A Generalized Top-N Dynamic Programming Approach to Optimal Discrete Grading

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Abstract

Educational assessment frequently requires mapping high-cardinality raw scores to a constrained discrete grade scale. This process is governed by two competing requirements: internal logical consistency (rank preservation) and external distributional mandates (mean or band quotas). This paper formalizes this reconciliation as a discrete optimization problem. We propose a Top-N Dynamic Programming algorithm that treats distributional targets as soft penalties to guarantee a solution even when tied score blocks exceed the width of institutional bands. Finally, we analyze the combinatorial boundaries of the solution space and identify the physical limitations of discrete block assignment.

1 Introduction

In many academic settings, the ”grading curve” is a mandatory instrument of policy. However, manual application often forces instructors into ad-hoc decisions that may violate the principle that identical performance should receive identical rewards. This paper proposes an algorithmic solution that decoupling the logical constraints of the exam from the statistical targets of the institution, providing a framework where ordinal performance is mapped to interval categories with mathematical rigor.

2 The Formal Framework

2.1 Ordinal and Logical Constraints

Let $S = \{s_1, s_2, \ldots, s_n\}$ be raw scores and $G = \{g_1, g_2, \ldots, g_m\}$ be the target grade values in descending order. We define the following ”hard” constraints:

1. **Monotonicity**: $s_i > s_j \implies f(s_i) \geq f(s_j)$.
2. **Tie-Preservation**: $s_i = s_j \implies f(s_i) = f(s_j)$.

To satisfy these, we aggregate S into K unique *score blocks*. The algorithm assigns each block to a single grade value; it is physically impossible for the algorithm to split a group of tied students between two different letter grades.

2.2 Objective Function and Soft Penalties

The "goodness" of an assignment is measured by a penalty function Z:

$$Z = \lambda|\bar{g} - \mu_{target}| + \sum_{b=1}^{B} \omega_b \cdot \text{Excursion}(p_b, [min_b, max_b])$$

In environments with narrow distribution bands, treating percentages as "hard" constraints often results in a null solution set. Consequently, the algorithm treats band-excursions as soft penalties. The weight ω_b is set high enough to minimize these departures, but the "soft" formulation ensures the algorithm always returns a valid monotonic assignment that is as close to policy as mathematically possible.

3 Algorithm: Top-N Dynamic Programming

The algorithm treats grading as a sequential decision problem through a layered trellis. At each grade level g_i, the algorithm decides how many score blocks to consume.

3.1 The Necessity of the Top-N Buffer

Because aggregate constraints like the mean create "long-range dependencies," a local assignment that appears sub-optimal regarding band percentages might be the only path capable of satisfying the final mean target. By maintaining a "Leaderboard" of N paths at each state (e.g., $N = 100$), we preserve a diverse set of candidate paths until the final state is reached.

4 Generalization to Arbitrary Systems

The algorithm is extensible to any system where ordinal performance must be mapped to discrete categories, including Pass/Fail systems, competency-based rubrics, and multi-section normalization.

5 Limitations and Edge Cases

5.1 Computational Intractability

The DP approach is efficient ($O(m \cdot K^2 \cdot N)$), but the search space is bounded by $\binom{K+m-1}{m-1}$. If the target mean is extremely precise and N is too small, the algorithm may prune the optimal solution early. Thus, N must be scaled with the desired precision of the aggregate statistic.

5.2 Incompatible Grading Schemes

The algorithm cannot accommodate non-ordinal metrics (e.g., categorical grouping) or multi-dimensional grading (e.g., two independent raw scores) without prior aggregation into a single ordinal index.

5.3 Quantization Error and Block Indivisibility

Because students are discrete units and tied scores form indivisible blocks, the mapping to a percentage-based band is quantized. The distribution behaves as a step function rather than a continuous curve. In cases where the size of a single score block exceeds the allowable width of a target distribution band (e.g., a block of 5 students in a 98-person class where a band is capped at 4%), a "perfect" solution is logically impossible. The Top-N DP resolves this by finding the path that minimizes the weighted excursion from the boundary.

6 Conclusion

The Generalized Top-N DP algorithm offers a principled way to navigate the tension between exam-level fairness and institutional policy. It ensures that the instructor respects the internal integrity of the exam while providing a transparent and defensible adherence to administrative mandates.