Flatland Geometry

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Chapter from Seth's dissertation

Flatland geometry is a fictional two-dimensional space where particles are constrained to the page [1, 2]. This differs from standard 2-D geometry, which represents a 3-D problem that is invariant in the z axis, where particles can travel at different polar angles out of the page. The constraint of living in the page reduces the phase space of the transport equation, as the flatland solution is only a function of the azimuthal angle rather than both azimuthal and polar angles. This reduction in phase space makes flatland geometry a computationally less burdensome testing ground for new methods.

Despite being easier computationally to solve, flatland has a few subtle quirks that need to be taken into account before implementing a solver in that geometry. Previous work in flatland [2, 3] has shown that the diffusion coefficient for flatland geometry $\frac{1}{2\sigma}$ is different from the physical diffusion coefficients $\frac{1}{3\sigma}$, but the correct formulation for flatland diffusion boundary conditions have remained an unanswered and indeed unasked question. This chapter answers that question in addition to providing other insights into this strange geometry.

1 Transport in flatland

Two-dimensional x-y geometry represents a three-dimensional problem that is invariant in the z direction.

The general-geometry steady-state transport equation with isotropic scattering and an isotropic source is

$$\mathbf{\Omega} \cdot \nabla \psi + \sigma \psi = \frac{\sigma_s}{\omega_0} \int_{\Omega} \psi \, \mathrm{d}\Omega + \frac{q}{\omega_0} \,. \tag{1}$$

Geometry	$\boldsymbol{\Omega}$	Domain Ω	$\mathrm{d}\Omega$	$\omega_0 \equiv \int_\Omega \mathrm{d}\Omega$	$\omega_1 \equiv \int_\Omega \lvert\boldsymbol{\Omega}\cdot\boldsymbol{i}\rvert\,\mathrm{d}\Omega$	$\omega_2 \equiv \int_\Omega (\boldsymbol{\Omega}\cdot\boldsymbol{i})^2\,\mathrm{d}\Omega$
1D	μ	$-1 \le \mu \le 1$	$\mathrm{d}\mu$	2	1	$\frac{2}{3}$
2D	$\sqrt{1-\mu^2}\cos\theta\,\boldsymbol{i} + \sqrt{1-\mu^2}\sin\theta\,\boldsymbol{j}$	$-1 \le \mu \le 1,\ 0 \le \theta < 2\pi$	$\mathrm{d}\mu\,\mathrm{d}\theta$	4π	2π	$\frac{4\pi}{3}$
Flatland	$\cos\theta\,\boldsymbol{i} + \sin\theta\,\boldsymbol{j}$	$0 \le \theta < 2\pi$	$\mathrm{d}\theta$	2π	4	π
3D	$\mu\boldsymbol{i} + \sqrt{1-\mu^2}\cos\theta\,\boldsymbol{j} + \sqrt{1-\mu^2}\sin\theta\,\boldsymbol{k}$	$-1 \le \mu \le 1,\ 0 \le \theta < 2\pi$	$\mathrm{d}\mu\,\mathrm{d}\theta$	4π	2π	$\frac{4\pi}{3}$

Table 1: Geometry descriptions and identities.

1.1 An insightful comparison problem

On paper, the problem looks like Fig. 2, where θ is the azimuthal angle in both flatland and 2-D geometry. However, in flatland, the line of length s is constrained to the plane, where in 2D, s is the length of the distance projected onto the plane. The line actually

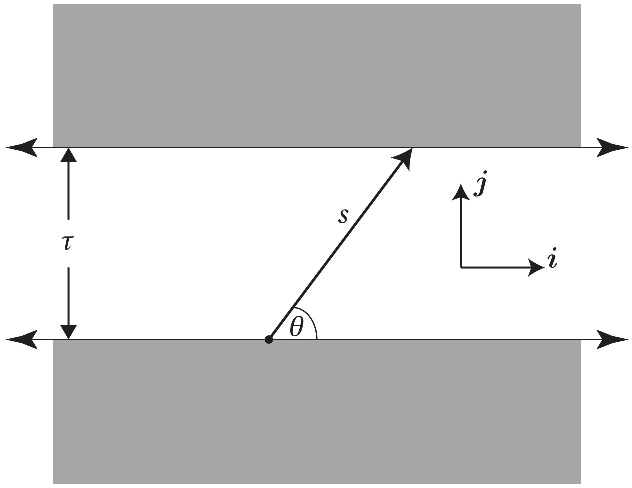

Figure 1: The chord length problem as represented on paper. The gap is τ mean free paths apart, θ is the azimuthal angle, and s is the distance across the gap.

Flatland probability of reaching other side:

$$p(\tau) = \frac{\int_0^\pi \mathrm{e}^{-\tau/\sin\theta} \sin\theta \,\mathrm{d}\theta}{\int_0^\pi \sin\theta \,\mathrm{d}\theta} = \frac{1}{2}\int_0^\pi \mathrm{e}^{-\tau/\sin\theta} \sin\theta \,\mathrm{d}\theta$$

2-D probability of reaching other side:

$$p(\tau) = \frac{\int_0^\pi \int_{-1}^1 \mathrm{e}^{-\tau/\left(\sqrt{1-\mu^2}\sin\theta\right)} \sqrt{1-\mu^2}\sin\theta \,\mathrm{d}\mu\,\mathrm{d}\theta}{\int_0^\pi \int_{-1}^1 \sqrt{1-\mu^2}\sin\theta \,\mathrm{d}\mu\,\mathrm{d}\theta} = \frac{\int_0^{2\pi} \int_0^1 \mathrm{e}^{-\tau/\mu} \mu \,\mathrm{d}\mu\,\mathrm{d}\theta}{\int_0^{2\pi} \int_0^1 \mu \,\mathrm{d}\mu\,\mathrm{d}\theta} = 2\int_0^1 \mathrm{e}^{-\tau/\mu} \mu \,\mathrm{d}\mu$$

Cylindrical probability of reaching other side:

$$p(\tau) = \frac{\int_0^\pi \int_{-1}^1 \mathrm{e}^{-\tau\sin\theta/\sqrt{1-\mu^2}} \sqrt{1-\mu^2}\sin\theta \,\mathrm{d}\mu\,\mathrm{d}\theta}{\int_0^\pi \int_{-1}^1 \sqrt{1-\mu^2}\sin\theta \,\mathrm{d}\mu\,\mathrm{d}\theta} = \frac{1}{\pi}\int_0^\pi \int_{-1}^1 \mathrm{e}^{-\tau\sin\theta/\sqrt{1-\mu^2}} \sqrt{1-\mu^2}\sin\theta \,\mathrm{d}\mu\,\mathrm{d}\theta$$

τ	r–z	Flatland	x–y
0.01	0.990	0.985	0.981
0.1	0.906	0.863	0.833
1	0.404	0.274	0.219
10	0.00772	1.63×10^{-5}	7.10×10^{-6}

Table 2: Comparison of the probability of crossing a channel τ mfp thick without colliding.

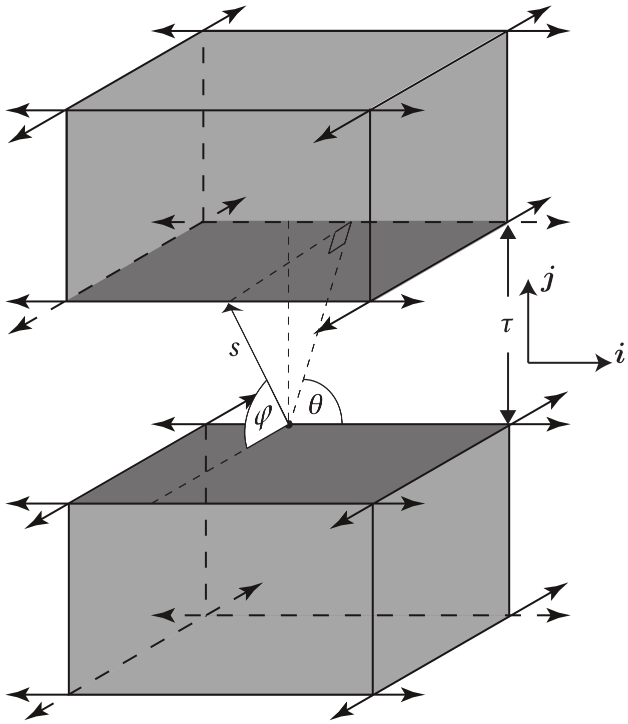

Figure 2: A more full view of the chord length problem in 2-D geometry. The polar angle cosine is $\mu = \cos\varphi$, and the azimuthal angle is θ.

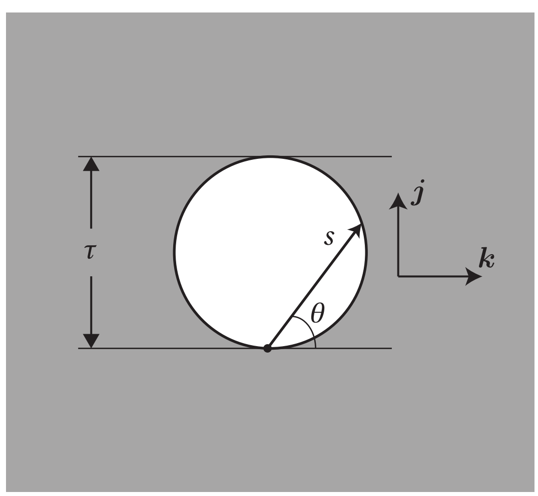

Figure 3: A cross-section of the chord length problem in r–z geometry. The orthogonal view looks like Fig. 2.

1.2 Monte Carlo sampling

[4, 5] Direct sampling probability distribution function (PDF) cumulative distribution function (CDF)

1.2.1 Isotropic volume source

An isotropic internal source, whether directly from an extraneous radiation source or indirectly from isotropic scattering, has an equal probability of entering any angle. The normalized that represents this process is

$$f(\theta)\,\mathrm{d}\theta = \frac{1}{2\pi}\,\mathrm{d}\theta\,, \quad \theta \in [0, 2\pi)\,.$$

To sample the angle that results from an isotropic event, we use the :

$$J(\theta) = \int_0^\theta f(\theta')\,\mathrm{d}\theta' = \frac{1}{2\pi}\theta\,.$$

Setting $\xi_1 = J(\theta)$ and solving for $\theta = J^{-1}(\xi_1)$ gives the simple result that

$$\theta = 2\pi\xi_1\,.$$

The particle's new angle is therefore

$$\boldsymbol{\Omega} = \cos\theta\,\boldsymbol{i} + \sin\theta\,\boldsymbol{j} = \cos 2\pi\xi_1\,\boldsymbol{i} + \sin 2\pi\xi_1\,\boldsymbol{j}\,.$$

1.2.2 Isotropic surface source

Particles emitted from an isotropic surface source have a cosine distribution [6], which makes constant the current in each differential angle:

$$f(\boldsymbol{\Omega})\,\mathrm{d}\Omega = c\,|\boldsymbol{\Omega}\cdot\boldsymbol{n}|\,\mathrm{d}\Omega\,, \boldsymbol{\Omega}\cdot\boldsymbol{n} < 0\,, \tag{2}$$

where c is a normalization constant.

In 3D, with $\boldsymbol{n} = \boldsymbol{i}$ so $|\boldsymbol{\Omega}\cdot\boldsymbol{n}| = \mu$, this has the form

$$f(\mu, \theta)\,\mathrm{d}\mu\,\mathrm{d}\theta = \frac{\mu\,\mathrm{d}\mu}{2}\frac{\mathrm{d}\theta}{2\pi}\,, \quad 0 \le \mu < 1\,,\ 0 \le \theta < 2\pi\,,$$

a separable distribution that gives $\mu = \sqrt{\xi_1}$ and $\theta = 2\pi\xi_2$.

For flatland geometry, the representation is different. Let us choose $\boldsymbol{n} = -\boldsymbol{j}$ so that incident directions are inside $\theta \in [0, \pi)$. Applying the flatland identities in Table 1 to Eq. (2) gives

$$f(\theta)\,\mathrm{d}\theta = c\,|-\sin\theta| = c\sin\theta\,, \quad 0 \le \theta < \pi\,.$$

Integrating this gives

$$J(\theta)\,\mathrm{d}\theta = c\,(1 - \cos\theta)\,, \quad 0 \le \theta < \pi\,,$$

The constant c should satisfy $J(\pi) = 1$:

$$1 = c(1 - (-1)) \quad \Longrightarrow \quad c = \frac{1}{2}\,.$$

Thus, the CDF for particle emission from a surface source in flatland geometry is

$$J(\theta)\,\mathrm{d}\theta = \frac{1}{2}\,(1 - \cos\theta)\,, \quad 0 \le \theta < \pi\,.$$

Solving for $\theta = J^{-1}(\xi_1)$ gives a sampled angle for a surface source in flatland:

$$\theta = \cos^{-1}(1-2\xi_1)\,.$$

Using the identity $\cos^2\theta + \sin^2\theta = 1$, the particle's new angle is

$$\begin{aligned}\boldsymbol{\Omega} &= \cos\theta\,\boldsymbol{i} + \sin\theta\,\boldsymbol{j}\\ &= \cos[\cos^{-1}(1-2\xi_1)]\,\boldsymbol{i} + \sin[\cos^{-1}(1-2\xi_1)]\,\boldsymbol{j}\\ &= (1-2\xi_1)\,\boldsymbol{i} + \sqrt{1-(1-2\xi_1)^2}\,\boldsymbol{j}\,.\end{aligned}$$

2 Diffusion in flatland

In 2-D geometry, particles diffuse in three dimensions, although their density is projected onto the 2-D plane. In contrast, flatland particles have one fewer dimension into which to leak: as we will see, the result is a larger diffusion coefficient.

The differences in the identities shown in Table 1 result not only in a different diffusion coefficient but also different boundary conditions. In this section, we derive boundary conditions from a steady-state transport equation with isotropic scattering. There is no loss of generality in ignoring the time dependence because of the quasi-static approximation made in the derivation of the diffusion coefficient (see §**??**).

To aid the reader in understanding whence the different coefficients in the diffusion equation arise, we use the geometric constants given in Table 1:

$$\omega_n \equiv \int_\Omega |\boldsymbol{\Omega}\cdot\boldsymbol{i}|^n \,\mathrm{d}\Omega\,,$$

where Ω is the domain of the angular variable $\boldsymbol{\Omega}$. In 2-D, $\boldsymbol{\Omega}$ sweeps the 4π unit sphere and is projected to the plane, but in flatland, $\boldsymbol{\Omega}$ lives on a 2π unit circle.

2.1 Interior diffusion approximation

The steady-state transport equation with isotropic scattering is

$$\boldsymbol{\Omega}\cdot\nabla\psi(\boldsymbol{x},\boldsymbol{\Omega}) + \sigma(\boldsymbol{x})\psi(\boldsymbol{x},\boldsymbol{\Omega}) = \frac{1}{\omega_0}\phi(\boldsymbol{x})\,,\quad \boldsymbol{x}\in V\,,\ \boldsymbol{\Omega}\in\Omega\,. \tag{3a}$$

Here, the scalar angular flux is $\int_\Omega \psi\,\mathrm{d}\Omega$, and the quantities ω_0 and Ω are defined in Table 1: in flatland, $\omega_0 = 2\pi$; in 2D, $\omega_0 = 4\pi$. We consider a specified incident boundary condition:

$$\psi(\boldsymbol{x},\boldsymbol{\Omega}) = \psi^b(\boldsymbol{x},\boldsymbol{\Omega})\quad \boldsymbol{x}\in\partial V\,,\ \boldsymbol{\Omega}\cdot\boldsymbol{n} < 0\,. \tag{3b}$$

The diffusion approximation begins by assuming that ψ is linear in angle:

$$\psi(\boldsymbol{x},\boldsymbol{\Omega}) \approx f(\boldsymbol{x}) + \boldsymbol{\Omega}\cdot\boldsymbol{g}(\boldsymbol{x})\,.$$

The zeroth angular moment of ψ determines f:

$$\phi = \int_\Omega \psi\,\mathrm{d}\Omega = \int_\Omega \left(f + \boldsymbol{\Omega}\cdot\boldsymbol{g}\right)\mathrm{d}\Omega = \int_\Omega \mathrm{d}\Omega\, f + 0 = \omega_0 f\,,$$

so $f = \phi/\omega_0$. Similarly, the first moment of ψ gives g:

$$\boldsymbol{J} = \int_\Omega \boldsymbol{\Omega}\psi \,\mathrm{d}\Omega = f\int_\Omega \boldsymbol{\Omega}\,\mathrm{d}\Omega + \boldsymbol{g}\cdot\int_\Omega \boldsymbol{\Omega\Omega}\,\mathrm{d}\Omega = \omega_2 \boldsymbol{g}\,,$$

so $\boldsymbol{g} = \boldsymbol{J}/\omega_2$. This is the P_1 approximation for ψ in the different geometries:

$$\psi(\boldsymbol{x},\boldsymbol{\Omega}) \approx \frac{1}{\omega_0}\phi(\boldsymbol{x}) + \frac{1}{\omega_2}\boldsymbol{\Omega}\cdot\boldsymbol{J}(\boldsymbol{x})\,. \tag{4}$$

The diffusion approximation is a closure for the first angular moment of the transport equation, so we now operate on Eq. (3a) with $\int_\Omega \boldsymbol{\Omega}(\cdot)\,\mathrm{d}\Omega$ and substitute Eq. (4):

$$\begin{aligned}
\nabla\cdot\int_\Omega \boldsymbol{\Omega\Omega}\psi\,\mathrm{d}\Omega + \sigma\int_\Omega \boldsymbol{\Omega}\psi\,\mathrm{d}\Omega &= \frac{1}{\omega_0}\phi(\boldsymbol{x})\int_\Omega \boldsymbol{\Omega}\,\mathrm{d}\Omega\\
\nabla\cdot\int_\Omega \boldsymbol{\Omega\Omega}\psi(\boldsymbol{x},\boldsymbol{\Omega})\,\mathrm{d}\Omega + \sigma\boldsymbol{J} &= 0\\
\nabla\cdot\int_\Omega \boldsymbol{\Omega\Omega}\left(\frac{1}{\omega_0}\phi + \frac{1}{\omega_2}\boldsymbol{\Omega}\cdot\boldsymbol{J}\right)\mathrm{d}\Omega + \sigma\boldsymbol{J} &= 0\\
\frac{1}{\omega_0}\nabla\cdot\int_\Omega \boldsymbol{\Omega\Omega}\,\mathrm{d}\Omega\phi + \sigma\boldsymbol{J} &= 0\\
\frac{\omega_2}{\omega_0}\nabla\phi + \sigma\boldsymbol{J} &= 0\,.
\end{aligned}$$

Solving for $\boldsymbol{J}$ gives Fick's law, expressed in the general-geometry form:

$$\boldsymbol{J}(\boldsymbol{x}) = -\frac{\omega_2}{\omega_0}\frac{1}{\sigma(\boldsymbol{x})}\nabla\phi(\boldsymbol{x}) \equiv -D(\boldsymbol{x})\nabla\phi(\boldsymbol{x})\,. \tag{5}$$

In 2-D and 3-D, $\omega_2/\omega_0 = (4\pi/3)/(4\pi) = 1/3$; however, in flatland, $\omega_2/\omega_0 = \pi/(2\pi) = 1/2$. Thus, $D = (3\sigma)^{-1}$ in 2-D but $D = (2\sigma)^{-1}$ in flatland.

Substituting Fick's law back into the linear-in-angle approximation, Eq. (4), we get the diffusion approximation to the angular angular flux:

$$\begin{aligned}
\psi(\boldsymbol{x},\boldsymbol{\Omega}) &\approx \frac{1}{\omega_0}\phi(\boldsymbol{x}) + \frac{1}{\omega_2}\boldsymbol{\Omega}\cdot\left[-\frac{\omega_2}{\omega_0}\frac{1}{\sigma(\boldsymbol{x})}\nabla\phi(\boldsymbol{x})\right].\\
\psi(\boldsymbol{x},\boldsymbol{\Omega}) &= \frac{1}{\omega_0}\left[\phi(\boldsymbol{x}) - \frac{1}{\sigma(\boldsymbol{x})}\boldsymbol{\Omega}\cdot\nabla\phi(\boldsymbol{x})\right]. \qquad (6)
\end{aligned}$$

In 2-D, Eq. (6) evaluates to the standard result

$$\psi(\boldsymbol{x},\boldsymbol{\Omega}) = \frac{1}{4\pi}\left[\phi(\boldsymbol{x}) - \frac{1}{\sigma(\boldsymbol{x})}\boldsymbol{\Omega}\cdot\nabla\phi(\boldsymbol{x})\right],$$

and in flatland, it is

$$\psi(\boldsymbol{x},\boldsymbol{\Omega}) = \frac{1}{2\pi}\left[\phi(\boldsymbol{x}) - \frac{1}{\sigma(\boldsymbol{x})}\boldsymbol{\Omega}\cdot\nabla\phi(\boldsymbol{x})\right]. \tag{7}$$

2.2 Marshak boundary condition

The Marshak boundary condition [7] preserves the incident radiation current on the boundary. It is derived by substituting the approximate diffusion angular flux from Eq. (6) into the boundary condition, Eq. (3b), multiplying by $|\boldsymbol{\Omega}\cdot\boldsymbol{n}|$, and integrating over incident directions.

$$\int_{\boldsymbol{\Omega}\cdot\boldsymbol{n}<0} |\boldsymbol{\Omega}\cdot\boldsymbol{n}|\,\psi^b\,\mathrm{d}\Omega = \int_{\boldsymbol{\Omega}\cdot\boldsymbol{n}<0} |\boldsymbol{\Omega}\cdot\boldsymbol{n}|\,\frac{1}{\omega_0}\left[\phi - \frac{1}{\sigma}\boldsymbol{\Omega}\cdot\nabla\phi\right]\mathrm{d}\Omega$$

$$\begin{aligned}
J^- &= \frac{1}{\omega_0}\phi\left(\int_{\boldsymbol{\Omega}\cdot\boldsymbol{n}<0} |\boldsymbol{\Omega}\cdot\boldsymbol{n}|\,\mathrm{d}\Omega\right) - \frac{1}{\omega_0}\frac{1}{\sigma}\int_{\boldsymbol{\Omega}\cdot\boldsymbol{n}<0} (-\boldsymbol{\Omega}\cdot\boldsymbol{n})\boldsymbol{\Omega}\,\mathrm{d}\Omega\cdot\nabla\phi \\
J^- &= \frac{1}{\omega_0}\phi\left(\frac{\omega_1}{2}\right) + \frac{1}{\omega_0}\frac{1}{\sigma}\boldsymbol{n}\cdot\int_{\boldsymbol{\Omega}\cdot\boldsymbol{n}<0} \boldsymbol{\Omega}\boldsymbol{\Omega}\,\mathrm{d}\Omega\cdot\nabla\phi \\
J^- &= \frac{\omega_1}{2\omega_0}\phi + \frac{1}{\omega_0}\frac{1}{\sigma}\boldsymbol{n}\cdot\left(\int_{\boldsymbol{\Omega}\cdot\boldsymbol{n}<0} \boldsymbol{\Omega}\boldsymbol{\Omega}\,\mathrm{d}\Omega\right)\cdot\nabla\phi \\
J^- &= \frac{\omega_1}{2\omega_0}\phi + \frac{1}{\omega_0}\frac{1}{\sigma}\boldsymbol{n}\cdot\left(\frac{\omega_2}{2}\mathbf{I}\right)\nabla\phi \\
J^- &= \frac{\omega_1}{2\omega_0}\phi + \frac{\omega_2}{2\omega_0}\frac{1}{\sigma}\boldsymbol{n}\cdot\nabla\phi .
\end{aligned}$$

Rearranging gives an expression for the Marshak boundary condition for

$$\frac{2\omega_0}{\omega_1}J^- = \phi + \frac{\omega_2}{\omega_1}\frac{1}{\sigma}\boldsymbol{n}\cdot\nabla\phi \tag{8}$$

The value

$$z_0 \equiv \frac{\omega_2}{\omega_1} = \begin{cases} \frac{2}{3} \approx 0.6667 & \text{1D, 2D, 3D,} \\ \frac{\pi}{4} \approx 0.7854 & \text{Flatland,} \end{cases}$$

is the Marshak extrapolation distance.

Substituting the diffusion coefficient D from Eq. (5) gives

$$\frac{2\omega_0}{\omega_1}J^- = \phi + \frac{\omega_0}{\omega_1}D\boldsymbol{n}\cdot\nabla\phi .$$

In 1D, 2D, and 3D, this evaluates to

$$4J^- = \phi + 2D\boldsymbol{n}\cdot\nabla\phi ,$$

but in flatland it is

$$\pi J^- = \phi + \frac{\pi}{2}D\boldsymbol{n}\cdot\nabla\phi .$$

2.3 Variational boundary condition

It is well known that the Marshak boundary condition is not consistent with the true transport boundary condition. A lengthy asymptotic boundary layer matching analysis [8] shows that the correct weighting of the boundary condition is not $|\boldsymbol{\Omega}\cdot\boldsymbol{n}|$ but rather $W(|\boldsymbol{\Omega}\cdot\boldsymbol{n}|)$, where W is related to Chandrasekhar's H-function [9]:

$$W(\mu) = \frac{\sqrt{3}}{2}\mu H(\mu) . \tag{9}$$

In general geometries, this leads to the extrapolation distance of

$$z_0 = \frac{\int_0^1 \mu W(\mu)\,\mathrm{d}\mu}{\int_0^1 W(\mu)\,\mathrm{d}\mu} \approx 0.7104 .$$

In real-world geometries, a variational analysis [10] can be used to derive a very accurate approximation to W:

$$W(\mu) \approx \mu + \tfrac{3}{2}\mu^2 ,$$

which gives the approximate extrapolation distance of

$$z_0 = \frac{\int_0^1 \mu(\mu + \frac{3}{2}\mu^2)\,\mathrm{d}\mu}{\int_0^1 (\mu + \frac{3}{2}\mu^2)\,\mathrm{d}\mu} = \frac{17}{24} \approx 0.7083 .$$

Now we repeat that analysis for flatland rather than real space. The first step is to consider a homogeneous, purely scattering transport problem in a semi-infinite plane. The transport equation (1) becomes

$$\cos\theta \frac{\partial \psi}{\partial x} + \sin\theta \frac{\partial \psi}{\partial y} + \sigma\psi = \frac{\sigma}{2\pi}\int_0^{2\pi} \psi \,\mathrm{d}\theta' , \quad -\infty < x < \infty,\ 0 \le y < \infty,\ 0 \le \theta < 2\pi . \tag{10a}$$

It has a uniform incident boundary condition,

$$\psi(x,0,\theta) = \psi^b(\theta) , \quad -\infty < x < \infty,\ 0 \le \theta < \pi . \tag{10b}$$

Also, because there is no variation in the boundary condition or σ along the x axis, $\partial\psi/\partial x = 0$, and Eq. (10a) becomes the one-dimensional transport equation

$$\sin\theta \frac{\partial}{\partial y}\psi(y,\theta) + \sigma\psi(y,\theta) = \frac{\sigma}{2\pi}\int_0^{2\pi} \psi(y,\theta')\,\mathrm{d}\theta' .$$

This is *not* the same one-dimensional transport equation as in slab geometry.

We define the y components of the angular moments of ψ as

$$\phi_m(y) = \int_0^{2\pi} (\boldsymbol{\Omega}\cdot\boldsymbol{j})^m \psi(y,\theta)\,\mathrm{d}\theta = \int_0^{2\pi} (\sin\theta)^m \psi(y,\theta)\,\mathrm{d}\theta . \tag{11}$$

As $y \to \infty$, the angular flux ψ will approach a constant $\varphi/2\pi$, which gives $\phi_0(\infty) = \varphi$. Concordantly, $\phi_1(\infty) = 0$.

Operating on the transport equation with $\int_0^{2\pi}(\sin\theta)^m(\cdot)\,\mathrm{d}\theta$ gives the mth angular moment in the y direction:

$$\begin{aligned}
\frac{\partial}{\partial y}\int_0^{2\pi}(\sin\theta)^{m+1}\psi\,\mathrm{d}\theta + \sigma\int_0^{2\pi}(\sin\theta)^m\psi\,\mathrm{d}\theta &= \frac{\sigma}{2\pi}\int_0^{2\pi}\psi\,\mathrm{d}\theta' \int_0^{2\pi}(\sin\theta)^m\,\mathrm{d}\theta \\
\frac{\partial\phi_{m+1}}{\partial y} + \sigma\phi_m &= \frac{\sigma}{2\pi}\phi_0\int_0^{2\pi}(\sin\theta)^m\,\mathrm{d}\theta .
\end{aligned} \tag{12}$$

For $m = 0$, the conservation equation, Eq. (12) evaluates to

$$\frac{\partial\phi_1}{\partial y} + \sigma\phi_0 = \frac{\sigma}{2\pi}\phi_0(2\pi) \quad\Longrightarrow\quad \frac{\partial\phi_1}{\partial y} = 0 .$$

That means the current is a constant, and because $\phi_1(\infty) = 0$, that constant is zero.

Evaluating Eq. (12) for $m = 1$ and using the result that $\phi_1 = 0$, we find

$$\frac{\partial\phi_2}{\partial y} + \sigma\phi_1 = \frac{\sigma}{2\pi}\phi_0(0) \quad\Longrightarrow\quad \frac{\partial\phi_2}{\partial y} = 0 .$$

Thus ϕ_2 is also a constant. As $y \to \infty$, $\psi \to \varphi/2\pi$, so

$$\phi_2 = \int_0^{2\pi}(\sin\theta)^2\frac{\varphi}{2\pi}\,\mathrm{d}\theta = \frac{1}{2}\varphi .$$

Now, since $\phi_1 = 0$, we can add $\alpha\phi_1$ to this equation for any α:

$$\begin{aligned}
\alpha\phi_1 + \phi_2 &= \frac{\varphi}{2} \\
\int_0^{2\pi}(\alpha\sin\theta + \sin^2\theta)\psi(y,\theta)\,\mathrm{d}\theta &= \frac{\varphi}{2} .
\end{aligned}$$

At the boundary $y = 0$, $\psi = \psi^b$ for incident angles $0 \le \theta < \pi$. The variational analysis makes the reasonable approximation that, to leading order, the exiting particles are isotropically distributed, $\psi(0,\theta) = \psi^{\text{out}}$.

$$\int_0^{\pi} (\alpha \sin\theta + \sin^2\theta)\psi^b(\theta)\,\mathrm{d}\theta + \int_{\pi}^{2\pi} (\alpha \sin\theta + \sin^2\theta)\,\mathrm{d}\theta \psi^{\text{out}} = \frac{\varphi}{2}.$$

The value $\alpha = \pi/4$ eliminates the integral over outgoing directions and gives a relation between moments of the incident angular flux and the magnitude of the angular flux as $y \to \infty$:

$$\int_0^{\pi} \left(\frac{\pi}{4}\sin\theta + \sin^2\theta\right)\psi^b(\theta)\,\mathrm{d}\theta = \frac{\varphi}{2}. \tag{13}$$

Diffusion in flatland approximates the angular flux as

$$\psi(\boldsymbol{x},\boldsymbol{\Omega}) \approx \frac{1}{2\pi}\left[\phi(\boldsymbol{x}) - \frac{1}{\sigma(\boldsymbol{x})}\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi(\boldsymbol{x})\right]$$

In our simple transport problem, $\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\phi = \sin\theta\frac{\partial\phi_0}{\partial y}$. We wish our boundary condition to preserve the value of φ when the diffusion method is used:

$$\begin{aligned}
\varphi = 2\int_0^{\pi}\left(\frac{\pi}{4}\sin\theta + \sin^2\theta\right)\psi^b(\theta)\,\mathrm{d}\theta &= 2\int_0^{\pi}\left(\frac{\pi}{4}\sin\theta + \sin^2\theta\right)\left(\frac{1}{2\pi}\phi - \frac{1}{\sigma}\sin\theta\frac{\partial\phi_0}{\partial y}\right)\mathrm{d}\theta \\
\int_0^{\pi}\left(\frac{\pi}{2}\sin\theta + 2\sin^2\theta\right)\psi^b(\theta)\,\mathrm{d}\theta &= \frac{1}{2\pi}\int_0^{\pi}\left(\frac{\pi}{2}\sin\theta + 2\sin^2\theta\right)\mathrm{d}\theta\phi - \frac{1}{2\pi}\int_0^{\pi}\left(\frac{\pi}{2}\sin^2\theta + 2\sin^3\theta\right)\mathrm{d}\theta\frac{1}{\sigma}\frac{\partial\phi_0}{\partial y} \\
&= \frac{1}{2\pi}\left(\frac{\pi}{2}[2] + 2\frac{\pi}{2}\right)\phi - \frac{1}{2\pi}\left(\frac{\pi}{2}\left[\frac{\pi}{2}\right] + 2\left[\frac{4}{3}\right]\right)\frac{1}{\sigma}\frac{\partial\phi_0}{\partial y} \\
&= \phi - \left(\frac{\pi}{8} + \frac{4}{3\pi}\right)\frac{1}{\sigma}\frac{\partial\phi_0}{\partial y}.
\end{aligned}$$

In this problem, we chose a boundary surface normal of $\boldsymbol{n} = -\boldsymbol{j}$. Now replacing $\sin\theta$ with $-\boldsymbol{\Omega}\cdot\boldsymbol{n}$, we get the general boundary condition for flatland diffusion,

$$\int_{\boldsymbol{\Omega}\cdot\boldsymbol{n}<0}\left[\frac{\pi}{2}|\boldsymbol{\Omega}\cdot\boldsymbol{n}| + 2(\boldsymbol{\Omega}\cdot\boldsymbol{n})^2\right]\psi^b(\boldsymbol{x},\boldsymbol{\Omega})\,\mathrm{d}\Omega = \phi(\boldsymbol{x}) - \left(\frac{\pi}{8} + \frac{4}{3\pi}\right)\frac{1}{\sigma}\boldsymbol{n}\cdot\boldsymbol{\nabla}\phi(\boldsymbol{x}). \tag{14}$$

Essentially, we have shown that the flatland equivalent of the W function, which we shall call V, can be approximated by

$$V(\theta) \approx \frac{1}{4}\sin\theta + \frac{1}{\pi}\sin^2\theta, \quad 0 \le \theta < \pi, \tag{15}$$

which satisfies

$$\int_0^{\pi} V(\theta)\,\mathrm{d}\theta = 1$$

and gives the extrapolation distance

$$z = \frac{\int_0^{\pi}\sin\theta V(\theta)\,\mathrm{d}\theta}{\int_0^{\pi} V(\theta)\,\mathrm{d}\theta} = \frac{\pi}{8} + \frac{4}{3\pi}.$$

"Marshak" extrapolation distance:

$$z = \frac{\pi}{4} \approx 0.78540$$

Variational extrapolation distance:

$$z = \frac{\pi}{8} + \frac{4}{3\pi} \approx 0.81711$$

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