THE VARIETY OF REDUCTIONS FOR A REDUCTIVE SYMMETRIC PAIR

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ABSTRACT. We define and study the variety of reductions for a reductive symmetric pair (G, θ), which is the natural compactification of the set of the Cartan subspaces of the symmetric pair. These varieties generalize the varieties of reductions for the Severi varieties studied by Iliev and Manivel, which are Fano varieties.

We develop a theoretical basis to the study these varieties of reductions, and relate the geometry of these variety to some problems in representation theory. A very useful result is the rigidity of semi-simple elements in deformations of algebraic subalgebras of Lie algebras.

We apply this theory to the study of other varieties of reductions in a companion paper, which yields two new Fano varieties.

1. INTRODUCTION

The problem of classifying all complex analytic compactifications of $\mathbf{C}^n$ which have second Betti number $b_2 = 1$, also known as *irreducible compactifications*, was stated by Hirzebruch [1]. An irreducible compactification is always a Fano variety, and classifiying the former ones is, surprisingly, not easier than classifying the later ones. Indeed, the classifications of the irreducible compactifications of dimension 1, 2 and 3 came as specializations of the classifications of the Fano varieties of the corresponding dimension. For the dimension 4 or above, there is at the current time no classification of the irreducible compactifications available and no classification of the Fano varieties. The irreducible compactifications of the affine spaces of dimension less than 3 are the projective spaces $\mathbf{P}^1$, $\mathbf{P}^2$ and $\mathbf{P}^3$, the quadric $\mathbf{Q}^3$, and two Fano varieties which have $b_3 = 0$ and respective Fano index 1 and 2. See Müller-Stach [2] for a short description of these varieties and an account on other results about compactifications of affine spaces.

As pointed out in [1], the fundamental homogenous spaces of reductive groups are examples of compactifications of the affine space, but new examples are very hard to find and seem to always show up as finite families. This lack of examples hinders the efforts aimed at the classification of Fano varieties or even that of irreducible compactifications of affine spaces. In this paper we associate to any *reductive symmetric pair* (G, θ), where G is a reductive complex group and θ an involution of G, its *variety of reductions* $\mathfrak{R}$. Some of these varieties were previously obtained by different means, and studied by Ranestad and Schreyer [3] and Iliev and Manivel [4]. At the present time, the study of seven of these varieties of reductions $\mathfrak{R}$ has been carried out, revealing that all of them are normal Fano varieties, and the smooth ones are even compactifications of the affine space which

have $b_2 = 1$. This observation is our main motivation for defining the varieties of reductions: the study of many of the low-dimensional ones may be carried out, and yield other compactifications of affine spaces or Fano varieties.

We study here general properties of varieties of reductions for symmetric pairs, and use this theory in a companion paper [5] to study three more examples of variety of reductions (two of them count in the number seven mentioned above). While we were primarily interested in developing tools and methods suited to the practical study of examples, it turned out that the general theory presents interesting aspects in its own right and relates to other problems in the representation theory of complex Lie groups. We now define the varieties of reductions and then outline the results of our study.

1.1. **Variety of reductions for a symmetric pair.** A *reductive symmetric* pair (G, θ) has a reductive group G as its first member and an involution θ of G as its second member. These pairs occur in the study of real forms of complex reductive groups and symmetric spaces, and they were classified using many different invariants, see S. Araki [6], A. G. Helminck [7], and T. Springer [8], for instance. To such a pair we attach its *connected fixed point group* $K = (G^\theta)^\circ$, which is reductive, and the decomposition of the Lie algebra $\mathfrak{g}$ of G as eigenspaces for the involution θ' tangent to θ at the unit element of G:

$$\mathfrak{g} = \mathfrak{g}(\theta')_1 \oplus \mathfrak{g}(\theta')_{-1} = \mathfrak{k} \oplus \mathfrak{p}.$$

This decomposition is called the Cartan decomposition of $\mathfrak{g}$, and $\mathfrak{p}$ the *anisotropic space* of the symmetric pair. We will use [9] by Kostant and Rallis and [10] by Tauvel and Yu as references for results about the operation of K in the anisotropic space $\mathfrak{p}$.

1.2. DEFINITION. A *Cartan subspace* of $\mathfrak{p}$ is a linear subspace $\mathfrak{a}$ of $\mathfrak{g}$ that is contained in $\mathfrak{p}$ and in some Cartan subalgebra of $\mathfrak{g}$, and that is maximal in the family of such subspaces ordered by the inclusion.

Any two Cartan subspaces of $\mathfrak{p}$ are *K*-conjugated [9, Theorem 1]. Their common dimension r is the *rank* of the symmetric pair (G, θ) and their set $\mathfrak{R}_o$ is a *K*-orbit in the Grassmann variety $\mathcal{G}(r, \mathfrak{p})$ of r-planes in $\mathfrak{p}$.

1.3. DEFINITION. The variety of reductions $\mathfrak{R}$ for the symmetric pair (G, θ) is the closure in $\mathcal{G}(r, \mathfrak{p})$ of the set $\mathfrak{R}_o$ of all Cartan subspace of $\mathfrak{p}$.

It is customary, while sometimes ambiguous, to write (G, K) for (G, θ) when referring to a particular symmetric pair. We stick to this usage, and emphasize that the Cartesian square $\mathfrak{G} \times \mathfrak{G}$ of a reductive group $\mathfrak{G}$ is turned into a symmetric pair by the involution swapping its two factors. Ranestad and Schreyer [3] have shown that the variety of reductions for the symmetric pairs $(\mathbf{SL}_n, \mathbf{SO}_n)$ are smooth only for $n \leq 5$. Iliev and Manivel [4] studied the varieties of reductions for the symmetric pairs $(\mathbf{SL}_3, \mathbf{SO}_3)$, $(\mathbf{SL}_3 \times \mathbf{SL}_3, \mathbf{SL}_3)$, $(\mathbf{SL}_6, \mathbf{Sp}_6)$ and (E_6, F_4). These four symmetric pairs occur as structure symmetries for the four simple Jordan algebras of rank 3 [4]. In [5] we study the varieties of reductions for $(\mathbf{SL}_4, \mathbf{SO}_4)$, and for the Cartesian squares of $\mathbf{Sp}_4$ and G_2.

1.4. **Abelian subalgebras.** In the small rank cases [4, 11, 5] the variety $\mathfrak{A}$ of all r-dimensional subalgebras of $\mathfrak{g}$ contained in $\mathfrak{p}$ is not larger than $\mathfrak{R}$, but we show the

THEOREM (5.3, 9.7). *Every point in $\mathfrak{R}$ is the Lie algebra of a subgroup of G. If G has large enough rank, then $\mathfrak{A}$ contains a point that is not the Lie algebra of a subgroup of G.*

In general $\mathfrak{R}$ is a strict irreducible component of $\mathfrak{A}$. We also show that $\mathfrak{A}$ consists of infinitely many orbits, while there is still no evidence that the same can happen for $\mathfrak{R}$.

Abelian subalgebras of $\mathfrak{g}$ have been extensively studied, by Schur, Malcev, Panyushev and others, but very little is known about the geometry of $\mathfrak{A}$. An enumeration of its irreducible components is not even at hand. Further investigations may confirm that the study of $\mathfrak{R}$ is easier than the one of $\mathfrak{A}$ is.

1.5. **Rigidity of anisotropic subtori.** One of our main result (4.6) states that, if $\mathfrak{a}_0$ is the degeneration of the conjugates of a subalgebra $\mathfrak{a}_1$ of $\mathfrak{g}$ contained in $\mathfrak{p}$ under the operation of K, then the semi-simple elements of $\mathfrak{a}_0$ are rigid. This means that they are the limits of the semi-simple elements in the conjugates of $\mathfrak{a}_1$. This rigidity theorem enables us to study varieties of reductions through their *subvarieties of reductions* (see below), but also to contribute to the theory of decomposition classes. Decomposition classes for a reductive symmetric pair generalize Jordan types for $\mathbf{GL}_n$. They were introduced by Bohro and Kraft [12] to study sheets in Lie algebras. As an application of our rigidity theorem, we show the

PROPOSITION (B.1). *The closure of a decomposition class is a union of decomposition classes.*

1.6. **Orbit theory for the varieties of reductions.** While the theory is still incomplete, there is some open subset of $\mathfrak{R}$ whose orbits we understand well, by comparing them to *decomposition classes* in $\mathfrak{g}$.

Sending a point $x \in \mathfrak{g}$ to its centralizer, we define a rational map $\mathcal{C}$ from $\mathfrak{g}$ to $\mathfrak{R}$.

PROPOSITION (3.7, 3.15, 3.16). *The map $\mathcal{C}$ enjoys the following properties:*
  (1) *Its pointwise image is an open subset $\mathfrak{R}_r$ of the smooth locus of $\mathfrak{R}$.*
  (2) *The pre-image of an orbit $\mathcal{O}_0$ in $\mathfrak{R}_r$ is a decomposition class $\mathcal{D}_0$ in $\mathfrak{g}$.*
  (3) *If $\mathcal{D}_1$ is a decomposition class in $\mathfrak{g}$ containing $\mathcal{D}_0$ in its closure, then $\mathcal{C}$ is defined at any point of $\mathcal{D}_1$ and the image of $\mathcal{D}_1$ is an orbit containing $\mathcal{O}_0$ in its closure.*

The problem of describing the genericity relations between decomposition classes is much easier than the analogous problem for orbits.

The previous proposition describes a bunch of orbits of low-codimension in $\mathfrak{R}$, but two important questions remain: What is the codimension of the complement of the image of $\mathcal{C}$? How intricated are the combinatorics of the orbits in this complement? The variety of reductions for $(\mathbf{SL}_4, \mathbf{SO}_4)$ is smooth, while the centralizer map is not surjective. In the varieties of reductions for the Cartesian squares of $\mathbf{Sp}_4$ and G_2, the image of the centralizer map equals the smooth locus, and its complement has codimension 2.

While we are not yet able to answer these questions, we noticed an interesting structure in the family of varieties of reductions, described in terms of *subvarieties of reductions*.

Let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{a}'$ a subalgebra of $\mathfrak{a}$ and G' its centralizer. Let $\mathfrak{R}'$ denote the closure of $G'\mathfrak{a}$ in $\mathfrak{R}$. As the data $(\mathfrak{a}, \mathfrak{a}')$ runs through all its possible values, $\mathfrak{R}'$ describes the set of subvarieties of reductions of $\mathfrak{R}$. Note that if G' is smaller than G, the variety $\mathfrak{R}'$ is isomorphic to the variety of Cartan

reductions for the derived group of G', which has smaller rank than G. We can now state our

THEOREM (6.4 and its corollaries). *Let x_0 be a point of $\mathfrak{R}$. Then:*

(1) *x_0 is contained in a strict subvariety of reductions of $\mathfrak{R}$ if, and only if, it contains a non-nilpotent element of $\mathfrak{g}$.*

(2) *A point x_1 is more general than x_0 in $\mathfrak{R}$ if, and only if, any subvariety of reductions $\mathfrak{R}'$ containing x_0 also contains a G-conjugate of x_1 that is more general than x_0 in $\mathfrak{R}'$.*

Hence, if we are able to describe orbits in all the subvarieties of reductions of $\mathfrak{R}$, and there is only a finite number of isomorphism classes of them, we can as well describe orbits in the open subset of $\mathfrak{R}$ whose complement is

$$\mathfrak{R}_n = \left\{\, \mathfrak{u} \in \mathfrak{R} \mid \text{every } u \in \mathfrak{u} \text{ is nilpotent} \,\right\}.$$

PROPOSITION (5.5). *The subvariety $\mathfrak{R}_n$ of $\mathfrak{R}$ contains the closed orbits of $\mathfrak{R}$.*

We may learn soon how to enumerate the closed orbits of $\mathfrak{R}$, but the detailed orbit theory of $\mathfrak{R}_n$ remains very mysterious. In particular we do not know whether it can contain infinitely many K-orbits or not.

1.7. **Partial positivity of the anticanonical class.** The minimal rational curves in $\mathfrak{R}$ are the lines of the natural projective embedding of $\mathfrak{R}$ that are contained in $\mathfrak{R}$. We can describe the generic ones in terms of the roots of the operation of $\mathfrak{a}$ on $\mathfrak{g}$. This description is precise enough to let us study the deformations of such a line, find explicit free curves contained in the smooth locus of $\mathfrak{R}$ and compute the intersection of the anticanonical class on these lines:

COROLLARY (8.10). *Let $\mathfrak{a} \in \mathfrak{R}_o$ be a general point of $\mathfrak{R}$ and Δ a line through $\mathfrak{a}$ contained in $\mathfrak{R}$. Let m be the dimension of the maximal linear subspace of $\mathfrak{R}$ through $\mathfrak{a}$ containing Δ. If Δ is contained in the smooth locus of $\mathfrak{R}$, then*

$$-K_r \cdot \Delta = m + 1$$

where K_r is the canonical class of the smooth locus of $\mathfrak{R}$.

This intersection number is always positive, it equals 3 when $\mathfrak{R}$ is a variety of Cartan reductions. When $\mathfrak{R}$ has Picard number one and its canonical class is a Cartier divisor, this yields the Fano index of $\mathfrak{R}$.

The open orbit in $\mathfrak{R}$ is affine since the stabilizer of a point therein is reductive, its complement is therefore a union of divisors. In the examples we studied [5], the complement of the image $\mathfrak{R}_o$ of $\mathcal{C}$ has codimension at least 2—the codimension 2 occuring for the variety of Cartan reductions for G_2. This means that the image of the centralizer map carries enough information to describe the Picard group of $\mathfrak{R}$. In the general theory, it is possible to bound by above the dimension of the orbit of a point outside of $\mathfrak{R}_o$, but narrowing our attention to $\mathfrak{R}_o$ we would miss a divisor in $\mathfrak{R} \setminus \mathfrak{R}_o$ swept out by a continuous family of small dimensional orbits, if such a divisor exists. Hence the flaws in our orbit theory of $\mathfrak{R}$ obstructs our understanding of the Picard group of $\mathfrak{R}$.

CONTENTS

2. Variety of reductions for a symmetric pair

2.1. **Variety of reductions.** Let (G, θ) be a reductive symmetric pair with rank r and $\mathcal{G}(r, \mathfrak{p})$ the Grassmann variety of r-planes in $\mathfrak{p}$.

2.2. Definition. We call the set $\mathfrak{R}_o$ of Cartan subspaces of $\mathfrak{p}$ the set of *ordinary reductions* for the symmetric pair (G, θ). Its closure $\mathfrak{R}$ in $\mathcal{G}(r, \mathfrak{p})$ is the *variety of reductions* for (G, θ), and $\mathfrak{R}_s = \mathfrak{R} \setminus \mathfrak{R}_o$ is the set of special reductions.

The variety of reductions depends only of the isogeny class of G, not of its fundamental group.

Recall that Grassmann varieties are embedded in projective spaces of some exterior powers, in particular a reduction $\mathfrak{a} \in \mathfrak{R}$ of which $a \in \mathfrak{a}^r$ is a basis has image $[a_1 \wedge \cdots \wedge a_r] \in \mathbf{P}(\Lambda^r \mathfrak{p})$.

2.3. **Variety of anisotropic subalgebras.** The variety $\mathfrak{R}$ of reductions for a reductive symmetric pair is a subvariety of the variety $\mathfrak{A}$ of all anisotropic subalgebras of $\mathfrak{g}$ of dimension r.

Let $A_r : \Lambda^r \mathfrak{p} \to \mathfrak{k} \otimes \Lambda^{r-2} \mathfrak{p}$ be the linear map whose value on the decomposable r-multivector $x_1 \wedge \cdots \wedge x_r$ is

$$A_r(x_1 \wedge \cdots \wedge x_r) = \sum_{1 \leq i < j \leq l} [x_i, x_j] \otimes x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_r$$

where terms with a hat above shall be discarded. The kernel $\operatorname{Ker} A_r$ of A_r is a linear subspace of $\mathbf{P}(\Lambda^r \mathfrak{p})$ meeting $\mathcal{G}(r, \mathfrak{p})$ along the set of subalgebras of $\mathfrak{g}$ of dimension r contained in $\mathfrak{p}$.

2.4. Definition. The variety of anistropic subalgebras for the reductive symmetric pair (G, θ) is the section $\mathfrak{A}$ of $\mathcal{G}(r, \mathfrak{p})$ with $\operatorname{Ker} A_r$ in $\mathbf{P}(\Lambda^r \mathfrak{p})$.

When the symmetric pair associated to a reductive group is under consideration, we call $\mathfrak{A}$ the variety of abelian subalgebras for the given group. We illustrate the presumable complexity of $\mathfrak{A}$ in section (9).

2.5. **Roots relative to an ordinary reduction.** Let $\mathfrak{a}$ be an ordinary reduction and let Φ be the set of weights of $\mathfrak{a}$ in $\mathfrak{g}$. The decomposition of $\mathfrak{g}$ into weight spaces relative to $\mathfrak{a}$ is

$$(2.1) \qquad \mathfrak{g} = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{a}) \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}(\mathfrak{a})_\alpha$$

where $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$ and $\mathfrak{g}(\mathfrak{a})_\alpha$ the weight space for the character α of $\mathfrak{a}$. The following propositions enable us to work with the decomposition (2.1), for a proof, see Tauvel and Yu [10, 37.5.3, 36.2.1, 38.2.7, 38.7.2].

2.6. PROPOSITION. *The centralizer of $\mathfrak{a}$ in $\mathfrak{g}$ is the direct sum of $\mathfrak{a}$ and the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$.*

2.7. PROPOSITION. *The set Φ is a root system.*

We emphasize that this root system may not be reduced, however the root system relative to the symmetric pair of a reductive group is the root system of the reductive group itself, and is thus always reduced.

Since θ' swaps $\mathfrak{g}(\mathfrak{a})_\alpha$ and $\mathfrak{g}(\mathfrak{a})_{-\alpha}$, the Cartan decomposition and the weight spaces decomposition are not comparable. It is thus useful to introduce the subspaces

$$(2.2) \qquad \mathfrak{p}(\mathfrak{a})_\alpha = (\mathfrak{g}(\mathfrak{a})_\alpha \oplus \mathfrak{g}(\mathfrak{a})_{-\alpha}) \cap \mathfrak{p} \quad \text{and} \quad \mathfrak{k}(\mathfrak{a})_\alpha = (\mathfrak{g}(\mathfrak{a})_\alpha \oplus \mathfrak{g}(\mathfrak{a})_{-\alpha}) \cap \mathfrak{k}.$$

These subspaces satisfy the relations $\mathfrak{p}(\mathfrak{a})_\alpha = \mathfrak{p}(\mathfrak{a})_{-\alpha}$, $\mathfrak{k}(\mathfrak{a})_\alpha = \mathfrak{k}(\mathfrak{a})_{-\alpha}$, and $\mathfrak{p}(\mathfrak{a})_\alpha \oplus \mathfrak{k}(\mathfrak{a})_\alpha = \mathfrak{g}(\mathfrak{a})_\alpha \oplus \mathfrak{g}(\mathfrak{a})_{-\alpha}$. We can then decompose $\mathfrak{k}$ and $\mathfrak{p}$ in the following manner:

$$(2.3) \qquad\qquad \mathfrak{k} = \mathfrak{m} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{k}(\mathfrak{a})_\alpha$$

where $\mathfrak{m} = \mathfrak{c}_{\mathfrak{k}}(\mathfrak{a})$ is the reductive Lie algebra centralizing $\mathfrak{a}$ in $\mathfrak{k}$;

$$(2.4) \qquad\qquad \mathfrak{p} = \mathfrak{a} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{p}(\mathfrak{a})_\alpha.$$

The spaces $\mathfrak{k}(\mathfrak{a})_\alpha$ and $\mathfrak{p}(\mathfrak{a})_\alpha$ are not stable under $\mathfrak{a}$ but are swapped by it: we have $\operatorname{ad}(\mathfrak{a})\mathfrak{k}(\mathfrak{a})_\alpha = \mathfrak{p}(\mathfrak{a})_\alpha$ and $\operatorname{ad}(\mathfrak{a})\mathfrak{p}(\mathfrak{a})_\alpha = \mathfrak{k}(\mathfrak{a})_\alpha$. This allows us to compute the dimension of $\mathfrak{R}$:

2.8. PROPOSITION. *The dimension of $\mathfrak{R}$ is $\dim \mathfrak{p} - r$.*

3. REGULAR ORBITS AND THE INCIDENCE DIAGRAM

3.1. **Incidence variety.** Let $\mathcal{J}$ be the incidence variety associated to $\mathfrak{R}$:

$$\mathcal{J} = \left\{ (\mathfrak{u}, u) \in \mathfrak{R} \times \mathfrak{p} \mid u \in \mathfrak{u} \right\}.$$

The projections of $\mathcal{J}$ to $\mathfrak{p}$ and $\mathfrak{R}$ are respectively denoted by π and τ. Notice that, since the tautological fibre bundle on $\mathcal{G}(r, \mathfrak{p})$ is locally trivial, the morphism τ is open.

3.2. **Centralizer map.** An element $u \in \mathfrak{p}$ is *regular* when the dimension of its centralizer $\mathcal{C}(u)$ in $\mathfrak{p}$ is minimal. It is then an r-dimensional subalgebra of $\mathfrak{g}$ and the map $\mathcal{C}$ is a well-defined morphism from $\mathcal{R}$ to $\mathfrak{A}$. We call *regular* a reduction in $\mathfrak{R}_r = \mathcal{C}(\mathcal{R})$, or a K-orbit through a regular reduction.

We write $\mathcal{S}$ for the set of semi-simple elements in $\mathfrak{p}$, and $\mathcal{N}$ for the set of nilpotent ones. The set $\mathcal{R}_o = \mathcal{R} \cap \mathcal{S}$ of *ordinary elements* in $\mathfrak{p}$ that are both regular and semi-simple is an open subset of $\mathcal{R}$, dense in $\mathfrak{p}$. The set of r-dimensional subspaces of $\mathfrak{p}$ containing an ordinary element is an open subspace of $\mathcal{G}(r, \mathfrak{p})$. This implies the following

3.3. PROPOSITION. *The variety $\mathfrak{R}$ of reductions for a reductive symmetric pair is an irreducible component of the variety $\mathfrak{A}$ of anisotropic subalgebras for this pair.*

3.4. COROLLARY. *The centralizer of any regular element is a reduction, that is, C maps $\mathcal{R}$ to $\mathfrak{R}$.*

3.5. **Incidence diagram.** The incidence variety and the centralizer map fit together in the *incidence diagram*:

$$
\begin{array}{ccc}
& \mathcal{J} & \\
\swarrow & & \searrow \\
\mathbf{P}(\mathfrak{p}) & \dashrightarrow & \mathfrak{R}
\end{array}
$$

and the centralizer map C equals $\tau \circ \pi^{-1}$ above $\mathcal{R}$. We use these constructions to explore the variety of reductions.

3.6. **Regular orbits are contained in the smooth locus.** The centralizer map parametrizes an open and smooth subset of $\mathfrak{R}$:

3.7. THEOREM. *The set $\mathfrak{R}_r = C(\mathcal{R})$ of regular reductions is an open subset of $\mathfrak{R}$, contained in the smooth locus.*

Proof. The subset of $\mathcal{G}(r, \mathfrak{p})$ of subspaces meeting $\mathcal{R}$ is open, which implies the truth of the first statement. We now study the smoothness.

Let $u_0 \in \mathcal{R}$, we study the tangent space to $\mathfrak{A}$ at $\mathfrak{u} = C(u_0)$. We choose a supplementary subspace $\mathfrak{v}$ to $\mathfrak{u}$ in $\mathfrak{p}$ and identify the affine neighbourhood of $\mathfrak{u}$ in the Grassmann variety $\mathcal{G}(r, \mathfrak{p})$ consisting of points admitting $\mathfrak{v}$ for supplementary with the space $A = \mathcal{L}(\mathfrak{u}, \mathfrak{v})$ of linear maps from $\mathfrak{u}$ to $\mathfrak{v}$. A point $a \in A$ belongs to $\mathfrak{A}$ if, and only if for all $(u_1, u_2) \in \mathfrak{u}^2$

$$[u_1 + a(u_1), u_2 + a(u_2)] = 0;$$

the linear equations of $T_{\mathfrak{u}}\mathfrak{A}$ are therefore

(3.1) $$[u_1, a(u_2)] = [u_2, a(u_1)]$$

for all $(u_1, u_2) \in \mathfrak{u}^2$. We narrow our attention to the subset of the linear equations of $T_{\mathfrak{u}}\mathfrak{A}$ obtained by letting $u_2 = u_0$. Note that $\mathfrak{u}$ is precisely the kernel of the restriction of $\mathrm{ad}(u_0)$ to $\mathfrak{p}$, hence, once u_1 is given there is at most one value of $a(u_1)$ satisfying (3.1). The tangent space is thus parametrised by a subspace of the set of images $a(u_0)$, when a varies in A and $T_{\mathfrak{u}}\mathfrak{A} \subset \mathfrak{v}$. But $\dim T_{\mathfrak{u}}\mathfrak{A} \geq \dim_{\mathfrak{u}} \mathfrak{A} \geq \dim \mathfrak{R}$ and $\dim \mathfrak{R} = \dim \mathfrak{v}$ (2.8). We conclude that $\dim T_{\mathfrak{u}}\mathfrak{A} = \dim_{\mathfrak{u}} \mathfrak{A} = \dim \mathfrak{R}$. The regular reductions are thus contained in the smooth locus of $\mathfrak{R}$. $\qquad\square$

3.8. **Review of decomposition classes.** The centralizer map puts the K-orbits in its image in correspondence with decomposition classes in $\mathfrak{R}$. Decomposition classes generalize the familiar Jordan types in $\mathfrak{sl}_n$ to every symmetric pairs. They are discussed by Bohro and Kraft [12], Broer [13] and Tauvel and Yu [10]. We briefly review elements of this discussion, referring to [10] for further details.

3.9. DEFINITION. Let u and v be two elements in $\mathfrak{p}$ with respective Chevalley-Jordan decomposition $u = u_s + u_n$ and $v = v_s + v_n$. They have the same *decomposition class* when there exists $g \in K$ such that $g \cdot \mathfrak{c}_{\mathfrak{p}}(u_s) = \mathfrak{c}_{\mathfrak{p}}(v_s)$ and $g u_n = v_n$.

This defines an equivalence relation on $\mathfrak{p}$. We write $\mathcal{D}$ for the associated partition and $\mathcal{D}(u)$ for the element of $\mathcal{D}$ containing u. Notice that decomposition classes are punctured cones in $\mathfrak{p}$, so that their projectivizations define a partition of the projective space $\mathbf{P}(\mathfrak{p})$.

This relation is equivalently described with *double centralizers*. The double centralizer of $u \in \mathfrak{p}$ is

$$\mathfrak{c}_{\mathfrak{p}}^2(u) = \{\, v \in \mathfrak{p} \mid \forall w \in \mathfrak{c}_{\mathfrak{p}}(u) \; [v, w] = 0 \,\}.$$

This is the set of elements in $\mathfrak{p}$ whose centralizer in $\mathfrak{p}$ contains $\mathfrak{c}_{\mathfrak{p}}(u)$. Chevalley's theorem implies that

(3.2) $$\mathfrak{c}_{\mathfrak{p}}^2(u)_o = \{\, v \in \mathfrak{p} \mid \mathfrak{c}_{\mathfrak{p}}(v) = \mathfrak{c}_{\mathfrak{p}}(u) \,\}$$

is a dense open subset of $\mathfrak{c}_{\mathfrak{p}}^2(u)$.

3.10. PROPOSITION ([10, 38.8.3, 39.5.1 and 39.5.4]). *For two elements in $\mathfrak{p}$, the following statements are equivalent.*

  (1) *They have the same decomposition class.*
  (2) *Their centralizers in $\mathfrak{p}$ are K-conjugate.*
  (3) *Their double centralizers in $\mathfrak{p}$ are K-conjugate.*

3.11. COROLLARY ([10, 39.5.5]). *The decomposition class $\mathcal{D}(u)$ of $u \in \mathfrak{p}$ is $K \cdot \mathfrak{c}_{\mathfrak{p}}^2(u)_o$. This is an irreducible locally closed subvariety of $\mathfrak{p}$.*

3.12. PROPOSITION ([10, 39.5.6]). *The set $\mathcal{D}$ of decomposition classes in $\mathfrak{p}$ is finite.*

Thus, decomposition classes form a finite partition of $\mathfrak{p}$ into locally closed sets. It follows that any irreducible subvariety X of $\mathfrak{p}$ or $\mathbf{P}(\mathfrak{p})$ is the closure of $X \cap \mathcal{D}_X$ in X, for a unique decomposition class $\mathcal{D}_X \in \mathcal{D}$. We say $\mathcal{D}_X$ is the *dominant decomposition class* in X.

If u is regular, semi-simple or nilpotent, elements of $\mathcal{D}(u)$ are regular, semi-simple or nilpotent, and we call $\mathcal{D}(u)$ a regular, semi-simple or nilpotent decomposition class.

In B.1 we use our rigidity theorem 4.6 to prove that the closure of a decomposition class is a union of decomposition classes.

3.13. **Genericity relation for regular orbits.** We want to describe the genericity relation for orbits in $\mathfrak{R}$, that is, which orbits lie in the closure of which. An orbit $\mathcal{O}_1$ is *more general* than an orbit $\mathcal{O}_0$ when $\mathcal{O}_0$ lies in the closure of $\mathcal{O}_1$; similarly, a decomposition class $\mathcal{D}_1$ is more general than a decomposition class $\mathcal{D}_0$ when $\mathcal{D}_0$ is contained in the closure of $\mathcal{D}_1$.

Through the centralizer map, the genericity relation for regular orbits reduces to the genericity relation for decomposition classes.

3.14. PROPOSITION. *The image of a regular decomposition class through the centralizer map is a regular K-orbit of $\mathfrak{R}$.*

This is a reformulation of the equivalence between (1) and (2) in (3.10). We are then allowed to speak of the decomposition class associated with a regular orbit: it is its inverse image through the centralizer map.

3.15. PROPOSITION. *Let $\mathfrak{R}_{\mathrm{I}}$ and $\mathfrak{R}_{\mathrm{II}}$ be orbits in $\mathfrak{R}$. If $\mathfrak{R}_{\mathrm{I}}$ is more general than $\mathfrak{R}_{\mathrm{II}}$ and $\mathfrak{R}_{\mathrm{II}}$ is regular, then $\mathfrak{R}_{\mathrm{I}}$ is regular and its decomposition class is more general than the one of $\mathfrak{R}_{\mathrm{II}}$.*

Proof. Let $\mathfrak{p}_{\mathrm{I}}$ be the projection $\tau(\pi^{-1}(\mathfrak{R}_{\mathrm{I}}))$ of $\mathfrak{R}_{\mathrm{I}}$ through the incidence diagram of $\mathfrak{R}$ (3.1), and $\mathfrak{p}_{\mathrm{II}}$ the projection of $\mathfrak{R}_{\mathrm{II}}$. Note that $\mathfrak{R}_{\mathrm{II}}$ being regular, $\mathfrak{p}_{\mathrm{II}}$ is a decomposition class and is thus irreducible (3.11).

The morphism π is continuous, so that $\pi^{-1}(\mathfrak{R}_{\mathrm{I}})$ contains $\pi^{-1}(\mathfrak{R}_{\mathrm{II}})$ in its closure; the morphism τ is open, so that $\mathfrak{p}_{\mathrm{I}} = \tau(\pi^{-1}(\mathfrak{R}_{\mathrm{I}}))$ contains $\mathfrak{p}_{\mathrm{II}}$ in its closure. Let $\mathfrak{q}$ be an irreducible component in $\mathfrak{p}_{\mathrm{I}}$ degenerating onto $\mathfrak{p}_{\mathrm{II}}$. The decomposition class $\mathcal{D}_{\mathfrak{q}}$ dominant in $\mathfrak{q}$ is more general than $\mathfrak{p}_{\mathrm{II}}$, it is thus regular and $\mathfrak{R}_{\mathrm{I}}$ is the regular K-orbit $\mathcal{C}(\mathfrak{q})$ in $\mathfrak{R}$. $\square$

3.16. COROLLARY. *Two regular orbits compare in the same way than their corresponding regular decomposition classes do.*

Proof. One implication is the previous proposition, the converse follows from the equivalence of (1) and (2) (3.10) and the continuity of the centralizer map $\mathcal{C}$. $\square$

To put it another way, the centralizer map $\mathcal{C}$ induces an increasing isomorphism between the ordered set of regular decomposition classes in $\mathfrak{p}$ and the ordered set of regular orbits in the variety of reductions for a symmetric pair.

3.17. **Irregular locus.** The irregular locus $\mathcal{W}$ is the complement $\mathfrak{p} \setminus \mathcal{R}$ of the set of regular elements. It was proven by Veldkamp [14] for the symmetric pair associated to the Cartesian square of a group that the irregular locus $\mathcal{W}$ is the set of points where the reductive quotient $\phi : \mathfrak{p} \to \mathfrak{p} /\!/ K$ fails to be submersive. However it remains unknown wether these equations span the ideal of $\mathcal{W}$ or not. We noticed that these equations are geometrically realized by the centralizer map (3.18 below). This could be helpful to determine if the ideal they span is reduced or not, or if the caracterization of the irregular locus given by Veldkamp extends to the symmetric setting.

We define the Jacobian morphism $J\phi$ associated to the reductive quotient $\phi : \mathfrak{p} \to \mathfrak{p} /\!/ K$ by

$$
\begin{array}{cccc}
J\phi : & \mathfrak{p} \times \Lambda^r \mathfrak{p} & \to & \mathbf{C} \\
& (x, \xi) & \mapsto & \Lambda^r T_x \phi(\xi).
\end{array}
$$

It is a homogeneous morphism, thus $J\phi \in S^N \mathfrak{p}^* \otimes \Lambda^r \mathfrak{p}^*$ for some N. The restriction to $\mathfrak{p}$ of a non degenerate bilinear form on $\mathfrak{g}$ that is invariant under G and θ is definite and K-invariant. Hence we can see $J\phi$ as a K-invariant rational map

$$
J\phi : \mathbf{P}(\mathfrak{p}) \dashrightarrow \mathbf{P}(\Lambda^r \mathfrak{p})
$$

by identifying $\mathfrak{p}$ with its dual.

3.18. PROPOSITION. *The Jacobian morphism $J\phi$ coincides with the centralizer map $\mathcal{C}$.*

Proof. Let $\mathfrak{a}$ be a Cartan subspace of $\mathfrak{p}$, $a \in \mathfrak{a}^r$ a basis of $\mathfrak{a}$ and x a regular element of $\mathfrak{a}$. We write $\mathfrak{g}'$ the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$ and $\mathfrak{k}' = \mathfrak{g}' \cap \mathfrak{k}$, $\mathfrak{p}' = \mathfrak{g}' \cap \mathfrak{p}'$, so that $\mathrm{ad}\, x$ is a linear automorphism of $\mathfrak{g}'$ exchanging $\mathfrak{k}'$ and $\mathfrak{p}'$.

Now let $\xi = \xi_1 \wedge \cdots \wedge \xi_r$ be a decomposed r-vector divisible by a vector ξ_1 belonging to $\mathfrak{p}'$. If f is a K-invariant function and $\epsilon \to 0$, we have

$$
f(\exp(\epsilon(\mathrm{ad}\, x)^{-1}\xi_1)x) = f(x) = f(x) - \epsilon T_x f(\xi_1) + o(\epsilon).
$$

Hence $T_x f(\xi_1) = 0$ and $J\phi(x, \xi) = 0$ when ξ is divisible by an element of $\mathfrak{p}'$. Now the reductive quotient is submersive at the regular element x [9, Theorem 13] so that $J\phi(x, a_1 \wedge \cdots \wedge a_r) \neq 0$. This shows that $J\phi$ and $\mathcal{C}$ agree on $\mathcal{R}$. $\square$

4. RIGIDITY OF ANISOTROPIC TORI

4.1. **Review of stability.** Concepts familiar to the geometric invariant theory show up in the study of degeneracies of semi-simple elements. Let us recall the appropriate definitions and facts.

4.2. DEFINITION. Let K be a reductive group, V a finite dimensional representation of V and $X \subset \mathbf{P}(V)$ a quasi-projective variety stable under the operation of K. A point $[v]$ of X is called

- *semi-stable* if there exists a K-invariant non-constant homogeneous polynomial that does not vanish at v;
- *poly-stable* if it is semi-stable and the K-orbit of $[v]$ is closed in the set of semi-stable elements of X;
- *stable* if it is poly-stable and has finite stabilizer in K;
- *unstable* if it is not semi-stable.

When X is closed subvariety of $\mathbf{P}(V)$, its semi-stable, poly-stable, stable or unstable points are the points of $\mathbf{P}(V)$ of the same kind, belonging to X. We say a non-zero vector $v \in V$ is semi-stable, poly-stable, stable or unstable, according to the nature of $[v]$. The following proposition is a basic result in geometric invariant theory:

4.3. PROPOSITION. *Let X be a closed subvariety of $\mathbf{P}(V)$ stable under the operation of K. Then:*

(1) *The point $[v]$ is semi-stable if, and only if, 0 does not belong to the closure of the K-orbit of v in V.*

(2) *The point $[v]$ is poly-stable if, and only if, the K-orbit of v is closed in V.*

According to [9] these propositions can be rephrased the following way in the setting of reductive symmetric pairs:

- the set of unstable elements in $\mathfrak{p}$ is precisely the set $\mathcal{N}$ of nilpotent elements, with 0 removed;
- the set of poly-stable elements is the set $\mathcal{S}$ of semi-simple elements, with 0 removed;
- the set of semi-stable but not poly-stable elements is precisely the set of elements with non-zero semi-simple part and non-zero nilpotent part;
- the set of stable elements is not empty if, and only if, the rank of the symmetric pair (G, θ) equals the rank of reductive group G; in this case the set of stable elements is the set of anisotropic semi-simple elements whose centralizer in $\mathfrak{p}$ is a Cartan subspace of $\mathfrak{g}$.

The following property of anisotropic algebras is remarkable with respect to stability theory: for any anisotropic algebra $\mathfrak{a}$, the set $\mathcal{S}(\mathfrak{a})$ of poly-stable elements of $\mathfrak{a}$ (with zero added), and the set $\mathcal{N}(\mathfrak{a})$ of unstable elements of $\mathfrak{a}$ (*idem*) are vector subspaces of $\mathfrak{a}$.

4.4. **Degeneracies of anisotropic algebras.** We introduce a language suited to our study of degeneracies. Let K be a group acting on a complete variety X. We consider a point $x_1 \in X$ and a *degeneration* x_0 of x_1, that is, a point belonging to the closure in X of the orbit Kx_1. There exists an irreducible smooth curve C^o in K such that x_0 belongs to the closure of $C^o x_1$, and we only consider degenerations

along such curves. In the study of the degeneration along the curve C^o, it is convenient to introduce the points at infinity of C^o.

Let C be the smooth completion of C^o. The points in $C \setminus C^o$ are called points at infinity. Since X is complete, the rational map sending $c \in C^o$ to cx_1 extends to a regular map on C and, by a slight abuse of notation, we write cx_1 for the image of c under this map even for c lying in $C \setminus C^0$. Now let c_0 be a point at infinity whose image is x_0. We say that C^o is an *arc in K pushing x_1 toward x_0*, and c_0 is a *point at infinity taking x_1 to x_0*.

The way semi-simple elements of an anisotropic algebra behave when it is deformed by the operation of K is described by the following theorem. It has many interesting consequences, such as 4.6, 6.4 and B.1.

4.5. THEOREM. *Let $\mathfrak{a}_1$ be an l-dimensional anisotropic subalgebra and $\mathfrak{a}_0 \in \mathcal{G}(l, \mathfrak{p})$ lying in the closure of the K-orbit through $\mathfrak{a}_1$. Let C^o be an arc in K pushing $\mathfrak{a}_1$ toward $\mathfrak{a}_0$ and c_0 a point at infinity taking $\mathfrak{a}_1$ to $\mathfrak{a}_0$.*

Then, unless $\mathfrak{a}_0$ only contains nilpotent elements, there exists a semi-stable element s_0 in $\mathfrak{a}_0$ and a semi-stable element s_1 in $\mathfrak{a}_1$ pushed toward s_0 by C^o and brought to s_0 by c_0.

Moreover, if $\mathfrak{a}_1$ is closed under the Chevalley-Jordan decomposition, then $\mathfrak{a}_0$ is closed under the Chevalley-Jordan decomposition as well, and for each semi-simple element s_0 in $\mathfrak{a}_0$ there exists a semi-simple element s_1 in $\mathfrak{a}_1$ that is brought to s_0 by c_0.

Proof. We denote by R_0 the local ring of C^o at c_0, $\mathfrak{m}_0$ its maximal ideal and ϵ a local parameter of C^o at c_0. According to (A.8) there exists a basis x of $\mathfrak{a}_1$, a basis y of $\mathfrak{a}_0$, a r-uple n of integers and a r-uple $\eta \in (\mathfrak{m}_0 \otimes \mathfrak{p})^r$ such that for all k

$$cx_k = \epsilon^{n_k}(y_k + \eta_k).$$

If all of the y_k are nilpotent, then $\mathfrak{a}_0$ purely consists of nilpotent elements. Hence, we assume that for some integer k the vector y_k is semi-stable and let f be a K-invariant function, homogeneous of degree $N > 0$, that does not vanish at y_k. We compute

$$f(x_k) = \epsilon^{Nn_k}f(y_k + \eta_k),$$

and show that $n_k = 0$. On the one hand, the function $f(x_k)/f(y_k + \eta_k)$ is regular at c_0, which implies $n_k \geq 0$. On the other hand, the non zero function $f(x_k)\epsilon^{-Nn_k}$ is regular at c_0, which implies $n_k \leq 0$. Thus n_k equals zero and f does not vanish at x_k. We conclude that cx_k converges to y_k when c approaches c_0 and x_k is semi-stable.

We now proceed to the proof of the second part of the statement and assume that $\mathfrak{a}_1$ is closed under the Chevalley-Jordan decomposition. According to (A.9) we may assume that each of the x_k is either semi-simple or nilpotent. We show that each of the y_k is either semi-simple or nilpotent.

The image of $\mathcal{N}$ in $\mathbf{P}(\mathfrak{p})$ is closed, hence $x_k \in \mathcal{N}$ implies $y_k \in \mathcal{N}$. We now assume that x_k is semi-simple. If y_k is not nilpotent, then $n_k = 0$ as before and we have $x_k = y_k + \eta_k$. Hence cx_k converges toward y_k as c approaches c_0. But x_k is semi-simple and its orbit is closed, so y_k is semi-simple and conjugated to x_k. Thus each y_k is either semi-simple or nilpotent, which implies that the anisotropic subalgebra $\mathfrak{a}_0$ of dimension l spanned by the y_k is closed under the Chevalley-Jordan decomposition.

Last, if $s_0 \in \mathfrak{a}_0$ is semi-simple, it is a linear combination $s_0 = \sum_{k \in K_s} Y_0^k y_k$ of the semi-simple y_k's, whose set of indice is $K_s \subset \{1, \ldots, l\}$. Hence the translates cs_1 of $s_1 = \sum_{k \in K_s} Y_0^k x_k$ converge toward s_0 as c approaches c_0.   □

If a l-dimensional anisotropic subalgebra $\mathfrak{a}_1$ of $\mathfrak{g}$ degenerates on $\mathfrak{a}_0$, the semi-simple elements subsisting in $\mathfrak{a}_0$ can be *rigidified*, that is, there exists an equivalent degeneration leaving these semi-simple elements untouched:

4.6. COROLLARY (Rigidity of anisotropic tori). *Let $\mathfrak{a}_1$ be a l-dimensional anisotropic subalgebra closed under Chevalley-Jordan decomposition, $\mathfrak{a}_0 \in \mathcal{G}(l, \mathfrak{p})$ lying in the closure of the K-orbit through $\mathfrak{a}_1$ and C^o be an arc in K pushing $\mathfrak{a}_1$ toward $\mathfrak{a}_0$ and c_0 a point at infinity taking $\mathfrak{a}_1$ to $\mathfrak{a}_0$. Let $\mathcal{S}(\mathfrak{a}_0)$ be the linear subspace of $\mathfrak{a}_0$ spanned by its semi-simple elements. Then there exists a K-conjugate $\mathfrak{a}_1'$ of $\mathfrak{a}_1$ containing $\mathcal{S}(\mathfrak{a}_0)$ and a degeneracy curve in the centralizer $C_K(\mathcal{S}(\mathfrak{a}_0))$ of $\mathcal{S}(\mathfrak{a}_0)$ in K pushing $\mathfrak{a}_1'$ toward $\mathfrak{a}_0$.*

This corollary is a consequence of 4.5 and the following general lemma dealing with degeneracies of orbits:

4.7. LEMMA (Straightening of degeneracies). *Let K be a connected group and X and Y be two K-varieties. Let x_1 be a point in X and x_0 a point belonging to the closure of the K-orbit through x_1, and let y_1 a point in Y. Let C^o be a degeneracy curve in K pushing x_1 toward x_0 and c_0 a point at infinity taking x_1 to x_0. If c_0 takes y_1 to a conjugate $y_0 = hy_1$, then there exists a degeneracy curve in the identity component of the stabilizer of y_0 in K pushing hx_1 to x_0, and a point at infinity taking hx_1 to x_0.*

Proof. The rational map sending c to cy_1 is regular at c_0 and c_0y_1 is a conjugate $y_0 = hy_1$ of y_1. Let $C(c_0) = C^o \cup \{c_0\}$ and define

$$Z = \{(g, c) \in K \times C(c_0) \mid gcy_1 = y_0\}.$$

The map cy_1 is regular at c_0, and its value belongs to the orbit of y_1, thus $(1, c_0) \in Z$ belongs to the closure of the inverse image W of C^o under the projection of Z to $C(c_0)$. Let D be a degeneracy curve in W containing $(1, c_0)$ in its closure. Then the image $(C^o)'$ of W under the map sending $(g, c) \in Z$ to $gch^{-1} \in K$ is contained in the centralizer of $y_0 \in K$ and pushes hx_1 toward x_0. This map extends to a regular map at $(1, c_0)$ whose image is a point at infinity for $(C^o)'$ taking hx_1 to x_0.   □

5. ALGEBRAICITY OF REDUCTIONS

5.1. **Degeneracies of subgroups.** The fact that each point in the variety of reductions is the Lie algebra of an algebraic subgroup of G is a consequence of the following general observation:

5.2. PROPOSITION. *Let G be an algebraic group and $\mathcal{H} \subset G \times B$ an irreducible family of subvarieties of G over a basis B. If the generic member of $\mathcal{H}$ is a subgroup of G, then each member of $\mathcal{H}$ is a subgroup of G.*

Proof. The algebraic subgroups of G are precisely the subvarieties H of G such that the restriction of the morphism $\psi : G \times G \to G$ defined by $\psi(g_1, g_2) = g_1 g_2^{-1}$ maps $H \times H$ in H. Let

$$B^o = \{b \in B \mid \mathcal{H}_b \text{ is a subgroup of } G\}$$

and $\mathcal{H}^o$ the restriction of $\mathcal{H}$ to B^o, and assume that B^o is dense in B. Since $\mathcal{H}$ is irreducible, $\mathcal{H}^o$ is dense therein. Hence $\psi^{-1}(\mathcal{H})$ contains $\mathcal{H} \times \mathcal{H}$, the closure of $\mathcal{H}^o \times \mathcal{H}^o$. $\square$

5.3. COROLLARY. *Any point of the variety of reductions is the Lie algebra of an algebraic subgroup of G.*

Proof. Let $\mathfrak{a}_0$ be a point of $\mathfrak{R}$, $\mathfrak{a}_1$ a Cartan subspace of $\mathfrak{p}$ and C^o a degeneracy curve in K pushing $\mathfrak{a}_1$ toward $\mathfrak{a}_0$. We denote by A_1 be the connected subgroup of G whose Lie algebra is $\mathfrak{a}_1$, by $\bar{G}$ a projective completion of G and by C a smooth completion of C^o. We consider the family $\mathcal{V} \subset \bar{G} \times \mathbf{P}(\mathfrak{p}) \times C$ obtained by taking the closure of

$$\mathcal{V}^o = \{\, (g, [x], c) \in G \times \mathbf{P}(\mathfrak{p}) \times C^o \mid g \in c A_1 \text{ and } x \in c\mathfrak{a}_1 \,\}.$$

Let c_0 be a point of C taking $\mathfrak{a}_1$ to $\mathfrak{a}_0$. The fiber $\mathcal{V}_{c_0}$ projects on $\bar{G}$ as a set meeting G along a subgroup A_0 of G (5.2), and on $\mathbf{P}(\mathfrak{p})$ as the projectivization of $\mathfrak{a}_0$. Hence the Lie algebra of A_0 contains $\mathfrak{a}_0$. But the family $\mathcal{V}$ over C is flat, since the curve C is smooth [15, 9.7], and the dimensions of A_1 and $\mathfrak{a}_1$ agree [15, 9.5]. $\square$

5.4. **Nilpotence of closed orbits.** Degeneracies in the varieties of reductions ultimately turn each semi-simple dimension into a nilpotent one, except of course those belonging to the center of $\mathfrak{g}$:

5.5. PROPOSITION. *If G is semi-simple, then a point in a closed orbit of $\mathfrak{R}$ is contained in the nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$.*

Proof. Let $\mathfrak{u}$ be a reduction belonging to a closed K-orbit in $\mathfrak{R}$, and P its stabilizer in K. The Lie algebra $\mathfrak{u}$ is algebraic (5.3) and abelian, it follows from the Chevalley-Jordan-decomposition that the set $\mathcal{S}(\mathfrak{u})$ of semi-simple elements of $\mathfrak{u}$ and the set $\mathcal{N}(\mathfrak{u})$ of nilpotent elements of $\mathfrak{u}$ are supplementary linear subspaces of $\mathfrak{u}$. The stabilizer in K of $\mathcal{S}(\mathfrak{u})$ hence contains P. It is thus both a reductive and a parabolic subgroup of K, so it equals K. This yields $\mathcal{S}(\mathfrak{u}) = 0$. $\square$

6. SUBVARIETIES OF REDUCTIONS

6.1. **Variety of reductions and derivation.** To the reductive symmetric pair (G, θ) corresponds a symmetric pair whose group is the derived group G' of G. Since G' is stabilized by any automorphism of G, the involution θ restricts to G', turning (G', θ) into a semi-simple symmetric pair.

6.2. PROPOSITION. *Let (G, θ) be a symmetric pair and G' the derived group of G. Let K' be the adjoint form of the identity component of the fixed points group of (G', θ). Let $\mathfrak{R}$ be the variety of reductions for (G, θ) and $\mathfrak{R}'$ the one for (G', θ). Let $\mathfrak{p}'$ be the space of anisotropic vectors of (G, θ) belonging to the Lie algebra of G', and $\mathfrak{p}_Z$ those belonging to the Lie algebra of the center of G. Then:*

(1) *The groups K and K' have the same orbits in $\mathfrak{R}$.*
(2) *The maps $p : \mathfrak{R} \to \mathfrak{R}'$ and $j : \mathfrak{R}' \to \mathfrak{R}$ defined by*

$$p(\mathfrak{u}) = \mathfrak{u} \cap \mathfrak{p}' \qquad and \qquad j(\mathfrak{v}) = \mathfrak{v} \oplus \mathfrak{p}_Z$$

are inverse K'-invariant isomorphisms.

Proof. To establish the first claim, we replace G with its adjoint form so that G is the direct product of its derived group G' and its center Z_G. Now $G^\theta = (G')^\theta \times Z_G^\theta$, and since Z_G^θ lies in the kernel of the map $G^\theta \to \mathrm{Aut}(\mathfrak{R})$, the map $K \to \mathrm{Aut}(\mathfrak{R})$ factors through K'.

To prove the second claim, it is sufficient to show that for any $\mathfrak{u} \in \mathfrak{R}$,

(6.1) $$\mathfrak{u} = (\mathfrak{u} \cap \mathfrak{p}') \oplus \mathfrak{p}_Z$$

holds. We notice that any reduction contains $\mathfrak{p}_Z$. The infinitesimal anisotropic center $\mathfrak{p}_Z$ is pointwise fixed by K, thus it is is enough to remark that any ordinary reduction contains it, which follows from the maximality condition. Now $\mathfrak{p} = \mathfrak{p}' \oplus \mathfrak{p}_Z$ and (6.1) holds. $\square$

We may narrow our study of varieties of reductions for reductive symmetric pairs down to semi-simple symmetric pairs. However, reductive symmetric pairs appear naturally as symmetric pairs associated with centralizers of anisotropic tori.

6.3. **Subvarieties of reductions.** Let $\mathfrak{a} \in \mathfrak{R}_o$ an ordinary reduction and $\tilde{\mathfrak{a}} \subset \mathfrak{a}$ a subspace of $\mathfrak{a}$. The centralizer $C_G(\tilde{\mathfrak{a}})$ is a θ-stable reductive subgroup of G, thus $(C_G(\tilde{\mathfrak{a}}), \theta)$ is a reductive symmetric pair whose rank equals the rank of (G, θ). Since $\mathfrak{c}_\mathfrak{g}(\tilde{\mathfrak{a}}) \subset \mathfrak{g}$, the variety $\mathfrak{R}(\tilde{\mathfrak{a}})$ of reductions for the reductive symmetric pair $(C_G(\tilde{\mathfrak{a}}), \theta)$ is a natural subvariety of the variety $\mathfrak{R}$ of reductions for the reductive symmetric pair (G, θ). It is naturally isomorphic to the variety of reductions $(C_G'(\tilde{\mathfrak{a}}), \theta)$ whose group is the derived group of $C_G(\tilde{\mathfrak{a}})$.

6.4. PROPOSITION. *Let $\tilde{\mathfrak{a}}$ be a subspace of an ordinary reduction and $\mathfrak{R}(\tilde{\mathfrak{a}}) \subset \mathfrak{R}$ the variety of reductions for the symmetric pair $(C_G(\tilde{\mathfrak{a}}), \theta)$. Then*

$$\mathfrak{R}(\tilde{\mathfrak{a}}) = \{\, \mathfrak{u} \in \mathfrak{R} \mid \mathfrak{u} \supset \tilde{\mathfrak{a}} \,\}.$$

Proof. The variety of reductions $\mathfrak{R}(\tilde{\mathfrak{a}})$ is *a priori* a subset of the right hand side. The reciprocal inclusion follows from (4.6). $\square$

6.5. DEFINITION. The *reductive symmetric pair centralizing* $\tilde{\mathfrak{a}}$ is the reductive symmetric pair $(C_G(\tilde{\mathfrak{a}}), \theta)$. The *variety of reductions containing* $\tilde{\mathfrak{a}}$ is the variety of reductions $\mathfrak{R}(\tilde{\mathfrak{a}})$ of $(C_G(\tilde{\mathfrak{a}}), \theta)$. The subvarieties of reductions of $\mathfrak{R}$ are the subvarieties of the form $\mathfrak{R}(\tilde{\mathfrak{a}})$. We write $K(\tilde{\mathfrak{a}})$ the adjoint form of the identity component of the fixed points group $C_G(\tilde{\mathfrak{a}})^\theta$. Recall that this is the group whose action on $\mathfrak{R}(\tilde{\mathfrak{a}})$ we are interested in.

Note that the space $\tilde{\mathfrak{a}}$ is tangent to the center of $C_G(\tilde{\mathfrak{a}})$. Hence the semi-simple symmetric pair associated to $(C_G(\tilde{\mathfrak{a}}), \theta)$ (6.2) has smaller rank than (G, θ). The following propositions are immediate consequences of the rigidity of $\tilde{\mathfrak{a}}$ (4.6).

6.6. PROPOSITION. *Let $\mathfrak{R}(\tilde{\mathfrak{a}})$ be a subvariety of reductions of $\mathfrak{R}$ and $\mathcal{O}$ an orbit of K in $\mathfrak{R}$, whose codimension is k. Then $\mathcal{O} \cap \mathfrak{R}(\tilde{\mathfrak{a}})$ is an orbit of $C_K(\tilde{\mathfrak{a}})$ in $\mathfrak{R}(\tilde{\mathfrak{a}})$ whose codimension is k. In particular $\mathcal{O} \cap \mathfrak{R}(\tilde{\mathfrak{a}})$ is a finite union of orbits of $K(\tilde{\mathfrak{a}})$.*

6.7. PROPOSITION. *Let $\mathfrak{R}(\tilde{\mathfrak{a}})$ be a subvariety of reductions of $\mathfrak{R}$ and $\mathfrak{a}_0$ a point of $\mathfrak{R}(\tilde{\mathfrak{a}})$. If $\mathcal{O}$ is a K-orbit in $\mathfrak{R}$ containing $\mathfrak{a}_0$ in its closure, then $\mathcal{O}$ contains an orbit of $K(\tilde{\mathfrak{a}})$ in $\mathfrak{R}(\tilde{\mathfrak{a}})$ containing $\mathfrak{a}_0$ in its closure.*

6.8. PROPOSITION. *Let $\mathfrak{a}_1$ and $\mathfrak{a}_0$ be two reductions, and let us assume that $\mathfrak{a}_0$ is not contained in the nilpotent cone $\mathcal{N}$ of $\mathfrak{p}$. Then, the following statements are equivalent.*

(1) *The K-orbit in $\mathfrak{R}$ containing $\mathfrak{a}_1$ is more general than the one containing $\mathfrak{a}_0$.*
(2) *For any subvariety of reductions $\mathfrak{R}(\tilde{\mathfrak{a}})$ containing $\mathfrak{a}_0$, there exists a K-conjugate of $\mathfrak{a}_1$ in $\mathfrak{R}(\tilde{\mathfrak{a}})$ whose $K(\tilde{\mathfrak{a}})$-orbit is more general than the one of $\mathfrak{a}_0$.*
(3) *There exists a subvariety of reductions $\mathfrak{R}(\tilde{\mathfrak{a}})$ containing $\mathfrak{a}_0$ and a K-conjugate of $\mathfrak{a}_1$ in $\mathfrak{R}(\tilde{\mathfrak{a}})$ whose $K(\tilde{\mathfrak{a}})$-orbit is more general than the one of $\mathfrak{a}_0$.*

7. Special reductions

The set $\mathfrak{R}_s = \mathfrak{R} \setminus \mathfrak{R}_o$ of special reductions is the complement of an affine open set in a projective variety, hence it is a divisor in $\mathfrak{R}$. This divisor is hard to describe in the general setting: we can not even count its irreducible components. We are however able to show that it is cut out by a smooth quadric of the ambiant space (7).

7.1. **Bilinear algebra and exterior algebra.** A symmetric bilinear form b on a finite dimensional vector space E induces a symmetric bilinear form $\Lambda^r b$ on the r-th exterior power of E. On two decomposed r-vectors $u_1 \wedge \cdots \wedge u_r$ and $v_1 \wedge \cdots \wedge v_r$ this bilinear form evaluates to the determinant of the matrix with coefficients $b(u_i, v_j)$. When b is regular, so is $\Lambda^r b$.

7.2. **Special reductions.** The Killing form b of the reductive Lie algebra $\mathfrak{g}$ is preserved by automorphisms of $\mathfrak{g}$. Consequently, the characteristic spaces $\mathfrak{k}$ and $\mathfrak{p}$ of θ' are orthogonal, since they are also supplementary, the restriction of b to any of them is regular. The quadratic form q associated to $\Lambda^r b$ on $\mathbf{P}(\Lambda^r \mathfrak{p})$ defines a smooth quadric Q.

7.3. PROPOSITION. *The set $\mathfrak{R}_s$ of special reductions is the intersection of the variety of reductions $\mathfrak{R}$ with the quadric Q.*

To begin with, we state a proposition and two lemmas.

7.4. PROPOSITION ([10, 37.5.2]). *The bilinear form induced by the Killing form on a Cartan subspace of $\mathfrak{p}$ is regular.*

7.5. LEMMA. *The bilinear form induced by the Killing form on an anisotropic subalgebra of $\mathfrak{p}$ containing a nilpotent element is degenerate.*

Proof. Let $\mathfrak{u}$ be an anisotropic subalgebra of $\mathfrak{p}$ containing a nilpotent element n. Since $\mathfrak{u}$ is abelian, endomorphisms $\operatorname{ad} u \circ \operatorname{ad} n$ are nilpotent for each $u \in \mathfrak{u}$, so that $b(u, n) = 0$. Hence n lies in the kernel of the restriction of b to $\mathfrak{u}$. $\square$

7.6. LEMMA. *An anisotropic algebra contains a nilpotent element, unless it is an ordinary reduction.*

Proof of the lemma. Let $\mathfrak{u} \in \mathfrak{A}$ be an anisotropic subalgebra of $\mathfrak{g}$ of dimension r and let $\mathfrak{a}$ be a Cartan subspace of $\mathfrak{g}$ containing the semi-simple parts of the elements of $\mathfrak{u}$. The Chevalley-Jordan-decomposition induces a linear map $\mathfrak{u} \to \mathfrak{a}$ whose kernel is the set of nilpotent elements in $\mathfrak{u}$. Since $\mathfrak{c}_\mathfrak{p}(\mathfrak{a}) = \mathfrak{a}$ (2.6) the image of this linear map has rank r only if $\mathfrak{u} = \mathfrak{a}$. $\square$

Proof of 7.3. It follows from the previous lemmas that the set of reductions $\mathfrak{u}$ for which the restriction of the Killing form to $\mathfrak{u}$ is degenerate is exactly the set of special reductions. These points are also the ones where q vanishes, so that $\mathfrak{R}_s = \mathfrak{R} \cap Q$. $\square$

8. Partial positivity of the anticanonical class

8.1. **Linear subspaces of Grassmann varieties.** Recall that linear subspaces of $\mathbf{P}(\Lambda^r \mathfrak{p})$ contained in $\mathcal{G}(r, \mathfrak{p})$ are precisely the Grassmann subvarieties of $\mathcal{G}(r, \mathfrak{p})$ which are also projective spaces. Grassmann subvarieties of $\mathcal{G}(r, \mathfrak{p})$ are the sets

$$\Gamma(\mathfrak{v}, \mathfrak{w}) = \{\, \mathfrak{u} \in \mathcal{G}(r, \mathfrak{p}) \mid \mathfrak{v} \subset \mathfrak{u} \subset \mathfrak{w} \,\}$$

where $\mathfrak{v}$ and $\mathfrak{w}$ are linear subspaces of $\mathfrak{p}$. Its dimension is $\dim(\mathfrak{u}/\mathfrak{v})\dim(\mathfrak{w}/\mathfrak{u})$, and it is a linear space precisely when $\dim(\mathfrak{u}/\mathfrak{v}) = 1$ or when $\dim(\mathfrak{w}/\mathfrak{u}) = 1$.

8.2. **Maximal linear subspaces of varieties of reductions.** Let us recall that a singular torus of S is a torus whose centralizer in G is not a maximal torus. We say an anisotropic reductive algebra is *singular* when its centralizer in $\mathfrak{p}$ is not a Cartan subspace. Maximal singular anisotropic reductive algebras are the kernels of the roots of $\mathfrak{g}$ relative to some Cartan subspace $\mathfrak{a}$ (2.1).

8.3. PROPOSITION. *Let $\mathfrak{z}$ be a maximal singular anisotropic reductive algebra. The subspace $\Gamma(\mathfrak{z}) = \Gamma(\mathfrak{z}, \mathfrak{c}_\mathfrak{p}(\mathfrak{z}))$ of $\mathcal{G}(r, \mathfrak{p})$ is contained in $\mathfrak{R}$, it pass through all points in $\mathfrak{R}_o$ containing $\mathfrak{z}$.*

Proof. Let $\mathfrak{a}$ be a Cartan subspace containing $\mathfrak{z}$ and α a root of $\mathfrak{g}$ relative to $\mathfrak{a}$ whose kernel is $\mathfrak{z}$. The open subset $\mathfrak{R}_o$ of the irreducible component $\mathfrak{R}$ of $\mathfrak{A}$ meets at $\mathfrak{a}$ the linear space $\Gamma(\mathfrak{z})$ contained in $\mathfrak{A}$. This linear space is therefore contained in $\mathfrak{R}$. □

8.4. THEOREM. *The linear subspaces $\Gamma(\mathfrak{z})$ are maximal among the linear subspaces of $\mathfrak{R}$ passing through a general point.*

Proof. Let $\mathfrak{a} \in \mathfrak{R}_o$ be a general point of $\mathfrak{R}$, and $\Gamma = \Gamma(\mathfrak{v}, \mathfrak{w})$ a linear subspace of $\mathfrak{R}$ passing through $\mathfrak{a}$. We shall see that $\mathfrak{v}$ needs to be a maximal singular anisotropic reductive algebra in order to let Γ be maximal.

Assume that $\mathfrak{v}$ has codimension greater than two in $\mathfrak{a}$, each pair in $\mathfrak{a} \times \mathfrak{v}$ has its members belonging to a point of Γ, now $\Gamma \subset \mathfrak{A}$, hence $\mathfrak{v} \subset \mathfrak{c}_\mathfrak{p}(\mathfrak{a})$. But $\mathfrak{c}_\mathfrak{p}(\mathfrak{a}) = \mathfrak{a}$ (2.6) and $\Gamma = \{\, \mathfrak{a} \,\}$ is not maximal.

Assume now that $\mathfrak{v}$ has codimension one in $\mathfrak{a}$. Since $\Gamma \subset \mathfrak{A}$, we have $\mathfrak{v} \subset \mathfrak{c}_\mathfrak{p}(\mathfrak{w})$. But this centralizer is

$$\mathfrak{a} \oplus \bigoplus \mathfrak{p}(\mathfrak{a})_\alpha$$

where the sum extends over the set of positive roots α relative to $\mathfrak{a}$ which vanish on $\mathfrak{v}$. In order to let $\mathfrak{w}$ be strictly bigger than $\mathfrak{a}$, this set of roots must not be empty. There also exists a root α whose kernel contains $\mathfrak{v}$, and for dimension reasons $\mathfrak{v} = \operatorname{Ker}\alpha$ is a maximal singular anisotropic reductive subalgebra. Thus $\Gamma \subset \Gamma(\mathfrak{v})$, but Γ is maximal, it equals $\Gamma(\mathfrak{v})$. In case $\mathfrak{c}_\mathfrak{p}(\mathfrak{v}) = \mathfrak{a}$ for each codimension one subspace $\mathfrak{v}$ in $\mathfrak{a}$, the root system of $\mathfrak{p}$ relative to $\mathfrak{a}$ is empty and $\mathfrak{p} = \mathfrak{a}$: $\mathfrak{R}$ is a point and $\Gamma(\mathfrak{v})$ is maximal. □

8.5. COROLLARY. *Through a general point $\mathfrak{a}$ of $\mathfrak{R}$ passes a finite number of maximal linear subspaces of $\mathfrak{R}$, meeting transversally in $\mathfrak{a}$. The intersection of any two of these subspaces is $\mathfrak{a}$.*

This follows from the decomposition (2.4). Notice that these linear subspaces do not need to share a common dimension. However, we can make this picture more accurate in the case of the variety of Cartan reductions for a reductive group.

8.6. COROLLARY. *Let $\mathfrak{R}$ be the variety of Cartan reductions for a reductive group of rank r and dimension $2m + r$. Through a general point of $\mathfrak{R}$ passes m projective planes, any two of them meeting transversally in this point.*

8.7. PROPOSITION. *A general line contained in $\mathfrak{R}$ is contained in the smooth locus of $\mathfrak{R}$.*

Proof. We show that a general line is contained in the image of the centralizer map (3.7). Such a line is contained in a space $\Gamma(\mathfrak{z})$, hence we can replace $\mathfrak{R}$ by the variety of reductions for the pair $(C'_G(\mathfrak{z}), \theta)$, whose group is the derived group of the centralizer of $\mathfrak{z}$ in G (see 6.4, 6.2). We are then reduced to the case of a reductive symmetric pair of rank one, where $\mathfrak{R} = \mathbf{P}(\mathfrak{p})$. The generic nilpotent element is regular [9, Theorem 3], so that the irregular locus has codimension at least 2 in $\mathfrak{R} = \mathbf{P}(\mathfrak{p})$: a generic line will miss it. $\qquad\square$

8.8. **Canonical class.**

8.9. PROPOSITION. *Let X be a smooth algebraic variety quasi homogeneous under the action of an algebraic group G. For any smooth rational curve C in X touching the open orbit of X, the Hilbert scheme $\mathcal{H}$ parametrising deformations of C in X is smooth at $[C]$.*

Proof. We show that the second cohomology group $H^1(N_{C/X})$ of the normal bundle to C in X vanishes, which occurs only if the Hilbert scheme is smooth at $[C]$ [16, Theorem 2.6].

Let $\theta : \mathfrak{g} \times X \to TX$ be the morphism obtained by restricting the map tangent to the group action $G \times X \to X$. (Recall that X is embedded in TX *via* the zero section.) Partial application of θ gives a global section $\theta_a : X \to TX$ of the tangent bundle from any $a \in \mathfrak{g}$, and at any point x in the open orbit X_o of X under G, the stalk $(TX)_x$ of the tangent bundle is generated by these global sections. Thus, the restriction of the tangent bundle of X to $C_o = X_o \cap C$ is globally generated and $N_{C/X}|_{C_o}$ is generated by global sections of $N_{C/X}$.

Since C and X are smooth, the normal sheaf to C in X is locally free. It splits as

$$N_{C/X} = \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r)$$

for some integer vector $a \in \mathbf{Z}^r$, and $\mathcal{O}(1)$ being the tautological bundle on $C \simeq \mathbf{P}^1$. If a_i is negative, the bundle $\mathcal{O}(a_i)$ has no global sections, and at any point $c \in C_o$, global sections span at most a hyperplane in $N_{C/X}|_c$. Therefore the the a_i's are non negative and thus $H^1(N_{C/X}) = 0$. $\qquad\square$

8.10. COROLLARY. *Let $\mathfrak{a} \in \mathfrak{R}_o$ be a general point of $\mathfrak{R}$ and Δ a line through $\mathfrak{a}$ contained in $\mathfrak{R}$. Let m be the dimension of the maximal linear subspace of $\mathfrak{R}$ through $\mathfrak{a}$ containing Δ. If Δ is contained in the smooth locus of $\mathfrak{R}$, then*

$$-K_r \cdot \Delta = m + 1$$

where K_r is the canonical class of the smooth locus of $\mathfrak{R}$.

8.11. REMARK. The hypothesis on Δ is always satisfied when this line is sufficiently general (8.7). When the variety of Cartan reductions for a reductive group is under consideration, we have $-K_r \cdot \Delta = 3$, for any such line.

Proof. According to the Riemann-Roch formula,

$$\chi(N_{\Delta/\mathfrak{R}}) = c_1(N_{\Delta/\mathfrak{R}}) + \mathrm{rk}(N_{\Delta/\mathfrak{R}})(1 - g(\Delta)),$$

the rank $\mathrm{rk}(N_{\Delta/\mathfrak{R}})$ is $\dim \mathfrak{R} - \dim \Delta$ and the genus $g(\Delta)$ of $\Delta \simeq \mathbf{P}^1$ vanishes. We compute:

$$
\begin{aligned}
c_1(N_{\Delta/\mathfrak{R}}) &= c_1(T\mathfrak{R}|_\Delta) - c_1(T\Delta) \\
&= c_1(\det T\mathfrak{R}|_\Delta) - c_1(\mathcal{O}(2)) \\
&= -K_r \cdot \Delta - 2.
\end{aligned}
$$

By (8.9) the Hilbert scheme $\mathcal{H}_{[\Delta]}$ parametrising deformations of Δ in $\mathfrak{R}$ is smooth at $[\Delta]$. Its dimension at $[\Delta]$ is $h^0(N_{\Delta/\mathfrak{R}})$ and $h^1(N_{\Delta/\mathfrak{R}}) = 0$. The Riemann-Roch formula eventually yields

$$
h^0(N_{\Delta/\mathfrak{R}}) = -K_r \cdot \Delta + \dim \mathfrak{R} - 3.
$$

We compute the dimension $h^0(N_{\Delta/\mathfrak{R}})$ of $\mathcal{H}_{\Delta]}$ another way. Put

$$
Z = \left\{ (\mathfrak{a}, A) \in \mathfrak{R} \times \mathcal{H}_{[\Delta]} \; \middle| \; \mathfrak{a} \in A \right\}.
$$

The Z component $(\mathfrak{a}, \Delta)$ belongs to, projects to a subset of $\mathfrak{R}$ containing $\mathfrak{R}_o$ with general fiber of dimension $m - 1$, while this same component projects to $\mathcal{H}_{[\Delta]}$ with general fiber of dimension 1. We can then compute

$$
\dim \mathfrak{R} + m - 1 = 1 + h^0(N_{\Delta/\mathfrak{R}})
$$

hence $K_r \cdot \Delta = -m - 1$. $\square$

9. Variety of anisotropic algebras

In this section, the symmetric pair associated to the Cartesian square of a simple group G of rank r is under consideration.

9.1. **Rough estimate of the dimension of a nilpotent orbit.** We say an orbit $\mathcal{O}$ in $\mathfrak{A}$ is *nilpotent* if it is the orbit of an abelian algebra contained in the nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$.

9.2. PROPOSITION. *The dimension of a nilpotent orbit in $\mathfrak{A}$ is less than $\dim G - r - 2$, unless G has type A_1.*

Proof. Let B be a Borel subgroup of G, $\mathfrak{u}$ the Lie algebra of its unipotent radical and $\mathfrak{v}$ an abelian subalgebra of $\mathfrak{u}$ with dimension r—any nilpotent orbit in $\mathfrak{A}$ contains such a $\mathfrak{v}$. We bound from below the dimension of the normalizer of $\mathfrak{v}$ in $\mathfrak{g}$ by the dimension of the normalizer $\mathfrak{n}_{\mathfrak{u}}(\mathfrak{v})$ of $\mathfrak{v}$ in $\mathfrak{u}$, thus obtaining a bound from above for the dimension of the orbit through $\mathfrak{v}$.

According to the Lie-Kolchin theorem, endomorphisms of $\mathfrak{u}/\mathfrak{v}$ adjoint to elements in $\mathfrak{v}$ share a common nilvector, so $\dim \mathfrak{n}_{\mathfrak{u}}(\mathfrak{v})/\mathfrak{v} \geq 1$. Assume that this dimension is 1, and let v be the generic element of $\mathfrak{v}$, and v' the endomorphism of $\mathfrak{u}/\mathfrak{v}$ adjoint to v. Since v is generic, $\mathrm{Ker}\, v' = \mathfrak{n}_{\mathfrak{u}}(\mathfrak{v})/\mathfrak{v}$ has dimension 1 and v' is a cyclic nilpotent endomorphism, whose Jordan normal form consists of a single block. The dimension of $\mathfrak{n}_{\mathfrak{u}}(\mathfrak{v})/\mathfrak{v}$ thus equals the degree of the minimal polynomial of v'. The endomorphism of $\mathfrak{u}$ adjoint to the generic element of $\mathfrak{u}$ has minimal polynomial of degree $h - 1$, where h is the Coxeter number of G. Since this minimal polynomial also vanishes at v', we have

$$
\dim \mathfrak{n}_{\mathfrak{u}}(\mathfrak{v})/\mathfrak{v} \leq h - 1.
$$

TABLE 1. Number of positive roots and Coxeter numbers

Type	$n/2$	h	$h+r-1$
A_r	$r(r+1)/2$	$r+1$	$2r$
B_r	r^2	$2r$	$3r-1$
C_r	r^2	$2r$	$3r-1$
D_r	$r(r-1)$	$2r-2$	$3r-3$
E_6	36	12	17
E_7	63	12	18
E_8	120	30	37
F_4	24	12	15
G_2	6	6	7

Let $n/2 = \dim \mathfrak{u}$ be the number of positive roots of G, the former inequality gives $n/2 \leq h+r-1$ and from the table 1 (see Bourbaki [17]) we infer this inequality is only possible when G has type A_1, A_2, A_3, B_2 or G_2. Type A_1 is not to be considered, types A_2 and A_3 have their variety of Cartan reductions studied by Iliev and Manivel [4, 11], and the remaining ones are studied in [5]. $\square$

9.3. **Infinitely many orbits.** Iliev and Manivel [11] established that $\mathfrak{A}$ consists of infinitely many orbits, using the Grassmann variety associated with maximal nilpotent abelian subalgebras in $\mathfrak{gl}_n$ for $n \geq 6$. Malcev [18] classified abelian subalgebras of maximal dimension in simple Lie algebras, allowing us to adapt the previous argument to show the

9.4. PROPOSITION. *The variety $\mathfrak{A}$ of abelian algebras consists of infinitely many G-orbits if G has one of the following types:*

$$A_r \ (r \geq 5) \quad B_r \ (r \geq 4) \quad C_r \ (r \geq 5) \quad D_r \ (r \geq 6) \quad E_7 \quad E_8.$$

Proof. Let r be the rank of our semi-simple group, n its number of roots, m the largest dimension an abelian unipotent subalgebra of $\mathfrak{g}$ can have, and $\mathfrak{m}$ an abelian unipotent subalgebra with this dimension. Let $\mathfrak{v}$ be an abelian subalgebra of $\mathfrak{g}$ contained in $\mathfrak{m}$. On the one hand, $\mathfrak{v}$ lies in the nilpotent cone of $\mathfrak{g}$, according to (9.2) the G-orbit through $\mathfrak{v}$ in $\mathfrak{A}$ has dimension less than $r-2$. On the other hand, the set of all possible $\mathfrak{v}$ is the Grassmann variety $\mathcal{G}(r, \mathfrak{m})$, which has dimension $r(m-r)$. We conclude by an explicit computation that whenever $r(m-r) > r-2$, the action of G on $\mathfrak{A}$ must have infinitely many orbits. $\square$

9.5. THEOREM (Malcev). *Each simple Lie algebra with the exclusion of B_4, D_4 and G_2 has up to automorphisms only one commutative subalgebra of maximal dimension with nilpotent elements. This dimension equals $[\frac{1}{4}(r-1)^2]$ for the algebra A_r $(r > 2)$ (brackets stand for the integer part of their argument), $\frac{1}{2}r(r-1)+1$ for B_r $(r > 4)$, $\frac{1}{2}r(r+1)$ for C_n, $\frac{1}{2}r(r-1)$ for D_n, and 16, 29, 36, 9 and 5 respectively for E_6, E_7, E_8, F_4 and B_3. The algebra B_4 has two classes of conjugate abelian subalgebras of maximal dimension 7, D_4 has two classes of dimension 6 and G_2 has three classes of dimension 3.*

9.6. **Non algebraic anisotropic algebras.** While the irreducible component $\mathfrak{R}$ of $\mathfrak{A}$ contains only algebraic subalgebras of $\mathfrak{g}$ (5.3), this is not the case of $\mathfrak{A}$.

9.7. PROPOSITION. *If a semi-simple group has rank large enough, its variety $\mathfrak{A}$ of abelian algebras contains non algebraic elements.*

Proof. Let G be a semi-simple group of rank r, and s a degenerate semi-simple element of its Lie algebra $\mathfrak{g}$. If r is large enough, the set of nilpotent elements in $\mathfrak{g}$ commuting with s contains a r-dimensional linear subspace: this is most easily seen when s is the coroot associated with an *extremal* node in the Dynkin diagram of G, in this case the centralizer of s has semi-simple part a simple group of rank $r-1$ and one can readily use Malcev theorem (9.5). If the semi-simple part of the centralizer of s in G has multiple simple ideals, one has to apply Malcev theorem on each of them, to conclude. Let m be a basis of such a space, the Lie algebra spanned by $s + m_1$, m_2, ..., m_r, is non algebraic, since it is not closed under the Chevalley-Jordan decomposition. □

APPENDIX A. TEMPERED MOVING FRAMES ABOVE DEGENERACY CURVES IN GRASSMANN VARIETIES

We study moving frames along degeneracy curves in Grassmann varieties and prove the technical result (A.8). We use here the language of degeneracies introduced in 4.4.

A.1. **Notations.** Let E be a vector space of finite dimension, A_1 a r-dimensional subspace of E and C^o a smooth and irreducible curve in $\mathbf{GL}(E)$. We let A_0 be a r-dimensional subspace lying in the closure of the curve $C^o A_1$ in the Grassmann variety $\mathcal{G}(r, E)$ of r-dimensional subspaces of E. We denote by $\mathcal{T}(r, E)$ the tautological bundle $\mathcal{T}(r, E) \to \mathcal{G}(r, E)$. The principal bundle $\mathcal{B}(r, E) \to \mathcal{G}(r, E)$ whose fiber at A is the set of all basis of A is a subbundle of the r-th bundle power of $\mathcal{T}(r, E)$. It maps onto the principal bundle $\mathcal{F}(r, E) \to \mathcal{G}(r, E)$ whose fiber at A is the set of complete flags of A. If x is a basis of A, we call its image in $\mathcal{F}(r, E)$ the complete flag corresponding to x. Given a complete flag $\mathcal{F}$ in A, we say that a basis x of A corresponds to $\mathcal{F}$ when its image in $\mathcal{F}(r, E)$ is $\mathcal{F}$.

A.2. **Tempered moving frames along degeneracy curves.** Let A_1 be a r-dimensional subspace of E and K an algebraic group acting linearly on E. We choose a point A_0 lying in the closure of the orbit $K A_1$ of K in $\mathcal{G}(r, E)$ containing A_1, a degeneracy curve C^o in K and c_0 a point at infinity taking A_1 to A_0. If x is a basis of A_1, we obtain a moving frame along $C^o A_1$ by sending c to the point (cA_1, cx) of $\mathcal{B}(r, E) \subset \mathcal{G}(r, E) \times E^r$.

This moving frame will usually not extend at c_0. In the first place, it will not do because the vectors cx_k may become infinitely small or infinitely large. In the second place, the lines defined by two vectors cx_i and cx_j may degenerate to a single line. If cx_i is asymptotically smaller than cx_j, no linear combination of cx_i and cx_j can yield a second direction at c_0.

The first difficulty is circumvented by replacing cx by the associated tempered moving frame: it is obtained from cx by rescaling its terms (A.6). We are able to characterize the basis in A_1 for which the second difficulty does not occur. They are the basis of A_1 whose associated flag is a refinement of some partial flag in A_1 determined by C^o. We call this partial flag the *magnitude orders flag*, it is defined in (A.4).

From now on we denote by R_0 the local ring of the smooth completion C of C^o at c_0, by $\mathfrak{m}_0$ its maximal ideal, by L_0 its ring of fractions and by $\epsilon \in \mathfrak{m}_0$ a local parameter of C at c_0. As the irreducible smooth curve C^o uniquely determines its smooth completion C, we call R_0 the local ring of C^o at c_0 and ϵ a local parameter of C^o at c_o.

A.3. **Filtration by the magnitude order.** We consider a linear representation V of K, for instance an exterior power of E. For any vector $v \in V$, the rational map cv from C to V is an element of $L_0 \otimes V$. Thus, the asymptotic behaviour of vectors in V under the operation of c near c_0 is related to the structure of the R_0-module $L_0 \otimes V$.

The L_0-vector space $L_0 \otimes V$ is a R_0-module filtered by the submodules $\mathfrak{m}_0^k(R_0 \otimes V)$ for $k \in \mathbf{Z}$. For any $v \in L_0 \otimes V \setminus \{0\}$ we call the number

$$\omega(v) = \sup \left\{ k \in \mathbf{Z} \;\middle|\; \mathfrak{m}_0^k(R_0 \otimes V) \ni v \right\}$$

the *magnitude order* of v near c_0, or briefly the magnitude order of v. Note that the zero vector has no magnitude order. For any v_1, v_2 in $L_0 \otimes V \setminus \{0\}$ we have

$$\omega(v_1 + v_2) \geq \min \left\{ \omega(v_1), \omega(v_2) \right\},$$

unless $v_1 + v_2 = 0$.

A.4. **Magnitude orders flag and corresponding basis.** We describe a filtration on A_1 associated to the filtration of $L_0 \otimes E$ by the magnitude order, which yields a partial flag in A_1. In the theory of the module $L_0 \otimes E$ over the principal ideal domain R_0, this flag is related to the invariant factors of the submodule $R_0 \otimes A_1$ of the finitely generated module $R_0 \otimes E$.

To any integer k we associate the subspace $(A_1)_k$ of A_1 defined by

$$(A_1)_k = \left\{ x \in A_1 \setminus \{0\} \mid \omega(cx) \geq k \right\} \cup \{0\}.$$

Let $(a_1, \ldots, a_m)$ be the increasing sequence of integers k such that $(A_1)_k$ is a strict superset of $(A_1)_{k+1}$. We thus have $(A_1)_{a_1} = A_1$ and $(A_1)_{1+a_m} = \{0\}$, and it is convenient to put $a_{m+1} = 1 + a_m$. This yields a partial flag

$$\left\{ (A_1)_{a_1} \supset \cdots \supset (A_1)_{a_m} \right\}$$

of A_1.

A.5. Definition. The partial flag $\mathcal{F}_1^\omega = \left\{ (A_1)_{a_1} \supset \cdots \supset (A_1)_{a_m} \right\}$ of A_1 is the *magnitude orders flag*. A *magnitude orders basis* of A_1 is a basis $x \in A_1^r$ of A_1 corresponding to a complete flag finer than the magnitude orders flag.

The natural map $\mathcal{B}(r, E) \to \mathcal{F}(r, E)$ is onto, which implies the existence of magnitude orders basis of A_1.

A.6. Definition. Let x be basis of A_1. The tempered moving frame associated to x is the map sending $c \in C^o$ to $(cA_1, \epsilon^{-\omega(cx_1)}cx_1, \ldots, \epsilon^{-\omega(cx_r)}cx_r)$.

A.7. **Extension of tempered moving frames.** Our characterization of tempered moving frames that extend at infinity is the following

A.8. Theorem. *Let x be a basis of A_1. The following conditions are equivalent:*

   (1) *x is a magnitude orders basis of A_1;*

(2) *the finite sequence* $(\omega(cx_k))_{1 \leq k \leq r}$ *is non-decreasing and for any* $1 \leq k \leq r$ *we have*

$$\omega(cx_1 \wedge \cdots \wedge cx_k) = \sum_{i=1}^{k} \omega(cx_i)$$

for magnitude orders in the R_0-*module* $\Lambda^k(L_0 \otimes E)$.

(3) *the tempered moving frame associated to* x *extends at* $A_0 = c_0 A_1$, *and its value at* c_0 *is a basis of* A_0.

We will only give a few indications about its proof, for it is rather lengthy and has little to do with the geometry of varieties of reductions. The hard work is the proof that (2) implies (3). This can be done by using a variation of Smith's algorithm computing the normal form of a matrix whose coefficients are in the principal ideal domain R_0. From this standpoint (2) means that the principal minors of the matrix whose columns are the coordinates of the tempered frame associated to x are invertible in R_0.

We also state without proof the following technical result:

A.9. PROPOSITION. *Let* $A_1 = B^1 \oplus \cdots \oplus B^l$ *be a direct sum decomposition of* A_1. *There exists a magnitude order basis of* A_1 *whose terms are in* $B^1 \cup \cdots \cup B^l$.

APPENDIX B. CLOSURES OF DECOMPOSITION CLASSES

We use the rigidity theorem 4.6 to show that the closure of a decomposition class is a union of decomposition class. It seems that this was yet only proved for the reductive symmetric pairs associated to the Cartesian square of a reductive group [10, 39.2.7].

B.1. THEOREM. *Let* (G, θ) *be a reductive symmetric pair, and* $\mathfrak{p}$ *its anisotropic space. For any* x *in* $\mathfrak{p}$, *the closure of the decomposition class* $\mathcal{D}(x)$ *of* x *in* $\mathfrak{p}$ *is a union of decomposition classes.*

Proof. Let $K = (G^\theta)^o$ be the connected component of the fixed point group of θ. According to [10, 39.5.2] we have:

$$\mathcal{D}(x) = K \cdot (\mathfrak{c}_\mathfrak{p}^2(x_s)_o + x_n) = K \cdot (\mathfrak{c}_\mathfrak{p}^2(x)_o)$$

(see (3.2) for the definition of $\mathfrak{c}_\mathfrak{p}^2(x_s)_o$ and $\mathfrak{c}_\mathfrak{p}^2(x)_o$). Let y be a point in the closure $\bar{\mathcal{D}}(x)$ of $\mathcal{D}(x)$ in $\mathfrak{p}$. We show that $\mathcal{D}(y)$ is contained in $\bar{\mathcal{D}}(x)$. Since this closure is K-stable, it is enough to prove that $\mathfrak{c}_\mathfrak{p}^2(y_s) + y_n \subset \bar{\mathcal{D}}(x)$.

Let $\Gamma^o \subset K \times \mathfrak{c}_\mathfrak{p}^2(x)_o$ be a smooth curve such that y lies in the closure of the image of Γ^o by the map sending (g, z) to gz. For all (g, z) in Γ^o we have $\mathfrak{c}_\mathfrak{p}^2(gz) = g\mathfrak{c}_\mathfrak{p}^2(x)$ so that the projection C^o of Γ^o in K pushes the anisotropic algebra $\mathfrak{c}_\mathfrak{p}^2(x)$ toward an anisotropic algebra containing y. A double centralizer is closed under Chevalley-Jordan decomposition, hence we can assume by the rigidity theorem 4.6 that y_s belongs to $\mathfrak{c}_\mathfrak{p}^2(x)$ and C^o is a subset of the centralizer of y_s in K. But then $\mathfrak{c}_\mathfrak{p}^2(y_s) \subset \mathfrak{c}_\mathfrak{p}^2(x_s) \subset \mathfrak{c}_\mathfrak{p}^2(x)$ so we are done. $\square$

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