UNIVERSAL FAMILY OF THE SUBGROUPS OF AN ALGEBRAIC GROUP

MICHAËL LE BARBIER GRÜNEWALD

ABSTRACT. We construct a moduli space for the connected subgroups of an algebraic group and the corresponding universal family. Morphisms from an algebraic variety to this moduli space correspond to flat families of connected algebraic subgroups parametrised by this variety. This moduli space is obtained by gluing together infinitely many irreducible projective varieties of bounded dimension along closed subvarieties.

Regarding families of non-connected subgroups of an algebraic group, we show that, given sich a family, the corresponding family of identity components is an irreducible component of the former, and the quotient of a family of groups group by the family of their identity components exists.

1. INTRODUCTION

Let G be a complex connected algebraic group, a flat family of subgroups of G parametrised by a variety P is a subvariety $\mathcal{H} \subset G \times P$ such that the projection $\mathcal{H} \to P$ is a flat morphism and whose scheme-theoretic fiber $\mathcal{H}_p$ at any $p \in P$ is a group subscheme of G. We construct a moduli space and a universal family for the connected subgroups of a complex algebraic group and discuss a few examples: tori, abelian varieties, $\mathbf{SL}_2$ and $\mathbf{SL}_3$.

1.1. **Notations.** Let G be a complex connected algebraic group, and $\mathfrak{g}$ its Lie algebra. We fix an integer k and denote by Γ the set of k-dimensional connected subgroups of G. A connected subgroup H of G is determined by its Lie algebra $\mathcal{L}(H)$, as a subalgebra of $\mathfrak{g}$, so we denote by $\mathbf{Gr}(k, \mathfrak{g})$ the Grassmann variety of k-planes in $\mathfrak{g}$, by $\Lambda \subset \mathbf{Gr}(k, \mathfrak{g})$ the variety of k-dimensional Lie subalgebras of $\mathfrak{g}$ and by A the set

$$A = \{ \mathcal{L}(H) \mid H \in \Gamma \}$$

of *algebraic k-dimensional Lie subalgebras of G*. The terminology *algebraic* was introduced by Chevalley [2]. We will take advantage in our construction of the natural operation of G on these sets.

1.2. **Main results and structure.** It belongs to moduli space problems that we have to allow more general objects than algebraic varieties in order to obtain a satisfying construction. Namely, we introduce the category of *bouquets of algebraic varieties* (4.1), that contains all algebraic varieties and objects obtained by gluing together possibly infinitely many algebraic varieties along closed subvarieties. These bouquets are similar to *ind-varieties* introduced by Shafarevich [11] and Kumar [7].

We describe in Section 4 a structure $\mathfrak{P}$ of bouquet on Γ and show that the set-theoretic universal family

$$\mathfrak{H} = \{\, (H,g) \in G \times \mathfrak{P} \mid g \in H \,\}$$

is actually a bouquet-theoretic universal family (4.7). The irreducible components of $\mathfrak{P}$ are projective varieties and the ones of $\mathfrak{H}$ are quasi-projective.

THEOREM (4.7). *Let P be an algebraic variety, and denote by*

(1) *F the set of flat families of k-dimensional connected subgroups of G parametrised by P;*
(2) *M the set of morphisms from P to $\mathfrak{P}$.*

Then the map $F \to M$ sending a family $\mathcal{H}$ to the morphism to Λ mapping $p \in \mathcal{H}$ to $\mathcal{L}(\mathcal{H}_p)$ and the map from $M \to F$ sending a morphism ψ to the family $\psi^* \mathfrak{H}$ define a natural correspondance between F and M.

The bouquet structure on $\mathfrak{P}$ is *a posteriori* defined by the fact that $\mathfrak{H}$ is a final object in the category of flat families of k-dimensional subgroups of G. The Lie functor $\mathcal{L}$ puts Γ and A in a one-to-one correspondence, but the topology induced by $\mathbf{Gr}(k, \mathfrak{g})$ on Γ is much coarser than it needs to be, with respect to the universal property of $\mathfrak{H}$. From the three stages we need to go through before get to the proof of Theorem 4.7—the construction of the moduli space $\mathfrak{P}$, of the universal family $\mathfrak{H}$ and the proof of the universal property—the second one draws most of our efforts. For this we need to show the

THEOREM (3.6). *Let P be an irreducible constructible subset of Λ. If P is contained in A, then the closure $\bar{P}$ of P in Λ is also contained in A and $\mathfrak{H}(\bar{P})$ equals $\bar{\mathfrak{H}}(P)$.*

Note that restriction of the tautological family $\mathfrak{H}(\bar{P})$ mentioned in the statement of our Theorem 3.6 is flat. The tautological family can be described through to the exponential map of G:

COROLLARY (3.8). *The set $\exp \times \mathrm{id}_\Lambda(\mathfrak{h}(\bar{P}))$ is actually $\mathfrak{H}(\bar{P})$, it is in particular algebraic.*

This Corollary is actually equivalent to Theorem 3.6, and we first thought at it as an intermediary result on the road to Theorem 3.6. However we were not able to overcome the difficulties tied to the analytic nature of the exponential map, and had to put things the other way around. Section 3 is devoted to its proof, that is built on two principles: First we do not need to show that all of $\mathfrak{H}(\bar{P})$ is an algebraic variety but we are allowed to work on a smaller set of parameters (see 3.5). Second we observe that some discrete invariants of groups behave semi-continuously in families (see 3.15 and 3.19). This observation allows us to find the suitable small subset of parameters over which we are able to prove the algebraicity of the set-theoretic family.

While representation theory of algebraic groups provide us with a wealth of examples of families of algebraic groups (see Example 2.4) it is unwise to expect them having irreducible fibres (see Example 2.5) even in the case of the family is a connected variety. In Section 5, we first show that the set-theoretic family of subgroups of G deduced from an algebraic one by replacing each fibre by its identity component is algebraic as well 5.3. Second, we show how to construct a quotient of a family of subgroups by the family of its identity components 5.7.

CONTENTS

2. ELEMENTARY PROPERTIES OF FAMILIES OF SUBGROUPS

2.1. **Definitions and examples.** The definition of family of subgroups that we introduce is a close variation on the definition of families of subvarieties (see [6, III.9.10]). However, we allow disconnected fibres. We then show two results allowing us to construct new families of subgroups, by pushing a family forward or backward through a morphism or intersecting its fibres with a fixed group. Less elementary construction theorems must be deferred until we have gathered the necessary prerequisites (section 3).

2.2. DEFINITION. Let X and P be two algebraic varieties. A family of subschemes of X parametrised by P is a subvariety $\mathcal{V}$ of $X \times P$ such that the morphism $\pi : \mathcal{V} \to P$ obtained by restricting the projection of $X \times P$ on P to $\mathcal{V}$ is surjective. The fibre $\mathcal{V}_p$ of the family at p is the subscheme $\pi^{-1}(p)$ of X.

The family is *irreducible* when $\mathcal{V}$ is irreducible. The family is *reduced* at $p \in P$ if the maximal ideal $\mathfrak{m}_p$ of $p \in P$ generates the ideal of $\mathcal{V}_p$. The family is reduced when all its fibre are reduced. The family is *flat* when the morphism π is flat.

We often refer to a subset $\mathcal{V}$ of $X \times P$ under the term "set-theoretic family" when it needs not to be a subvariety of $G \times P$. In this context we refer to the families defined above under the term "algebraic family."

2.3. DEFINITION. Let G be an algebraic group and P a variety. A family $\mathcal{H}$ of loose subgroups of G is a family of subschemes of G such that each fibre $\mathcal{H}_p$ is supported by an algebraic subgroup of G. A family of subgroups of G is a loose family whose fibres are reduced.

A family $\mathcal{H}$ of loose subgroups of G parametrised by P is trivial when it is of the form $H \times P$, where H is a subgroup of G. Such a family is reduced.

We denote by $\iota : G \to G$ the inverse map of G, and by $\mu : G \times G \to G$ its product map. These morphisms extend to families: we define

$$\iota_P : \begin{array}{ccc} G \times P & \to & G \times P \\ (g, p) & \mapsto & (\iota(g), p) \end{array} \quad \text{and} \quad \mu_P : \begin{array}{ccc} G \times G \times P & \to & G \times P \\ (g_1, g_2, p) & \mapsto & (\mu(g_1, g_2), p). \end{array}$$

If $\mathcal{H}$ is a family of loose subgroups of G parametrised by a variety P, we denote by $\mathcal{H} \times_P \mathcal{H}$ the fibred Cartesian square of $\mathcal{H}$, it is a family of loose subgroups of $G \times G$ parametrised by P whose fibre at p is $\mathcal{H}_p \times \mathcal{H}_p$. The morphism ι_P restricts to $\mathcal{H} \to \mathcal{H}$ and the morphism μ_P to $\mathcal{H} \times_P \mathcal{H} \to \mathcal{H}$.

2.4. EXAMPLE (General form of families). Let X and P be algebraic varieties and $\mathcal{V}$ a family of subschemes of X parametrised by P. The stabiliser $G_\mathcal{V}$ of $\mathcal{V}$ in G defined by

$$G_\mathcal{V} = \left\{ (g, p) \in G \times P \mid \forall x \in \mathcal{V}_p \quad gx \in \mathcal{V}_p \right\}$$

is a family of subgroups of G parametrised by P and each family of subgroups of G can be presented as a stabiliser.

2.5. EXAMPLE (A connected and reducible family). Let T be a torus in $G = \mathbf{SL}_2$ and U a unipotent subgroup of $\mathbf{SL}_2$ such that TU is a Borel subgroup of $\mathbf{SL}_2$. Let also

$$\mathbf{P}^1 = U \cup \{ 0, \infty \}$$

be a projective line compactifying U. Consider the operation of $\mathbf{SL}_2$ on itself defined by conjugation and construct the closure $\mathcal{V}$ in $G \times \mathbf{P}^1$ of

$$\mathcal{V}^o = \left\{ (utu^{-1}, u) \in G \times \mathbf{P}^1 \mid (t, u) \in T \times U \right\}.$$

For each $u \in U$ the fiber $\mathcal{V}_u$ is a torus of G and at $u \in \{ 0, \infty \}$ the fibre is a Borel subgroup of G. Now let $\mathcal{H} = G_\mathcal{V}$ be the stabiliser of $\mathcal{V}$ in G. At each $u \in U$ the fibre $\mathcal{H}_u$ is the normaliser of the torus $\mathcal{V}_u$ in G, that is an extension of the Weyl group of G by $\mathcal{V}_u$. It has two connected components. At $u \in \{ 0, \infty \}$ the fibre $\mathcal{H}_u$ is the normaliser of the Borel subgroup $\mathcal{V}_u$ of G, thus $\mathcal{H}_u = \mathcal{V}_u$. Hence $\mathcal{V}$ has two irreducible components that dominates $\mathbf{P}^1$. We show that is possible to do the quotient of a group by its identity component in families (5.7). Here the quotient space consists of two copies of $\mathbf{P}^1$ glued together along $\{ 0, \infty \}$.

2.6. EXAMPLE (A loose family). We now give an example of a family of loose subgroups of $\mathbf{C}^2$ that is not a family of subgroups. Let C be the plane cuspidal cubic, whose algebra of regular functions is

$$\mathbf{C}[p_1, p_2] / (p_1^2 - p_2^3),$$

and consider the family $\mathcal{H}_0$ of lines in $\mathbf{C}^2$ meeting C in $(0,0)$ and a distinct point:

$$\mathcal{H}_0 = \left\{ (x, y, p) \in \mathbf{C}^2 \times C \setminus \{ (0,0) \} \mid p_1 y - p_2 x = 0 \right\}.$$

The closure $\mathcal{H}$ of $\mathcal{H}_0$ in $\mathbf{C}^2 \times C$ is defined by the ideal generated by

$$p_1 y - p_2 x \quad \text{and} \quad y^2 - p_1 x^2.$$

In this family, each fibre is supported by a line in $\mathbf{C}^2$, but the fibre at $(0,0)$ fails to be reduced. Hence this family of loose subgroups of $\mathbf{C}^2$ is not a family of subgroups of $\mathbf{C}^2$.

3. DIFFERENTIATION AND INTEGRATION OF FAMILIES

3.1. **Differentiation of families of groups.** Let G be an algebraic group, P an algebraic variety and let $\mathcal{H}$ be a family of subgroups of G parametrised by P. We construct the corresponding family of Lie algebras. For this, we consider the trivial family $\mathfrak{g} \times P$ as a subvariety of the tangent bundle to $G \times P$.

3.2. DEFINITION. The family $\mathcal{L}(\mathcal{H})$ of Lie algebras associated to $\mathcal{H}$ is

$$\mathcal{L}(\mathcal{H}) = T\mathcal{H} \cap (\mathfrak{g} \times P).$$

Serre [10] studied the analytic structure of algebraic varieties. We recall some of his results in Appendix A. They allow us to relate the flatness of $\mathcal{H}$ and of $\mathcal{L}(\mathcal{H})$ through the exponential map of G:

3.3. PROPOSITION. *Let $\mathcal{H}$ be a family of connected loose algebraic subgroups of G parametrised by P. This family is flat if, and only if, $\mathcal{L}(\mathcal{H})$ is flat.*

Proof. The exponential $\exp : \mathfrak{g} \to G$ induces a biholomorphism from a neighbourhood W_0 of 0 in $\mathfrak{g}$ and a neighbourhood W_1 of the identity e element in G. It follows from A.3 that the flatness of $\mathcal{H}$ at each point of $\mathcal{H} \cap W_1 \times P$ is equivalent to the flatness of $\mathcal{L}(\mathcal{H})$ at each point of $\mathcal{L}(\mathcal{H}) \cap W_0 \times P$. □

3.4. COROLLARY. *Let $\mathcal{H}$ be a family of connected k-dimensional algebraic subgroups of G parametrised by P. If $\mathcal{H}$ is flat, then there is a morphism $\psi : P \to \mathbf{Gr}(k, \mathfrak{g})$ such that $\mathcal{L}(\mathcal{H})$ is obtained by pulling back the tautological bundle $\mathcal{T} \to \mathbf{Gr}(k, \mathfrak{g})$ through ψ.*

3.5. **Integration of families of Lie algebras.** We study the possibility of going backwards in the process of differentiating a flat family of connected subgroups of G. With the notations of the previous section, $\psi(P)$ is a constructible subset of Λ contained into the set A of algebraic subalgebras of $\mathfrak{g}$. We show that the integration of a flat family of algebraic Lie algebras is possible, let us introduce some notations before we state our main theorem.

For any subset P of A we write $\mathfrak{h}(P)$ for the set-theoretic family of subalgebras of $\mathfrak{g}$ obtained by restricting to P the tautological bundle $\mathcal{T} \to \mathbf{Gr}(k, \mathfrak{g})$. For all $p \in$ A we denote by $\mathfrak{H}_p$ the connected subgroup of G whose Lie algebra is $\mathfrak{h}_p$ and by $\mathfrak{H}(P)$ the set-theoretic family of connected subgroups of G defined by

$$\mathfrak{H}(P) = \left\{ (g, p) \in G \times P \mid g \in \mathfrak{H}_p \right\}.$$

We can now state our main

3.6. THEOREM. *Let P be an irreducible constructible subset of Λ. If P is contained in A, then the closure $\bar{P}$ of P in Λ is also contained in A and $\mathfrak{H}(\bar{P})$ equals $\bar{\mathfrak{H}}(P)$.*

This theorem follows from Proposition 3.12 and Proposition 3.25 below. For now, we give two corollaries of 3.6:

3.7. COROLLARY. *With the notations of 3.6, $\mathfrak{H}(\bar{P})$ is a flat family of connected subgroups of G whose corresponding family of Lie subalgebras of $\mathfrak{g}$ is $\mathcal{L}(\mathfrak{H}(\bar{P})) = \mathfrak{h}(\bar{P})$.*

Proof. We have $\mathcal{L}(\mathfrak{H}(\bar{P})) = \mathfrak{h}(\bar{P})$, but the right hand side is the restriction to $\bar{P}$ of the tautological bundle $\mathcal{T} \to \mathbf{Gr}(k, \mathfrak{g})$, so it is a flat family of linear subspaces of $\mathfrak{g}$. By A.3, the corresponding family of groups is flat. □

3.8. COROLLARY. *The set $\exp \times \mathrm{id}_\Lambda(\mathfrak{h}(\bar{P}))$ is actually $\mathfrak{H}(\bar{P})$, it is in particular algebraic.*

Proof. For complex connected algebraic groups, the exponential map is always surjective, so this result follows from 3.6. □

3.9. **Proof of 3.6 — Construction of families.** We present some elementary constructions of families of groups.

3.10. PROPOSITION. *Let P be an irreducible algebraic variety and $\mathcal{H} \subset G \times P$ a family of subschemes of G parametrised by $\mathcal{H}$. If there exists an open subset V of P such that the restriction of $\mathcal{H}$ to V is a family of loose subgroups of G, then $\mathcal{H}$ itself is a family of loose subgroups of G.*

Proof. Let $\theta : G \times G \times P \to G \times P$ be the morphism defined by $\theta(g_1, g_2, p) = (g_1 g_2^{-1}, p)$. Note that a family $\mathcal{H} \subset G \times P$ of subvarieties of G is a family of subgroups of G if, and only if, $\theta(\mathcal{H} \times_P \mathcal{H}) \subset \mathcal{H}$.

Since $\mathcal{H}$ is an irreducible subvariety of $G \times P$, it is the closure in $G \times P$ of the restriction $\mathcal{H}(V)$ of $\mathcal{H}$ to V. The relation $\theta(\mathcal{H}(V) \times_P \mathcal{H}(V)) \subset \mathcal{H}(V)$ and the continuity of θ imply $\theta(\mathcal{H} \times_P \mathcal{H}) \subset \mathcal{H} \times_P \mathcal{H}$. Hence $\mathcal{H}$ is a family of loose subgroups of G. $\qquad\square$

3.11. PROPOSITION. *Let P be an irreducible algebraic variety and let $\mathcal{H} \subset G \times P$ be a set-theoretic family of subgroups of G. If $\mathcal{H}$ is an irreducible constructible subset of $G \times P$, then there is a dense open subset V of P, such that the restriction of $\mathcal{H}$ to V is an algebraic family of loose subgroups of G parametrised by V.*

Proof. Let U be the largest open subset of the closure of $\mathcal{H}$ in $P \times G$ contained in $\mathcal{H}$. Since $\mathcal{H}$ is constructible, U is dense in the closure of $\mathcal{H}$.

Take any fibre $\mathcal{H}_p$ meeting U. The set $U_p = U \cap \mathcal{H}_p$ is open and dense in $\mathcal{H}_p$, thus for any g in $\mathcal{H}_p$ the open dense subsets $\iota(U_p)g$ and U_p have a common point. This shows that g belongs to $\mu(U_p \times U_p)$, that must be equal to $\mathcal{H}_p$. It implies that $\mu_P(U \times_P U)$ contains U. But the morphism μ_P is flat, hence $\mu_P(U \times_P U)$ is open in $\bar{\mathcal{H}}$ ([6, III, ex. 9.1]) and contained in $\mathcal{H}$. From the maximality of U follows that $U = \mu_P(U \times_P U)$.

The set of parameters p corresponding to a fibre meeting U is constructible and dense in P, hence it contains a dense open subset V of P. The restriction of the set-theoretic family $\mathcal{H}$ to V equals $U \cap (G \times V)$, hence this family is algebraic. $\quad\square$

3.12. PROPOSITION. *Let P be a constructible subset of Λ contained in* A*. If there is an open subset U of $\bar{P}$ contained in P such that $\mathfrak{H}(U)$ is algebraic, then $\bar{P}$ is contained in* A *and $\mathfrak{H}(\bar{P}) = \bar{\mathfrak{H}}(P)$, in particular it is algebraic.*

Proof. Let $\mathcal{V} = \bar{\mathfrak{H}}(U)$ the closure of $\mathfrak{H}(U)$ in $G \times \Lambda$. By 3.10 this is a family of loose subgroups of G. Considering the zero section of $\mathfrak{H}(U)$ makes it clear that the projection of $\mathcal{V}$ to Λ is $\bar{P}$, hence $\mathcal{V}$ is parametrised by $\bar{P}$. We only have to show that the Lie algebra of the reduced fibre of $\mathcal{V}$ at p is $\mathfrak{h}_p$, since this implies at once that p belongs to A and that $\mathcal{V}_p = \mathfrak{H}_p$.

To study $\mathcal{V}$ closely we consider neighbourhoods W_0 of 0 in $\mathfrak{g}$ and W_1 of e in G that are biholomorphic through the exponential map of G. Now let $p \in \bar{\mathcal{H}} \setminus U$ and x a closed point in $\mathcal{V}_p \cap W_1$. Since x belongs to the closure of $\mathfrak{H}(U)$, there is a curve C in $\mathfrak{H}(U)$ such that $x \in \bar{C}$. Looking through the exponential we see that $\exp^{-1}(C \cap W_1)$ is contained in $\mathfrak{h}(U)$. But the family $\mathfrak{h}$ is locally trivial and the closure of $\mathfrak{h}(U)$ in $\mathfrak{g} \times \mathbf{Gr}(k, \mathfrak{g})$ is $\mathfrak{h}(\bar{U}) = \mathfrak{h}(\bar{P})$. It follows that $\exp^{-1} x$ belongs to $\mathfrak{h}_p$ and that the Lie algebra of the group supported by $\mathcal{V}_p$ must be equal to $\mathfrak{h}_p$. $\qquad\square$

3.13. Corollary. *Let P be a constructible subset of Λ contained in* A. *The set-theoretic family $\mathfrak{H}(\bar{P})$ of connected subgroups of G is algebraic if the set-theoretic family $\mathfrak{H}(Q)$ is algebraic, where Q is*

  (1) *either a dense open subset of $\bar{P}$;*
  (2) *or the closure of $G \cdot P$ in Λ;*
  (3) *or the intersection $\bar{P} \cap T$ of $\bar{P}$ with a subvariety T of Λ such that*

$$P \cap (G \cdot (P \cap T))$$

  is dense in $\bar{P}$.

3.14. Corollary. *Let G be an algebraic group, P a variety, and $\mathcal{H}^i$ ($i \in \{1,2\}$) two families of loose subgroups of G parametrised by P. If for each p in P the group $\mathcal{H}^1_p$ normalises $\mathcal{H}^2_p$ and $\mathcal{H}^1_p \cap \mathcal{H}^2_p$ is finite, then the restriction of the set-theoretic family $\mathcal{H}^1\mathcal{H}^2$ defined by*

$$(\mathcal{H}^1\mathcal{H}^2)_p = \mathcal{H}^1_p\mathcal{H}^2_p$$

to a suitable dense open subset of P is algebraic.

3.15. **Proof of 3.6 — Levi-Malcev decomposition.** We study the behaviour of Levi-Malcev decomposition in families of Lie algebras. The rigidity of semi-simple Lie algebras proved by Richardson [9] implies the semi-continuity of the application sending a Lie algebra to the isomorphism class of its semi-simple part. Let us recall this rigidity theorem:

3.16. Theorem (Richardson [9, 9.2 and 9.6]). *Let P be a subvariety of Λ and $p \in P$. We denote by $\mathfrak{s}$ a semi-simple Lie algebra contained in $\mathfrak{H}_p$. Then there is an analytic neighbourhood W of p in Λ and an analytic map $\psi : W \to \mathrm{Aut}(\mathfrak{g})$ such that $\psi(p)$ is the identical transformation and $\psi(q)$ maps $\mathfrak{s}$ into $\mathfrak{H}_q$ for all $q \in W$.*

Let S be the set of isomorphism classes of semi-simple Lie algebras. We denote the isomorphism class of a semi-simple Lie algebra $\mathfrak{s}$ by $S(\mathfrak{s})$. The set S is countable, ordered by the relation

$$S(\mathfrak{s}_1) \leq S(\mathfrak{s}_2) \quad \Longleftrightarrow \quad \text{there is an injection } \mathfrak{s}_1 \to \mathfrak{s}_2.$$

Each non-empty subset of S has at least one minimal element. For each Lie algebra $\mathfrak{h}$ we denote by $S(\mathfrak{h})$ the isomorphism class of a semi-simple part of its Levi-Malcev decomposition.

3.17. Proposition. *Let $P \subset \Lambda$ be a locally closed subvariety of Λ. If $S(\mathfrak{s})$ is a minimal element of*

$$\left\{ S(\mathfrak{h}_p) \mid p \in P \right\},$$

then the set

$$P' = \left\{ p \in P \mid S(\mathfrak{h}_p) = S(\mathfrak{s}) \right\}$$

is Zariski closed in P.

Proof. Consider the variety

$$\mathrm{Hom}_{\mathcal{L}}(\mathfrak{s}, \mathfrak{h}(P)) = \left\{ (u, p) \in \mathrm{Hom}_{\mathcal{L}}(\mathfrak{s}, \mathfrak{g}) \times P \mid \mathrm{Im}\, u \subset \mathfrak{h}_p \right\}.$$

Since $S(\mathfrak{s})$ is minimal along the isomorphism classes of the semi-simple factors occuring in Levi-Malcev decomposition of fibres of $\mathfrak{h}(P)$, a morphism from $\mathfrak{s}$ to $\mathfrak{h}_p$ is either zero or injective. The set P' is thus the projection on P of

$$\mathrm{Hom}_{\mathcal{L}}(\mathfrak{s}, \mathfrak{h}(P)) \setminus \{0\} \times P$$

and is constructible. By Richardson's Theorem the sub-levels of the map from P to S defined by $p \mapsto S(\mathfrak{h}_p)$ are closed in the analytic topology. Hence P' is closed in the analytic topology and constructible, it must also be closed in the Zariski topology. $\qquad\square$

3.18. COROLLARY. *The map $\Lambda \to S$ sending a Lie algebra to the isomorphism class of its semi-simple factors is lower semi-continuous, in the Zariski topology.*

3.19. **Proof of 3.6 — Integration of solvable families.**

3.20. PROPOSITION. *Let P be a subvariety of Λ. If P is contained in A and if each fibre of $\mathfrak{h}(P)$ is a solvable subgroup of G, then the map sending $p \in P$ to the rank of $\mathfrak{h}_p$ is lower semi-continuous, in the Zariski topology.*

We start with two lemmas:

3.21. LEMMA. *If $\mathfrak{h}$ is an algebraic Lie subalgebra of $\mathfrak{g}$, then any maximal torus of $\mathfrak{h}$ is algebraic.*

Proof. The algebraic Lie algebra $\mathfrak{h}$ contains the algebraic hull of any of its Lie subalgebra, and the algebraic hull of a torus is again a torus. $\qquad\square$

3.22. LEMMA. *Let $\mathfrak{s}$ be a semi-simple Lie algebra and $\mathfrak{a}$ a subalgebra of a maximal torus of $\mathfrak{s}$. If $\mathfrak{a}$ is algebraic, then the restriction to $\mathfrak{a}$ of the Killing form of $\mathfrak{s}$ is regular.*

Proof. Let Φ be the roots of $\mathfrak{s}$ with respect to a maximal torus of $\mathfrak{s}$ containing $\mathfrak{a}$. The Killing quadratic form evaluates on $a \in \mathfrak{a}$ to

$$\sum_{\alpha \in \Phi} \alpha^2(a).$$

But $\mathfrak{a}$ is algebraic, so there is a basis of it such that the $\alpha \in \Phi$ take integral values on its members. On the rational linear subspace of $\mathfrak{a}$ spanned by this basis, the Killing quadratic form is positive definite, hence regular. The space $\mathfrak{a}$ is the complexification of this rational linear subspace $\mathfrak{a}_{\mathbf{Q}}$ and the Killing form is then the complexification of a regular form on $\mathfrak{a}_{\mathbf{Q}}$, it must be regular as well. $\qquad\square$

Proof of 3.20. By Ado's Theorem we can assume that our Lie algebras are subalgebras of some simple Lie algebra $\mathfrak{s}$. It follows from the two previous lemmas that for $p \in P$ the unipotent radical of $\mathfrak{h}_p$ is the kernel of the map $\mathfrak{h}_p \to \mathfrak{h}_p^*$ defined by the Killing form of $\mathfrak{s}$. The conclusion the follows from Chevalley's Theorem on the dimension of the fibres of a morphism. $\qquad\square$

3.23. THEOREM. *Let P a locally closed sub-variety of Λ. If P is contained in A and each fibre $\mathfrak{h}_p$ is solvable, then the set-theoretic family $\mathfrak{H}(P)$ of subgroups of G is algebraic.*

Proof. We may assume that $G \cdot P = P$ (3.13 point 2). Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$. The set T of parameters $p \in P$ such that $\mathfrak{h}_p$ is contained in $\mathfrak{b}$ is constructible and meets all orbit of P under the operation of G. By 3.13 point 3, we only need to prove the proposition in the case where $P = T$. The Lie subalgebra $\mathfrak{b}$ of G is algebraic, let B be an algebraic subgroup of G such that $\mathcal{L}(B) = \mathfrak{b}$, note that $B \cdot P = P$.

Let $\mathfrak{s}$ be a maximal torus of $\mathfrak{b}$, $\mathfrak{u}$ its unipotent radical, and let m be the maximal rank of the solvable algebras $\mathfrak{h}_p$ for $p \in P$. Since the rank is semi-continuous (3.20) we may use 3.13 a last time to restrict ourselves to the case where each p in P corresponds to a fibre $\mathfrak{h}_p$ meeting $\mathfrak{s}$ along a m-dimensional subspace.

Since $\mathfrak{s} \cap \mathfrak{h}_p$ is an algebraic subtorus of $\mathfrak{s}$, it can not move and there is a subtorus $\mathfrak{s}'$ of $\mathfrak{s}$ such that $\mathfrak{s} \cap \mathfrak{h}_p = \mathfrak{s}'$ for all $p \in P$ and S' a connected subgroup of G whose Lie algebra is $\mathfrak{s}'$.

Now we denote by $\mathfrak{h}^u(P)$ the family of unipotent radical of the fibres of $\mathfrak{h}(P)$, it is defined by

$$\mathfrak{h}^u(P) = \mathfrak{h}(P) \cap (\mathfrak{u} \times P).$$

The exponential map restricts to an algebraic morphism on $\mathfrak{u}$, thus

$$\mathfrak{H}^u(P)(\exp \times \operatorname*{id}_P(\mathfrak{h}^u(P)))$$

is an algebraic family of subgroups of G. We conclude with 3.14 that

$$\mathfrak{H}_p = \mathfrak{H}_p^u S'$$

defines an algebraic family of subgroups of G parametrised by P, and $\mathcal{L}(\mathfrak{H}(P)) = \mathfrak{h}(P)$. □

3.24. **Proof of 3.6 — Integration of general families.**

3.25. Proposition. *Let P be a quasi-projective subvariety of Λ. If P is contained in A then the set-theoretic family $\mathfrak{H}(P)$ is an algebraic subvariety of $G \times P$.*

Proof. Taking 3.13, point 1 and 3.18 into account, we may freely assume that all the semi-simple factors of the fibres $\mathfrak{h}(P)$ are isomorphic to a fixed semi-simple algebra $\mathfrak{s}$. Moreover all semi-simple factors of a Lie algebra are equivalent under the operation of the unipotent isomorphisms of this algebra. We may invoke 3.11, point 3 to restrict ourselves to the case where $\mathfrak{s}$ is a subalgebra of each fibre of $\mathfrak{h}(P)$. The unipotent radical of a Lie algebra is the kernel of its Killing form, hence the set-theoretic family $\mathfrak{h}^r(P)$ of Lie subalgebras of $\mathfrak{g}$ whose fibre at p is the unipotent radical of $\mathfrak{h}_p$ is algebraic. By 3.23 there is an algebraic family $\mathcal{H}$ of connected subgroups of G parametrised by P such that $\mathcal{L}(\mathcal{H}) = \mathfrak{h}^r(P)$. Let S be the connected subgroup of G whose Lie algebra is $\mathfrak{s}$, we conclude by 3.14 that

$$\mathfrak{H}(P) = (S \times P)\mathcal{H},$$

hence it is algebraic. □

4. Moduli space and universal family

4.1. **Bouquet of algebraic varieties.** The moduli space we construct is not an algebraic variety, but is a ringed space obtained by gluing together possibly infinitely many algebraic varieties along closed subsets.

4.2. Definition. Let $\mathcal{X}$ be a set of irreducible algebraic varieties. A *gluing pattern* (F, h) for $\mathcal{X}$ assigns to each pair (X, Y) of elements of $\mathcal{X}$ a closed subset F_{YX} of X and an isomorphism $h_{YX} : F_{YX} \to F_{XY}$, tied to satisfy the following properties:

(1) For all $x \in \mathcal{X}$, one has $f_{XX} = X$ and h_{XX} is the identity map.
(2) For any triple (X, Y, Z) of elements of $\mathcal{X}$, the composition $h_{ZY}h_{YX}$ agrees with h_{ZX} whenever it can be defined.

A gluing pattern on a set of algebraic varieties defines an equivalence relation on the disjoint sum of the elements of $\mathcal{X}$. The quotient space $\mathcal{X}/(F, h)$ of this equivalence relation carries a natural structure of ringed space.

The Zariski topology on $\mathcal{X}/(F,h)$ is the final topology on the set $\mathcal{X}/(F,h)$, with respect to the natural injections $X \to \mathcal{X}/(F,h)$. Since the various F_{YX} are closed, the maps $X \to \mathcal{X}/(F,h)$ are closed immersions, the set of irreducible components of $\mathcal{X}/(F,h)$ is $\mathcal{X}$, and a subset U of $\mathcal{X}/(F,h)$ is open if, and only if, for all $X \in \mathcal{X}$, the trace $U \cap X$ is open in X.

If U is such an open subset of $\mathcal{X}/(F,h)$, the ring $\mathcal{O}(U)$ of regular functions on U is the subring of the ring

$$\prod_{X \in \mathcal{X}} \mathcal{O}_X(U \cap X)$$

spanned by the elements that define a function on the quotient $\mathcal{X}/(F,h)$.

4.3. DEFINITION. A *bouquet of algebraic varieties* is a ringed space $\mathcal{X}/(F,h)$.

The name is suggested by the analogy between the gluing pattern of the moduli space for 1-dimensional connected subgroups of $\mathbf{SL}_3$ described in 6.12 and the one of a *bouquet of circles* in Topology.

Defining a morphism between two bouquets is the same as defining a set of morphisms between their irreducible components, that are compatible with their gluing patterns. In the category of bouquets, one can again glue together a family of bouquets, and obtain a new bouquet.

Note that if E is a vector space and k an integer, the tautological bundle over the Grassmann variety of k-dimensional subspaces of E is a final object in the category of flat families of k-dimensional subspaces of E over a bouquet.

4.4. REMARK. Bouquets of algebraic varieties are called *infinite-dimensional varieties* by Shafarevich [11] or *ind-varieties* by Kumar [7]. However, these authors seem to have been interested by the case where the variety obtained after gluing is irreducible and infinite dimensional at its general point, while the varieties we are interested in are finite dimensional at their general point, and have infinitely many irreducible components. We thus felt that it would be misleading to refer to the varieties we encountered under one of the afore mentioned names.

4.5. **Construction.** Let $\mathcal{X}_{\mathfrak{P}}$ be the set of all irreducible subvarieties of Λ that are contained in A and $\mathcal{X}_{\mathfrak{H}}$ the set

$$\mathcal{X}_{\mathfrak{H}} = \{ \mathfrak{H}(P) \mid P \in \mathcal{X}_{\mathfrak{P}} \}$$

of irreducible subvarieties of $G \times \Lambda$ (3.6). We define a gluing pattern $(A,h)_{\mathfrak{P}}$ (resp. $(A,h)_{\mathfrak{H}}$) for $\mathcal{X}_{\mathfrak{P}}$ (resp. $\mathcal{X}_{\mathfrak{H}}$) by assigning to each pair (P_1, P_2) of elements of $\mathcal{X}_{\mathfrak{P}}$ (resp. $\mathcal{X}_{\mathfrak{H}}$) the closed subset $P_1 \cap P_2$ of P_1 and the morphism from $P_1 \cap P_2$ to $P_2 \cap P_1$ obtained by restricting the identical transform of Λ (resp. $G \times \Lambda$).

4.6. DEFINITION. The bouquet of algebraic varieties $\mathfrak{P} = \mathcal{X}_{\mathfrak{P}}/(A,h)_{\mathfrak{P}}$ is the *moduli space for connected subgroups of G* and $\mathfrak{H} = \mathcal{X}_{\mathfrak{H}}/(A,h)_{\mathfrak{H}}$ is the *universal family for connected subgroups of G.*

The universality of these objects lies in the following

4.7. THEOREM. *Let P be an algebraic variety, and denote by*

(1) *F the set of flat families of k-dimensional connected subgroups of G parametrised by P;*

(2) *M the set of morphisms of ringed spaces from P to $\mathfrak{P}$.*

Then the map $F \to M$ sending a family $\mathcal{H}$ to the morphism to Λ associated with $\mathcal{L}(\mathcal{H})$ and the map from $M \to F$ sending a morphism ψ to the family $\psi^* \mathfrak{H}$ define a natural correspondance between F and M.

4.8. REMARK. One can introduce the category of flat families of connected sub-schemes of G parametrised by bouquets of algebraic varieties. It follows from Theorem 4.7 that $\mathfrak{H} \to \mathfrak{P}$ is a final object in this category.

5. FAMILY OF IDENTITY COMPONENTS

The set of connected components of an algebraic group is naturally identified with the quotient group of the group by its identity component. We show that the set-theoretic family $\mathcal{H}^o$ of the identity components of the fibres of $\mathcal{H}$ is an algebraic family, and the family of quotients $\mathcal{H}/\mathcal{H}^o$ can be constructed for the general parameter.

5.1. **Closedness.**

5.2. PROPOSITION. *Let G be an algebraic group, P a quasi-projective variety, and $\mathcal{H}$ a family of loose subgroups of G parametrised by P. For each $p \in P$ there is a closed subset F_p of $\mathcal{H}$ that contains all the identity components of the fibres of $\mathcal{H}$ and meets $\mathcal{H}_p$ exactly along its identity component $\mathcal{H}_p^o$.*

Proof. Let K_p be any projective completion of G such that the connected components of the closure of $\mathcal{H}_p$ in K are the closures of the connected components of $\mathcal{H}_p$ (see B.4), and let Q be any projective completion of P. We consider the inclusion of $G \times P$ in $K \times Q$ and the Stein factorisation of the projection to Q of the closure X of the image of $\mathcal{H}$ in $K \times Q$:

$$X \xrightarrow{\ f\ } Q' \to Q.$$

Now let $F_p = \mathcal{H} \cap f^{-1}(f(\{\, \theta_p(e) \,\} \times Q))$, this is a closed subset of $\mathcal{H}$ that contains the identity component of each fibre of $\mathcal{H}$ and meets $\mathcal{H}_p$ precisely along $\mathcal{H}_p^o$. $\quad\square$

5.3. COROLLARY. *Let G be an algebraic group, P a variety, and $\mathcal{H}$ a family of (resp. loose) subgroups of G parametrised by P. The set-theoretic family*

$$\mathcal{H}^o = \left\{\, (g, p) \in \mathcal{H} \ \middle|\ g \in \mathcal{H}_p^o \,\right\}$$

of the identity components of the fibres of $\mathcal{H}$ is closed in $\mathcal{H}$ and defines an algebraic family of (resp. loose) subgroups of G parametrised by P.

Proof. Being closed is a local property, hence we may assume that P is an affine variety. By 5.2 we have

$$\mathcal{H}^o = \bigcap_{p \in P} F_p$$

which shows that $\mathcal{H}^o$ is a closed subset of $G \times P$. Since each fibre $\mathcal{H}_p^o$ of $\mathcal{H}^o$ is open in $\mathcal{H}_p$, the former is reduced when the latter is. $\quad\square$

5.4. **Quotient.**

5.5. PROPOSITION. *Let G be an algebraic group and P be an affine variety. For all family $\mathcal{H}$ of loose subgroups of G parametrised by P, there exists a projective variety X, an embedding of $G \times P$ in $X \times P$ and an open subset U of P such that for all $p \in U$ the connected components of the closure of $\mathcal{H}_p$ in K are the closures of the connected components of $\mathcal{H}_p$.*

In other words, there exists a compactification of $\mathcal{H}$ such that the various connected components of a general fibre in $\mathcal{H}$ have no common points at infinity.

5.6. LEMMA. *Let L be an affine group and P an affine variety. For all family $\mathcal{H}$ of loose subgroups of L parametrised by P, there is*

(1) *a finite partition $\mathcal{Q}$ of P;*
(2) *representations $(V_Q)_{Q \in \mathcal{Q}}$ of G;*
(3) *morphisms $d_Q : P \to V_Q$;*

such that for all $Q \in \mathcal{Q}$ and all $p \in Q$ we have $d_Q(p) \neq 0$ and the stabiliser of $[d_Q(p)] \in \mathbf{P}(V_Q)$ in G is $\mathcal{H}_p^o$.

Proof. Let E be a finite dimensional L-stable linear subspace of $\mathcal{O}_L \otimes \mathcal{O}_P$ containing generators of the ideal I of the family $\mathcal{H}^o$ of identity components of $\mathcal{H}$, and let $W = E \cap I$. For each p in P, we denote by $\epsilon(p)$ the partial evaluation morphism from $\mathcal{O}_L \otimes \mathcal{O}_P$ to $\mathcal{O}_L$ sending a function f to the function $\epsilon(p)(f)$ defined by $\epsilon(p)(f)(g) = f(g, p)$. The linear subspace $E = \epsilon(P)(E)$ of $\mathcal{O}_L$ spanned by partial evaluations is finite dimensional, and contains all the spaces

$$W_p = \epsilon(p)(W), \qquad p \in P.$$

Let $\delta : P \to \mathbf{N}$ be the function defined by $p \mapsto \dim W_p$ and put $m = \max \{ \delta(P) \}$. For each p in P there is a subsystem of $(\epsilon(p)(w_1), \ldots, \epsilon(p)(w_m))$ that is a basis of W_p. Let γ be a function choosing for each $p \in P$ indices in $\{ 1, \ldots, m \}$ of such a subsystem. Let $\mathcal{Q}$ be the partition of P defined by the function $\delta \times \gamma$. For each $Q \in \mathcal{Q}$ we set

$$V_Q = \Lambda^{\delta(Q)} E$$

and, with $\gamma(Q) = \left\{ i_1 < \cdots < i_{\delta(Q)} \right\}$,

$$d_Q(p) = \epsilon(p)(w_{i_1}) \wedge \cdots \wedge \epsilon(p)(w_{i_{\delta(Q)}}).$$

When $p \in P$ belongs to $Q \in \mathcal{Q}$, the map d_Q does not vanish at p. The ideal of $\mathcal{O}_G$ spanned by

$$\left\{ \epsilon(p)(w_{i_1}), \ldots, \epsilon(p)(w_{i_{\delta(Q)}}) \right\}$$

is the ideal of $\mathcal{H}_p^o$, hence the stabiliser of $[d_Q(p)] \in \mathbf{P}(V_Q)$ in G is $\mathcal{H}_p^o$ (see [1, 3.8 and 5.1]). $\square$

Proof of 5.5. According to the Chevalley structure Theorem [4], G is an extension of an abelian variety A by an affine group L. We apply the lemma to L and put

$$V = \bigoplus_{Q \in \mathcal{Q}} V_Q \quad \text{and} \quad d = \bigoplus_{Q \in \mathcal{Q}} d_Q.$$

We denote by pr_Q the projection from $\mathbf{P}(V) \times \mathbf{P}(V_Q)$ to $\mathbf{P}(V_Q)$ of the closure of the graph of the projection from $\mathbf{P}(V)$ to $\mathbf{P}(V_Q)$. The latter is a rational G-invariant map that is defined at $[d(p)]$ for each $p \in Q$.

Starting with K_0 any projective compactification of G we let $X = K_0 \times \mathbf{P}(V)$ and define the embedding

$$\theta : \begin{array}{ccc} G \times P & \to & X \times P \\ (g, p) & \mapsto & (g, gd(p), p). \end{array}$$

Now choose $p \in P$, $Q \in \mathcal{Q}$ containing p and consider two different connected components $h_1 \mathcal{H}_p^o$ and $h_2 \mathcal{H}_p^o$, where $\{ h_1, h_2 \} \subset L$, of $\mathcal{H}_p$. The stabilizer of $[d_Q(p)]$ in L is $L \cap \mathcal{H}_p^o$ (B.3), hence the two subvarieties

$$h_1 \mathcal{H}_p^o d_Q(p) \quad \text{and} \quad h_2 \mathcal{H}_p^o d_Q(p)$$

of $\mathbf{P}(V_Q)$ are disjoint. Since they are isomorphic to the abelian variety $\mathcal{H}_p^o / (\mathcal{H}_p^o \cap L)$, they are furthermore closed in $\mathbf{P}(V_Q)$. We conclude that $h_1 \mathcal{H}_p^o$ and $h_2 \mathcal{H}_p^o$ are respectively mapped by $(\mathrm{id}_G \times \mathrm{pr}_Q \times \mathrm{id}_P) \circ \theta$ into the two disjoint closed subvarieties

$$K_0 \times \left\{ [h_1 \mathcal{H}_p^o d_Q(p)] \right\} \times \{ p \} \quad \text{and} \quad K_0 \times \left\{ [h_2 \mathcal{H}_p^o d_Q(p)] \right\} \times \{ p \}$$

of $G \times \mathbf{P}(V_Q) \times P$, so their closures in $X \times P$ remain disjoint. □

5.7. COROLLARY. *Let G be an algebraic group, P a variety, and $\mathcal{H}$ a family of loose subgroups of G parametrised by P. The projection $\mathcal{H} \to P$ admits a Stein factorisation*

$$\mathcal{H} \xrightarrow{\pi_0} \pi_0(\mathcal{H}) \to P$$

of $\mathcal{H} \to P$: the morphism $\pi_0(\mathcal{H})$ is an algebraic variety, the morphism $\mathcal{H} \to \pi_0(\mathcal{H})$ has connected fibres and the morphism $\pi_0(\mathcal{H}) \to P$ is proper and finite.

The fibre of $\pi_0(\mathcal{H})$ at $p \in P$ is the group $\pi_0(\mathcal{H}_p)$, so we were able to perform the quotient $\mathcal{H}_p / \mathcal{H}_p^o$ in all the fibres of the family $\mathcal{H}$ simultaneously.

Proof. Let $j : \mathcal{H} \to X \times P$ the embedding provided by 5.5, it is enough to define π_0 on an open cover of P so we may restrict ourselves to the case where P is affine, and admits a projective compactification Q. Now the wished Stein factorisation is obtained by restricting to $\mathcal{H} \to P$ the Stein factorisation of the projection on Q of the closure of $j(\mathcal{H})$ in $X \times Q$. □

6. EXAMPLES

6.1. **Families of subgroups of tori.** Let S be a torus. A 1-parameter subgroup λ of S is characterized by its initial tangent vector $\lambda'(1) \in \mathfrak{s}$. The correspondence between 1-parameter subgroups and initial tangent vectors induces a group isomorphism between the group of all 1-parameter subgroups of S and the lattice $Y \subset \mathfrak{s}$ of their initial tangent vectors. A connected subgroup of S is a torus as well, and is characterized by the set of its 1-parameter subgroups, hence we have a natural bijection between the set Γ of k-dimensional connected subgroups of S and $\mathbf{Gr}(k, Y) \subset \mathbf{Gr}(k, \mathfrak{s})$.

6.2. PROPOSITION. *Let S be a torus, P a variety and $\mathcal{H}$ a family of loose subgroups of S parametrised by P. If each irreducible component of $\mathcal{H}$ dominates P, then $\mathcal{H}$ is a trivial family.*

Proof. We first assume that $\mathcal{H}$ is irreducible, so that each fibre is connected (5.3). Let then $\mathcal{L}(\mathcal{H}) \subset \mathfrak{s} \times P$ be the algebraic family whose fiber at $p \in P$ is the Lie algebra of $\mathcal{H}_p$. This family is equidimensional, hence there is an open subset P^o of P such that its restriction to P^o is flat. The universal property of the Grassmann variety yields a morphism $\psi : P^o \to \mathbf{Gr}(k, \mathfrak{g})$ such that $\mathcal{L}(\mathcal{H})$ is the pullback of the tautological bundle on $\mathbf{Gr}(k, \mathfrak{g})$. But ψ takes its values in $\mathbf{Gr}(k, Y)$ which, in the euclidean topology, is a totally discontinuous topological space. Hence ψ is constant and $\mathcal{H}$ is a trivial family above P^o, it must then be trivial above all of P.

We now treat the case where $\mathcal{H}$ may have multiple connected components. By the case where $\mathcal{H}$ is irreducible there is a subtorus S' of S such that $\mathcal{H}^o = S' \times P$. By Baire's Theorem there is an exponant $N \in \mathbf{N}$ that kills all the fibres of the image of $\mathcal{H}$ in $S/S' \times P$. The subgroup C of S consisting of elements killed by N is finite and if $C' = \left\{ s \in C \mid \exists p \in P \ s \in \mathcal{H}_p \right\}$ then $\mathcal{H} = C'S' \times P$, so it is a trivial family. $\square$

6.3. **Families of subgroups of abelian varieties.** A variation (6.8) on the *rigidity lemma* for projective morphisms [8] and the Stein factorization enable us to show that families of subgroups of abelian varieties are constant (6.7).

6.4. PROPOSITION. *Let A be an abelian variety, P a variety and $\mathcal{H}$ a family of (loose) subgroups of A parametrised by P. The set-theoretic family*

$$\mathcal{H}^o = \left\{ (a, p) \in A \times P \ \middle| \ a \in \mathcal{H}_p^o \right\}$$

of subgroups of A is an algebraic family of (loose) subgroups of A.

Proof. We show that $\mathcal{H}^o$ is a closed subset of $\mathcal{H}$. Since being closed in $A \times P$ is a local property, we may assume that P is affine, and therefore admits a projective completion X.

Now consider the Stein factorisation

$$\bar{\mathcal{H}} \xrightarrow{\ f\ } X' \to X$$

of the projection to X of the closure $\bar{\mathcal{H}}$ of $\mathcal{H}$ in $A \times X$: the variety X' is projective, the morphism f has connected fibers and $X' \to X$ is finite. We have then

$$\mathcal{H}^o = \mathcal{H} \cap f^{-1}(f(\{\, e \,\} \times X),$$

which shows that this set is closed. For each p, the map $\mathcal{H}_p^o \to \mathcal{H}_p$ is an open immersion, hence each $\mathcal{H}_p^o$ is reduced when each $\mathcal{H}_p$ is. $\square$

6.5. COROLLARY. *Let A be an abelian variety, P a variety and $\mathcal{H}$ a family of (loose) subgroups of A parametrised by P. If $\mathcal{H}$ is irreducible, then each of its fibre is connected.*

6.6. COROLLARY. *Let A be an abelian variety, P a variety and $\mathcal{H}$ a family of loose subgroups of A parametrised by P. If each irreducible component of $\mathcal{H}$ dominates P, then $\mathcal{H}^o$ is irreducible.*

Proof. The fibres of $\mathcal{H}^o \to P$ are irreducible, let n be their minimal dimension. By Chevalley's Theorem for the dimensions of the fibres of a morphism, the general fibre of $\mathcal{H}^o \to P$ has dimension n. There exists an irreducible component $\mathcal{H}^1$ of $\mathcal{H}^o$ whose general fibre has dimension n. Hence $\mathcal{H}^1$ and $\mathcal{H}^o$ agree above a dense open subset of P. Since each irreducible $\mathcal{H}^2$ component of $\mathcal{H}^o$ dominates P,

it meets $\mathcal{H}^1$ along a dense open subset of $\mathcal{H}^2$, so that $\mathcal{H}^2 \subset \mathcal{H}^1$. We conclude that $\mathcal{H}^o$ is irreducible. $\qquad\square$

6.7. PROPOSITION. *Let A be an abelian variety, P a variety and $\mathcal{H}$ a family of loose subgroups of A parametrised by P. If each irreducible component of $\mathcal{H}$ dominates P, then $\mathcal{H}$ is a trivial family.*

We start with two lemmas:

6.8. LEMMA (Rigidity lemma variant). *Let X be a complete variety, Y and Z any varieties, $\mathcal{V} \subset X \times Y$ an irreducible family of connected subvarieties of X parametrised by Y admitting a section, and $f : \mathcal{V} \to Z$ a morphism. If there exists a member of $\mathcal{V}$ contracted to a point by f, then there is a morphism $g : Y \to Z$ such that $f = g \circ \mathrm{pr}_Y$, where pr_Y is the projection of $\mathcal{V}$ to Y.*

Proof. Let $s : Y \to \mathcal{V}$ be a section of $\mathcal{V}$ and let $g : Y \to Z$ be the map defined by $g(y) = f(s(y), y)$. Since $\mathcal{V}$ is irreducible, we only need to show that f and $g \circ \mathrm{pr}_Y$ agree on a dense open subset of $\mathcal{V}$.

Let y_0 be a parameter such that $\mathcal{V}_{y_0}$ is contracted to a point z_0 by f, and let U an affine neighbourhood of z_0 in Z. Let $F = Z \setminus U$ its complement and $G = \mathrm{pr}_Y f^{-1}(F)$. Since $f^{-1}(F)$ is a closed subset of $X \times Y$ and X is complete, the projection G of $f^{-1}(F)$ on Y is closed. Further $y_0 \notin G$ since $f(\mathcal{V}_{y_0}) = z_0$. Therefore $V = Y - G$ is a non-empty open subset of Y, for each $y \in V$ the complete connected variety $\mathcal{V}_y$ gets mapped by f into the affine variety U, hence to a single point of U. But this means that for any $x \in X$, $y \in V$, we have $f(x, y) = f(s(y), y) = g \circ \mathrm{pr}_Y(x, y)$. $\qquad\square$

6.9. LEMMA. *Let A be an abelian variety, P a variety and $\mathcal{H}$ a family of loose subgroups of A parametrised by P. If $\mathcal{H}$ is irreducible, then it is a trivial family.*

Proof. Since $\mathcal{H}$ is irreducible, it has connected fibres (6.5). Let p be a point of P dominated by a connected fiber $\mathcal{H}_p$ of $\mathcal{H}$. According to a theorem of Chow [3, Theorem 1] there exists an abelian variety Z and a group homomorphism $f_0 : A \to Z$ whose kernel is $\mathcal{H}_p$. Let f be the restriction to $\mathcal{H}$ of $f_0 \times \mathrm{id}_P$. The fiber at p is contracted by f to the unit element of Z, hence it follows from Lemma 6.8 that f can be factorized out through the projection pr_P of $\mathcal{H} \to P$. Consequently, each fiber $\mathcal{H}_q$ of $\mathcal{H}$ is contained in the kernel $\mathcal{H}_p$ of f_0. We conclude with the theorem on the dimension of the fibers of a morphism that there is an open subset V of P above which each fiber actually equals $\mathcal{H}_p$. But $\mathcal{H}$ is irreducible and equals the closure of $\mathcal{H}_p \times V$ in $A \times P$ that is, that the family $\mathcal{H}$ is trivial. $\quad\square$

6.10. LEMMA. *Let A be an abelian variety, P a variety and $\mathcal{H}$ a family of loose finite subgroups of A parametrised by P. If each irreducible component of $\mathcal{H}$ dominates P, then $\mathcal{H}$ is a trivial family.*

Proof. By Baire's Theorem, there is a common exponent $N \in \mathbf{N}$ that kills all the fibres of $\mathcal{H}$. But the subgroup B of A consisting of its elements of order N is finite, and $\mathcal{H}$ is a subvariety of $B \times P$. Since each irreducible component of $\mathcal{H}$ dominates P, this subvariety is of the form $B' \times P$ for some subgroup B' of B. $\quad\square$

Proof of 6.7. By 6.6 the family $\mathcal{H}^o$ of connected components of $\mathcal{H}$ is irreducible, and by 6.9 it must be trivial. Let B be the connected closed subgroup of A such

that $\mathcal{H}^o = B \times P$. According to Chow [3, Theorem 1] there is an abelian variety Z and a morphism of algebraic groups $f : A \to Z$ whose kernel is B.

We denote by f_P the morphism $A \times P \to Z \times P$ associated to f and by $f_P(\mathcal{H})$ the image of $\mathcal{H}$ through this morphism. It is a closed subvariety of $Z \times P$, and defines a family of loose finite subgroups of Z whose irreducible components dominate P. We conclude by 6.10 that his family is trivial, of the form $B' \times P$. Hence $\mathcal{H} = f^{-1}(B') \times P$ is also a trivial family. □

6.11. **Universal family for $\mathbf{SL}_2$.** We proceed in two steps, analysing first the one-dimensional subgroups and second the two-dimensional subgroups. We let $G = \mathbf{SL}_2$.

One-dimensional subgroups. The operation of G on $\mathbf{P}(\mathfrak{g}) = \mathbf{Gr}(k, \mathfrak{g})$ has two orbits: the closed orbit is the set of nilpotent elements in $\mathbf{P}(\mathfrak{g})$ and the open one is the set of semi-simple elements. Thus each one-dimensional Lie subalgebra of $\mathfrak{g}$ is algebraic: $A_1 = \mathbf{P}(\mathfrak{g})$ and the universal family of one-dimensional subgroups of G is a smooth irreducible variety of dimension 3.

Two dimensional subgroups. A 2-dimensional subgroup of G is a Borel subgroup B. According to the theory of reductive groups, their set Γ is the variety $G/B \simeq \mathbf{P}^1$. The universal family of 2-dimensional subgroups of G is the saturation with respect to the operation of G of the set $\{\,(B, g) \mid g \in B\,\}$ in $G/B \times G$. This is a smooth irreducible algebraic variety of dimension 3.

6.12. **Universal family of one-dimensional subgroups of $\mathbf{SL}_3$.** This example demonstrates the hairy structure of the moduli space, while remaining tractable. We let $G = \mathbf{SL}_3$ and describe the moduli space of one-dimensional subgroups of G.

Our first task is to describe A_1 as a set, to do this we take advantage of the operation of G on $\Lambda = \mathbf{P}(\mathfrak{g})$. It follows from the Chevalley-Jordan decomposition that the tangent space to a one-dimensional subgroup of G is either spanned by a semi-simple element of $\mathfrak{g}$ or by a nilpotent element. The group G has two nilpotent orbits $\{\,\mathcal{N}^6, \mathcal{N}^3\,\}$ in $\mathbf{P}(\mathfrak{g})$ whose dimensions are respectively 6 and 3, each of them is included in A. In order to parametrize the semi-simple orbits of G in A, we introduce a torus S of G and $A(S)_1 \subset A_1$ the countable set of the algebraic Lie 1-dimensional subalgebras of $\mathcal{L}(S)$. Since any semi-simple element in $\mathbf{P}(\mathfrak{g})$ is G-conjugated to an element of $\mathcal{L}(S)$, we may conclude that

$$\{\,Gp \mid p \in A(S)\,\} \cup \{\,\mathcal{N}^6, \mathcal{N}^3\,\}$$

is the partition of A_1 in G-orbits. An irreducible subvariety of Λ_1 contained in A_1.

APPENDIX A. OBSERVATION OF FLATNESS THROUGH A BIHOLOMORPHISM

Let X be an algebraic variety and $\mathcal{O}_X$ its structural sheaf. We denote by X^{an} the corresponding analytic variety and by $\mathcal{O}_X^{\mathrm{an}}$ the sheaf of holomorphic functions on X^{an}. Note that X is endowed with its Zariski topology while X^{an} is with its euclidean one. We denote by ϕ the natural map $X^{\mathrm{an}} \to X$. It is is a morphism of ringed spaces, that enables us to define the analytic extension $\mathcal{F}^{\mathrm{an}}$ of a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules by

$$\mathcal{F}^{\mathrm{an}} = \phi^{-1}\mathcal{F} \otimes_{\phi^{-1}\mathcal{O}_X} \mathcal{O}_X^{\mathrm{an}},$$

this is a sheaf of $\mathcal{O}_X^{\mathrm{an}}$-modules. Serre [10] introduced the notion of *flat pair* of rings (*ibid.* définition 4). The following statement results immediately from [10, Corollaire 1, p. 11] and [10, Proposition 22, p. 36].

A.1. PROPOSITION. *For all sheaf of $\mathcal{O}_X$-module $\mathcal{F}$ and all $x \in X$, the natural map $\mathcal{F}_x \to \mathcal{F}_x^{\mathrm{an}}$ is injective.*

We use it to compare flatness of $\mathcal{F}$ and $\mathcal{F}^{\mathrm{an}}$ through a biholomorphism.

A.2. PROPOSITION. *Let $f : X \to P$ a morphism of algebraic variety and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules on X. For all x, $\mathcal{F}_x$ is a flat $(\mathcal{O}_P)_x$-module if, and only if, $\mathcal{F}_x^{\mathrm{an}}$ is a flat $(\mathcal{O}_P^{\mathrm{an}})_x$-module.*

Proof. Recall that, when A is a ring, an A-module M is flat if, and only if, for any finitely generated ideal $\mathfrak{a}$ of A the natural map $\mathfrak{a} \otimes M \to M$ is injective.

If $\mathcal{F}_x$ is not a flat $(\mathcal{O}_P)_x$-module, the kernel of the map $(\mathcal{O}_P)_x \otimes \mathcal{F}_x \to \mathcal{F}_x$ contains a non-zero element. This element also belongs to the kernel of the map $(\mathcal{O}_P^{\mathrm{an}})_x \otimes \mathcal{F}_x^{\mathrm{an}} \to \mathcal{F}_x^{\mathrm{an}}$ and is non-zero by A.1, so $\mathcal{F}_x^{\mathrm{an}}$ is not a flat $(\mathcal{O}_P^{\mathrm{an}})_x$-module.

The converse assertion is implied by the fact that a flat module remains flat after base extension [6, 9.1A]. $\square$

A.3. COROLLARY. *Let P be a variety and for $i \in \{0, 1\}$ a morphism $f_i : X_i \to P$, a sheaf $\mathcal{F}_i$ of $\mathcal{O}_{X_i}$-modules, and a point $x_i \in X_i$. If there is a local biholomorphism $\psi : W_0 \to W_1$ mapping a neighbourhood W_0 of x_0 in X_0 to a neighbourhood W_1 of x_1 in X_1 and such that $f_1|_{W_1} = \psi \circ f_1|_{W_1}$ and $\psi^* \mathcal{F}_0^{\mathrm{an}}|_{W_0} = \mathcal{F}_1^{\mathrm{an}}|_{W_1}$, then $\mathcal{F}_0$ is flat at x_0 if, and only if, $\mathcal{F}_1$ is flat at x_1.*

APPENDIX B. CONSTRUCTION OF CERTAIN COMPACTIFICATIONS

B.1. PROPOSITION. *Let X be a variety, G be an algebraic group and L be a subgroup of G operating on X. There exists a G-variety $G \times_L X$ and a closed immersion of L-varieties $j : X \to G \times_L X$ such that, for all G-variety Y and all immersion of L-varieties $j_Y : X \to Y$, there is a unique morphism of G-varieties $f : G \times_L X \to Y$ such that $j = f \circ j_Y$.*

Furthermore, there is a G-invariant projection $G \times_L X \to G/L$ whose fibre above the co-set L is $j(X)$, and this projection is proper if, and only if, X is complete.

Proof. Let L operate on $G \times X$ by $l \cdot (g, x) = (gl^{-1}, lx)$. This is a free action and the geometric quotient $G \times_L X$ exists [5]. It fulfills the conclusions of the proposition. $\square$

B.2. COROLLARY. *When X and G/L are complete, so is $G \times_L X$.*

B.3. COROLLARY. *If $\mathcal{O}_1$ and $\mathcal{O}_2$ are two distinct orbits of L in X, then $G\mathcal{O}_1$ and $G\mathcal{O}_2$ are two distinct orbits of G in $G \times_L X$.*

Proof. Let x_1 and x_2 two points in X and assume that Gx_1 and Gx_2 have a common point. These two orbits are then equal, and there is $g \in G$ sending x_1 to x_2. Using the projection $G \times_L X \to G/L$, one sees that g belongs to L, so that $Lx_1 = Lx_2$. $\square$

Let G be an algebraic group, by Chevalley structure Theorem [4], there is an exact sequence of algebraic groups

$$1 \to L \to G \xrightarrow{\alpha} A \to 0$$

where L is an affine group and A an abelian variety. The group L is the maximal affine subgroup of G. Remark that a subgroup H of G is connected if, and only if, $\alpha(H)$ and $H \cap L$ are connected.

B.4. Proposition. *Let G be an algebraic group and H a subgroup of G. There is a projective completion K of G such that the connected components of the closure of H in K are the closures of the connected components of H.*

Proof. Let L be the maximal affine subgroup of G. By Chevalley's Theorem on homogeneous spaces of affine groups [1, 5.1], there is a linear representation E of L containing a line D whose stabilizer in L is the identity component M^o of $M = L \cap H$. The orbit of D under the operation of M on $\mathbf{P}(E)$ is a finite subset $F \simeq M/M^o$ of $\mathbf{P}(E)$.

Let $X = G \times_L \mathbf{P}(E)$ and K_0 be any projective completion of G. We consider the morphism

$$\begin{array}{rccc} \theta: & G & \to & K_0 \times X \\ & g & \mapsto & (g, gD) \end{array}$$

and claim that the closure K of $\theta(G)$ in $K_0 \times X$ is a suitable compactification.

Assume that H has multiple connected components and let $h_1 H^o$ and $h_2 H^o$ be two of them, where $\{\, h_1, h_2 \,\} \subset H$. The connected components $h_i H^o D$ of the orbit HD are isomorphic to the abelian variety H^o/M^o, hence are complete varieties. They are furthermore disjoint by B.3. We can conclude that the connected components $h_i H^o$ are mapped under θ into the disjoint complete subvarieties $K_0 \times h_i H^o D$ of $K_0 \times X$, so their closures are disjoint. □

References

[1] Armand Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[2] Claude Chevalley. *Théorie des groupes de Lie. Tome II. Groupes algébriques*. Actualités Sci. Ind. no. 1152. Hermann & Cie., Paris, 1951.

[3] Wei-Liang Chow. On the quotient variety of an Abelian variety. *Proc. Nat. Acad. Sci. U. S. A.*, 38:1039–1044, 1952.

[4] Brian Conrad. A modern proof of Chevalley's theorem on algebraic groups. *J. Ramanujan Math. Soc.*, 17(1):1–18, 2002.

[5] John Fogarty. Geometric quotients are algebraic schemes. *Adv. in Math.*, 48(2):166–171, 1983.

[6] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[7] Shrawan Kumar. *Kac-Moody groups, their flag varieties and representation theory*, volume 204 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 2002.

[8] David Mumford. *Abelian varieties*, volume 5 of *Tata Institute of Fundamental Research Studies in Mathematics*. Published for the Tata Institute of Fundamental Research, Bombay, 2008. With appendices by C. P. Ramanujam and Yuri Manin, Corrected reprint of the second (1974) edition.

[9] R. W. Richardson, Jr. Deformations of subalgebras of Lie algebras. *J. Differential Geometry*, 3:289–308, 1969.

[10] Jean-Pierre Serre. Géométrie algébrique et géométrie analytique. *Ann. Inst. Fourier, Grenoble*, 6:1–42, 1955–1956.

[11] I. R. Shafarevich. On some infinite-dimensional groups. II. *Izv. Akad. Nauk SSSR Ser. Mat.*, 45(1):214–226, 240, 1981.

Michaël Le Barbier Grünewald, Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53 111 Bonn

E-mail address: michi@mpim-bonn.mpg.de

URL: http://guests.mpim-bonn.mpg.de/michi