Capacity – Info Theo 2

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- ▶ Fundamental limits : best performance.
- ▶ Tool: information theory.
- ▶ Capacity: max rate s.t. $P_{err} \to 0$
- ▶ First AWGN, then fading channels (calculate capacity)
- ▶ Derive diversity results
- ▶ Will give insight: allocate W and $\bar{P}$

- ▶ Inventor of IT: Shannon (1948).
  - › $\exists R > 0$ s.t. $P_{err} \to 0$ for long T
  - › $C = \max R$: $P_{err}(R) \to 0$ as $T \to \infty$
  - › and $P_{err} \to 1 \; \forall R > C$

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► Consider AWGN (Additive White Gaussian Noise)

$$y[m] = x[m] + w[m]$$

> gives sense to $P_{err} \to 0$
> explain capacity

► 1) T=1
  > Since $x[m] = \pm\sqrt{P}$, means

$$P_e = Q(\frac{half - distance}{\sigma_w}) = Q(\frac{\sqrt{P}}{\sigma_w})$$

  > Not reliable: P_e fails to go 0

- ▶ 2) T large: Consider Repetition Coding:

$$\mathbf{x}_A = \sqrt{P}[1, 1, ..., 1], \quad \mathbf{x}_B = -\sqrt{P}[1, 1, ..., 1]$$

- ˃ Let $\mathbf{x} = \mathbf{x}_A$

$$\Rightarrow P_e = Q(\sqrt{\frac{||\mathbf{x}_A - \mathbf{x}_B||^2}{4\sigma^2}})$$

$$= Q(\sqrt{\frac{||\sqrt{P}(2, 2, 2, ..., 2)||^2}{4\sigma^2}})$$

$$= Q(\sqrt{\frac{||4\sqrt{P}(1, 1, 1, ..., 1)||^2}{4\sigma^2}}) = Q(\sqrt{\frac{P}{\sigma^2} \sum_{i=1}^{N} 1}) = Q(\sqrt{\frac{NP}{\sigma^2}})$$

$$\approx e^{-\frac{NP}{\sigma^2}} \xrightarrow{N \to \infty} 0$$

- ˃ but

$$R(rate) = \frac{1}{N}$$

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- **M-PAM**

  $$\mathbf{x} = \sqrt{P}[q, q, ..., q], \quad q \in \{-(M-1), ..., -1, 1, 3, ..., M-3, M-1\}$$

  > example: $M = 6$
  $$q \in \{-5, -3, -1, 1, 3, 5\}$$

- Say

  $$||\mathbf{x}|| \leq PN, \ \forall \mathbf{x}$$
  $$\Rightarrow ||\sqrt{P}\theta[5, 5, .., 5]||^2 \leq NP$$
  $$||\theta[5, 5, ..., 5]||^2 \leq N \Rightarrow \theta = \frac{1}{M-1}$$

  half distance between $\sqrt{P}\theta[1, 1, 1, ..., 1] \rightarrow \sqrt{P}\theta[-1, -1, ..., -1]$

  $$||\sqrt{P}\frac{1}{M-1}[1, 1, 1, ..., 1]||^2 = \frac{PN}{(M-1)^2}$$

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► Probability of error

$$\Rightarrow P_e = Q(\frac{half - dist}{\sigma_w}) = Q(\sqrt{\frac{PN}{(M-1)^2}})$$

► Reliable communications if
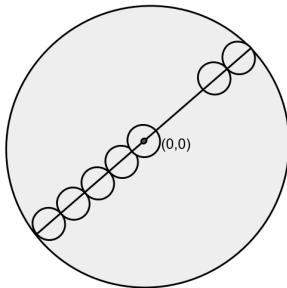
$$\Rightarrow (M-1)^2 < N \Rightarrow M < \sqrt{N}$$

► Bound on rate

$$\Rightarrow R = \frac{\log M}{N} = \frac{\log \sqrt{N}}{N} \to 0$$

- ▶ Repetition code packed every codeword onto single dimension
- ▶ N-dimensional sphere. ($N = 2$ here, but generally very large)

- ▶ But we have N dimensions
  - › Codewords must not be close because noise $\rightarrow$ error.
  - › Must decide on a distance between codewords
- ▶ Try to pack as many such spheres as possible
- ▶ A sphere packing problem in N-dim space

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1 Sphere packing

▶ Packing sphere → to reach capacity.

▶ Sphere packing problem: pack max number of codewords in sphere of certain radius

▶ recall:

$$\mathbf{y} = \mathbf{x} + \mathbf{w}, \quad ||\mathbf{x}||^2 \leq NP$$

▶ as $N \to \infty$

$$\frac{1}{N} \sum_{i=1}^{N} ||w_i||^2 \to \sigma^2 \Rightarrow ||\mathbf{w}||^2 \to N\sigma^2 \Rightarrow ||\mathbf{y}||^2 = ||\mathbf{x}+\mathbf{w}||^2 \leq NP+N\sigma^2$$

▶ thus

$$||\mathbf{y}||^2 \leq N(P+\sigma^2) \Rightarrow \mathbf{y} \in Ball(rad = \sqrt{N(P+\sigma^2)}), \quad \text{with Prob} \to 1$$

▶ Also given $\mathbf{X}_A$ (transmit codeword) and given $||\mathbf{w}|| \to \sqrt{N\sigma^2}$, then

▶ $\mathbf{y}$ is on the periphery of a sphere, centered on $\mathbf{X}_A$, with radius $\sqrt{N\sigma^2}$

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1 Sphere packing

▶ Try to pack max number of sphere of radius $\sqrt{N\sigma^2}$ in a bigger sphere of radius $\sqrt{N(P+\sigma^2)}$

▶ This max number is

$$\frac{Vol(Ball(rad = \sqrt{N(P+\sigma^2)}))}{Vol(Ball(rad = \sqrt{N\sigma^2}))} \approx \frac{(\sqrt{N(P+\sigma^2)})^N}{(\sqrt{N\sigma^2})^N} = \left(\frac{N(P+\sigma^2)}{N\sigma^2}\right)^{\frac{N}{2}}$$

▶ Thus max rate $R = \frac{1}{N}\log\left(\frac{N(P+\sigma^2)}{N\sigma^2}\right)^{\frac{N}{2}} = \frac{1}{2}\log(1+\frac{P}{\sigma^2})$

▶ The above is per <u>real</u> space. But we have complex space, thus

$$C = \log(1 + \frac{P}{\sigma^2}) \text{ bits/complex dimension}$$

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- ▶ Cap gives the challenge.
- ▶ Codes : another Story
- ▶ Decoders : another story.
- ▶ People relatively happy with various practical codes
  - > turbo codes (interesting story)
  - > LPPC codes (also interesting story)
- ▶ Commonly used soft decoders (complicated).

- ▶ Capacity analysis helps us allocate resources
- ▶ Total Power $\bar{P}$ (Watts)
- ▶ Total bandwidth W (Hz)
- ▶ Thus $\frac{\bar{P}}{W}$ $(J/s/Hz) = (Watts/Hz)$
- ▶ Noise: $\sigma_w^2 = \frac{N_0}{2}$
- ▶ Sample at rate $\frac{1}{W}$ gives

$$y[m] = x[m] + w[x]$$

  - > Complex baseband representation (two independent uses of the real channel)
- ▶ $\frac{N_0}{2}$ noise power per real symbol $\Rightarrow N_0$ power of $w[x]$
- ▶ $\frac{\bar{P}}{2W}$ power constraint per real symbol.

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- $C_{real} = \frac{1}{2} \log(1 + \frac{P_{real}}{\sigma_{real}^2}) = \frac{1}{2} \log(1 + \frac{\frac{\bar{P}}{w}}{\frac{N_0}{2}}) = \frac{1}{2} \log(1 + \frac{\bar{P}}{wN_0})$

$$\Rightarrow C_{AWGN} = 2C_{real} = \log(1 + \frac{\bar{P}}{wN_0})$$

- Consider $SNR = \frac{\bar{P}}{wN_0}$ (note: SNR wrt complex dimension)

$$\Rightarrow C_{complex} = \log(1 + SNR) \quad \text{(bits/complex symbol)}$$

- C_{AWGN}: called spectral efficiency.
- Total capacity

$$C_{AWGN}(\bar{P}, W) = WC_{complex} = W\log(1+SNR) = W\log(1+\frac{\bar{P}}{WN_0}) \quad bits/sec$$

▶ Recall

$$C(\bar{P}, w) = W \log(1 + \frac{\bar{P}}{WN_0})$$

> Choose (invest in) $\bar{P}$ and/or W

▶ Note:
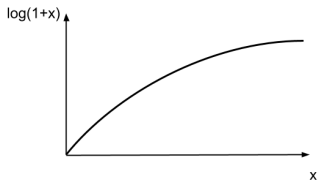
> log is concave, i.e., $F''(SNR) \leq 0$

>

$$\log(1 + x) \approx x \log_2 e, \text{ as } x \to 0, \quad \text{(linear)}$$

>

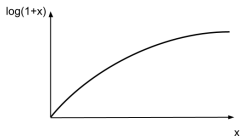$$\log(1 + x) \to \log(x), \text{ as } x \to \infty, \quad \text{(logarithmic)}$$

▶ low SNR: $\bar{P} \uparrow \Rightarrow C \uparrow$ linearly $\Rightarrow$ invest in $\bar{P}$

▶ high SNR: $\bar{P} \uparrow \Rightarrow C \uparrow$ logarithmically $\Rightarrow$ do NOT invest in $\bar{P}$

- ▶ Fix $\bar{P}$
- ▶ C increasing function of W
- ▶ if W small $\Rightarrow$ SNR large $\Rightarrow W \uparrow \Rightarrow \log(\cdot) \downarrow$ a little

  $\Rightarrow W \log(\cdot) \uparrow$ almost linearly $\Rightarrow$ invest in W (bandwidth limited)

- ▶ if W large $\Rightarrow$ SNR small

  $\Rightarrow \log(1 + x) \downarrow$ linearly $\Rightarrow$ compensates for $W \uparrow$ linearly

  $$W \log(1 + \frac{\bar{P}}{N_0 W}) \approx W \log_2 e \frac{\bar{P}}{N_0 W} = \log_2 e \frac{\bar{P}}{N_0}$$

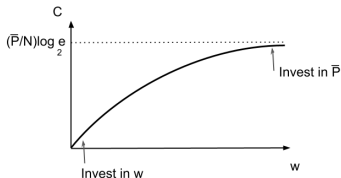 $>$ do not invest in W, but invest in $\bar{P}$ (linear increase) (power limited)

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▶ Minimize $\frac{E_b}{N_0}$: Power efficient region $\bar{P} \to 0$ (turns out)

$$\left(\frac{E_b}{N_0}\right)_{min} = \lim_{\bar{P} \to 0} \frac{\bar{P}}{N_0 C_{AWGN}(\bar{P}, W)} = \frac{\bar{P}}{N_0 W \log_2(1 + \frac{\bar{P}}{N_0 W})}$$

$$\approx \frac{\bar{P}}{N_0 W \log_2 e \frac{\bar{P}}{N_0 W}} = \frac{1}{\log_2 e}$$

$$\Rightarrow 10 \log_{10} \frac{1}{\log_2 e} = -1.59 dB$$

> Minimum possible power to send a single bit
> Orthogonal codes from Hadamard sequences

Capacity of LTI Gaussian channels

- ▶ The SIMO channel (*L*-receive antennas)

$$y_\ell[m] = h_\ell x[m] + w_\ell[m], \quad \ell = 1, ..., L, \quad w_\ell \sim \mathbb{C}N(0, N_0)$$

  - > Channel vector $\mathbf{h} = [h_1, ..., h_L]^T$ is fixed

- ▶ Capacity achieving policy: <u>receive beamforming</u>

$$\tilde{y}[m] = \frac{\mathbf{h}^\dagger}{||\mathbf{h}||}\mathbf{y}[m] = \frac{||\mathbf{h}^\dagger||^2}{||\mathbf{h}||}x[m] + \overbrace{\frac{\mathbf{h}^\dagger}{||\mathbf{h}||}\mathbf{w}[m]}^{AWGN \ \sigma^2=1}$$

  - > $x[m]$ from a capacity achieving AWGN code
  - > Optimal since $\tilde{y}[m]$ is <u>sufficient statistic</u>

- ▶ Thus

$$SNR = ||\mathbf{h}^\dagger||^2\frac{P}{N_0}, \quad E[||x[m]||^2] \leq P \ (Watts/symbol)$$

- ▶ Thus capacity takes form

$$C_{\text{SIMO}} = \log(1 + \frac{P}{N_0}||\mathbf{h}||^2)$$

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1 Capacity of Multiple Input Single Output channels

▶ The MISO channel (L-transmit antennas)

$$y[m] = \mathbf{h}^T \mathbf{x}[m] + w[m], \quad w[m] \sim \mathbb{C}N(0, N_0), \quad \mathbf{h} = [h_1, ..., h_L]^T \text{ fixed}$$

▶ Capacity achieving policy: transmit beamforming optimal (proof skipped)

$$\mathbf{x}[m] = \frac{\mathbf{h}^T}{||\mathbf{h}||} \tilde{x}[m]$$

  > $\tilde{x}[m]$ from opt. AWGN code. $E[||\tilde{x}[m]||^2] \leq P$

▶ Thus

$$y[m] = ||\mathbf{h}|| \tilde{x}[m] + w[m]$$

▶ Thus MISO capacity is

$$C_{tx-beam} = C_{MISO} = \log(1 + \frac{P||\mathbf{h}||^2}{N_0}) \ \ bits/s/Hz$$

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▶

$$y[m] = \sum_{\ell=0}^{L-1} h_{[}m - \ell] + w[m], \quad E[|x[m]|^2] \leq P \quad (1)$$

> L taps (convolution)

▶ Solution: Convert to N_c parallel channels (N_c info symbols $+ L$ taps)

▶ ith block

$$\tilde{y}_n[i] = \tilde{h}_n \tilde{d}_n[i] + \tilde{w}_n[i] \quad n = 0 \to N_c - 1 \quad (2)$$

$\tilde{\mathbf{d}}[i] = [\tilde{d}_0[i] \to \tilde{d}_{N_c-1}[i]]$, DFT of input of ith block, $||\tilde{\mathbf{d}}[i]||^2 \leq N_c P$ (Pars

$\tilde{\mathbf{w}}[i] = [\tilde{w}_0[i] \to \tilde{w}_{N_c-1}[i]]$, DFT of noise: $\tilde{\mathbf{w}}[i] \sim \mathbb{C}N(0, N_0 I)$

$\tilde{\mathbf{y}}[i] = [\tilde{y}_0[i] \to \tilde{y}_{N_c-1}[i]]$, DFT of output of ith block (after some remova

$\tilde{\mathbf{h}}_{1 \times N_c} = [\tilde{h}_0 \to \tilde{h}_{N_c-1}]^T = \mathbf{F}_{fourier}\mathbf{h}$, ($N_c -$ point DFT)

▶ As $N_c \to \infty \Rightarrow N_c \gg L$
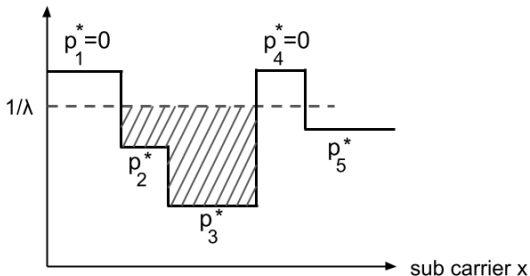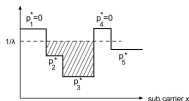
$$\Rightarrow C_{(1)} \approx C_{(2)}$$

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▶ Power allocation, then use capacity-achieving code over each parallel channel

$$C_{N_c} = \max_{P_0 \to P_{N_c-1}} \sum_{n=0}^{N_c-1} \log(1 + \frac{P_n|\tilde{h}_n|^2}{N_0}), \quad s.t. \sum_{n=0}^{N_c-1} P_n = N_c P, \ P_n \geq 0, \forall n$$

▶ Optimal $\{P_n\}$ via water-filling (y-axis is $\frac{N_0}{|\tilde{h}_n|^2}$)

- Write Lagrangian $\mathcal{L}(\lambda) = \sum_{n=0}^{N_c-1} \log(1 + \frac{P_n|\tilde{h}_n|^2}{N_0}) - \lambda \sum_{n=0}^{N_c-1} P_n$

- Then evaluate $\frac{\delta \mathcal{L}}{\delta P_n} = \begin{cases} 0 & \text{if } P_n > 0 \\ \leq 0 & \text{if } P_n = 0 \end{cases}$

- Optimal power

$$P_n^* = \left(\frac{1}{\lambda} - \frac{N_0}{|\tilde{h}_n|^2}\right)^+, \quad \text{where } \lambda : \sum_{n=0}^{N_c-1} \overbrace{\left(\frac{1}{\lambda} - \frac{N_0}{|\tilde{h}_n|^2}\right)^+}^{P_n^*} = PN_c$$

- Coding across parallel channels does not improve capacity (turns out)
  - > but gives better performance if block length is small

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Capacities of Fading Channels

▶ Fading channel

$$y[m] = h[m]x[m] + w[m], \quad w[m] \sim \mathbb{C}N(0, N_0), \quad SNR = \frac{P}{N_0}$$

▶ h randomly drawn, with $E[|h[m]|^2] = 1$
▶ For now only CSIR (No CSIT)
▶ Slow fading "channel randomly drawn but here to stay"

$$\Rightarrow y[m] = hx[m] + w[m], \Rightarrow SNR_{Rx} = |h|^2 SNR$$

$$C(h) = \log(1 + SNR|h|^2)$$

▶ No lower-limit ($C(h) \to 0$ maybe) (NO CSIT)
▶ Consider <u>outage</u> event

▶ ...Consider <u>outage</u> event

$$\text{Error} \iff \text{Capacity}(h) < \text{Rate}$$

▶ Need another measure: $P(C(h) < R) = P(outage)$

   > applies when R is fixed

$$P_{out}(R) = P(\log(1 + SNR|h|^2) < R) \Rightarrow P(|h|^2 < \frac{2^R - 1}{SNR})$$

   > Note

$$P(|h|^2 < \epsilon) \xrightarrow{\epsilon \to 0} \epsilon$$

$$\Rightarrow P_{out}(R) = P(|h|^2 < \frac{2^R - 1}{SNR}) \xrightarrow{SNR \to \infty} \frac{2^R - 1}{SNR}$$

$$\Rightarrow P_{out}(R) \to \frac{1}{SNR}$$

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▶ Another measure of interest - ϵ-outage capacity

$$y = hx + w$$

▶ Recall probability of outage

$$P_{out}(R) = P(\log(1 + \rho|h|^2) < R) = P(|h|^2 < \overbrace{\frac{2^R - 1}{\rho}}^{x}) = \epsilon$$

▶ Definition: ϵ-outage capacity

$$C_\epsilon = \max_R \ : \ P_{out}(R) \le \epsilon$$

▶ Evaluate C_ϵ for any SNR

$$\text{Let} \quad x \triangleq \frac{2^R - 1}{\rho} \ \Rightarrow \ P(|h|^2 < x) = \epsilon \ \Rightarrow \ P(|h|^2 \geq x) = 1 - \epsilon$$

▶ and denote

$$F(x) \triangleq P(|h|^2 > x) \ \Rightarrow \ F(x) = 1 - \epsilon \Rightarrow x = F^{-1}(1 - \epsilon)$$

$$\Rightarrow \ \frac{2^R - 1}{\rho} = F^{-1}(1 - \epsilon) \Rightarrow 2^R = 1 + \rho F^{-1}(1 - \epsilon)$$

▶ Thus

$$C_\epsilon = \log(1 + SNR \cdot F^{-1}(1 - \epsilon))$$

▶ We have

$$C_\epsilon = \log(1 + SNR \cdot F^{-1}(1 - \epsilon))$$

▶ At high SNR

$$C_\epsilon \approx \log(SNR \cdot F^{-1}(1 - \epsilon)) = \log SNR + \log(F^{-1}(1 - \epsilon))$$

▶ Thus

$$C_\epsilon = \log SNR - \overbrace{\log(\frac{1}{F^{-1}(1 - \epsilon)})}^{\text{constant offset}}$$

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▶ Low SNR

$$C_\epsilon \approx \log_2 eSNR \cdot F^{-1}(1-\epsilon)) \approx F^{-1}(1-\epsilon)C_{AWGN}$$

▶ For Rayleigh fading

$$P(|h|^2 < \epsilon) \approx \epsilon \Rightarrow P(|h|^2 > \epsilon) \approx 1-\epsilon \Rightarrow F(\epsilon) \approx 1-\epsilon \Rightarrow F^{-1}(1-\epsilon) \approx \epsilon$$

$$C_\epsilon \approx \epsilon SNR \approx \epsilon C_{AWGN}$$

▶ Big problem: Example

$$\epsilon \approx 1\% \rightarrow C_\epsilon \approx \frac{C_{AWGN}}{100}$$

▶ Thus diversity is important

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▶ Consider SIMO (R_x- diversity) to boost C_ϵ at low SNR

$$P_{out}(R) = P\big(\log(1 + \rho|\mathbf{h}|^2) < R\big) \xrightarrow{\rho \to 0} P(\log_2 e \cdot SNR \cdot |\mathbf{h}|^2 < R)$$

$$\Rightarrow P_{out}(R) \approx P\left(|\mathbf{h}|^2 < \overbrace{\frac{R}{SNR \cdot \log_2 e}}^{\text{small: } R << \rho}\right) \overset{\text{want}}{=} \epsilon$$

$$P_{out}(R) \approx P(|h_0|^2 < \frac{R}{SNR}, ..., |h_{L-1}|^2 < \frac{R}{SNR}) = \left(P(|h_0|^2 < \frac{R}{SNR})\right)^L = \epsilon$$

$$\Rightarrow P(|h_0|^2 < \frac{R}{SNR}) = \epsilon^{\frac{1}{L}} =: \delta$$

$$\Rightarrow \frac{R}{SNR} = \delta \quad (\text{since } P(|h_0|^2 < \delta) \approx \delta)$$

$$\frac{R}{SNR} = \epsilon^{\frac{1}{L}} \Rightarrow R = \epsilon^{\frac{1}{L}} SNR \Rightarrow C_\epsilon = \epsilon^{\frac{1}{L}} C_{AWGN}$$

$$y_0[m] = h_0 x_0[m] + w_0[m]$$

$$\vdots$$

$$y_{L-1}[m] = h_{L-1} x_{L-1}[m] + w_{L-1}[m]$$

$$C(\mathbf{h}) = \sum_{\ell=0}^{L-1} \log(1 + |h_\ell|^2 SNR) \to 0$$

$$P_{out}(R) = P\left(\frac{1}{L} \sum_{\ell=0}^{L-1} \log(1 + |h_\ell|^2 SNR) < R\right)$$

▶ Fast Fading: $T_{code} \gg T_c \quad \Rightarrow L \to \infty$

$$\frac{1}{L} \sum_{\ell=0}^{L-1} \log(1 + |h_\ell|^2 SNR) \to E_{\mathbf{h}}[\log(1 + |h|^2 SNR)]$$

▶ Fast-fading (ergodic) capacity

$$C_{FF} = E_h\{\log(1 + |h|^2 SNR)\}$$

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▶ Employ Jensen's inequality $E\{f(u)\} \le f(E\{u\})$

$$E_h\{\log(1 + \rho|h|^2)\} = \log(1 + \rho E\{|h|^2\}) \Rightarrow C_{FF} \le C_{AWGN}$$

▶ But generally very close

▶ Low SNR:

$$C_{FF} = E\{\log(1+\rho|h|^2)\} \approx E\{\log_2 e\rho|h|^2\} = \log_2 e\rho \overbrace{E\{|h|^2\}}^{1} = C_{AWGN}$$

▶ High SNR:

$$C_{FF} \approx E\{\log(\rho|h|^2)\} = \log\rho + E\{\log|h|^2\} \approx C_{AWGN} - \overbrace{0.83}^{2.5dB}$$

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- ▶ TDD reciprocity, FDD feedback (expensive)
- ▶ Slow fading: could do channel inversion, or rate adaptation i.e

$$\log(1 + \frac{P_t}{N_0}|h|^2) < R_t$$

> $\Rightarrow$ change R_t or P_t (When bad channel)

▶ Fast Fading: Waterfilling

$$y_0 = h_0 x_0 + w_0$$

$$y_1 = h_1 x_1 + w_1$$

$$\vdots$$

$$y_{L-1} = h_{L-1} x_{L-1} + w_{L-1}$$

$$\Rightarrow \max_{P_0 \to P_{L-1}} \frac{1}{L} \sum_{\ell=0}^{L-1} \log(1 + \frac{P_\ell}{N_0}|h_\ell|^2), \qquad \sum P_\ell = LP$$

▶ Seen

$$P_\ell^* = (\frac{1}{\lambda} - \frac{N_0}{|h_\ell|^2})^+, \quad for \quad \lambda \quad : \quad \frac{1}{L}\sum_{\ell=0}^{L-1}(\frac{1}{\lambda} - \frac{N_0}{|h_\ell|^2})^+ = P$$

▶ Causality an issue, BUT for large L

$$\frac{1}{L}\sum_{\ell=0}^{L-1}(\frac{1}{\lambda} - \frac{N_0}{|h_\ell|^2})^+ \to E\{(\frac{1}{\lambda} - \frac{N_0}{|h|^2})^+\}$$

$$\Rightarrow P_\ell^*(\mathbf{h}) = (\frac{1}{\lambda} - \frac{N_0}{|h_\ell|^2})^+ \text{ for } \lambda : E\{(\frac{1}{\lambda} - \frac{N_0}{|h_\ell|^2})^+\} = P$$

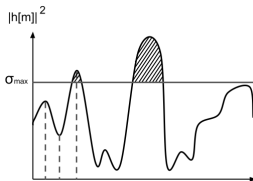$$\Rightarrow C = E\{\log(1 + P^*(h)\frac{|h|^2}{N_0})\}$$

▶ In the end: For most SNR

$$C_{AWGN} \approx C_{FF} \approx C_{CSIT}$$

▶ except (in theory) at low SNR where $C_{CSIT} >> C_{AWGN}$ WARNING!!

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▶ Send only if $|h[m]|^2 \approx G_{max}$, and store power while not sending

$$C \approx P\left(|h|^2 \approx G_{max}\right) \log\left(1 + \frac{G_{max} SNR}{P(|h|^2 \approx G_{max})}\right) \approx G_{max} \rho \log_2 e \approx G_{max} C_{AWGN}$$
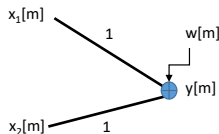
▶ Would not work in practice!!

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▶ Up-link AWGN (multiple-access channel)

$$y[m] = x_1[m] + x_2[m] + w[m]$$

> $E\{|x_k[m]|^2 = P_k, k = 1, 2, \quad w[m] \sim \mathbb{CN}(0, N_0)$ iid

▶ Capacity region: Best possible R_1, R_2. First derive outer bounds:

$$R_1 < \log(1 + \frac{P_1}{N_0})$$

$$R_2 < \log(1 + \frac{P_2}{N_0}) \quad \text{single user bounds}$$

$$R_1 + R_2 < \log(1 + \frac{P_1 + P_2}{N_0}), \quad \text{MISO bound with fixed } h_1 = h_2 = 1$$

> $E\{|\underline{x}|^2\} = P_1 + P_2$, where $\underline{x} = [x_1, x_2]^T$.

- Let us achieve these outer bounds.
- Use capacity-achieving codes for $x_1[m]_{m=1}^{\cdots}$, $x_2[m]_{m=1}^{\cdots}$
- First R_x decodes message of user 2 (x_2) by treating x_1 as noise:

$$\Rightarrow R_2 = \log(1 + \frac{P_2}{P_1 + N_0})$$

- Now x_2 removed from y to decode x_1 without interference

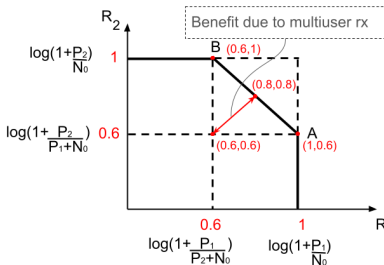$$\Rightarrow R_1 = \log(1 + \frac{P_1}{N_0}) \quad \text{naturally optimal}$$

- but note that

$$R_1 + R_2 = \log[1 + \frac{P_1}{N_0}] + \log[1 + \frac{P_2}{P_1 + N_0}] = \log[1 + \frac{P_1 + P_2}{N_0}]$$

is again optimal

- but if R_1 is optimal & $R_1 + R_2$ is optimal then R_2 is also optimal.

To achieve point $A = [\log(1 + \frac{P_1}{N_0}), \log(1 + \frac{P_2}{P_1+N_0})]$

> decode x_2 first by treating x_1 as noise $\log(1 + \frac{P_2}{P_1+N_0})$
> Remove x_2 and decode x_1 without interference

Similarly (reverse order) for point $B = [\log(1 + \frac{P_1}{P_2+N_0}), \log(1 + \frac{P_2}{N_0})]$

> decode x_1 first by treating x_2 as noise $\log(1 + \frac{P_1}{P_2+N_0})$
> Remove x_1 and decode x_2 without interference

Line AB contains all optimal points (optimal sum-rate)

Example $P_1 = P_2 = 1$

▶ Considering the non-symmetric setting, say $P_1 >> P_2$.
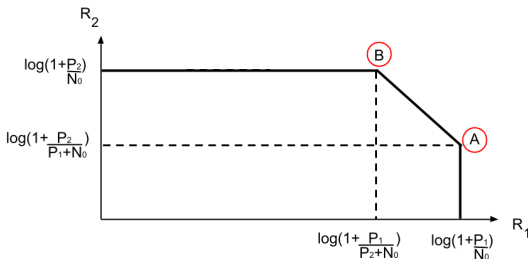
▶ Whom should we decode first?
   › decode strongest first (small interference from weaker)
   › then fully decode weaker (point B)
   › Slightly bother U1, no interference for U2 (Reverse would kill U2)

▶ Strong users close to BS transmit fast, with no trouble to rest
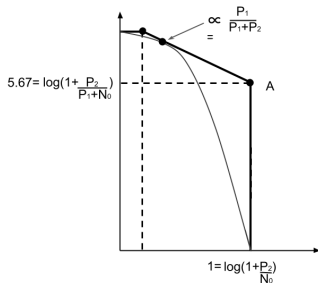   › No power control. Turns near-far issue into advantage.

▶ Orthogonal Multiple access techniques Optimal? $(\alpha, 1 - \alpha)$

$$R_1 = \alpha W \log(1 + \frac{P_1}{\alpha N_0}), \quad R_2 = (1 - \alpha) W \log(1 + \frac{P_2}{(1 - \alpha) N_0}) \text{ bits/s}$$

▶ $\exists$ only one optimal α : $\alpha = \frac{P_1}{P_1 + P_2}$ that achieves opt sum-capacity

  > Penalizes weak users $(P_1 \gg P_2 \Rightarrow \alpha \approx 1, \Rightarrow 1 - \alpha \approx 0)$
  > Logarithmic increase vs. linear decrease
  > Giving high rate to weak user $\Rightarrow$ entirely sacrifice rate for strong user
  > Example $P_2 = 0dB, P_1 = 20dB$.

▶ for $P_i = P_j = P$

$$C_{sum} = \log(1 + \frac{kP}{N_0})$$

$$C_{sym} = \frac{1}{k} \log(1 + \frac{kP}{N_0}) \quad \text{max rate when all have the same rate}$$

▶ achievable with ?? (say k=2 , $\alpha = \frac{1}{2}$, $\alpha \log(1 + \frac{P}{\alpha N_0})$, $(1 - \alpha) \log(1 + \frac{P}{(1-\alpha)N_0})$) under equal received power

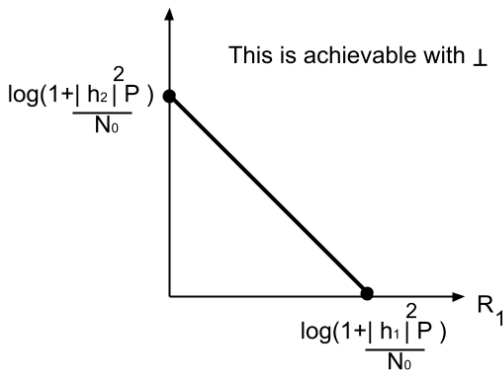▶ OFDM ¿ CDMA , CDMA treats all others as noise

$$C_{sum} = k \log(1 + \frac{P}{(k-1)P + N_0}) \approx k \frac{P}{(k-1)P + N_0} \log_2 e \approx l \log_2 e \approx 1.44$$

$$\log(1 + \frac{kP}{N_0}) \uparrow \infty$$

- $y_k[m] = h_k \quad x[m] + w_k[m] \quad k = 1?? ?? $ k fixed
- assume CSTIR $|X[m]|^2 \leq P \quad \frac{Joules}{symbol}$
- Capacity region R_1, R_2
- $R_k \leq \log(1 + \frac{P|h_k|^2}{N_0})$ all power for user K and rate

$$X[m] = X_1[m] + X_2[m] \quad \textit{iid Gaussian codes}$$

$$R_1 = \log(1 + \frac{P_1|h_1|^2}{P_2|h_1|^2 + N_0}) = \log(1 + \frac{(P_1 + P_2)|h_1|^2}{N_0}) - \log(1 + \quad)$$

$$R_2 = \log(1 + \frac{P_2|h_2|^2}{N_0})$$
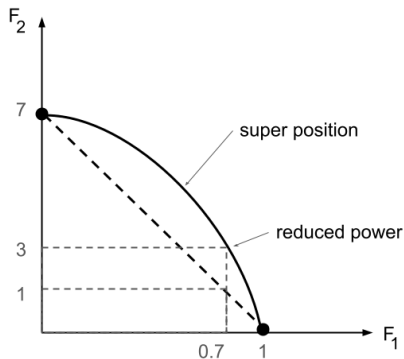
▶ User 2 can decode anything ??/

$$x = x_1 + x_2$$

$$R_1 = \log(1 + \frac{P_1|h_1|^2}{|h_1|^2 P_2 + N_0}) \quad \textit{weakerseesinterference}$$

$$R_2 = \log(1 + \frac{P_2|h_2|^2}{N_0}) \quad \textit{userseesnointerference}$$

▶ first, decode user 1 data and then Which is generally better than ?? (for all power splits $P_1 + P_2 = P$)

$$R_1 = \alpha \log(1 + \frac{P_1|h_1|^2}{\alpha N_0})$$

$$R_2 = (1 - \alpha) \log(1 + \frac{P_2|h_2|^2}{(1-\alpha)N_0})$$

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▶ In General

$$R_k = \log(1 + \frac{P_k |h_k|^2}{N_0 + \sum_{J=k+1}^{K} P_J |h_k|^2}) \quad P = \sum_{k=1}^{K} p_k$$

▶ sum capacity here (unlike uplink which all user R_x)

▶ Have seen that for parallel channel

$$y_1 = h_1 x_1 + w_1$$

$$y_2 = h_2 x_2 + w_2$$

$$R = \log(1+??|h_1|^2) + \log(1+??|h_2|^2) = \log[(1+??|h_1|^2)(1+??|h_2|^2)]$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$= \log det[I_2 + ?? \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \times \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}^H]$$

▶ Consider the mimo channel

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = H\underline{x} + \underline{w} = u.\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} u^H \underline{x} + \underline{w}$$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \times \underline{x}' + w$$

▶ in General, Mimo capacity is in order of (higher SNR)

$$\log det[I_{n_r} + ?? HH^H]$$

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