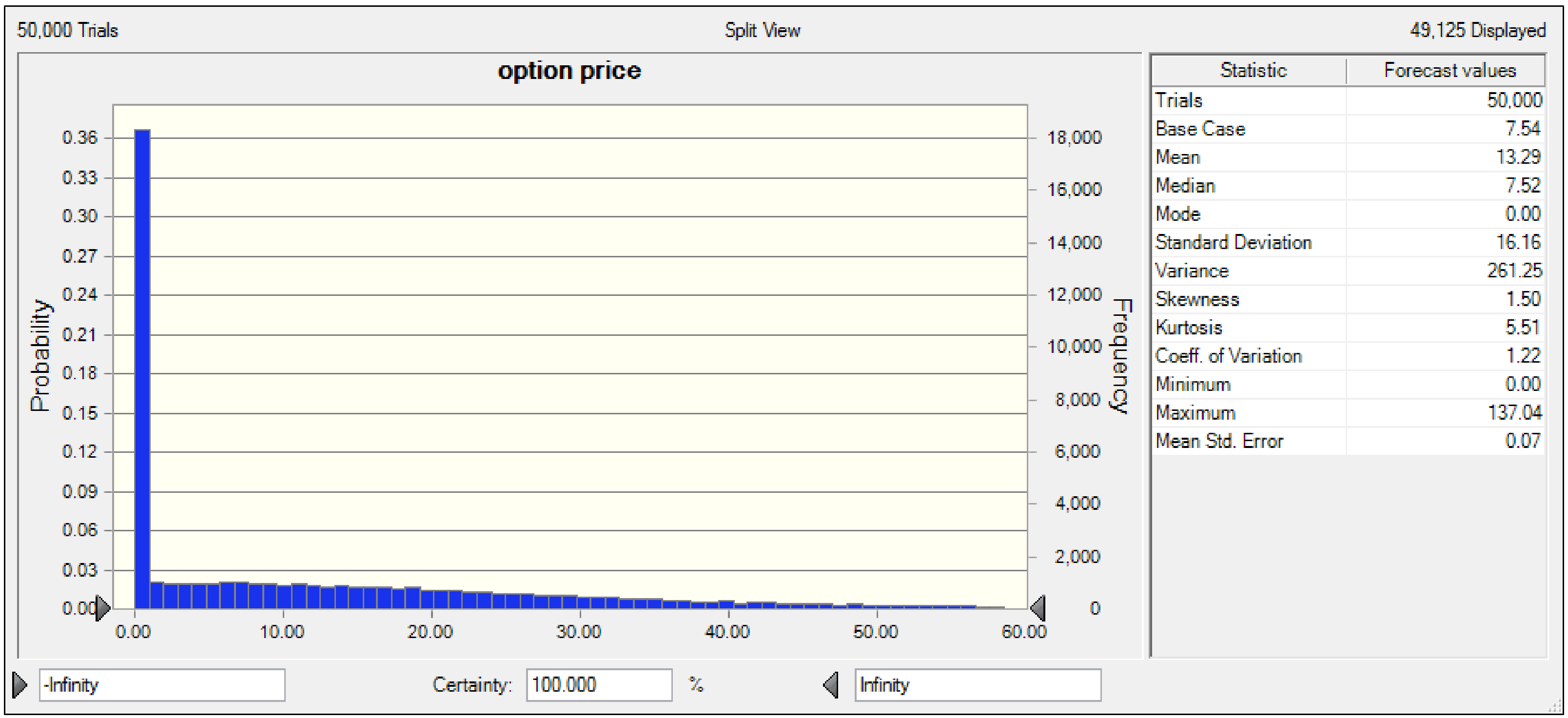


Lognormal distribution of the stock price



Statistic	Fit: Lognorma	Forecast values
Trials	---	50,000
Base Case	---	108.33
Mean	110.49	110.49
Median	108.29	108.31
Mode	104.00	---
Standard Dev	22.42	22.42
Variance	502.63	502.74
Skewness	0.6171	0.6179
Kurtosis	3.68	3.69
Coeff. of Vari	0.2029	0.2029
Minimum	0.00	45.97
Maximum	Infinity	251.45
Mean Std. Err	---	0.10

EUROPEAN CALL BASED ON A LOG-NORMALLY DISTRIBUTED STOCK PRICE



Summary

- **Monte Carlo simulation can be used to price path-dependent options**
- The basic concept is that games of chance can be played to approximate solutions to real world problems
- **The steps to value options are:**

1. Model stock price distribution
2. Simulate stock paths
3. Compute payoffs for each trial. Mean payoff is price.

- **Exotic option can provide unique payoff structures, meet specific hedging needs, or help with tax, regulatory or accounting reasons.**
- Asian option
- Lookback option
- Knock-out option

Finance 2325

Alternative Project: Monte Carlo Simulation Excel exercise

Deadline to submit: December 8

Create a histogram of stock prices at maturity

Compute the value of a European call & put through simulation, and make a histogram of the put option payoffs

Value ten exotic options

Submit your Excel file as well for verification. This is a group project.