1 Introduction

We be MIG'n and hourglassin'.

2 Theory

2.1 MIG

Use least-squares to show $\lim_{n \to \inf} err = 0$.

2.2 Hourglassin'

2.2.1 Motivation

Because available sensor models (such as RPC) often lack a meaningful error model to input to MIG, and for computational simplicity, we introduce a heuristic approach as an alternative to rigorous least-squares MIG. Although when visualized 'up close' to the true answer, any particular tangle of rays may seem unclear about where the ground point should be localized, from a global perspective, the ray bundle will be something like a cone, widening to the cluster of satellite (or airborne) perspective centers, narrowing at the ground point, and widening again beyond the ground point. We seek the answer at the narrowest point of the cone. In an ideal case, the ray bundle will intersect at a single point, which has cross-sectional area of 0. In a real-world case, the bundle will appear as a cone with a 'fat' intersection. Thus the name 'hourglassing' to motivate the heuristic technique.

Given a bundle of rays, we can intersect the bundle with planes of various heights, compute the collection of intersections of the ray bundle with each height plane, measure the spread of the 2-D distribution of points, and choose the plane with the least spread to be the solution for the height of the desired ground point. For the horizontal location of the ground point, the natural choice is the mean of the intersection points in the chosen plane.

Rather than attacking this problem with a brute force search for the spread-minimizing by computing intersection sets at very many heights, and slicing height space sufficiently thin to achieve a desired vertical resolution, we attempt some theoretical underpinnings for this technique that will allow a more efficient and precise solution, and help motivate a meaningful estimate of the error of the resulting ground point.

2.2.2 Computing height of minimum spread

Assume two heights $z_+ > z_-$, and assume a set of N 3D lines L_i, none of which is horizontal. Specify the lines by their intersections with the planes of heights $z_\pm$ at points (x_+^i, y_+^i, z_+) and (x_-^i, y_-^i, z_-), for $i = 1 \ldots N$.

Define

$$x^i(\lambda) = \lambda x_+^i + (1 - \lambda) x_-^i$$

$$y^i(\lambda) = \lambda y^i_+ + (1-\lambda) y^i_-$$

$$z(\lambda) = \lambda z_- + (1-\lambda) z_-$$

For any λ, $(x^i(\lambda), y^i(\lambda), z(\lambda))$ is a point on line L_i; whether $0 \leq \lambda \leq 1$ determines whether the point is an interpolation between or extrapolation beyond the $z_\pm$ planes (in either case, we will just use the term interpolated). Together, all the points $(x^i(\lambda), y^i(\lambda), z(\lambda))$ for $i = 1 \ldots N$ represent the intersection of lines L_i with the plane with height $z(\lambda)$.

Note that the set of all means $(\bar{x}(\lambda), \bar{y}(\lambda), z(\lambda))$ comprise a line. For the plane at height $z(\lambda)$ that yields the smallest spread of points $(x^i(\lambda), y^i(\lambda)$ will be the point on that line which we choose as our answer.

The two-dimensional spread of the intersection set at a particular $z(\lambda)$ is a symmetric, positive definite, 2x2 covariance matrix

$$M(\lambda) = \begin{bmatrix} var(x^i(\lambda)) & covar(x^i(\lambda), y^i(\lambda)) \\ - & var(y^i(\lambda)) \end{bmatrix}$$

It can be shown that the covariance of the interpolalated points $(x^i(\lambda), y^i(\lambda))$ can be expressed in terms of variances of and covariances between the four elementary datasets $x^i_+, x^i_-, y^i_+, y^i_-$ as follows:

$$var(x^i(\lambda)) = \lambda^2 \sigma^2_{x+x+} + \lambda(1-\lambda)(\sigma^2_{x+x-} + \sigma^2_{x-x+}) + (1-\lambda)^2 \sigma^2_{x-x-} \quad (1)$$

$$var(y^i(\lambda)) = \lambda^2 \sigma^2_{y+y+} + \lambda(1-\lambda)(\sigma^2_{y+y-} + \sigma^2_{y-y+}) + (1-\lambda)^2 \sigma^2_{y-y-} \quad (2)$$

$$covar(x^i(\lambda), y^i(\lambda)) = \lambda^2 \sigma^2_{x+y+} + \lambda(1-\lambda)(\sigma^2_{x+y-} + \sigma^2_{x-y+}) + (1-\lambda)^2 \sigma^2_{x-y-} \quad (3)$$

where

$$\sigma^2_{x+x+} = var(x^i_+),$$

$$\sigma^2_{x+y-} = covar(x^i_+, y^i_-),$$

etc. Note in particular that, since all the σ^2 are functions only of constants $x^i_\pm, y^i_\pm$, and N, each of the 4 terms of $M(\lambda)$ is a quadratic function of λ.

If we denote the determinant of a matrix with $|\cdot|$, the area of the 1-sigma error ellipse of $M(\lambda)$ is

$$a(\lambda) = \pi \sqrt{|M(\lambda)|},$$

i.e. the square root of a quartic (4th degree) polynomial.

We seek to minimize $a(\lambda)$, which is equivalent to minimizing the quartic polynomial

$$d(\lambda) = |M(\lambda)| = var(x^i(\lambda)) var(y^i(\lambda)) - covar(x^i(\lambda), y^i(\lambda))^2$$

Note that, for efficient computation, the quadratic, linear, and constant coefficients of equations (1-3) can be computed to yield

$$var(x^i(\lambda)) = a_x \lambda^2 + b_x \lambda + c_x \quad (4)$$

$$var(y^i(\lambda)) = a_y\lambda^2 + b_y\lambda + c_y \tag{5}$$

$$covar(x^i(\lambda), y^i(\lambda)) = a_{xy}\lambda^2 + b_{xy}\lambda + c_{xy} \tag{6}$$

Given coefficients computed in equations (4-6), $d(\lambda)$ can then be expressed as

$$\begin{aligned} d(\lambda) &= (a_x\lambda^2 + b_x\lambda + c_x)(a_y\lambda^2 + b_y\lambda + c_y) - (a_{xy}\lambda^2 + b_{xy}\lambda + c_{xy})^2 \\ &= (a_x a_y - a_{xy}^2)\lambda^4 + (\ldots)\lambda^3 + (\ldots)\lambda^2 + (\ldots)\lambda + (\ldots) \end{aligned} \tag{7}$$

The value λ_{min} which yields the minimum value of $d(\lambda)$ will also determine the height $z(\lambda_{min})$ of the output ground point, and then λ_{min} can be used to interpolate the intersection set $(x^i(\lambda_{min}), y^i(\lambda_{min}))$, of which the mean $(\bar{x}(\lambda_{min}), \bar{y}(\lambda_{min}))$ constitutes the horizontal component of the output ground point.

2.2.3 Non-uniqueness

It is desirable that this quartic polynomial $d(\lambda)$ have no local minima, but only a single, global minimum. Equivalently, the cubic derivative should have a single real root and two complex roots. Unfortunately, there can be degenerate arrangements of image rays with three real roots of $d'(\lambda$ and multiple minima for $d(\lambda)$.

Consider the bimodal situation depicted in Fig. **??**: images of the ground point are captured from satellite positions spaced equally around a horizontal circle, with orientations perfectly intersecting at a ground point at height z_+. To this bundle add a duplicate bundle, shifted both horizontally and downwards, to intersect at a lower height z_-. Clearly, the spread of the joint bundle at heights $z(\lambda = 0) = z_-$ and $z(\lambda = 1) = z_+$ are the same. And the spread at $z(\lambda = 1/2)$ is somewhat larger (note that the shape of $d(\lambda)$ is depicted sideways to the right). Thus $\lambda = 0, 1$ present two minima of $d(\lambda)$, and the hourglassing procedure in this case cannot provide a clear answer.

At least we can compute the exact form of $d(\lambda)$, and with standard techniques understand clearly whether such a degenerate situation were ever to present itself. (TBD: In our empirical testing in section 3, we [never/seldom] encountered such a degenerate case, in NNNN hourglassing computations involving from 4 to 1000 images.)

2.2.4 Estimated horizontal error

Following section 2.1, we seek to provide a meaningful estimate of error that is informed by the central limit theorem's general principle of uncertainty that decreases by a factor of $\sqrt{N}$, and thus grows arbitrarily small as the number of images N increases.

We chose as horizontal location for our point, the mean $(\bar{x}(\lambda_{min}), \bar{y}(\lambda_{min}))$ of spread-minimizing λ_{min}. If we consider the true answer $(\hat{x}, \hat{y})$ to be the goal we are attempting to measure with a sample of N possible imaging rays out of an infinity of possibilities, then we intuit that the estimator $(\bar{x}(\lambda_{min}), \bar{y}(\lambda_{min}))$ should be proportional to $\sqrt{N}a(\lambda_{min})$. In section 3 we evaluate this measure of accuracy using a simulation with known ground truth (and find it's wickid awesome).

We also want to verify that our estimator for horizontal error responds properly to the width of the ray bundle. Wider bundles should yield larger horizontal error, and narrower bundles should yield smaller horizontal error.

2.2.5 Estimated vertical error

For vertical error, an estimator for the empirical technique of hourglassing does not readily present itself. Apart from matching reality in simulations with ground truth, and behaving comparably to MIG, the estimator needs to have the fundamental property that, for a wider bundle must have a lower estimate of vertical error, and a narrower bundle must have a higher estimate of vertical error.

Consider the (continuous? smooth?) surface formed by stacking the 1-sigma error ellipse for every height $z(\lambda)$ (for $\lambda \in \Re$). Its intersection with a horizontal plane of height $z(\lambda)$ is an ellipse centered on $(\bar{x}(\lambda), \bar{y}(\lambda), z(\lambda)$, with area $a(\lambda)$, and major and minor axes with direction and length defined by the eigenvectors and square-roots of eigenvalues of $M(\lambda)$. (Can we somehow generate a visualization of this from a concrete example?)

Narrowness of this roughly conical surface corresponds to a low amount of curvature of $a(\lambda)$, and a shallow minimum which yields uncertainty as to which height plane is really the best estimate. Wideness corresponds to sharp curvature of $a(\lambda)$ and more certainty that the chosen height is close to correct.

Sharpness of curvature can be measured by a large 2nd derivative, so we can consider various measures that increase with $a''(\lambda_{min})$ or $d''(\lambda_{min})$.

However, we also want our estimate of vertical uncertainty to improve with N, and $a''(\lambda_{min})$ by itself will simply more accuracetly estimate the width of the whole bundle as N increases. So, similar to horizontal error above, we will evaluate combining a factor of $1/\sqrt{N}$ in our estimate to give it behavior consistent with the central limit theorem.

2.2.6 Weighting

So far, our development of the hourglassing heuristic has applied no weights, which is to say, all image rays have been weighted equally. If some source of information provides distinct weights to various image rays, it is possible to incorporate them.

Obviously, it is possible to compute $\bar{x}(\lambda), \bar{y}(\lambda)$ with a weighted average, which will pull the average closer to more heavily weighted image rays.

As for measuring spread, it is clear that if the weights are all integral, the spread calculation could be modified by including duplicates of each ray according to its weight, and the same method applies. The technique extends also to fractional weights. (adjusted formulas?)

3 Results with simulated sensor models

In this section we develop a large simulated testbed, and evaluate MIG, and hourglassing with various experimental measures for estimated error, on samples of randomly-selected subsets of images, of size ranging from 4 to 1000.

To develop the test set, we start with a large as possible set of real, unadjusted sensor models which have common overlap. From (**??**) we have 10 Worldview-1 sensor models which all view the ground location 36N 117.5W 1700mHAE (WGS84). This point is set as Ground Truth.

We then extend the set of real sensor models by random perturbation. We repeatedly add random amounts of correction to position and orientation adjustable parameters. The random corrections are sampled from uniform distributions with bounds many times the nominal triangulation defaults, with the aim of spreading the simulated sensors as uniformly as possible, minimizing clustering around the original sensor models. (See resulting clustering in Fig **??**). Most large random perturbations of this kind will steer the real sensor models away from viewing our truth point. Sensor models that still contain the truth point within their image bounds are retained, until we have 99 perturbations for each original, real sensor model, for a total of 1000 sensor models.

The simulated set of 1000 images thus constructed is considered the Truth set of sensor models, with position and orientation parameters that perfectly represent conditions at image collection, to which actual sensor models would be imperfect estimates. The sensor model's ground to image function is used to project the truth point into image coordinates for each sensor model. These image measurements are retained as truth as well. The ray bundle emanating from these truth image points intersect perfectly (to within computational precision of the image to ground function) at the truth point in ground space. Thus the MIG and Hourglassing approaches would both yield the ground point, for 1000 images or for any subset.

Once the idealized bundle of 1000 image rays is assembled, experiments can be conducted by adding a controlled amount of error (randomly sampled from a known distribution) to each sensor model. The 1000 sensor models thus perturbed represent a possible realization of 1000 images with position and orientation parameters being different from their actual truth. The idealized image measurements can be used (because that is where the visualization of the ground feature actually appears), a controlled amount of error can be added in image space (to represent a desired amount of image measurement error, or unmodeled sensor error). Using perturbed image measurements/sensor models, the resulting ray bundle will represent a realistic spread of image rays around the truth point, to which MIG and Hourglassing algorithms can be applied and evaluated. This procedure can be carried out for any subset of N images, in fact for any number of N-image samples.

Lots of tables and figures.

4 Results with real images

If we can get enough, like 50+.

5 Bibliography