# Advanced Statistical Inference
# Bayesian Classifier

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# Bayes classifier



- Our first probabilistic classifier is based on Bayes rule:

$$P(t_{\text{new}} = k | \mathbf{X}, \mathbf{t}, \mathbf{x}_{\text{new}}) = \frac{P(\mathbf{x}_{\text{new}} | t_{\text{new}} = k, \mathbf{X}, \mathbf{t}) P(t_{\text{new}} = k)}{\sum_j p(\mathbf{x}_{\text{new}} | t_{\text{new}} = j, \mathbf{X}, \mathbf{t}) P(t_{\text{new}} = j)}$$

- We need to define a likelihood and a prior and we're done!

# Bayes classifier – likelihood

$$p(\mathbf{x}_{\text{new}} | t_{\text{new}} = k, \mathbf{X}, \mathbf{t})$$

- How likely is $\mathbf{x}_{\text{new}}$ if it is in class $k$? (not necessarily a probability...)

# Bayes classifier – likelihood



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- We are free to define this class-conditional distribution as we like.
- Will depend on type of data.
- e.g.
  - Data are $D$-dimensional vectors of real values – Gaussian likelihood.
  - Data are number of heads in $N$ coin tosses – Binomial likelihood.

# Bayes classifier – likelihood



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- In both cases, training data with $t = k$ used to determine parameters of likelihood for class $k$ (e.g. Gaussian mean and covariance).

# Bayes classifier – prior



$$P(t_{\text{new}} = k)$$

- $\mathbf{x}_{\text{new}}$ not present.
- Used to specify prior probabilities for different classes.
- e.g.
  - There are far fewer instances of class 0 than class 1: $P(t_{\text{new}} = 1) > P(t_{\text{new}} = 0)$.
  - No prior preference: $P(t_{\text{new}} = 0) = P(t_{\text{new}} = 1)$.
  - Class 0 is very rare: $P(t_{\text{new}} = 0) \ll P(t_{\text{new}} = 1)$.

# Naive-Bayes

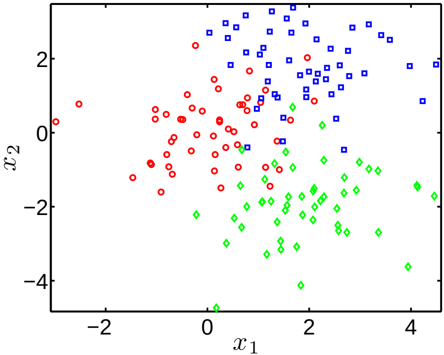


- Naive-Bayes makes the following additional likelihood assumption:
- The components of $\mathbf{x}_{\text{new}}$ are independent for a particular class:

$$p(\mathbf{x}_{\text{new}} | t_{\text{new}} = k, \mathbf{X}, \mathbf{t}) = \prod_{d=1}^{D} p(x_d^{\text{new}} | t_{\text{new}} = k, \mathbf{X}, \mathbf{t})$$

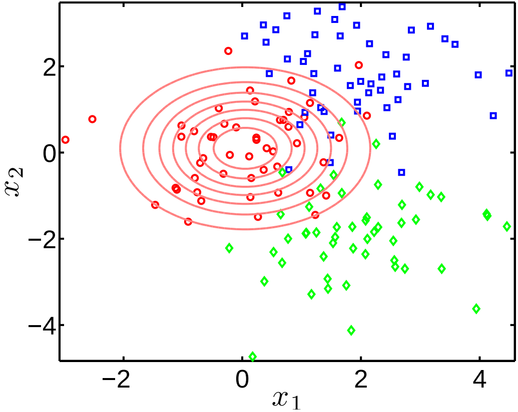
- Where $D$ is the number of dimensions and $x_d^{\text{new}}$ is the value of the $d$th one.
- Often used when $D$ is high:
  - Fitting $D$ uni-variate distributions is easier than fitting one $D$-dimensional one.

# Bayes classifier, example 1




- Each object has two attributes: $\mathbf{x} = [x_1, x_2]^\mathsf{T}$.
- $K = 3$ classes.
- We'll use Gaussian class-conditional distributions (with Naive-Bayes assumption).
- $P(t_{\text{new}} = k) = 1/K$ – uniform prior.

# Step 1: fitting the class-conditional densities




$$p(\mathbf{x}|t=k,\mathbf{X},\mathbf{t}) = \prod_{d=1}^{2} \mathcal{N}(\mu_{kd},\sigma^2_{kd})$$

$$\mu_{kd} = \frac{1}{N_k}\sum_{n:t_n=k} x_{nd} \qquad \sigma^2_{kd} = \frac{1}{N_k}\sum_{n:t_n=k}(x_{nd}-\mu_{kd})^2$$

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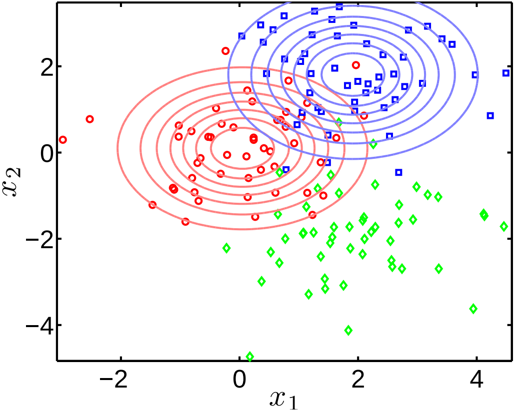


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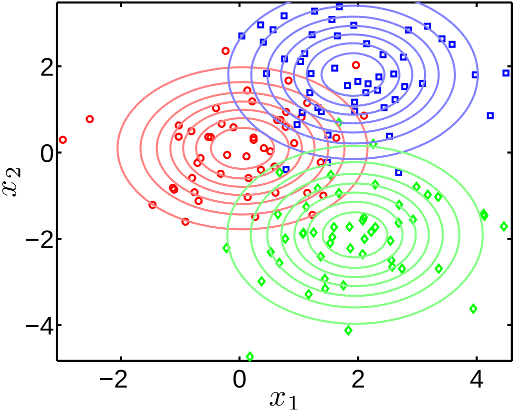


$$p(\mathbf{x}|t = k, \mathbf{X}, \mathbf{t}) = \prod_{d=1}^{2} \mathcal{N}(\mu_{kd}, \sigma^2_{kd})$$

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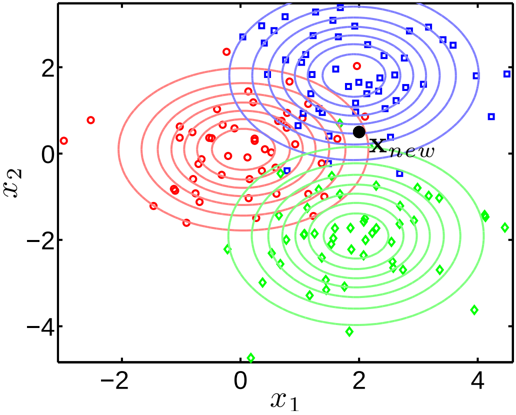
# Step 2: Evaluate densities at test point



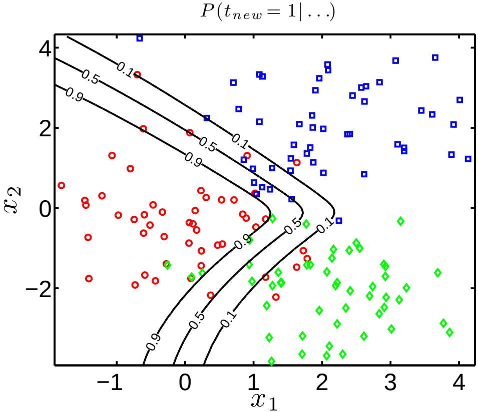

$$p(\mathbf{x}_{\text{new}} | t_{\text{new}} = k, \mathbf{X}, \mathbf{t}) = \prod_{d=1}^{D} \mathcal{N}(\mu_{kd}, \sigma_{kd}^2)$$

# Compute predictions



- Remember that we assumed $P(t_{\text{new}} = k) = 1/K$.

$$P(t_{\text{new}} = k | \mathbf{x}_{\text{new}}, \mathbf{X}, \mathbf{t}) = \frac{p(\mathbf{x}_{\text{new}} | t_{\text{new}} = k, \mathbf{X}, \mathbf{t}) p(t_{\text{new}} = k)}{\sum_j p(\mathbf{x}_{\text{new}} | t_{\text{new}} = j, \mathbf{X}, \mathbf{t}) P(t_{\text{new}} = j)}$$



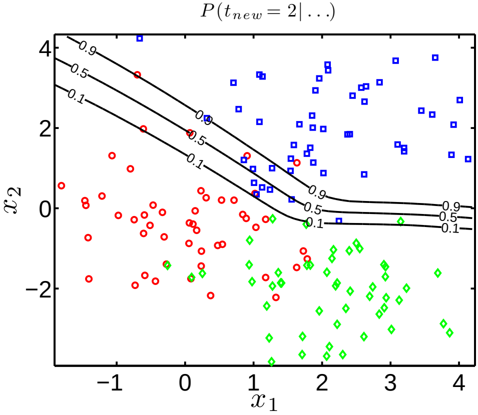

Contours of $P(t_{\text{new}} = 1 | \mathbf{x}_{\text{new}}, \mathbf{X}, \mathbf{t})$

# Compute predictions



- Remember that we assumed $P(t_{\text{new}} = k) = 1/K$.

$$P(t_{\text{new}} = k|\mathbf{x}_{\text{new}}, \mathbf{X}, \mathbf{t}) = \frac{p(\mathbf{x}_{\text{new}}|t_{\text{new}} = k, \mathbf{X}, \mathbf{t})p(t_{\text{new}} = k)}{\sum_j p(\mathbf{x}_{\text{new}}|t_{\text{new}} = j, \mathbf{X}, \mathbf{t})P(t_{\text{new}} = j)}$$

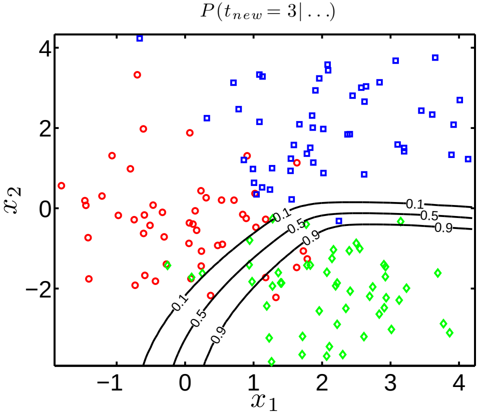



Contours of $P(t_{\text{new}} = 2|\mathbf{x}_{\text{new}}, \mathbf{X}, \mathbf{t})$

# Compute predictions

- Remember that we assumed $P(t_{\text{new}} = k) = 1/K$.

$$P(t_{\text{new}} = k|\mathbf{x}_{\text{new}}, \mathbf{X}, \mathbf{t}) = \frac{p(\mathbf{x}_{\text{new}}|t_{\text{new}} = k, \mathbf{X}, \mathbf{t})P(t_{\text{new}} = k)}{\sum_j p(\mathbf{x}_{\text{new}}|t_{\text{new}} = j, \mathbf{X}, \mathbf{t})P(t_{\text{new}} = j)}$$




Contours of $P(t_{\text{new}} = 3|\mathbf{x}_{\text{new}}, \mathbf{X}, \mathbf{t})$

# Bayes classifier, example 2



- Data are number of heads in 20 tosses (repeated 50 times for each) from one of two coins:
  - Coin 1 ($t_n = 0$): $x_n = 4,\ 7,\ 7,\ 7,\ 4, \ldots$
  - Coin 2 ($t_n = 1$): $x_n = 18,\ 16,\ 18,\ 14,\ 17, \ldots$
- Use binomial class conditional densities:

$$P(x_n | r_k) = \left( \begin{array}{c} 20 \\ x_n \end{array} \right) r^{x_n} (1 - r)^{20 - x_n}$$

- Where $r_k$ is the probability that coin $k$ lands heads on any particular toss.
- Problem – predict the coin, $t_{\text{new}}$ given a new count, $x_{\text{new}}$.
- (Again assume $P(t_{\text{new}} = k) = 1/K$)

# Fit the class conditionals...

- Fitting is just finding $r_k$:

$$r_k = \frac{1}{20N_k} \sum_{n:t_n=k} x_n$$

- $r_0 = 0.287$, $r_1 = 0.706$.

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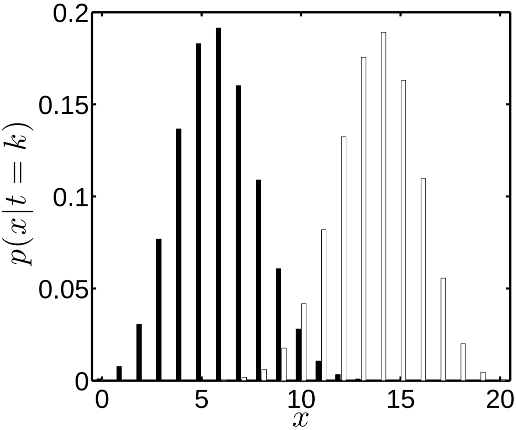


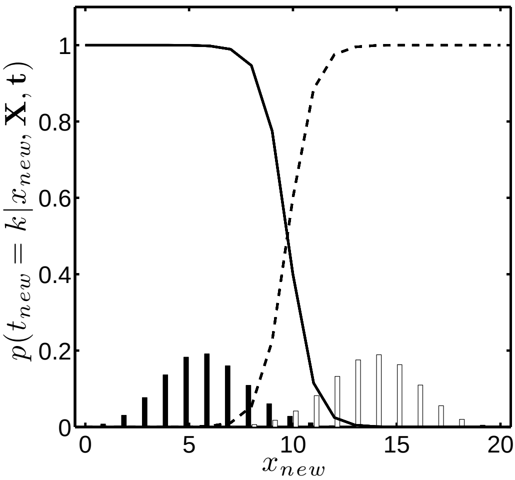
# Compute predictions

$$P(t_{\text{new}} = k | x_{\text{new}}, \mathbf{X}, \mathbf{t}) = \frac{p(x_{\text{new}} | t_{\text{new}} = k, \mathbf{X}, \mathbf{t}) P(t_{\text{new}} = k)}{\sum_j p(x_{\text{new}} | t_{\text{new}} = j, \mathbf{X}, \mathbf{t}) P(t_{\text{new}} = j)}$$

# Compute predictions



$$P(t_{\text{new}} = k | x_{\text{new}}, \mathbf{X}, \mathbf{t}) = \frac{p(x_{\text{new}} | t_{\text{new}} = k, \mathbf{X}, \mathbf{t}) P(t_{\text{new}} = k)}{\sum_j p(x_{\text{new}} | t_{\text{new}} = j, \mathbf{X}, \mathbf{t}) P(t_{\text{new}} = j)}$$



# Bayes classifier – summary

- Decision rule based on Bayes rule.
- Choose and fit class conditional densities.
- Decide on prior.
- Compute predictive probabilities.
- Naive-Bayes:
  - Assume that the dimensions of $\mathbf{x}$ are independent within a particular class.
  - Our Gaussian used the Naive Bayes assumption (could have written $p(\mathbf{x}|t = k, \ldots)$ as product of two independent Gaussians).