An anisotropic diffusion approximation to nonlinear radiation transport

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Outline

Thermal radiative transfer

- TRT is the dominant heat transfer process in very hot materials
- Photons born isotropically via black body emission ($q_{\text{rad}} \propto \sigma T^4$)
- Cold material heats up and becomes relatively transparent ($\sigma \propto T^{-3}$)

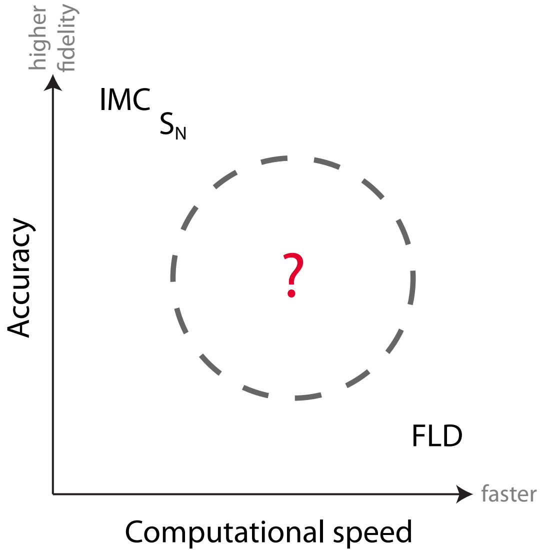
Difficulties in solving:

- High dimensionality of solution phase space $(\boldsymbol{x}, \boldsymbol{\Omega}, h\nu, t)$
- Highly nonlinear coupled partial differential equations for radiation field $I(\boldsymbol{x}, \boldsymbol{\Omega}, h\nu, t)$ and material energy

Particular application of this work: CRASH project

- Center for RAdiative Shock Hydrodynamics program: "Assessment of Predictive Capability"
- Simulate laser-driven shock in a xenon-filled tube
- Uncertainty quantification: hundreds of solution instances needed

Motivation


higher
fidelity
IMC
S_N
Accuracy
?
FLD
faster
Computational speed

Gray TRT equations

Common approximations for radiation transport methods development:

- work in a fixed medium, disregarding material advection;
- assume local thermodynamic equilibrium (LTE), which uses a single material temperature;
- neglect thermal conduction in material;
- average over all photon energies $h\nu$ (gray).

Radiation transfer equation, intensity $I(\boldsymbol{x}, \boldsymbol{\Omega}, t)$:

$$\frac{1}{c}\frac{\partial I}{\partial t} + \boldsymbol{\Omega} \cdot \boldsymbol{\nabla} I + \sigma I = \frac{\sigma c a T^4}{4\pi} + \frac{cQ}{4\pi} \tag{1a}$$

Material energy balance equation:

$$\frac{1}{c_v}\frac{\partial T}{\partial t} = \sigma \int_{4\pi} I \,\mathrm{d}\Omega - \sigma c a T^4 \tag{1b}$$

Anisotropic diffusion

Previous work:

- Steady-state infinite medium VHTR-like problem with analytically calculated coefficients [1]
- Non-local tensor diffusion [2] for steady-state radiative transfer, no further development or analysis in literature

Current work:

- Formulates boundary conditions and time-dependent terms
- Uses transport-calculated anisotropic diffusion tensors
- Applies to nonlinear, time-dependent problems with isotropic sources

Potential applications:

- Extends diffusion theory to new regimes of applicability
- Variance reduction with shielding problems that have voids

Outline

1. Define the anisotropic intensity as $\Psi = I - \frac{1}{4\pi}\phi$. To handle boundary conditions, define $\Psi \equiv \tilde{\Psi} + \Psi_{\mathrm{bl}}$. We will approximate $\tilde{\Psi}$ rather than I, and use Ψ_{bl} to determine matched boundary conditions.
2. From the radiation transport equation and conservation equation, we get a differential transport equation for $\tilde{\Psi}$ and Ψ_{bl}. Transform the former to an *integral* transport equation for $\tilde{\Psi}$.
3. Assume $I = O(1)$, $\frac{1}{c}\frac{\partial}{\partial t} = O(\epsilon^2)$, $\boldsymbol{\nabla} = O(\epsilon)$, $\int_{4\pi} \boldsymbol{\Omega}(\cdot)\,\mathrm{d}\Omega = O(\epsilon)$.
4. Use Taylor series to approximate nonlocal unknowns with local unknowns, discarding small terms. This yields

$$\tilde{\Psi}(\boldsymbol{x}, \boldsymbol{\Omega}) \approx -f(\boldsymbol{x}, \boldsymbol{\Omega})\boldsymbol{\Omega} \cdot \boldsymbol{\nabla}\phi\,.$$

5. Apply standard transport-matching procedure to Ψ_{bl}. Use the identity $\int_{4\pi} \Psi\,\mathrm{d}\Omega = 0$ to find the boundary condition for f.
6. Take the first angular moment of $\tilde{\Psi}$ to get $\boldsymbol{F} = -\mathbf{D} \cdot \boldsymbol{\nabla}\phi$
7. Substitute $\boldsymbol{F}$ into the time-dependent particle conservation equation to get time-dependent anisotropic diffusion.

Transport equation

Inside a time step, with "frozen" opacities:

$$\frac{1}{c}\frac{\partial I}{\partial t}(\boldsymbol{x},\boldsymbol{\Omega},t) + \boldsymbol{\Omega}\cdot\boldsymbol{\nabla} I(\boldsymbol{x},\boldsymbol{\Omega},t) + \sigma^*(\boldsymbol{x}) I(\boldsymbol{x},\boldsymbol{\Omega},t)$$
$$= \frac{1}{4\pi}\sigma^*(\boldsymbol{x}) ac[T(\boldsymbol{x},t)]^4 + \frac{1}{4\pi} q_r(\boldsymbol{x},t) \equiv \frac{1}{4\pi} Q(\boldsymbol{x},t)\,,$$
$$x \in V,\ 0 \le t < \Delta_t,\ \boldsymbol{\Omega} \in 4\pi, \quad \text{(2a)}$$

with the boundary condition

$$I(\boldsymbol{x},\boldsymbol{\Omega},t) = I^b(\boldsymbol{x},\boldsymbol{\Omega},t)\,, \quad \boldsymbol{x} \in \partial V,\ \boldsymbol{\Omega}\cdot\boldsymbol{n} < 0,\ 0 \le t < \Delta_t \quad \text{(2b)}$$

and the initial condition

$$I(\boldsymbol{x},\boldsymbol{\Omega},0) = I^i(\boldsymbol{x},\boldsymbol{\Omega},t)\,, \quad \boldsymbol{x} \in V,\ \boldsymbol{\Omega} \in 4\pi\,. \quad \text{(2c)}$$

Conservation equations

Operating on Eq. (2a) by $\int_{4\pi} (\cdot)\, \mathrm{d}\Omega$ gives

$$\frac{1}{c}\frac{\partial \phi}{\partial t}(\boldsymbol{x}, t) + \boldsymbol{\nabla} \cdot \boldsymbol{F}(\boldsymbol{x}, t) + \sigma^* \phi(\boldsymbol{x}, t) = Q(\boldsymbol{x}, t)\,. \tag{3a}$$

and on the initial condition, Eq. (2c),

$$\phi(\boldsymbol{x}, 0) = \int_{4\pi} I^i(\boldsymbol{x}, \boldsymbol{\Omega})\, \mathrm{d}\Omega = \phi^i(\boldsymbol{x})\,. \tag{3b}$$

Add $\boldsymbol{\Omega} \cdot \boldsymbol{\nabla}\phi$ to both sides of Eq. (3a) and multiply by $\frac{1}{4\pi}$:

$$\frac{1}{4\pi}\frac{1}{c}\frac{\partial \phi}{\partial t} + \frac{1}{4\pi}\boldsymbol{\Omega} \cdot \boldsymbol{\nabla}\phi + \frac{1}{4\pi}\sigma^* \phi = \frac{1}{4\pi}Q(\boldsymbol{x}, t) + \frac{1}{4\pi}\boldsymbol{\Omega} \cdot \boldsymbol{\nabla}\phi - \frac{1}{4\pi}\boldsymbol{\nabla} \cdot \boldsymbol{F} \tag{4}$$

Anisotropic intensity equations

Define "anisotropic intensity":

$$\Psi(\boldsymbol{x}, \boldsymbol{\Omega}) \equiv I(\boldsymbol{x}, \boldsymbol{\Omega}) - \frac{1}{4\pi}\phi(\boldsymbol{x})\,. \tag{5}$$

(This satisfies $\int_{4\pi} \Psi = 0$ and $\int_{4\pi} \boldsymbol{\Omega}\Psi = \boldsymbol{F}$.)
Subtract Eq. (4) from Eq. (2a); the isotropic source cancels:

$$\frac{1}{c}\frac{\partial}{\partial t}\left[I - \frac{\phi}{4\pi}\right] + \boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\left[I - \frac{\phi}{4\pi}\right] + \sigma^*(\boldsymbol{x})\left[I - \frac{\phi}{4\pi}\right] = \frac{1}{4\pi}\boldsymbol{\nabla}\cdot\boldsymbol{F} - \frac{1}{4\pi}\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi$$

Subtract $\phi/4\pi$ from the transport boundary condition:

$$I - \frac{\phi}{4\pi} = I^b - \frac{\phi}{4\pi}$$

Subtract Eq. (3b) from Eq. (2c):

$$I(\boldsymbol{x}, \boldsymbol{\Omega}, 0) - \frac{1}{4\pi}\phi(\boldsymbol{x}, 0) = I^i - \frac{\phi^i}{4\pi}$$

Anisotropic intensity equations

Transport equation:

$$\frac{1}{c}\frac{\partial}{\partial t}\Psi + \boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\Psi + \sigma^*(\boldsymbol{x})\Psi = \frac{1}{4\pi}\boldsymbol{\nabla}\cdot\boldsymbol{F} - \frac{1}{4\pi}\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi \equiv \hat{Q}(\boldsymbol{x},\boldsymbol{\Omega},t)$$

Boundary condition:

$$\Psi = \Psi^b = I^b - \frac{\phi}{4\pi}$$

Initial condition:

$$\Psi(\boldsymbol{x},\boldsymbol{\Omega},0) = \Psi^i = I^i - \frac{\phi^i}{4\pi}\,.$$

The exact solutions for I, ϕ, $\boldsymbol{F}$ satisfy these equations: still no approximations.

Boundary layer equations

In anticipation of approximating $\tilde{\Psi} = -f\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi$, separate Ψ into a boundary layer plus an internal solution:

$$\Psi(\boldsymbol{x},\boldsymbol{\Omega},t) \equiv \tilde{\Psi}(\boldsymbol{x},\boldsymbol{\Omega},t) + \Psi_{\mathrm{bl}}(\boldsymbol{x},\boldsymbol{\Omega},t)\,.$$

The exact equations for $\tilde{\Psi}$:

$$\frac{1}{c}\frac{\partial}{\partial t}\tilde{\Psi} + \boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\tilde{\Psi} + \sigma^*(\boldsymbol{x})\tilde{\Psi} = \hat{Q}(\boldsymbol{x},\boldsymbol{\Omega},t)$$

with new boundary condition for $\boldsymbol{x}\in\partial V$, $\boldsymbol{\Omega}\cdot\boldsymbol{n}<0$:

$$\tilde{\Psi} = -\zeta\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi\,.$$

Therefore the corresponding boundary layer equation is:

$$\frac{1}{c}\frac{\partial}{\partial t}\Psi_{\mathrm{bl}} + \boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\Psi_{\mathrm{bl}} + \sigma^*(\boldsymbol{x})\Psi_{\mathrm{bl}} = 0$$

with boundary condition for $\boldsymbol{x}\in\partial V$, $\boldsymbol{\Omega}\cdot\boldsymbol{n}<0$:

$$\Psi_{\mathrm{bl}} = I^b - \frac{1}{4\pi}\phi + \zeta\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi\,.$$

Integral transport equation

Streaming path from $(\boldsymbol{x}, t)$ backward along $-\boldsymbol{\Omega}$, accumulate sources and attenuate:

$$\begin{aligned}\tilde{\Psi}(\boldsymbol{x}, \boldsymbol{\Omega}, t) = {} & \tilde{\Psi}^b(\boldsymbol{x} - s_b\boldsymbol{\Omega}, \boldsymbol{\Omega}, t - s_b/c)\,\mathrm{e}^{-\tau(\boldsymbol{x}, \boldsymbol{x} - s_b\boldsymbol{\Omega})} U(ct - s_b) \\ & + \Psi^i(\boldsymbol{x} - ct\boldsymbol{\Omega}, \boldsymbol{\Omega})\,\mathrm{e}^{-\tau(\boldsymbol{x}, \boldsymbol{x} - ct\boldsymbol{\Omega})} U(s_b - ct) \\ & + \int_0^{s_b} \left[\hat{Q}(\boldsymbol{x} - s\boldsymbol{\Omega}, \boldsymbol{\Omega}, t - s/c)\right] \mathrm{e}^{-\tau(\boldsymbol{x}, \boldsymbol{x} - s\boldsymbol{\Omega})}\,\mathrm{d}s\,. \end{aligned} \tag{6a}$$

$$\equiv \mathcal{I}_b\left[\tilde{\Psi}^b\right] + \mathcal{I}_i\left[\Psi^i\right] + \mathcal{I}_v\left[\hat{Q}\right] \tag{6b}$$

s_b is the distance to the boundary, $U(\cdots)$ is the heaviside function, and the optical thickness is

$$\tau(\boldsymbol{x}, \boldsymbol{x}') = \int_0^{\|\boldsymbol{x} - \boldsymbol{x}'\|} \sigma^*(\boldsymbol{x} - s\boldsymbol{\Omega})\,\mathrm{d}s\,. \tag{6c}$$

These are nonlocal unknowns; we will approximate them with local unknowns.

Time for some approximations

Asymptotic ansatz: assume weak spatial gradients, mildly anisotropic intensity, very small time derivative:

$$I = O(1), \qquad \boldsymbol{\nabla} I = O(\epsilon) \qquad \int_{4\pi} \boldsymbol{\Omega} I \, \mathrm{d}\Omega = O(\epsilon) \qquad \frac{1}{c}\frac{\partial}{\partial t} = O(\epsilon^2)$$

Our first approximation: $\mathcal{I}_i[\cdot] = O(\epsilon^2)$ and $\boldsymbol{\nabla} \cdot \boldsymbol{F} = O(\epsilon^2)$:

$$\tilde{\Psi} = \mathcal{I}_i\left[\Psi^i\right] - \mathcal{I}_b[\zeta \boldsymbol{\Omega} \cdot \boldsymbol{\nabla}\phi] + \mathcal{I}_v\left[\frac{1}{4\pi}\boldsymbol{\nabla} \cdot \boldsymbol{F}\right] - \mathcal{I}_v\left[\frac{1}{4\pi}\boldsymbol{\Omega} \cdot \boldsymbol{\nabla}\phi\right]$$

$$\tilde{\Psi} \approx -\mathcal{I}_b[\zeta \boldsymbol{\Omega} \cdot \boldsymbol{\nabla}\phi] - \mathcal{I}_v\left[\frac{1}{4\pi}\boldsymbol{\Omega} \cdot \boldsymbol{\nabla}\phi\right] + O(\epsilon^2)$$

Taylor series expansion:

$$\phi(\boldsymbol{x} - s\boldsymbol{\Omega}, t - s/c) \sim \phi(\boldsymbol{x}, t) - s\left(\frac{1}{c}\frac{\partial}{\partial t} + \boldsymbol{\Omega} \cdot \boldsymbol{\nabla}\right)\phi(\boldsymbol{x}, t) + O(\epsilon^2)$$

$$\phi(\boldsymbol{x} - s\boldsymbol{\Omega}, t - s/c) = \phi(\boldsymbol{x}, t) + O(\epsilon) \qquad (7)$$

Taylor series applied

If ϕ is smooth like the ansatz hypothesizes, the volumetric term becomes:

$$-\mathcal{I}_v\left[\frac{1}{4\pi}\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi\right] = -\int_0^{s_b}\left[\frac{1}{4\pi}\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi\right]_{(\boldsymbol{x}-s\boldsymbol{\Omega},t-s/c)}\mathrm{e}^{-\tau(\boldsymbol{x},\boldsymbol{x}-s\boldsymbol{\Omega})}\,\mathrm{d}s$$
$$\sim -\int_0^{s_b}\left[\frac{1}{4\pi}\right]\mathrm{e}^{-\tau(\boldsymbol{x},\boldsymbol{x}-s\boldsymbol{\Omega})}\,\mathrm{d}s\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi(\boldsymbol{x},t)+O(\epsilon^2)$$
$$= -\mathcal{I}_v\left[\frac{1}{4\pi}\right]\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi(\boldsymbol{x},t)\,. \tag{8}$$

The boundary term similarly is

$$-\mathcal{I}_b[\zeta\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi] = -\int_0^{s_b}[\zeta\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi]_{(\boldsymbol{x}-s_b\boldsymbol{\Omega},t-s_b/c)}\mathrm{e}^{-\tau(\boldsymbol{x},\boldsymbol{x}-s\boldsymbol{\Omega})}\,\mathrm{d}s$$
$$\sim -\mathcal{I}_b[\zeta]\,\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi(\boldsymbol{x},t)\,. \tag{9}$$

Thus,

$$\tilde{\Psi}(\boldsymbol{x},\boldsymbol{\Omega},t)\approx-\left[\mathcal{I}_b[\zeta]+\mathcal{I}_v\left[\frac{1}{4\pi}\right]\right]\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi(\boldsymbol{x},t)\equiv-f(\boldsymbol{x},\boldsymbol{\Omega})\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi(\boldsymbol{x},t)$$

Transport matched boundary

Transport theory: boundary solution decays quickly if we enforce the relation $0 = \int_{\boldsymbol{\Omega}\cdot\boldsymbol{n}<0} W(|\boldsymbol{\Omega}\cdot\boldsymbol{n}|)\Psi_{\mathrm{bl}}\,\mathrm{d}\Omega$ on the boundary, where $W(\mu) \approx 2\mu + 3\mu^2$ is related to the Chandrasekhar function. The transport extrapolation distance is $\int_0^1 \mu W\,\mathrm{d}\mu / \int_0^1 W\,\mathrm{d}\mu$.

$$
\begin{aligned}
0 &= \int_{\boldsymbol{\Omega}\cdot\boldsymbol{n}<0} W(|\boldsymbol{\Omega}\cdot\boldsymbol{n}|)\Psi_{\mathrm{bl}}\,\mathrm{d}\Omega \\
&= \int_{\boldsymbol{\Omega}\cdot\boldsymbol{n}<0} W\left[I^b - \frac{1}{4\pi}\phi + \zeta\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi\right]\mathrm{d}\Omega
\end{aligned}
$$

or

$$
\int_{\boldsymbol{\Omega}\cdot\boldsymbol{n}<0} W I^b\,\mathrm{d}\Omega = \phi - \int_{\boldsymbol{\Omega}\cdot\boldsymbol{n}<0} W\zeta\boldsymbol{\Omega}\,\mathrm{d}\Omega\cdot\boldsymbol{\nabla}\phi \tag{11}
$$

One more equation is needed to determine ζ.

Determining ζ

Recall that in the exact anisotropic transport equation, $\int_{4\pi} \Psi \, \mathrm{d}\Omega = 0$. So we choose to enforce $\int_{4\pi} \tilde{\Psi} \, \mathrm{d}\Omega = 0$ on the boundary:

$$0 = \int_{4\pi} \tilde{\Psi} \, \mathrm{d}\Omega = \int_{\boldsymbol{\Omega} \cdot \boldsymbol{n} < 0} (-\zeta \boldsymbol{\Omega} \cdot \boldsymbol{\nabla} \phi) \, \mathrm{d}\Omega + \int_{\boldsymbol{\Omega} \cdot \boldsymbol{n} > 0} (-f \boldsymbol{\Omega} \cdot \boldsymbol{\nabla} \phi) \, \mathrm{d}\Omega$$

Or:

$$\int_{\boldsymbol{\Omega} \cdot \boldsymbol{n} < 0} \boldsymbol{\Omega} \zeta \, \mathrm{d}\Omega \cdot \boldsymbol{\nabla} \phi = \int_{\boldsymbol{\Omega} \cdot \boldsymbol{n} > 0} [-\boldsymbol{\Omega}] f \, \mathrm{d}\Omega \cdot \boldsymbol{\nabla} \phi$$

$$\int_{\boldsymbol{\Omega} \cdot \boldsymbol{n} < 0} \boldsymbol{\Omega} \zeta(\boldsymbol{x}, \boldsymbol{\Omega}) \, \mathrm{d}\Omega \cdot \boldsymbol{\nabla} \phi = \int_{\boldsymbol{\Omega} \cdot \boldsymbol{n} < 0} \boldsymbol{\Omega} f(\boldsymbol{x}, -\boldsymbol{\Omega}) \, \mathrm{d}\Omega \cdot \boldsymbol{\nabla} \phi$$

One possible way to satisfy this is:

$$\zeta(\boldsymbol{x}, \boldsymbol{\Omega}) = f(\boldsymbol{x}, -\boldsymbol{\Omega})$$

for $\boldsymbol{x} \in \partial V$, $\boldsymbol{\Omega} \cdot \boldsymbol{n} < 0$. This is a reflecting boundary condition!

Summary of boundary layer analysis

Approximate expression for anisotropic intensity:

$$\begin{aligned}\tilde{\Psi}(\boldsymbol{x}, \boldsymbol{\Omega}, t) &\approx -\left\{\mathcal{I}_b[f(\boldsymbol{x}, -\boldsymbol{\Omega})] + \mathcal{I}_v\left[\frac{1}{4\pi}\right]\right\} \boldsymbol{\Omega} \cdot \boldsymbol{\nabla}\phi(\boldsymbol{x}, t) \\ &\equiv -\left\{f(\boldsymbol{x}, \boldsymbol{\Omega})\right\} \boldsymbol{\Omega} \cdot \boldsymbol{\nabla}\phi(\boldsymbol{x}, t)\end{aligned}$$

Low-order boundary condition (after substituting $\zeta(\boldsymbol{x}, -\boldsymbol{\Omega})$):

$$\begin{aligned}&\int_{\boldsymbol{\Omega}\cdot\boldsymbol{n}<0} W(|\boldsymbol{\Omega} \cdot \boldsymbol{n}|) I^b(\boldsymbol{x}, \boldsymbol{\Omega}, t)\, \mathrm{d}\Omega \\ &\qquad = \phi(\boldsymbol{x}, t) - \int_{\boldsymbol{\Omega}\cdot\boldsymbol{n}>0} \boldsymbol{\Omega} W(|\boldsymbol{\Omega} \cdot \boldsymbol{n}|) f(\boldsymbol{x}, \boldsymbol{\Omega})\, \mathrm{d}\Omega \cdot \boldsymbol{\nabla}\phi(\boldsymbol{x}, t)\end{aligned}$$

Boundary condition for f:

$$f(\boldsymbol{x}, \Omega) = f(\boldsymbol{x}, -\boldsymbol{\Omega})\,.$$

An analogy to Fick's law

To get an expression for the radiation flux use the identity $\boldsymbol{F} = \int_{4\pi} \boldsymbol{\Omega}\tilde{\Psi}\,\mathrm{d}\Omega$, which gives

$$\begin{aligned}\boldsymbol{F}(\boldsymbol{x},t) &= \int_{4\pi} \boldsymbol{\Omega}\left\{-f\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi(\boldsymbol{x},t)\right\}\mathrm{d}\Omega \\ &= -\left[\int_{4\pi}\boldsymbol{\Omega}\boldsymbol{\Omega}f\,\mathrm{d}\Omega\right]\cdot\boldsymbol{\nabla}\phi(\boldsymbol{x},t) \\ &\equiv -\mathbf{D}\cdot\boldsymbol{\nabla}\phi\,.\end{aligned}$$

Substitute into radiation energy conservation equation:

$$\frac{1}{c}\frac{\partial\phi}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{F} + \sigma^*\phi = \sigma acT^4 + cQ$$

Couple with the material energy balance equation:

$$\frac{1}{c_v}\frac{\partial T}{\partial t} = \sigma^*\phi - \sigma^* acT^4$$

Approximate the red terms semi-implicitly.

The transport problem used to calculate $\mathbf{D}$ is

$$\boldsymbol{\Omega} \cdot \boldsymbol{\nabla} f + \sigma^* f = \frac{1}{4\pi}\,, \boldsymbol{x} \in V,\ \boldsymbol{\Omega} \in 4\pi\,,$$

with boundary condition

$$f(\boldsymbol{x}, \boldsymbol{\Omega}) = f(\boldsymbol{x}, -\boldsymbol{\Omega})\,, \boldsymbol{x} \in \partial V,\ \boldsymbol{\Omega} \cdot \boldsymbol{n} < 0\,.$$

- Takes only one transport sweep to solve if the boundaries are many mean free paths apart
- Only needs to be calculated once per time step (because of changing σ^*) in a nonlinear problem
- Requires no storage of the angular intensity, just accumulation of second moment, $D_{ij} = \int_{4\pi} \Omega_i \Omega_j f \,\mathrm{d}\Omega$
- Has the solution $f = 1/4\pi\sigma$ if σ is a constant. Then, $\int_{4\pi} \boldsymbol{\Omega} f \boldsymbol{\Omega} \,\mathrm{d}\Omega = \mathbf{I}/3\sigma$.

Properties of anisotropic diffusion

The anisotropic diffusion tensor $\mathbf{D}(\boldsymbol{x}, t)$:

- Does not "blow up" in void regions
- Has a greater "action" along the direction of a voided channel than across it
- Reduces to $\mathbf{I}/3\sigma$ for a homogeneous medium, which gives standard diffusion solution (and boundary conditions reduce to transport-corrected diffusion BCs)
- Is continuous in $\boldsymbol{x}$, so the approximate AD-calculated ϕ has continuous first derivatives (i.e., ϕ is smooth like our ansatz requires)

Outline

Compared methods

- Implicit Monte Carlo (IMC) [3] implemented with variance reduction methods, 10^7 particles per time step
- Flux-limited diffusion (FLD) with Larsen limiter [4], with semi-implicit treatment of diffusion coefficient and radiation:

$$\boldsymbol{F}^{n+1} = -D^n \boldsymbol{\nabla}\phi^{n+1} = -\left[(3\sigma^n)^2 + \left(\frac{\|\boldsymbol{\nabla}\phi^n\|}{\phi^n}\right)^2\right]^{-1/2} \boldsymbol{\nabla}\phi^{n+1}$$

- Standard diffusion, with semi-implicit treatment of nonlinearities:

$$\boldsymbol{F}^{n+1} = -D^n \boldsymbol{\nabla}\phi^{n+1} = -\frac{1}{3\sigma^n} \boldsymbol{\nabla}\phi^{n+1}$$

- Anisotropic diffusion, with semi-implicit treatment of nonlinearities:

$$\boldsymbol{F}^{n+1} = -\mathbf{D}^n \cdot \boldsymbol{\nabla}\phi^{n+1}$$

AD implementation

Approximations in the theory

- Assume weak gradients and angular moments for I (*don't* assume that I is a linear function of $\mathbf{\Omega}$!)
- Apply semi-implicit approximation for nonlinear material coupling and radiation

$\mathbb{D}$ transport equation

- S_N angular approximation
- DD spatial approximation
- One source iteration per time step

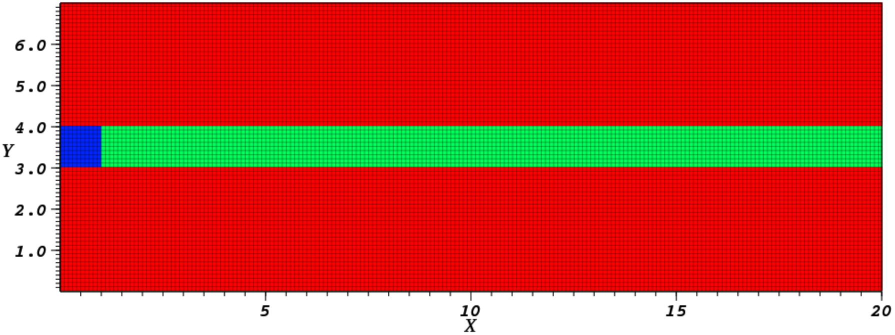
AD equation

- 9-point cell-centered finite difference spatial approximation

Problem description

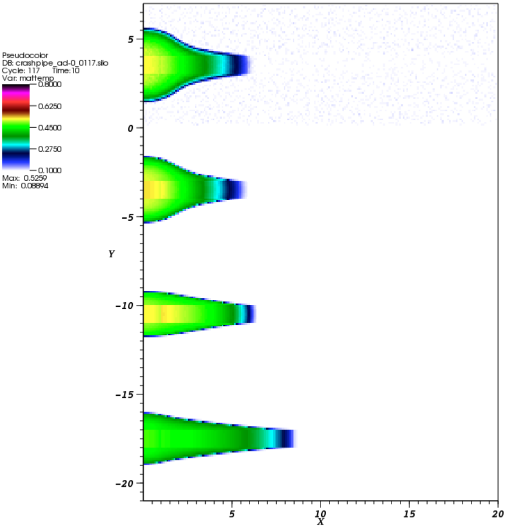
Flatland geometry!
Uniform spatial grid: $\Delta_x = 0.1$
Piecewise linear time grid: $\Delta_t = 0.1$ for $t \geq 1$
Reflecting bndy on left, others vacuum

Source: $c_v = 0.5$, $\sigma = 0.5$; $Q = 1$ for $0 \leq t \leq 1$, $Q = 0$ for $t > 1$.
Diffusive: $c_v = 0.1$, $\sigma = T^{-3}$
Channel: $c_v = 0.1$, $\sigma = 0.01T^{-3}$
Initial condition: $T = T_{\text{rad}} = 0.1$

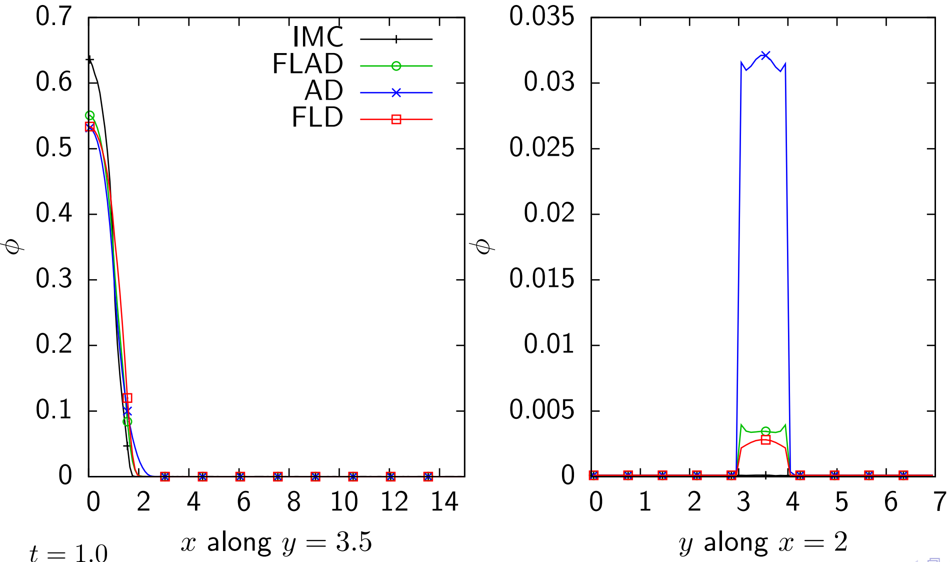


Time evolution of material temperature

(Replaced with movie)

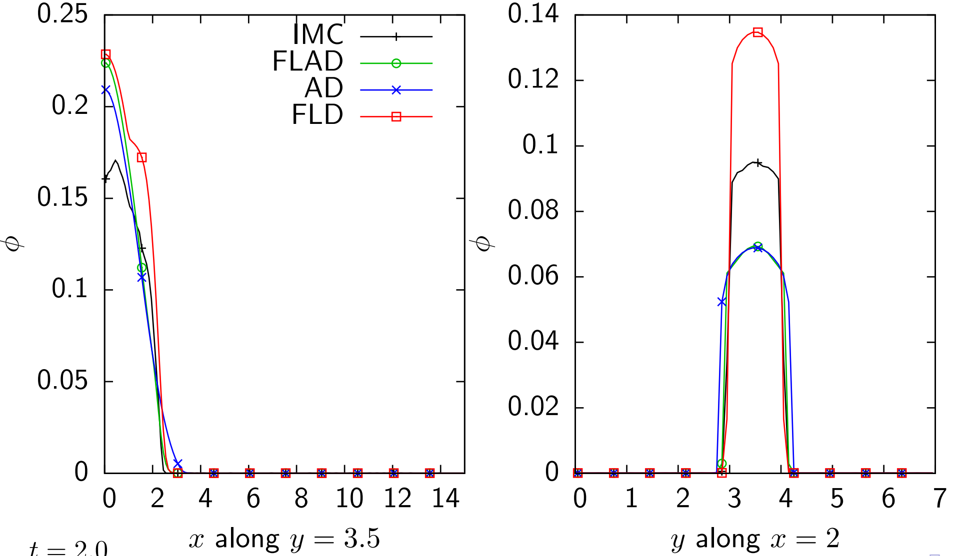


Time evolution of radiation energy density



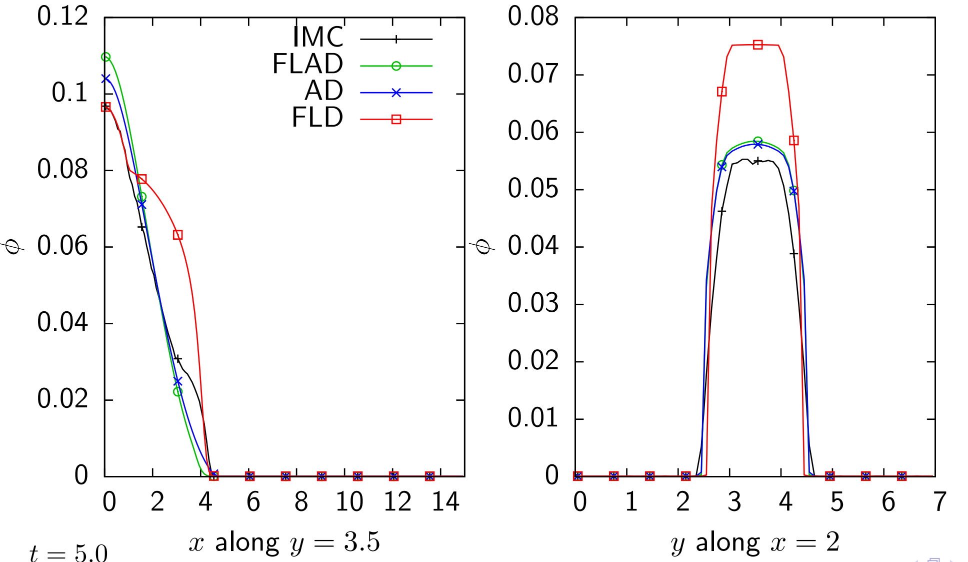
$t = 1.0$

Time evolution of radiation energy density



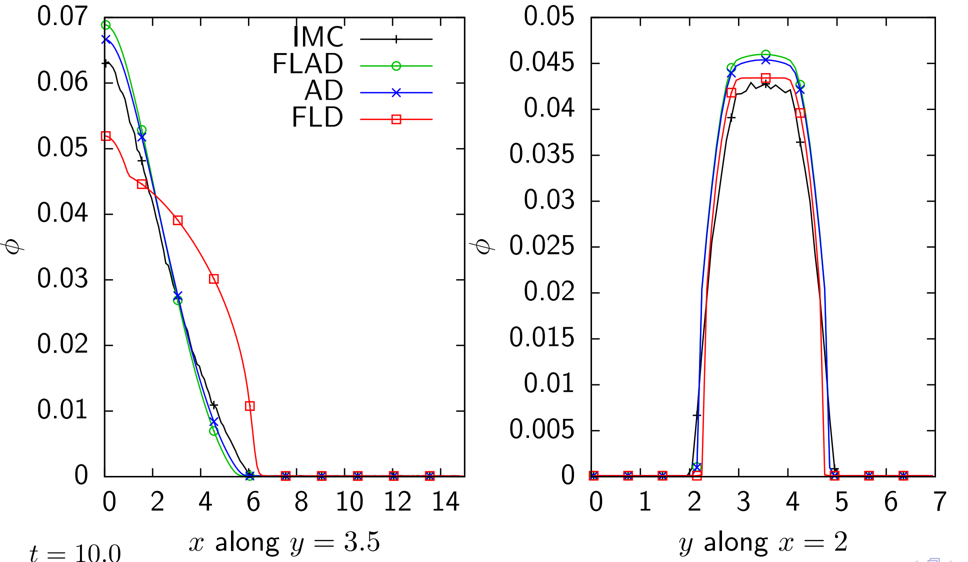
$t = 2.0$

Time evolution of radiation energy density



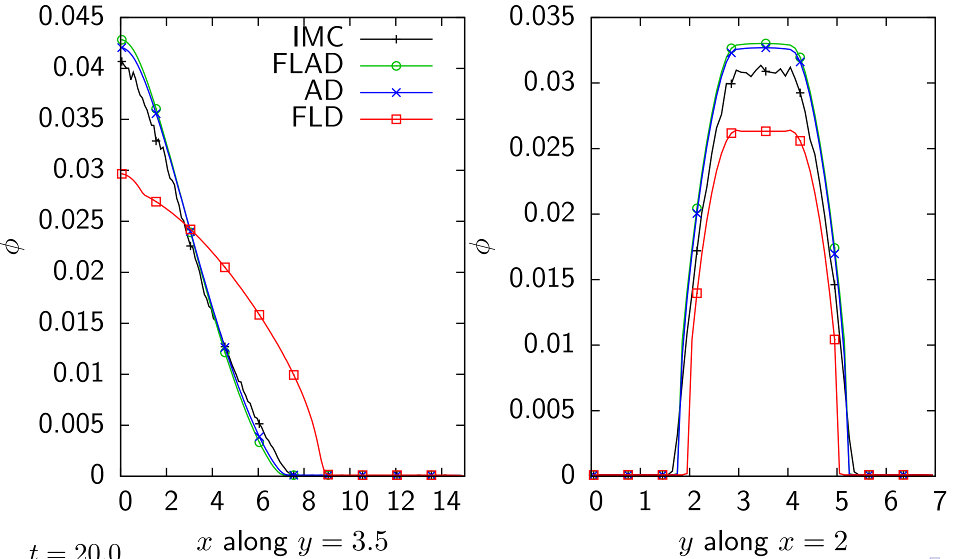
$t = 5.0$

Time evolution of radiation energy density



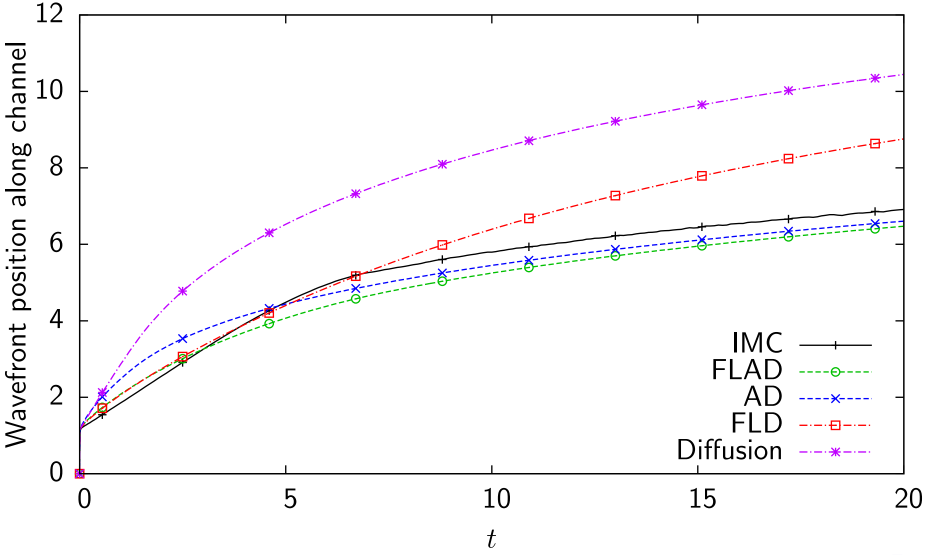
$t = 10.0$

Time evolution of radiation energy density

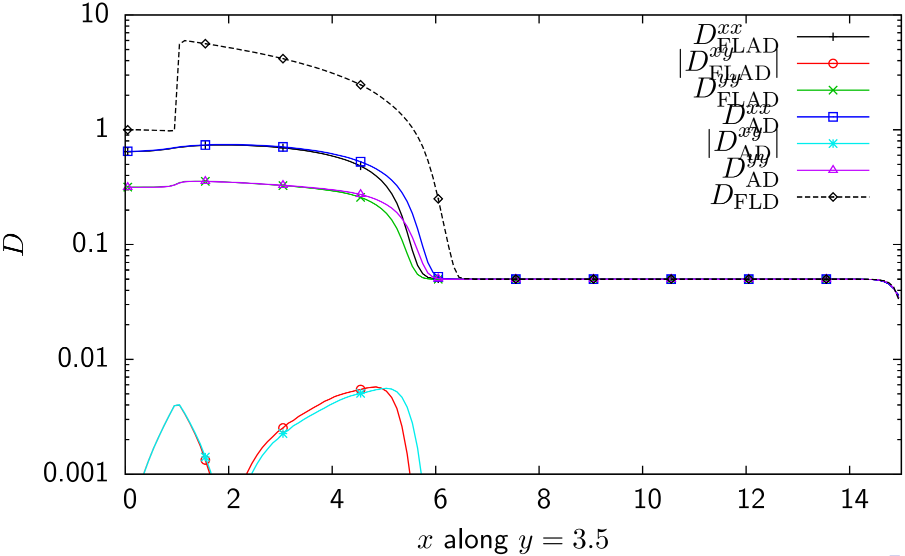


$t = 20.0$

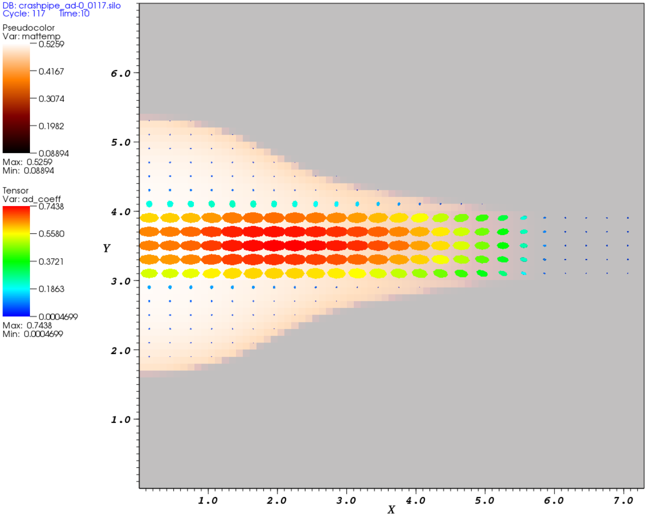
Time evolution of radiation temperature wavefront



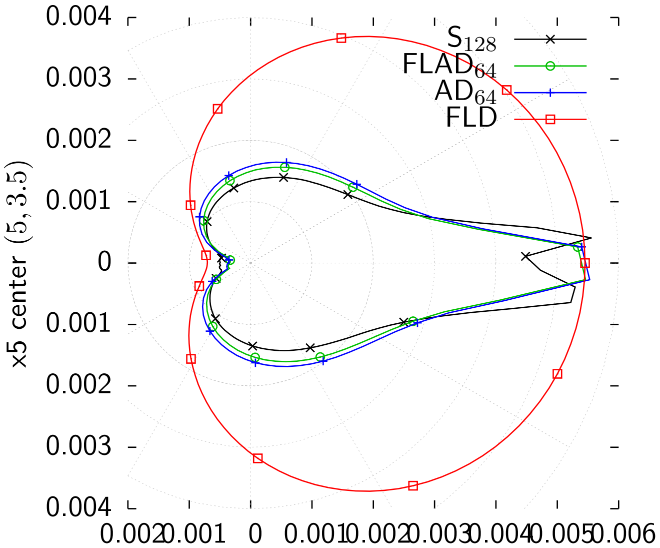
Diffusion coefficients ($t = 10$)



Anisotropic diffusion tensor visualization



Approximate representations of the intensity



Timing results

Method	Wall time (s)
IMC	2730
FLD	21
D	20
AD_{64}	36
AD_{128}	59

Table 1: Approximate run times for pipe test problem with $\Delta_x = 0.1$.

Outline

Conclusions

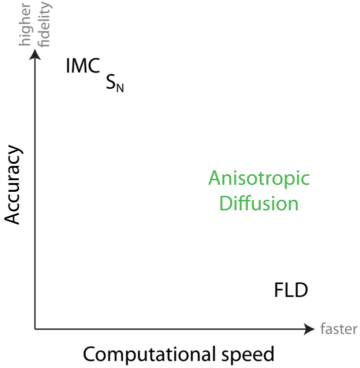
Anisotropic diffusion:

- Accounts for some amount of arbitrary anisotropy in angular intensity, unlike standard or flux-limited diffusion, by preserving some transport physics
- Works best in problems with weaker derivatives, as suggested by theory and borne out by numerical experiments
- Accurately treats the nonlinear time-dependent flow of radiation through a tube like that found in CRASH experiments

Future work

- Further analysis of boundary conditions
- Implement and test "Anisotropic P_1" ($\frac{1}{c}\frac{\partial}{\partial t} = O(\epsilon)$ instead of $O(\epsilon^2)$)
- Extend method to anisotropic internal sources
- Keep the $\boldsymbol{\nabla} \cdot \boldsymbol{F}$ term by ignoring assumption of $\int_{4\pi} \boldsymbol{\Omega}(\cdot)\,\mathrm{d}\Omega = O(\epsilon)$

Questions?



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References

E. W. LARSEN and T. J. TRAHAN, "2-D Anisotropic Diffusion in Optically Thin Channels," *Trans. Am. Nucl. Soc.*, **101**, 387–389 (2009), [CD-ROM].

J. E. MOREL, "A Non-Local Tensor Diffusion Theory," Tech. Rep. LA-UR-07-5257, Los Alamos National Laboratory (2007).

J. A. FLECK, JR. and J. D. CUMMINGS, "An Implicit Monte Carlo Scheme for Calculating Time and Frequency Dependent Nonlinear Radiation Transport," *J. Comput. Phys.*, **8**, 3, 313–342 (1971).

G. L. OLSON, L. H. AUER, and M. L. HALL, "Diffusion, P_1, and other approximate forms of radiation transport," *J. Quant. Spectrosc. Radiat. Transfer*, **64**, 619–634 (2000).