An anisotropic diffusion approximation to nonlinear radiation transport

Seth Johnson   Ed Larsen

University of Michigan, Ann Arbor

May 6, 2011

MICHIGAN

Nuclear Energy University Programs
U.S. Department of Energy

Outline

Thermal radiative transfer

- TRT is the dominant heat transfer process in very hot materials
- Photons born isotropically via black body emission ($q_{\text{rad}} \propto \sigma T^4$)
- Cold material heats up and becomes relatively transparent ($\sigma \propto T^{-3}$)

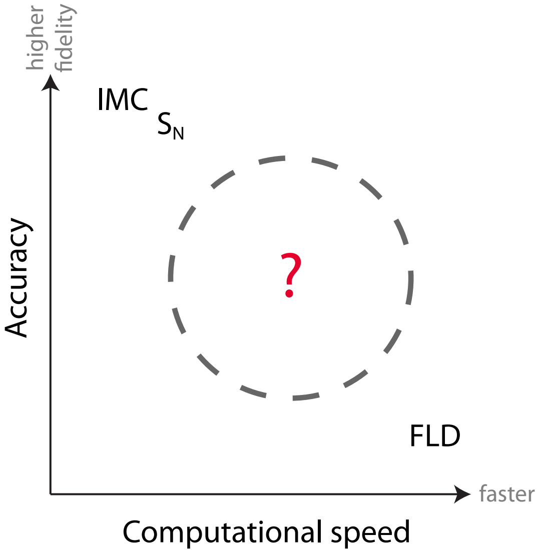
Difficulties in solving:

- High dimensionality of solution phase space $(\boldsymbol{x}, \boldsymbol{\Omega}, h\nu, t)$
- Highly nonlinear coupled partial differential equations for radiation field $I(\boldsymbol{x}, \boldsymbol{\Omega}, h\nu, t)$ and material energy

Particular application of this work: CRASH project

- Center for RAdiative Shock Hydrodynamics program: "Assessment of Predictive Capability"
- Simulate laser-driven shock in a xenon-filled tube
- Uncertainty quantification: hundreds of solution instances needed

Motivation


higher
fidelity
Accuracy
IMC
S_N
?
FLD
faster
Computational speed

Gray TRT equations

Common approximations for radiation transport methods development:

- work in a fixed medium, disregarding material advection;
- assume local thermodynamic equilibrium (LTE), which uses a single material temperature;
- neglect thermal conduction in material;
- average over all photon energies $h\nu$ (gray).

Radiation transfer equation, intensity $I(\boldsymbol{x}, \boldsymbol{\Omega}, t)$:

$$\frac{1}{c}\frac{\partial I}{\partial t} + \boldsymbol{\Omega} \cdot \boldsymbol{\nabla} I + \sigma I = \frac{\sigma c a T^4}{4\pi} + \frac{cQ}{4\pi} \tag{1a}$$

Material energy balance equation:

$$\frac{1}{c_v}\frac{\partial T}{\partial t} = \sigma \int_{4\pi} I \, \mathrm{d}\Omega - \sigma c a T^4 \tag{1b}$$

Anisotropic diffusion

Previous work:

- Steady-state infinite medium VHTR-like problem with analytically calculated coefficients [1]
- Non-local tensor diffusion [2] for steady-state radiative transfer, no further development or analysis in literature

Current work:

- Formulates boundary conditions and time-dependent terms
- Uses transport-calculated anisotropic diffusion tensors
- Applies to nonlinear, time-dependent problems with isotropic sources

Potential applications:

- Extends diffusion theory to new regimes of applicability
- Variance reduction with shielding problems that have voids

Outline

1. Define the anisotropic intensity as $\Psi = I - \frac{1}{4\pi}\phi$. To handle boundary conditions, define $\Psi \equiv \tilde{\Psi} + \Psi_{\mathrm{bl}}$. We will approximate $\tilde{\Psi}$ rather than I, and use Ψ_{bl} to determine matched boundary conditions.
2. From the radiation transport equation and conservation equation, we get a differential transport equation for $\tilde{\Psi}$ and Ψ_{bl}. Transform the former to an *integral* transport equation for $\tilde{\Psi}$.
3. Assume $I = O(1)$, $\frac{1}{c}\frac{\partial}{\partial t} = O(\epsilon^2)$, $\boldsymbol{\nabla} = O(\epsilon)$, $\int_{4\pi} \boldsymbol{\Omega}(\cdot)\,\mathrm{d}\Omega = O(\epsilon)$.
4. Use Taylor series to approximate nonlocal unknowns with local unknowns, discarding small terms. This yields

$$\tilde{\Psi}(\boldsymbol{x}, \boldsymbol{\Omega}) \approx -f(\boldsymbol{x}, \boldsymbol{\Omega})\boldsymbol{\Omega} \cdot \boldsymbol{\nabla}\phi \,.$$

5. Apply standard transport-matching procedure to Ψ_{bl}. Use the identity $\int_{4\pi} \Psi \,\mathrm{d}\Omega = 0$ to find the boundary condition for f.
6. Take the first angular moment of $\tilde{\Psi}$ to get $\boldsymbol{F} = -\mathbf{D} \cdot \boldsymbol{\nabla}\phi$
7. Substitute $\boldsymbol{F}$ into the time-dependent particle conservation equation to get time-dependent anisotropic diffusion.

Transport equation

Inside a time step, with "frozen" opacities:

$$\frac{1}{c}\frac{\partial I}{\partial t}(\boldsymbol{x},\boldsymbol{\Omega},t) + \boldsymbol{\Omega}\cdot\boldsymbol{\nabla} I(\boldsymbol{x},\boldsymbol{\Omega},t) + \sigma^*(\boldsymbol{x}) I(\boldsymbol{x},\boldsymbol{\Omega},t)$$
$$= \frac{1}{4\pi}\sigma^*(\boldsymbol{x}) ac[T(\boldsymbol{x},t)]^4 + \frac{1}{4\pi} q_r(\boldsymbol{x},t) \equiv \frac{1}{4\pi} Q(\boldsymbol{x},t)\,,$$
$$x \in V,\ 0 \le t < \Delta_t,\ \boldsymbol{\Omega} \in 4\pi, \quad \text{(2a)}$$

with the boundary condition

$$I(\boldsymbol{x},\boldsymbol{\Omega},t) = I^b(\boldsymbol{x},\boldsymbol{\Omega},t)\,, \quad \boldsymbol{x} \in \partial V,\ \boldsymbol{\Omega}\cdot\boldsymbol{n} < 0,\ 0 \le t < \Delta_t \qquad \text{(2b)}$$

and the initial condition

$$I(\boldsymbol{x},\boldsymbol{\Omega},0) = I^i(\boldsymbol{x},\boldsymbol{\Omega},t)\,, \quad \boldsymbol{x} \in V,\ \boldsymbol{\Omega} \in 4\pi\,. \qquad \text{(2c)}$$

Conservation equations

Operating on Eq. (2a) by $\int_{4\pi}(\cdot)\,\mathrm{d}\Omega$ gives

$$\frac{1}{c}\frac{\partial\phi}{\partial t}(\boldsymbol{x},t)+\boldsymbol{\nabla}\cdot\boldsymbol{F}(\boldsymbol{x},t)+\sigma^{*}\phi(\boldsymbol{x},t)=Q(\boldsymbol{x},t)\,. \tag{3a}$$

and on the initial condition, Eq. (2c),

$$\phi(\boldsymbol{x},0)=\int_{4\pi}I^{i}(\boldsymbol{x},\boldsymbol{\Omega})\,\mathrm{d}\Omega=\phi^{i}(\boldsymbol{x})\,. \tag{3b}$$

Add $\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi$ to both sides of Eq. (3a) and multiply by $\frac{1}{4\pi}$:

$$\frac{1}{4\pi}\frac{1}{c}\frac{\partial\phi}{\partial t}+\frac{1}{4\pi}\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi+\frac{1}{4\pi}\sigma^{*}\phi=\frac{1}{4\pi}Q(\boldsymbol{x},t)+\frac{1}{4\pi}\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi-\frac{1}{4\pi}\boldsymbol{\nabla}\cdot\boldsymbol{F} \tag{4}$$

Anisotropic intensity equations

Define "anisotropic intensity":

$$\Psi(\boldsymbol{x}, \boldsymbol{\Omega}) \equiv I(\boldsymbol{x}, \boldsymbol{\Omega}) - \frac{1}{4\pi}\phi(\boldsymbol{x}) \,. \tag{5}$$

(This satisfies $\int_{4\pi} \Psi = 0$ and $\int_{4\pi} \boldsymbol{\Omega}\Psi = \boldsymbol{F}$.)
Subtract Eq. (4) from Eq. (2a); the isotropic source cancels:

$$\frac{1}{c}\frac{\partial}{\partial t}\left[I - \frac{\phi}{4\pi}\right] + \boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\left[I - \frac{\phi}{4\pi}\right] + \sigma^*(\boldsymbol{x})\left[I - \frac{\phi}{4\pi}\right] = \frac{1}{4\pi}\boldsymbol{\nabla}\cdot\boldsymbol{F} - \frac{1}{4\pi}\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi$$

Subtract $\phi/4\pi$ from the transport boundary condition:

$$I - \frac{\phi}{4\pi} = I^b - \frac{\phi}{4\pi}$$

Subtract Eq. (3b) from Eq. (2c):

$$I(\boldsymbol{x}, \boldsymbol{\Omega}, 0) - \frac{1}{4\pi}\phi(\boldsymbol{x}, 0) = I^i - \frac{\phi^i}{4\pi}$$

Anisotropic intensity equations

Transport equation:

$$\frac{1}{c}\frac{\partial}{\partial t}\Psi + \boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\Psi + \sigma^*(\boldsymbol{x})\Psi = \frac{1}{4\pi}\boldsymbol{\nabla}\cdot\boldsymbol{F} - \frac{1}{4\pi}\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi \equiv \hat{Q}(\boldsymbol{x},\boldsymbol{\Omega},t)$$

Boundary condition:

$$\Psi = \Psi^b = I^b - \frac{\phi}{4\pi}$$

Initial condition:

$$\Psi(\boldsymbol{x},\boldsymbol{\Omega},0) = \Psi^i = I^i - \frac{\phi^i}{4\pi}\,.$$

The exact solutions for I, ϕ, $\boldsymbol{F}$ satisfy these equations: still no approximations.

Boundary layer equations

In anticipation of approximating $\tilde{\Psi} = -f\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi$, separate Ψ into a boundary layer plus an internal solution:

$$\Psi(\boldsymbol{x},\boldsymbol{\Omega},t) \equiv \tilde{\Psi}(\boldsymbol{x},\boldsymbol{\Omega},t) + \Psi_{\mathrm{bl}}(\boldsymbol{x},\boldsymbol{\Omega},t)\,.$$

The exact equations for $\tilde{\Psi}$:

$$\frac{1}{c}\frac{\partial}{\partial t}\tilde{\Psi} + \boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\tilde{\Psi} + \sigma^*(\boldsymbol{x})\tilde{\Psi} = \hat{Q}(\boldsymbol{x},\boldsymbol{\Omega},t)$$

with new boundary condition for $\boldsymbol{x}\in\partial V$, $\boldsymbol{\Omega}\cdot\boldsymbol{n} < 0$:

$$\tilde{\Psi} = -\zeta\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi\,.$$

Therefore the corresponding boundary layer equation is:

$$\frac{1}{c}\frac{\partial}{\partial t}\Psi_{\mathrm{bl}} + \boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\Psi_{\mathrm{bl}} + \sigma^*(\boldsymbol{x})\Psi_{\mathrm{bl}} = 0$$

with boundary condition for $\boldsymbol{x}\in\partial V$, $\boldsymbol{\Omega}\cdot\boldsymbol{n} < 0$:

$$\Psi_{\mathrm{bl}} = I^b - \frac{1}{4\pi}\phi + \zeta\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi\,.$$

Integral transport equation

Streaming path from $(\boldsymbol{x}, t)$ backward along $-\boldsymbol{\Omega}$, accumulate sources and attenuate:

$$\begin{aligned}\tilde{\Psi}(\boldsymbol{x}, \boldsymbol{\Omega}, t) = \tilde{\Psi}^b(\boldsymbol{x} - s_b\boldsymbol{\Omega}, \boldsymbol{\Omega}, t - s_b/c)\, \mathrm{e}^{-\tau(\boldsymbol{x}, \boldsymbol{x} - s_b\boldsymbol{\Omega})} U(ct - s_b) \\ + \Psi^i(\boldsymbol{x} - ct\boldsymbol{\Omega}, \boldsymbol{\Omega})\, \mathrm{e}^{-\tau(\boldsymbol{x}, \boldsymbol{x} - ct\boldsymbol{\Omega})} U(s_b - ct) \\ + \int_0^{s_b} \left[\hat{Q}(\boldsymbol{x} - s\boldsymbol{\Omega}, \boldsymbol{\Omega}, t - s/c)\right] \mathrm{e}^{-\tau(\boldsymbol{x}, \boldsymbol{x} - s\boldsymbol{\Omega})}\, \mathrm{d}s\,. \end{aligned} \tag{6a}$$

$$\equiv \mathcal{L}_b^{-1}\left[\tilde{\Psi}^b\right] + \mathcal{L}_i^{-1}\left[\Psi^i\right] + \mathcal{L}_v^{-1}\left[\hat{Q}\right] \tag{6b}$$

s_b is the distance to the boundary, $U(\cdots)$ is the heaviside function, and the optical thickness is

$$\tau(\boldsymbol{x}, \boldsymbol{x}') = \int_0^{\|\boldsymbol{x} - \boldsymbol{x}'\|} \sigma^*(\boldsymbol{x} - s\boldsymbol{\Omega})\, \mathrm{d}s\,. \tag{6c}$$

These are nonlocal unknowns; we will approximate them with local unknowns.

Time for some approximations

Asymptotic ansatz: assume weak spatial gradients, mildly anisotropic intensity, very small time derivative:

$$I = O(1), \qquad \boldsymbol{\nabla} I = O(\epsilon) \qquad \int_{4\pi} \boldsymbol{\Omega} I \,\mathrm{d}\Omega = O(\epsilon) \qquad \frac{1}{c}\frac{\partial}{\partial t} = O(\epsilon^2)$$

Our first approximation: $\mathcal{L}_i^{-1}[\cdot] = O(\epsilon^2)$ and $\boldsymbol{\nabla} \cdot \boldsymbol{F} = O(\epsilon^2)$:

$$\tilde{\Psi} = \mathcal{L}_i^{-1}\left[\Psi^i\right] - \mathcal{L}_b^{-1}[\zeta \boldsymbol{\Omega} \cdot \boldsymbol{\nabla} \phi] + \mathcal{L}_v^{-1}\left[\frac{1}{4\pi} \boldsymbol{\nabla} \cdot \boldsymbol{F}\right] - \mathcal{L}_v^{-1}\left[\frac{1}{4\pi} \boldsymbol{\Omega} \cdot \boldsymbol{\nabla} \phi\right]$$

$$\tilde{\Psi} \approx -\mathcal{L}_b^{-1}[\zeta \boldsymbol{\Omega} \cdot \boldsymbol{\nabla} \phi] - \mathcal{L}_v^{-1}\left[\frac{1}{4\pi} \boldsymbol{\Omega} \cdot \boldsymbol{\nabla} \phi\right] + O(\epsilon^2)$$

Taylor series expansion:

$$\phi(\boldsymbol{x} - s\boldsymbol{\Omega}, t - s/c) \sim \phi(\boldsymbol{x}, t) - s\left(\frac{1}{c}\frac{\partial}{\partial t} + \boldsymbol{\Omega} \cdot \boldsymbol{\nabla}\right)\phi(\boldsymbol{x}, t) + O(\epsilon^2)$$

$$\phi(\boldsymbol{x} - s\boldsymbol{\Omega}, t - s/c) = \phi(\boldsymbol{x}, t) + O(\epsilon) \tag{7}$$

Taylor series applied

If ϕ is smooth like the ansatz hypothesizes, the volumetric term becomes:

$$\begin{aligned}
-\mathcal{L}_v^{-1}\left[\frac{1}{4\pi}\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi\right] &= -\int_0^{s_b}\left[\frac{1}{4\pi}\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi\right]_{(\boldsymbol{x}-s\boldsymbol{\Omega},t-s/c)} \mathrm{e}^{-\tau(\boldsymbol{x},\boldsymbol{x}-s\boldsymbol{\Omega})}\,\mathrm{d}s \\
&\sim -\int_0^{s_b}\left[\frac{1}{4\pi}\right]\mathrm{e}^{-\tau(\boldsymbol{x},\boldsymbol{x}-s\boldsymbol{\Omega})}\,\mathrm{d}s\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi(\boldsymbol{x},t) + O(\epsilon^2) \\
&= -\mathcal{L}_v^{-1}\left[\frac{1}{4\pi}\right]\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi(\boldsymbol{x},t)\,. \qquad (8)
\end{aligned}$$

The boundary term similarly is

$$\begin{aligned}
-\mathcal{L}_b^{-1}[\zeta\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi] &= -\int_0^{s_b}[\zeta\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi]_{(\boldsymbol{x}-s_b\boldsymbol{\Omega},t-s_b/c)}\,\mathrm{e}^{-\tau(\boldsymbol{x},\boldsymbol{x}-s\boldsymbol{\Omega})}\,\mathrm{d}s \\
&\sim -\mathcal{L}_b^{-1}[\zeta]\,\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi(\boldsymbol{x},t)\,. \qquad (9)
\end{aligned}$$

Thus,

$$\tilde{\Psi}(\boldsymbol{x},\boldsymbol{\Omega},t) \approx -\left[\mathcal{L}_b^{-1}[\zeta] + \mathcal{L}_v^{-1}\left[\frac{1}{4\pi}\right]\right]\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi(\boldsymbol{x},t) \equiv -f(\boldsymbol{x},\boldsymbol{\Omega})\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi(\boldsymbol{x},t)$$

Transport matched boundary

Transport theory: boundary solution decays quickly if we enforce the relation $0 = \int_{\boldsymbol{\Omega}\cdot\boldsymbol{n}<0} W(|\boldsymbol{\Omega}\cdot\boldsymbol{n}|)\Psi_{\mathrm{bl}}\,\mathrm{d}\Omega$ on the boundary, where $W(\mu) \approx 2\mu + 3\mu^2$ is related to the Chandrasekhar function. The transport extrapolation distance is $\int_0^1 \mu W\,\mathrm{d}\mu / \int_0^1 W\,\mathrm{d}\mu$.

$$
\begin{aligned}
0 &= \int_{\boldsymbol{\Omega}\cdot\boldsymbol{n}<0} W(|\boldsymbol{\Omega}\cdot\boldsymbol{n}|)\Psi_{\mathrm{bl}}\,\mathrm{d}\Omega \\
&= \int_{\boldsymbol{\Omega}\cdot\boldsymbol{n}<0} W\left[I^b - \frac{1}{4\pi}\phi + \zeta\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi\right]\mathrm{d}\Omega
\end{aligned}
$$

or

$$
\int_{\boldsymbol{\Omega}\cdot\boldsymbol{n}<0} W I^b\,\mathrm{d}\Omega = \phi - \int_{\boldsymbol{\Omega}\cdot\boldsymbol{n}<0} W\zeta\boldsymbol{\Omega}\,\mathrm{d}\Omega\cdot\boldsymbol{\nabla}\phi \qquad (11)
$$

One more equation is needed to determine ζ.

Determining ζ

Recall that in the exact anisotropic transport equation, $\int_{4\pi} \Psi \, \mathrm{d}\Omega = 0$. So we choose to enforce $\int_{4\pi} \tilde{\Psi} \, \mathrm{d}\Omega = 0$ on the boundary:

$$0 = \int_{4\pi} \tilde{\Psi} \, \mathrm{d}\Omega = \int_{\boldsymbol{\Omega}\cdot\boldsymbol{n}<0} (-\zeta \boldsymbol{\Omega} \cdot \boldsymbol{\nabla}\phi) \, \mathrm{d}\Omega + \int_{\boldsymbol{\Omega}\cdot\boldsymbol{n}>0} (-f \boldsymbol{\Omega} \cdot \boldsymbol{\nabla}\phi) \, \mathrm{d}\Omega$$

Or:

$$\int_{\boldsymbol{\Omega}\cdot\boldsymbol{n}<0} \boldsymbol{\Omega}\zeta \, \mathrm{d}\Omega \cdot \boldsymbol{\nabla}\phi = \int_{\boldsymbol{\Omega}\cdot\boldsymbol{n}>0} [-\boldsymbol{\Omega}] f \, \mathrm{d}\Omega \cdot \boldsymbol{\nabla}\phi$$

$$\int_{\boldsymbol{\Omega}\cdot\boldsymbol{n}<0} \boldsymbol{\Omega}\zeta(\boldsymbol{x}, \boldsymbol{\Omega}) \, \mathrm{d}\Omega \cdot \boldsymbol{\nabla}\phi = \int_{\boldsymbol{\Omega}\cdot\boldsymbol{n}<0} \boldsymbol{\Omega} f(\boldsymbol{x}, -\boldsymbol{\Omega}) \, \mathrm{d}\Omega \cdot \boldsymbol{\nabla}\phi$$

One possible way to satisfy this is:

$$\zeta(\boldsymbol{x}, \boldsymbol{\Omega}) = f(\boldsymbol{x}, -\boldsymbol{\Omega})$$

for $\boldsymbol{x} \in \partial V$, $\boldsymbol{\Omega} \cdot \boldsymbol{n} < 0$. This is a reflecting boundary condition!

Summary of boundary layer analysis

Approximate expression for anisotropic intensity:

$$\begin{aligned}\tilde{\Psi}(\boldsymbol{x}, \boldsymbol{\Omega}, t) &\approx -\left\{\mathcal{L}_b^{-1}[f(\boldsymbol{x}, -\boldsymbol{\Omega})] + \mathcal{L}_v^{-1}\left[\frac{1}{4\pi}\right]\right\}\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi(\boldsymbol{x}, t) \\ &\equiv -\{f(\boldsymbol{x}, \boldsymbol{\Omega})\}\,\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi(\boldsymbol{x}, t)\end{aligned}$$

Low-order boundary condition (after substituting $\zeta(\boldsymbol{x}, -\boldsymbol{\Omega})$):

$$\begin{aligned}&\int_{\boldsymbol{\Omega}\cdot\boldsymbol{n}<0} W(|\boldsymbol{\Omega}\cdot\boldsymbol{n}|) I^b(\boldsymbol{x}, \boldsymbol{\Omega}, t)\,\mathrm{d}\Omega \\ &\qquad = \phi(\boldsymbol{x}, t) - \int_{\boldsymbol{\Omega}\cdot\boldsymbol{n}>0} \boldsymbol{\Omega} W(|\boldsymbol{\Omega}\cdot\boldsymbol{n}|) f(\boldsymbol{x}, \boldsymbol{\Omega})\,\mathrm{d}\Omega\cdot\boldsymbol{\nabla}\phi(\boldsymbol{x}, t)\end{aligned}$$

Boundary condition for f:

$$f(\boldsymbol{x}, \Omega) = f(\boldsymbol{x}, -\boldsymbol{\Omega})\,.$$

An analogy to Fick's law

To get an expression for the radiation flux use the identity $\boldsymbol{F} = \int_{4\pi} \boldsymbol{\Omega}\tilde{\Psi}\,\mathrm{d}\Omega$, which gives

$$\begin{aligned} \boldsymbol{F}(\boldsymbol{x},t) &= \int_{4\pi} \boldsymbol{\Omega}\left\{-f\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi(\boldsymbol{x},t)\right\}\mathrm{d}\Omega \\ &= -\left[\int_{4\pi} \boldsymbol{\Omega}\boldsymbol{\Omega} f\,\mathrm{d}\Omega\right]\cdot\boldsymbol{\nabla}\phi(\boldsymbol{x},t) \\ &\equiv -\mathbf{D}\cdot\boldsymbol{\nabla}\phi\,. \end{aligned}$$

Substitute into radiation energy conservation equation:

$$\frac{1}{c}\frac{\partial\phi}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{F} + \sigma^*\phi = \sigma a c T^4 + cQ$$

Couple with the material energy balance equation:

$$\frac{1}{c_v}\frac{\partial T}{\partial t} = \sigma^*\phi - \sigma^* a c T^4$$

Approximate the red terms semi-implicitly.

The transport problem used to calculate $\mathbf{D}$ is

$$\boldsymbol{\Omega} \cdot \boldsymbol{\nabla} f + \sigma^* f = \frac{1}{4\pi}\,, \boldsymbol{x} \in V,\ \boldsymbol{\Omega} \in 4\pi\,,$$

with boundary condition

$$f(\boldsymbol{x}, \boldsymbol{\Omega}) = f(\boldsymbol{x}, -\boldsymbol{\Omega})\,, \boldsymbol{x} \in \partial V,\ \boldsymbol{\Omega} \cdot \boldsymbol{n} < 0\,.$$

- Takes only one transport sweep to solve if the boundaries are many mean free paths apart
- Only needs to be calculated once per time step (because of changing σ^*) in a nonlinear problem
- Requires no storage of the angular intensity, just accumulation of second moment, $D_{ij} = \int_{4\pi} \Omega_i \Omega_j f \, \mathrm{d}\Omega$
- Has the solution $f = 1/4\pi\sigma$ if σ is a constant. Then, $\int_{4\pi} \boldsymbol{\Omega} f \boldsymbol{\Omega} \, \mathrm{d}\Omega = \mathbf{I}/3\sigma$.

Properties of anisotropic diffusion

The anisotropic diffusion tensor $\mathbf{D}(\boldsymbol{x}, t)$:

- Does not “blow up” in void regions
- Has a greater “action” along the direction of a voided channel than across it
- Reduces to $\mathbf{I}/3\sigma$ for an infinite homogeneous medium, which gives standard diffusion solution
- Is continuous in $\boldsymbol{x}$, so the approximate AD-calculated ϕ has continuous first derivatives (i.e., ϕ is smooth like our ansatz requires)

Outline

Compared methods

- Implicit Monte Carlo (IMC) [3] implemented with variance reduction methods, 10^7 particles per time step
- Flux-limited diffusion (FLD) with Larsen limiter [4], with semi-implicit treatment of diffusion coefficient and radiation:

$$\boldsymbol{F}^{n+1} = -D^n \boldsymbol{\nabla}\phi^{n+1} = -\left[(3\sigma^n)^2 + \left(\frac{\|\boldsymbol{\nabla}\phi^n\|}{\phi^n}\right)^2\right]^{-1/2} \boldsymbol{\nabla}\phi^{n+1}$$

- Standard diffusion, with semi-implicit treatment of nonlinearities:

$$\boldsymbol{F}^{n+1} = -D^n \boldsymbol{\nabla}\phi^{n+1} = -\frac{1}{3\sigma^n}\boldsymbol{\nabla}\phi^{n+1}$$

- Anisotropic diffusion, with semi-implicit treatment of nonlinearities:

$$\boldsymbol{F}^{n+1} = -\mathbf{D}^n \cdot \boldsymbol{\nabla}\phi^{n+1}$$

AD implementation

Approximations in the theory

- Assume weak gradients and angular moments for I (*don't* assume that I is a linear function of $\mathbf{\Omega}$!)
- Apply semi-implicit approximation for nonlinear material coupling and radiation

$\mathbb{D}$ transport equation

- S_N angular approximation
- DD spatial approximation
- One source iteration per time step

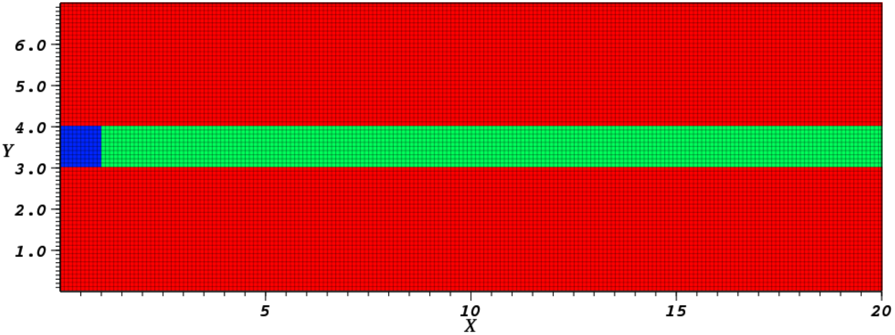
AD equation

- 9-point cell-centered finite difference spatial approximation

Problem description

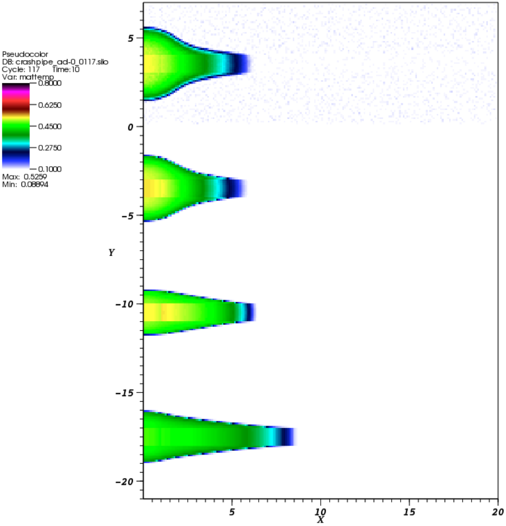
Flatland geometry!
Uniform spatial grid: $\Delta_x = 0.1$
Piecewise linear time grid: $\Delta_t = 0.1$ for $t \geq 1$
Reflecting bndy on left, others vacuum

Source: $c_v = 0.5$, $\sigma = 0.5$; $Q = 1$ for $0 \leq t \leq 1$, $Q = 0$ for $t > 1$.
Diffusive: $c_v = 0.1$, $\sigma = T^{-3}$
Channel: $c_v = 0.1$, $\sigma = 0.01T^{-3}$
Initial condition: $T = T_{\text{rad}} = 0.1$

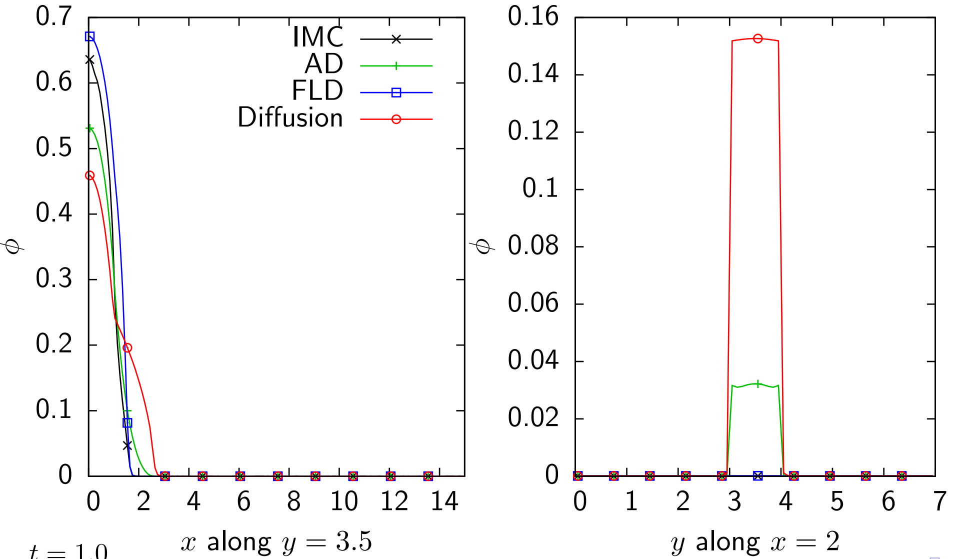


Time evolution of material temperature

(Replaced with movie)

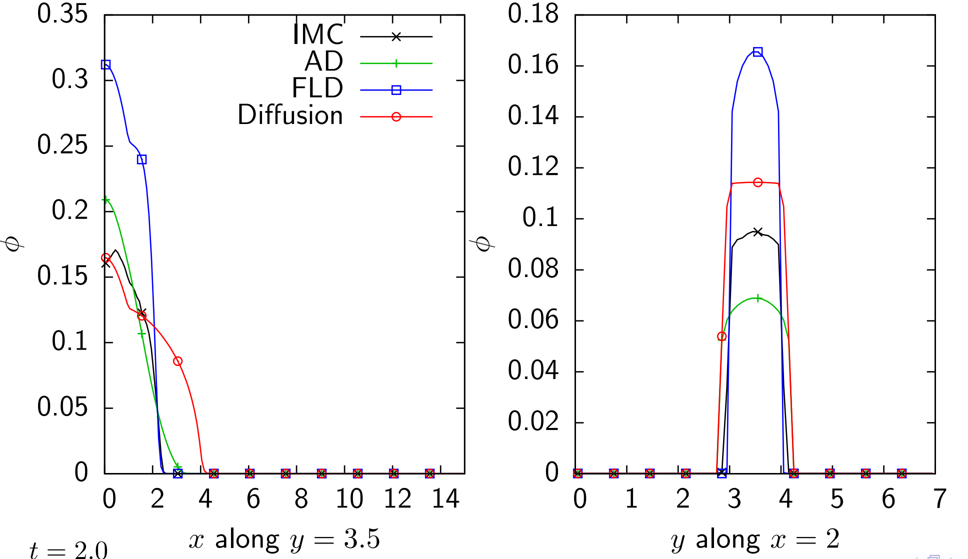


Time evolution of radiation energy density



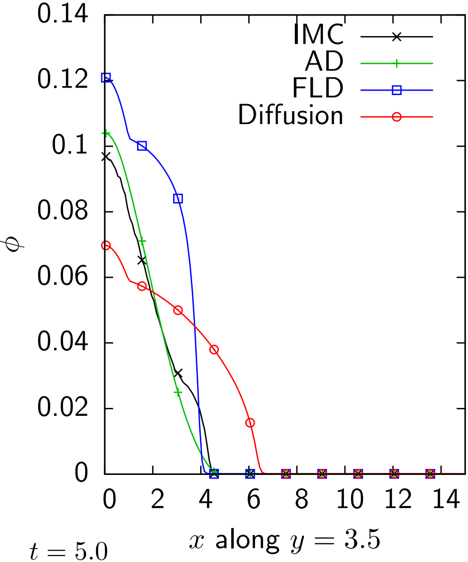
$t = 1.0$

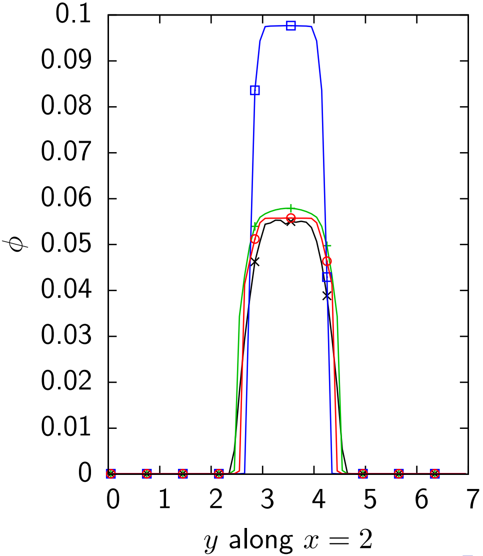
Time evolution of radiation energy density



$t = 2.0$

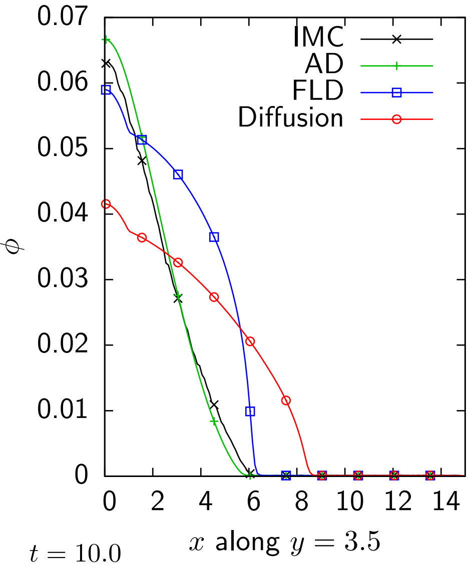
Time evolution of radiation energy density

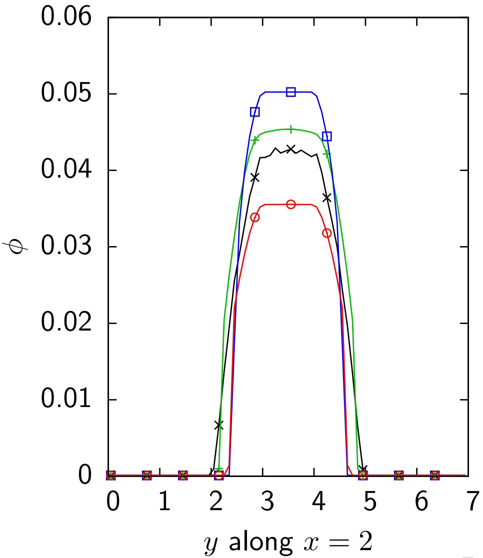




$t = 5.0$

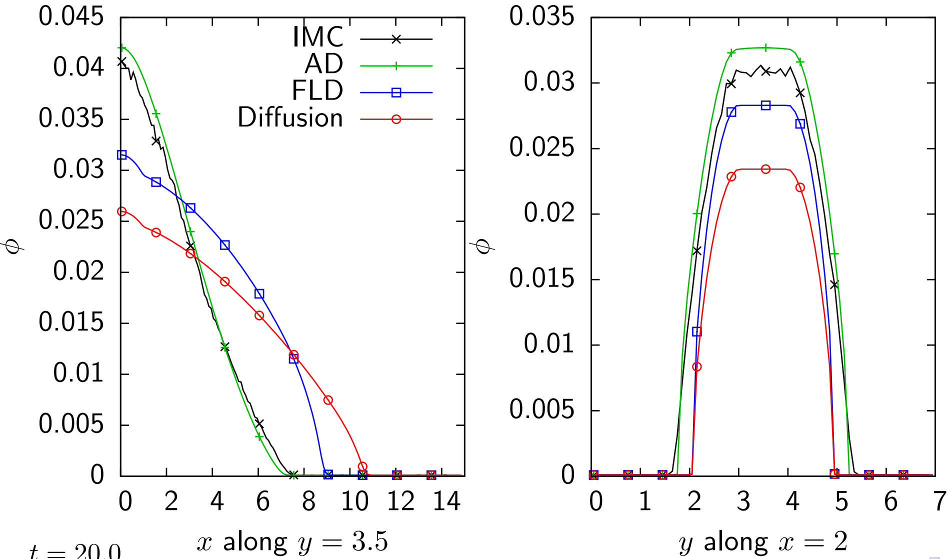
Time evolution of radiation energy density





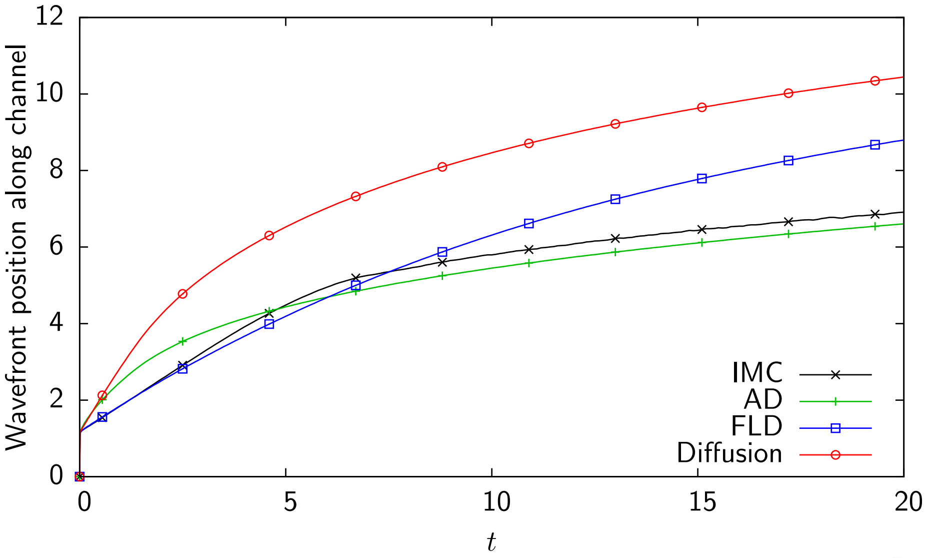
$t = 10.0$

Time evolution of radiation energy density

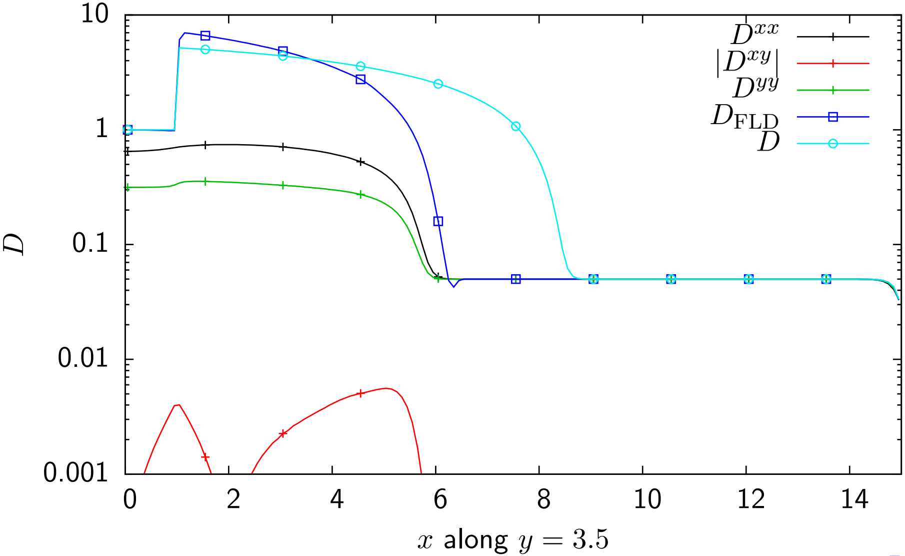


$t = 20.0$

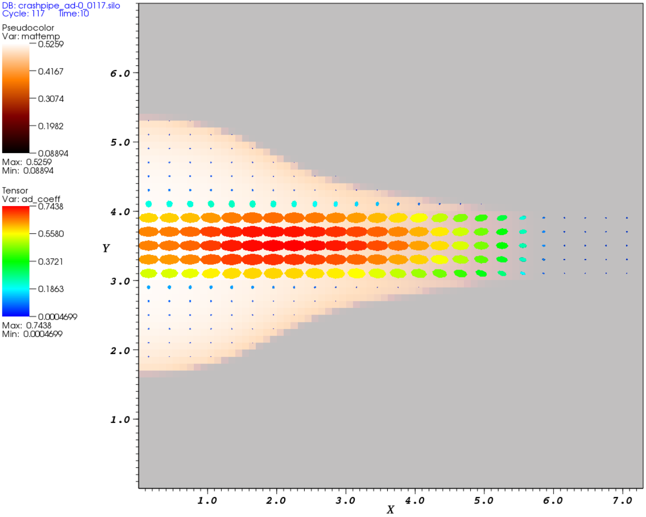
Time evolution of radiation temperature wavefront



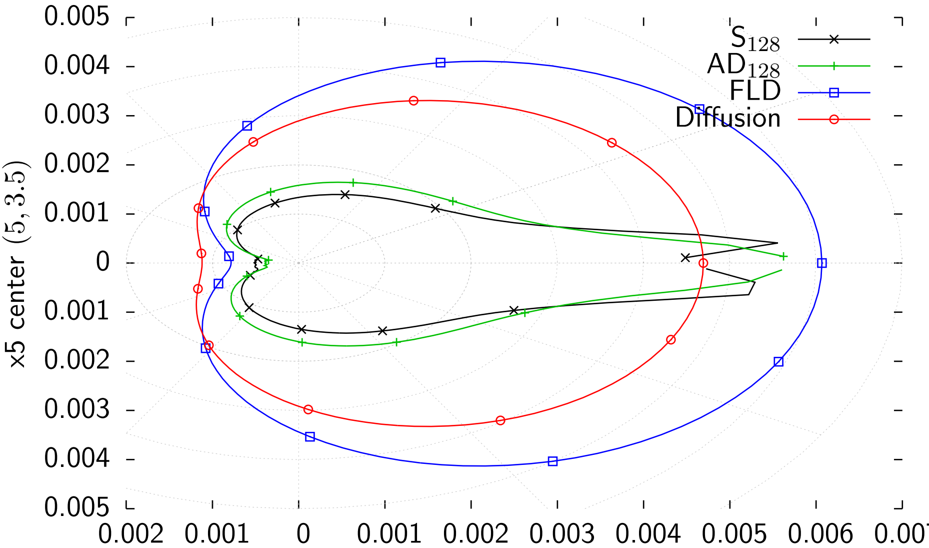
Diffusion coefficients ($t = 10$)



Anisotropic diffusion tensor visualization



Approximate representations of the intensity



Timing results

Method	Wall time (s)
IMC	2730
FLD	21
D	20
AD_{64}	36
AD_{128}	59

Table 1: Approximate run times for pipe test problem with $\Delta_x = 0.1$.

Outline

Conclusions

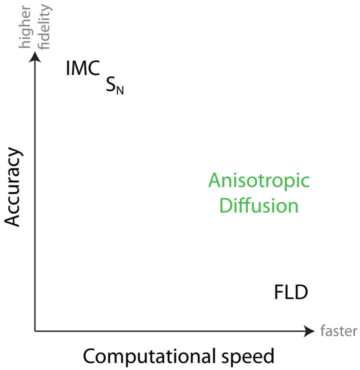
Anisotropic diffusion:

- Accounts for some amount of arbitrary anisotropy in angular intensity, unlike standard or flux-limited diffusion, by preserving some transport physics
- Works best in problems with weaker derivatives, as suggested by theory and borne out by numerical experiments
- Accurately treats the nonlinear time-dependent flow of radiation through a tube like that found in CRASH experiments

Future work

- Further analysis of boundary conditions
- Implement and test "Anisotropic P_1" ($\frac{1}{c}\frac{\partial}{\partial t} = O(\epsilon)$ instead of $O(\epsilon^2)$)
- Extend method to anisotropic internal sources
- Keep the $\boldsymbol{\nabla} \cdot \boldsymbol{F}$ term by ignoring assumption of $\int_{4\pi} \boldsymbol{\Omega}(\cdot)\, \mathrm{d}\Omega = O(\epsilon)$

Questions?



This material is based upon work supported under a National Science Foundation Graduate Research Fellowship and a Department of Energy Nuclear Energy University Programs Graduate Fellowship. Any opinions, findings, conclusions or recommendations expressed in this publication are those of the author and do not necessarily reflect the views of the National Science Foundation or the Department of Energy Office of Nuclear Energy.

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