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Essential Mathematical Methods for Engineers (MathEng)

Linear algebra A

Tutorial Questions
(from course textbook “Introduction to linear algerbra”, by G. Strang)

1. Which 2 by 2 matrices:

   (a) permute $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} y \\ x \end{bmatrix}$;

   (b) rotate every vector by 90º, e.g. transforms $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} y \\ -x \end{bmatrix}$, and

   (c) rotate every vector by 180º?

2. Which matrix E subtract the first components from the second components in the following:

$$E\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \text{and} \quad E\begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}?$$

3. Reduce this system to upper triangular form:

$$\begin{aligned} 2x + 3y + z &= 1 \\ 4x + 7y + 5z &= 7 \\ -2y + 2z &= 6 \end{aligned}$$

   Circle the pivots. Solve by back substitution for z, y, x. Two row operations are enough if a zero coefficient appears in which positions?

4. Which number q makes this system singular and which right side t gives it infinitely many solutions? Find the solution that has $z = 1$.

$$\begin{aligned} x + 4y - 2z &= 1 \\ x + 7y - 6z &= 6 \\ 3y + qz &= t \end{aligned}$$

5. Apply elimination to the 2 by 3 augmented matrix A'. What is the triangular system $Ux=c$? What is the solution x?

$$A=\begin{bmatrix}2&3\\4&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}1\\17\end{bmatrix}.$$

6. Choose the numbers a, b, c, d in this augmented matrix so that there is (a) no solution and (b) infinitely many solutions.

$$A'=\begin{bmatrix}1&2&3&a\\0&4&5&b\\0&0&d&c\end{bmatrix}$$

Which of the numbers a, b, c, or d have no effect on the solvability?

7. Invert the following matrices by the Gauss-Jordan method starting with $[A\ \ I]$:

$$A=\begin{bmatrix}1&0&0\\2&1&3\\0&0&1\end{bmatrix}\quad\text{and}\quad A=\begin{bmatrix}1&1&1\\1&2&2\\1&2&3\end{bmatrix}$$

8. What three elimination matrices put A into upper triangular form so that $E_{32}E_{31}E_{21}A=U$? Multiply by E_{32}^{-1}, E_{31}^{-1} and E_{21}^{-1} to factor A into LU where $L=E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$. Find L and U:

$$A=\begin{bmatrix}1&0&1\\2&2&2\\3&4&5\end{bmatrix}$$

9. What are L and D for this matrix A? What is U in $A=LU$ and what is the new U in $A=LDU$?

$$A=\begin{bmatrix}2&4&8\\0&3&9\\0&0&7\end{bmatrix}$$

10. Compute L and U for the symmetric matrix:

$$A=\begin{bmatrix}a&a&a&a\\a&b&b&b\\a&b&c&c\\a&b&c&d\end{bmatrix}$$

Find four conditions on a, b, c, d to get $A=LU$ with four pivots.

11. Factor the following symmetric matrix into $A=LDL^{T}$. The pivot matrix is diagonal.

$$A=\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix}$$

12. Describe the column spaces (lines or planes) of these particular matrices:

$$A=\begin{bmatrix}1&2\\0&0\\0&0\end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix}1&0\\0&2\\0&0\end{bmatrix} \quad \text{and} \quad A=\begin{bmatrix}1&0\\2&0\\0&0\end{bmatrix}$$

13. For which vectors (b_1, b_2, b_3) do these systems have a solution?

$$\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}1&1&1\\0&1&1\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

$$\text{and} \quad \begin{bmatrix}1&1&1\\0&0&1\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}.$$

14. Construct a 3 by 3 matrix whose column space contains (1, 1, 0) and (1, 0, 1) but not (1, 1, 1).

15. Reduce the following matrices to their ordinary echelon forms U, identify the pivot variables and the free variables and find a special solution for each free variable. Then, by combining the special solutions describe every solution to $Ax=0$.

$$A=\begin{bmatrix}1&2&2&4&6\\1&2&3&6&9\\0&0&1&2&3\end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix}2&4&2\\0&4&4\\0&8&8\end{bmatrix}.$$

16. Suppose an m by n matrix has r pivots. The number of special solutions is ___. The nullspace contains only $x = 0$ when r = ___. The column space is all of R^m when r = ___.

17. Contruct a matrix whose nullspace consists of all combinations of (2, 2, 1, 0) and (3, 1, 0, 1).

18. Construct a matrix whose column space contains (1, 1, 0) and (0, 1, 1) and whose nullspace contains (1, 0, 1) and (0, 0, 1).

19. Construct a matrix whose column space contains (1, 1, 1) and whose nullspace is the line of multiples of (1, 1, 1, 1).

20. What is the nullspace matrix N (containing special solutions) for A, B, C?

$$A=[I \;\; I] \quad \text{and} \quad B=\begin{bmatrix}I&I\\0&0\end{bmatrix} \quad \text{and} \quad C=I.$$

21. If the nullspace of A consists of all multiples of x = (2, 1, 0, 1), how many pivots appear in U?

22. What are the special solutions to $Rx = 0$ and $y^{\mathrm{T}}R = 0$ for these R?

$$R=\begin{bmatrix}1&0&2&3\\0&1&4&5\\0&0&0&0\end{bmatrix} \quad \text{and} \quad R=\begin{bmatrix}0&1&2\\0&0&0\\0&0&0\end{bmatrix}$$

23. Write the complete solution as x_p plus any multiple of s:

$$\begin{array}{rcrcrcr} x & + & 3y & + & 3z & = & 1 \\ 2x & + & 6y & + & 9z & = & 5. \\ -x & - & 3y & + & 3z & = & 5 \end{array}$$

24. Find the complete solution (also called the general solution) to:

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

25. Construct a 2 by 3 system $Ax = b$ with particular solution $x_p = (2, 4, 0)$ and homogenous solution x_n = any multiple of (1, 1, 1).

26. Apply Gauss –Jordan elimination to $Ux = 0$ and $Ux = c$. Reach $Rx = 0$ and $Rx = d$:

$$\begin{bmatrix} U & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} U & c \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 4 & 8 \end{bmatrix}.$$

Solve $Rx = 0$ to find x_n (its free variable is $x_2 = 1$). Solve $Rx = d$ to find x_p (its free variable is $x_2 = 0$).

27. Reduce $Ax = b$ to $Ux = c$ (Gaussian elimination) and then to $Rx = d$ (Gauss-Jordan):

$$Ax = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix} = b.$$

Find a particular solution x_p and all homongenous solutions x_n.

28. Find the largest possible number of independent vectors among:

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, v_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, v_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

29. Find a basis for each of the four subspaces associated with:

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$