1 Introduction

We're hourglassin'.

2 Theory

2.1 MIG

Use least-squares to show $\lim_{n\to\inf} err = 0$.

2.2 Hourglassin'

Assume two heights $z_+ > z_-$, and assume a set of N 3D lines L_i, none of which is horizontal. Specify the lines by their intersections with the planes of heights $z_\pm$ at points (x_+^i, y_+^i, z_+) and (x_-^i, y_-^i, z_-), for $i = 1 \ldots N$.

Define

$$x^i(\lambda) = \lambda x_+^i + (1-\lambda)x_-^i$$
$$y^i(\lambda) = \lambda y_+^i + (1-\lambda)y_-^i$$
$$z(\lambda) = \lambda z_- + (1-\lambda)z_-$$

For any λ, $(x^i(\lambda), y^i(\lambda), z(\lambda))$ is a point on line L_i; whether $0 \leq \lambda \leq 1$ determines whether the point is an interpolation between or extrapolation beyond the $z_\pm$ planes (in either case, we will just use the term interpolated). Together, all the points $(x^i(\lambda), y^i(\lambda), z(\lambda))$ for $i = 1 \ldots N$ represent the intersection of lines L_i with the plane with height $z(\lambda)$.

The two-dimensional spread of the intersection set at a particular $z(\lambda)$ is a symmetric, positive definite, 2x2 covariance matrix

$$M(\lambda) = \begin{bmatrix} var(x^i(\lambda)) & covar(x^i(\lambda), y^i(\lambda)) \\ - & var(y^i(\lambda)) \end{bmatrix}$$

It can be shown that the covariance of the interpolalated points $(x^i(\lambda), y^i(\lambda))$ can be expressed in terms of variances of and covariances between the four elementary datasets $x_+^i, x_-^i, y_+^i, y_-^i$ as follows:

$$var(x^i(\lambda)) = \lambda^2\sigma^2_{x+x+} + \lambda(1-\lambda)(\sigma^2_{x+x-} + \sigma^2_{x-x+}) + (1-\lambda)^2\sigma^2_{x-x-}$$
$$var(y^i(\lambda)) = \lambda^2\sigma^2_{y+y+} + \lambda(1-\lambda)(\sigma^2_{y+y-} + \sigma^2_{y-y+}) + (1-\lambda)^2\sigma^2_{y-y-}$$
$$covar(x^i(\lambda), y^i(\lambda)) = \lambda^2\sigma^2_{x+y+} + \lambda(1-\lambda)(\sigma^2_{x+y-} + \sigma^2_{x-y+}) + (1-\lambda)^2\sigma^2_{x-y-}$$

where

$$\sigma^2_{x+x+} = var(x_+^i),$$
$$\sigma^2_{x+y-} = covar(x_+^i, y_-^i),$$

etc. Note in particular that, since all the σ^2 are functions only of constants $x^i_\pm, y^i_\pm$, and N, each of the 4 terms of $M(\lambda)$ is a quadratic function of λ.

If we denote the determinant of a matrix with $|\cdot|$, the area of the 1-sigma error ellipse of $M(\lambda)$ is

$$a(\lambda) = \pi\sqrt{|M(\lambda)|},$$

i.e. the square root of a quartic (4th degree) polynomial.

We seek to minimize $a(\lambda)$, which is equivalent to minimizing the quartic polynomial

$$|M(\lambda)| = var(x^i(\lambda))var(y^i(\lambda) - covar(x^i(\lambda), y^i(\lambda))^2$$

For that polynomial to have no local minima except for a unique global minimum, it is sufficient to show that it's derivative, a cubic polynomial, has one real and two complex roots.