# Essential Mathematical Methods for Engineers

Lecture 1b:
Time domain description and convolution

# Outline

- time domain description and convolution
  - the impulse response
    - the impulse
    - signal representation
    - system response to an impulse
  - convolution
  - properties of convolution
    - time delay



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# The impulse response

- time and frequency domain descriptions of the input signals denoted $x(t)$ and $X(s)$ respectively
- identical descriptions of the output: $y(t)$ and $Y(s)$
- what about the system?
  - we've seen a frequency domain description $H(s)$
  - we also have a time domain description $h(t)$

$$h(t) = L^{-1}\left[H(s)\right]$$

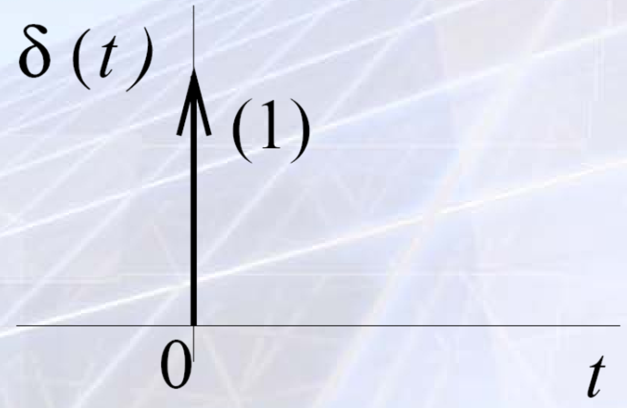
  this is the impulse response and is an important time domain description of the system

# The impulse

- the unit impulse
  - occurs at time $t = 0$
  - has infinite height
  - elsewhere it is zero
  - the area under the impulse function is equal to 1

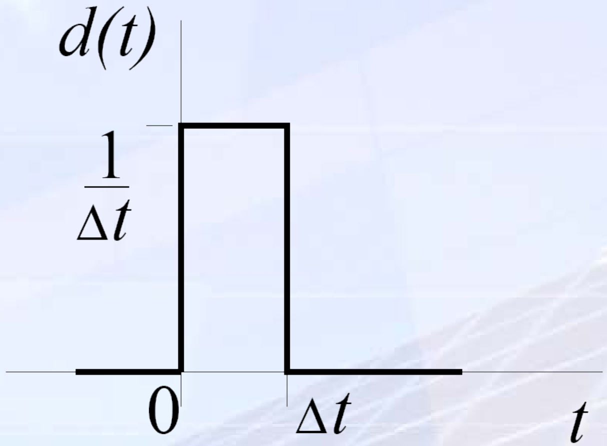
$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$



# The impulse

- to aid our understanding of the impulse consider a function $d(t)$
  - starts at time $t = 0$
  - lasts for $\Delta t$ seconds
  - has height $1/\Delta t$
  - an area of $1$




- if we reduce the width and increase the height and maintain

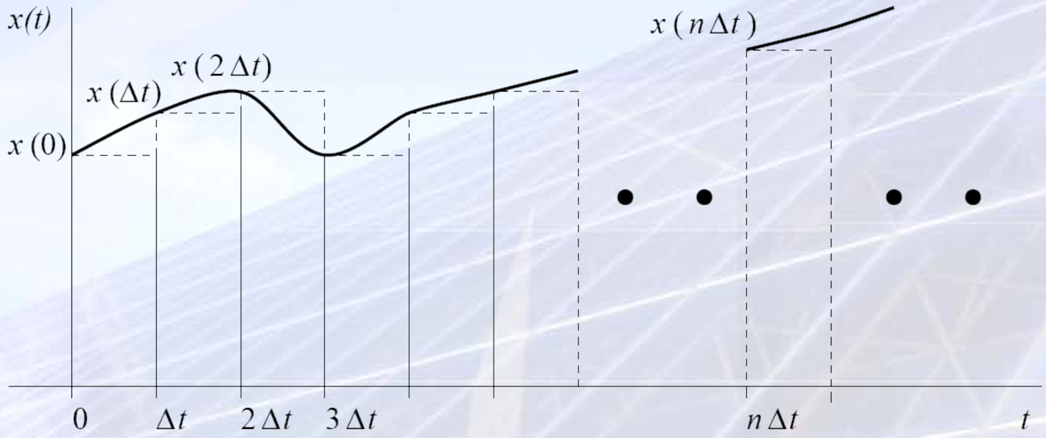
$$\int_{-\infty}^{\infty} d(t)dt = 1$$

$\Delta t$ tends toward zero and $d(t)$ tends toward $\delta(t)$

$$\delta(t) = \lim_{\Delta t \to 0} d(t)$$

# Signal representation

- we've seen how we can represent signals as a sum (or integral) of exponential basis functions
- we can also represent a signal as a sum (or integral) of impulses
- the representation improves as $\Delta t$ tends toward zero





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# Signal representation

- the pulse at time $t = n\Delta t$ has height $x(n\Delta t)$ so each pulse can be written as:

$$x(n\Delta t)d(t - n\Delta t)\Delta t$$

- adding together all of these impulses we have

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(n\Delta t)d(t - n\Delta t)\Delta t$$

- and as $\Delta t$ tends toward zero the
  - pulse becomes an impulse at time $t = n\Delta t$
  - product $n\Delta t$ becomes the continuous time variable $\tau$
  - time step $\Delta t$ becomes the differential $d\tau$
  - summation becomes an integration

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau$$

where $\tau$ is the time at which each impulse occurs



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# System response to an impulse

- we need to apply the Laplace transform

$$L[\delta(t)] = \int_{0^-}^{\infty} \delta(t)\exp(-st)dt$$

but at time $t = 0$ we have $\exp(-st) = 1$ so

$$L[\delta(t)] = \int_{0^-}^{\infty} \delta(t)\,1\,dt = 1$$

- now to find the transform of the output

$$\begin{aligned} Y(s) &= H(s)\,1 \\ &= H(s) \end{aligned}$$

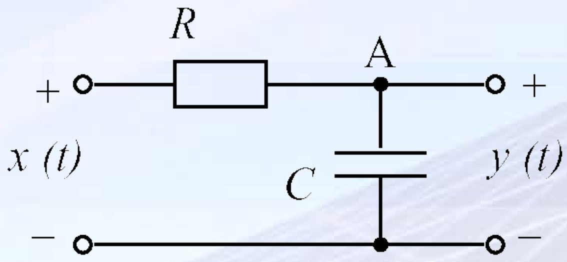
and to find the time domain description

$$\begin{aligned} y(t) &= L^{-1}[Y(s)] \\ &= L^{-1}[H(s)] \\ &= h(t) \end{aligned}$$

# System response to an impulse

***Example***

Calculate the impulse response of the following RC circuit using both a transform domain and time domain approach.



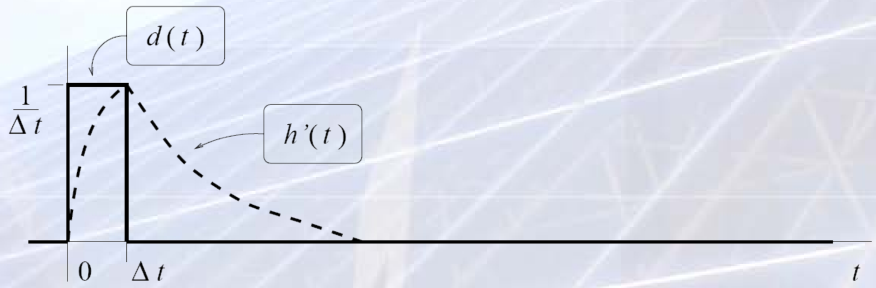

$$H(s) = \frac{1}{1 + RCs}$$

# Convolution

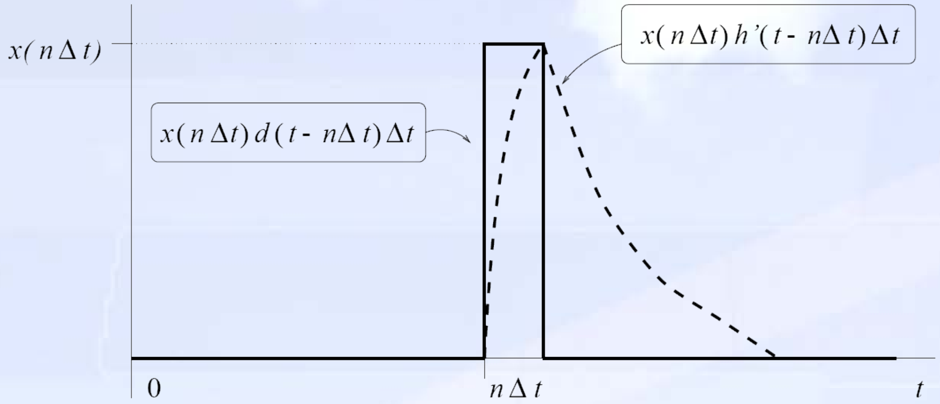
- in order to calculate the response of a system to an input $x(t)$ using the impulse response $h(t)$ recall that
  - the input $x(t)$ can be represented as a summation (integration) of impulses
  - the system is linear and superposition applies

- therefore we can evaluate the response to one of the impulses then integrate the responses to all such impulses to obtain the output $y(t)$

- as $\Delta t$ tends toward zero $d(t)$ tends toward an impulse and $h'(t)$ tends toward the impulse response



# Convolution




- illustrated is the response to a delayed and scaled pulse

$$x(n\Delta t)d(t-n\Delta t)\Delta t$$

- the response is given by

$$x(n\Delta t)h'(t-n\Delta t)\Delta t$$

- using superposition the response to the input

$$\hat{x}(t)=\sum_{n=-\infty}^{\infty}x(n\Delta t)d(t-n\Delta t)\Delta t$$

is given by

$$\hat{y}(t)=\sum_{n=-\infty}^{\infty}x(n\Delta t)h'(t-n\Delta t)\Delta t$$

# Convolution

$$\hat{y}(t) = \sum_{n=-\infty}^{\infty} x(n\Delta t)h'(t - n\Delta t)\Delta t$$

- and as $\Delta t \rightarrow 0$ the
  - pulse becomes an impulse at time $n\Delta t$
  - $n\Delta t$ becomes continuous time variable $\tau$
  - $h'(.)$ becomes the impulse response $h(.)$
  - time step $\Delta t$ becomes the differential $d\tau$
  - the summation becomes an integral

$$y(t) = \int_{-\infty}^{\infty} h(t - \tau)x(\tau)d\tau$$

or alternatively and equivalently

$$y(t) = \int_{-\infty}^{\infty} x(t - \tau)h(\tau)d\tau$$

# Convolution

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau \qquad y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$

- $t$ is the time for which we calculate the output
- $\tau$ is the variable used for integration

- notation for convolution:

$$y(t) = x(t) * h(t)$$

- an important relationship between the frequency and time domain

*“convolution in the time domain is equivalent to multiplication in the frequency domain”*

# Convolution

- we can verify the convolution by considering a complex phasor input $x(t) = \exp(st)$ to a system with impulse response $h(t)$

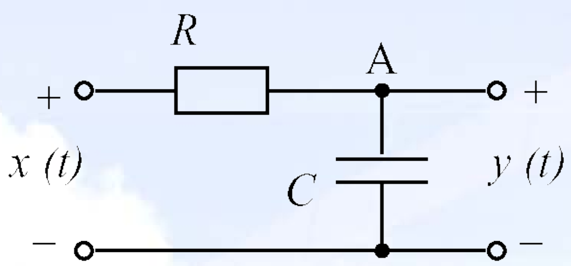
$$
\begin{aligned}
y(t) &= \int_0^{\infty} x(t-\tau)h(\tau)d\tau \\
&= \int_0^{\infty} \exp\big(s(t-\tau)\big)h(\tau)d\tau \\
&= \exp(st)\int_0^{\infty} h(\tau)\exp(-s\tau)d\tau \\
&= \exp(st)H(s)
\end{aligned}
$$

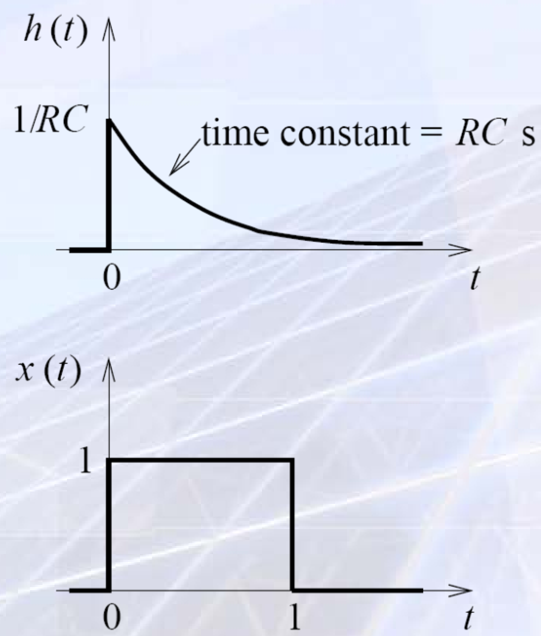
i.e. the response to a complex phasor is a complex phasor scaled by the transfer function $H(s)$

# Convolution

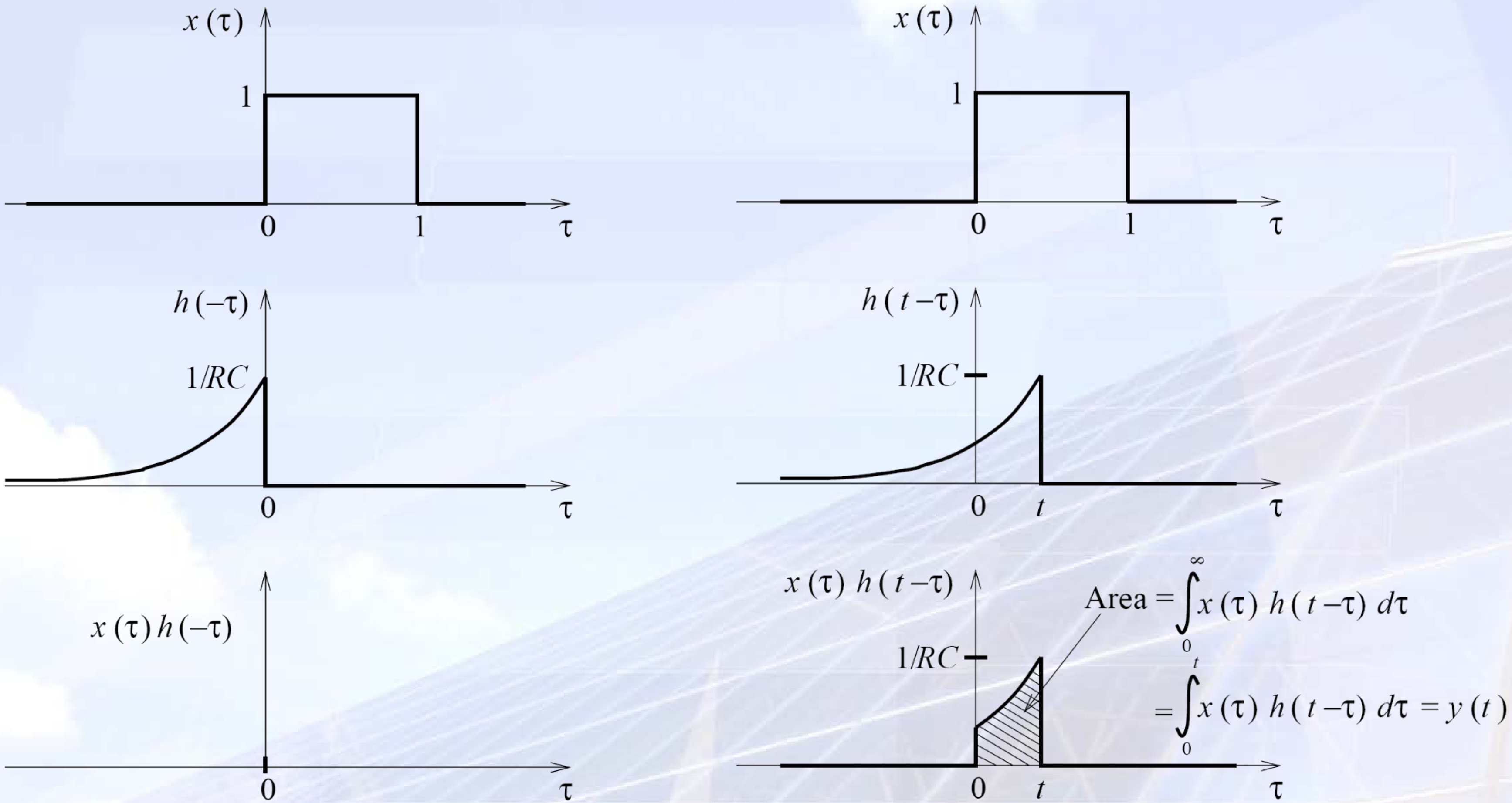
***Example***

Using convolution, calculate the response of the RC circuit to the rectangular pulse $x(t)$.

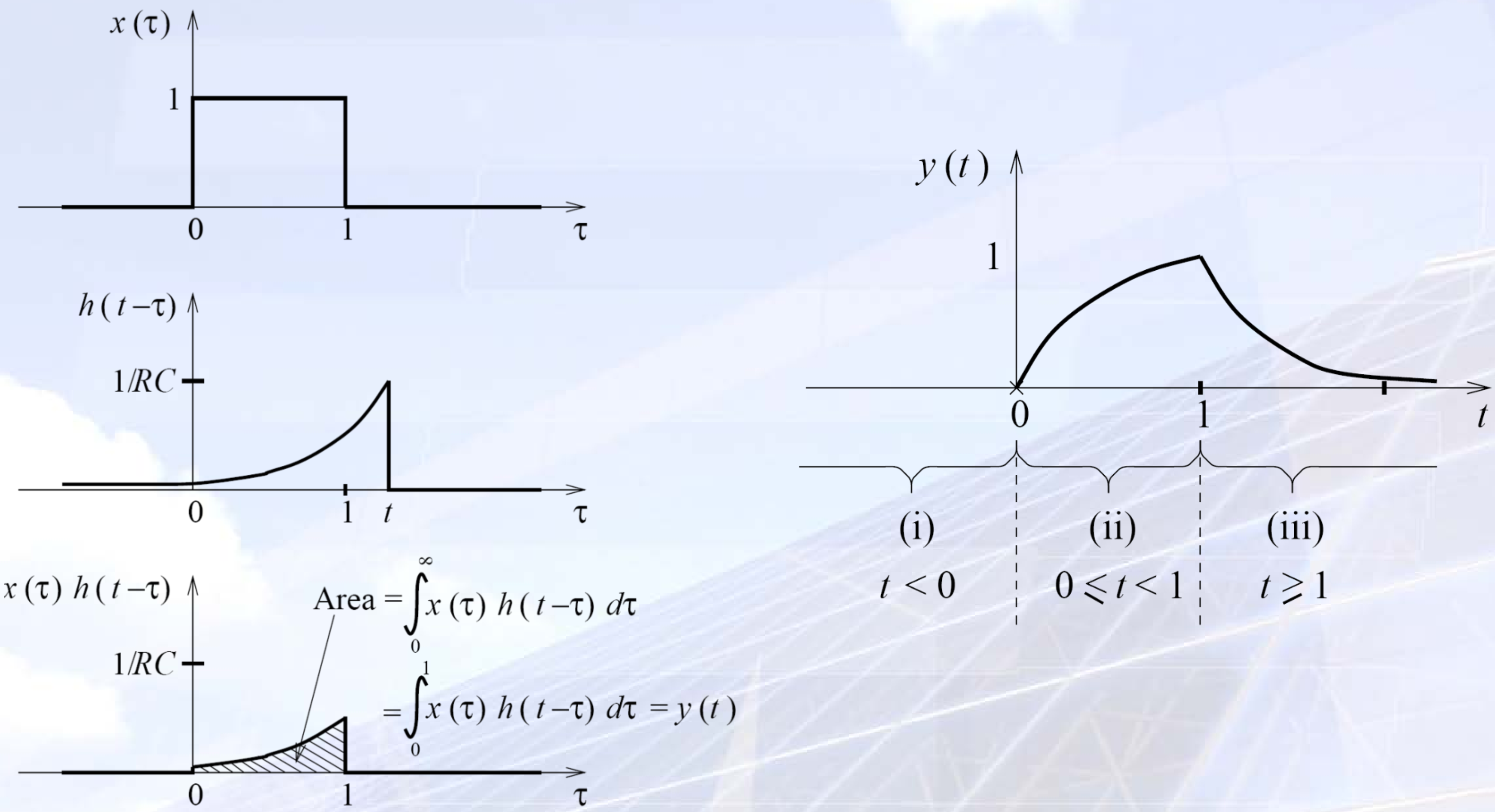


# Convolution



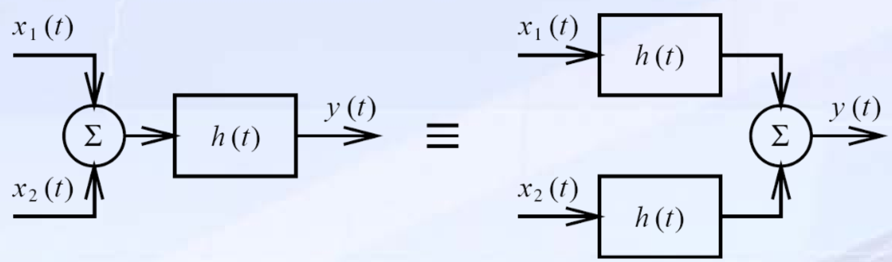
# Convolution



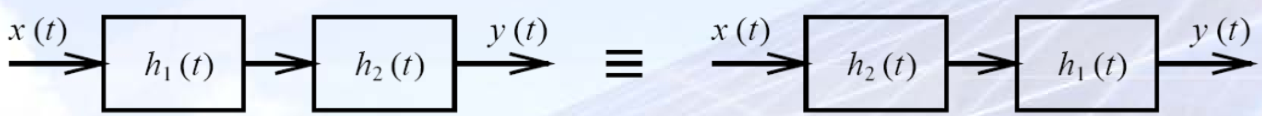
# Properties of convolution

- commutative $x(t) * h(t) = h(t) * x(t)$


- distributive over addition $h(t) * [x_1(t) + x_2(t)] = [h(t) * x_1(t)] + [h(t) * x_2(t)]$




- associative $h_2(t) * [h_1(t) * x(t)] = h_1(t) * [h_2(t) * x(t)]$




- multiplication

$$L[x_1(t)\, x_2(t)] = \frac{1}{2\pi j} X_1(s) * X_2(s) \qquad F[x_1(t)\, x_2(t)] = \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$

relevant to the study of modulation

*“multiplication in the time domain is equivalent to convolution in the frequency domain”*

# Properties of convolution

## Time delay

- in some cases the transform domain is easier to work with than the time domain – e.g. differential equations of *RC* circuits
- for some problems it is the time domain that is better suited, e.g. time delay
- consider a unit impulse $\delta(t)$ applied to a pure time delay of $a$ seconds

$$h(t) = \delta(t-a)$$

- and if we apply $x(t)$ to the same system

$$\begin{aligned} y(t) &= h(t) * x(t) \\ &= \delta(t-a) * x(t) \\ &= x(t-a) \end{aligned}$$

# Properties of convolution

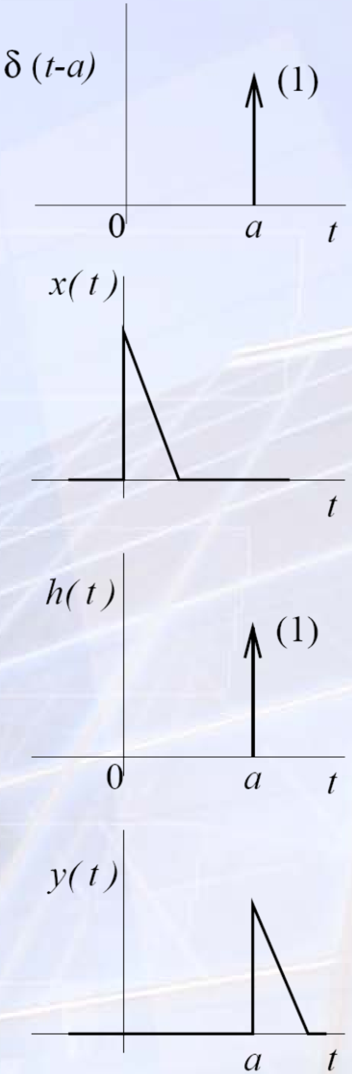
## Time delay

- therefore a system with impulse response $\delta(t-a)$ just delays the input by $a$ seconds

- if we have the impulse response we can obtain the transfer function:

$$\begin{aligned} H(s) &= L[\delta(t-a)] \\ &= \exp(-as)\,1 \\ &= \exp(-as) \end{aligned}$$

- we have a complete description of a system from its
  - impulse response
  - transfer function





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# Summary

You should be:

- familiar with the impulse function and its properties;
- able to work out the impulses response of a simple linear system;
- familiar with the convolution integral and its properties;
- able to evaluate the response of simple linear system to a simple waveform using the integral;
- calculate the response of a simple system to a simple waveform using transform techniques.