Essential Mathematical Methods for Engineers (MathEng)
EXAM
December 2023
Duration: 2 hrs, all documents and calculators permitted
ATTEMPT ALL QUESTIONS – ANSWER IN ENGLISH

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1. Using Euler's identity (or any other appropriate method), write down an expression for the complex Fourier series of the signal $x(t)$:

$$x(t) = 3\cos(5t) + 4\sin(10t)$$

[5 marks]

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2. Develop an expression for the Fourier Transform of the signal $x(t)$ illustrated in Figure Q2 below.

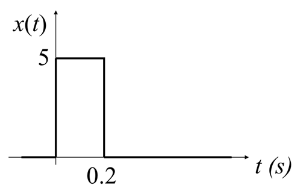

Figure Q2

[6 marks]

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3. A linear, time-invariant system has the following transfer function:

$$H(s) = \frac{10(s+100)}{s^2+2s+100}$$

(a) Derive an expression for $H(s)$ in the usual, normal form.
(b) Determine the frequency-invariant gain K and the position of any poles and zeros.
(c) Sketch a Bode plot of the magnitude-frequency response.
(d) Sketch a Bode plot of the phase-frequency response.

[8 marks]

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4. Sketch magnitude and phase responses for a sampled data system with a pair of complex conjugate zeros and two poles at the origin.

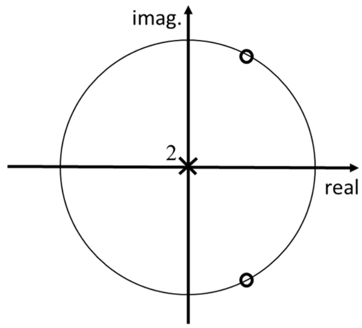

Figure Q4

[4 marks]

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5. A random variable X is uniformly distributed between x = 0 and x = 1. Via any appropriate method, determine the expected value $E[Y]$ of $Y = \exp(X)$.

[5 marks]

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6. Identify the pivots and free variables of the following two matrices A and B. Following the method which we studied in class, find the special solution corresponding to each free variable and, by combining the special solutions, describe every solution to Ax=0 and Bx=0.

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}$$

[7 marks]

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7. For a projection matrix $P = A(A^TA)^{-1}A^T$, show that $P^2 = P$ and then explain, in terms of the column space of P, why projections Pb and $P(Pb)$ give identical results.

[5 marks]

Table of selected Laplace transforms

$$X(s) = \int_{-\infty}^{\infty} x(t)\exp(-st)\,dt$$

N.B.: lower limit is 0 for one-sided Laplace transform

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\exp(st)\,ds$$

$x(t)\quad (t \geq 0)$	$X(s)$
$\delta(t)$	1
$\delta(t-\alpha)$	$\exp(-\alpha s)$
1 (unit step)	$\frac{1}{s}$
t (unit ramp)	$\frac{1}{s^2}$
$\exp(-\alpha t)$	$\frac{1}{s+\alpha}$
$t\exp(-\alpha t)$	$\frac{1}{(s+\alpha)^2}$
$\sin(\alpha t)$	$\frac{\alpha}{s^2+\alpha^2}$
$\cos(\alpha t)$	$\frac{s}{s^2+\alpha^2}$
$e^{-\alpha t}\sin(\omega t)$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
$e^{-\alpha t}\cos(\omega t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$

Bode plots

Poles or zeros on the real axis:

$$(s+a) = a\left(\frac{s}{a}+1\right) = \frac{1}{\tau}(\tau s+1)$$

Complex conjugate poles (or zeros):

$$(s^2+As+B) = (s^2+2\zeta\omega_0 s+\omega_0^2) = \omega_0^2((s/\omega_0)^2+2\zeta(s/\omega_0)+1)$$

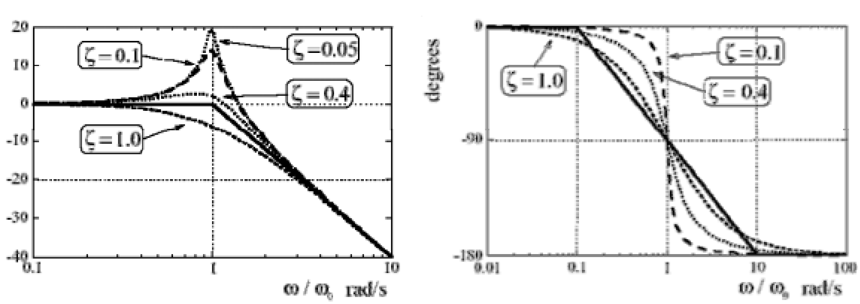

Table of selected z-transforms

$$x_c(t) = \sum_{n=0}^{\infty} x(n\Delta t)\delta(t - n\Delta t)$$

$$X_c(s) = \sum_{n=0}^{\infty} x(n\Delta t)\exp(-n\Delta ts)$$

$$X_c(s) = X(z)|_{z=e^{\Delta ts}}$$

$$X_c(\omega) = X_s(s)|_{s=j\omega}$$

$$X(\omega) = X(z)|_{z=\exp(\Delta tj\omega)}$$

$$X(z) = \sum_{n=0}^{\infty} x(n\Delta t)z^{-n}$$

$x(n)\quad (n \geq 0)$	$X(z)$
$\delta(n)$ unit pulse	1
$\delta(n-m)$	z^{-m}
1 (unit step)	$\dfrac{z}{z-1}$
n (unit ramp)	$\dfrac{z}{(z-1)^2}$
$\exp(-\alpha n)$	$\dfrac{z}{(z-e^{-\alpha})}$
$n\exp(-\alpha n)$	$\dfrac{e^{-\alpha}z}{(z-e^{-\alpha})^2}$
$\sin(\beta n)$	$\dfrac{z\sin(\beta)}{z^2-2z\cos(\beta)+1}$
$\cos(\beta n)$	$\dfrac{z^2-z\cos(\beta)}{z^2-2z\cos(\beta)+1}$
$e^{-\alpha n}\sin(\beta n)$	$\dfrac{ze^{-\alpha}\sin(\beta)}{z^2-2ze^{-\alpha}\cos(\beta)+e^{-2\alpha}}$
$e^{-\alpha n}\cos(\beta n)$	$\dfrac{z^2-ze^{-\alpha}\cos(\beta)}{z^2-2ze^{-\alpha}\cos(\beta)+e^{-2\alpha}}$

Table of selected Fourier transform pairs

Function	$x(t)$	$X(\omega)$
Rectangular function of width τ	$\Pi(t/\tau)$	$\tau\ \mathrm{sinc}(\omega\tau/2)$
Triangular function of width 2τ	$\Lambda(t/\tau)$	$\tau\ \mathrm{sinc}^2(\omega\tau/2)$
Train of impulses every Δt	$\delta_T(t)$	$2\pi/\Delta t\, \Sigma_n \delta(\omega - 2\pi n/\Delta t)$

NB: $\mathrm{sinc}(x) = \sin(\pi x)/\pi x$
NB: $\mathrm{sa}(x) = \sin(x)/x$

Euler's identity

$$\exp(j\theta) = \cos\theta + j\sin\theta$$

$$\cos(n\omega_0 t) = \frac{\exp(jn\omega_0 t) + \exp(-jn\omega_0 t)}{2}$$

$$\sin(n\omega_0 t) = \frac{\exp(jn\omega_0 t) - \exp(-jn\omega_0 t)}{2j}$$

Fourier series and transforms

Trigonometric Fourier series

$$x(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t)$$

$$A_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(n\omega_0\, t)dt$$

$$B_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(n\omega_0\, t)dt$$

Complex Fourier series – periodic and continuous in time, discrete in frequency

$x(t) = \sum_{n=-\infty}^{+\infty} X_n \exp(jn\omega_0 t)$	$X_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\exp(-jn\omega_0 t)\, dt$

Fourier transform – continuous in time, continuous in frequency

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\exp(j\omega t)\, d\omega$	$X(\omega) = \int_{-\infty}^{\infty} x(t)\exp(-j\omega t)\, dt$

Discrete-time Fourier transform – discrete in time, continuous and periodic in frequency

$x(n\Delta t) = \frac{\Delta t}{2\pi}\int_{-\pi/\Delta t}^{\pi/\Delta t} X(\omega)\exp(jn\Delta t\omega)\, d\omega$	$X(\omega) = \sum_{n=-\infty}^{\infty} x(n\Delta t)\exp(-j\omega n\Delta t)$

Discrete Fourier transform – discrete and periodic in time and in frequency

$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\exp\left(\frac{jnk2\pi}{N}\right)$	$X(k) = \sum_{n=0}^{N-1} x(n)\exp\left(\frac{-jnk2\pi}{N}\right)$

Transformation of random variables

$$f_Y(y) = \sum_{i=1}^{N} f_X(x_i) \left|\frac{dx_i}{dy}\right|_{x_i = g_i^{-1}(y)}$$

$$f_{UV}(u,v) = f_{XY}(x,y) \left|\frac{\partial(x,y)}{\partial(u,v)}\right|_{\substack{x = g_1^{-1}(u,v) \\ y = g_2^{-1}(u,v)}}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$