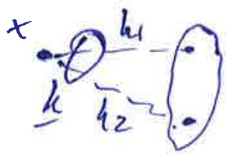
Channel Models.

Point to Point

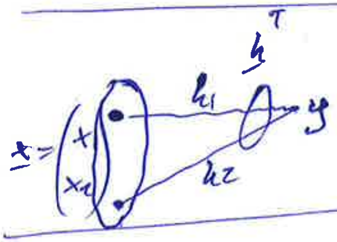
$y = x + w$ (AWGN)

$y = hx + w$ → fixed h indoor
→ outdoor $h \sim CN(0,1)$ changes 100's of times per sec.

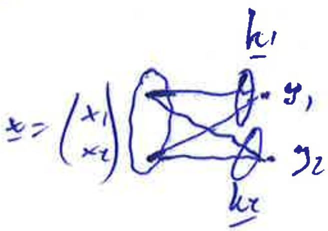


$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \underline{h}\cdot x + \underline{w} \Rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} x + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$\underline{y} = \underline{h}\, x + \underline{w}$$

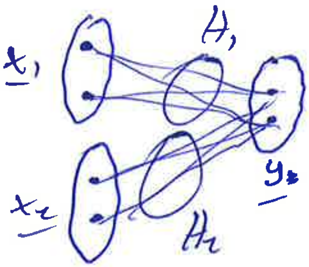


$$y = \underline{h}^T \underline{x} + w = \begin{pmatrix} h_1 & h_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + w.$$



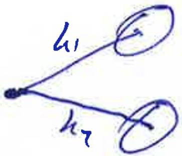
$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad H = \begin{pmatrix} \underline{h}_1^T \\ \underline{h}_2^T \end{pmatrix}.$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} h_1^T \\ h_2^T \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underline{w} \qquad \underline{y} = H\,\underline{x} + \underline{w}.$$



$$\underline{y} = \begin{pmatrix} H_1 & H_2 \end{pmatrix} \begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \end{pmatrix} + \underline{w}$$

Download.



like SIMO just y_1, y_2 differently treated

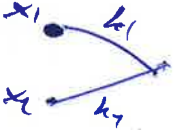
$$y_1 = h_1 . x + w_1$$
$$y_2 = h_2 . x + w_2 .$$

(effectively one user at a time)

↓

(").

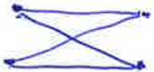
Uplink



like MISO
just x_1, x_2
uncoordinated.

$$y = h_1 x_1 + h_2 x_2 + w$$
$$= \underline{h}^T . \underline{x} + \underline{w}$$

Downlink MISO
MU-MISO



$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = H . \underline{x} + \underline{w} .$$

Need for diversity

$$y = hx + w \qquad h \sim \mathcal{CN}(0,1) \qquad P = E\{|x|^2\} \qquad \text{SNR} = \frac{P}{E\{|w|^2\}} = \frac{P}{N_0}.$$

$$\text{error} \approx |h|^2 < P^{-1} \quad , \quad |h|^2 \sim \exp(1)$$

$$\Rightarrow P(|h|^2 < \varepsilon) \overset{\varepsilon \downarrow 0}{\approx} \varepsilon \quad \Rightarrow P_{err} \approx P(|h|^2 < P^{-1}) \approx P^{-1}$$ too high.

Time diversity

for example

$$\underline{x} = (x_1 \; x_2 \cdots x_T) = (x \; x \cdots x)$$

$$(y_1 \; y_2 \cdots y_T) = (h_1 x_1 \; h_2 x_2 \cdots h_T x_T) + (w_1 \; w_2 \cdots w_T)$$

$$= \underbrace{(h_1 \; h_2 \cdots h_T)}_{\underline{h}^T} \cdot x + (\underline{w}^T)$$

$$\Rightarrow \underline{y} = \underline{h} x + \underline{w}$$

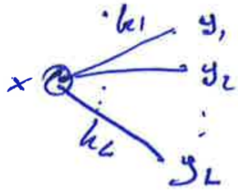
$$y' = \frac{\underline{h}^H}{\|\underline{h}\|} \cdot \underline{y} = \frac{\underline{h}^H}{\|\underline{h}\|} \cdot \underline{h} x + \frac{\underline{h}^H}{\|\cdot\|} \cdot \underline{w} \Rightarrow y' = |\underline{h}| x + w'$$

$$P_{err} = P\left(|\underline{h}|^2 < \frac{1}{P}\right) = P\left(\sum_{i=1}^{T} |h_i|^2 < P^{-1}\right)$$

$$\text{as } P \uparrow \quad P^{-1} \downarrow 0 \quad \overset{\approx}{\Rightarrow} P\left(|h_i|^2 < P^{-1}, \; \forall i\right)$$

$$h_i \text{ indep} \Rightarrow \approx \left[P(|h_i|^2 < P^{-1})\right]^T = P^{-T}$$

Rx diversity.

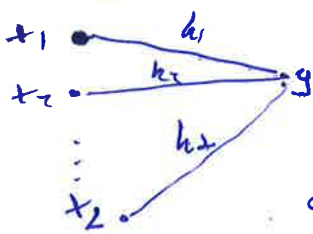


$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_L \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_L \end{pmatrix} \cdot x + \underline{w}$$

$$\Rightarrow \underline{y} = \underline{h}\,x + \underline{w} \quad \text{like before}$$

$$y' = \frac{\underline{h}^H}{\|\underline{h}\|} \cdot \underline{y} = \underline{|\underline{h}| \cdot x + w = y'} \quad \Rightarrow P_{err} \approx P\left(|\underline{h}|^2 \leq \bar{P}^{-1}\right) = \bar{P}^{-L}.$$

Tx Diversity



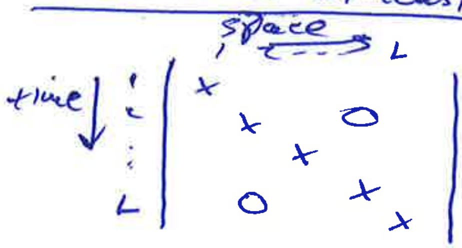
$$y = h_1 x_1 + \cdots h_L x_L + w = \underline{h}^T \cdot \underline{x} + \underline{w}$$

$$y = \underline{h}^T \underline{x} + \underline{w} \qquad y = \sum_{i=1}^{L} h_i x_i + w$$

even if $x_i = x \ \forall x \Rightarrow y = \underbrace{\left(\sum_i h_i\right)}_{\sim \mathcal{N}(0,\cdot)} x + w$ again Gaussian

but again $P(err) \approx \bar{P}^{-1}$ bad (no diversity).

$\Rightarrow$ need time dimension



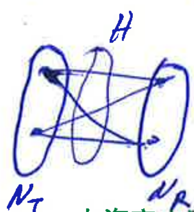
$$\Rightarrow (y_1\ y_2 \cdots y_L)^T = (h_1\ h_2 \cdots h_L)^T \cdot x + w$$

$$\underline{y} = \underline{h}\,x + \underline{w} \quad (\text{same model as before})$$

$$y' = \frac{\underline{h}^H}{\|\underline{h}\|} \cdot \underline{y} = |h|\,x + \underline{w}' \Rightarrow P_{err} \approx P\left(|h|^2 < \bar{P}^{-1}\right)$$

$$\approx \bar{P}^{-L} \ (\text{diversity})$$

Tx & Rx diversity.



$$\underline{Y}_{N_R \times N_T} = \underbrace{\begin{pmatrix} h_1^T \\ \vdots \\ h_{N_R}^T \end{pmatrix}}_{H} X_{N_T \times N_T} + \underline{w}$$

easy to see (Kronecker expansion)

$$P(err) \approx \bar{P}^{-N_T \cdot N_R} \quad (\text{verify check})$$

Capacity

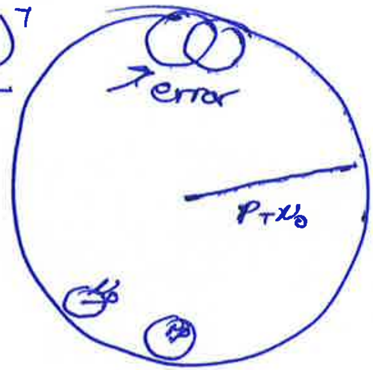
$y = x + w$ encode in time $(x_1\ x_2 \cdots x_T)$

$\Rightarrow\ \underline{y} = \underline{x} + \underline{w}$ $\quad E\{\|\underline{x}\|^2\} = T.P$ $\quad E\{\|\underline{w}\|^2\} = T.N_0.$

$\underline{y}$ (the set of all $\underline{y}$) will be w.h.prob in ball of radius $P + N_0$

Each codewords can you fit in this ball.

$$\#\text{ of codewords} = \frac{\text{Vol (big)}}{\text{Vol (small)}} = \frac{(P+N_0)^T}{(N_0)^T}$$



$$= \left(1 + \frac{P}{N_0}\right)^T$$

$\Rightarrow$ Can Communicate $T.\log\left(1 + \frac{P}{N_0}\right)$

$=$ Capacity AWGN channel.

$y = hx + w$ $\quad$ h fixed. $\quad E\{\|\underbrace{h\underline{x}}_{\text{signal part.}}\|^2\} = |h|^2.P$

$$C = \log\left(1 + \frac{P.|h|^2}{N_0}\right) = \log\left(1 + |h|^2\rho\right).$$

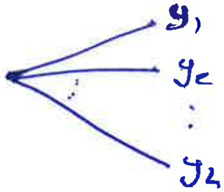
Power Gain.

$y = hx + w \qquad y' = \frac{h^*}{|h|} \cdot y = \frac{h^*}{|h|} \cdot (hx + w) = |h| \cdot x + w'$ → as before

best thing to do. $\Rightarrow \text{SNR} = \frac{E\{\|hx\|^2\}}{E\{\|w\|^2\}} = \frac{|h|^2 \cdot P}{N_0}$

Higher dimension Power Gain.

~~As before.~~ Rx-antenna power gain



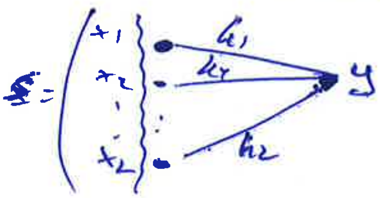
$\underline{y} = \underline{h}x + \underline{w}$

$y' = \frac{\underline{h}^*}{\|h\|}\,\underline{y} = |\underline{h}|x + w' = y$

$y' = \underbrace{|\underline{h}|}_{\text{power up}} x + w \qquad$ as $L \uparrow \Rightarrow \|\underline{h}\|^2 \to L \Rightarrow \text{SNR} \uparrow$ L times

Can see this wrt Capacity $\quad C_{SIMO} = \log\left(1 + \frac{|h|^2 \cdot P}{N_0}\right) \xrightarrow{L \nearrow} \log\left(1 + \frac{P \cdot L}{N_0}\right)$

Tx antenna Power Gain



want so send $s \qquad E\{|s|^2\} = P$

$\underline{x} = \frac{\underline{h}}{\|\underline{h}\|} \cdot s \qquad y = \underline{h}^T \cdot \underline{x} + w$

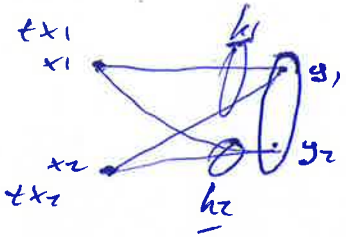
$\qquad = \underline{h}^T \frac{\underline{h}}{\|\underline{h}\|} s + w \qquad \Rightarrow y = \underbrace{|\underline{h}|}_{\text{up to } L \text{ power gain}} s + w$

$C = \log\left(1 + P \cdot \frac{|\underline{h}|^T}{N_0}\right)$

Removing Interference

At Rx

Need multiple Rx-Antennas.



$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \underline{y} = \underline{h}_1 x_1 + \underline{h}_2 x_2 = \begin{bmatrix} \underline{h}_1 & \underline{h}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix}.$$

Interested only in x_1 $\Rightarrow$ $\underline{y}_1' = \frac{\underline{h}_2^{T\perp}}{\|\underline{h}_2^{\perp}\|} \cdot \underline{y} = \frac{\underline{h}_2^{T\perp}}{\|\cdot\|} \underline{h}_1 x_1 + \cancelto{0}{\frac{\underline{h}_2^{T\perp}}{\|\cdot\|} \underline{h}_2} x_2 + w_1'$

$\Rightarrow$ $y_1' = \frac{\underline{h}_2^{T\perp}}{\|\underline{h}_2^{\perp}\|} \underline{h}_1 x_1 + w_1'$ $\Rightarrow$ $\underline{y_1' = h_1' x_1 + w_1}$ removed Interference

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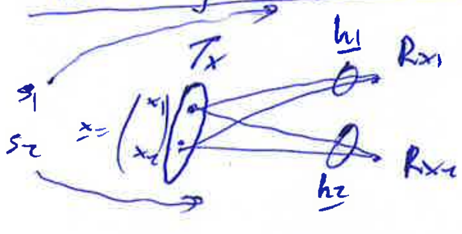
Removing Interference at Tx.

Need multiple Tx-antennas and CSIT.



$$y_1 = \underline{h}_1^{T} \cdot \underline{x} + w_1$$

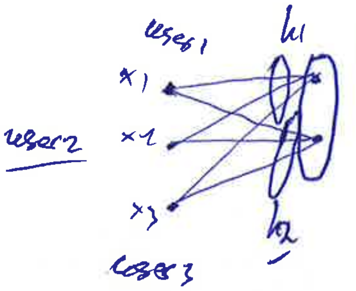
Let $\underline{x} = \frac{\underline{h}_2^{\perp}}{\|\underline{h}_2^{\perp}\|} \cdot s_1 + \frac{\underline{h}_1^{\perp}}{\|\underline{h}_1^{\perp}\|} s_2$

$\Rightarrow$ $y_1 = \underline{h}_1^{T} \frac{\underline{h}_2^{\perp}}{\|\cdot\|} s_1 + \cancelto{0}{\underline{h}_1^{T} \cdot \frac{\underline{h}_1^{\perp}}{\|\cdot\|}} s_2 + w_1'$ $\Rightarrow$ $\boxed{y_1 = h_1' s_1 + w_1'}$ similarly for s_2, y_2.

or Similarly $\underline{y} = \underbrace{\begin{bmatrix} \underline{h}_1 & \underline{h}_2 \end{bmatrix}}_{H} \underline{x} + \underline{w}$ $\quad$ $\underline{x} = \frac{H^{-1}}{\|\cdot\|} \cdot \underline{s}$ $\Rightarrow$ $\underline{y} = H H^{-1} \underline{s} + \underline{w}$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}}{\|\cdot\|} + \underline{w}$$

Multiple Users Removing Interference at Rx



$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

I want x_1

Can ignore user 3, & as before remove out User 2.
only

$$\underline{y} = H\underline{x} + \underline{w} = \begin{bmatrix} \underline{h}_1 & \underline{h}_2 & \underline{h}_3 \end{bmatrix}_{3\times 3} \underline{x} + \underline{w} = \sum_{i=1}^{3} \underline{h}_i x_i$$

Rx only wants x_1.

Consider $\mathbb{R}^n = \mathbb{R}^3$, let S = subspace of dimensionality $m=1$. $\downarrow \underline{h}_1$

$\Rightarrow N \triangleq \{\underline{v} \in \mathbb{R}^n : \underline{v} \perp S\}$ ← null space of S.

$\Rightarrow \dim(N) = n - \dim(S) = n - m$

~~Here $\dim(N) = 3-1 = 2$.~~

$\exists$ $\underline{v} \in N$ s.t $\quad y' = \ . \underline{y}$

looking for vector $\perp \underline{h}_2$ & $\perp \underline{h}_3$ $\quad V_{23} = \text{span}(h_2, h_3) \quad \dim(V) = 2$

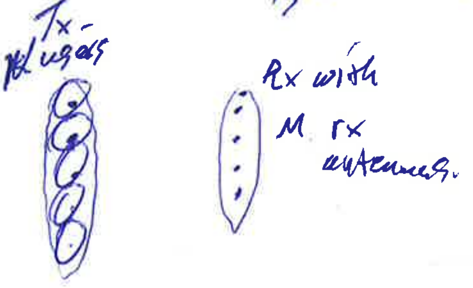
$\text{Null}(V_{23}) \Rightarrow \dim(\text{Null}(V_{23})) = 3-2 = 1$

$\Rightarrow \exists$ vector $\underline{v} \perp h_2 \& h_3$

$\Rightarrow y_1' = \frac{\underline{v}^H \underline{y}}{\|\cdot\|} \Rightarrow \frac{\underline{v}^H}{\|\cdot\|} \sum_{i=1}^{3} \underline{h}_i x_i = \frac{\underline{v}}{\|\cdot\|} \underline{h}_1 x_1 = h_1' x_1 + w = y_1'$

Similarly if I simply wanted x_2 at Rx

$\Rightarrow V_{13} = \text{span}(h_1, h_3) \Rightarrow \underline{v}_2 \in \text{Null}(V_{13}) \Rightarrow y_2' = \underline{v}_2^H \underline{y} \ldots$



$N > M$

Reduce system to

$N \times M$ & Remove $M-1$ users as above

rest $N-M+1$ stay interfering.

T_x antennas, bigger case.
interference cancelation

y_1 - Rx_1
y_2 - Rx_2
y_3 - Rx_3

s_i . For Rx i

$y_i = \underline{h_i}\ \underline{x} + w_i \quad i = 1 \to 3.$

$V_{23} = \text{span}(h_2, h_3)$ ~~$v_1 \in V_{23}$~~ $\underline{v_1} \in Null(V_{23})$

$\underline{v_2} \in Null(V_{13})$, $\underline{v_3} \in Null(V_{12})$.

$\Rightarrow \underline{x} = \underline{v_1}\ s_1 + s_2 v_2 + s_3 v_3$

$\Rightarrow y_1 = \underline{h_1}^T . \underline{x} = \underline{h_1}^T . \underline{v_1} s_1 + \cancel{h_1^T v_2}^{\to 0} s_2 + \cancel{h_1^T v_3}^{\to 0} s_3 + w$ ✓

similarly for the rest.

N tx antennas K users $K \le N$ $\forall$ interference removed

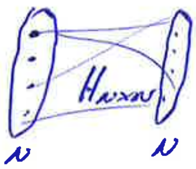
$K > N$ $\forall$ except $N-K$ users

$N \ge K \quad \underline{x} = V . \underline{s} \quad V$ = pseudo inverse.

$N = K \Rightarrow \underline{x} = \dfrac{H^{-1}}{\|\cdot\|} . \underline{s} \quad \Rightarrow \underline{y} = \dfrac{H H^{-1}}{\|H^{-1}\|} . \underline{s} + \underline{w} = \dfrac{\underline{s}}{\|H^{-1}\|} + w$

Similar Concept with MIMO. (1 T_x, 1 R_x, N antennas each)

Multiplexing Gain.



$\underline{y} = H\, \underline{x} + \underline{w} \qquad \underline{\underline{ZF}} \quad \underline{x} = \dfrac{H^{-1}}{\|\cdot\|} \underline{s} \Rightarrow \underline{y} = \dfrac{\underline{s}}{\|\cdot\|} + w$

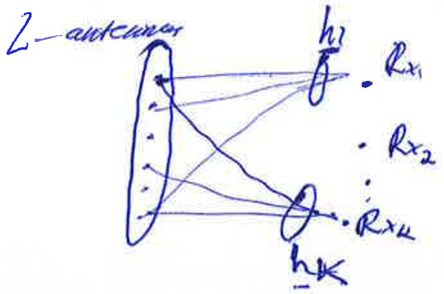
$y_i = d_i . s_i + w_i \quad \forall i$

$\Rightarrow$ $N \times N$ MIMO allows us to send to N users at a time

$C_{\substack{N \times N \\ MIMO}} \approx N . C_{SISO} \quad \text{as } \dfrac{P}{N_0} \uparrow \infty.$

CSIT Needed

Massive Mimo.



$\underline{h}_i \in \mathbb{C}^L$

as $L \uparrow$ and as $\frac{L}{K} \uparrow$

$\Rightarrow \underline{h}_i^T \underline{h}_j \to 0.$

$$\Rightarrow \underline{x} = \sum_{i=1}^{K} \frac{\underline{h}_i^*}{\|\cdot\|} \cdot s_i$$

$$\Rightarrow \underline{y_i \approx |h_i| s_i + \ldots}$$

simplicity in encoding
no inverses.

Space Time Codes for T_x-diversity. (MISO)

Recall $X = \underset{\text{space}}{\downarrow} \overset{\xrightarrow{\text{time}}}{\begin{pmatrix} x & & 0 \\ & \ddots & \\ 0 & & x \end{pmatrix}} \Rightarrow \underline{y}^T = \underline{h}^T \cdot X + \underline{w}^T \Rightarrow \underline{y} = \underline{h}\, x + \underline{w}$

$\Rightarrow y' = \frac{\underline{h}^H}{\|\cdot\|} \underline{y} = |\underline{h}| x + w' = y'$

$\Rightarrow P_{err} \approx \underset{\text{as } P\uparrow}{P^{-L}}$ (div gain of L).

But too slow. 1 symbol per L time-slots.

Consider $L=2$.

$$\begin{pmatrix} s_1 & -s_2^* \\ s_2 & s_1^* \end{pmatrix}$$

$$(y_1 \;\; y_2) = (h_1 \;\; h_2) \begin{pmatrix} s_1 & -s_2^* \\ s_2 & s_1^* \end{pmatrix} + w_1 \; w_2 \quad \Rightarrow \underbrace{\begin{pmatrix} y_1 \\ y_2^* \end{pmatrix}}_{\underline{y}'} = \left(\overset{\underline{h}_1}{\begin{pmatrix} h_1 \\ h_2^* \end{pmatrix}} \overset{\underline{h}_2}{\begin{pmatrix} -h_2^* \\ h_1 \end{pmatrix}} \right) \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} + \begin{pmatrix} w_1' \\ w_2' \end{pmatrix}$$

$y_1'' = \underline{h}_1^T \cdot \underline{y}' = |h| s_1 + w'$, $\quad y_2'' = \underline{h}_2^T \underline{y}' = |\underline{h}| \cdot s_2 + w_2$.

Again full diversity but also sending faster.

Degrees of Freedom

What is problem when repeating?

$(y_1\ y_2\ \cdots\ y_L) = (h_1\ h_2\ \cdots\ h_L) \cdot x + (w_1\ \cdots\ )$

signal in one dimension, span be $\underline{h} = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_L \end{pmatrix}$.

L-dimensional space $\mathbb{C}^L$ available

BUT only using one dimension.

Whereas (e.g $L=2$).

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = H \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ space is two dimensional
(L) - " .

not repetition

Give example of room

Higher # of dimensions - can pack things better
" " " " - increase distance between codewords.

Gains from multiple antennas

diversity gains = L i.e. $P_{err} \approx P^{-1} \longrightarrow P^{-L}$

$\Rightarrow$ for same SNR $= P$ $\Rightarrow$ $P_{err} \downarrow$ exponentially

or same P_{err} with Power : $P \downarrow \rightarrow P^{1/L}$

Power gain. $(L \uparrow)$ ~~Rx~~ Power : $P \rightarrow L.P$ ✓

(humble but important).

Multiplexing gain (MIMO & MU-MISO / MIMO).

1 symbol served at a time $\longrightarrow$ L symbols

gain nice seen by capacity.

($\approx$ $C \longrightarrow L.C$ (as $P \uparrow$ at high SNR).

$\Rightarrow$ Power $P \rightarrow P^{1/L}$

(treat carefully).