Transmission Coeffecient of a Finite Potential Well Derivation

We compute the transmission coefficient (T) by definition ($T^{-1} = \frac{|A|^2}{|F|^2}$), substituting F with equivalent expression in terms of the energy of the wave (E) and the depth of the potential well. We choose units such that $\hbar = 1$ and asuume $m = 1$.

Suppose a potential well has a width of $2a$ and is centered at $x = 0$, where the potential $V(x)$ is equal to $-V_0$ for $-a \leq x \leq a$ and 0 for $x > a$ and $x < -a$. We label the region corresponding to $x < -a$ as region I, to $-a < x < a$ as region II, to $x > a$ as region III. During our meetings we solved for the one-dimentional time-independent Schrödinger equation for when $E > V(x)$ and $E < V(x)$, obtaining

$$\begin{aligned}
\Psi_{\text{I}}(x) &= Ae^{ikx} + Be^{-ikx} \\
\Psi_{\text{II}}(x) &= C\sin(lx) + D\cos(lx) \\
\Psi_{\text{III}}(x) &= Fe^{ikx}
\end{aligned}$$

where $l = \sqrt{2(E + V_0)}$, $k = \sqrt{2E}$ and the subscript of the denotation of each function indicates over which region the function is defined. The requirements to stitch the three functions into one function follows the properties of a valid solution to the Schrödinger equation:

$$Ae^{-ika} + Be^{ika} = -C\sin(la) + D\cos(la) \tag{1}$$

$$C\sin(la) + D\cos(la) = Fe^{ika} \tag{2}$$

$$ik(Ae^{-ika} - Be^{ika}) = l(C\cos(la) + D\sin(la)) \tag{3}$$

$$l(C\cos(la) - D\sin(la)) = ikFe^{ika} \tag{4}$$

Now we solve for F in terms of l, k, a, and A. From equation (2), we solve

$$C = \frac{Fe^{ika} - D\cos(la)}{\sin(la)} \tag{5}$$

and substitute into equation (4) to solve for

$$\begin{aligned}
Fe^{ika}\left(\frac{ik}{l}\sin(la) - \cos(la)\right) &= -D(\cos^2(la) + \sin^2(la)\Big) \\
D = Fe^{ika}\left(\cos(la) - \frac{ik}{l}\sin(la)\right)
\end{aligned}$$

Then we substitute D in terms of F into equation (5) and simplify. Note $1 - \cos^2(la) = 1 - 1 - \frac{1}{2}(1 + \cos(2la)) = 1 - \frac{1}{2}(1 + 1 - 2\sin^2(la)) = \sin^2(la)$.

$$C = \frac{Fe^{ika}(1 - \cos^2(la) + \frac{ik}{l}\cos(la)\sin(la))}{\sin(la)} = Fe^{ika}\left(\frac{1 - \cos^2(la)}{\sin(la)} + \frac{ik}{l}\cos(la)\right)$$

$$C = Fe^{ika}\left(\sin(la) + \frac{ik}{l}\cos(la)\right)$$

Note $\cos^2(la) - \sin^2(la) = \cos(2la)$ and $2\cos(la)\sin(la) = \sin(2la)$. Substituting D and C in terms of F into equation (1), we get

$$\begin{aligned} Ae^{-ika} &= Fe^{ika}\left(-\sin^2(la) - \frac{ik}{l}\cos(la)\sin(la) + \cos^2(la) - \frac{ik}{l}\sin(la)\cos(la)\right) - Be^{ika} \\ &= Fe^{ika}\left(\cos(2la) - \frac{ik}{l}\sin(2la)\right) - Be^{ika}. \end{aligned} \tag{6}$$

Now we substitute this result and D and C in terms of F into equation (3), and solve for B.

$$\begin{aligned} Fe^{ika}\left(-\frac{ik}{l}\sin(2la) + \cos(2la)\right) - 2Be^{ika} &= \frac{l}{ik}Fe^{ika}\left(2\cos(la)\sin(la) + \frac{ik}{l}\cos^2(la) - \frac{ik}{l}\sin^2(la)\right) \\ -2B &= F\sin(2la)\left(\frac{l}{ik} + \frac{ik}{l}\right) \\ B &= i\frac{\sin(2la)}{2kl}(l^2 - k^2)F \end{aligned}$$

To solve for F in terms of A we substitute B, C, and D into equation (6).

$$\begin{aligned} Ae^{-ika} &= Fe^{ika}\left(\cos(2la) - \frac{ik}{l}\sin(2la) - i\frac{\sin(2la)}{2kl}(l^2 - k^2)\right) \\ Ae^{-2ika} &= Fe^{ika}\left(\cos(2la) - \left(\frac{ik}{l} + \frac{i(l^2 - k^2)}{2lk}\right)\sin(2la)\right) \\ F &= \frac{e^{-2ika}A}{\cos(2la) - i\left(\frac{k^2+l^2}{2kl}\right)\sin(2la)} \end{aligned}$$

Finally, we compute T^{-1} and rewrite it in terms of E and V_0.

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\begin{aligned}
T^{-1} = \frac{|A|^2}{|F|^2} = \frac{|A|^2}{\frac{\left|e^{-2ika}\right|^2|A|^2}{\left|\cos^2(2la)+i\left(\frac{k^2+l^2}{2kl}\right)\sin(2la)\right|^2}} = \cos^2(2la) + \left(\frac{(k^2+l^2)^2}{4k^2l^2}\right)\sin^2(2la) \\
= 1 - \sin^2(2la) + \left(\frac{(k^2+l^2)^2}{4k^2l^2}\right)\sin^2(2la) = 1 + \left(\frac{k^4 - 2k^2l^2 + l^4}{4k^2l^2}\right)\sin^2(2la) \\
= 1 + \left(\frac{4E^2 - 8E(E+V_0) + 4(E^2 + 2EV_0 + V_0^2)}{16E(E+V_0)}\right)\sin^2(2a\sqrt{2(E+V_0)}
\end{aligned}
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\boxed{T = \frac{1}{1 + \left(\frac{V_0}{4E(E+V_0)}\right)\sin^2(2a\sqrt{2(E+V_0)}}}
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