Example

A typical transfer function that might be obtained by applying Kirchhoff's laws to a circuit is:

$$H(s) = \frac{s^2 + 2s + 2}{s^2 + 4s + 13}$$

Determine the positions of its poles and zeros.

Transfer function and system characterisation 149

Zeros are obtained by setting the top line to zero:

$$s^2 + 2s + 2 = 0$$

$$s = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)} = \frac{-2 \pm \sqrt{-4}}{2}$$

$$= \underline{-1 \pm j}$$

Poles are obtained by setting the bottom line to zero:

$$s^2 + 4s + 13 = 0$$

$$s = \frac{-4 \pm \sqrt{4^2 - 4(1)(13)}}{2(1)} = \frac{-4 \pm \sqrt{-36}}{2}$$

$$= \underline{-2 \pm j3}$$

Example

If the Laplace transfer function of a system is:

$$H(s) = \frac{1}{s+7}$$

what is its frequency response?

Transfer function and system characterisation 154

$$H(\omega) = \frac{1}{j\omega + 7}$$

Example

Sketch the amplitude frequency response of each of the following 2 systems:

(a) $(s^2 + 9)/(s^2 + 0.6s + 4.09)$

(b) $(s^2 + 0.6s + 4.09)/(s^3 + 0.8s^2 + 9.16s)$

Transfer function and system characterisation 159

(a) The zeros are solutions of $(s^2+9)=0$ ie. $s = \pm 3j$.
The poles are solutions of $(s^2+0.6s+4.09)=0$ ie. $s = -0.3 \pm j2$

There is thus a peak in the frequency response at $\omega = 2$ rad/s and a trough at $\omega = 3$ rad/s. Here the trough in the frequency response goes to zero because the length of the vector from $3j$ is zero at a frequency of 3 rad/s.

At 0 rad/s the gain is determined by the ratio of the distances from the zeros over the distances from the poles to the origin ie:

$$(3 \times 3)/\left(\sqrt{0.3^2+2^2} \times \sqrt{0.3^2+2^2}\right) = 2.2$$

At high frequencies as $\omega \to \infty$ the distances from the poles and zeros are equal and since there are the same number of poles as zeros the gain is 1

The full amplitude frequency response is illustrated in the top-right graph on slide 160.

(b) The zeros are solutions of $(s^2 + 0.6s + 4.09) = 0$ i.e. $s = {}^-0.3 \pm j2$

The poles are solutions of $(s^3 + 0.8s^2 + 9.16s) = 0$

$$= s(s^2 + 0.8s + 9.16) = 0$$

so there are poles at $s = 0$ & $s = {}^-0.4 \pm 3j$

There is thus a peak in the frequency response when $\omega = 0$ rad/s. Since the length of the vector from the pole at $s = 0$ is zero the gain will be infinite. A trough occurs when $\omega = 2$ rad/s. Finally another peak occurs at $\omega = 3$ rad/s.

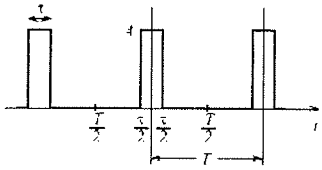
At high frequencies as $\omega \to \infty$ the distances from the poles and zeros are equal but since there are more poles than zeros the gain is zero.

(1) *Example*

Determine the Fourier transform of a sine wave $A\sin(\omega a t)$

(2) *Example*

Derive an expression for the Fourier transform of the following signal:



Note the similarity to the example on slide 102.

Transfer function and system characterisation 163

(1) Given that $F[e^{j\omega_a t}] = 2\pi\delta(\omega-\omega_a)$ & expanding the sine wave as a sum of phasors using Euler's identity, we have

$$F[A\sin(\omega_a t)] = A F\left[\frac{e^{j\omega_a t} - e^{-j\omega_a t}}{2j}\right] = \frac{A}{2j}\left[F(e^{j\omega_a t}) - F(e^{-j\omega_a t})\right]$$

$$= \frac{A}{2j}\left[2\pi\delta(\omega-\omega_a) - 2\pi\delta(\omega+\omega_a)\right] = -jA\pi\left[\delta(\omega-\omega_a) - \delta(\omega+\omega_a)\right]$$

(2) From the example on slide 102 we have

$$X_n = \frac{A\tau}{T}\,\mathrm{sa}(n\omega_0\tau/2)$$

Given that

$$x(t) = \sum_{n=-\infty}^{\infty} X_n \exp(jn\omega_0 t)$$

we have $x(t) = \sum_{n=-\infty}^{\infty} A\tau/T \times \mathrm{sa}(n\omega_0\tau/2)\exp(jn\omega_0 t)$

Taking the Fourier transform of both sides we have

$$X(\omega) = \frac{A\tau}{T}\sum_{n=-\infty}^{\infty}\mathrm{sa}(n\omega_0\tau/2)\,F[\exp(jn\omega_0 t)]$$

$$= \frac{2\pi A\tau}{T}\sum_{n=-\infty}^{\infty}\mathrm{sa}(n\omega_0\tau/2)\,\delta(\omega-n\omega_0)$$

Example

Sketch the Bode plot of a system with the following transfer function:

$$H(s) = \frac{s + 20}{s + 2000}$$

The frequency response is obtained by replacing s with $j\omega$

$$H(\omega) = \frac{j\omega + 20}{j\omega + 2000}$$

normalising the numerator & denominator to obtain recognisable terms

$$H(\omega) = \frac{j\omega/20 + 1}{(j\omega/2000 + 1)100}$$

The individual terms are plotted overleaf as dashed lines

- the constant gain term with a gain of $20\log_{10}(1/100) = -40\text{dB}$ and a phase shift of zero
- the zero term $(j\omega/20 + 1)$ with a cut-in frequency of 20 rad/s
- the pole term $(j\omega/2000 + 1)$ with a cut-off frequency of 2000 rad/s

The total response (asymptotic) is formed by adding the individual responses. The actual bode plots evaluated numerically are also shown for comparison.

Example

Is the transfer function $H(s) = s/(s^2 + 4s + 68)$ stable or unstable? What is the period of oscillation and the time constant of its impulse response?

Transfer function and system characterisation 179

The poles are given by solutions to $s^2 + 4s + 68 = 0$

i.e. $s = -2 \pm j8$

Both are in the left-hand side of the s-plane so the system is stable. The partial fraction expansion of the transfer function will have the form:

$$H(s) = \frac{A_1}{s + 2 - 8j} + \frac{A_2}{s + 2 + 8j}$$

which will lead to an impulse response of the form

$$h(t) = A_1 e^{-2t} e^{j8t} + A_2 e^{-2t} e^{-j8t}$$

The envelope is controlled by e^{-2t}. Thus the time constant is 0.5 seconds. The oscillation comes from e^{j8t} which has a frequency of 8 rad/s and hence a period of 0.79 seconds.

Example

What is the damping factor and undamped natural frequency of a system with transfer function:

$$H(s) = 7s/(12s^2 + 118.8s + 2700)?$$

In normalised form

$$H(s) = \frac{7s}{12s^2 + 118.8s + 2700} = \frac{7s}{12(s^2 + 9.9s + 225)}$$

thus $s^2 + 9.9s + 225 = s^2 + 2\zeta\omega_0 s + \omega_0^2$

Equating terms $\omega_0^2 = 225$ & hence $\omega_0 = 15$. Likewise $2\zeta\omega_0 = 9.9$ and hence $\zeta = 0.33$.

Example

A low pass, second order system has a peak time of 2 s and an overshoot of 10%. Estimate its bandwidth.

The percentage overshoot is related to the damping factor by:-

$$PO = 10 = 100 \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right) \Rightarrow \ln\left(\frac{1}{10}\right) = \frac{-\zeta\pi}{\sqrt{1-\zeta^2}}$$

$$\therefore \quad (1-\zeta^2)\ln^2\left(\frac{1}{10}\right) = \zeta^2\pi^2$$

Rearranging gives $\zeta = 0.59$

The peak time is related to the damping factor & undamped natural frequency by

$$T_p = \frac{\pi}{\omega_0\sqrt{1-\zeta^2}}$$

$$\therefore \quad \omega_0 = \frac{\pi}{T_p\sqrt{1-\zeta^2}} = 1.95 \text{ rad/s} = 0.31 \text{ Hz}$$

Referring to the graph on slide 196 the bandwidth will be a little greater than 0.31 Hz