Machine Learning and Intelligent Systems

K-Nearest Neighbors

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Definition

Intuition

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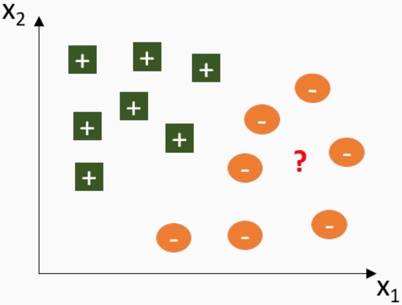

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- What drove you to make that choice? What is the assumption behind your choice?

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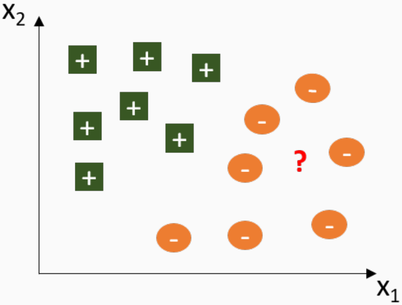

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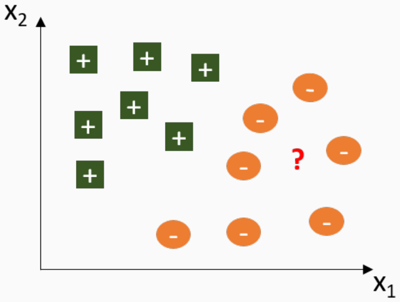
Assumption: Similar inputs have similar outputs

k Nearest Neighbors: Formalization

Data Assumptions:

Similar $\mathbf{x}$ have similar y



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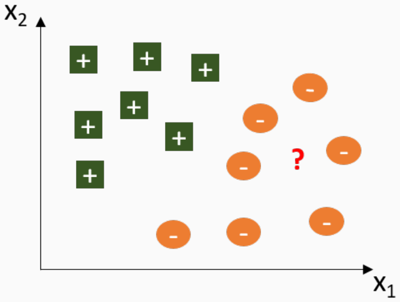
$$y \in \mathbb{R} \text{ - regression}$$

$$y \in \mathcal{C} \text{ - classification}$$



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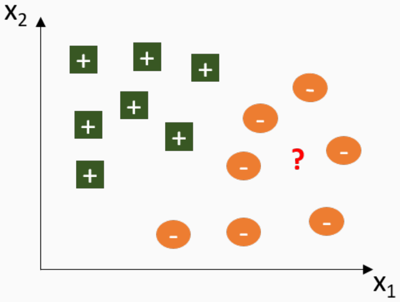

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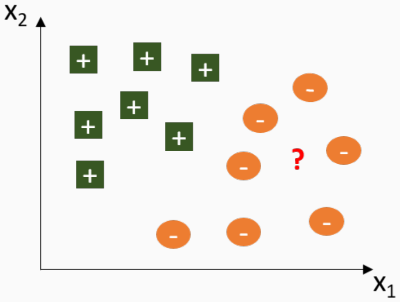

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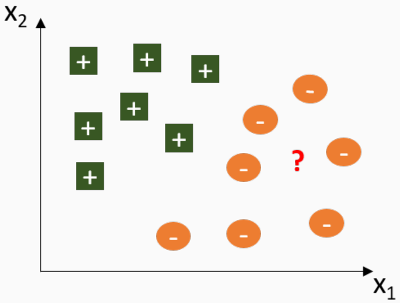
$$\mathcal{S}_{\mathbf{x}^*} \subseteq \mathcal{D} \text{ .s.t. } |\mathcal{S}_{\mathbf{x}^*}| = k$$

$$\forall (\mathbf{x}', y') \in \mathcal{D} \setminus \mathcal{S}_{\mathbf{x}^*} \ \mathrm{dist}(\mathbf{x}^*, \mathbf{x}') \geq \max_{(\mathbf{x}'', y'') \in \mathcal{S}_{\mathbf{x}^*}} \mathrm{dist}(\mathbf{x}^*, \mathbf{x}'')$$

k Nearest Neighbors: Hypothesis



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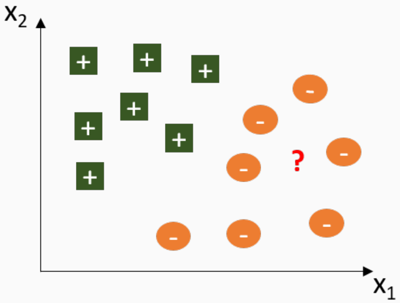
Regression: The output is the average of the values of k nearest neighbors

$$\hat{y} = h(\mathbf{x}^*) = \frac{1}{k} \sum_{(\mathbf{x}_i'', y_i'') \in \mathcal{S}_{\mathbf{x}^*}} y_i'' \tag{1}$$



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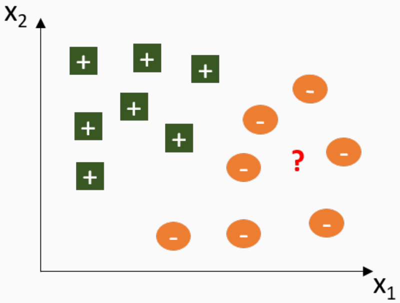
Classification: An unseen point $\mathbf{x}^*$ is classified by a majority vote of its k nearest neighbors:

$$\hat{y} = h(\mathbf{x}^*) = \text{mode}(\{y_i'' : (\mathbf{x}_i'', y_i'') \in \mathcal{S}_{\mathbf{x}^*}\}) \qquad (2)$$



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$$\text{dist}(\mathbf{x}, \mathbf{z}) = \left(\sum_{j=1}^{D} |x_j - z_j|^p \right)^{1/p} \tag{3}$$

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The main advantage of this metric is its generality. It contain many well-known distances as special cases.

Question: Can you identify what case is $p = 2$?

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This type of approach is referred as **instance-based learning** or **non-generalizing learning**.

Question: Can you spot the differences?

Placement Exam

Wrap-Up

Summary: Notation

Symbol	Reads as
X	Input variable ($\mathbb{R}^D$)
$\mathbf{x}_i$	i^{th} feature vector. Observed value of X.
$\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)^T$	Matrix of N input $D-$dimensional vectors $\mathbf{x}_i$
x_j	j^{th} element of the i^{th} input vector $\mathbf{x}_i$, i.e. x_i^j
Y	Output variable ($\mathcal{C}$)
y_i	i^{th} output label
$\mathbf{y} = (y_1, \ldots, y_N)^T$	Observed vector of outputs y_i
$\mathbf{x}^*$	Test point (unseen data)
$\hat{y}$	Prediction for $\mathbf{x}^*$

Table 1: Different notation for the input and output variables

Further Reading and Useful Material

Source	Notes
The Elements of Statistical Learning	Ch. 2