1 Loopy Belief Propagation

1.1 Introduction

Given a Factorial Hidden Markov Model (FHMM) with:

- M independent hidden state chains (factors)
- K possible states per factor
- T time steps
- Observation sequence $O = (o_1, o_2, \ldots, o_T)$
- Transition potentials $\phi_i(s_t^i, s_{t-1}^i)$ for each factor i
- Emission potentials $\phi_O(o_t, s_t^1, s_t^2, \ldots, s_t^F)$

Compute marginal probabilities $P(s_t^i)$ for each variable s_t^i.

1.2 Algorithm

1.3 Message Passing Scheme

1.3.1 Transition Factor to Variable Messages

For each transition factor between times t and $t+1$:

$$m_{T_{t,f} \to V_{t,f}}(s_t^f) = \sum_{s_{t+1}^f} \phi_f(s_{t+1}^f, s_t^f) \cdot m_{V_{t+1,f} \to T_{t,f}}(s_{t+1}^f) \tag{1}$$

$$m_{T_{t,f} \to V_{t+1,f}}(s_{t+1}^f) = \sum_{s_t^f} \phi_f(s_{t+1}^f, s_t^f) \cdot m_{V_{t,f} \to T_{t,f}}(s_t^f) \tag{2}$$

1.3.2 Observation Factor to Variable Messages

For each time slice t, compute messages from the observation factor to each variable:

$$m_{O_t \to V_{t,f}}(s_t^f) = \sum_{\mathbf{s}_t \setminus \{\mathbf{s}_t^f\}} \phi_O(o_t, \mathbf{s}_t) \prod_{g \neq f} m_{V_{t,g} \to O_t}(s_t^g) \tag{3}$$

1.3.3 Variable to Factor Messages

Each variable sends messages to its neighboring factors, excluding the target factor:

$$m_{V_{t,f}\to O_t}(s_t^f) = \prod_{T' \in \mathcal{N}(V_{t,f})\setminus O_t} m_{T'\to V_{t,f}}(s_t^f) \tag{4}$$

$$m_{V_{t,f}\to T_{t,f}}(s_t^f) = \prod_{F' \in \mathcal{N}(V_{t,f})\setminus T_{t,f}} m_{F'\to V_{t,f}}(s_t^f) \tag{5}$$

1.4 Time Complexity

1.4.1 Per-Iteration Complexity Breakdown

- **Transition Factor Updates:** $O((T-1) \times M \times K \times K) = O(T \times M \times K^2)$
- **Observation Factor Updates:** $O(T \times K^M \times M \times M) = O(T \times K^M \times M^2)$
- **Variable Message Updates:** $O(T \times M \times K)$

1.4.2 Overall Complexity

- **Per iteration:**

$$\begin{aligned} C_{\text{iter}} &= O(T \times M \times K^2) + O(T \times K^M \times M^2) && (6) \\ &= O(T \times M \times K^2 + T \times K^M \times M^2) && (7) \end{aligned}$$

- **Final Belief Computation:** $O(T \times M \times K)$

Total algorithm:

$$\begin{aligned} C_{\text{total}} &= C_{\text{iter}} \times \text{max\_iterations} + O(T \times M \times K) && (8) \\ &= O(\text{max\_iterations} \times T \times M \times (K^2 + K^M \times M)) && (9) \\ &= O(T \times M^2 \times K^M) && (10) \end{aligned}$$

1.5 References

- https://www.cs.cmu.edu/~epxing/Class/10708-14/scribe_notes/scribe_note_lecture13.pdf
- https://arxiv.org/pdf/1301.6725
- https://www.cs.ubc.ca/~murphyk/papers/loopy_uai99.pdf
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