Exercise 1 (Proof of Lemma 11) Let L be a lattice, and $S \subseteq \mathrm{Span}_R(L)$ be a real subvector space such that $\dim(S) = \dim(S \cap L)$. Prove that $(S \cap L)^\vee = \pi_S(L^\vee)$ where π_S is the orthogonal projection onto S.

Exercise 2 (Sublattices) Let L be a lattice, and $S \subset L$ a full rank sublattice.

(a) Prove that $L^\vee \subset S^\vee$, and it is a full rank inclusion.

(b) Prove that $|S^\vee / L^\vee| = |L/S|$.

Exercise 3 (Transference) The term transference refers to the geometric relations between the primal and the dual. Here are some examples.

(a) Prove that $\lambda_1(L) \cdot \lambda_n(L^\vee) \geq 1$. (Hint: We almost did it in Lemma 2.)

(b) Prove that $\lambda_1(L) \cdot \lambda_1(L^\vee) \leq \gamma_{n-1}^{\frac{n-1}{n-2}} \leq n$. (Hint: We've just seen that quantity in Lecture 6!)

Toward Harmonic Analysis. The consideration of the exercise below is the foundation for "harmonic analysis", namely the generalization of Fourier analysis. In the same way that periodic function over $\mathbb{R}$ can be decomposed as a sum of trigonometric function, harmonic analysis is concerned with decomposition of periodic function over $\mathbb{R}^n$. The period of a periodic function is now defined by a lattice L, and each term of such a decomposition is associated with a vector of the dual lattice $L^\vee$.

Exercise 4 (Duality) Lattices are also abelian groups, and abelian groups already have a notion of a dual. More precisely, for a compact abelian group G, the dual group G^* is the group of continuous morphism from G to the complex unit circle $\mathbb{C}^\times = \{c \in \mathbb{C} \mid |c| = 1\}$. Such morphisms are also called characters of G, typically denoted with the symbol χ.

Let L be a full-rank lattice of dimension n. Consider $T = \mathbb{R}^n/L$ is the torus define by L. One can think of the torus T as a fundamental parraleliped $\mathcal{P}(\mathbf{B})$, with its opposing facets glued back together. More formally, a metric can be defined on T by the distance function $d(a,b) = \min\{\|\mathbf{a} - \mathbf{b}\| \mid \mathbf{a} \in a + L, \mathbf{b} \in b + L\}$, i.e., the minimal distance between any representative of each coset. This induces a topology on T.

We are going to establish an isomorphism between the dual group of the torus T^* and the dual lattice $L^\vee$. For any $\mathbf{v} \in L^\vee$ we defined $\chi_\mathbf{v} : \mathbb{R}^n \to \mathbb{C}^\times$ via $\chi_\mathbf{v}(\mathbf{w}) = \exp(2\imath\pi\langle \mathbf{v}, \mathbf{w}\rangle)$

(a) Prove that for all $\mathbf{v} \in L^\vee$, $\chi_\mathbf{v}$ is a character, *i.e.* a continuous group morphism.

(b) Prove that it is L-periodic: $\chi_\mathbf{v}(\mathbf{t} + \mathbf{w}) = \chi_\mathbf{v}(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{R}^n$ and $\mathbf{w} \in L$.

(c) Define $\chi'_\mathbf{v} : T \to \mathbb{C}^\times$ by $\chi'_\mathbf{v}(x) := \chi'_\mathbf{v}(\mathbf{x})$ for some $\mathbf{x} \in x + L$. This is well defined via the previous question. Prove that $\chi'_\mathbf{v}$ is a continuous morphism. (Hint: Continuity is the tricky bit. No need to bother with topology, the metric continuity ("ε and δ") is applicable here.)

(d) Define $\phi : L^\vee \to T^*$ by $\phi(\mathbf{v}) = \chi'_\mathbf{v}$. Prove that ϕ is an injective morphism. (Hint: At this point, $\chi'_{\mathbf{v}+\mathbf{w}} = \chi'_\mathbf{v} \cdot \chi'_\mathbf{w}$ is a matter of rewriting. For injectivity, consider evaluating $\chi_\mathbf{v}$ at $\mathbf{v}/(2\|\mathbf{v}\|^2)$.)

We now move to surjectivity. That is, we are given some character $\chi' \in T^*$ and want to reconstruct a $\mathbf{v} \in L^\vee$ such that $\chi' = \chi'_\mathbf{v}$. We also consider the extension χ of χ' by given by L-periodicity $\chi(\mathbf{x}) := \chi'(\mathbf{x} \bmod L)$.

(e) Show that if, for some $\varepsilon > 0$, it holds that $\chi(\mathbf{x}) = \chi_{\mathbf{v}}(\mathbf{x})$ for all $\mathbf{x} \in \varepsilon \cdot \mathfrak{B}$, then $\chi(\mathbf{x}) = \chi_{\mathbf{v}}(\mathbf{x})$ for any $\mathbf{x} \in \mathbb{R}^n$. (Hint: If $\chi(\mathbf{x})$ is determined, so is $\chi(m\mathbf{x})$ for any integer m.)

(f) Show that there exists $\varepsilon > 0$ and a linear function $f : \mathbb{R}^n \mapsto \mathbb{R}$ such that $\chi'(\mathbf{x}) = \exp(2\imath\pi f(\mathbf{x}))$ for all $\mathbf{x} \in \varepsilon \cdot \mathfrak{B}$. (Hint: Take the complex logarithm, at least where continuity lets you do so.)

(g) Deduce that there exists $\mathbf{v} \in \mathbb{R}^n$ such that $\chi = \chi_{\mathbf{v}}$. (Hint: Recall from linear algebra: a finite dimensional vector space is isomorphic to its dual.)

(h) Show that in fact, the above $\mathbf{v}$ belongs to $L^\vee$. (Hint: We haven't used L-periodicity yet.)