Tutorial Sheet 6

① (a) $\mathcal{Z}\{2^k\} = \sum_{k=0}^{\infty} \frac{2^k}{z^k} = \sum_{k=0}^{\infty} (2/z)^k$

which is a geometric series with common ratio $r = 2/z$ between successive terms. The series thus converges for $|z| > 2$, when

$$\sum_{k=0}^{\infty} \left(2/z\right)^k = \lim_{z \to \infty} \frac{1-(2/z)^k}{1-(2/z)} = \frac{1}{1-2/z}$$

leading to

$$\mathcal{Z}\{2^k\} = \frac{z}{z-2} \qquad (|z| > 2)$$

We can think of this result as a generating function for the sequence $\{2^k\}$ in the sense that the coefficient of z^{-k} in the expansion of $X(z)$ in powers of $1/z$ generates the k^{th} term of the sequence $\{2^k\}$. This can easily be verified, since

$$\frac{z}{z-2} = \frac{1}{1-2/z} = \left(1 - \frac{2}{z}\right)^{-1}$$

and since $|z| > 2$ we can expand this as:

$$\left(1 - 2/z\right)^{-1} = 1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \dots + \left(\frac{2}{z}\right)^k + \dots$$

and we see that the coefficient of z^{-k} is indeed 2^k, as expected.

Generalising for $\mathcal{Z}\{a^k\}$ we have

$$\mathcal{Z}\{a^k\} = \frac{z}{z-a} \qquad (|z| > |a|)$$

Differentiating we have

$$\frac{d}{da}\mathcal{Z}\{a^k\} = \mathcal{Z}\left\{\frac{da^k}{da}\right\} = \frac{d}{da}\left(\frac{z}{z-a}\right)$$

which gives

$$\mathcal{Z}\{ka^{k-1}\} = \frac{z}{(z-a)^2} \qquad (|z| > |a|)$$

(b) For the last case above, when $a = 1$ we have

$$\mathcal{Z}\{k\} = \{0, 1, 2, \ldots\} = \sum_{k=0}^{\infty}\frac{k}{z^k} = \frac{z}{(z-1)^2}$$

thus

$$\mathcal{Z}\{0, 2, 4, \ldots\} = 0 + \frac{2}{z} + \frac{4}{z^2} + \ldots = 2\sum_{k=0}^{\infty}\frac{k}{z^k}$$

so that

$$\mathcal{Z}\{2k\} = 2\mathcal{Z}\{k\} = \frac{2z}{(z-1)^2}$$

(2) Sampling the causal signal $f(t)$ generates the sequence

$$\{f(kT)\} = \{f(0), f(T), f(2T), \ldots, f(nT), \ldots\}$$
$$= \{1, e^{-T}, e^{-2T}, e^{-3T}, \ldots, e^{-nT}, \ldots\}$$

$$\therefore \mathcal{Z}\{f(kT)\} = \sum_{k=0}^{\infty}\frac{e^{-kT}}{z^k} = \sum_{k=0}^{\infty}\left(\frac{e^{-T}}{z}\right)^k$$

so that

$$\mathcal{Z}\{e^{-kT}\} = \frac{z}{z-e^{-T}} \qquad (|z| > e^{-T})$$

(3) Since $\cos k\omega T = \frac{1}{2}\left(\exp(jk\omega T) + \exp(-jk\omega T)\right)$

using the linearity property we have

$$\mathcal{Z}\{\cos k\omega T\} = \mathcal{Z}\{\tfrac{1}{2}\exp(jk\omega T) + \tfrac{1}{2}\exp(-jk\omega T)\}$$

$$= \mathcal{Z}\{\tfrac{1}{2}\exp(jk\omega T)\} + \mathcal{Z}\{\tfrac{1}{2}\exp(-jk\omega T)\}$$

Using the result from Q1 & noting that
$|\exp(j\omega kT)| = |\exp(-j\omega kT)| = 1$

$$\mathcal{Z}\{\cos k\omega T\} = \frac{1}{2}\frac{z}{z-e^{j\omega T}} + \frac{1}{2}\frac{z}{z-e^{-j\omega T}} \qquad (|z|>1)$$

$$= \frac{1}{2}\,\frac{z(z-e^{-j\omega T}) + z(z-e^{j\omega T})}{z^2-(e^{j\omega T}+e^{-j\omega T})z+1}$$

leading to the z-transform pair

$$\mathcal{Z}\{\cos k\omega T\} = \frac{z(z-\cos\omega T)}{z^2-2z\cos\omega T+1} \qquad (|z|>1)$$

Similarly

$$\mathcal{Z}\{\sin k\omega T\} = \frac{1}{2j}\frac{z}{z-e^{j\omega T}} - \frac{1}{2j}\frac{z}{z-e^{-j\omega T}}$$

$$= \frac{z\sin\omega T}{z^2-2z\cos\omega T+1}$$

(4) $\mathcal{Z}\{(1/2)^k\} = \dfrac{2z}{2z-1}$

$\therefore \; \mathcal{Z}\{y_k\} = \dfrac{1}{z^3} \times \dfrac{2z}{2z-1} = \dfrac{2}{z^2(2z-1)}$

Proceeding directly

$$\mathcal{Z}\{y_k\} = \sum_{k=0}^{\infty} \frac{x_{k-3}}{z^k} = \sum_{k=0}^{\infty} \frac{x_r}{z^{r+3}} = \frac{1}{z^3} \times \mathcal{Z}\{x_k\} = \frac{2}{z^2(2z-1)}$$

(5) (a) Directly from the table of transforms

$$\mathcal{Z}^{-1}\left\{\frac{z}{z-2}\right\} = \{2^k\}$$

(b) $\mathcal{Z}^{-1}\left\{\dfrac{z}{(z-1)(z-2)}\right\} \Rightarrow$ let $\dfrac{Y(z)}{z} = \dfrac{1}{(z-1)(z-2)}$

resolving into partial fractions

$$\frac{Y(z)}{z} = \frac{1}{z-2} - \frac{1}{z-1}$$

so that

$$Y(z) = \frac{z}{z-2} - \frac{z}{z-1}$$

Using the result $\mathcal{Z}^{-1}[z/(z-a)] = \{a^k\}$ & the linearity property, we have

$$\mathcal{Z}^{-1}[Y(z)] = \mathcal{Z}^{-1}\left[\frac{z}{z-2} - \frac{z}{z-1}\right] = \mathcal{Z}^{-1}\left[\frac{z}{z-2}\right] - \mathcal{Z}^{-1}\left[\frac{z}{z-1}\right]$$

$$= \{2^k\} - \{1^k\} \quad (k \geq 0)$$

$$= \{2^k - 1\} \quad (k \geq 0)$$

Noting that $\sin\theta = e^{j\theta} - e^{-j\theta}/2j$

$$Y(z) = \frac{z}{(j2\sin\frac{1}{3}\pi)(z - e^{j\pi/3})} - \frac{z}{(j2\sin\frac{1}{3}\pi)(z - e^{-j\pi/3})}$$

$$= \frac{1}{j\sqrt{3}}\frac{z}{(z - e^{j\pi/3})} - \frac{1}{j\sqrt{3}}\frac{z}{(z - e^{-j\pi/3})}$$

& using the result $\mathcal{Z}^{-1}[z/(z-a)] = \{a^k\}$

$$\mathcal{Z}^{-1}[Y(z)] = \frac{1}{j\sqrt{3}}\left(e^{jk\pi/3} - e^{-jk\pi/3}\right) = \left\{2\sqrt{\tfrac{1}{3}}\sin\tfrac{1}{3}k\pi\right\}$$

6 (a) When $a = 0.4$ we have $\{y_{\delta k}\} = \{0.4^k - 0.5^k\}$

As $k \to \infty$ both $0.4^k \to 0$, $0.5^k \to 0$ $\Rightarrow$ a stable response which tends to zero

(b) When $a = 1.2$ we have $\{y_{\delta k}\} = \{1.2^k - 0.5^k\}$

As $k \to \infty$ $1.2^k \to \infty$ $\Rightarrow$ an unstable response which 'blows up' as $k \to \infty$

For the step response we need the transfer function

$$G(z) = Y_\delta(z) = \mathcal{Z}\{a^k - 0.5^k\}$$

$$\& \quad G(z) = \frac{z}{z-a} - \frac{z}{z-0.5}$$

For the step response $\mathcal{Z}\{h_k\} = \dfrac{z}{z-1}$ $\quad (\{h_k\} = \{1, 1, 1, \ldots\})$

Thus $Y(z) = G(z)\mathcal{Z}\{h_k\} = \left(\dfrac{z}{z-a} - \dfrac{z}{z-0.5}\right)\dfrac{z}{z-1}$

so that $\dfrac{Y(z)}{z} = \dfrac{z}{(z-a)(z-1)} - \dfrac{z}{(z-0.5)(z-1)}$

$$= \frac{a}{a-1}\frac{1}{z-a} - \frac{1}{z-0.5} + \left(-2 + \frac{1}{1-a}\right)\frac{1}{z-1}$$

which gives

$$Y(z) = \frac{a}{a-1}\frac{z}{z-a} - \frac{z}{z-0.5} + \left(-2 + \frac{1}{1-a}\right)\frac{z}{z-1}$$

which on taking inverse transforms gives

$$\{y_k\} = \left\{\frac{a}{a-1}a^k - (0.5)^k + \left(-2 + \frac{1}{1-a}\right)\right\}$$

& when $a = 0.4$ since $(0.4)^k \to 0$ as $k \to \infty$ the
$(0.5)^k \to 0$

output sequence terms tend to the constant value

$$-2 + \frac{1}{1-0.4} = 0.\dot{3}$$

In the case of $a = 1.2$, $(1.2)^k \to \infty$ as $k \to \infty$ so once again the output is unbounded and 'blows up' as $k \to \infty$

(7) The S/H is modelled by multiplication in the time domain.

$$x_c(t) = x(t)\,\delta_T(t)$$

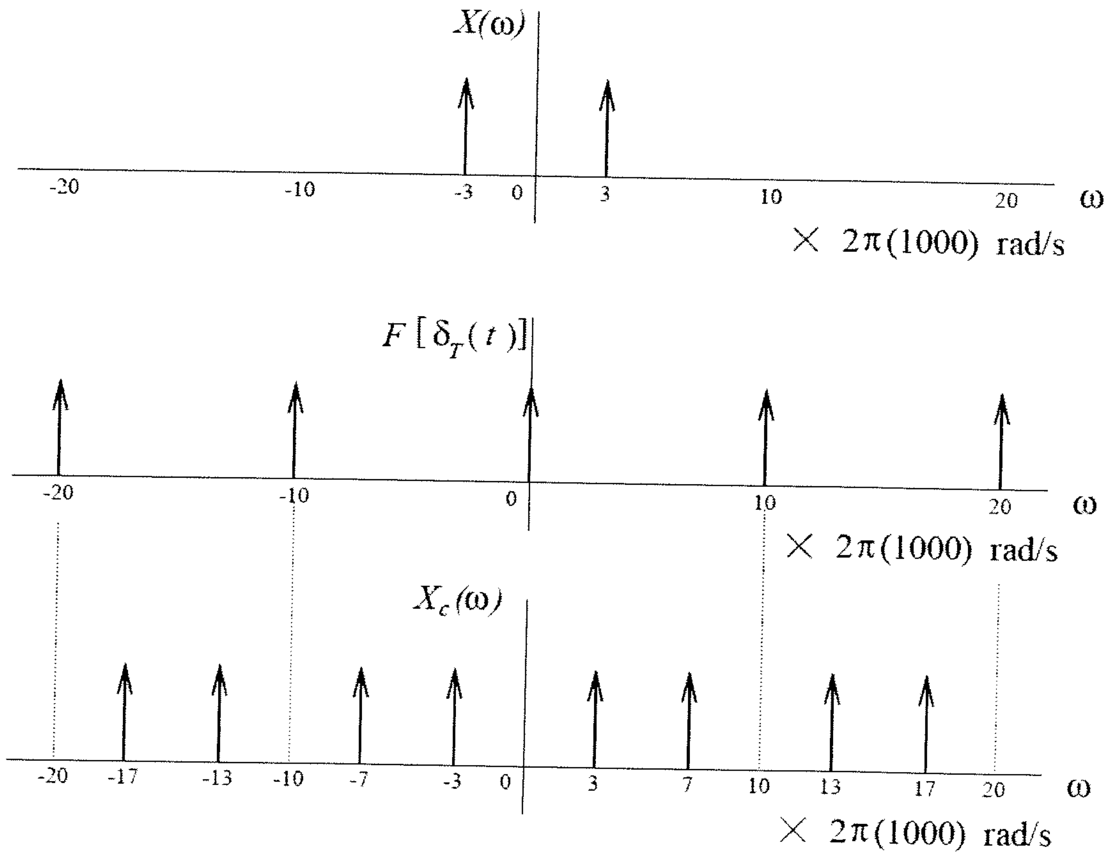
which is equivalent to convolution in the frequency domain;

$$X_c(\omega) = \frac{1}{2\pi} X(\omega) * F[\delta_T(t)]$$

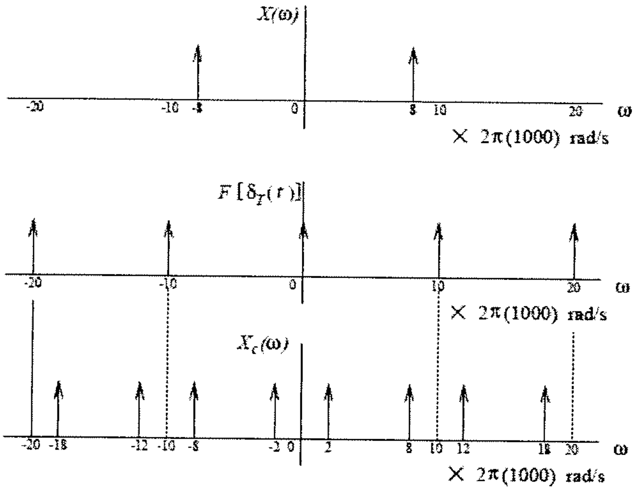
(a) The Fourier transform of a 3kHz cosine wave is

$$F\left[A\cos(2\pi \times 10^3 t)\right] = A\pi\,\delta(\omega - 3\times10^3) + A\pi\,\delta(\omega + 3\times10^3)$$

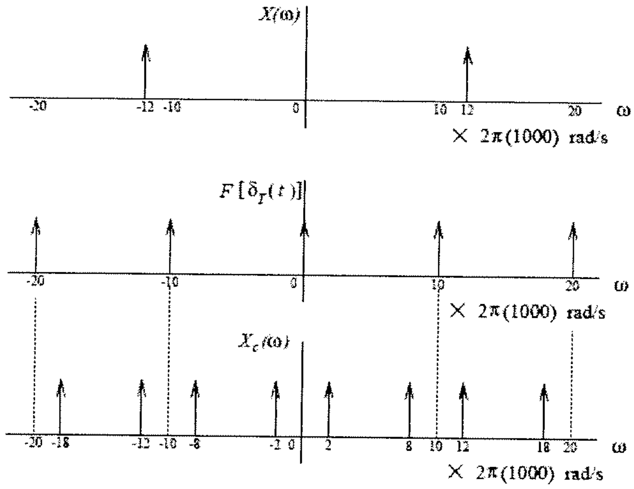
The Fourier transform of the impulse train is a train of impulses at multiples of 10kHz – the sampling frequency. The Fourier transform of the sampled signal is obtained by convolving the two Fourier transforms. A perfect low pass filter will remove all frequency components apart from a cosine wave of 3kHz.



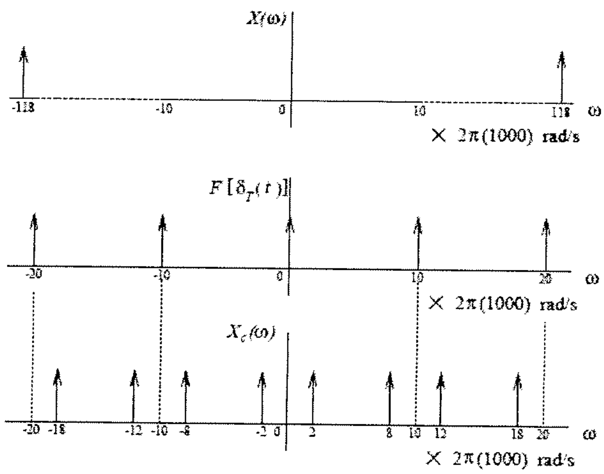
(b) The LPF will remove all but the 2kHz cosine wave.



(c) ... all but the 2kHz cosine wave.



(d) ... all but the 2kHz cosine wave.



(c) $\frac{Y(z)}{z} = \frac{1}{z^2+a^2} = \frac{1}{(z+ja)(z-ja)} = \frac{1}{j2a}\left[\frac{1}{z-ja} - \frac{1}{z+ja}\right]$

Using the result $\mathcal{Z}^{-1}[z/(z-a)] = \{a^k\}$ we have

$$\mathcal{Z}^{-1}\left[\frac{z}{z-ja}\right] = \{(ja)^k\} = \{j^k a^k\}$$

$$\mathcal{Z}^{-1}\left[\frac{z}{z+ja}\right] = \{(-ja)^k\} = \{(-j)^k a^k\}$$

From the relation $e^{j\theta} = \cos\theta + j\sin\theta$ we have

$$j = e^{j\pi/2} \quad \& \quad -j = e^{-j\pi/2}$$

so that

$$\mathcal{Z}^{-1}\left[\frac{z}{z-ja}\right] = \{a^k(e^{j\pi/2})\} = \{a^k e^{jk\pi/2}\} = \{a^k(\cos\tfrac{1}{2}k\pi + j\sin\tfrac{1}{2}k\pi)\}$$

$$\mathcal{Z}^{-1}\left[\frac{z}{z+ja}\right] = \{a^k(\cos\tfrac{1}{2}k\pi - j\sin\tfrac{1}{2}k\pi)\}$$

& using the linearity property we have

$$\mathcal{Z}^{-1}[Y(z)] = \left\{\frac{a^k}{j2a}(\cos\tfrac{1}{2}k\pi + j\sin\tfrac{1}{2}k\pi - \cos\tfrac{1}{2}k\pi + j\sin\tfrac{1}{2}k\pi)\right\}$$

$$= \{a^{k-1}\sin\tfrac{1}{2}k\pi\}$$

(d) The denominator may be factorised as

$$z^2 - z + 1 = \left(z - \tfrac{1}{2} - j\tfrac{\sqrt{3}}{2}\right)\left(z - \tfrac{1}{2} + j\tfrac{\sqrt{3}}{2}\right)$$

In exponential form we have

$$\tfrac{1}{2} \pm j\tfrac{\sqrt{3}}{2} = e^{\pm j\pi/3}$$

$$\therefore \quad z^2 - z + 1 = (z - e^{j\pi/3})(z - e^{-j\pi/3})$$

So $\frac{Y(z)}{z} = \frac{1}{(z - e^{j\pi/3})(z - e^{-j\pi/3})}$

which, resolved into partial fractions, gives

$$= \frac{1}{(e^{j\pi/3} - e^{-j\pi/3})(z - e^{j\pi/3})} + \frac{1}{(e^{-j\pi/3} - e^{j\pi/3})(z - e^{-j\pi/3})}$$

(8) If the highest frequency present is 15.8kHz then the minimum sampling frequency is $2 \times 15.8 \times 10^3 = 31.6$ kHz. The time between samples is thus 31.6μs. For a delay of 0.5 seconds we require $0.5/\Delta t = 15800$ memory locations.

The cheaper system requires a sampling frequency of $3 \times 2 \times 2 \times 10^6$ kHz $= 12$ kHz. This would require 6000 memory locations.

If the 4th harmonic is present it will be aliased to 4kHz when sampled at 12kHz. The second harmonic is at 4kHz and hence this distortion may be pleasing to the ear.

(9) The impulse response is given in your notes.

For the transfer function, $H(s) = L[h(t)]$

$$= \int_0^\infty h(t)\exp(-st)\,dt = \int_0^{\Delta t} \exp(-st)\,dt = \left[1 - \exp(-s\Delta t)\right]/s$$

The frequency response is given by:

$$H(\omega) = (1 - \exp(-j\omega\Delta t))/(j\omega)$$
$$= \exp(-j\omega\Delta t/2)\left(\exp(j\omega\Delta t/2) - \exp(-j\omega\Delta t/2)\right)/(j\omega)$$
$$= \Delta t \exp(-j\omega\Delta t/2)\,\mathrm{sinc}(\omega\Delta t/2)$$

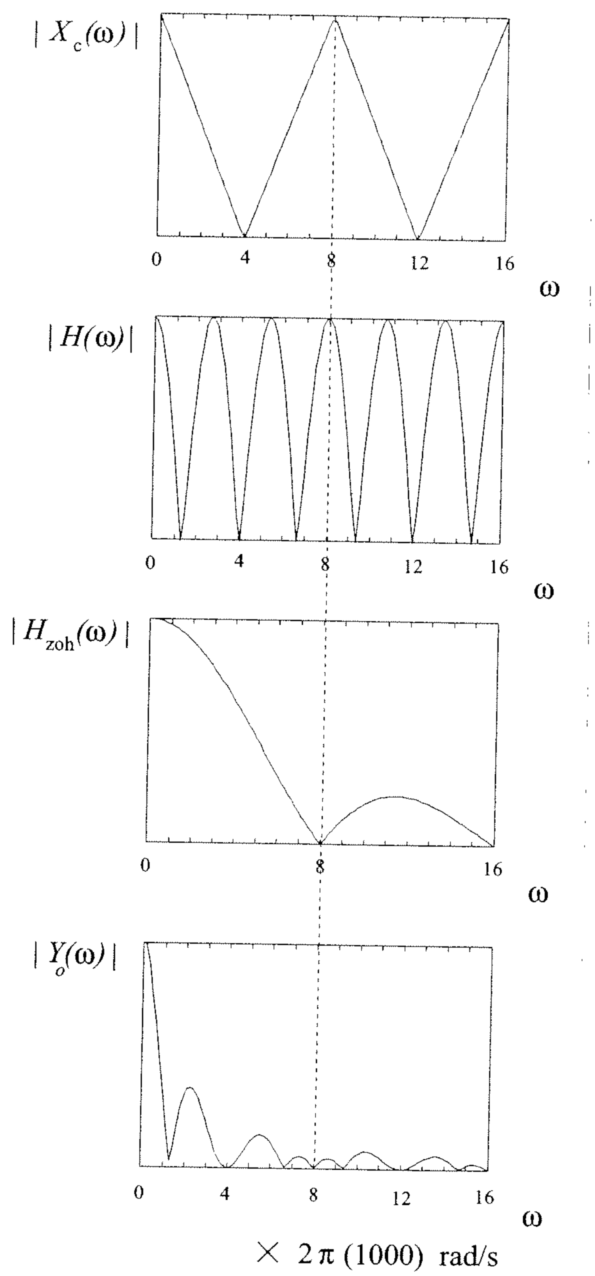
The amplitude response is given by:

$$|H(\omega)| = \Delta t\,|\mathrm{sinc}(\omega\Delta t/2)|$$

and the phase response by:

$$\angle H(\omega) = -\omega\Delta t/2 + \angle\mathrm{sinc}(\omega\Delta t/2)$$

(10) The Fourier transform $X_c(\omega)$ of the sampled signal is obtained by convolving the Fourier transform $X(\omega)$ with the Fourier transform of the impulse train as illustrated:



Taking z-transforms of the difference equation gives

$$Y(z) = X(z) + z^{-3}X(z)$$

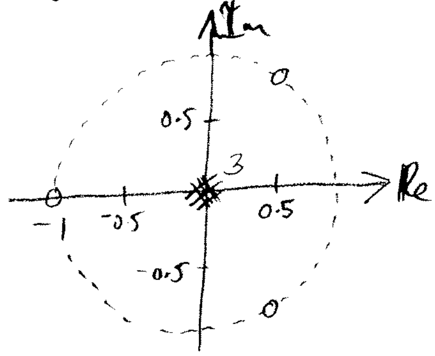
Thus the transfer function is given by

$$H(z) = \frac{Y(z)}{X(z)} = 1 + z^{-3} = \frac{z^3+1}{z^3}$$

The poles are the roots of the denominator polynomial, ie $z^3 = 0$ - there are three poles at $z = 0$

The zeros are the roots of the numerator polynomial, ie $z^3 = -1$. The poles are thus the 3 cube roots of -1 ie -1, $\exp(j\pi/3)$ & $\exp(-j\pi/3)$.

The pole/zero diagram is



The 3 poles have no effect on the amplitude response since they are by definition always the same distance away from any point on the unit circle. The zeros produce zero gain at 1/6th, 1/2 & 5/6th of the sampling frequency.

Here it is possible to develop an expression for the frequency response. By replacing z in the transfer function by $\exp(j\omega\Delta t)$ we obtain the frequency response.

$$H(\omega) = 1 + \exp(-3j\omega\Delta t)$$

$$= \exp(-3j\omega\Delta t/2)\left(\exp(3j\omega\Delta t/2) + \exp(-3j\omega\Delta t/2)\right)$$

$$= \exp(-3j\omega\Delta t/2)\, 2\cos(3\omega\Delta t/2)$$

Thus the amplitude response looks like a rectified cosine wave:

$$|H(\omega)| = 2|\cos(3\omega\Delta t/2)|$$

The amplitude response of the ZOH is given by

$$|H(\omega)| = \Delta t|\mathrm{sinc}(\omega\Delta t/2)|$$

The fourier transform $Y_o(\omega)$ of the analogue output is obtained by multiplying the three transforms at every frequency

$$Y_o(\omega) = H_{ZOH}(\omega)H(\omega)X_c(\omega)$$

thus

$$|Y_o(\omega)| = |H_{ZOH}(\omega)||H(\omega)||X_c(\omega)|$$

(11) (a) The impulse response sequence:

$$\{h(n)\} = 1, 1, 0, 0 \ldots, 0, \ldots$$

The transfer function

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} = 1 + z^{-1} = \frac{z+1}{z}$$

There is a zero at $z = -1$ & a pole at $z = 0$

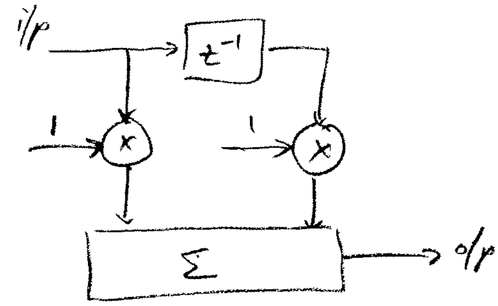
For the difference equation:

$$H(z) = 1 + z^{-1} = \frac{Y(z)}{X(z)}$$

$$Y(z) = X(z) + z^{-1} X(z)$$

Inverse transform both sides:

$$y(n) = x(n) + x(n-1)$$



(b) The impulse response sequence:

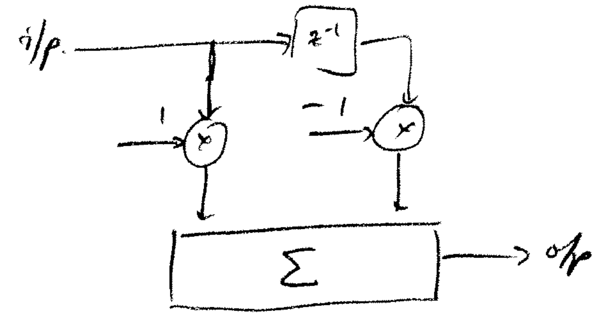
$$\{h(n)\} = 1, -1, 0, 0 \ldots, 0, \ldots$$

The transfer function

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} = 1 - z^{-1} = \frac{z-1}{z}$$

There is a zero at $z = 1$ & a pole at $z = 0$

Similarly we obtain $y(n) = x(n) - x(n-1)$



(c) The impulse response sequence.

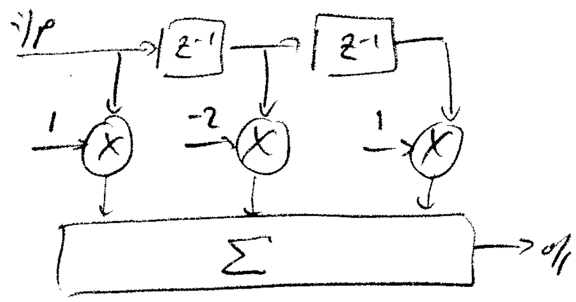
$$\{h(n)\} = 1, -2, 1, 0, 0 \ldots, 0, \ldots$$

The transfer function

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} = 1 - 2z^{-1} + z^{-2} = \frac{z^2 - 2z + 1}{z^2}$$

There are two zeros at $z = 1$ & two poles at $z = 0$.

for the difference equation



$$H(z) = 1 - 2z^{-1} + z^{-2} = \frac{Y(z)}{X(z)}$$

thus $\quad Y(z) = X(z) - 2z^{-1}X(z) + z^{-2}X(z)$

& taking inverse transforms

$$y(n) = x(n) - 2x(n-1) + x(n-2)$$

(d) The impulse response sequence:

$$h(n) = r^n \sin(w_0 n)$$

The transfer function

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} = \frac{1}{2j}\left[\sum_{n=0}^{\infty} r^n \exp(jw_0 n) z^{-n} - \sum_{n=0}^{\infty} r^n \exp(-jw_0 n) z^{-n}\right]$$

$$= \frac{1}{2j}\left[\sum_{n=0}^{\infty} (r \exp(jw_0) z^{-1})^n - \sum_{n=0}^{\infty} (r \exp(jw_0) z^{-1})^n\right]$$

$$= \frac{1}{2j}\left(\frac{1}{1 - r\exp(jw_0) z^{-1}} - \frac{1}{1 - r\exp(-jw_0) z^{-1}}\right)$$

$$= \frac{z^{-1} r (\exp(jw_0) - \exp(-jw_0))/2j}{1 - r(\exp(jw_0) + \exp(-jw_0)) z^{-1} + r^2 z^{-2}}$$

$$= \frac{z^{-1} r \sin(w_0)}{1 - r 2\cos(w_0) z^{-1} + r^2 z^{-2}}$$

$$= \frac{z r \sin(\omega_0)}{z^2 - r 2\cos(\omega_0) z^{-1} + r^2} = \frac{0.344 z}{z^2 - 1.663 z + 0.81}$$

There is one zero at $z = 0$ & complex conjugate poles at

$$z = 0.9 \exp(\pm j\pi/8) = 0.8315 \pm j0.344$$

For the difference equation

$$H(z) = \frac{Y(z)}{X(z)} = \frac{0.344 z}{z^2 - 1.663 z + 0.81}$$

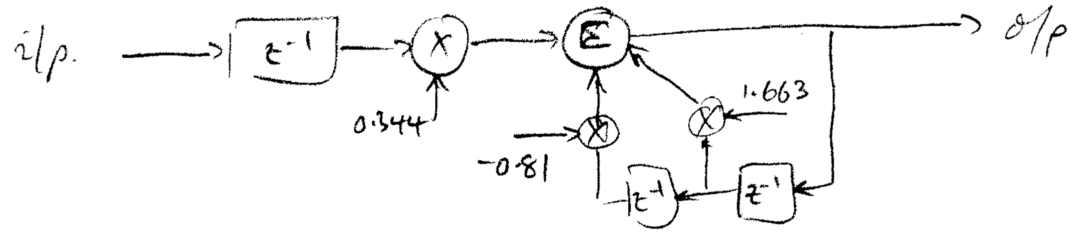
$$Y(z)\left(1 - 1.663 z^{-1} + 0.81 z^{-2}\right) = X(s)\, 0.344 z^{-1}$$

Inverse z-transform both sides

$$y(n) - 1.663\, y(n-1) + 0.81\, y(n-2) = 0.344\, x(n-1)$$

Rearranging gives

$$y(n) = 0.344\, x(n-1) + 1.663\, y(n-1) - 0.81\, y(n-2)$$



(12) Taking z-transforms of both sides of the difference equations gives

$$Y(z) = 1.6z^{-1}Y(z) - 0.8z^{-2}Y(z) + X(z)$$

Thus $$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 1.6z^{-1} + 0.8z^{-2}} = \frac{z^2}{z^2 - 1.6z + 0.8}$$

there are two zeros at $z=0$ & complex conjugate poles at $z = 0.8 \pm j0.4$.

For a unit pulse the first eight outputs are given by:

$$y(0) = 1.6y(-1) - 0.8y(-2) + x(0) = 0 - 0 + 1 = 1$$ } zero initial conditions!

$y(1) = 1.6$, $y(2) = 1.76$, $y(3) = 1.54$

$y(4) = 1.05$, $y(5) = 0.45$, $y(6) = -0.12$, $y(7) = -0.55$

$$\frac{H(z)}{z} = \frac{z}{z^2 - 1.6z + 0.8} = \frac{z}{(z - r\exp(j\phi))(z - r\exp(-j\phi))}$$

where $r = 0.8944$ & $\phi = 0.4636$ rad.

$$H(z) = \frac{\exp(j\phi)z}{2j\sin\phi(z - r\exp(j\phi))} - \frac{\exp(-j\phi)z}{2j\sin\phi(z - r\exp(-j\phi))}$$

thus $$h(n) = \frac{\exp(j\phi)}{2j\sin\phi}(r\exp(j\phi))^n - \frac{\exp(-j\phi)}{2j\sin\phi}(r\exp(-j\phi))^n$$

$$= \frac{r^n}{\sin\phi}\sin(\phi(n+1))$$

$$= 2.236(0.8944)^n \sin(0.4636(n+1)) \quad \text{for } n \geq 0.$$

Discrete convolution gives

$$y(0) = h(0)x(0) + h(1)x(-1) + h(2)x(2) + \ldots$$
$$= 1(0) + 1.6(0) + 1.76(0) + \ldots = 0$$

$y(1) = h(0)x(1) + h(1)x(0) + h(2)x(-1) + \dots$

$= 1(0.25) + 1.6(0) + 1.76(0) + \dots = 0.25$

$y(2) = 0.9, \quad y(3) = 1.99, \quad y(4) = 3.46$

$y(5) = 3.95, \quad y(6) = 3.55, \quad y(7) = 2.52$

(13) There are two zeros at $z=-1$ & three poles at $z=0$.

$$H(\omega) = \frac{(\exp(j\omega\Delta t)+1)^2}{(\exp(j\omega\Delta t))^3}$$

We expect a notch in the amplitude response at half the sampling frequency. From graphical arguments it is also straightforward to calculate the gain and phase shift at 0 Hz and at a quarter of the sampling frequency. Thus at $\omega = 0$

$$|H(0)| = \frac{2^2}{1^2} = 4 \qquad \angle H(0) = 2(0) - 3(0) = 0^\circ$$

At $\omega = \pi/(2\Delta t)$ – a quarter of the sampling frequency

$$|H(\pi/(2\Delta t))| = \frac{(\sqrt{2})^2}{1^3} = 2 \qquad \angle H(\pi/(2\Delta t)) = 2(45) - 3(90) = -180^\circ$$

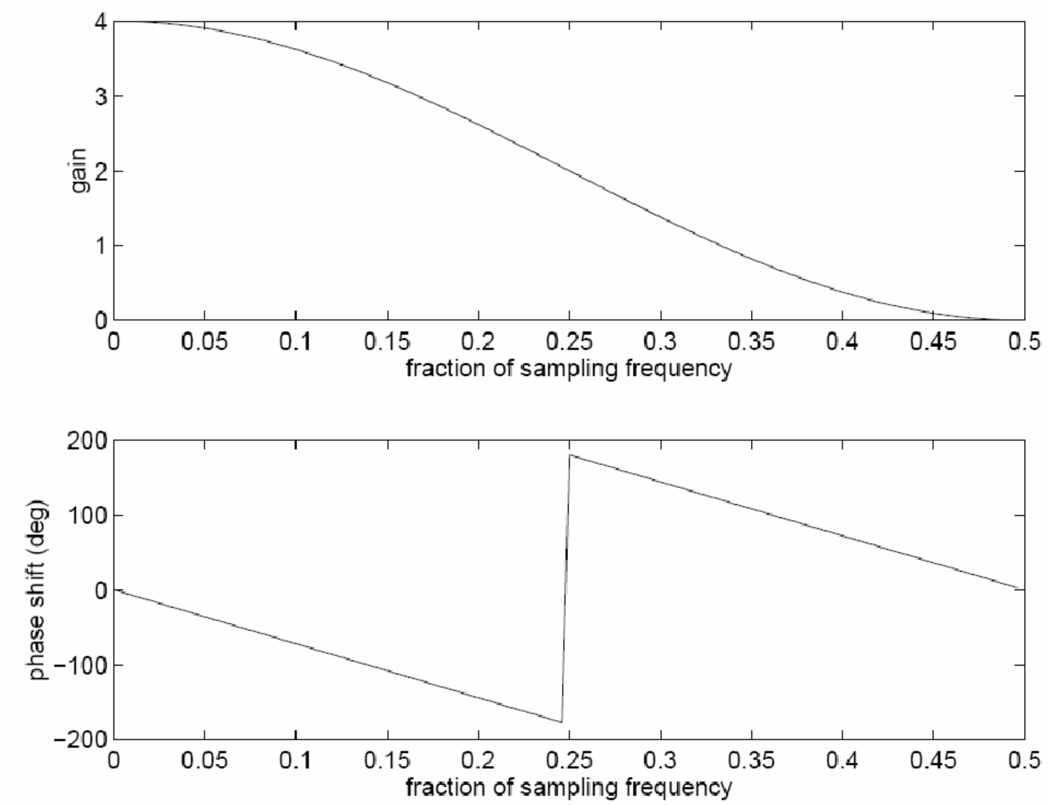
At $\omega = \pi/(\Delta t)$ – half the sampling frequency

$$|H(\pi/(\Delta t))| = \frac{0^2}{1^3} = 0$$

& taking a frequency slightly lower than half the sampling frequency

$$\angle H(\pi/(\Delta t)) = 2(90) - 3(180) = 0^\circ$$

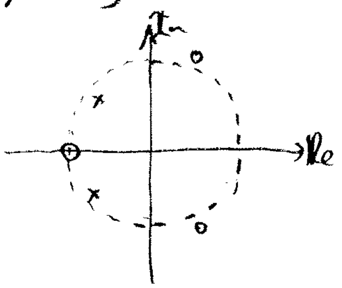
The full amplitude & phase responses are as illustrated.

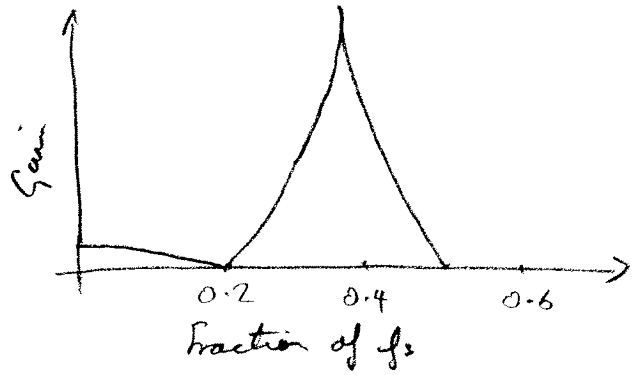


(14) There is a zero at $z=-1$ & at $z=1.05\exp(\pm 0.4j\pi)$.

There are complex conjugate poles at $z=0.95\exp(\pm j3\pi/4)$ & since they are inside the unit circle the filter is stable.

We expect a notch in the frequency response at 1/5th the sampling frequency with a small gain (not zero). There will be a peak in the response at 3/8th the sampling frequency and a further notch at 1/2 the sampling frequency where the gain will be zero.





$$H(z)=\frac{(z+1)(z-1.05e^{0.4j\pi})(z-1.05e^{-0.4j\pi})}{(z-0.95e^{3j\pi/4})(z-0.95e^{-3j\pi/4})}$$

Therefore the frequency response is given by

$$H(\omega)=\frac{(e^{j\omega\Delta t}+1)(e^{j\omega\Delta t}-1.05e^{0.4j\pi})(e^{j\omega\Delta t}-1.05e^{-0.4j\pi})}{(e^{j\omega\Delta t}-0.95e^{3j\pi/4})(e^{j\omega\Delta t}-0.95e^{-3j\pi/4})}$$

& the amplitude response by

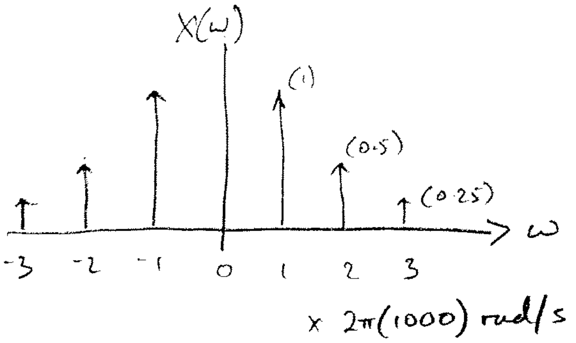
$$|H(\omega)|=\frac{|(e^{j\omega\Delta t}+1)||(e^{j\omega\Delta t}-1.05e^{0.4j\pi})||(e^{j\omega\Delta t}-1.05e^{-0.4j\pi})|}{|(e^{j\omega\Delta t}-0.95e^{3j\pi/4})||(e^{j\omega\Delta t}-0.95e^{-3j\pi/4})|}$$

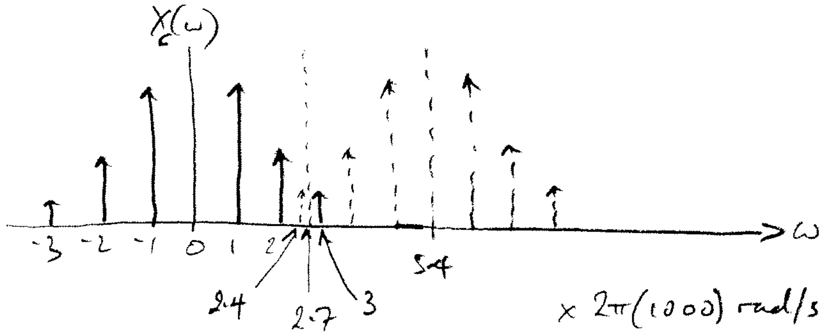
The maximum occurs at $\omega=3(2\pi/\Delta t)/8$ rad/s. Hence $\omega\Delta t=\frac{3\pi}{4}$ and the maximum gain is

$$|H(3(2\pi/\Delta t)/8|=\frac{0.7654\times 1.072\times 1.9934}{0.05\times 1.3793}$$

$$=23.71=\underline{\underline{27.5\,dB}}$$

(15) The Fourier transform, $X(\omega)$, of the periodic signal is





The strengths of the impulses are relative

The anti-aliasing filter has frequency response

$$H(\omega) = \frac{2000\pi}{2000\pi + j\omega} = \frac{1}{1 + j\omega/(2000\pi)}$$

hence it is a first order LPF with cut-off frequency of 2000π rad/s which is equivalent to 1kHz. At frequencies above this it will roll off at 20dB/decade. The gain is:

$$|H(\omega)| = \frac{1}{|1 + j\omega/(2000\pi)|}$$

The effects of the anti-aliasing filter on each harmonic component are summarised below

f (kHz)	amp. in	gain	amp. out
1	1	0.7071	0.7071
2	0.5	0.4472	0.2236
3	0.25	0.3162	0.0791

The effect of sampling at 5.4 kHz is illustrated in the lower half of the figure above and summarised below.

f (kHz)	amp.
1	0.7071
2	0.2236
2.4	0.0791

The difference equation

$$y(n) = x(n) - 0.7922\,x(n-1) + x(n-2)$$

gives a transfer function of

$$H(z) = 1 - 0.7992\,z^{-1} + z^{-2}$$

There are two poles at $z=0$ & two zeros at

$$z = 0.3961 \pm j\,0.9182 = \exp(\pm j\,1.1635)$$

ie on the unit circle.

The amplitude response is given by

$$|H(\omega)| = |\exp(j\omega\Delta t) - \exp(j1.1635)| \times |\exp(j\omega\Delta t) - \exp(-j1.1635)|$$

$\Delta t = 1/(5.4\times10^3)$ - the effects of the digital filter on the three frequencies are summarised in the following table.

freq. (kHz)	amp. in	gain	amp. out
1	0.7071	0.0	0.0
2	0.2236	2.1647	0.4840
2.4	0.0791	2.6716	0.2112

Between 0Hz & half the sampling frequency there are two frequency components present at the output of the digital filter at 2kHz & 2.4kHz with relative amplitudes of 0.48 & 0.21 respectively. The relative powers are thus 0.12 & 0.02 respectively. The components outside this range are easily obtained by symmetry.