STATS 2021 Exam

First name, family name

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(Please also write your name at the end of each page)

December 9, 2021

1 Mean Estimation

Let P be a probability distribution on $\mathbb{R}$ with mean $\mu = 1$ and variance $\sigma^2 = 10$. Let $X_1, \dots, X_n \in \mathbb{R}$ i.i.d. data points from P, where $n = 9$.

Let $c > 0$ be a constant, and consider the following estimator of μ:

$$\hat{\mu} = c \sum_{i=1}^{n} X_i.$$

Q1 (1pt). Derive the mean of $\hat{\mu}$ in terms of c.

Answer:

Q2 (1pt). Derive the variance of $\hat{\mu}$ in terms of c.

Answer:

Q3 (1pt). Derive the mean square error of $\hat{\mu}$ in terms of c.

Answer:

Q4 (1pt). Consider the estimator $\hat{\mu}$ in the following two cases: (i) $c = 0$ and (ii) $c = 1/n$. Answer which of (i) and (ii) has a smaller mean square error.

Answer:

2 Maximum Likelihood Estimation

Let $\mathcal{P}_\Theta := \{p_\theta \mid \theta \in \Theta\}$ be a parametric model, where $\Theta \subset \mathbb{R}^q$ is a set of parameters and $p_\theta(x)$ is a probability density function on $\mathbb{R}^d$ indexed by $\theta \in \Theta$. Suppose that we are given i.i.d. data $X_1, \dots, X_n \in \mathbb{R}^d$ from an unknown probability density function $p(x)$ on $\mathbb{R}^d$. Suppose we perform Maximum Likelihood Estimation (MLE) using the data $X_1, \dots, X_n$ with the parametric model $\mathcal{P}_\Theta$.

Q1 (1pt). Write the definition of the **likelihood function**.

Answer:

Q2 (1pt). Write the definition of the **Maximum Likelihood Estimator**.

Answer:

Q3 (1pt). Consider the misspecified setting, i.e., $p \notin \mathcal{P}_\Theta$. Explain how MLE can be justified in this setting.

Answer:

3 Hypothesis Testing

Consider coin flipping, the result of which is either "Head" or "Tail". Let H and T denote the event that Head or Tail appears, respectively, as a result of coin flipping. Suppose we want to perform a statistical hypothesis testing regarding the following null and alternative hypotheses:

- Null hypothesis H_0: The coin is fair (i.e., the probabilities of head and tail are equal, i.e., $\Pr(H) = \Pr(T) = 1/2$)
- Alternative hypothesis H_1: The coin is not fair (i.e., the probabilities of head and tail are not equal, i.e., , $\Pr(H) \neq \Pr(T) \neq 1/2$).

We conduct the following experiment: flip the coin 6 times independently, and record the results as $\omega = (\omega_1, \dots, \omega_6) \in \{H, T\}^6 =: \Omega$, where $w_i \in \{H, T\}$ for $i = 1, \dots, 6$.

Q1. (1pt) Let P_0 be the probability distribution on $\Omega = \{H, T\}^6$ under the null hypothesis. What is the probability of observing $\omega_a := (H, T, H, H, T, H) \in \Omega$ or $\omega_b := (H, T, H, H, H, H) \in \Omega$, i.e., $\underline{P_0(\{\omega_a, \omega_b\})}$?

Answer:

Q2. (1pt) Define a critical region with significance level $\alpha = 0.05$. (Hint: $1/64 \approx 0.0156$))

Answer:

Q3. (1pt) Suppose you have observed $\omega^* = (H, T, T, T, T, T)$ as a result of the experiment. Answer whether you should reject the null hypothesis according to **your** critical region from Q2 (yes or no).

Answer:

4 Bayesian Inference

Let $\mathcal{P}_\Theta := \{p_\theta \mid \theta \in \Theta\}$ be a parametric model, where $\Theta \subset \mathbb{R}^q$ is a set of parameters and $p_\theta(x)$ is a probability density function on $\mathbb{R}^d$ indexed by $\theta \in \Theta$. Suppose that we are given i.i.d. data $X_1, \dots, X_n \in \mathbb{R}^d$ from an unknown probability density function $p(x)$ on $\mathbb{R}^d$.

Let $\pi(\theta)$ be a prior density function on Θ. Suppose we want to perform Bayesian inference regarding the parameters of the parametric model $\mathcal{P}_\Theta$ using the data $X_1, \dots, X_n$ and the prior density function $\pi(\theta)$.

Q1. (1pt) Write the definition of the **posterior density function** on Θ in terms of the prior density function $\pi(\theta)$ and the likelihood function (define also this), using Bayes' theorem.

Answer:

Q2. (1pt) Define the **Maximum a Posteriori (MAP)** estimator.

Answer:

Q3. (1pt) Explain what will happen with the posterior density function when the sample size n goes to infinity.

Answer:

5 Bayesian Hypothesis Testing

Let $\mathcal{P}_\Theta := \{p_\theta \mid \theta \in \Theta\}$ be a parametric model, where $\Theta \subset \mathbb{R}^q$ is a set of parameters and $p_\theta(x)$ is a probability density function on $\mathbb{R}^d$ indexed by $\theta \in \Theta$. Suppose that we are given i.i.d. data $X_1, \dots, X_n \in \mathbb{R}^d$ from an unknown probability density function $p(x)$ on $\mathbb{R}^d$. Suppose there exists $\theta^* \in \Theta$ such that $p(x) = p_{\theta^*}(x)$.

Define a null hypothesis H_0 and alternative hypothesis H_1 as

$$H_0 : \theta^* \in \Theta_0, \quad H_1 : \theta^* \in \Theta_1,$$

where $\Theta_0, \Theta_1 \subset \Theta$ are disjoint subsets, $\Theta_0 \cap \Theta_1 = \phi$.

Let $\pi_0(\theta)$ be a prior density function on Θ_0 under H_0, and let $\pi_1(\theta)$ be a prior density function on Θ_1 under H_1. Let $\pi(H_0)$ and $\pi(H_1)$ be prior probabilities of the hypotheses H_0 and H_1, respectively, and suppose that they are equal: $\pi(H_0) = \pi(H_1) = 1/2$.

Q1. (0.5pt) Write the definition of the **marginal likelihood** of observing $X_1, \ldots, X_n$ under H_0.

Answer:

Q2. (0.5pt) Write the definition of the **marginal likelihood** of observing $X_1, \ldots, X_n$ under H_1.

Answer:

Q3. (1pt) Write the **Bayes factor** in favor of H_1 against H_0. (Hint: this Bayes factor supports H_1 when it is small).

Answer:

Calculation sheet

You can use this space for calculation. Note that anything written here will **not** be evaluated.

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