REPORT

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Final Exam for ATW Fall-2022

Petros Elia, elia@eurecom.fr
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Instructions

- Exercises fall in categories of 1-point and 2-point exercises.
- Total of $(14 \times 1 + 3 \times 3 + 1 \times 4)$ +one extra credit exercise for 5 points = total $= 27 + 5 = 32$ points (where naturally the 100% grade corresponds to 27 points).
- The last exercise is for extra credit.
- Open book and open class notes are allowed (including notes taken by students during exam). No other notes are allowed.
- Each answer should be clearly written, and the solution should be developed in detail.
- Mathematical derivations need to show all steps that lead to the answer.
- Partial credit will be given for incomplete solutions.
- There is NO penalty for incorrect solutions.

Hints - equations - conventions:

- Notation
  - R represents the rate of communication in bits per channel use (b.p.c.u),
  - ρ represents the SNR (signal to noise ratio),
  - w will denote additive noise which will be distributed as a circularly symmetric Gaussian random variable $\mathbb{C}\mathcal{N}(0, N_0)$. If N_0 is not specified, then set $N_0 = 1$,
  - Remember: for a given signal-to-noise ratio (SNR), then SNR in dB is simply $10 \log_{10}$ SNR
  - SISO stands for single-input single-output, MISO stands for multiple-input single output, SIMO stands for single-input multiple output, MIMO stands for multiple input multiple output.
  - CSIT stands for channel state information at the transmitter, while CSIR stands for channel state information at the receiver.
  - AWGN stands for additive white Gaussian noise
- GOOD LUCK!!

1) ***(1 point)***. Consider a SISO setting, with no fading. Consider that the maximum possible rate (i.e., the capacity) is equal to 3 bpcu. What is the minimum SNR required to achieve this rate?

The capacity formula for a SISO channel without fading is given by the **Shannon capacity** equation: $C = \log_2(1 + \text{SNR})$

Given $C = 3\,\text{bpcu}$, solve for SNR: $3 = \log_2(1 + \text{SNR})$

Convert to linear form: $1 + \text{SNR} = 2^3 = 8$

Thus, the minimum SNR required is: $\text{SNR} = 8 - 1 = 7$

Final Answer: The minimum SNR required is **7** (or **8.45 dB** if expressed in dB).

2) ***(1 point)***. Consider a SISO fading channel with no CSIT. Imagine that we are not happy with a probability of error $P_{\text{err}} \approx (SNR)^{-1}$. Suggest various ways we can decrease this down to $P_{\text{err}} \approx (SNR)^{-4}$.

To achieve $P_{\text{err}} \approx \text{SNR}^{-4}$, increase diversity using:

1. **Time Diversity:** Transmit over multiple independent time slots (e.g., through retransmissions or coding).

2. **Frequency Diversity:** Use multiple independent frequency channels (e.g., OFDM).

3. **Spatial Diversity:** Employ multiple antennas (e.g., MISO, SIMO) or space-time coding (e.g., Alamouti).

4. **Error-Correcting Codes:** Apply powerful FEC codes (e.g., LDPC, turbo codes).

3) ***(1 point)***. What are some of the advantages of **MISO** vs. **SIMO**?

- **Advantages of MISO over SIMO:**
  1. **Beamforming gain:** MISO enables the transmitter to perform **beamforming**, focusing the signal in the direction of the receiver, improving SNR and range.

  2. **Power efficiency:** MISO can distribute transmission power across multiple antennas, reducing overall energy consumption while maintaining performance.

  3. **Interference control:** In multi-user scenarios, MISO allows for **precoding** to mitigate interference between users.
- **SIMO advantage:** SIMO provides **receive diversity** without requiring complex transmission strategies, which is simpler to implement but lacks beamforming capabilities.

4) ***(1 point)***. In multi-user **MISO** settings, describe some of the advantages and some of the disadvantages of matched filtering vs. zero forcing.

- **Matched Filtering:**

  - **Advantages:** Simple to implement, low computational complexity, maximizes signal power at each user.

  - **Disadvantages:** Does not effectively handle inter-user interference, especially in high-user scenarios.
- **Zero Forcing (ZF):**
  - **Advantages:** Eliminates inter-user interference by orthogonalizing users' signals.

  - **Disadvantages:** Higher computational complexity and sensitivity to noise amplification in low-SNR conditions.

5) ***(1 point)***. Provide a real-life scenario that entails an AWGN channel and one that entails fast fading.

- **AWGN Channel:** Communication between two stationary ground stations with minimal interference and obstacles (e.g., satellite-to-ground communication in clear weather).

- **Fast Fading Channel:** Communication in a mobile environment, such as between a moving car and a base station, where rapid changes in the channel occur due to mobility and multipath effects.

6) ***(1 point)***. What is the main reason for using OFDM? (1 line)

The main reason for using **OFDM** is to convert a **frequency-selective fading** channel into multiple **flat-fading** subchannels, simplifying equalization and improving data transmission over broadband channels.

7) ***(1 point)***. What is the complication that prevents interference alignment from being widely used in practice? (1 line)

The main complication is the requirement for **perfect and global channel state information (CSI)** at all transmitters, which is difficult to achieve due to **feedback delays**, **channel estimation errors**, and **dynamic channel conditions**.

8) ***(1 point)***. What does channel hardening mean in Massive **MIMO**? (1-2 lines)

Channel hardening in Massive MIMO refers to the phenomenon where the effective channel gain becomes nearly **deterministic** as the number of antennas increases, reducing the impact of small-scale fading and improving **link reliability**.

9) ***(1 point)***. What does favorable propagation mean in Massive MIMO and what does it imply for receiver and transmitter design? (2-3 lines)

Favorable propagation in Massive MIMO refers to the condition where the channel vectors between different users are **nearly orthogonal**. This implies: - **Simpler receiver design** with

minimal interference, allowing linear detectors like **ZF** or **MMSE** to perform well.
- **Simpler transmitter design** with efficient **spatial multiplexing** and **precoding**, improving system capacity and user separation.

10) ***(1 point)***. How would Cell-Free Massive MIMO allow us to reduce energy consumption? (1 line)

Cell-Free Massive MIMO reduces energy consumption by using **distributed antennas** closer to users, which lowers path loss and enables **low-power beamforming** with improved energy efficiency.

11) ***(1 point)***. In a single-user MIMO channel, how much diversity gain would we be able to get if we employed a transmitter with 4 transmit antennas and a receiver with 2 receive antennas?

The **diversity gain** in a **single-user MIMO** channel is given by:

Diversity gain $= N_t \cdot N_r$ Where: - $N_t = 4$ (transmit antennas), - $N_r = 2$ (receive antennas).

Thus, the **diversity gain** is: Diversity gain $= 4 \cdot 2 = \boxed{8}$

12) ***(1 point)***. In a single-user MISO channel, how much multiplexing gain would we be able to get if we employed a transmitter with 2 transmit antennas?

In a **single-user MISO** channel, **multiplexing gain** refers to the maximum number of independent data streams that can be transmitted simultaneously.

- With **2 transmit antennas** and **1 receive antenna**, the system can only transmit **1 spatial stream**, since there is only **1 receive antenna** to distinguish and decode the streams.

Thus, the **multiplexing gain** is **1**, even with multiple transmit antennas.

13) ***(1 point)***. Consider a SISO fading channel with coherence time equal to $T_c = 10$ milli-seconds.

Imagine that our boss imposes a delay constraint of 50ms. - What type of diversity do we expect to get here: time diversity or spatial diversity? - What is the maximal such diversity?

- **Type of Diversity:**
  Time diversity – Since it's a SISO (Single Input Single Output) channel, there are no multiple antennas to provide spatial diversity. However, multiple independent channel realizations can occur within the allowed delay period, enabling time diversity.

- **Maximal Time Diversity:**
  The number of independent channel realizations within the delay constraint is: Max time diversity $= \frac{\text{Delay constraint}}{T_c} = \frac{50\,\text{ms}}{10\,\text{ms}} = \boxed{5}$

14) ***(1 point)***. Consider communication over a quasi-static 2×1 **MISO** fading channel. Assume that you must draw symbols from 16-QAM. - What is the diversity and the rate (in bpcu) given by the Alamouti code? Justify your answer.

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1. **Diversity:**
The **Alamouti code** is a **space-time block code** for a **2 × 1 MISO** system. It provides full **diversity gain**, which is equal to the product of transmit and receive antennas: Diversity gain $= N_t \cdot N_r = 2 \cdot 1 = \boxed{2}$

2. **Rate (in bpcu):**
The **rate** of a code is given by: $R = \frac{\text{Number of information symbols per codeword}}{\text{Number of time slots}}$

   - In the Alamouti code, 2 information symbols are transmitted over **2 time slots**.
   - Since 16-QAM encodes **4 bits per symbol**, the rate is: $R = \frac{2 \cdot 4\,\text{bits}}{2\,\text{time slots}} = \boxed{4\,\text{bpcu}}$

15) ***(3 points)***. In the context of various strategies, answer if each of the following statements are true or false, justifying briefly your answers. - In a **MISO** channel, we can get transmitter beamforming gain even without CSIT. - A base station equipped with 5 antennas in the downlink, can simultaneously serve up to 5 users (single receive antenna each). - A base station equipped with 5 antennas in the downlink, can simultaneously serve up to 10 users (two receive antennas each). - A base station equipped with 4 antennas in the downlink, can simultaneously serve up to 2 users (two receive antennas each). - Line of sight channels are detrimental for spatial multiplexing in both single-user and multiuser **MIMO**. - For a **MIMO** receiver using spatial multiplexing, the complexity of ZF receiver is more than the complexity of the maximum-likelihood receiver. - CSIT is easier to obtain than CSIR. - CSIT is of cardinal importance in multi-user **MIMO**.

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1. **In a MISO channel, we can get transmitter beamforming gain even without CSIT:**
$\boxed{\text{False}}$ – Beamforming requires **channel state information at the transmitter (CSIT)** to direct the signal towards the receiver effectively.
2. **A base station with 5 antennas can simultaneously serve up to 5 users (single antenna each):**
$\boxed{\text{True}}$ – With **5 antennas**, the base station can form **5 spatial streams**, allowing it to serve up to **5 single-antenna users** simultaneously through spatial multiplexing.
3. **A base station with 5 antennas can simultaneously serve up to 10 users (two antennas each):**
$\boxed{\text{False}}$ – The base station can support a maximum of **5 spatial streams**, so it cannot serve **10 users** simultaneously, even if each user has two antennas.
4. **A base station with 4 antennas can simultaneously serve up to 2 users (two antennas each):**
$\boxed{\text{True}}$ – The base station can support **4 spatial streams**. Two users with **2 antennas each** can be served if the channel supports full spatial rank.
5. **Line of sight channels are detrimental for spatial multiplexing in both single-user and multi-user MIMO:**
$\boxed{\text{True}}$ – In line-of-sight (LOS) channels, **high correlation** between antenna signals can limit the spatial multiplexing gain due to poor channel rank.
6. **For a MIMO receiver using spatial multiplexing, the complexity of a ZF receiver is more than the complexity of a maximum-likelihood receiver:**
$\boxed{\text{False}}$ – **Zero-forcing (ZF)** has lower complexity than **maximum-likelihood (ML)**, which involves exhaustive search over all possible transmitted symbol combinations.
7. **CSIT is easier to obtain than CSIR:**

False – **Channel state information at the receiver (CSIR)** is naturally obtained through pilot signals. **CSIT** requires feedback from the receiver, which introduces additional complexity and latency.

8. **CSIT is of cardinal importance in multi-user MIMO:**
   True – CSIT enables **precoding** and **interference management** in multi-user MIMO, which are crucial for efficiently serving multiple users simultaneously.

16) ***(3 points)***. - What are some reasons for transporting OFDM symbols in the frequency-domain in the O-RAN 7.2 fronthaul-protocol? (1-2 lines) - What is the purpose of the "control-plane" in a fronthaul protocol? (1 line) - Why should radio units be time and frequency synchronized? (1 line) - What are the main differences between CPRI and ECPRI?

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1. **Reasons for frequency-domain OFDM symbol transport in O-RAN 7.2:**
   Reduces bandwidth and fronthaul latency by sending only **subcarrier data**, enabling efficient processing and scalability at the distributed unit (DU).

2. **Purpose of the control-plane in a fronthaul protocol:**
   It manages **configuration, scheduling, and coordination** of radio resources and data transport between DU and RU.

3. **Need for time and frequency synchronization:**
   Ensures **coordinated transmissions** across RUs, reducing interference and supporting **features like beamforming** and **network-wide synchronization**.

4. **Differences between CPRI and eCPRI:**

   - **CPRI:** Transports raw time-domain IQ samples, leading to high bandwidth requirements.

   - **eCPRI:** Transports **processed data (e.g., frequency-domain)**, significantly reducing bandwidth and enabling packet-based transport over Ethernet/IP networks.

17) ***(3 points)***. In the context of various strategies, answer the following with brief justification of your answers. - How the capacity of a SU-MIMO channel is achieved when the channel is known at both the transmitter and the receiver? - In a MU-MIMO channel, if I double the number of users I simultaneously serve, must I always half the individual rate to each user? Justify your answer

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1. **SU-MIMO Capacity Achievement:**
   The capacity of a **Single-User MIMO (SU-MIMO)** channel is achieved through **singular value decomposition (SVD)** and **water-filling power allocation** across parallel eigenmodes of the channel. When the channel is fully known at both the transmitter and receiver, the transmitter uses **beamforming** along the channel's eigenvectors to maximize the achievable data rate.

2. **MU-MIMO User Doubling and Rate:**
   No, doubling the number of users does **not always halve** the individual rate.

   - With **perfect channel state information (CSI)** and proper **precoding** (e.g., zero-forcing or MMSE), users can be spatially separated, allowing the system to maintain higher total throughput without strictly halving rates.

- However, if CSI or precoding is inefficient or suboptimal, interference increases, potentially reducing user rates. Thus, it depends on system conditions and precoding strategies.

18) ***(4 points)***. As in the matlab session, consider communication over the 3×1 quasi-static fading **MISO** channel, using a diagonal code (see below for details) such that the channel model is given by

$$\overbrace{(y_1\ y_2)}^{\underline{y}} = \theta \overbrace{(h_1\ h_2)}^{\underline{h}} \overbrace{\begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}}^{X_{\text{tr}}} + \overbrace{(w_1\ w_2)}^{\underline{w}}$$

where $h_i \sim \mathbb{C}\mathcal{N}(0,1)$ and $w_i \sim \mathbb{C}\mathcal{N}(0,1)$ and where θ is the power normalization factor that lets you regulate SNR. - Describe the ML decoding rule. - Describe the cardinality[1] of code $\mathcal{X}_{tr}$ if you wish a rate of $R = 4$ bpcu. - For a desired rate of $R = 8$ bpcu, and a desired $SNR = 10$ dB (where by SNR we mean the AVERAGE signal power divided by the noise unit power, under QAM) then what is the normalizing factor θ? - Imagine that what you transmit (x_1, x_2) are independently chosen from 16-PAM, then - What is the rate of your code (in bpcu)? - What is the slope of your probability of error, in high SNR, if you plot on the y-axis the probability of error, in log scale ($\log_{10}$(Prob)), and the x-axis is the SNR, in dB? - Imagine now that $(x_1\ x_2) = (s_1\ s_2) \cdot \mathrm{Q}$, where $s1, s2$ are independently chosen from a **64-QAM** constellation, where the matrix Q is a randomly chosen orthogonal matrix. Then - What is the rate of your code? - What is the aforementioned slope of your probability of error?

[1] By cardinality we mean the number of matrices that the code has.

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1. ML Decoding Rule: Given the model: $\underline{y} = \theta\, \underline{h} X_{\text{tr}} + \underline{w}$

The **Maximum Likelihood (ML)** decoding rule is: $\hat{X}_{\text{tr}} = \arg\min_{X_{\text{tr}} \in \mathcal{X}_{\text{tr}}} \|\underline{y} - \theta\, \underline{h} X_{\text{tr}}\|^2$

2. Cardinality of $\mathcal{X}_{\text{tr}}$ for $R = 4$ bpcu: The rate equation is: $R = \frac{\log_2(\text{Cardinality})}{T}$

- Here $T = 2$ (as X_{tr} is a 2×2 diagonal matrix).
- For $R = 4$ bpcu: $4 = \frac{\log_2(\text{Cardinality})}{2} \quad \Rightarrow \quad \text{Cardinality} = 2^8 = 256$

Thus, the cardinality is **256**.

3. Normalizing factor θ for $R = 8$ bpcu and $SNR = 10$ dB:

- With $R = 8$ bpcu, each x_1 and x_2 is chosen from a 256-QAM constellation ($2^{8/2} = 256$).
- Average power per x_i for a 256-QAM constellation is approximately: $P_{\text{signal}} = \frac{256^2-1}{3} \approx 21845$ (normalized to unity grid spacing)
- SNR (in linear scale) is $10^{10/10} = 10$, and $\text{SNR} = \frac{\theta^2 P_{\text{signal}}}{2}$.

Solving for θ: $10 = \frac{\theta^2 \cdot 21845}{2} \quad \Rightarrow \quad \theta^2 = \frac{20}{21845} \quad \Rightarrow \quad \theta = \sqrt{\frac{20}{21845}} \approx 0.03$

4. **Rate and Slope for 16-PAM:**

   - **Rate:**
     For 16-PAM on each antenna, $R = \log_2(16) \times 2 = 8\,\text{bpcu}$.
   - **Slope:**
     In high SNR, the slope of the probability of error curve is determined by **diversity gain** d.
     - In a 3×1 MISO system, $d = N_t = 3$.
       Therefore, the slope is -3 when plotted in log-log scale.

5. **Rate and Slope with 64-QAM and Orthogonal Matrix:**

   - **Rate:**
     Since s_1, s_2 are from a 64-QAM constellation, each symbol contributes 6 bits. With 2 time slots: $R = 6 \times 2 = 12\,\text{bpcu}$
   - **Slope:**
     The slope of the probability of error in high SNR is still given by the diversity gain $d = 3$. Thus, the slope remains -3.

19) ***(Extra Credit: 5 points)***. Consider communication over a quasi-static 2×2 **MIMO** channel, utilizing the space-time code $\mathcal{X} = \{X_1, X_2, X_3, X_4\}$, where

$$X_1 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, X_2 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, X_3 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \text{ and } X_4 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$$

- What is the average SNR?
- What is the rate of the code in bpcu?
- What is the diversity gain of this code?
- What is the approximate (in the high SNR regime) probability of error of this code, if SNR is 30dB?

1) Average SNR The SNR (Signal-to-Noise Ratio) depends on the transmitted power and noise variance. The **average SNR** is given by: $\text{SNR}_{\text{avg}} = \frac{\mathbb{E}[\|X\|_F^2]}{2\sigma^2}$

Here: - $\|X\|_F^2$ is the Frobenius norm of the code matrices. - The code matrices have entries of either 1 or -1.

For any matrix X_i: $\|X_i\|_F^2 = 1^2 + (-1)^2 + 1^2 + 1^2 = 4$

Since each matrix has a Frobenius norm squared of 4, the average transmitted power is 4. Therefore, the average SNR is: $\text{SNR}_{\text{avg}} = \frac{4}{2\sigma^2}$

Given that SNR = 30 dB: $\text{SNR}_{\text{dB}} = 10\log_{10}\left(\frac{4}{2\sigma^2}\right) = 30$

Solving for σ^2, we get: $\frac{4}{2\sigma^2} = 10^3 \quad \Rightarrow \quad \sigma^2 = 2 \times 10^{-3}$

Thus, the average SNR is $30\,\text{dB}$.

2) Rate of the code (in bits per channel use, bpcu) The **rate** is given by: $R = \frac{\log_2(M)}{T}$

Here: - $M = 4$ (4 codewords), - $T = 2$ (each codeword matrix is transmitted over 2 time slots).

Thus, the rate is: $R = \frac{\log_2(4)}{2} = \frac{2}{2} = 1\,\text{bpcu}$

3) Diversity gain The **diversity gain** d measures the rate at which the probability of error decays with increasing SNR. For a 2×2 MIMO system and a full-rank space-time code, the diversity gain is: $d = N_t \cdot N_r = 2 \cdot 2 = 4$

Thus, the diversity gain is **4**.

4) Approximate probability of error (high SNR) At high SNR, the probability of error P_e is approximated by: $P_e \approx c \cdot \left(\frac{1}{\mathrm{SNR}}\right)^d$

Here: - $d = 4$ is the diversity gain, - SNR (linear) is $10^3 = 1000$, - c is a constant that depends on the modulation and code.

Assuming $c \approx 1$ for simplicity, we have: $P_e \approx \left(\frac{1}{1000}\right)^4 = 10^{-12}$

Thus, the approximate probability of error is 10^{-12}.

0.0.1 Final Answers

1. **Average SNR:** $30\,\mathrm{dB}$
2. **Rate:** $1\,\mathrm{bpcu}$
3. **Diversity gain:** 4
4. **Approximate probability of error:** 10^{-12}