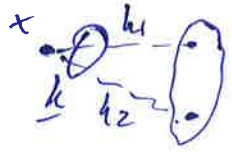
Channel Models.

Point to Point

$y = x + w$ (AWGN)

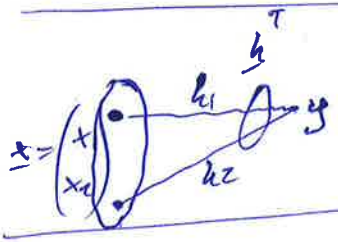
$y = hx + w$ → fixed h indoor

→ outdoor $h \sim \mathcal{CN}(0,1)$ changes 100^s of times per sec.

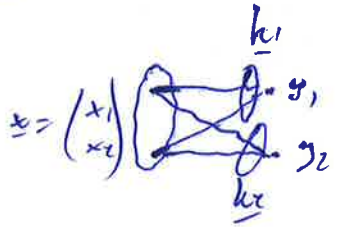


$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \underline{h} \cdot x + \underline{w} \Rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} x + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$\underline{y} = \underline{h}\, x + \underline{w}$$

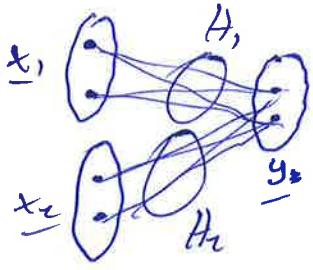


$$y = \underline{h}^T \underline{x} + w = \begin{pmatrix} h_1 & h_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + w.$$



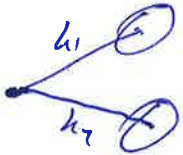
$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad H = \begin{pmatrix} \underline{h_1}^T \\ \underline{h_2}^T \end{pmatrix}.$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} h_1^T \\ h_2^T \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underline{w} \qquad \underline{y} = H\,\underline{x} + \underline{w}.$$



$$\underline{y} = \begin{pmatrix} H_1 & H_2 \end{pmatrix} \begin{pmatrix} \underline{x_1} \\ \underline{x_2} \end{pmatrix} + \underline{w}$$

Download.



like SIMO just y_1, y_2 differently treated

$y_1 = h_1 \cdot x + w_1$

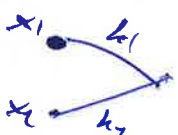
$y_2 = h_2 \cdot x + w_2$.

(effectively one user at a time)

↓

(»).

Uplink



like MISO
Just x_1, x_2 uncoordinated.

$y = h_1 x_1 + h_2 x_2 + w$

$= \underline{h}^T \cdot \underline{x} + \underline{w}$

Downlink MISO
MU-MISO

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = H \cdot \underline{x} + \underline{w}$.

Need for diversity

$y = hx + w \qquad h \sim \mathcal{CN}(0,1) \qquad P = E\{|x|^2\} \qquad SNR = \frac{P}{E\{|w|^2\}} = \frac{P}{N_0}.$

fading

error $\approx |h|^2 < \bar{P}^1$, $|h|^2 \sim \exp(1)$

$\Rightarrow P(|h|^2 < \varepsilon) \overset{\varepsilon \to 0}{\approx} \varepsilon \Rightarrow P_{err} \approx P(|h|^2 < \bar{P}^1) \approx \bar{P}^1$ too high.

Time diversity

$\underline{x} = (x_1 \; x_2 \cdots x_T) = (x \; x \cdots x)$ ← for example

$(y_1, y_2 \cdots y_T) = (h_1 x_1 \quad h_2 x_2 \cdots h_T x_T) + (w_1, w_2 \cdots w_T)$

$= (\underbrace{h_1 + h_2 \cdots h_T}_{\underline{h}^T}) \cdot x + (\underline{w}^T)$

* $\underline{y} = \underline{h}\, x + \underline{w}$

$y' = \frac{\underline{h}^H}{\|\underline{h}\|} \cdot \underline{y} = \frac{\underline{h}^H}{\|\underline{h}\|} \cdot \underline{h}\, x + \frac{\underline{h}^H}{\|\cdot\|} \cdot \underline{w} \Rightarrow y' = \|\underline{h}\| x + w'$

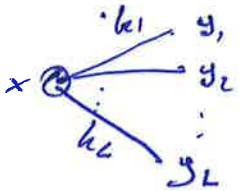
$P_{err} = P\left(|\underline{h}|^2 < \frac{1}{P}\right) = P\left(\sum_{i=1}^{T} |h_i|^2 < \bar{P}^1\right)$

as $P \uparrow$ $\bar{P}^1 \downarrow 0 \overset{\approx}{\Rightarrow} P\left(|h_i|^2 \leq \bar{P}^1, \; \forall i\right)$

h_i indep $\Rightarrow \approx \left[P\left(|h_i|^2 < \bar{P}^1\right)\right]^T = \bar{P}^T$

Rx diversity

SIMO



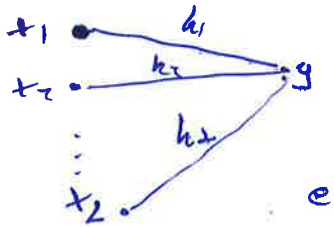
$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_L \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_L \end{pmatrix} \cdot x + \underline{w}$$

$$\Rightarrow \underline{y} = \underline{h}\, x + \underline{w} \quad \text{like before}$$

$$y' = \frac{\underline{h}^H}{\|\underline{h}\|}\,\underline{y} = \|\underline{h}\|\, x + w = y' \quad \Rightarrow P_{err} \approx P\left(\|\underline{h}\|^2 \le \bar{P}^1\right) = \bar{P}^L.$$

Tx Diversity

MISO



$$y = h_1 x_1 + h_2 x_2 + w = \underline{h}^T \cdot x + \underline{w}$$

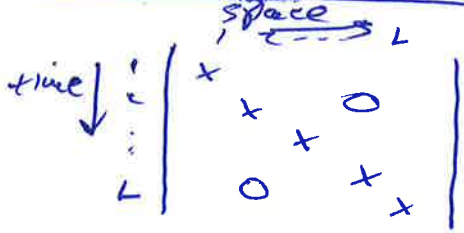
$$y = \underline{h}^T x + \underline{w} \qquad y = \sum_{i=1} h_i x_i + w$$

even if $x_i = x\ \forall x \Rightarrow y = \left(\sum_i h_i\right) x + w$

$w(0,..)$ again Gaussian

but again $P(err) \approx \bar{P}^1$ bad (no diversity).

$\Rightarrow$ need time dimension



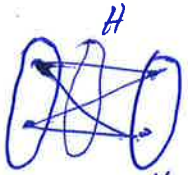
$$\Rightarrow (y_1, y_2 \cdots y_L)^T = (h_1, h_2 \cdots h_L)^T \cdot x + w$$

$$\underline{y} = \underline{h}\, x + \underline{w}$$ (same model as before)

$$y' = \frac{\underline{h}^H}{\|\underline{h}\|} \cdot \underline{y} = \|h\|\, x + w' \Rightarrow P_{err} \approx P\left(\|h\|^2 \ge \bar{P}^1\right) \approx \bar{P}^L \text{ (diversity)}$$

Tx & Rx diversity

MIMO



$$\underset{N_R \times N_T}{\underline{H}} = \begin{pmatrix} h_1^T \\ \vdots \\ h_{N_R}^T \end{pmatrix}$$

$$\underset{N_T \times N_T}{X} + \underline{w}$$

easy to see (Kronecker expansion)

$$P(err) \approx \bar{P}^{-N_T \cdot N_R}$$ (very high)

Capacity $\quad \underline{y = x + w}$ encode in time $(x_1, x_2 \cdots x_T)$

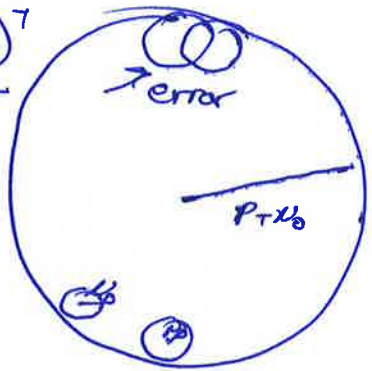
If $\underline{y} = \underline{x} + \underline{w}$ $\quad E\{\|\underline{x}\|^2\} = T \cdot P \quad E\{\|\underline{w}\|^2\} = T \cdot N_0$.

$\underline{y}$ (the set of all $\underline{y}$) will be w.h.prob in ball of radius $\sqrt{T(P+N_0)}$

Each codewords can you fit in this ball.

$$\#\text{ of codewords} = \frac{\text{Vol}(\text{Big})}{\text{Vol}(\text{small})} = \frac{(P+N_0)^T}{(N_0)^T}$$

$$= \left(1 + \frac{P}{N_0}\right)^T$$



$\Rightarrow$ Can Communicate $T \cdot \log\left(1 + \frac{P}{N_0}\right)$

= Capacity AWGN channel.

$\underline{y = hx + w}$ $\underline{h \text{ fixed.}}$ $\quad E\{\|\underbrace{hx}_{\text{signal part.}} + \|^2\} = |h|^2 \cdot P$

$$C = \log\left(1 + \frac{P \cdot |h|^2}{N_0}\right) = \log\left(1 + |h|^2 e\right).$$

Power Gain

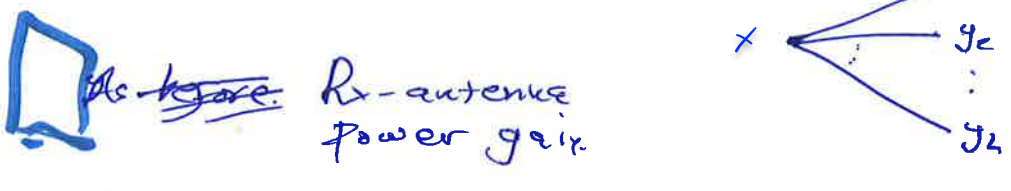
$y = hx + w$ $\quad y' = \frac{h^*}{|h|} \cdot y = \frac{h^*}{|h|}(hx + w) = |h| \cdot x + w'$ → as before

matched filter

best thing to do $\Rightarrow$ SNR $= \frac{\mathcal{E}\{\|hx\|^2\}}{\mathcal{E}\{\|w\|^2\}} = \frac{|h|^2 \cdot P}{N_0}$

Higher dimension Power Gain

SIMO



$\underline{y} = \underline{h}x + \underline{w}$

$\underline{y}' = \frac{\underline{h}^*}{\|\underline{h}\|}\underline{y} = \|\underline{h}\|x + w'$

as before: Rx-antenna power gain

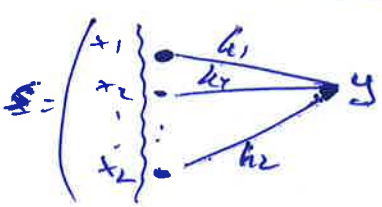
$y' = \|\underline{h}\|x + w$ as $L \uparrow \Rightarrow \|\underline{h}\|^2 \to L \Rightarrow$ SNR $\uparrow$ L times

↓ power up

Can see this wrt Capacity $C_{SIMO} = \log\left(1 + \|\underline{h}\|^2 \frac{P}{N_0}\right)$

$L \uparrow \Rightarrow \log\left(1 + \frac{P \cdot L}{N_0}\right)$

Tx antenna Power Gain



$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_L \end{pmatrix}$

want to send s $\quad \mathcal{E}\{|s|^2\} = P$

$\underline{x} = \frac{\underline{h}}{\|\underline{h}\|} \cdot s$ $\quad y = \underline{h}^T \cdot \underline{x} + w = \underline{h}^T \frac{\underline{h}^*}{\|\underline{h}\|} s + w$ $\quad \Rightarrow y = \|\underline{h}\| s + w$

L Tx antennas normalized

↓ up to L Power gain

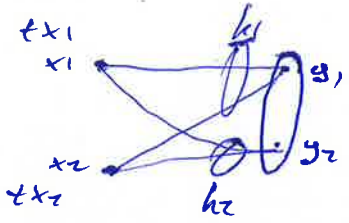
$$C = \log\left(1 + P \cdot \frac{\|\underline{h}\|^2}{N_0}\right)$$

Removing Interference

TDMA

At Rx

Need multiple Rx-Antennas.



$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \underline{y} = \underline{h}_1 x_1 + \underline{h}_2 x_2 = \begin{bmatrix} \underline{h}_1 & \underline{h}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

Interested only in x_1 $\Rightarrow \underline{y}_1' = \frac{\underline{h}_2^{\perp}}{\|\underline{h}_2^{\perp}\|} \cdot \underline{y} = \frac{\underline{h}_2^{\perp}}{\|\cdot\|} \underline{h}_1 x_1 + \frac{\underline{h}_2^{\perp}}{\|\cdot\|} \underline{h}_2 x_2 + w'$ ($\underline{h}_2^{\perp} \perp \underline{h}_2 \Rightarrow 0$)

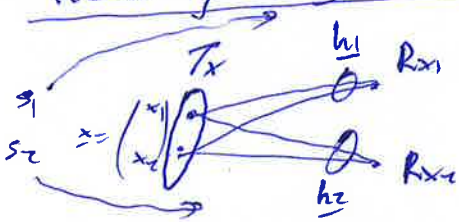
$$\Rightarrow y_1' = \frac{\underline{h}_2^{\perp}}{\|\underline{h}_2^{\perp}\|} \underline{h}_1 x_1 + w_1' \Rightarrow \underline{y_1' = h' x_1 + w_1}$$ removed Interference

---

Removing Interference at Tx.

Need multiple Tx-antennas and CSIT.

TDMA



$$y_1 = \underline{h}_1^T \cdot \underline{x} + w_1$$

Let $\underline{x} = \frac{\underline{h}_2^{\perp}}{\|\underline{h}_2^{\perp}\|} \cdot s_1 + \frac{\underline{h}_1^{\perp}}{\|\underline{h}_1^{\perp}\|} s_2$

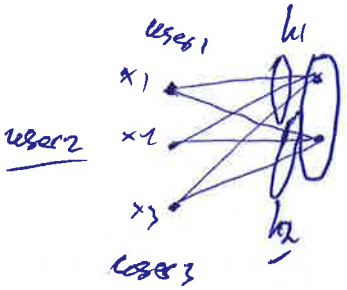
$$\Rightarrow y_1 = \underline{h}_1^T \frac{\underline{h}_2^{\perp}}{\|\cdot\|} s_1 + \underline{h}_1^T \frac{\underline{h}_1^{\perp}}{\|\cdot\|} s_2 + w_1' \quad (\underline{h}_1^T \perp \underline{h}_1^{\perp} \Rightarrow 0)$$

$$\Rightarrow \boxed{y_1 = h_1' s_1 + w_1'}$$ similarly for s_2, y_2.

or similarly $\underline{y} = \underbrace{\begin{bmatrix} \underline{h}_1 & \underline{h}_2 \end{bmatrix}}_{H} \underline{x} + \underline{w}$ $\quad \underline{x} = \frac{H^{-1}}{\|\cdot\|} \cdot \underline{s} \Rightarrow \underline{y} = H H^{-1} \underline{s} + \underline{w}$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}}{\|\cdot\|} + \underline{w}$$

Multiple Users Removing Interference at Rx



I want x_1

Can ignore user 3, & as before remove out User 2.

only

$$\underline{y} = H\underline{x} + \underline{w} = \begin{bmatrix} \underline{h}_1 & \underline{h}_2 & \underline{h}_3 \end{bmatrix}_{3\times 3} \underline{x} + \underline{w} = \sum_{i=1}^{3} \underline{h}_i x_i$$

Rx only wants x_1.

Consider $R^n = R^3$. Let S = subspace of dimension 1. $\dim = 1$.

$\Rightarrow N \triangleq \{\underline{v} \in R^n : \underline{v} \perp S\}$ ← null space of S. $\perp \underline{h}_1$

$\Rightarrow \dim(N) = n - \dim(S) = n - u$

~~Here $\dim(N) = 3 - 1 = 2$.~~

~~$\exists \underline{v} \in N$~~ s.t. $y' = \underline{v}^H \underline{y}$

looking for vector $\perp \underline{h}_2$ & $\perp \underline{h}_3$

$V_{23} = \text{span}(\underline{h}_2, \underline{h}_3)$ $\dim(V) = 2$

$\text{Null}(V_{23})$ ~~=~~ $\dim(\text{Null}(V_{23})) = 3 - 2 = 1$

$\Rightarrow \exists$ vector $\perp \underline{h}_2 \& \underline{h}_3$

$\Rightarrow y_1' = \frac{\underline{v}^H \underline{y}}{\|\underline{v}\|} \Rightarrow \frac{\underline{v}^H}{\|\cdot\|} \sum_{i=1}^{3} \underline{h}_i x_i = \frac{\underline{v}^H \underline{h}_1}{\|\cdot\|} x_1 = h_1' x_1 + w' = y_1'$

Similarly if I simply wanted x_2 at Rx

$\Rightarrow V_{13} = \text{span}(\underline{h}_1, \underline{h}_3) \Rightarrow \underline{v}_2 \in \text{Null}(V_{13}) \Rightarrow y_2' = \underline{v}_2^H \underline{y}$ ....

$N > M$

Reduce system to $M \times M$ & Remove $M-1$ users as above

~~keep~~ rest $N - M + 1$ stay interfering.

Tx antennas, bigger case. interference cancelation

y_1 - Rx1
y_2 - Rx2
y_3 - Rx3

$y_i = \underline{h}_i \underline{x} + \omega_i \quad i = 1 \to 3.$

$V_{23} = \text{span}(h_2, h_3)$ ~~$\underline{v}_1 \in V_{23}$~~ $\underline{v}_1 \in \text{Null}(V_{23})$

$\underline{v}_2 \in \text{Null}(V_{13})$, $\underline{v}_3 \in \text{Null}(V_{12})$.

s_i : for Rx i

$\Rightarrow \underline{x} = \underline{v}_1 s_1 + \underline{v}_2 s_2 + \underline{v}_3 s_3$

$\Rightarrow y_1 = \underline{h}_1^T \cdot \underline{x} = \underline{h}_1^T \cdot \underline{v}_1 s_1 + \underline{h}_1^T \underline{v}_2 s_2 + \underline{h}_1^T \underline{v}_3 s_3 + \omega$ ✓ (the $\underline{h}_1^T \underline{v}_2 s_2$ and $\underline{h}_1^T \underline{v}_3 s_3$ terms $\to 0$)

similarly for the rest.

N tx antennas K users $K \leq N$ ∀ interference removed

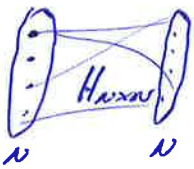
$K \geq N$ ∀ except $N-K$ users

$N \geq K$ $\underline{x} = V \cdot \underline{s}$ V = pseudo inverse.

$N = K$ ⇒ $\underline{x} = \frac{\vec{H}^{-1}}{\|\cdot\|} \cdot \underline{s}$ ⇒ $\underline{y} = H \frac{\vec{H}^{-1}}{\|\vec{H}^{-1}\|} \underline{s} + \underline{\omega} = \frac{\underline{s}}{\|\vec{H}\|} + \omega$

Similar concept with MIMO. (1 Tx, 1 Rx, N antennas each)

Multiplexing Gain.



$\underline{y} = H \underline{x} + \underline{\omega}$ If $\underline{x} = \frac{\vec{H}^{-1}}{\|\cdot\|} \underline{s} \Rightarrow \underline{y} = \frac{\underline{s}}{\|\cdot\|} + \omega$

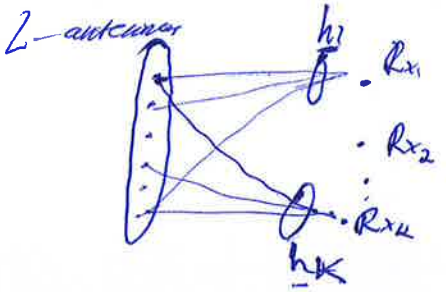
note $y_i = d_i \cdot s_i + \omega_i \quad \forall i$

$\Rightarrow$ $N \times N$ MIMO allows us to send to N users at a time

$C_{\substack{N\times N \\ \text{MIMO}}} \approx N \cdot C_{\text{SISO}}$ as $\frac{P}{P_0} \to \infty$.

CSIT Needed

Massive Mimo.



$\underline{h}_i \in \mathbb{C}^L$

as $L \uparrow$ and as $\frac{L}{K} \uparrow$

$\Rightarrow \underline{h}_i^T \underline{h}_j \to 0.$

$\Rightarrow \underline{x} = \sum_{i=1}^{K} \frac{\underline{h}_i^*}{\|\cdot\|} \cdot s_i$

$\Rightarrow \underline{y_i \approx |h_i| s_i + 0}$ simplicity & encoding no inverses.

Space Time Codes For Tx-diversity. (MISO)

Recall $X = \underset{\text{space}}{\downarrow} \begin{pmatrix} \xrightarrow{\text{time}} \\ x & \cdots & 0 \\ \vdots & \ddots & \\ 0 & & x \end{pmatrix} \rightarrow \underline{y}^T = \underline{h}^T \cdot X + \underline{\omega}^T \Rightarrow \underline{y} = \underline{h}\, x + \underline{\omega}$

$\Rightarrow y' = \frac{\underline{h}^H}{\|\cdot\|} \underline{y} = \underline{|h| x + \omega' = y'}$

$\Rightarrow P_{err} \approx \bar{P}^{-L}$ as $P \uparrow$ (div gain of L).

But too slow. 1 symbol per L time slots.

Consider $L=2$.

$$(y_1\ y_2) = (h_1\ h_2) \begin{pmatrix} s_1 & -s_2^* \\ s_2 & s_1^* \end{pmatrix} + \omega_1\ \omega_2 \quad \Rightarrow \quad \underbrace{\begin{pmatrix} y_1 \\ y_2^* \end{pmatrix}}_{\underline{y}'} = \begin{pmatrix} \underline{h}_1 & \underline{h}_2 \\ h_2^* & -h_1^* \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} + \begin{pmatrix} \omega_1 \\ \omega_2^* \end{pmatrix}$$

$y_1'' = \underline{h}_1^T \cdot \underline{y}' = |h| s_1 + \omega'$, $y_2'' = \underline{h}_2^T \underline{y}' = |h| \cdot s_2 + \omega_2$.

Again Full diversity but also scaling Faster.

Degrees of Freedom.

What is problem when repeating?

$(y_1\ y_2 \cdots y_L) = (h_1\ h_2 \cdots h_L) \cdot x + \tau\ (w_1 \ldots)$

signal in one dimension, span be $\underline{h} = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_L \end{pmatrix}$.

L-dimensional space $\mathbb{C}^L$ available

BUT only using one dimension.

Whereas (e.g. $L=2$).

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = H \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

↑ not repetition

space is two dimensional (L) →.

Give example of room

Higher # of dimensions → can pack things better.

» » » » → increase distance between codewords.

Gains from multiple antennas

diversity gain = L i.e $P_{err} \approx P^{-1} \longrightarrow P^{-L}$

⇒ for same SNR = P ⇒ P_{err} ↓ exponentially

or same P_{err} with Power : P ↓ → $P^{1/L}$

Power gain (L ↑). Power : P → L.P ✓

(humble but important).

Multiplexing gain (MIMO & MU-MISO / MIMO).

1 symbol served at a time → L symbols

gain MIMO sees by capacity

($\sim$ C → L.C (as P ↑ at high SNR).

⇒ Power P → $P^{1/L}$

(treat carefully).