Using the Telegrapher's Equations to Model the Anolis Lizard Interaural Cavity

Seth Peacock

October 2024

1 Introduction

The anolis lizard is an animal with a rather interesting pair of ears - interesting not necessarily because of the ears themselves, but rather because the ears are *coupled together*. Between the left and right tympanic membranes is an air filled space we call an interaural cavity (IAC). We would like to find a way to model this cavity, and see how it might allow the tympani to "communicate" with one another. We are primarily interested in how this communication affects the generation of spontaneous otoacoustic emissions (SOAEs), which are vibrations (sounds) that the anolis' ears make on their own in the absence of external stimuli. For our purposes, you can imagine that the tympanum is driven by a complicated internal motor which causes it to oscillate at various frequencies. For more information on how we model this internal motor, see [4]. We will be modeling these oscillations as occurring in one dimension, and will assume that at any given time t we know the displacement and velocity of both the left and right tympanic oscillators. Our task is to find out how to model the interaction between these tympani through the IAC – more specifically, we would like to find the force on each tympanum due to the air pressure in the cavity as a function of time.

We will use the same approach as Roongthumskul et al. in [3]: we model the transfer of air through the IAC via a set of partial differential equations commonly known as the telegrapher's equations. As the name suggests, these equations were originally developed to describe how an electrical signal travels along a long stretch of telegraph line. We will be making use of an analogy between acoustic systems (air flow) and electric circuits (current flow). The equations which describe current flow in electric circuits can be shown to be directly analogous to those which describe air flow. For a full account of the analogy between these equations, see [2]. For more on designing electrical circuits to represent acoustical systems, see [1]. A quick guide to this analogy is given below.

Air Particles $\rightarrow$ Charge

Particle Velocity $\rightarrow$ Current

Air Pressure $\rightarrow$ Voltage

Large Cavity $\rightarrow$ Capacitor

Long Pipe $\rightarrow$ Inductor

Thin Pipe / Thermoviscous Losses $\rightarrow$ Resistor

2 The Telegrapher's Equations

Imagine a long stretch of cable (which we will analogize with the air cavity between the anolis' ears) which is tasked with transmitting a signal. Since this is an electrical system, the cable must have two conductors – one for the outward current and one for the return current. We model this cable as being made up of an infinite series of infintesimally small circuits, each representing an infintesimally small section of the cable. The cable isn't made of electrical components, of course, but any real cable inevitably has some resistance/inductance/capacitance. Therefore, each of these circuits consists of:

- A series resistor for the natural resistance of the conductor
- A series inductor for the self-inductance of the conductor
- A shunt (from one conductor to the other) capacitor, for the self-capacitance of the pair of conductors
- A shunt resistor, representing the inevitable small amount of conductivity between the pair of conductors in the cable.

See Figure 2 for a diagram of one of these circuits.

*Note: We should mention that when we use the symbols R, L, C, G in the following analysis these represented **distributed** quantities – i.e. R (for example) is the resistance **per unit length** of cable. That is, technically what we write as "R" is really a derivative $\frac{dR}{dx}$ and has units of $\frac{resistance}{length}$. Since we're not going into the details of deriving the telegrapher's equations, this subtlety will not affect our analysis – "R" / $\frac{dR}{dx}$ is just some constant quantity which does not change with x or t. The cable is uniform down its length and unchanging in time, so the total resistance in any section of length l at any time t is always $\frac{dR}{dx} * l \equiv R * l$.*

Now, as we move down the length of cable, we have to treat V and I as continuous in both time *and* space. I smell a partial differential equation! In fact, it leads to two PDEs - here are the telegrapher's equations (TE):

$$\begin{aligned} \frac{\partial}{\partial x}V(x,t) &= -L\frac{\partial}{\partial t}I(x,t) - RI(x,t) \\ \frac{\partial}{\partial x}I(x,t) &= -C\frac{\partial}{\partial t}V(x,t) - GV(x,t) \end{aligned} \tag{1}$$

We can read each equation as answering the question: "At a given time t and position along the cable x, if we take a tiny step down the cable dx, how does (V or I) change?" Let's see how each distributed quantity $\mathbf{L}, \mathbf{C}, \mathbf{R}, \mathbf{G}$ has an effect on the electric (air) current and voltage (pressure) and thereby on the corresponding electrical (acoustical) wave.

- Inductance: $\frac{\partial}{\partial x}V(x,t) = -\mathbf{L}\frac{\partial}{\partial t}I(x,t)$
  - Inductance gives the current *inertia*, just like a mass – it makes it difficult to quickly change the current flow at a given moment.
  - This makes a wave travel slower, for the same reason that a wave traveling down a heavy string travels slower than a wave traveling down a light string.
  - In the acoustic analogy, if air is flowing in one direction down a pipe with a pressure source at one end but then the pressure source is cut off, the air won't immediately stop flowing. It will gradually slow to a stop, like a ball you were pushing along and then suddenly stopped pushing.
- Resistance: $\frac{\partial}{\partial x}V(x,t) = -\mathbf{R}I(x,t)$

- As current flows through the conductor, some of the energy is lost as heat. This energy loss is proportional to the resistance.
- We see this show up in the TE just like in Ohm's law $V = IR$ - as we take a tiny step down the cable, the voltage *drop* (hence the minus sign) is just $I * R$.
- In the acoustical case, one reason for this energy loss could be if the surrounding chamber isn't perfectly reflective, and so some energy is lost to vibrating the walls. As current flows through the chamber, pressure drops a bit as some of the energy is deposited in the walls.

- Capacitance: $\frac{\partial}{\partial x} I(x,t) = -\mathbf{C}\frac{\partial}{\partial t} V(x,t)$
  - The distributed capacitance doesn't make a lot of sense to me. In the acoustic analogy, it appears to be much different than the way a single, large capacitor is used to model a large cavity of air (as in [2]).
  - From [7]: "The capacitance couples voltage to the energy stored in the electric field. It controls how much the bunched-up electrons within each conductor repel, attract, or divert the electrons in the other conductor. By deflecting some of these bunched up electrons, the speed of the wave and its strength (voltage) are both reduced. With a larger capacitance, C, there is less repulsion, because the other line (which always has the opposite charge) partly cancels out these repulsive forces within each conductor. Larger capacitance equals weaker restoring forces, making the wave move slightly slower, and also gives the transmission line a lower surge impedance (less voltage needed to push the same AC current through the line)."

- Conductance: $\frac{\partial}{\partial x} I(x,t) = -\mathbf{G}V(x,t)$
  - This represents current leaking from one of the two lines in the cable to the other.
  - Note that conductance has units of 1/Ohms and is essentially the reciprocal of R; in light of that, this equation $\frac{\partial}{\partial x} I(x,t)\frac{1}{\mathbf{G}} = -V(x,t)$ also looks very similar to Ohm's law. The difference between this and the way R shows up in the TE is that in this case the derivative if on $I(x,t)$ – it affects how $I(x,t)$ changes as we take a step down the line. I'm not entirely sure how to interpret this difference though.
  - I'm not sure what the acoustical analogy is.

To find the acoustic analogs of these quantities, we'll look at [3] where they use the following equations:

$$\begin{aligned} \frac{\partial p(x,t)}{\partial x} &= -\rho\left(R_V u(x,t) + \frac{\partial u(x,t)}{\partial t}\right) \\ \frac{\partial u(x,t)}{\partial x} &= -\beta\left(R_T p(x,t) + \frac{\partial p(x,t)}{\partial t}\right) \end{aligned} \tag{2}$$

Where:

- $p(x,t)$ is the pressure with respect to a reference state of zero velocity, pressure $P_{\mathrm{r}} = 101\mathrm{kPa}$, and temperature $T = 25°\mathrm{C}$
- $u(x,t)$ is the particle velocity (in the x direction) of the air
- R_V is the viscous damping rate
- R_T is the thermal damping rate

- ρ is the air's reference density
- β is its adiabatic compressibility

Matching coefficients between Equation 1 and Equation 2 we have:

$$\begin{aligned} L &\to \rho \\ R &\to \rho * R_V \\ C &\to \beta \\ G &\to \beta * R_T \end{aligned} \tag{3}$$

Though as noted above, these quantities L, R, C, G are actually the derivative with respect to x – for example, really $\rho = \frac{dL}{dx}$. Again, for now we will just call this L, though we will have to address this again in section 7.

3 Decoupling the Telegrapher's Equations

Now let's see what we can do with these equations. Note that in their current form, there's a V AND an I in both equations. This makes things tricky, so we'll start by *decoupling* them. First, we take the partial derivative of the I equation with respect to t and the partial derivative of the V equation with respect to x.

$$\begin{aligned} I_x = -V_t - GV &\Rightarrow I_{xt} = -CV_{tt} - GV_t \\ V_x = -LI_t - RI &\Rightarrow V_{xx} = -LI_{tx} - RI_x \end{aligned}$$

Then we can plug the equation for I_{xt} into the second equation:

$$V_{xx} = LCV_{tt} + LGV_t - RI_x$$

And finally plug in the original equation for I_x in to obtain:

$$\begin{aligned} V_{xx} - LCV_{tt} &= LCV_{tt} + LGV_t + RCV_t + RGV \\ &= (RC + GL)V_t + GRV \end{aligned}$$

We can do a similar thing to obtain a nearly identical (just with I instead of V) form for the I equation, and we now have two decoupled PDEs. The price we pay for this form is that now they are second order - but we'll take what we can get!

$$\begin{aligned} \frac{\partial^2}{\partial x^2}V(x,t) - LC\frac{\partial^2}{\partial t^2}V(x,t) &= (RC + GL)\frac{\partial}{\partial t}V(x,t) + GRV(x,t) \\ \frac{\partial^2}{\partial x^2}I(x,t) - LC\frac{\partial^2}{\partial t^2}I(x,t) &= (RC + GL)\frac{\partial}{\partial t}I(x,t) + GRI(x,t) \end{aligned} \tag{4}$$

4 Lossless Equations

In an ideal world, we could make cable with no resistance and with perfect insulation between the wires. While the world is not ideal, it may be a reasonable approximation (in the case of our anolis' IAC) to assume that R and G are very small. So small, in fact, that we can go ahead and neglect all terms involving R and G in (4). Let's see what this gives us:

$$\begin{aligned} \frac{\partial^2}{\partial x^2}V(x,t) - LC\frac{\partial^2}{\partial t^2}V(x,t) &= 0 \\ \frac{\partial^2}{\partial x^2}I(x,t) - LC\frac{\partial^2}{\partial t^2}I(x,t) &= 0 \end{aligned}$$

Now let's define $c = \frac{1}{\sqrt{LC}}$ and move the terms around a bit.

$$\begin{aligned}\frac{\partial^2}{\partial t^2}V(x,t) &= c^2\frac{\partial^2}{\partial x^2}V(x,t)\\ \frac{\partial^2}{\partial t^2}I(x,t) &= c^2\frac{\partial^2}{\partial x^2}I(x,t)\end{aligned} \tag{5}$$

Hey now – that's just our good friend the wave equation (x2)! Would-ya-look-at-that!

5 Solution

Let's see if we can figure out our initial and boundary conditions.

5.1 Initial Conditions

When we implement this model in code, we'll only be interested in the asymptotic behavior of the system. That is, we'll let the system run for a very long time before we start to take measurements. This should allow us to simulate what the animal's ears are doing *in vivo*. As $t \to \infty$, any effects from the initial conditions are going to be negligible compared to the effects from the boundary conditions (which we'll examine in a moment). This is good, because before running our simulation, we really have no way of knowing what accurate initial conditions should be! In code, we'd likely randomize our initial conditions (just to be sure that the initial conditions don't have any effect on the asymptotic behavior), but for the purposes of simplifying our analysis here, we'll just set them to zero.

5.2 Boundary Conditions

Here's where things get a little tricky. For V, we don't have boundary conditions. In fact, these boundary conditions for V are exactly what we're looking for, since they will give the force on the tympanum (see 6.2). As such, we'll ignore the V equation, and focus on I. Since V depends directly on I and its derivatives (in 1), we'll later be able to approximate a solution for V from our solution for I.

What are our boundary conditions for I? In the analogy, I is the particle velocity of the air particles. Think about what's happening right around the (say, left) tympanum: the air particles directly to the right of the tympanum *must be moving along with the tympanum.* Any alternative would require some sort of vacuum happening right next to the tympanum, which doesn't make much sense. Therefore, we can just identify the particle velocity of the air at each boundary with the *velocity of the nearby tympanum* $\dot{T}_{L,R}(t)$ – which is fantastic, because we have access to the velocity (and the position) of both tympani at any time t!

Let's define $\dot{T}_{L,R}(t) \equiv I_{L,R}(t)$. Now – if we assume that the left tympanum is at $x = 0$ and our right tympanum is at $x = l$ – we have our boundary conditions:

$$\begin{aligned}I(0,t) &= I_L(t)\\ I(l,t) &= I_R(t)\end{aligned}$$

5.3 Reformulating the Problem

Here's our problem:

$$\frac{\partial^2}{\partial t^2}I(x,t) = c^2\frac{\partial^2}{\partial x^2}I(x,t) \quad 0 < x < l,\, t > 0$$
$$I(0,t) = I_L(t) \quad t > 0$$
$$I(l,t) = I_R(t) \quad t > 0$$
$$I(x,0) = 0 \quad 0 < x < l$$
$$I_t(x,0) = 0 \quad 0 < x < l$$

First, we'll reformulate the problem so we're dealing with a source rather than boundary conditions that are a function of time.

Define $\phi(x,t)$ and $w(x,t)$ so that:

$$I(x,t) = \phi(x,t) + w(x,t)$$

where $\phi(x,t)$ is a function chosen to match the boundary conditions. Specifically, we want:

$$\phi(0,t) = I_L(t), \quad \phi(l,t) = I_R(t).$$

A simple choice for $\phi(x,t)$ is a linear interpolation between the boundary values:

$$\phi(x,t) = I_L(t) + \frac{x}{l}\left(I_R(t) - I_L(t)\right).$$

ϕ is known, so now it remains to solve for w. Let's see what problem w solves. Since $w(x,t) = I(x,t) - \phi(x,t)$, we have that $w(0,t) = w(l,t) = 0$. These homogeneous boundary conditions are looking nice!

The initial conditions look a little more complicated at first, but they're nothing we can't handle:

$$w(x,0) = I(x,0) - \phi(x,0) = 0 - (I_L(0) + \frac{x}{l}(I_R(0) - I_L(0)))$$
$$w_t(x,0) = I_t(x,0) - \phi_t(x,0) = 0 - (\dot{I}_L(0) + \frac{x}{l}(\dot{I}_R(0) - \dot{I}_L(0))$$

But hey, we're setting all the initial conditions inside the IAC to zero, so let's go ahead and set the initial tympanic velocities and accelerations (that is, $I_{L,R}(0)$ and $\dot{I}_{L,R}(0)$) to zero as well for the same reasons as before. (If we let our simulation run long enough, these initial conditions won't make a difference.) Then:

$$w(x,0) = w_t(x,0) = 0$$

Yippie!

Here's the only inhomogeneous bit. Let's substitute $I(x,t) = w(x,t) + \phi(x,t)$ into the original wave equation which we know $I(x,t)$ solves.

$$\begin{aligned}
\frac{\partial^2(w(x,t)+\phi(x,t))}{\partial t^2} &= c^2\frac{\partial^2(w(x,t)+\phi(x,t))}{\partial x^2} \\
\frac{\partial^2 w}{\partial t^2}+\frac{\partial^2 \phi}{\partial t^2} &= c^2\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 \phi}{\partial x^2}\right) \\
\frac{\partial^2 w}{\partial t^2} &= c^2\frac{\partial^2 w}{\partial x^2}+\left(c^2\frac{\partial^2 \phi}{\partial x^2}-\frac{\partial^2 \phi}{\partial t^2}\right) \\
&= c^2\frac{\partial^2 w}{\partial x^2}+\left(0-\frac{\partial^2 \phi}{\partial t^2}\right) \\
&= c^2\frac{\partial^2 w}{\partial x^2}-\frac{d^2 I_L(t)}{dt^2}-\frac{x}{l}\left(\frac{d^2 I_R(t)}{dt^2}-\frac{d^2 I_L(t)}{dt^2}\right) \\
&\equiv c^2\frac{\partial^2 w}{\partial x^2}+f(x,t)
\end{aligned}$$

Where $f(x,t) \equiv -\frac{d^2 I_L(t)}{dt^2}-\frac{x}{l}\left(\frac{d^2 I_R(t)}{dt^2}-\frac{d^2 I_L(t)}{dt^2}\right)$ is an inhomogeneous source term.

Note that used ChatGPT throughout this subsection to check my algebra and generate LaTeX code for the equations. It was very accurate!

5.4 Final Solution

It remains to solve for $w(x,t)$ which satisfies

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}w(x,t) &= c^2\frac{\partial^2}{\partial x^2}w(x,t)+f(x,t) \quad 0<x<l,\, t>0 \\
w(0,t) &= 0 \quad t>0 \\
w(l,t) &= 0 \quad t>0 \\
w(x,0) &= 0 \quad 0<x<l \\
w_t(x,0) &= 0 \quad 0<x<l
\end{aligned}$$

By Duhamel's Principle, the solution to this is going to be the sum of the homogeneous solution (the same problem just without the f term) and a convolution integral of the form shown below. But in our case with zero initial data, $u_h(x,t)$ is just going to be zero, so we have:

$$\begin{aligned}
u(x,t) &= u_h(x,t)+\int_0^t\int_0^L G\left(x,t-\tau;x'\right)f\left(x',\tau\right)dx'd\tau \\
&= \int_0^t\int_0^L G\left(x,t-\tau;x'\right)f\left(x',\tau\right)dx'd\tau
\end{aligned} \tag{6}$$

Now, how do we interpret the convolution integral – and what's G? G is a *Green's function*, which in this case is the solution of the wave equation with a simple source: just a single unit impulse located at x' and time τ. That is, it's the solution of the above problem with

$$f(x,t)=\delta(x-x')\delta(t-\tau)$$

which is zero everywhere and infinite at the point $(x,t)=(x',\tau)$.

The key observation here is that we can treat our *real* source f as an infinite collection of scaled unit impulses (one for each point in space x' and at every time τ). Then in Equation 6 we're summing (integrating) the system's response (G) to all of them - at least, all of them which are in the domain of dependence of the point in question (x,t) (as we'll see below).

If we were on the infinite real line, our Green's function would be (see [6]):

$$G\left(x, t-\tau ; x^{\prime}\right)=\frac{1}{2 c} \Theta\left(t-\tau-\frac{\left|x-x^{\prime}\right|}{c}\right)$$

where $\Theta(z)$ is the Heaviside step function. The $t-\tau$ is because that's how long it's been since the disturbance in question initially occurred at time τ, and the $x-x'$ is the distance from the point we care about to where the disturbance initially occurred. In this case, G's main job is basically to make sure that we only integrate over the disturbances that have had enough time to reach the point we care about (x,t). That is, we need to integrate over the domain of dependence for (x,t), which is $\{(x',\tau) : t-\tau-\frac{|x-x'|}{c}\}$ – precisely the set for which the Heaviside function is turned on.

Unfortunately, we're not on the infinite real line. In our case, the tricky thing is the finite length of our interval. We have reflections at *both* $x=0$ and $x=L$, and we have to somehow track how these disturbances can bounce back from either end. Thus we'll have a sum over all possible "orders" of reflection (how many times it's bounced back and forth) with a Heaviside step function ($\Theta(\xi)$) within the sum to make sure that higher order reflections only contribute when there has been enough time since the original disturbance.

To find the Green's function for our finite line, I went down a rabbit hole with ChatGPT. I was always able to find some kind of mistake in its responses, so I'm not at all confident about this answer. However, after fixing all of its mistakes and contradictions I was able to piece together what is at least a plausible result:

$$G\left(x, t-\tau ; x^{\prime}\right)=\frac{1}{2 c} \sum_{n}\left[\Theta\left(t-t^{\prime}-\frac{\left|x-x_{n}^{\prime}\right|}{c}\right)-\Theta\left(t-t^{\prime}-\frac{\left|x+x_{n}^{\prime}\right|}{c}\right)\right]$$

where x'_n are "image sources" to account for the reflections from the boundaries at $x=0$ and $x=L$. These image sources represent the multiple reflections of the wave between the boundaries. Rather than treat them as reflections, we treat them as a disturbance that was way out to the left or right, *outside* of the finite line. We place it at exactly the right distance so that (traveling at speed c) it would reach the boundary precisely when the reflection *would* bounce off that boundary. Very clever!

But what is "exactly the right distance" for a given reflection? This was one way that ChatGPT failed me; though the general expression it gave for x'_n provided some correct locations, it failed to account for all of them. Let's find the first few orders of reflections by hand:

- $x'_n = x'$ is the original source.
- The first reflection from $x=0$ has to travel a distance of x' before bouncing back (and currently being at) $x=0$, so we can treat this as if there were a source at $x'_n=-x'$. This source then propagates rightward (it also propagates leftward, but we never see that on our finite line) and it will reach $x=0$ moving rightward at the same time as our reflection would be doing exactly this.
- The first reflection from $x=l$ has to travel a distance of $l-x'$ before bouncing off (and currently being at) $x=l$, moving leftward. Therefore we can model this as a source at $x'_n=2l-x'$; our reflection is then accounted for by the *leftward* propagation of this image source (which reaches $x=l$ precisely when the reflection would be).
- $x'_n=-2l+x'$ is the second order reflection that started at x' moving to the right, bounced off of $x=l$ (after traveling $l-x'$), then bounces of $x=0$ (after traveling l) and is currently at $x=0$. $(l-x')+l=2l-x'$; this is how far it has to travel before being at $x=0$, so our

source is at $x'_n = -(2l - x') = -2l + x'$. The minus sign is because the image source starts way to the left of $x = 0$ and moves rightward.

- $x'_n = 2l + x'$ is the second order reflection that started at x' moving to the left, bounced off of $x = 0$ (after traveling x'), then bounces of $x = l$ (after traveling l) and is currently at $x = l$. $x' + l$ is how far it has to travel leftward (after starting way off to the right) before being at $x = l$, so our source is at $x'_n = 2l + x'$.

- $x'_n = 3l + x'$ is the third order reflection that started at x' moving to the right, bounced off of $x = l$ (after traveling $l - x'$), then bounces off $x = 0$ (after traveling l), then bounces off $x = l$ (after traveling l) and is currently at $x = l$. $(l - x') + l + l + l = 4l - x'$.

- $x'_n = -2l - x'$ is the third order reflection that started at x' moving to the left, bounced off of $x = 0$ (after traveling x'), then bounces of $x = l$ (after traveling l), then bounces off $x = 0$ (after traveling l) and is currently at $x = 0$. $x' + l + l = 2l + x'$ is the distance it has to travel before ending up at $x = 0$, so we place this at $x'_n = -(2l + x')$.

Unfortunately, I was not able to get this idea in a simple closed form formula for x'_n as a function of n. I spent some time scouring the internet and could not find one there either.

Now, what's up with the second Heaviside function in there? Supposedly, it's job is to make sure that the boundary conditions are respected. We definitely need *something* to play this role; without it, once we plug this expression into Equation 6, the first Heaviside function on its own could clearly create nonzero contributions to u at the boundaries. So we need something to subtract off these contributions. According to ChatGPT, that role is being played by the second Heaviside function: whenever there would be a nonzero contribution to u at one of the boundaries, this second term (from one of the n-terms in the sum) will subtract off that contribution. However, ChatGPT was *not* able to provide a reference for this claim. (It provided references, but none of them seemed to include the equation or anything like it!) I tried to work out if it was valid or not on my own, but I could not figure out how to show if it properly canceled out the contributions to u at the boundary from *all* of the infinite amount of reflections.

At this point, I have a strong suspicion that something is not quite right, and I would like to emphasize that blindly trusting ChatGPT for something like this is *extremely* ill-advised. So we won't. Never trust ChatGPT when it provides something you don't understand; only use it to remind you of things you *do* understand and help you walk through new applications where you can check its work step by step (such as in section 7). We will see in the following sections that this approach is unnecessarily clumsy in practice, and so I do not wish to spend more time than I already have deriving an expression we won't ultimately use. Still, for the sake of seeing what such a final expression would look like (to observe its cumbersome form if nothing else), let's plug this back into our original equation:

$$
\begin{aligned}
I(x,t) &= \phi(x,t) + w(x,t) \\
&= I_L(t) + \frac{x}{l}\left(I_R(t) - I_L(t)\right) + \int_0^t \int_0^L G\left(x, t-\tau; x'\right) f\left(x', \tau\right) dx' d\tau \\
&= I_L(t) + \frac{x}{l}\left(I_R(t) - I_L(t)\right) + \int_0^t \int_0^L G\left(x, t-\tau; x'\right) \\
&* \left(-\frac{d^2 I_L(\tau)}{d\tau^2} - \frac{x'}{l}\left(\frac{d^2 I_R(\tau)}{d\tau^2} - \frac{d^2 I_L(\tau)}{d\tau^2}\right)\right) dx' d\tau \\
&= I_L(t) + \frac{x}{l}\left(I_R(t) - I_L(t)\right) - \int_0^t \int_0^L G\left(x, t-\tau; x'\right) \\
&* \left(\frac{d^2 I_L(\tau)}{d\tau^2} + \frac{x'}{l}\left(\frac{d^2 I_R(\tau)}{d\tau^2} - \frac{d^2 I_L(\tau)}{d\tau^2}\right)\right) dx' d\tau
\end{aligned}
$$

$$
\begin{aligned}
I(x,t) =& I_L(t) + \frac{x}{l}\left(I_R(t) - I_L(t)\right) \\
& - \frac{1}{2c}\int_0^t \int_0^L \sum_n \left[\Theta\left(t - t' - \frac{|x - x_n'|}{c}\right) - \Theta\left(t - t' - \frac{|x + x_n'|}{c}\right)\right] \\
& \left(\frac{d^2 I_L(\tau)}{d\tau^2} + \frac{x'}{l}\left(\frac{d^2 I_R(\tau)}{d\tau^2} - \frac{d^2 I_L(\tau)}{d\tau^2}\right)\right) dx' d\tau
\end{aligned} \tag{7}
$$

Don't forget we'd also need to derive all the x_n'. Whoof.

6 Implementation

6.1 Solving for $I(x,t)$

All of these are known (or readily calculable) quantities, so we could integrate it numerically if we wished. At any time t we like, we can solve this integral (numerically) to solve for all of our x (well, we can't solve for all x on a continuous spectrum on a computer, but for as fine of a grid on x as we like). The infinite sum over n wouldn't be a problem either (assuming we could find a general formula for the image sources), since for any finite time t, there will only be a finite number of orders of reflection which can play a part in the solution.

6.2 Force on the Tympanum

Recall that our goal was to find the *force* on each tympanum at any time t. The force on a tympanum (say, the left one at $x = 0$) is due to the difference in pressure on the outside of the tympanum (which we assume is just at atmospheric pressure) and the pressure at the inside of the tympanum (the voltage $V(0,t)$). But in our analogy, we can just represent the atmospheric pressure as ground – zero voltage! So this difference is just $V(0,t) - 0 = V(0,t)$. Similarly, the pressure on the right tympanum is just $V(l,t)$.

As we solve the I solution, we should be able to numerically integrate a solution for V alongside it using the original equation $I_x = -V_t$ from (1) (recall we have assumed that any terms involving G can be ignored). At each time step we just go along our grid of x, approximate I_x at each point, and use that to approximate a timestep forward for V using $V_t = -I_x$. Of course, there's likely some fancy numerical methods we could use here to improve these approximations.

Speaking of numerical methods, now that we're just solving things numerically, why not drop the assumptions we made initially and just integrate our original telegrapher's equations (1) using a finite element method of some sort? This is a totally valid approach, and is exactly what Roongthumskul et. al did in [3] (though their tympanic oscillator was a simple one dimensional ODE, while ours will have 80+ dimensions as in [4]). Typically, numerical methods approaches require boundary conditions to solve a PDE to ensure stability and prevent numerical issues (failure to converge, multiple solutions, etc). On the one hand, we don't have boundary conditions for $V(x,t)$. However, since V depends directly on I and its derivatives, the boundary conditions on I should help stabilize the system by indirectly influencing the behavior of V. So this should work!

6.3 A Better Way?

Again, numerical integration via a finite element approach is totally valid. However, it could lead to a computational nightmare - especially given that our PDE is also coupled with a $2 * 80$ dimensional ODE system (the tympanic oscillators). Fortunately, I think there's a better way! Let's take advantage of our circuit analogy – there's two ways we can do this.

First, we could design a single electric circuit filter to model the IAC. We would have an input current $I_L(t)$ and $I_R(t)$ on either side of the circuit, and we could get the forcing term as just the voltage (relative to ground) as the leftmost/rightmost node on either side of the circuit. I believe the circuit would look something like Figure 1.

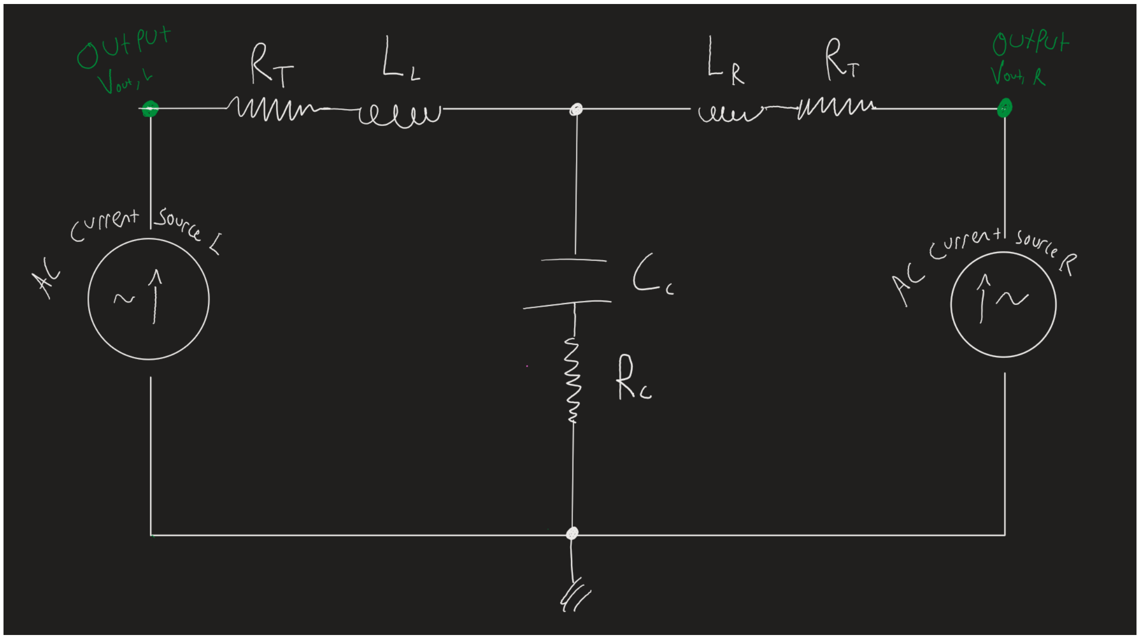

Figure 1: Single electric circuit model for the IAC

Note we have two inputs and two outputs to the circuit. Fortunately, by the linearity of the differential equations describing electrical circuits, we can just split this into two systems with one input each, and add the results. That is, say $v_{\text{out},L,L}$ is the output on the left side of the circuit due to the left input and $v_{\text{out},L,R}$ is the output on the left side of the circuit due to the right input; then (by linearity) the final left output is the sum

$$v_{\text{out},L} = v_{\text{out},L,L} + v_{\text{out},L,R}$$

and similarly the final right output is

$$v_{\text{out},R} = v_{\text{out},R,L} + v_{\text{out},R,R}$$

Therefore, we can split this problem into four single-input single-output problems. For each of these, we can calculate the transfer function that describes the input-output relation in the s-domain (continuous frequency domain). We then do a little algebra to get this as a transfer function in the z-domain (discrete time domain), which we can then put in the following form:

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

Finally, we can use this to very easily design a digital filter which replicates our circuit (approximately, since we're moving to discrete time):

$$y[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] - a_1 y[n-1] - a_2 y[n-2]$$

where $y[n]$ is the output and $x[n]$ the input at timestep n, and a_j and b_j are the poles and zeros (respectively) of the transfer function $H(z)$! This will be super easy to implement in code, and will be very computationally efficient – just a simple sum of a few known terms per timestep. Note these expressions assume that the highest order zero/pole is of order 2, but we can just add higher a_j and b_j to both expressions in the obvious way if that's not the case.

However, getting our electrical circuit model into a single form like this certainly requires making lots of simplifying assumptions about the propagation of the wave. Simplifying assumptions are fine, but I think we can do better without sacrificing much in terms of computational time or complexity! (Plus, I'm not sure at all how accurate my Figure 1 is!)

7 The Filter Chain Method

Let's take some inspiration from the idea of the telegrapher's equations representing (roughly speaking) an infinite chain of infinitesimally small circuit components, each of the form shown in Figure 2. We can't make this chain infinite, and we can't make them infinitesimally small, but we can most certainly approximate this! Let's chop our "transmission line" (the IAC) of length l into K pieces, each with length $\Delta x \equiv \frac{l}{K}$, and each having the form shown in Figure 2. We'll chain all of these pieces together, and take the output voltages (aka the pressure/force on the left/right tympanum) as the voltage at the leftmost / rightmost (non-ground) node of the chain. Each piece we'll be able to implement in code as a simple digital filter in the way described at the end of the last section, and then chain them all together by identifying each link's output to the next link's input.

Now, recall that the component values we were using above were really derivatives with respect to x – quantities *per unit length.* For example (from Equation 3) the quantity that mapped on to the ρ from [3] was $\frac{dL}{dx}$. So we will model our piece of length Δx as having an inductor with inductance $L \approx \frac{dL}{dx} * \Delta x = \rho * \Delta x$. Thus the symbols L, C, R, G we use in *this* section (up to the end) actually represent inductance/capacitance/etc. with units of Henries/Farads/etc unlike the quantities we we used previously – sorry if this creates any confusion!

Also, instead of G, we will use $\frac{1}{G} \equiv R_s$ to simplify intuition for the electronic analysis (as a resistor rather than a conductor).

Summarizing our definitions for this section:

$$\begin{aligned} L &\equiv \rho\Delta x \\ R &\equiv \rho R_V \Delta x \\ C &\equiv \beta\Delta x \\ G &\equiv \beta R_T \Delta x \\ R_s &\equiv \frac{1}{\beta R_T \Delta x} \end{aligned} \tag{8}$$

Reiterating one more time to line things up with Figure 2: really that figure should say "$\frac{dL}{dx}dx$" for the inductance (this point is indeed indicated on the Wikipedia page I took it from), and now we are approximating this inductance for a single tiny link as $\frac{dL}{dx}\Delta x$ which we define as $\frac{dL}{dx}\Delta x \equiv L$. In the acoustic analogy, this "inductance" becomes the quantity $L \to \rho\Delta x$. Ditto for R, C, G.

7.1 Transfer Functions

Again, we can take advantage of our good friend linearity: technically, our chain has two inputs (left and right), but due to linearity, we can just take each input one at a time and then superimpose the resulting outputs. The treatment for the right input will be exactly analogous to the left, so for expository purposes we'll just assume we're talking about the left input. The left current input

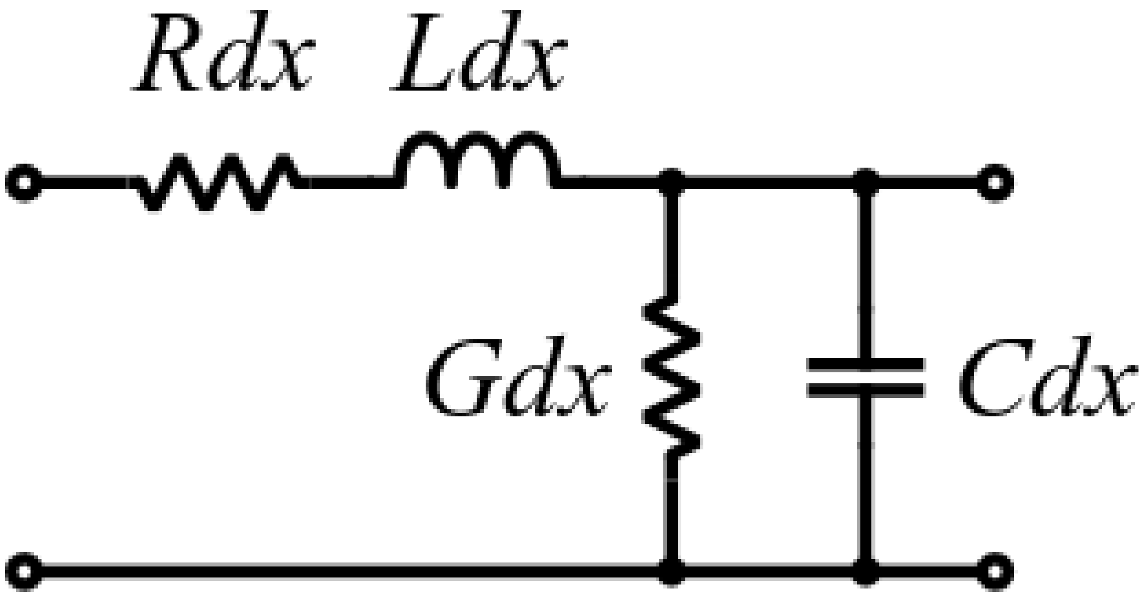

Figure 2: Infinitesimally small circuit link. (Source: Omegatron - Own work, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=1960030)

will be applied across the two nodes on the left side of a circuit just like the ones that make up our chain (Figure 2), and that current source will then induce a voltage across those two nodes. We'll find the transfer function that describes that induced voltage, and then apply that induced voltage as the voltage input to the first link in the chain. (This is just a little trick so that our true chain can start with a voltage input.) The voltage output of that first link will then be applied as an input to the next link, and so on until we reach the right tympanum where the final voltage output is just the force on the tympanum.

Therefore there are two transfer functions to calculate – one describing the output voltage (across the capacitor) of a given link due to an input voltage source:

$$H(s) = \frac{V_{\text{out}}(s)}{V_{\text{in}}(s)}$$

and one describing the voltage induced by the input *current* source:

$$H_0(s) = \frac{V_{\text{out}}(s)}{I_{\text{in}}(s)}$$

Note that these capital letters (and input (s)) indicate that we're in the Laplace domain. We'll use lowercase to represent the original voltage/current in the time domain.

7.1.1 Calculating $H(s)$

Let's start with $H(s)$. First, we use KVL to relate the voltages across all the components (note that $v_C = v_{R_s}$ since the capacitor and shunt resistor are in parallel):

$$v_{\text{in}}\,(t) = v_R(t) + v_L(t) + v_C(t) \tag{9}$$

Now we express these voltages in terms of currents, using the Ohm's law and properties of inductors/capacitors:

$$v_R(t) = i(t)Rv_L(t) \quad = L\frac{di(t)}{dt}$$

where $i(t)$ is the series current through the resistor/inductor. This current is then split up among the capacitor and the shunt resistor:

$$i(t) = i_C(t) + i_{R_s}(t) = C\frac{dv_C(t)}{dt} + \frac{v_C(t)}{R_s}$$

Now substitute these into 9:

$$\begin{aligned}
v_{\text{in}}\ (t) &= Ri(t) + L\frac{di(t)}{dt} + v_C(t) \\
&= R\left(C\frac{dv_C(t)}{dt} + \frac{v_C(t)}{R_s}\right) + L\frac{d}{dt}\left(C\frac{dv_C(t)}{dt} + \frac{v_C(t)}{R_s}\right) + v_C(t) \\
&= RC\frac{dv_C(t)}{dt} + \frac{R}{R_s}v_C(t) + LC\frac{d^2v_C(t)}{dt^2} + \frac{L}{R_s}\frac{dv_C(t)}{dt} + v_C(t) \\
&= LC\frac{d^2v_C(t)}{dt^2} + \left(RC + \frac{L}{R_s}\right)\frac{dv_C(t)}{dt} + \left(1 + \frac{R}{R_s}\right)v_C(t)
\end{aligned}$$

Now we can Laplace transform both sides of the equation (using zero initial conditions). Derivatives just become powers of s, like so:

$$\begin{aligned}
\mathcal{L}\left(\frac{d^2v_C(t)}{dt^2}\right) &= s^2V_C(s) \\
\mathcal{L}\left(\frac{dv_C(t)}{dt}\right) &= sV_C(s) \\
\mathcal{L}\left(v_C(t)\right) &= V_C(s) \\
\mathcal{L}\left(v_{\text{in}}(t)\right) &= V_{\text{in}}(s)
\end{aligned}$$

In total:

$$V_{\text{in}}\ (s) = LCs^2V_C(s) + \left(RC + \frac{L}{R_s}\right)sV_C(s) + \left(1 + \frac{R}{R_s}\right)V_C(s)$$

But hey now, $V_C(s)$ is just $V_{\text{out}}(s)$! So we can just move some things around for:

$$H(s) = \frac{V_{\text{out}}(s)}{V_{\text{in}}\ (s)} = \frac{1}{LCs^2 + \left(RC + \frac{L}{R_s}\right)s + \left(1 + \frac{R}{R_s}\right)} \tag{10}$$

7.1.2 Calculating $H_0(s)$

Now let's calculate $H_0(s)$. We have the same circuit, but this time with an input current source. We'd like to find the voltage induced by this current, aka the voltage across the current source which will be placed between the two nodes on the far left of Figure 2. Thus by KVL this voltage is just going to be the sum of all the voltage drops around the circuit:

$$v_{\text{out}}\ (t) = v_R(t) + v_L(t) + v_C(t)$$

We have

$$v_R(t) = i_{\text{in}}\ (t) \cdot R$$

$$v_L(t) = L\frac{di_{\text{in}}(t)}{dt}$$

Taking all of these equations to the Laplace domain gives

$$V_{\text{out}}\ (s) = V_R(s) + V_L(s) + V_C(s)V_R(s) \quad = I_{\text{in}}\ st) \cdot RV_L(s) = LsI_{\text{in}}(s)$$

Now we need to get V_c. The input current will go straight through the series branch (resistor and inductor) unchanged, and then will be split among the parallel branch like so:

$$\begin{aligned} i_{\text{in}}\ (t) &= i_C(t) + i_{R_s}(t) \\ &= C\frac{dv_C(t)}{dt} + \frac{v_C(t)}{R_s} \end{aligned}$$

Let's go ahead and take this equation to the Laplace domain so we can easily solve for $V_C(s)$:

$$C\left(sV_C(s)\right) + \frac{V_C(s)}{R_s} = I_{\text{in}}(s) \quad V_C(s) = \frac{I_{\text{in}}(s)}{Cs + \frac{1}{R_s}}$$

Plugging these all into our KVL equation:

$$V_{\text{out}}\ (s) = RI_{\text{in}}\ (s) + LsI_{\text{in}}\ (s) + \frac{I_{\text{in}}\ (s)}{Cs + \frac{1}{R_s}}$$

Now we can get our transfer function:

$$\begin{aligned} H_0(s) &= \frac{V_{\text{out}}\ (s)}{I_{\text{in}}\ (s)} \\ &= R + Ls + \frac{1}{Cs + \frac{1}{R_s}} \\ &= R + Ls + \frac{R_s}{R_sCs + R_s} \\ &= \frac{(R + Ls)\left(R_sCs + 1\right) + R_s}{R_sCs + 1} \end{aligned}$$

$$H_0(s) = \frac{RR_sCs + R + LR_sCs^2 + Ls + R_s}{R_sCs + 1} \tag{11}$$

7.2 Digital Filters

7.2.1 Filter for $H(s)$

Let's start designing a digital filter for our transfer function $H(s) = \frac{V_{\text{out}}(s)}{V_{\text{in}}\ (s)} = \frac{1}{LCs^2 + \left(RC + \frac{L}{R_s}\right)s + \left(1 + \frac{R}{R_s}\right)}$. The first step is to go from the s domain to the z domain via the bilinear transform (see [5]):

$$s = \frac{2}{T} \cdot \frac{1 - z^{-1}}{1 + z^{-1}}$$

where T is the sample spacing (inverse of sample rate). Our transfer function then becomes:

$$
\begin{aligned}
H(z) &= \frac{1}{LC\left(\frac{2}{T}\cdot\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + \left(RC+\frac{L}{R_s}\right)\cdot\left(\frac{2}{T}\cdot\frac{1-z^{-1}}{1+z^{-1}}\right) + \left(1+\frac{R}{R_s}\right)} \\
&= \frac{1}{\frac{4LC}{T^2}\cdot\frac{(1-z^{-1})^2}{(1+z^{-1})^2} + \frac{2\left(RC+\frac{L}{R_s}\right)}{T}\cdot\frac{1-z^{-1}}{1+z^{-1}} + \left(1+\frac{R}{R_s}\right)} \\
&= \frac{\left(1+z^{-1}\right)^2}{\frac{4LC}{T^2}\left(1-z^{-1}\right)^2 + \frac{2\left(RC+\frac{L}{R_s}\right)}{T}\left(1-z^{-1}\right)\left(1+z^{-1}\right) + \left(1+\frac{R}{R_s}\right)\left(1+z^{-1}\right)^2} \\
&= \frac{\left(1+z^{-1}\right)^2}{\frac{4LC}{T^2}\left(1-2z^{-1}+z^{-2}\right) + \frac{2\left(RC+\frac{L}{R_s}\right)}{T}\left(1-z^{-2}\right) + \left(1+\frac{R}{R_s}\right)\left(1+2z^{-1}+z^{-2}\right)} \\
&= \frac{\left(1+z^{-1}\right)^2}{\left(\frac{4LC}{T^2} + \frac{2\left(RC+\frac{L}{R_s}\right)}{T} + \left(1+\frac{R}{R_s}\right)\right) + \left(-\frac{8LC}{T^2} + 2\left(1+\frac{R}{R_s}\right)\right)z^{-1} + \left(\frac{4LC}{T^2} - \frac{2\left(RC+\frac{L}{R_s}\right)}{T} + \left(1+\frac{R}{R_s}\right)\right)z^{-2}} \\
&= \frac{1+2z^{-1}+z^{-2}}{A_0 + A_1 z^{-1} + A_2 z^{-2}}
\end{aligned}
$$

Where (using Equation 8 and $f_s \equiv \frac{1}{T}$ is the sample rate):

$$
\begin{aligned}
A_0 &= \frac{4LC}{T^2} + \frac{2\left(RC+\frac{L}{R_s}\right)}{T} + 1 + \frac{R}{R_s} \\
&= \frac{4(\rho\Delta x)(\beta\Delta x)}{\left(\frac{1}{f_s}\right)^2} + \frac{2\left(\left(\rho R_V\Delta x\right)\left(\beta\Delta x\right) + \frac{\rho\Delta x}{\frac{1}{\beta R_T\Delta x}}\right)}{\frac{1}{f_s}} + 1 + \frac{\rho R_V\Delta x}{\frac{1}{\beta R_T\Delta x}} \\
&= 4\rho\beta(\Delta x)^2 f_s^2 + 2\rho\beta(\Delta x)^2\left(R_V+R_T\right)f_s + 1 + \rho\beta R_V R_T(\Delta x)^2 \\
A_1 &= -\frac{8LC}{T^2} + 2 + \frac{2R}{R_s} \\
&= -\frac{8(\rho\Delta x)(\beta\Delta x)}{\left(\frac{1}{f_s}\right)^2} + 2 + \frac{2\rho R_V\Delta x}{\frac{1}{\beta R_T\Delta x}} \\
&= -8\rho\beta(\Delta x)^2 f_s^2 + 2 + 2\rho\beta R_V R_T(\Delta x)^2 \\
A_2 &= \frac{4LC}{T^2} - \frac{2\left(RC+\frac{L}{R_s}\right)}{T} + 1 + \frac{R}{R_s} \\
&= \frac{4(\rho\Delta x)(\beta\Delta x)}{\left(\frac{1}{f_s}\right)^2} - \frac{2\left(\left(\rho R_V\Delta x\right)\left(\beta\Delta x\right) + \frac{\rho\Delta x}{\frac{1}{\beta R_T\Delta x}}\right)}{\frac{1}{f_s}} + 1 + \frac{\rho R_V\Delta x}{\frac{1}{\beta R_T\Delta x}} \\
&= 4\rho\beta(\Delta x)^2 f_s^2 - 2\rho\beta(\Delta x)^2\left(R_V+R_T\right)f_s + 1 + \rho\beta R_V R_T(\Delta x)^2
\end{aligned}
$$

We're now ready to derive our digital filter. ChatGPT generated a nice writeup of how this is done (which I have verified elsewhere):

1. Write the transfer function in a general form:

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_N z^{-N}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_M z^{-M}}$$

2. Multiply both sides by the Denominator

$$\left(1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_M z^{-M}\right) Y(z) = \left(b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_N z^{-N}\right) X(z)$$

3. Convert the Equation Back to the Time Domain

Apply the inverse z-transform to convert this equation from the z-domain to the time domain. This step involves interpreting z^{-1} as a time delay (i.e., z^{-1} corresponds to $y[n-1], z^{-2}$ to $y[n-2]$, and so on). Doing this yields the difference equation:

$$y[n] + a_1 y[n-1] + a_2 y[n-2] + \cdots + a_M y[n-M] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + \cdots + b_N x[n-N]$$

$$y[n] = -a_1 y[n-1] - a_2 y[n-2] - \cdots - a_M y[n-M] + b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + \cdots + b_N x[n-N]$$

In our case, we'll start by multiplying by $1/A_0$ to get this in the proper form:

$$H(z) = \frac{\frac{1}{A_0} + \frac{2}{A_0} z^{-1} + \frac{1}{A_0} z^{-2}}{1 + \frac{A_1}{A_0} z^{-1} + \frac{A_2}{A_0} z^{-2}}$$

Thus:

$$\begin{aligned} y[n] &= -\frac{A_1}{A_0} y[n-1] - \frac{A_2}{A_0} y[n-2] + \frac{1}{A_0} x[n] + \frac{2}{A_0} x[n-1] + \frac{1}{A_0} x[n-2] \\ A_0 &= 4\rho\beta(\Delta x)^2 f_s^2 + 2\rho\beta(\Delta x)^2 \left(R_V + R_T\right) f_s + 1 + \rho\beta R_V R_T (\Delta x)^2 \\ A_1 &= -8\rho\beta(\Delta x)^2 f_s^2 + 2 + 2\rho\beta R_V R_T (\Delta x)^2 \\ A_2 &= 4\rho\beta(\Delta x)^2 f_s^2 - 2\rho\beta(\Delta x)^2 \left(R_V + R_T\right) f_s + 1 + \rho\beta R_V R_T (\Delta x)^2 \end{aligned} \tag{12}$$

Where x is the input voltage at the start of the link, y is the induced output voltage.

(Still need to double check this algebra since it was done by ChatGPT. All the coefficients on the difference equation look unitless though, which is good.)

7.2.2 Filter for H_0

Now let's do the same thing for $H_0(s) = \frac{RR_sCs + R + LR_sCs^2 + Ls + R_s}{R_sCs + 1}$. Converting to the z-domain gives:

$$
\begin{aligned}
H_0(z) &= \frac{LR_sC\left(\frac{2}{T}\cdot\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + (RR_sC+L)\left(\frac{2}{T}\cdot\frac{1-z^{-1}}{1+z^{-1}}\right) + (R+R_s)}{R_sC\left(\frac{2}{T}\cdot\frac{1-z^{-1}}{1+z^{-1}}\right)+1} \\
&= \frac{LR_sC\cdot\frac{4}{T^2}\left(1-2z^{-1}+z^{-2}\right) + \frac{2(RR_sC+L)}{T}\left(1-z^{-2}\right) + (R+R_s)\left(1+2z^{-1}+z^{-2}\right)}{\frac{2R_sC}{T}\left(1-z^{-2}\right)+\left(1+2z^{-1}+z^{-2}\right)} \\
&= \frac{\left(\frac{4LR_sC}{T^2}+\frac{2(RR_sC+L)}{T}+(R+R_s)\right) + \left(-\frac{8LR_sC}{T^2}+2\left(R+R_s\right)\right)z^{-1} + \left(\frac{4LR_sC}{T^2}-\frac{2(RR_sC+L)}{T}+(R+R_s)\right)z^{-2}}{\left(\frac{2R_sC}{T}+1\right)+2z^{-1}+\left(1-\frac{2R_sC}{T}\right)z^{-2}} \\
&= \frac{B_0+B_1z^{-1}+B_2z^{-2}}{C_0+C_1z^{-1}+C_2z^{-2}} \\
&= \frac{\frac{B_0}{C_0}+\frac{B_1}{C_0}z^{-1}+\frac{B_2}{C_0}z^{-2}}{1+\frac{C_1}{C_0}z^{-1}+\frac{C_2}{C_0}z^{-2}}
\end{aligned}
$$

Where

$$
\begin{aligned}
B_0 &= \frac{4LR_sC}{T^2} + \frac{2\left(RR_sC + L\right)}{T} + \left(R + R_s\right) \\
&= \frac{4(\rho\Delta x)\left(\frac{1}{\beta R_T \Delta x}\right)(\beta\Delta x)}{\left(\frac{1}{f_s}\right)^2} + \frac{2\left((\rho R_V \Delta x)\left(\frac{1}{\beta R_T \Delta x}\right)(\beta\Delta x) + \rho\Delta x\right)}{\frac{1}{f_s}} + \left(\rho R_V \Delta x + \frac{1}{\beta R_T \Delta x}\right) \\
&= \frac{4\rho f_s^2}{R_T} + 2\left(\frac{\rho R_V}{R_T} + \rho\Delta x\right) f_s + \rho R_V \Delta x + \frac{1}{\beta R_T \Delta x} \\
B_1 &= -\frac{8LR_sC}{T^2} + 2\left(R + R_s\right) \\
&= -\frac{8(\rho\Delta x)\left(\frac{1}{\beta R_T \Delta x}\right)(\beta\Delta x)}{\left(\frac{1}{f_s}\right)^2} + 2\left(\rho R_V \Delta x + \frac{1}{\beta R_T \Delta x}\right) \\
&= -\frac{8\rho f_s^2}{R_T} + 2\rho R_V \Delta x + \frac{2}{\beta R_T \Delta x} \\
B_2 &= \frac{4LR_sC}{T^2} - \frac{2\left(RR_sC + L\right)}{T} + \left(R + R_s\right) \\
&= \frac{4(\rho\Delta x)\left(\frac{1}{\beta R_T \Delta x}\right)(\beta\Delta x)}{\left(\frac{1}{f_s}\right)^2} - \frac{2\left((\rho R_V \Delta x)\left(\frac{1}{\beta R_T \Delta x}\right)(\beta\Delta x) + \rho\Delta x\right)}{\frac{1}{f_s}} + \left(\rho R_V \Delta x + \frac{1}{\beta R_T \Delta x}\right) \\
&= \frac{4\rho f_s^2}{R_T} - 2\left(\frac{\rho R_V}{R_T} + \rho\Delta x\right) f_s + \rho R_V \Delta x + \frac{1}{\beta R_T \Delta x} \\
C_0 &= \frac{2R_sC}{T} + 1 \\
&= \frac{2\left(\frac{1}{\beta R_T \Delta x}\right)(\beta\Delta x)}{\frac{1}{f_s}} + 1 \\
&= \frac{2f_s}{R_T} + 1 \\
C_1 &= 2 \\
C_2 &= 1 - \frac{2R_sC}{T} \\
&= 1 - \frac{2\left(\frac{1}{\beta R_T \Delta x}\right)(\beta\Delta x)}{\frac{1}{f_s}} \\
&= 1 - \frac{2f_s}{R_T}
\end{aligned}
$$

Matching coefficients with our general form:

$$
\begin{aligned}
y_0[n] &= -\frac{C_1}{C_0}y_0[n-1] - \frac{C_2}{C_0}y_0[n-2] + \frac{B_0}{C_0}x_0[n] + \frac{B_1}{C_0}x_0[n-1] + \frac{B_2}{C_0}x_0[n-2] \\
B_0 &= \frac{4\rho f_s^2}{R_T} + 2\left(\frac{\rho R_V}{R_T} + \rho\Delta x\right) f_s + \rho R_V \Delta x + \frac{1}{\beta R_T \Delta x} \\
B_1 &= -\frac{8\rho f_s^2}{R_T} + 2\rho R_V \Delta x + \frac{2}{\beta R_T \Delta x} \\
B_2 &= \frac{4\rho f_s^2}{R_T} - 2\left(\frac{\rho R_V}{R_T} + \rho\Delta x\right) f_s + \rho R_V \Delta x + \frac{1}{\beta R_T \Delta x} \\
C_0 &= \frac{2f_s}{R_T} + 1 \\
C_1 &= 2 \\
C_2 &= 1 - \frac{2f_s}{R_T}
\end{aligned}
\tag{13}
$$

Where x_0 is our input current and y_0 is the induced voltage at the beginning of the chain.

Again, we should probably double check the algebra here. Note that the coefficients on the delayed output terms are unitless as they should be (we'll see in subsection 7.8 that R_T has units of s^{-1}), but the coefficients $\frac{B_j}{C_0}$ on the input coefficents should be converting particle velocity to pressure. Since C_0 is unitless, we should check that all the B_j have units of this conversion. Assuming this is correct though, there we have it!

7.3 The Force on the Tympani

We can now set up two filter chains, one going to the left and the other going to the right. To get the force on a tympanum (say, the right one) we will add up the voltage (pressure) output in the final link of the rightward chain along with the voltage output from the "zeroth" link (which converted current to voltage) in the leftward chain. We're taking a 2 input 2 output system and using linearity to split the inputs into two different chains. Then at each end of the chain we just add the values at that location from each chain.

7.4 Reflections

What happens when a pressure wave from one side of the IAC hits the other side? It's exerting a force on the other tympanum, so it'll do some work and lose some energy there. But some of it should also reflect back and go the other way, right? This should be easy to model. We'll let $\gamma_r \in [0, 1]$ be a "reflectivity constant" that determines how much of the wave reflects. Then whatever the output is on the final link in the chain, we'll multiply it by γ_r and add it back into the first link in the other chain. (That is, the first "true" link which takes voltage input to voltage output).

7.5 Air Resistance

We'd like to somehow track the immediate "pushback" the IAC has on the tympanum as it moves back and forth. On the one hand, we could incorporate an air resistance term just by adding it on the internal dynamics of the tympanic oscillator. However, the advantage of the current approach is that it allows canceling out due to the pressure wave of the other oscillator. That is, if the pressure wave induced by the left tympanum is exactly the opposite as what was induced by the right tympanum (by the time it gets over to the left), then the left tympanum should feel no force from the IAC. That said, maybe I'm wrong to conflate these two things; perhaps we also need an air resistance term on top of this! But – outside of the context of "cancelling out" – doesn't it seem like if we're trying to describe the pressure force on the tympanum due to it's motion in and out of the IAC, that would just be air resistance? I'm not sure.

7.6 Time Delay

One thing to worry about is the question of the finite speed of the pressure wave as it crosses the IAC. In the PDE form, this is clearly baked into the equation – in the lossless limit, the PDEs just become wave equations, which have a finite speed of propagation. However, I believe that we've lost something in the process of discretizing space from dx to Δx. In the continuous case, each link communicated instantaneously accross the distance dx, but it had infinite links to go through – somehow leading to the finite speed of propagation c. Looking at the form of our difference equations, the output of each at timestep n depends instantaneously on the input at timestep n, and so a disturbance at one side of the IAC would be able to propagate instantaneously to the other side. This is no good, but it'll be easy to fix.

All we have to do is incorporate a time delay between each link. That is, the input that link k receives at timestep n is the output from link k at timestep $n-1$. In this way, if there are K links in the chain and our sample rate is $1/T$, it will take $K*T$ seconds for a disturbance to propagate. If we want the speed of propagation to be c and the length of the IAC is l, then we just need K such that $\frac{l}{c} = KT$ or $K = \frac{Tl}{c}$. In Equation 5 we saw that the lossless form of the telegrapher's equations use a speed of propagation of

$$c = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\rho\beta}} \approx 345.03$$

Where we used Equation 3 for L and C (these are the L and C used in the beginning of the paper). This is basically just the speed of sound (343 m/s), so we could just use that too.

Note that if this is too many K to be computationally feasible, we could half it and make the time delay 2 samples instead of 1.

7.7 Putting It All Together

We'll just look at one direction since the other is analogous. We create K filters plus an input current to voltage converter filter. That's $K+1$ new state variables which we need to keep track of. (Really, we just need to keep track of the last few samples to calculate the filter output at each step, so we can do that if we want to cut down on memory. It would be fun to keep them all so we can animate the waves traveling down the IAC though!) Updating them each timestep will be easy to calculate using the difference equations derived above.

The input to the zeroth link of the initial chain will be the velocity of the tympanum (which is then the particle velocity of the air immediately nearby.) No conversion factor is needed; following Roongthumskul our "current" described in the equations is literally just particle velocity and our "voltage" is literally just pressure. The output of the zeroth link is a voltage which is then the input to the first true link of the chain. Each link will be connected to the next link in the chain with a single sample delay in between. The exception will be the current to voltage converter at the beginning (the "zeroth" link), since that's all happening instantaneously at the same location. The force on the opposing tympanum will be the voltage output at the final link in the chain plus the voltage output of the zeroth link of the other chain (though this summed voltage will need to be multiplied by the area of the tympanum to convert pressure to force).

7.8 Parameters

Most of the parameter values we will be able to borrow directly from [3]. Their table of parameters is copied here in Figure 3. It does not include R_V or R_T, which are derived in the first passage from the supplementary section which I quote at the end of this subsection. First, I'll summarize their definitions:

$$R_V = \frac{2\sqrt{2\pi f_c \mu}}{r\sqrt{\rho}} R_T = \frac{2\gamma\sqrt{2\pi f_c \alpha}}{r(\gamma - 1)}$$

Note that R_V has units of $m^{-3}s^{-1}$ while R_T has units of s^{-1}.
There's also a couple new parameters we need to decide on ourselves:

- γ_r the reflectivity of the tympani
- Our sample rate ($128Hz$ seemed to be fine over the summer)
- The number of links K (probably just $K = \frac{Tl}{c}$)

"We model the internal coupling of a tokay gecko's ears by air in its mouth cavity and Eustachian tubes. The cavity is approximated to be a closed cylinder of length L, radius r, and cross-sectional area A[1]. For simplicity, the motion of the air is described in one dimension by its axial velocity $u(x,t)$ and pressure $p(x,t)$ with respect to a reference state of zero velocity, pressure $P_r = 101\text{kPa}$, and temperature $T = 25°\text{C}$. To account for damping associated with the air interacting with the sides of the cylinder we respectively write the conservation of momentum and conservation of mass equations as

$$\begin{aligned}\frac{\partial p}{\partial x} &= -\rho\left(R_V u + \frac{\partial u}{\partial t}\right) + \eta_V \\ \frac{\partial u}{\partial x} &= -\beta\left(R_T p + \frac{\partial p}{\partial t}\right) + \eta_T\end{aligned} \tag{14}$$

in which ρ is the air's reference density and β is the air's reference adiabatic compressibility. The viscous damping rate is described by

$$R_V = \frac{2\mu}{r\rho w_V},$$

in which μ is the dynamic viscosity of the air. The viscous boundary layer width is given by

$$w_V = \sqrt{\frac{\mu}{2\pi f_c \rho}},$$

in which f_c is a characteristic frequency [2]. Similarly, the thermal damping rate is described by

$$R_T = \frac{2\gamma\alpha}{(\gamma - 1) r w_T},$$

in which α is the thermal diffusion rate, γ is the air's adiabatic index, and the thermal boundary layer width is

$$w_T = \sqrt{\frac{\alpha}{2\pi f_c}}.$$

Because the boundary layer widths change slowly with frequency, we fix the characteristic frequency $f_c = 15\text{kHz}$. To a first approximation, overestimating the characteristic frequency accounts for an increase in the damping rates owing to the intricate anatomy of the head cavity."

7.9 Noise

If we want to incorporate noise, here's how Roongthumskul et. al calculated the noise terms (seen in Equation 14). I'm not sure how we would incorporate these; somehow divvy them up among the links? Anyhow, I they ended up killing them to save on computational time, so we'll probably just ignore them at first.

"According to the fluctuation-dissipation theorem, the Gaussian white noise terms $\eta_V(x,t)$ and $\eta_T(x,t)$ satisfy

$$\begin{aligned} \langle \eta_V(t,x) \rangle = 0, \quad \langle \eta_V(t,x)\eta_V\left(t',x'\right)\rangle = 2k_B T \rho R_V A^{-1} \delta\left(t-t'\right)\delta\left(x-x'\right) \\ \langle \eta_T(t,x) \rangle = 0, \quad \langle \eta_T(t,x)\eta_T\left(t',x'\right)\rangle = 2k_B T \beta R_T A^{-1} \delta\left(t-t'\right)\delta\left(x-x'\right) \end{aligned}$$

in which $\langle \ldots \rangle$ denotes the ensemble average and k_B is Boltzmann's constant [3–4]."

7.10 Open Questions

- Are we correct to treat current as particle velocity, or should it be volume velocity? Roongthumskul clearly says particle velocity, but I thought that canonically electric current is analogous to volume velocity. Check units to see which makes sense in our equations.
- Do we also want to incorporate an air resistance term on the dynamics of the tympanum(/papilla)?
- How should we incorporate noise in the IAC? Or should we not?
- Did ChatGPT do all the algebra correctly when deriving the filter difference equations?

References

[1] Brian E. Anderson and Scott D. Sommerfeldt. Solving one-dimensional acoustic systems using the impedance translation theorem and equivalent circuits: A graduate level homework assignment. *The Journal of the Acoustical Society of America*, 150(6):4155–4165, Dec 2021.

[2] N. H. Fletcher and Suszanne Thwaites. Physical models for the analysis of acoustical systems in biology. *Quarterly Reviews of Biophysics*, 12(1):25–65, Feb 1979.

[3] Yuttana Roongthumskul, Dáibhid Ó Maoiléidigh, and A.J. Hudspeth. Bilateral spontaneous otoacoustic emissions show coupling between active oscillators in the two ears. *Biophysical Journal*, 116(10):2023–2034, May 2019.

[4] Andrej Vilfan and Thomas Duke. Frequency clustering in spontaneous otoacoustic emissions from a lizard's ear. *Biophysical Journal*, 95(10):4622–4630, 2008.

[5] Wikipedia contributors. Bilinear transform — Wikipedia, the free encyclopedia, 2024. [Online; accessed 22-October-2024].

[6] Wikipedia contributors. Green's function — Wikipedia, the free encyclopedia, 2024. [Online; accessed 21-October-2024].

[7] Wikipedia contributors. Telegrapher's equations — Wikipedia, the free encyclopedia, 2024. [Online; accessed 6-October-2024].

TABLE S1 Parameter values used in the model. "Scale" indicates values rescaled from reference, in which the distance unit is $\mathrm{DU} = 22\ \mathrm{pm}$, the time unit $\mathrm{TU} = 50\ \mathrm{ms}$, and the force unit $\mathrm{FU} = 44\ \mathrm{nN}$.

parameters		values	units	ref
ρ	Air's reference density	1.2	$\mathrm{kg \cdot m^3}$	
β	Air's adiabatic compressibility	7×10^{-6}	$\mathrm{Pa^{-1}}$	
μ	Dynamic viscosity of air	18.6×10^{-6}	$\mathrm{Pa \cdot s}$	
α	Thermal diffusion rate	2.2×10^{-5}	$\mathrm{m^2 \cdot s^{-1}}$	
γ	Air's adiabatic index	1.4		
f_c	Characteristic frequency defining the viscous boundary layer width	15	kHz	
r	Eardrum radius	1.575	mm	
L	Effective interaural distance	4.7	cm	
γ_1, γ_2	Damping coefficient of active oscillator	0.4	$\mathrm{FU \cdot TU \cdot DU^{-1}}$	Scale
a_1, a_2	Negative stiffness of an oscillator	3.5	$\mathrm{FU \cdot DU^{-1}}$	Scale
b_1, b_2	Coupling stiffness	0.9	$\mathrm{FU \cdot DU^{-1}}$	Scale
c_1, c_2	Coupling compliance of an active oscillator	1	$\mathrm{FU^{-1} \cdot DU}$	Scale
N_1, N_2	Strength of the nonlinearity	1	$\mathrm{FU \cdot DU^{-3}}$	Scale
τ_1, τ_2	Time constant of the active force	10	TU	Scale
K_{H1} K_{H2}	Maximum stiffness for spontaneous oscillations	$a_1 - \gamma_1/\tau_1$ $a_2 - \gamma_2/\tau_2$	$\mathrm{FU \cdot DU^{-1}}$	Scale
K_1 K_2	Stiffness of an active oscillator	$0.99K_{H1}$ $0.99K_{H2}$	$\mathrm{FU \cdot DU^{-1}}$	Scale
c_1, c_2	Coupling compliance of a passive oscillator	0	$\mathrm{FU^{-1} \cdot DU}$	Scale
K_1 K_2	Stiffness of a passive oscillator	$1.01a_1$ $1.01a_2$	$\mathrm{FU \cdot DU^{-1}}$	Scale

Figure 3: Table S1 from Roongthumskul et al.