EURECOM

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Essential Mathematical Methods for Engineers (MathEng)

Sampled data systems and the z-transform

1. Determine the z-transform of the sequence:
   a. $\{x_k\} = \{2^k\}$ $(k \geq 0)$

   then by considering this result as a generating function derive a general expression for $\mathcal{Z}\{a^k\}$. Differentiate your result with respect to a and derive a general expression for $\mathcal{Z}\{ka^{k-1}\}$. Then determine the z-transform of the sequence:
   b. $\{x_k\} = \{2k\} = \{0, 2, 4, 6, 8, \ldots\}$.

2. The signal $f(t) = e^{-t}H(t)$ is sampled at intervals T. What is the z-transform of the resulting sequence of samples?

3. The continuous-time function $f(t) = \cos \omega t\, H(t)$, where ω is a constant, is sampled in the idealised sense at intervals T to generate the sequence $\{\cos k\omega T\}$. Using the linearity property, determine the z-transform of the sequence. Repeat the exercise where the cosine wave is replaced with a sine wave.

4. Use the first shift property to calculate the z-transform of the sequence $\{y_k\}$, where:

$$y_k = \begin{cases} 0 & (k < 3) \\ x_k & (k \geq 3) \end{cases}$$

   where $\{x_k\}$ is causal and $x_k = (½)^k$. Confirm your result by direct evaluation of $\mathcal{Z}(y_k)$ using the definition of the z-transform.

5. Determine the inverse z-transforms of:
   a. $Y(z) = z/(z-2)$,
   b. $Y(z) = z/[(z-1)(z-2)]$,
   c. $Y(z) = z/(z^2+a^2)$, and
   d. $Y(z) = z/(z^2-z+1)$.

6. A system has the impulse response sequence $\{y_{\delta k}\} = \{a^k - 0.5^k\}$ where $a > 0$ is a real constant. What is the nature of this response when:
   a. $a = 0.4$, and
   b. $a = 1.2$?

   Find the step response of the system in both cases.

7. A S/H operates at a sampling frequency of 10 kHz. Determine and sketch the Fourier transform of the sampled signal when: (a) a 3 kHz cosine wave is applied; (b) an 8 kHz cosine wave is applied; (c) a 12 kHz cosine wave is applied, and (d) a 118 kHz cosine wave is applied to the input. If the sampled signal is applied to a perfect low-pass filter with a cut-off frequency of 5 kHz, what frequency will be present at its output in the above four cases?

8. In recording studios it is possible to enhance a singer's performance by constructing a digital delay and adding the delayed signal to the original signal. A delay of T seconds is achieved by sampling the signal, storing each sample in random access memory (RAM), and reading the sample out of RAM T seconds later. If the highest frequency to be recorded is 15.8 kHz, what is the minimum sampling frequency which can be used? How many memory locations would be required to produce a delay of 0.5 s?

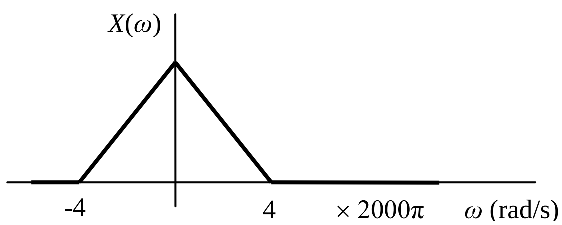
   A cheaper delay unit can be manufactured for the guitar because the highest note that the guitar produces has a fundamental frequency of 2 kHz and the only significant harmonic is the third. What is the minimum sampling frequency in this case? How many memory locations would be required to produce a delay of 0.5s? If the fourth harmonic of the highest note is present on some makes of guitar, at what frequency will it appear at the output of the cheaper unit? Will this be pleasing to the ear?

9. Derive the impulse response, transfer function, magnitude and frequency response of a ZOH. Sketch your results.

10. A signal, $x(t)$, has the magnitude Fourier transform sketched below. The signal is applied to a S/H-A/D combination with a sampling rate of 8 kHz. The samples are read in by a digital filter implemented on a microprocessor with the following algorithm:

$$y(n) = x(n) + x(n-3)$$

   Finally there is a D/A to produce an analogue output $y(t)$ from the sampled signal produced by the digital filter. Sketch the magnitude Fourier transform of the output.



11. Given the following unit sample response sequences, calculate $H(z)$ without using z-transform tables. Sketch the pole/zero pattern in the z-plane and the filter block diagram.
    a. $h(0) = 1$, $h(1) = 1$, $h(n) = 0$ otherwise;
    b. $h(0) = 1$, $h(1) = -1$, $h(n)= 0$ otherwise;
    c. $h(0) = 1$, $h(1) = -2$, $h(2) = 1$, $h(n)= 0$ otherwise;
    d. $h(n) = r^n \sin(\omega_0 n)$ where $\omega_0 = \pi/8$ and $r = 0.9$.

12. Determine the transfer function and the pole/zero map for the following digital filter:

$$y(n) = 1.6y(n-1) - 0.8y(n-2) + x(n)$$

   Determine the first eight terms of the unit pulse response $\{h(n)\}$ using the above algorithm. Derive a general expression forth unit pulse response using z-transform methods. Use discrete convolution to calculate the first weight samples of the output when the following sequence is applied to the filter:

   0, 0.25, 0.5, 0.75, 1.0, 0, 0, 0, … , 0.

13. For the transfer function:

$$H(z) = \frac{z^2 + 2z + 1}{z^3}$$

Make sketches of the magnitude and phase frequency response based on the position of the *z*-plane poles and zeros.

14. A digital filter is described by the following transfer function:

$$H(z) = \frac{(z+1)(z^2 - 0.6489z + 1.1025)}{(z^2 + 1.3435z + 0.9025)}$$

Is the filter stable? Sketch the frequency response. At what frequency is the maximum gain? Without using graphical methods, calculate the maximum gain in dB.

15. The music signal from an electronic synthesiser is a periodic waveform with a fundamental frequency of 1 kHz. All the power in the signal is contained in the first three harmonics. The amplitude of the second harmonic is half the amplitude of the fundamental and the amplitude of the third harmonic is half the amplitude of the second. The signal is applied to a digital signal processing system which includes an anti-aliasing filter with frequency response $2000\pi/(2000\pi + j\omega)$ and an analogue-to-digital converter which operates at a sampling frequency 5.4 kHz. The sampled signal is processed using a digital filter which is defined by the following difference equation:

$$y(n) = x(n) - 0.7922x(n-1) + x(n-2)\,.$$

What frequencies are present at the output of the digital filter and what are their relative magnitudes?

	Signal, *x*[*n*]	Z-transform, *X*(*z*)	ROC
1	$\delta[n]$	1	all z
2	$\delta[n-n_0]$	$\frac{1}{z^{n_0}}$	all z
3	$u[n]$	$\frac{z}{z-1}$	$\|z\| > 1$
4	$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
5	$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
6	$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
7	$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
8	$\cos(\omega_0 n)u[n]$	$\frac{1-z^{-1}\cos(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}$	$\|z\| > 1$
9	$\sin(\omega_0 n)u[n]$	$\frac{z^{-1}\sin(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}$	$\|z\| > 1$
10	$a^n\cos(\omega_0 n)u[n]$	$\frac{1-az^{-1}\cos(\omega_0)}{1-2az^{-1}\cos(\omega_0)+a^2z^{-2}}$	$\|z\| > \|a\|$
11	$a^n\sin(\omega_0 n)u[n]$	$\frac{az^{-1}\sin(\omega_0)}{1-2az^{-1}\cos(\omega_0)+a^2z^{-2}}$	$\|z\| > \|a\|$