Extended Kalman Filters

Slide credits: Sebastian Thrun, Wolfram Burgard, Dieter Fox, and others

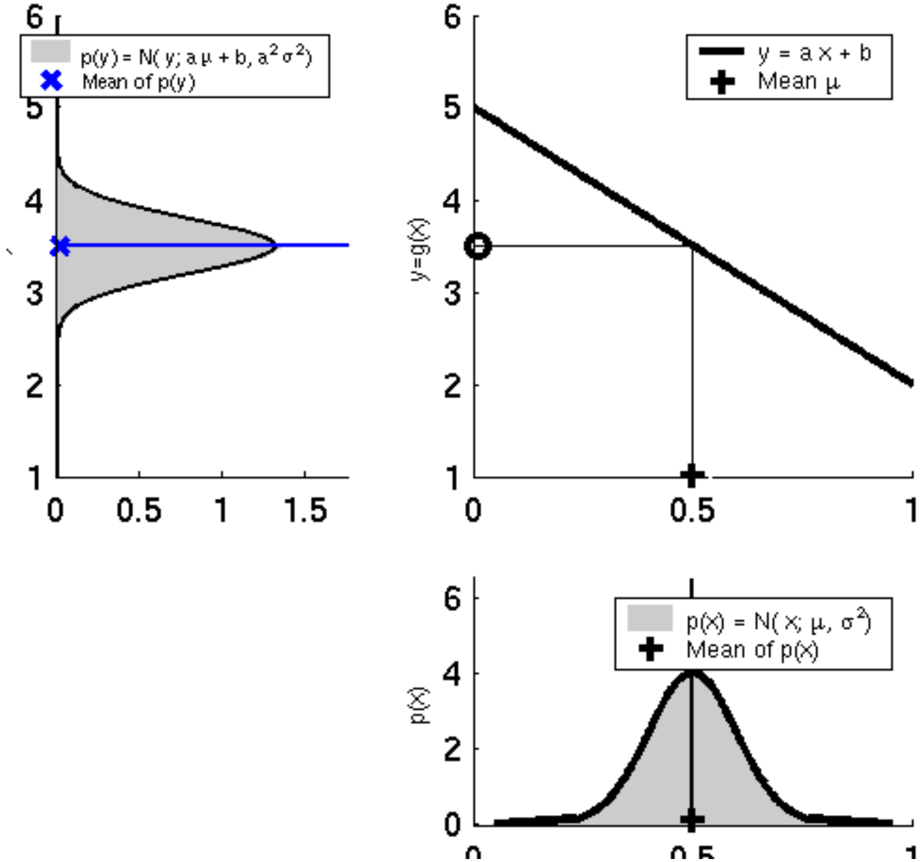
Nonlinear Dynamic Systems

- Most realistic robotic problems involve nonlinear functions

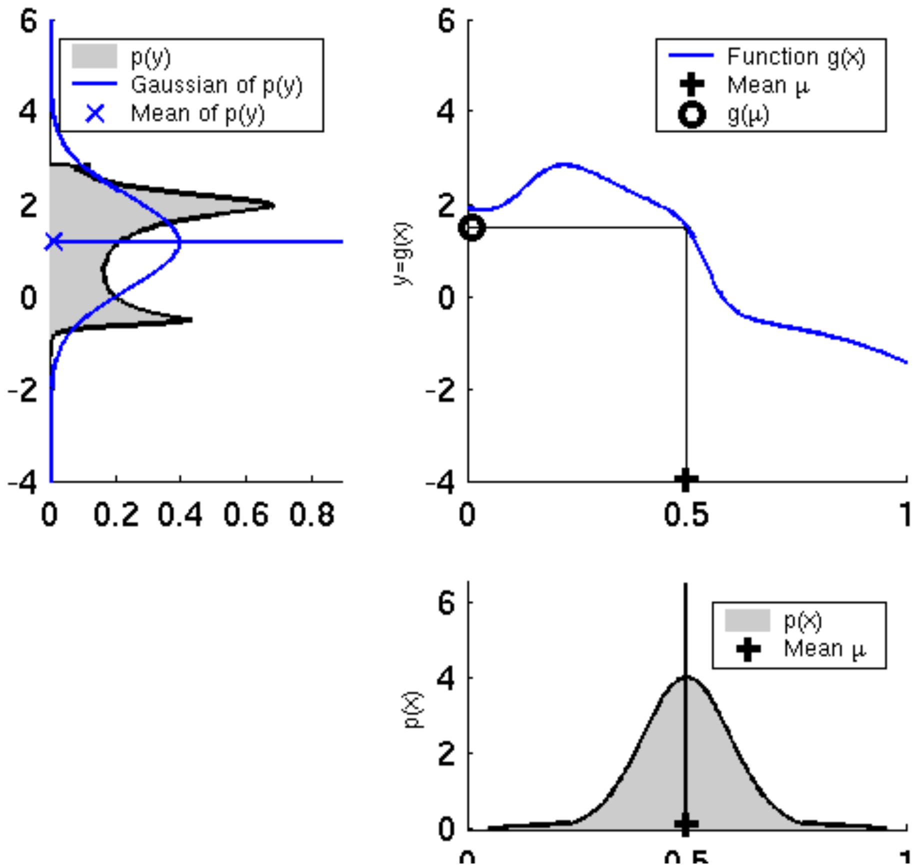
$$\boxed{x_t = g(u_t, x_{t-1})}$$

$$\boxed{z_t = h(x_t)}$$

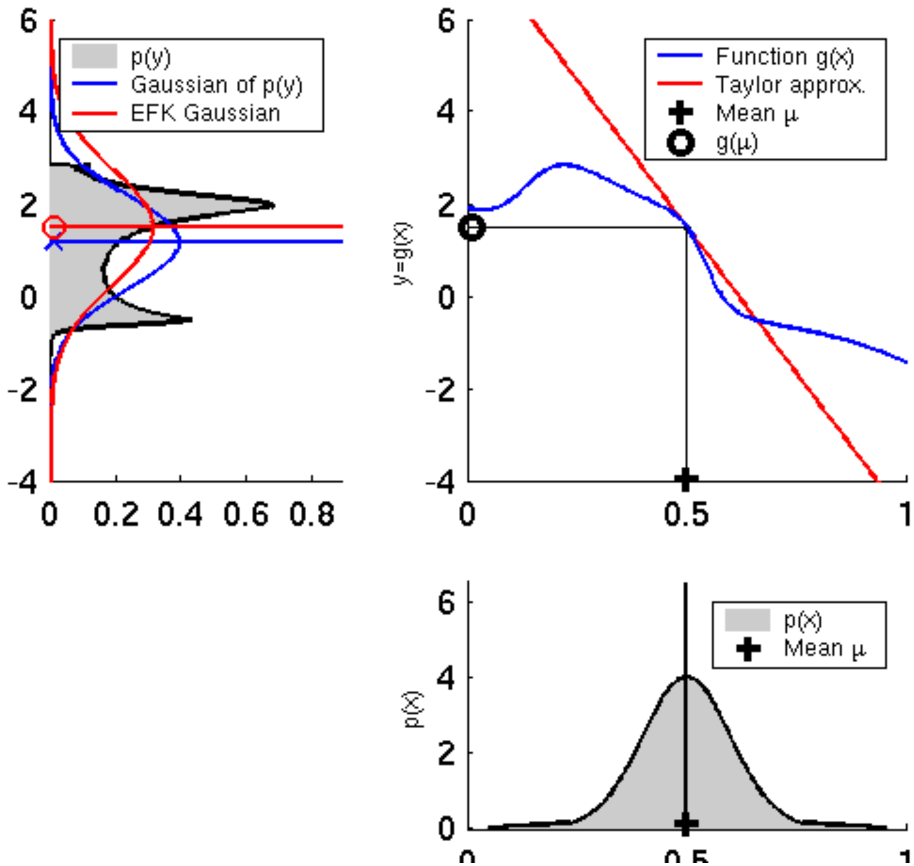
Linearity Assumption Revisited



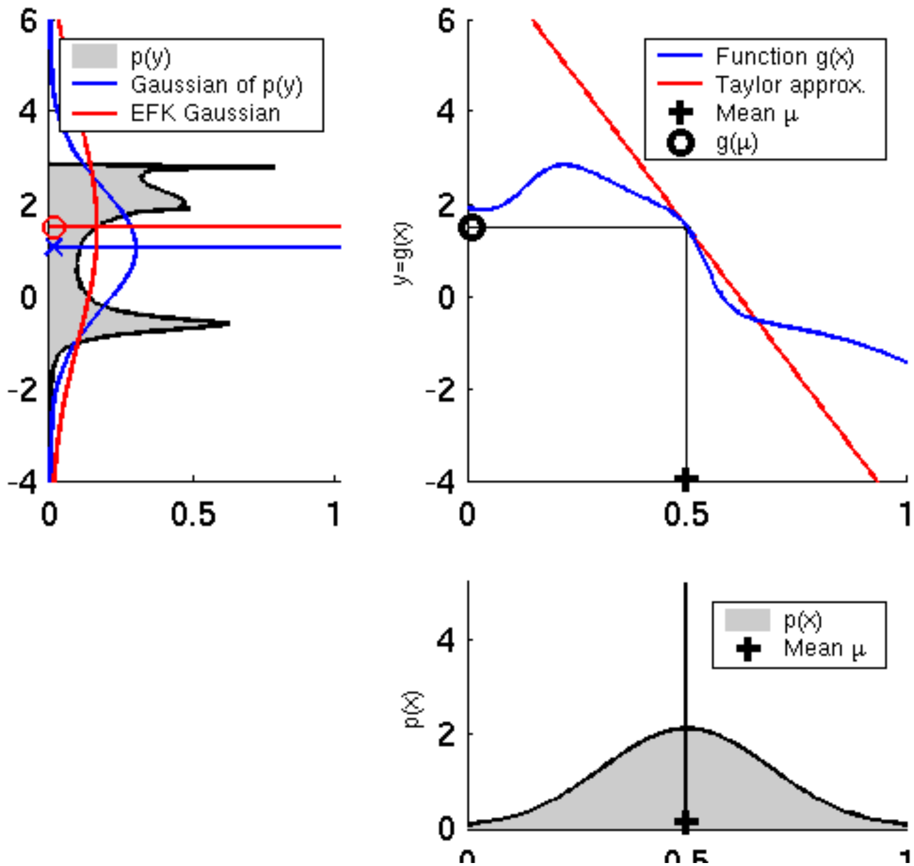
Non-linear Function



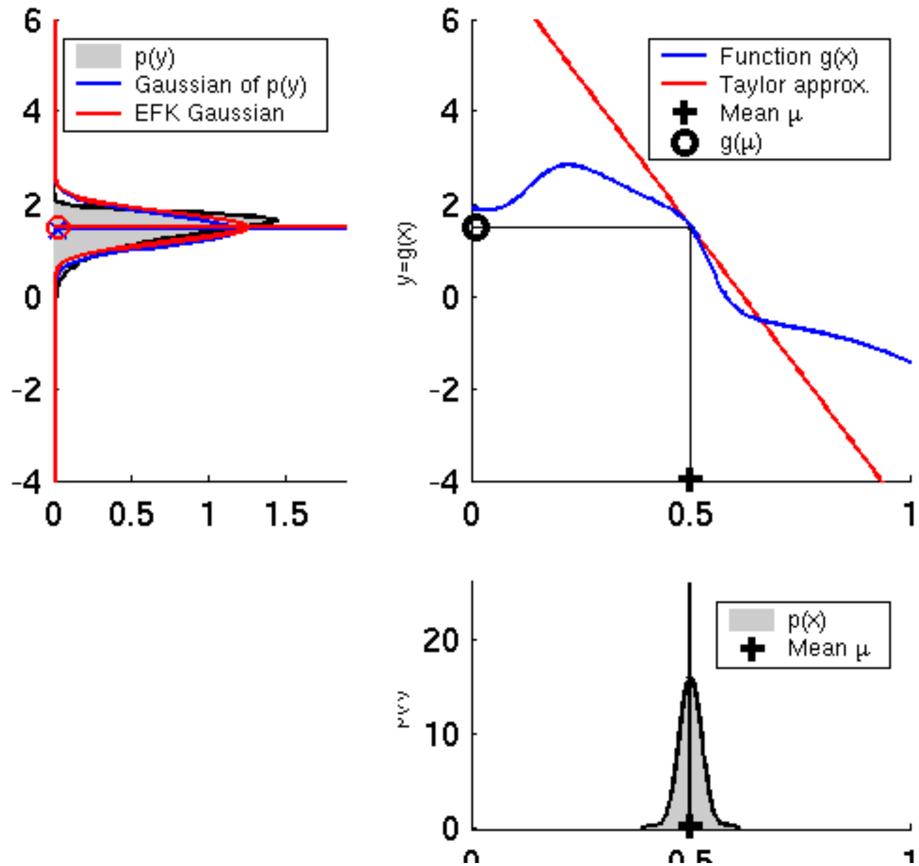
EKF Linearization (1)



EKF Linearization (2)



EKF Linearization (3)



EKF Linearization: First Order Taylor Series Expansion

- Prediction:

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}} (x_{t-1} - \mu_{t-1})$$

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + G_t \ (x_{t-1} - \mu_{t-1})$$

- Correction:

$$h(x_t) \approx h(\bar{\mu}_t) + \frac{\partial h(\bar{\mu}_t)}{\partial x_t} (x_t - \bar{\mu}_t)$$

$$h(x_t) \approx h(\bar{\mu}_t) + H_t \ (x_t - \bar{\mu}_t)$$

EKF Algorithm

1. **Extended_Kalman_filter**($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$):

2. Prediction:

3. $\bar{\mu}_t = g(u_t, \mu_{t-1})$ $\longleftarrow$ $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$

4. $\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + R_t$ $\longleftarrow$ $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$

5. Correction:

6. $K_t = \bar{\Sigma}_t H_t^T (H_t \bar{\Sigma}_t H_t^T + Q_t)^{-1}$ $\longleftarrow$ $K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$

7. $\mu_t = \bar{\mu}_t + K_t (z_t - h(\bar{\mu}_t))$ $\longleftarrow$ $\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$

8. $\Sigma_t = (I - K_t H_t) \bar{\Sigma}_t$ $\longleftarrow$ $\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$

9. Return μ_t, Σ_t

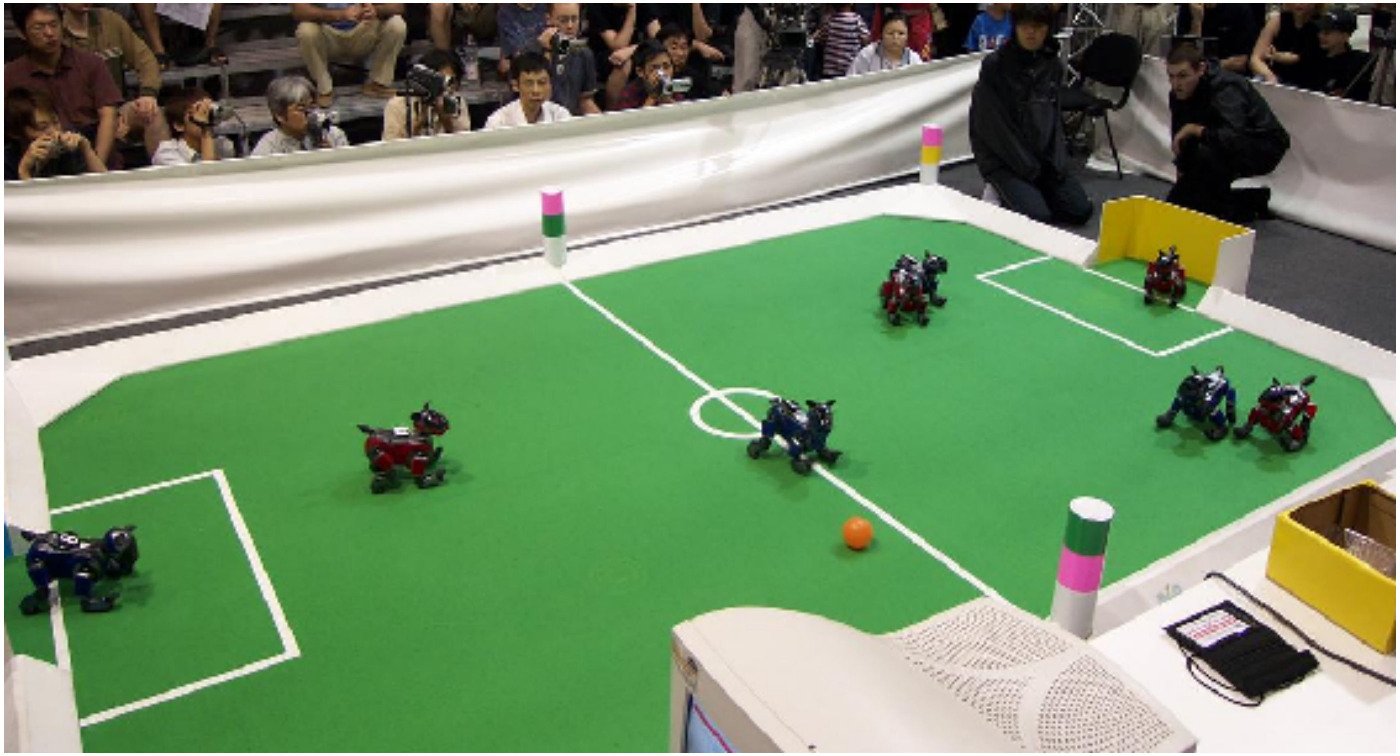
$$H_t = \frac{\partial h(\bar{\mu}_t)}{\partial x_t} \qquad G_t = \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}}$$

Localization

"Using sensory information to locate the robot in its environment is the most fundamental problem to providing a mobile robot with autonomous capabilities." [Cox '91]

- **Given**
  - Map of the environment
  - Sequence of sensor measurements
- **Wanted**
  - Estimate of the robot's position
- **Problem classes**
  - Position tracking
  - Global localization
  - Kidnapped robot problem (recovery)

Landmark-based Localization



EKF_localization ($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, m$):

Prediction:

2. $$G_t = \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}} = \begin{pmatrix} \frac{\partial x'}{\partial \mu_{t-1,x}} & \frac{\partial x'}{\partial \mu_{t-1,y}} & \frac{\partial x'}{\partial \mu_{t-1,\theta}} \\ \frac{\partial y'}{\partial \mu_{t-1,x}} & \frac{\partial y'}{\partial \mu_{t-1,y}} & \frac{\partial y'}{\partial \mu_{t-1,\theta}} \\ \frac{\partial \theta'}{\partial \mu_{t-1,x}} & \frac{\partial \theta'}{\partial \mu_{t-1,y}} & \frac{\partial \theta'}{\partial \mu_{t-1,\theta}} \end{pmatrix}$$ Jacobian of *g* w.r.t location

4. $$V_t = \frac{\partial g(u_t, \mu_{t-1})}{\partial u_t} = \begin{pmatrix} \frac{\partial x'}{\partial v_t} & \frac{\partial x'}{\partial \omega_t} \\ \frac{\partial y'}{\partial v_t} & \frac{\partial y'}{\partial \omega_t} \\ \frac{\partial \theta'}{\partial v_t} & \frac{\partial \theta'}{\partial \omega_t} \end{pmatrix}$$ Jacobian of *g* w.r.t control

5. $$M_t = \begin{pmatrix} (\alpha_1 |v_t| + \alpha_2 |\omega_t|)^2 & 0 \\ 0 & (\alpha_3 |v_t| + \alpha_4 |\omega_t|)^2 \end{pmatrix}$$ Motion noise

6. $\bar{\mu}_t = g(u_t, \mu_{t-1})$ Predicted mean

7. $\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + V_t M_t V_t^T$ Predicted covariance

EKF_localization ($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, m$):

Correction:

2. $$\hat{z}_t = \begin{pmatrix} \sqrt{(m_x - \bar{\mu}_{t,x})^2 + (m_y - \bar{\mu}_{t,y})^2} \\ \operatorname{atan2}(m_y - \bar{\mu}_{t,y}, m_x - \bar{\mu}_{t,x}) - \bar{\mu}_{t,\theta} \end{pmatrix}$$ Predicted measurement mean

4. $$H_t = \frac{\partial h(\bar{\mu}_t, m)}{\partial x_t} = \begin{pmatrix} \frac{\partial r_t}{\partial \bar{\mu}_{t,x}} & \frac{\partial r_t}{\partial \bar{\mu}_{t,y}} & \frac{\partial r_t}{\partial \bar{\mu}_{t,\theta}} \\ \frac{\partial \varphi_t}{\partial \bar{\mu}_{t,x}} & \frac{\partial \varphi_t}{\partial \bar{\mu}_{t,y}} & \frac{\partial \varphi_t}{\partial \bar{\mu}_{t,\theta}} \end{pmatrix}$$ Jacobian of *h* w.r.t location

5. $$Q_t = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_r^2 \end{pmatrix}$$

6. $$S_t = H_t \bar{\Sigma}_t H_t^T + Q_t$$ Pred. measurement covariance

7. $$K_t = \bar{\Sigma}_t H_t^T S_t^{-1}$$ Kalman gain

8. $$\mu_t = \bar{\mu}_t + K_t (z_t - \hat{z}_t)$$ Updated mean

9. $$\Sigma_t = (I - K_t H_t) \bar{\Sigma}_t$$ Updated covariance

EKF Localization

μ_t = estimate of $(x_t, y_t, \theta_t)^T$

$u_t = (v_t, \omega_t)^T$

1: **Algorithm EKF_localization_known_correspondences(**$\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, c_t, m$**):**

prediction

2: $\theta = \mu_{t-1,\theta}$ state estimate from prior time

3: $G_t = \begin{pmatrix} 1 & 0 & -\frac{v_t}{\omega_t}\cos\theta + \frac{v_t}{\omega_t}\cos(\theta + \omega_t \Delta t) \\ 0 & 1 & -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta + \omega_t \Delta t) \\ 0 & 0 & 1 \end{pmatrix}$ Jacobian of $g(u_t, x_{t-1})$ wrt state

4: $V_t = \begin{pmatrix} \frac{-\sin\theta + \sin(\theta+\omega_t\Delta t)}{\omega_t} & \frac{v_t(\sin\theta - \sin(\theta+\omega_t\Delta t))}{\omega_t^2} + \frac{v_t\cos(\theta+\omega_t\Delta t)\Delta t}{\omega_t} \\ \frac{\cos\theta - \cos(\theta+\omega_t\Delta t)}{\omega_t} & -\frac{v_t(\cos\theta - \cos(\theta+\omega_t\Delta t))}{\omega_t^2} + \frac{v_t\sin(\theta+\omega_t\Delta t)\Delta t}{\omega_t} \\ 0 & \Delta t \end{pmatrix}$

Jacobian of $g(u_t, x_{t-1})$ wrt inputs

5: $M_t = \begin{pmatrix} \alpha_1 v_t^2 + \alpha_2 \omega_t^2 & 0 \\ 0 & \alpha_3 v_t^2 + \alpha_4 \omega_t^2 \end{pmatrix}$

noise in control space

6: $\bar{\mu}_t = \mu_{t-1} + \begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta + \omega_t \Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\cos(\theta + \omega_t \Delta t) \\ \omega_t \Delta t \end{pmatrix}$ $\bar{\mu}_t = g(u_t, \mu_{t-1})$

7: $\bar{\Sigma}_t = G_t\ \Sigma_{t-1}\ G_t{}^T + V_t\ M_t\ V_t{}^T$ uncertainty in states due to motion noise

uncertainty in states at time $(t-1)$

8: $Q_t = \begin{pmatrix} \sigma_r^2 & 0 & 0 \\ 0 & \sigma_\phi^2 & 0 \\ 0 & 0 & \sigma_s^2 \end{pmatrix}$ uncertainty due to measurement noise

EKF Localization

s_t^i is signature of i^{th} landmark (we won't estimate this)

correction

for all observed features $z_t^i = (r_t^i \;\; \phi_t^i \;\; \cancel{s_t^i})^T$ *do*

$j = c_t^i$ — i^{th} feature at time t corresponds to j^{th} landmark in map

$q = (m_{j,x} - \bar{\mu}_{t,x})^2 + (m_{j,y} - \bar{\mu}_{t,y})^2$ — range squared (based on estimate)

$$\hat{z}_t^i = \begin{pmatrix} \sqrt{q} \\ \operatorname{atan2}(m_{j,y} - \bar{\mu}_{t,y}, m_{j,x} - \bar{\mu}_{t,x}) - \bar{\mu}_{t,\theta} \\ \cancel{m_{j,s}} \end{pmatrix}$$

estimate of measurement (range and bearing)

$$H_t^i = \begin{pmatrix} -\frac{m_{j,x}-\bar{\mu}_{t,x}}{\sqrt{q}} & -\frac{m_{j,y}-\bar{\mu}_{t,y}}{\sqrt{q}} & 0 \\ \frac{m_{j,y}-\bar{\mu}_{t,y}}{q} & -\frac{m_{j,x}-\bar{\mu}_{t,x}}{q} & -1 \\ \cancel{0} & \cancel{0} & \cancel{0} \end{pmatrix}$$

Jacobian of $h(x_t, j, m)$

uncertainty in $\hat{z}_t$

$S_t^i = H_t^i \, \bar{\Sigma}_t \, [H_t^i]^T + Q_t$

uncertainty due to measurement noise

uncertainty in measurement due to uncertainty in robot state

Kalman gain

$K_t^i = \bar{\Sigma}_t \, [H_t^i]^T [S_t^i]^{-1}$

$\bar{\mu}_t = \bar{\mu}_t + K_t^i (z_t^i - \hat{z}_t^i)$

Kalman gain maps innovation in measurement space into state space

$\bar{\Sigma}_t = (I - K_t^i \, H_t^i) \, \bar{\Sigma}_t$

Update state uncertainty based on measurement

endfor

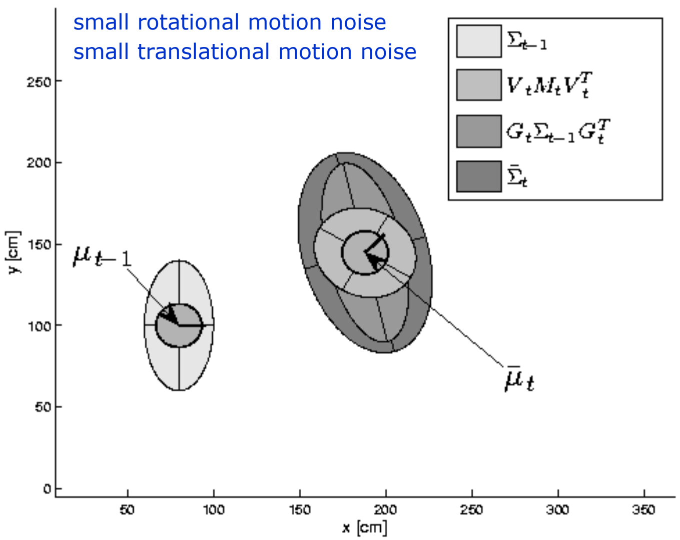
$\mu_t = \bar{\mu}_t$

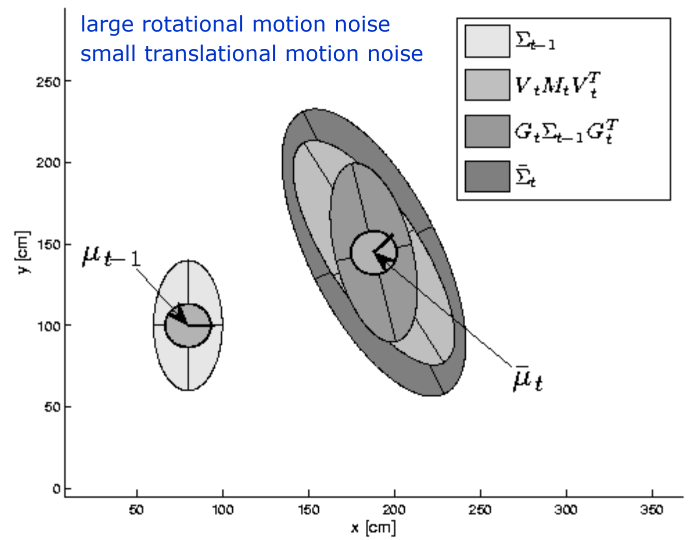
$\Sigma_t = \bar{\Sigma}_t$

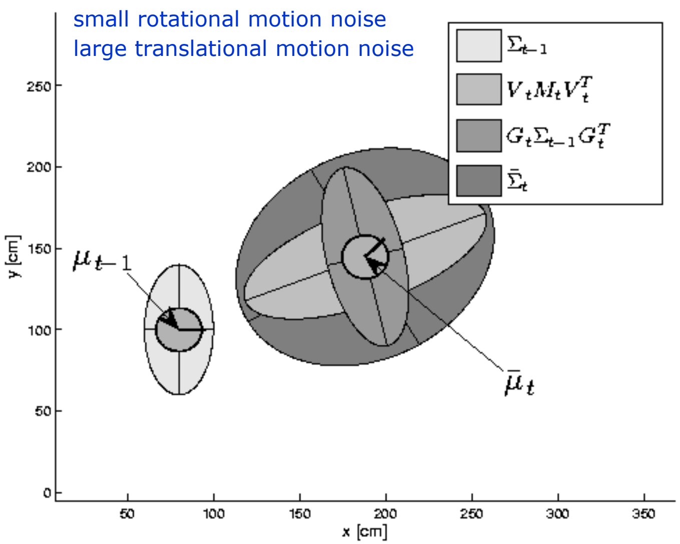
$p_{z_t} = \prod_i \det\left(2\pi S_t^i\right)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2} \, (z_t^i - \hat{z}_t^i)^T [S_t^i]^{-1} (z_t^i - \hat{z}_t^i)\right\}$

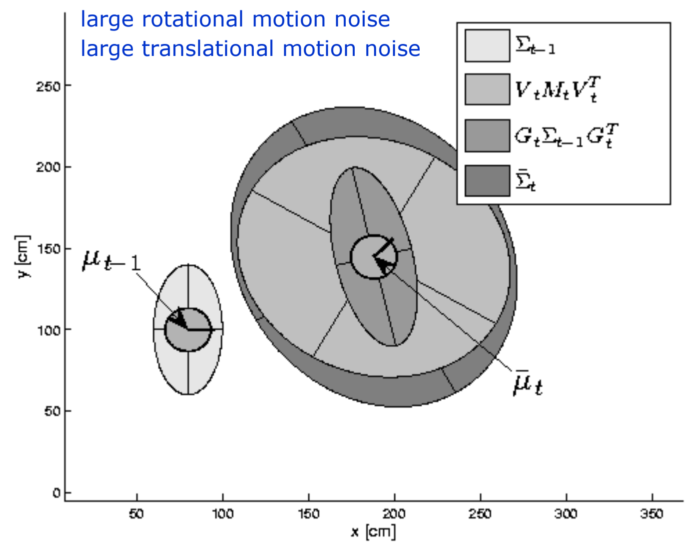
return μ_t, Σ_t, p_{z_t}

EKF Prediction Step

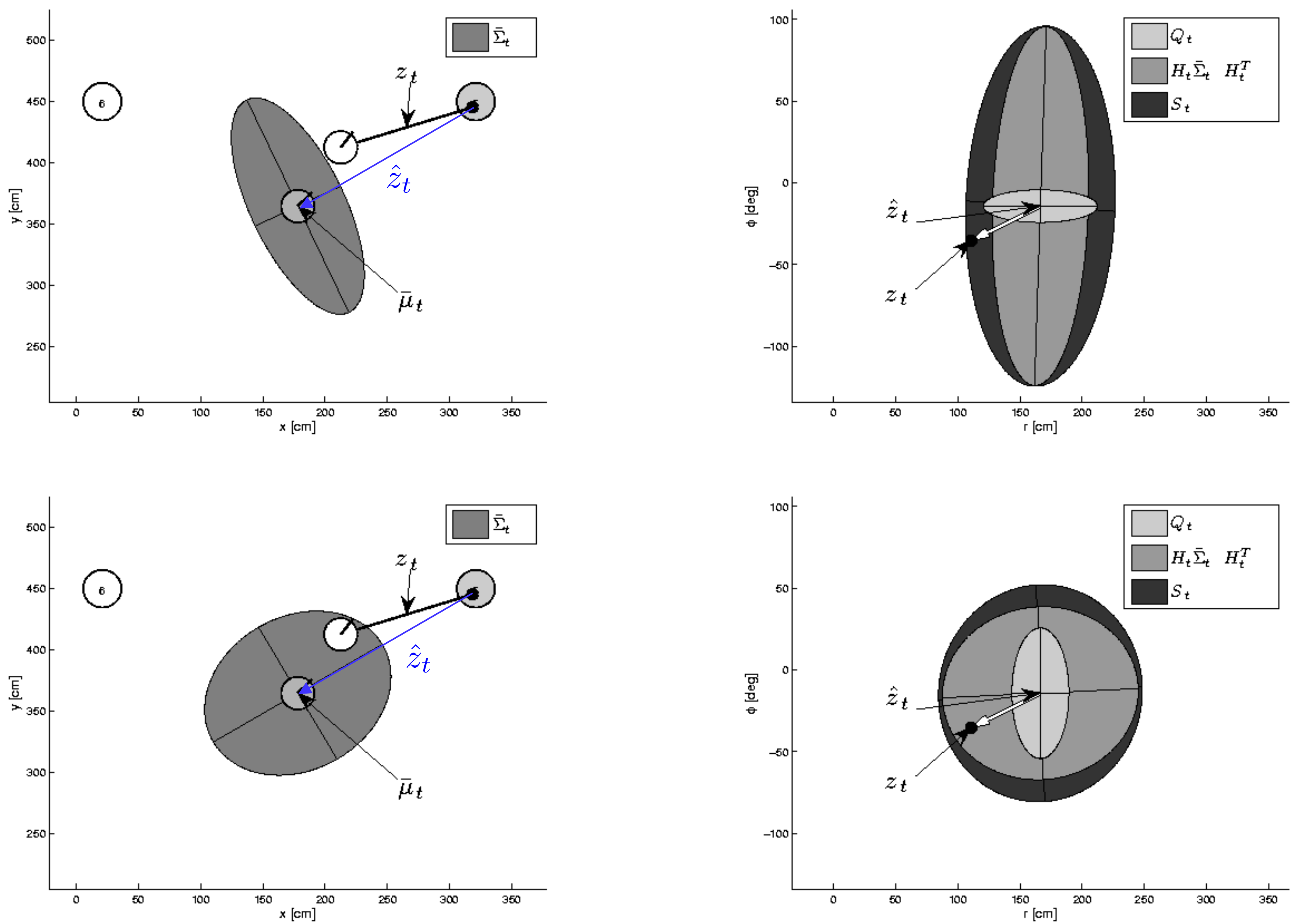








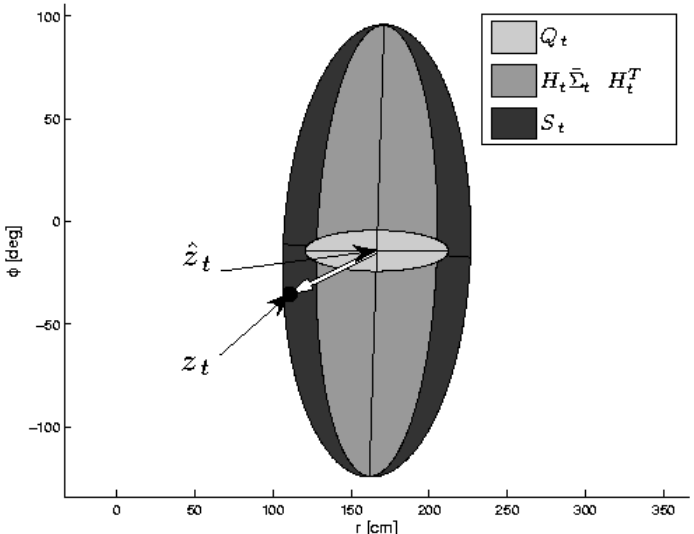
EKF Observation Prediction Step

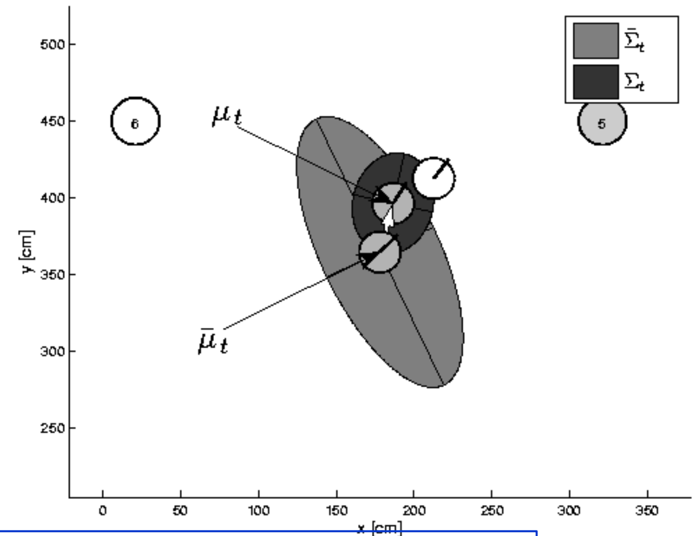


state space

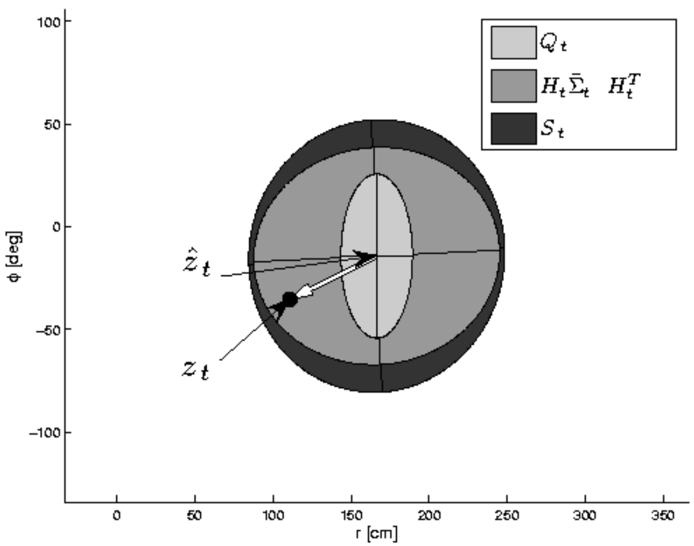
measurement space

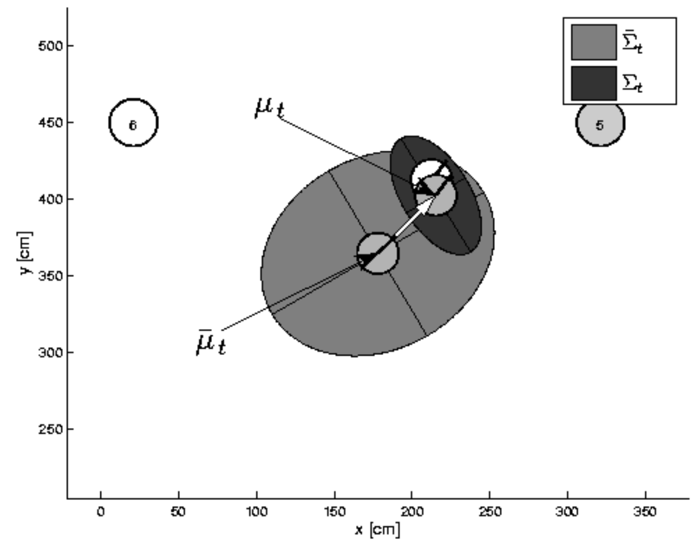
EKF Correction Step





Kalman gain maps innovation vector from measurement space to state space
The more likely the measurement the larger the correction step

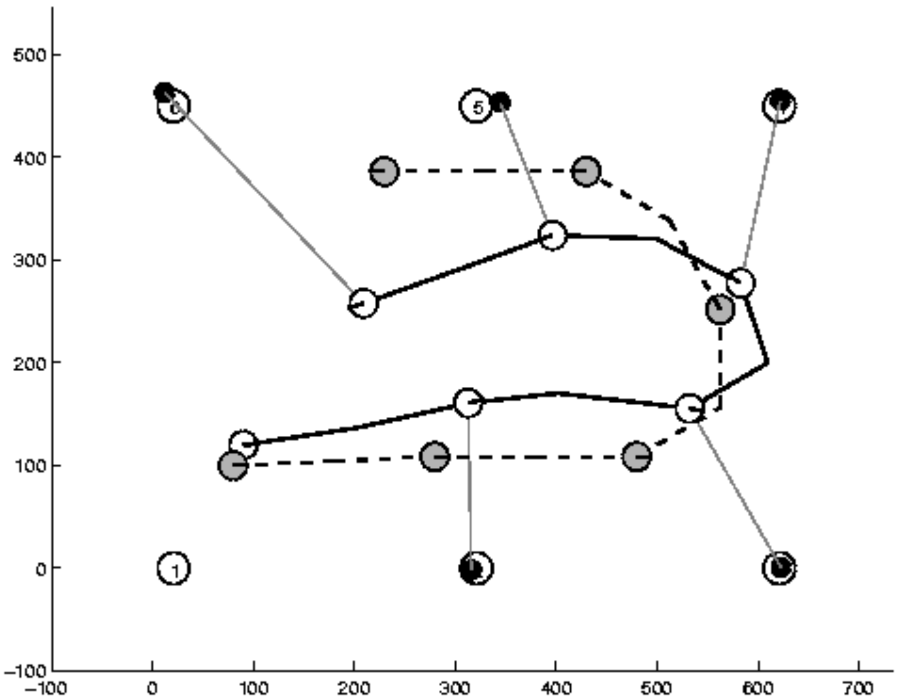


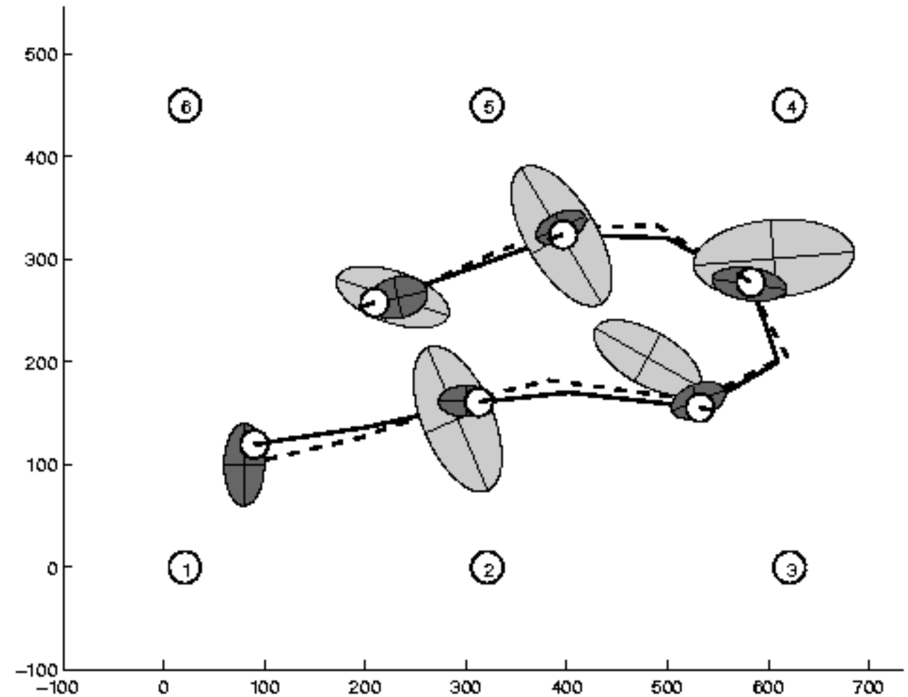


measurement space

state space

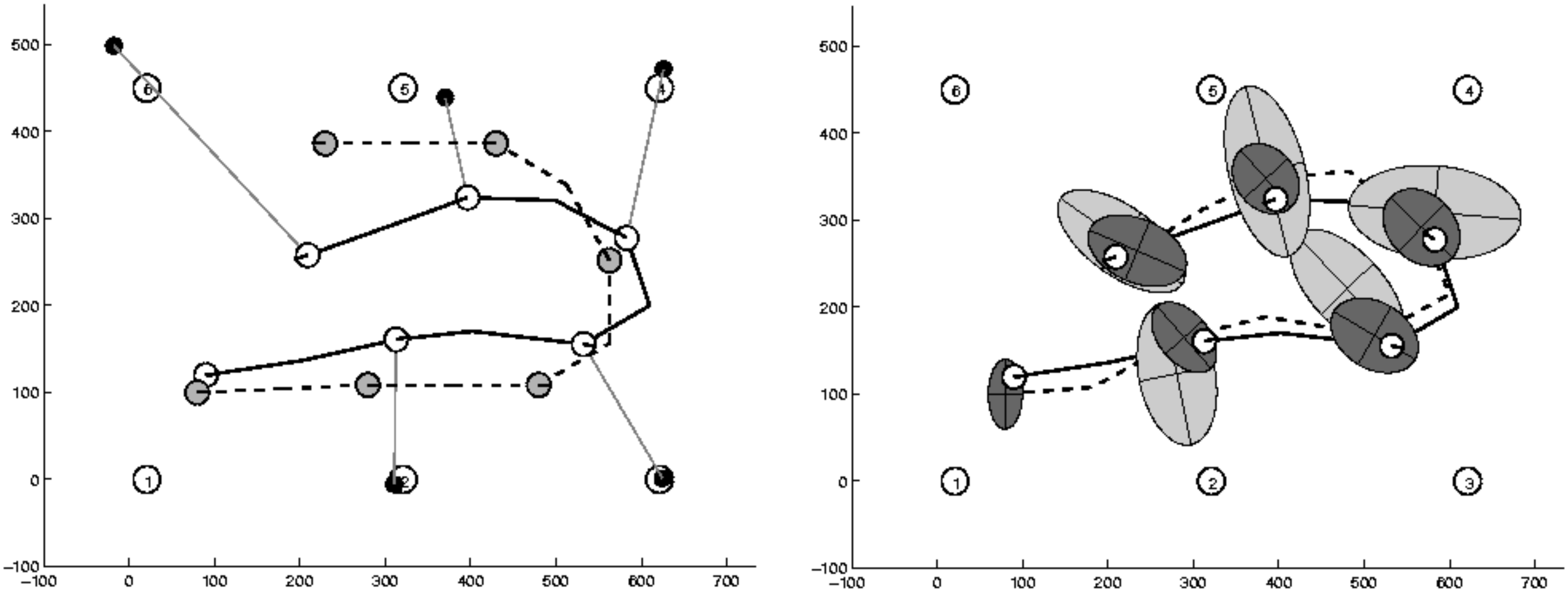
Estimation Sequence (1)





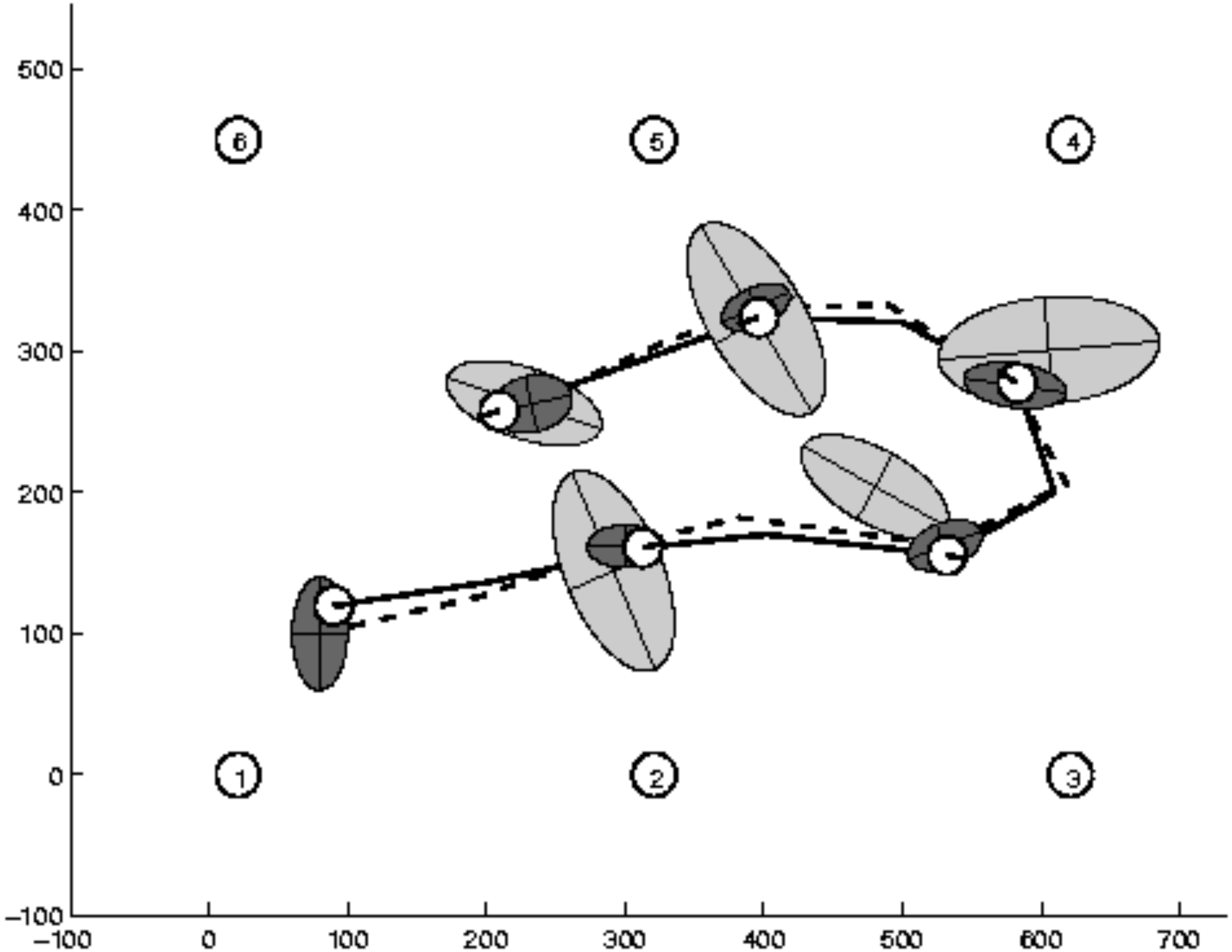
More accurate sequence

Estimation Sequence (2)



Less accurate sequence

Comparison to GroundTruth



EKF Summary

- Highly efficient: Polynomial in measurement dimensionality *k* and state dimensionality *n*:

$$O(k^{2.376} + n^2)$$

- Not optimal!
- Can diverge if nonlinearities are large!
- Works surprisingly well even when all assumptions are violated!

MATLAB IMPLEMENTATION

Modified measurement update

- Sometimes a well-implemented textbook EKF can struggle to estimate states well and it is necessary to apply the art and skill of design (aka hacks)
- One approach is to tune the filter by choosing noise covariance parameters that improve results
  - Often these are the process noise parameters since the plant model is often the source of greatest uncertainty
- Other methods are used to prevent the EKF from updating the state estimate too aggressively
- One method that has been necessary with the TurtleBot is to saturate the measurement residual

```
sat = 0.05;
residual = z - zhat;
residual = saturate(residual,-sat,sat);
update = K*residual;
mu = mu + update;
Sig = (eye(3) - K*H)*Sig;
```

- More rigorous approaches exist
  - E.g., Kevin M. Brink. "Partial-Update Schmidt–Kalman Filter", Journal of Guidance, Control, and Dynamics, Vol. 40, No. 9 (2017), pp. 2214-2228. https://doi.org/10.2514/1.G002808