

# An anisotropic diffusion approximation to nonlinear radiation transport

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# Outline

1 Introduction

2 Theory

3 Results

4 Conclusions

# Thermal radiative transfer

- TRT is the dominant heat transfer process in very hot materials
- Photons born isotropically via black body emission ( $q_{\text{rad}} \propto \sigma T^4$ )
- Cold material heats up and becomes relatively transparent ( $\sigma \propto T^{-3}$ )

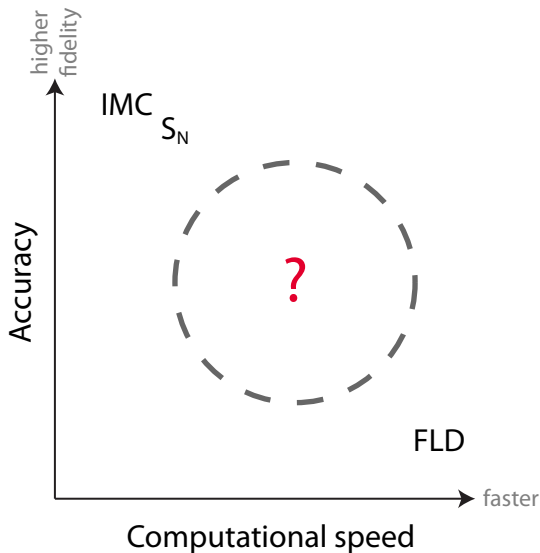
Difficulties in solving:

- High dimensionality of solution phase space ( $\mathbf{x}, \boldsymbol{\Omega}, h\nu, t$ )
- Highly nonlinear coupled partial differential equations for radiation field  $\psi(\mathbf{x}, \boldsymbol{\Omega}, h\nu, t)$  and material energy

Particular application of this work: CRASH project

- Center for RAdiative Shock Hydrodynamics program: “Assessment of Predictive Capability”
- Simulate laser-driven shock in a xenon-filled tube
- Uncertainty quantification: hundreds of solution instances needed

# Motivation



# Gray TRT equations

Common approximations for radiation transport methods development:

- work in a fixed medium, disregarding material advection;
- assume local thermodynamic equilibrium (LTE), which uses a single material temperature;
- neglect thermal conduction in material;
- average over all photon energies  $h\nu$  (gray).

Radiation transfer equation, angular flux  $\psi(\mathbf{x}, \boldsymbol{\Omega}, t)$ :

$$\frac{1}{c} \frac{\partial \psi}{\partial t} + \boldsymbol{\Omega} \cdot \nabla \psi + \sigma \psi = \frac{\sigma c a T^4}{4\pi} + \frac{cQ}{4\pi} \quad (1a)$$

Material energy balance equation:

$$\frac{1}{c_v} \frac{\partial T}{\partial t} = \sigma \int_{4\pi} \psi \, d\Omega - \sigma c a T^4 \quad (1b)$$

# Anisotropic diffusion

## Previous work:

- Steady-state infinite medium VHTR-like problem with analytically calculated coefficients [1]
- Non-local tensor diffusion [2] for steady-state radiative transfer, no further development or analysis in literature

## Current work:

- Formulates boundary conditions and time-dependent terms
- Uses transport-calculated anisotropic diffusion tensors
- Applies to nonlinear, time-dependent problems with isotropic sources

## Potential applications:

- Extends diffusion theory to new regimes of applicability
- Variance reduction with shielding problems that have voids

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- ① Define the anisotropic angular flux as  $\Psi = \psi - \frac{1}{4\pi}\phi$ . To handle boundary conditions, define  $\Psi \equiv \tilde{\Psi} + \Psi_{\text{bl}}$ . We will approximate  $\tilde{\Psi}$  rather than  $\psi$ , and use  $\Psi_{\text{bl}}$  to determine matched boundary conditions.
- ② From the radiation transport equation and conservation equation, we get a differential transport equation for  $\tilde{\Psi}$  and  $\Psi_{\text{bl}}$ . Transform the former to an *integral* transport equation for  $\tilde{\Psi}$ .
- ③ Assume  $\psi = O(1)$ ,  $\frac{1}{c} \frac{\partial}{\partial t} = O(\epsilon^2)$ ,  $\nabla = O(\epsilon)$ ,  $\int_{4\pi} \Omega(\cdot) d\Omega = O(\epsilon)$ .
- ④ Use Taylor series to approximate nonlocal unknowns with local unknowns, discarding small terms. This yields

$$\tilde{\Psi}(\mathbf{x}, \Omega) \approx -f(\mathbf{x}, \Omega) \Omega \cdot \nabla \phi.$$

- ⑤ Apply standard transport-matching procedure to  $\Psi_{\text{bl}}$ . Use the identity  $\int_{4\pi} \Psi d\Omega = 0$  to find the boundary condition for  $f$ .
- ⑥ Take the first angular moment of  $\tilde{\Psi}$  to get  $\mathbf{J} = -\mathbf{D} \cdot \nabla \phi$
- ⑦ Substitute  $\mathbf{J}$  into the time-dependent particle conservation equation to get time-dependent anisotropic diffusion.



# Transport equation

Inside a time step, with “frozen” cross sections (opacities):

$$\begin{aligned} \frac{1}{c} \frac{\partial \psi}{\partial t}(\mathbf{x}, \boldsymbol{\Omega}, t) + \boldsymbol{\Omega} \cdot \nabla \psi(\mathbf{x}, \boldsymbol{\Omega}, t) + \sigma^*(\mathbf{x}) \psi(\mathbf{x}, \boldsymbol{\Omega}, t) \\ = \frac{1}{4\pi} \sigma^*(\mathbf{x}) ac[T(\mathbf{x}, t)]^4 + \frac{1}{4\pi} q_r(\mathbf{x}, t) \equiv \frac{1}{4\pi} Q(\mathbf{x}, t), \\ x \in V, 0 \leq t < \Delta_t, \boldsymbol{\Omega} \in 4\pi, \end{aligned} \quad (2a)$$

with the boundary condition

$$\psi(\mathbf{x}, \boldsymbol{\Omega}, t) = \psi^b(\mathbf{x}, \boldsymbol{\Omega}, t), \quad \mathbf{x} \in \partial V, \boldsymbol{\Omega} \cdot \mathbf{n} < 0, 0 \leq t < \Delta_t \quad (2b)$$

and the initial condition

$$\psi(\mathbf{x}, \boldsymbol{\Omega}, 0) = \psi^i(\mathbf{x}, \boldsymbol{\Omega}, t), \quad \mathbf{x} \in V, \boldsymbol{\Omega} \in 4\pi. \quad (2c)$$

# Conservation equations

Operating on Eq. (2a) by  $\int_{4\pi}(\cdot) d\Omega$  gives

$$\frac{1}{c} \frac{\partial \phi}{\partial t}(\mathbf{x}, t) + \nabla \cdot \mathbf{J}(\mathbf{x}, t) + \sigma^* \phi(\mathbf{x}, t) = Q(\mathbf{x}, t). \quad (3a)$$

and on the initial condition, Eq. (2c),

$$\phi(\mathbf{x}, 0) = \int_{4\pi} \psi^i(\mathbf{x}, \Omega) d\Omega = \phi^i(\mathbf{x}). \quad (3b)$$

Add  $\Omega \cdot \nabla \phi$  to both sides of Eq. (3a) and multiply by  $\frac{1}{4\pi}$ :

$$\frac{1}{4\pi} \frac{1}{c} \frac{\partial \phi}{\partial t} + \frac{1}{4\pi} \Omega \cdot \nabla \phi + \frac{1}{4\pi} \sigma^* \phi = \frac{1}{4\pi} Q(\mathbf{x}, t) + \frac{1}{4\pi} \Omega \cdot \nabla \phi - \frac{1}{4\pi} \nabla \cdot \mathbf{J} \quad (4)$$

# Anisotropic angular flux equations

Define “anisotropic angular flux”:

$$\Psi(\mathbf{x}, \boldsymbol{\Omega}) \equiv \psi(\mathbf{x}, \boldsymbol{\Omega}) - \frac{1}{4\pi} \phi(\mathbf{x}). \quad (5)$$

(This satisfies  $\int_{4\pi} \Psi = 0$  and  $\int_{4\pi} \boldsymbol{\Omega} \Psi = \mathbf{J}$ .)

Subtract Eq. (4) from Eq. (2a); the isotropic source cancels:

$$\frac{1}{c} \frac{\partial}{\partial t} \left[ \psi - \frac{\phi}{4\pi} \right] + \boldsymbol{\Omega} \cdot \boldsymbol{\nabla} \left[ \psi - \frac{\phi}{4\pi} \right] + \sigma^*(\mathbf{x}) \left[ \psi - \frac{\phi}{4\pi} \right] = \frac{1}{4\pi} \boldsymbol{\nabla} \cdot \mathbf{J} - \frac{1}{4\pi} \boldsymbol{\Omega} \cdot \boldsymbol{\nabla} \phi$$

Subtract  $\phi/4\pi$  from the transport boundary condition:

$$\psi - \frac{\phi}{4\pi} = \psi^b - \frac{\phi}{4\pi}$$

Subtract Eq. (3b) from Eq. (2c):

$$\psi(\mathbf{x}, \boldsymbol{\Omega}, 0) - \frac{1}{4\pi} \phi(\mathbf{x}, 0) = \psi^i - \frac{\phi^i}{4\pi}$$

# Anisotropic angular flux equations

Transport equation:

$$\frac{1}{c} \frac{\partial}{\partial t} \Psi + \boldsymbol{\Omega} \cdot \boldsymbol{\nabla} \Psi + \sigma^*(\mathbf{x}) \Psi = \frac{1}{4\pi} \boldsymbol{\nabla} \cdot \mathbf{J} - \frac{1}{4\pi} \boldsymbol{\Omega} \cdot \boldsymbol{\nabla} \phi \equiv \hat{Q}(\mathbf{x}, \boldsymbol{\Omega}, t)$$

Boundary condition:

$$\Psi = \Psi^b = \psi^b - \frac{\phi}{4\pi}$$

Initial condition:

$$\Psi(\mathbf{x}, \boldsymbol{\Omega}, 0) = \Psi^i = \psi^i - \frac{\phi^i}{4\pi}.$$

The exact solutions for  $\psi$ ,  $\phi$ ,  $\mathbf{J}$  satisfy these equations: still no approximations.

# Boundary layer equations

In anticipation of approximating  $\tilde{\Psi} = -f\mathbf{\Omega} \cdot \nabla\phi$ , separate  $\Psi$  into a boundary layer plus an internal solution:

$$\Psi(\mathbf{x}, \mathbf{\Omega}, t) \equiv \tilde{\Psi}(\mathbf{x}, \mathbf{\Omega}, t) + \Psi_{\text{bl}}(\mathbf{x}, \mathbf{\Omega}, t).$$

The exact equations for  $\tilde{\Psi}$ :

$$\frac{1}{c} \frac{\partial}{\partial t} \tilde{\Psi} + \mathbf{\Omega} \cdot \nabla \tilde{\Psi} + \sigma^*(\mathbf{x}) \tilde{\Psi} = \hat{Q}(\mathbf{x}, \mathbf{\Omega}, t)$$

with new boundary condition for  $\mathbf{x} \in \partial V$ ,  $\mathbf{\Omega} \cdot \mathbf{n} < 0$ :

$$\tilde{\Psi} = -\zeta \mathbf{\Omega} \cdot \nabla \phi.$$

Therefore the corresponding boundary layer equation is:

$$\frac{1}{c} \frac{\partial}{\partial t} \Psi_{\text{bl}} + \mathbf{\Omega} \cdot \nabla \Psi_{\text{bl}} + \sigma^*(\mathbf{x}) \Psi_{\text{bl}} = 0$$

with boundary condition for  $\mathbf{x} \in \partial V$ ,  $\mathbf{\Omega} \cdot \mathbf{n} < 0$ :

$$\Psi_{\text{bl}} = \psi^b - \frac{1}{4\pi} \phi + \zeta \mathbf{\Omega} \cdot \nabla \phi.$$

# Integral transport equation

Streaming path from  $(\mathbf{x}, t)$  backward along  $-\mathbf{\Omega}$ , accumulate sources and attenuate:

$$\begin{aligned}\tilde{\Psi}(\mathbf{x}, \mathbf{\Omega}, t) = & \tilde{\Psi}^b(\mathbf{x} - s_b \mathbf{\Omega}, \mathbf{\Omega}, t - s_b/c) e^{-\tau(\mathbf{x}, \mathbf{x} - s_b \mathbf{\Omega})} U(ct - s_b) \\ & + \Psi^i(\mathbf{x} - ct \mathbf{\Omega}, \mathbf{\Omega}) e^{-\tau(\mathbf{x}, \mathbf{x} - ct \mathbf{\Omega})} U(s_b - ct)\end{aligned}\quad (6a)$$

$$\begin{aligned}& + \int_0^{s_b} \left[ \hat{Q}(\mathbf{x} - s \mathbf{\Omega}, \mathbf{\Omega}, t - s/c) \right] e^{-\tau(\mathbf{x}, \mathbf{x} - s \mathbf{\Omega})} ds. \\ \equiv & \mathcal{L}_b^{-1} \left[ \tilde{\Psi}^b \right] + \mathcal{L}_i^{-1} \left[ \Psi^i \right] + \mathcal{L}_v^{-1} \left[ \hat{Q} \right]\end{aligned}\quad (6b)$$

$s_b$  is the distance to the boundary,  $U(\dots)$  is the heaviside function, and the optical thickness is

$$\tau(\mathbf{x}, \mathbf{x}') = \int_0^{\|\mathbf{x} - \mathbf{x}'\|} \sigma^*(\mathbf{x} - s \mathbf{\Omega}) ds. \quad (6c)$$

These are nonlocal unknowns; we will approximate them with local unknowns.

# Time for some approximations

Asymptotic ansatz: assume weak spatial gradients, mildly anisotropic angular flux, very small time derivative:

$$\psi = O(1), \quad \nabla \psi = O(\epsilon) \quad \int_{4\pi} \boldsymbol{\Omega} \psi \, d\Omega = O(\epsilon) \quad \frac{1}{c} \frac{\partial}{\partial t} = O(\epsilon^2)$$

Our first approximation:  $\mathcal{L}_i^{-1}[\cdot] = O(\epsilon^2)$  and  $\nabla \cdot \mathbf{J} = O(\epsilon^2)$ :

$$\tilde{\Psi} = \mathcal{L}_i^{-1}[\Psi^i] - \mathcal{L}_b^{-1}[\zeta \boldsymbol{\Omega} \cdot \nabla \phi] + \mathcal{L}_v^{-1} \left[ \frac{1}{4\pi} \nabla \cdot \mathbf{J} \right] - \mathcal{L}_v^{-1} \left[ \frac{1}{4\pi} \boldsymbol{\Omega} \cdot \nabla \phi \right]$$

$$\tilde{\Psi} \approx -\mathcal{L}_b^{-1}[\zeta \boldsymbol{\Omega} \cdot \nabla \phi] - \mathcal{L}_v^{-1} \left[ \frac{1}{4\pi} \boldsymbol{\Omega} \cdot \nabla \phi \right] + O(\epsilon^2)$$

Taylor series expansion:

$$\phi(\mathbf{x} - s\boldsymbol{\Omega}, t - s/c) \sim \phi(\mathbf{x}, t) - s \left( \frac{1}{c} \frac{\partial}{\partial t} + \boldsymbol{\Omega} \cdot \nabla \right) \phi(\mathbf{x}, t) + O(\epsilon^2)$$

$$\phi(\mathbf{x} - s\boldsymbol{\Omega}, t - s/c) = \phi(\mathbf{x}, t) + O(\epsilon) \tag{7}$$

# Taylor series applied

If  $\phi$  is smooth like the ansatz hypothesizes, the volumetric term becomes:

$$\begin{aligned}
 -\mathcal{L}_v^{-1} \left[ \frac{1}{4\pi} \boldsymbol{\Omega} \cdot \nabla \phi \right] &= - \int_0^{s_b} \left[ \frac{1}{4\pi} \boldsymbol{\Omega} \cdot \nabla \phi \right]_{(\mathbf{x}-s\boldsymbol{\Omega}, t-s/c)} e^{-\tau(\mathbf{x}, \mathbf{x}-s\boldsymbol{\Omega})} ds \\
 &\sim - \int_0^{s_b} \left[ \frac{1}{4\pi} \right] e^{-\tau(\mathbf{x}, \mathbf{x}-s\boldsymbol{\Omega})} ds \boldsymbol{\Omega} \cdot \nabla \phi(\mathbf{x}, t) + O(\epsilon^2) \\
 &= -\mathcal{L}_v^{-1} \left[ \frac{1}{4\pi} \right] \boldsymbol{\Omega} \cdot \nabla \phi(\mathbf{x}, t).
 \end{aligned} \tag{8}$$

The boundary term similarly is

$$\begin{aligned}
 -\mathcal{L}_b^{-1} [\zeta \boldsymbol{\Omega} \cdot \nabla \phi] &= - \int_0^{s_b} [\zeta \boldsymbol{\Omega} \cdot \nabla \phi]_{(\mathbf{x}-s_b\boldsymbol{\Omega}, t-s_b/c)} e^{-\tau(\mathbf{x}, \mathbf{x}-s\boldsymbol{\Omega})} ds \\
 &\sim -\mathcal{L}_b^{-1} [\zeta] \boldsymbol{\Omega} \cdot \nabla \phi(\mathbf{x}, t).
 \end{aligned} \tag{9}$$

Thus,

$$\tilde{\Psi}(\mathbf{x}, \boldsymbol{\Omega}, t) \approx - \left[ \mathcal{L}_b^{-1} [\zeta] + \mathcal{L}_v^{-1} \left[ \frac{1}{4\pi} \right] \right] \boldsymbol{\Omega} \cdot \nabla \phi(\mathbf{x}, t) \equiv -f(\mathbf{x}, \boldsymbol{\Omega}) \boldsymbol{\Omega} \cdot \nabla \phi(\mathbf{x}, t)$$



# Transport matched boundary

Transport theory: boundary solution decays quickly if we enforce the relation  $0 = \int_{\mathbf{\Omega} \cdot \mathbf{n} \leq 0} W(|\mathbf{\Omega} \cdot \mathbf{n}|) \Psi_{\text{bl}} d\Omega$  on the boundary, where  $W(\mu) \approx 2\mu + 3\mu^2$  is related to the Chandrasekhar function. The transport extrapolation distance is  $\int_0^1 \mu W d\mu / \int_0^1 W d\mu$ .

$$\begin{aligned} 0 &= \int_{\mathbf{\Omega} \cdot \mathbf{n} < 0} W(|\mathbf{\Omega} \cdot \mathbf{n}|) \Psi_{\text{bl}} d\Omega \\ &= \int_{\mathbf{\Omega} \cdot \mathbf{n} < 0} W \left[ \psi^b - \frac{1}{4\pi} \phi + \zeta \mathbf{\Omega} \cdot \nabla \phi \right] d\Omega \end{aligned}$$

or

$$\int_{\mathbf{\Omega} \cdot \mathbf{n} < 0} W \psi^b d\Omega = \phi - \int_{\mathbf{\Omega} \cdot \mathbf{n} < 0} W \zeta \mathbf{\Omega} d\Omega \cdot \nabla \phi \quad (11)$$

One more equation is needed to determine  $\zeta$ .

# Determining $\zeta$

Recall that in the exact anisotropic transport equation,  $\int_{4\pi} \Psi \, d\Omega = 0$ . So we choose to enforce  $\int_{4\pi} \tilde{\Psi} \, d\Omega = 0$  on the boundary:

$$0 = \int_{4\pi} \tilde{\Psi} \, d\Omega = \int_{\Omega \cdot \mathbf{n} < 0} (-\zeta \mathbf{\Omega} \cdot \nabla \phi) \, d\Omega + \int_{\Omega \cdot \mathbf{n} > 0} (-f \mathbf{\Omega} \cdot \nabla \phi) \, d\Omega$$

Or:

$$\begin{aligned} \int_{\Omega \cdot \mathbf{n} < 0} \mathbf{\Omega} \zeta \, d\Omega \cdot \nabla \phi &= \int_{\Omega \cdot \mathbf{n} > 0} [-\mathbf{\Omega}] f \, d\Omega \cdot \nabla \phi \\ \int_{\Omega \cdot \mathbf{n} < 0} \mathbf{\Omega} \zeta(\mathbf{x}, \mathbf{\Omega}) \, d\Omega \cdot \nabla \phi &= \int_{\Omega \cdot \mathbf{n} < 0} \mathbf{\Omega} f(\mathbf{x}, -\mathbf{\Omega}) \, d\Omega \cdot \nabla \phi \end{aligned}$$

One possible way to satisfy this is:

$$\zeta(\mathbf{x}, \mathbf{\Omega}) = f(\mathbf{x}, -\mathbf{\Omega})$$

for  $\mathbf{x} \in \partial V$ ,  $\mathbf{\Omega} \cdot \mathbf{n} < 0$ . This is a reflecting boundary condition!

# Summary of boundary layer analysis

Approximate expression for anisotropic angular flux:

$$\begin{aligned}\tilde{\Psi}(\mathbf{x}, \mathbf{\Omega}, t) &\approx - \left\{ \mathcal{L}_b^{-1}[f(\mathbf{x}, -\mathbf{\Omega})] + \mathcal{L}_v^{-1} \left[ \frac{1}{4\pi} \right] \right\} \mathbf{\Omega} \cdot \nabla \phi(\mathbf{x}, t) \\ &\equiv - \{f(\mathbf{x}, \mathbf{\Omega})\} \mathbf{\Omega} \cdot \nabla \phi(\mathbf{x}, t)\end{aligned}$$

Low-order boundary condition (after substituting  $\zeta(\mathbf{x}, -\mathbf{\Omega})$ ):

$$\begin{aligned}\int_{\mathbf{\Omega} \cdot \mathbf{n} < 0} W(|\mathbf{\Omega} \cdot \mathbf{n}|) \psi^b(\mathbf{x}, \mathbf{\Omega}, t) \, d\mathbf{\Omega} \\ = \phi(\mathbf{x}, t) - \int_{\mathbf{\Omega} \cdot \mathbf{n} > 0} \mathbf{\Omega} W(|\mathbf{\Omega} \cdot \mathbf{n}|) f(\mathbf{x}, \mathbf{\Omega}) \, d\mathbf{\Omega} \cdot \nabla \phi(\mathbf{x}, t)\end{aligned}$$

Boundary condition for  $f$ :

$$f(\mathbf{x}, \mathbf{\Omega}) = f(\mathbf{x}, -\mathbf{\Omega}).$$

# An analogy to Fick's law

To get an expression for the current use the identity  $\mathbf{J} = \int_{4\pi} \boldsymbol{\Omega} \tilde{\Psi} \, d\Omega$ , which gives

$$\begin{aligned} \mathbf{J}(\mathbf{x}, t) &= \int_{4\pi} \boldsymbol{\Omega} \{ -f \boldsymbol{\Omega} \cdot \nabla \phi(\mathbf{x}, t) \} \, d\Omega \\ &= - \left[ \int_{4\pi} \boldsymbol{\Omega} \boldsymbol{\Omega} f \, d\Omega \right] \cdot \nabla \phi(\mathbf{x}, t) \\ &\equiv -\mathbf{D} \cdot \nabla \phi. \end{aligned}$$

Substitute into radiation energy conservation equation:

$$\frac{1}{c} \frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{J} + \sigma^* \phi = \sigma_{ac} T^4 + cQ$$

Couple with the material energy balance equation:

$$\frac{1}{c_v} \frac{\partial T}{\partial t} = \sigma^* \phi - \sigma_{ac} T^4$$

Approximate the red terms semi-implicitly.

The transport problem used to calculate  $\mathbf{D}$  is

$$\boldsymbol{\Omega} \cdot \nabla f + \sigma^* f = \frac{1}{4\pi}, \mathbf{x} \in V, \boldsymbol{\Omega} \in 4\pi,$$

with boundary condition

$$f(\mathbf{x}, \boldsymbol{\Omega}) = f(\mathbf{x}, -\boldsymbol{\Omega}), \mathbf{x} \in \partial V, \boldsymbol{\Omega} \cdot \mathbf{n} < 0.$$

- Takes only one transport sweep to solve if the boundaries are many mean free paths apart
- Only needs to be calculated once per time step (because of changing  $\sigma^*$ ) in a nonlinear problem
- Requires no storage of the angular flux, just accumulation of second moment,  $D_{ij} = \int_{4\pi} \Omega_i \Omega_j f \, d\Omega$
- Has the solution  $f = 1/4\pi\sigma$  if  $\sigma$  is a constant. Then,  $\int_{4\pi} \boldsymbol{\Omega} f \boldsymbol{\Omega} \, d\Omega = \mathbf{I}/3\sigma$ .

# Properties of anisotropic diffusion

The anisotropic diffusion tensor  $\mathbf{D}(\mathbf{x}, t)$ :

- Does not “blow up” in void regions
- Has a greater “action” along the direction of a voided channel than across it
- Reduces to  $\mathbf{I}/3\sigma$  for a homogeneous medium, which gives standard diffusion solution (and boundary conditions reduce to transport-corrected diffusion BCs)
- Is continuous in  $\mathbf{x}$ , so the approximate AD-calculated  $\phi$  has continuous first derivatives (i.e.,  $\phi$  is smooth like our ansatz requires)

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# Compared methods

- Implicit Monte Carlo (IMC) [3] implemented with variance reduction methods,  $10^7$  particles per time step
- Flux-limited diffusion (FLD) with Larsen limiter [4], with semi-implicit treatment of diffusion coefficient and radiation:

$$\mathbf{J}^{n+1} = -D^n \nabla \phi^{n+1} = - \left[ (3\sigma^n)^2 + \left( \frac{\|\nabla \phi^n\|}{\phi^n} \right)^2 \right]^{-1/2} \nabla \phi^{n+1}$$

- Standard diffusion, with semi-implicit treatment of nonlinearities:

$$\mathbf{J}^{n+1} = -D^n \nabla \phi^{n+1} = -\frac{1}{3\sigma^n} \nabla \phi^{n+1}$$

- Anisotropic diffusion, with semi-implicit treatment of nonlinearities:

$$\mathbf{J}^{n+1} = -\mathbf{D}^n \cdot \nabla \phi^{n+1}$$



# AD implementation

## Approximations in the theory

- Assume weak gradients and angular moments for  $\psi$  (*don't* assume that  $\psi$  is a linear function of  $\Omega$ !)
- Apply semi-implicit approximation for nonlinear material coupling and radiation

## D transport equation

- $S_N$  angular approximation
- DD spatial approximation
- One source iteration per time step

## AD equation

- 9-point cell-centered finite difference spatial approximation

# Problem description

Flatland geometry!

Uniform spatial grid:  $\Delta_x = 0.1$

Piecewise linear time grid:  $\Delta_t = 0.1$   
for  $t \geq 1$

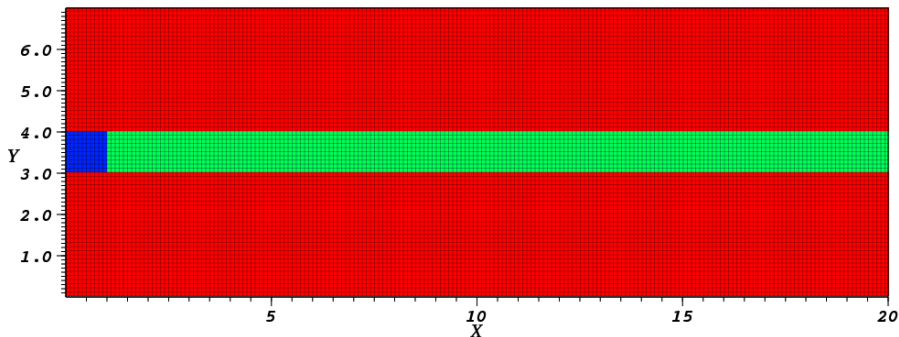
Reflecting bndy on left, others  
vacuum

**Source:**  $c_v = 0.5$ ,  $\sigma = 0.5$ ;  $Q = 1$   
for  $0 \leq t \leq 1$ ,  $Q = 0$  for  $t > 1$ .

**Diffusive:**  $c_v = 0.1$ ,  $\sigma = T^{-3}$

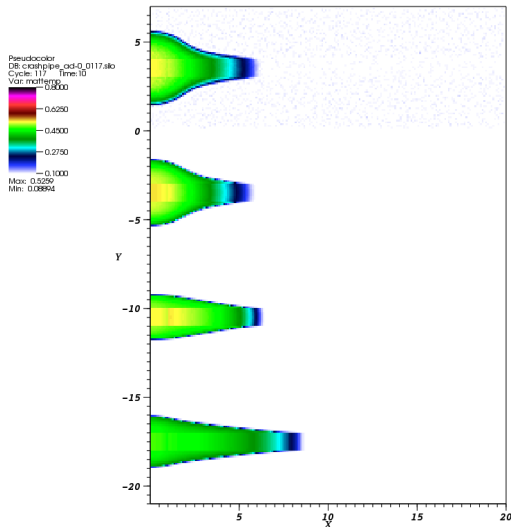
**Channel:**  $c_v = 0.1$ ,  $\sigma = 0.01T^{-3}$

Initial condition:  $T = T_{\text{rad}} = 0.1$

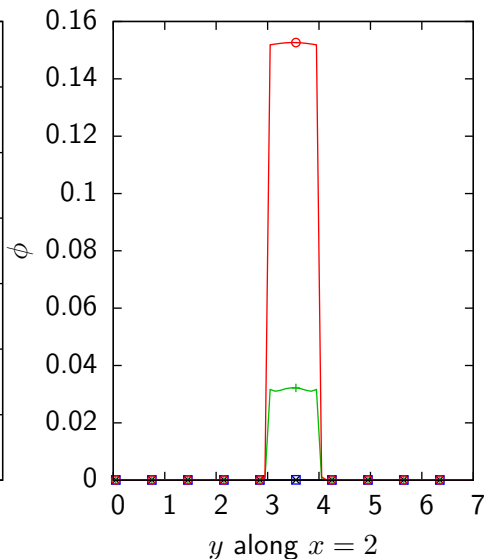
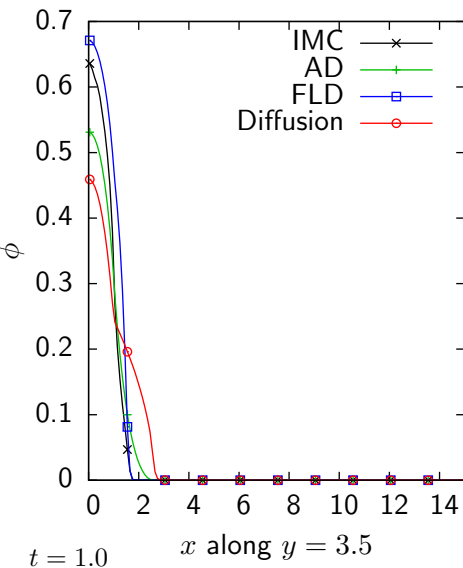


# Time evolution of material temperature

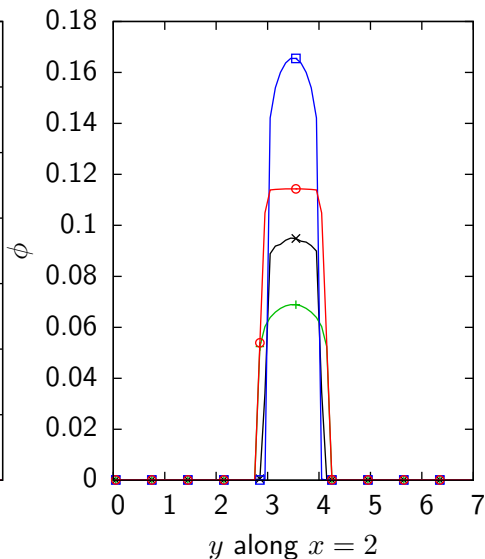
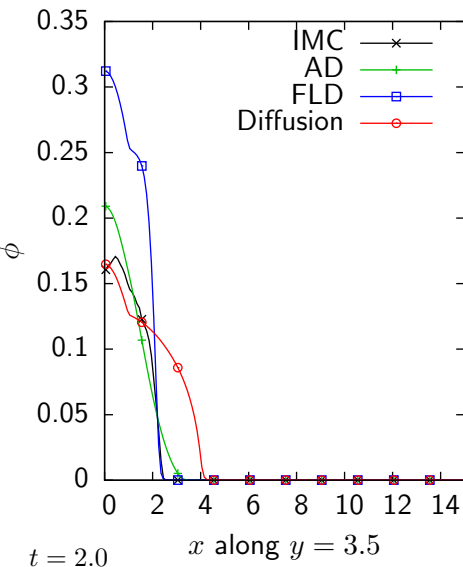
(Replaced with movie)



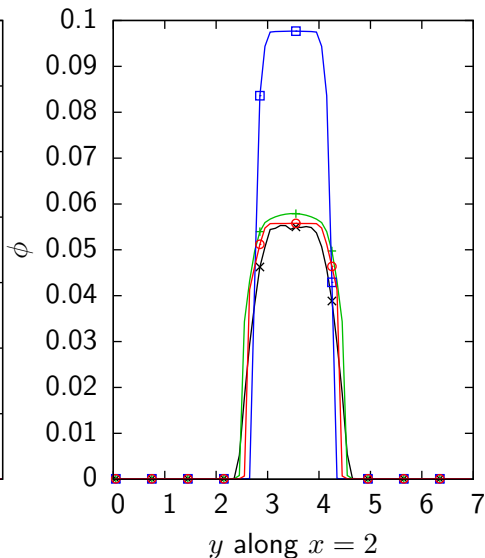
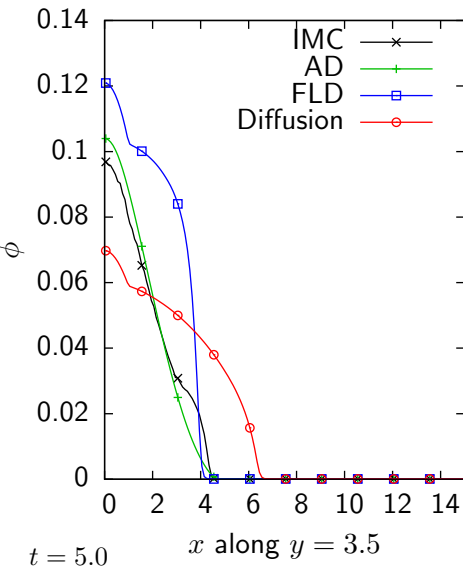
# Time evolution of radiation energy density



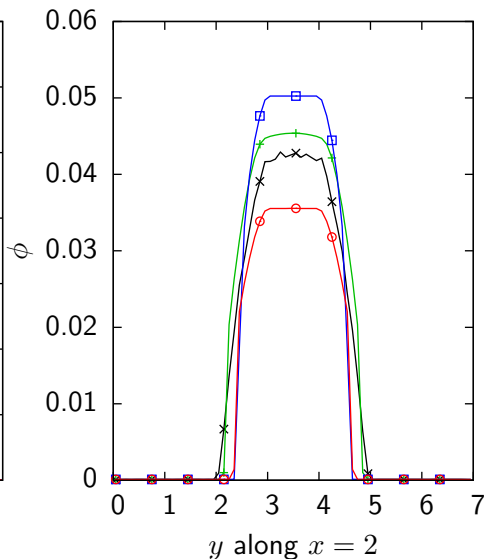
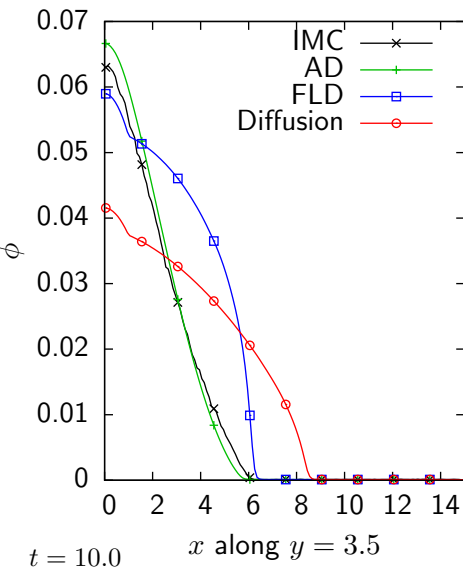
# Time evolution of radiation energy density



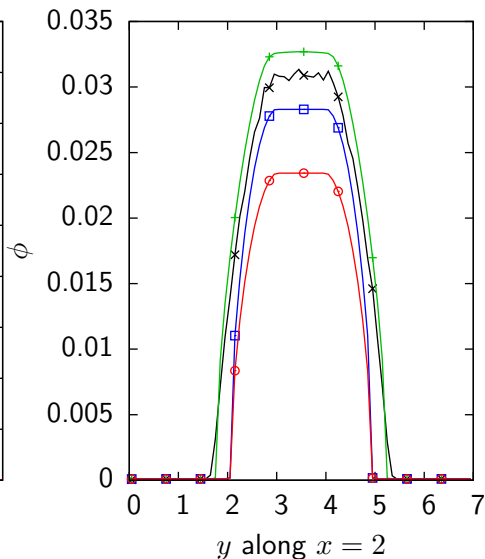
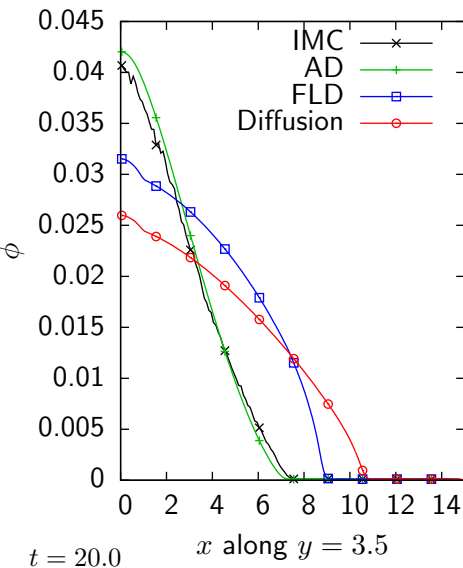
# Time evolution of radiation energy density



# Time evolution of radiation energy density

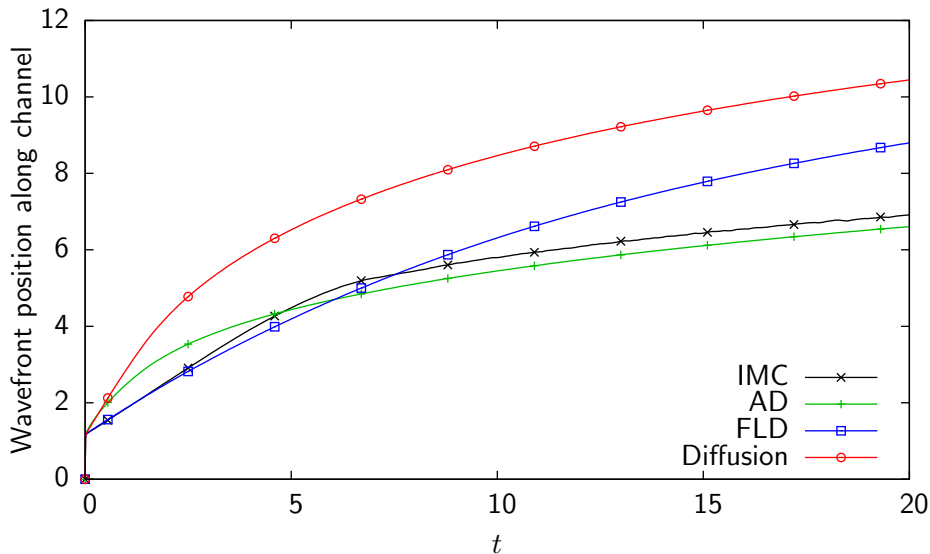


# Time evolution of radiation energy density

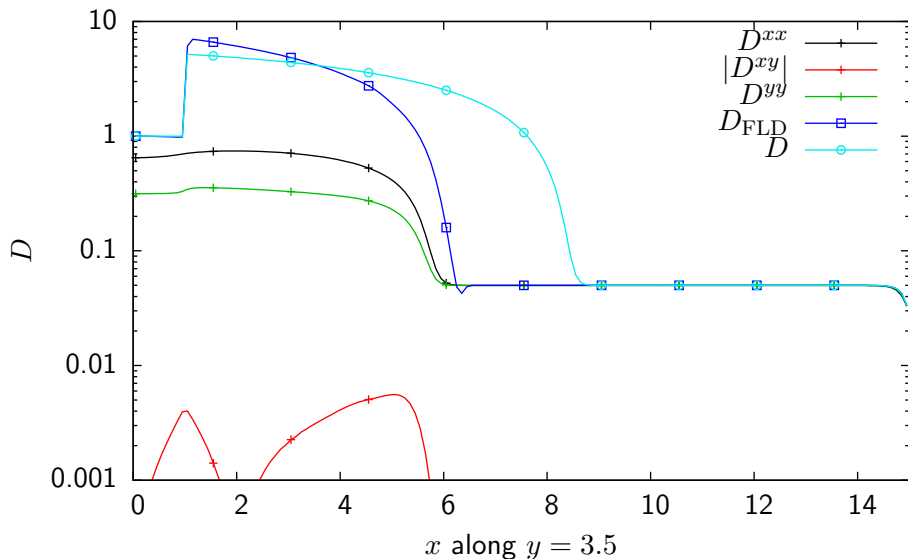




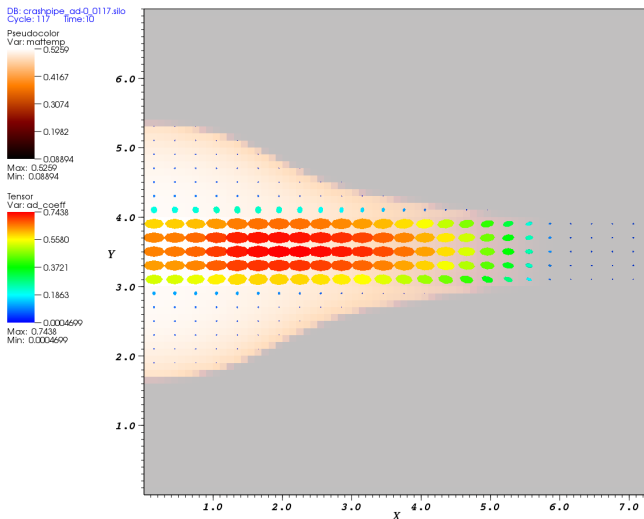
# Time evolution of radiation temperature wavefront



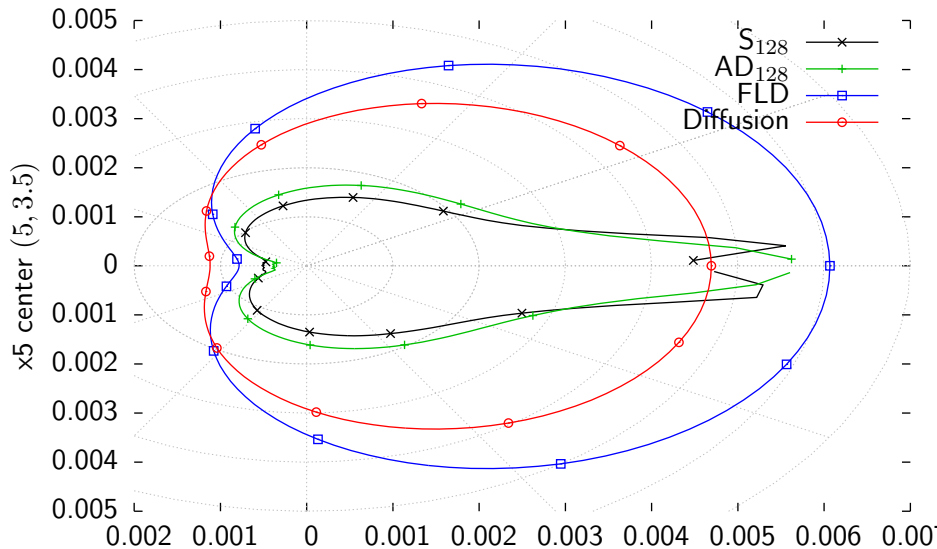
# Diffusion coefficients ( $t = 10$ )



# Anisotropic diffusion tensor visualization



# Approximate representations of the angular flux



# Timing results

Method	Wall time (s)
IMC	2730
FLD	21
D	20
AD <sub>64</sub>	36
AD <sub>128</sub>	59

Table 1: Approximate run times for pipe test problem with  $\Delta_x = 0.1$ .

# Outline

- 1 Introduction
- 2 Theory
- 3 Results
- 4 Conclusions**

# Conclusions

## Anisotropic diffusion:

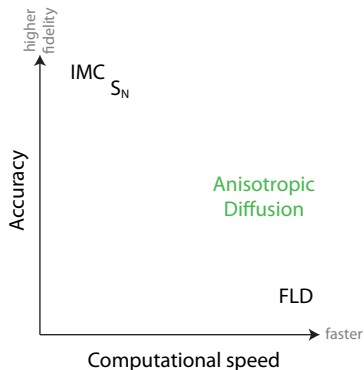
- Accounts for some amount of arbitrary anisotropy in angular flux, unlike standard or current-limited diffusion, by preserving some transport physics
- Works best in problems with weaker derivatives, as suggested by theory and borne out by numerical experiments
- Accurately treats the nonlinear time-dependent flow of radiation through a tube like that found in CRASH experiments

# Future work

- Further analysis of boundary conditions
- Implement and test “Anisotropic  $P_1$ ” ( $\frac{1}{c} \frac{\partial}{\partial t} = O(\epsilon)$  instead of  $O(\epsilon^2)$ )
- Extend method to anisotropic internal sources
- Keep the  $\nabla \cdot \mathbf{J}$  term by ignoring assumption of  $\int_{4\pi} \boldsymbol{\Omega}(\cdot) d\Omega = O(\epsilon)$







# Questions?



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