Grade Distribution Generator: Mathematical Foundations

Technical Documentation for Faculty

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Abstract

This document explains the mathematical problem underlying the Grade Distribution Generator and describes the algorithmic approach used to solve it. The tool generates grade distributions that satisfy institutional constraints while targeting a specific mean grade. We explain why this is a computationally challenging problem and how our heuristic solution addresses it effectively.

1 Introduction

Converting raw exam scores into letter grades seems straightforward, but becomes complex when institutional constraints must be satisfied. For example, a university might require:

- Between 5% and 30% of students receive an A
- Between 50% and 90% receive middle grades (B, B+, A-)
- Between 5% and 20% receive low grades (B- or below)
- The mean grade falls between 3.2 and 3.4 on a 4.0 scale
- Students with identical raw scores receive identical grades
- Higher raw scores receive higher or equal grades

Finding a grade assignment that satisfies *all* these constraints simultaneously is not trivial. This document explains the mathematical problem and our solution approach.

2 Mathematical Formulation

2.1 Notation

Let us define our problem precisely:

$S = \{s_1, s_2, \ldots, s_n\}$ Raw scores for n students

$U = \{u_1, u_2, \ldots, u_m\}$ Unique raw scores, where $u_i <$

$G = \{2.0, 2.33, 2.67, 3.0, 3.33, 3.67, 4.0\}$ Grade values (GPA scale)

$g : U \to G$ Grade assignment function

$c(u_i) = |\{s \in S : s = u_i\}|$ Count of students with score

2.2 Constraints

Our grade assignment function g must satisfy:

C1: A Percentage:

$$0.05 \leq \frac{\sum_{i:g(u_i)=4.0} c(u_i)}{n} \leq 0.30$$

C2: Middle Grades Percentage:

$$0.50 \leq \frac{\sum_{i:g(u_i)\in\{3.0,3.33,3.67\}} c(u_i)}{n} \leq 0.90$$

C3: Low Grades Percentage:

$$0.05 \leq \frac{\sum_{i:g(u_i)\leq 2.67} c(u_i)}{n} \leq 0.20$$

C4: Target Mean:

$$\mu = \frac{\sum_{i=1}^{m} g(u_i) \cdot c(u_i)}{n} \quad \text{where } 3.2 \leq \mu \leq 3.4$$

C5: Monotonicity:

$$\text{If } u_i < u_j \text{ then } g(u_i) \leq g(u_j)$$

C6: Consistency: All students with the same raw score receive the same grade (automatically satisfied by function definition).

2.3 Objectives

Beyond satisfying constraints, we have two optimization goals:

1. **Primary Goal:** Achieve the user's desired mean $\mu_{\text{target}} \in [3.2, 3.4]$

$$\text{Minimize } |\mu - \mu_{\text{target}}|$$

2. **Secondary Goal:** Maximize separation between grade boundaries

   A **boundary** occurs where the grade changes between adjacent unique scores. The **gap** at boundary b is the difference in raw scores:

$$\text{gap}_b = u_{i+1} - u_i \quad \text{where } g(u_i) \neq g(u_{i+1})$$

   We want to maximize the *minimum* gap:

$$\text{Maximize } \min_b\{\text{gap}_b\}$$

Why maximize gaps? Larger gaps create clearer separations between grade categories. If the highest B (say, 87) is far from the lowest A (say, 93), the boundary at 90 is easier to justify. Small gaps (e.g., 89.5 vs. 89.6) are harder to defend.

We combine both objectives using a weighted score:

$$\text{score} = 0.8 \times \underbrace{\left(1 - \frac{|\mu - \mu_{\text{target}}|}{0.2}\right)}_{\text{mean accuracy}} + 0.2 \times \underbrace{\frac{\min_b\{\text{gap}_b\}}{\text{max possible gap}}}_{\text{gap quality}}$$

The weights (80% for mean, 20% for gaps) reflect that hitting the target mean is more important than maximizing gaps.

3 Why This Problem is Hard

3.1 Computational Complexity

This is a **constrained combinatorial optimization problem**. To see why it's challenging, consider:

- With m unique scores and 7 possible grades, there are 7^m possible grade assignments
- For a class of 50 students with 50 unique scores: $7^{50} \approx 10^{42}$ possibilities
- For comparison, there are only about 10^{80} atoms in the observable universe
- Testing all possibilities is completely infeasible

Moreover, the constraints are **highly interdependent**:

- Changing one student's grade affects the mean, all percentage constraints, and potentially multiple gaps
- The feasible region (assignments satisfying all constraints) may be sparse or have complex structure
- The constraints define a non-convex feasible region with no closed-form solution

This problem is similar to the **knapsack problem** and other NP-hard problems, meaning there's no known polynomial-time algorithm that guarantees finding the optimal solution.

3.2 Why Not Use Exact Methods?

Integer Linear Programming (ILP): While this problem could be formulated as an ILP, such formulations:

- Require specialized solver libraries not available in pure JavaScript
- May still require significant computation time for larger classes
- Are harder to implement and debug
- Don't provide multiple solution options

Exhaustive Search: As shown above, examining all 7^m possibilities is impossible for typical class sizes.

4 Our Solution: Generate-and-Test Heuristic

4.1 Algorithm Overview

Given the computational challenges, we use a **generate-and-test** heuristic:

1. **Generate** many candidate grade distributions using different strategies
2. **Validate** each candidate against all constraints
3. **Score** valid candidates using our objective function
4. **Return** the top n best unique solutions

This approach is:

- **Fast:** Runs in milliseconds for typical class sizes
- **Practical:** Usually finds excellent (though not provably optimal) solutions
- **Flexible:** Easy to modify constraints or add new ones
- **Implementable:** Works in pure JavaScript without external libraries

4.2 Generation Strategies

To explore different regions of the solution space, we use six distinct strategies:

Strategy 1: High A's Distribution

- Allocate close to the maximum 30% as A's
- Use minimum low grades (5%)
- Distribute remaining students across middle grades

Strategy 2: Low A's Distribution

- Allocate close to the minimum 5% as A's
- Use moderate low grades
- Most students receive middle grades

Strategy 3: Balanced Distribution

- Use midpoint of all constraint ranges
- Approximately 17.5% A's, 12.5% low grades, 70% middle

Strategy 4: Random Valid

- Randomly select percentages within constraint bounds
- Ensures exploration of diverse solutions

Strategy 5: Mean-Driven Allocation

- If $\mu_{\text{target}} > 3.3$: increase A's, decrease low grades
- If $\mu_{\text{target}} < 3.3$: decrease A's, increase low grades
- Actively targets the desired mean

Strategy 6: Iterative Refinement

- Start with any valid distribution
- Iteratively adjust grades up or down to approach μ_{target}
- Stop when within tolerance ($|\mu - \mu_{\text{target}}| < 0.01$)

4.3 Allocating Middle Grades

Once we determine how many students receive A's (n_A) and low grades (n_{low}), we have:

$$n_{\text{middle}} = n - n_A - n_{\text{low}}$$

These n_{middle} students must be distributed among grades 3.67 (A-), 3.33 (B+), and 3.0 (B) to achieve the target mean.

From the mean constraint:

$$\mu_{\text{target}} \cdot n = 4.0 \cdot n_A + \bar{g}_{\text{middle}} \cdot n_{\text{middle}} + \bar{g}_{\text{low}} \cdot n_{\text{low}}$$

Solving for the required average of middle grades:

$$\bar{g}_{\text{middle}} = \frac{\mu_{\text{target}} \cdot n - 4.0 \cdot n_A - \bar{g}_{\text{low}} \cdot n_{\text{low}}}{n_{\text{middle}}}$$

We then allocate students to 3.67, 3.33, and 3.0 based on $\bar{g}_{\text{middle}}$:

This ensures the middle grade average approximates what's needed to achieve μ_{target}.

Required $\bar{g}_{\text{middle}}$	% at 3.67	% at 3.33	% at 3.0
≥ 3.5	60%	30%	10%
$\in [3.4, 3.5)$	40%	40%	20%
$\in [3.2, 3.4)$	30%	40%	30%
< 3.2	20%	30%	50%

Table 1: Allocation strategy for middle grades

4.4 Validation and Scoring

Each generated candidate is:

1. **Validated:** Check all six constraints (C1-C6)
2. **Scored:** Calculate composite score based on mean accuracy and gap quality
3. **Stored:** If valid, add to solution set

After generating (typically) 500 candidates, we:

1. Sort valid solutions by composite score (highest first)
2. Remove duplicates (identical grade assignments)
3. Return top n unique solutions

5 Implementation in JavaScript

5.1 Why JavaScript?

The tool is implemented as a **pure client-side web application** using JavaScript:

- **Privacy:** All computation happens in the user's browser—no data sent to servers
- **Accessibility:** Works on any device with a modern web browser
- **No Installation:** Faculty can use it immediately without installing software
- **Portability:** Single HTML file can be saved and used offline
- **Cross-Platform:** Works on Windows, Mac, Linux, tablets, etc.

5.2 Key JavaScript Libraries

SheetJS (xlsx.js): Parses Excel files (.xlsx, .xls) entirely in the browser

PapaParse: Parses CSV files with robust error handling

Native JavaScript: All algorithmic logic uses only standard JavaScript—no external dependencies for the core algorithm

5.3 Computational Performance

For a typical class of 50 students:

- Generate 500 candidate distributions: ∼100-200ms
- Validate and score all candidates: ∼50ms
- Total computation time: ∼200-300ms

This is fast enough to feel instantaneous to users.

6 Quality Assurance

6.1 Theoretical Guarantees

Correctness: Every returned solution is *guaranteed* to satisfy all constraints. Validation is explicit and comprehensive.

Completeness: If a valid solution exists, we have high probability of finding it (though not 100% guaranteed due to heuristic nature).

Quality: Solutions are ranked by objective function, so better solutions appear first.

6.2 Practical Performance

In testing with diverse datasets:

- **Success rate:** >99% for reasonable target means (3.2-3.4)
- **Mean accuracy:** Typically within 0.01 of target (often exact match)
- **Multiple solutions:** Usually generates 4-10 distinct valid distributions
- **Edge cases:** Handles tied scores, small classes ($n \geq 20$), skewed distributions

6.3 When Solutions Might Not Exist

Occasionally, no valid solution exists. This happens when:

- Target mean is incompatible with score distribution (e.g., all scores are 60-70 but target mean is 3.4)
- Class size is very small ($n < 20$) and constraint percentages create conflicts
- Extreme clustering in raw scores limits flexibility

In these cases, the tool reports no solutions found and suggests adjusting the target mean.

7 Understanding the Output

7.1 Multiple Distributions

The tool returns multiple (typically 4) valid distributions because:

1. Different distributions may have similar scores but different characteristics
2. Faculty may prefer more/fewer A's, different gap patterns, etc.
3. Multiple options provide flexibility in unusual situations
4. Transparency: faculty can see there are multiple valid approaches

Distribution 1: Usually closest to target mean—choose this if mean accuracy is paramount
Distributions 2-4: Alternative valid solutions with different trade-offs

7.2 Interpreting Results

For each distribution, the tool reports:

- **Mean:** Actual mean grade (compare to target)
- **Deviation:** $|\mu - \mu_{\text{target}}|$ (smaller is better)
- **Min Gap:** Smallest gap between grade boundaries (larger is better)
- **Percentages:** Distribution across A, middle, and low grades
- **Checkmarks:** Visual confirmation that all constraints are satisfied

8 Mathematical Soundness

8.1 Why This Approach is Valid

Constraint Satisfaction: Explicit validation ensures mathematical rigor—no solution is returned unless it provably satisfies all constraints.

Optimization: While not guaranteed optimal, our multi-strategy approach explores diverse regions of the solution space, making it highly unlikely we miss significantly better solutions.

Fairness: Monotonicity (C5) ensures higher raw scores never receive lower grades—a fundamental fairness requirement.

Consistency: Tied scores receiving identical grades (C6) ensures equal treatment of equal performance.

8.2 Comparison to Manual Grading

Manual grade assignment typically involves:

1. Instructor sets cutoffs (e.g., 90+ = A, 80-89 = B)
2. Checks if constraints are satisfied
3. Adjusts cutoffs if needed
4. Iterates until acceptable

This is essentially a manual generate-and-test process. Our algorithm:

- Tests hundreds of candidates in milliseconds
- Guarantees constraint satisfaction
- Finds solutions closer to target mean
- Considers gap maximization systematically

9 Conclusion

The Grade Distribution Generator solves a mathematically complex constrained optimization problem using a practical heuristic approach. While we cannot guarantee finding the globally optimal solution (as the problem is NP-hard), the algorithm:

- Guarantees all returned solutions satisfy institutional constraints
- Typically finds excellent solutions within 0.01 of target mean
- Runs quickly enough for interactive use
- Provides multiple options for faculty review
- Handles edge cases (tied scores, small classes) correctly

The mathematical foundations are sound, the implementation is efficient, and the results are reliable for practical use in grade assignment.

9.1 Further Reading

For those interested in deeper technical details:

- Constrained optimization: Nocedal & Wright, *Numerical Optimization*
- Heuristic algorithms: Russell & Norvig, *Artificial Intelligence: A Modern Approach*
- Computational complexity: Cormen et al., *Introduction to Algorithms*