Exercise 1 Prove the following statement. For any two lattices L', L of same rank such that $L' \subset L$, we have $|L/L'| = \det(L')/\det(L) = |L \cap T'|$ where T' is any tiling of L'.

Exercise 2 (Variation on Proposition 13 for a shift of a lattice) Prove the following statement. Let T be a bounded tiling for a lattice L, and $\mu = \sup_{\mathbf{x} \in T} \|x\|$. Then, for any radius $r > 2\mu$ and any shift $\mathbf{t} \in \mathrm{Span}_{\mathbb{R}}(L)$, it holds that $\frac{(r-2\mu)^n}{\det(L)} \leq \frac{|r\mathcal{B} \cap (L+\mathbf{t})|}{\mathrm{vol}(\mathcal{B})} \leq \frac{(r+2\mu)^n}{\det(L)}$.

Exercise 3 (easy) Search online for the volume of the n-dimensional ℓ_p-ball for $p \geq 1$ (and be amazed by how surprisingly simple it is). Specialize Minkowski's bound to any ℓ_p norm.

Exercise 4 (easy) Show that there is no such lower bound akin to Minkowski's upper bound for the normalized minimal distance. Namely, for a arbitrary dimension n, and for arbitrary $\varepsilon > 0$, exhibit a lattice L of dimension n, determinant 1, such that $\lambda_1(L) \leq \varepsilon$.

Exercise 5 (Planetary Alignment) A planetary system with $n+1$ planets, all orbit a star in perfect circles. The planets have constant angular frequencies $\sigma_0, \ldots, \sigma_n$ respectively (i.e. planet i moves through an angle 2π in time $1/\sigma_i$). We say that the planets are in ε-planetary alignment when the angle of planet $1, \ldots, n$ with planet 0 is at most $2\pi\varepsilon$.

Suppose the planetary system is at 0-planetary alignment at time $t = 0$. Show that the next ε-planetary alignment occurs at time at most $2^n / (\varepsilon^{n-1} \max_{i \neq 0} |\sigma_i - \sigma_0|)$.

Exercise 6 (hard) Generalize Minkowski's first theorem to any symmetric convex set $S \subseteq \mathrm{Span}_{\mathbb{R}}(L)$ of arbitrary volume, but proving the bound $|S \cap L| \geq \left\lceil \frac{\mathrm{vol}(S)}{2^n \cdot \det(L)} \right\rceil$.
(Hint: What is the *average* number of point in $(S + \mathbf{r}) \cap L'$ for a uniformly random $\mathbf{r} \in \mathcal{P}(\mathbf{B}')$?)