Tutorial Sheet 4.

(1) Applying KCL at the output we have

$$\frac{e-v}{R_1} + C\frac{d}{dt}(e-v) = \frac{v}{R_2}$$

Rearranging gives

$$\frac{e}{R_1} + C\frac{de}{dt} = \frac{v}{R_1} + \frac{v}{R_2} + C\frac{dv}{dt}$$

Taking Laplace transforms of both sides

$$\frac{E}{R_1} + CsE = \frac{V}{R_1} + \frac{V}{R_2} + CsV$$

The transfer function is thus

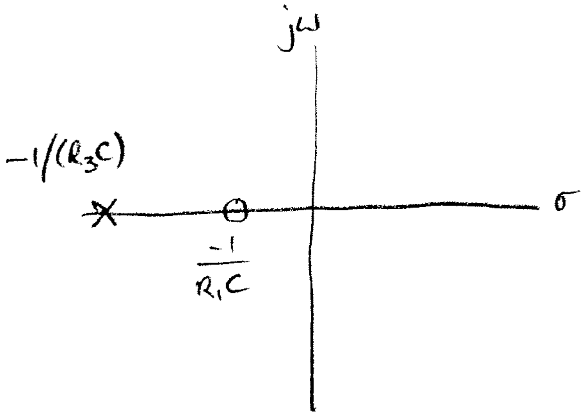
$$H(s) = \frac{V(s)}{E(s)} = \frac{(1+R_1Cs)R_2}{R_1 + R_2 + CR_1R_2s}$$

Zeros are solutions of $(1+R_1Cs)R_2 = 0 \Rightarrow s = \frac{-1}{RC}$

Poles are solutions to $R_1 + R_2 + CR_1R_2s = 0 \Rightarrow s = \frac{-R_1+R_2}{CR_1R_2} = \frac{-1}{CR_3}$

where $R_3 = R_1 /\!/ R_2$. Since $R_3 < R_1$

we have $\frac{1}{CR_3} > \frac{1}{CR_1}$ therefore the pole/zero diagram is:



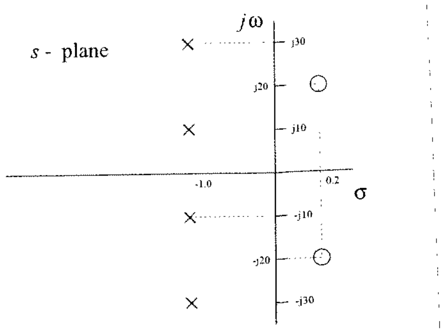
(2) Zeros are solution of $s^2 - 0.4s + 400.04 = 0$

i.e. $s = 0.2 \pm j20.0$

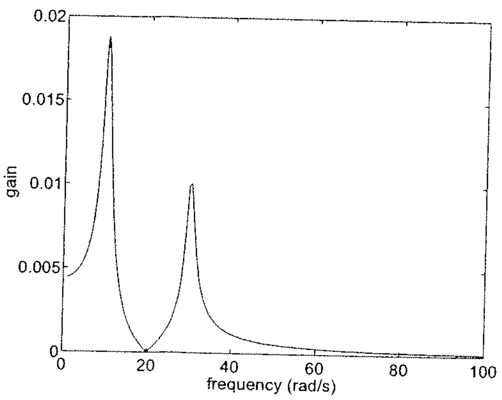
Poles are solutions of $(s^2 + 2s + 101)(s^2 + 2s + 901) = 0$

i.e. $s = -1.0 \pm j30.0$ or $s = -1.0 \pm j10.0$

The corresponding pole/zero map is:



and the frequency response is:



(3) Given that $F(e^{j\omega_a t}) = 2\pi\delta(\omega-\omega_a)$ and by expanding the sinewave as a sum of phasors using Euler's identity, we have

$$F(A\sin(\omega_a t)) = A F\left(\frac{e^{j\omega_a t} - e^{-j\omega_a t}}{2j}\right)$$

$$= \frac{A}{2j} F(e^{j\omega_a t}) - \frac{A}{2j} F(e^{-j\omega_a t})$$

$$= \frac{A}{2j} 2\pi\delta(\omega-\omega_a) - \frac{A}{2j} 2\pi\delta(\omega+\omega_a)$$

$$= -jA\pi\left[\delta(\omega-\omega_a) - \delta(\omega+\omega_a)\right]$$

(4) The transmitted signal is given by

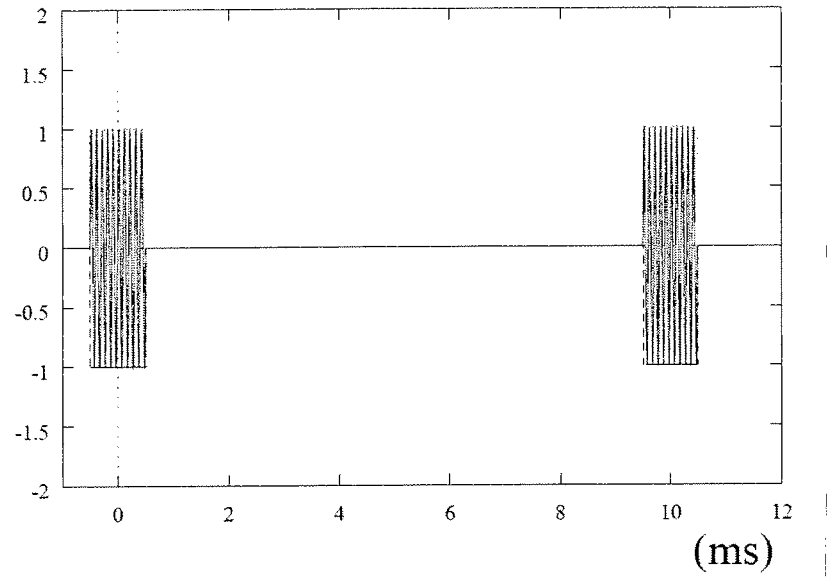
$$x_w(t) = x_c(t)\, x_p(t)$$

where $x_c(t)$ is the 10 kHz carrier & $x_p(t)$ is the on-off keyed pulse train.

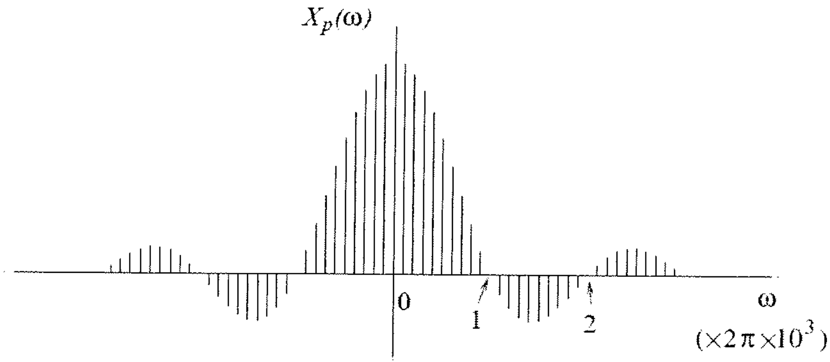
$X_w(\omega)$ may be obtained by convolving $X_c(\omega)$ & $X_p(\omega)$

$$X_w(\omega) = \frac{1}{2\pi} X_c(\omega) * X_p(\omega)$$

The following is an illustration of the transmitted waveform.



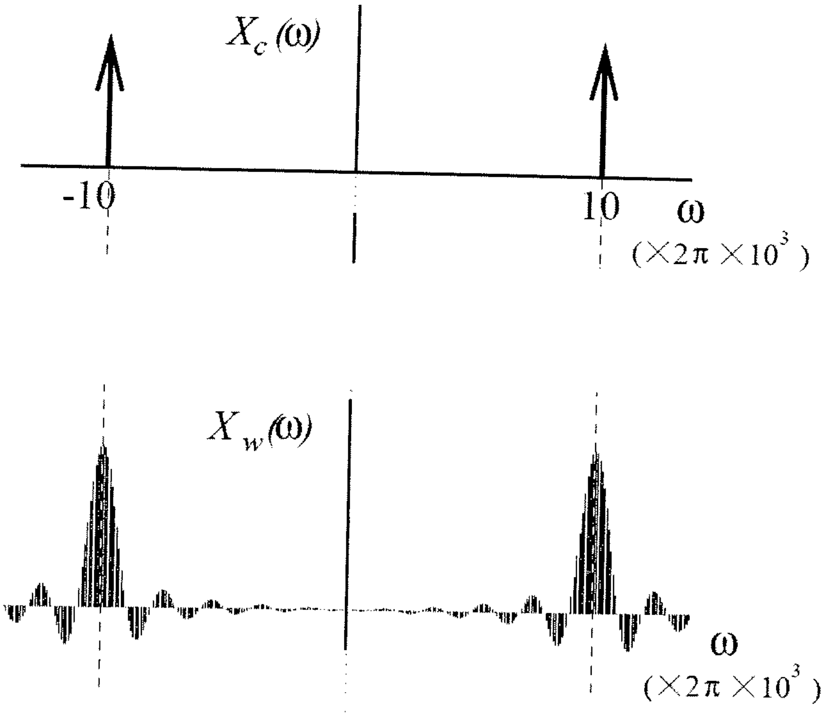
x_p(t) has a Fourier series expansion



where the harmonics are spaced at $1/(10\times10^{-3}) = 0.1\,kHz$ and the first null of the $\sin(x)/x$ envelope is at $1\,kHz$.

The Fourier transform of the carrier is two impulses. Since the carrier and the pulse train are multiplied in the time domain their Fourier transforms are convolved in the frequency domain. Convolution with each impulse is a matter of setting down $X_p(\omega)$ everywhere there is an impulse.

Thus we have



$X_c(\omega)$
-10
10
ω
$(\times 2\pi \times 10^3)$
$X_w(\omega)$
ω
$(\times 2\pi \times 10^3)$

(5) The transfer function may be written as

$$H(s) = \frac{2500(s+10)}{s(s+2)(s^2+30s+2500)}$$

The frequency response is obtained from

$$H(\omega) = \frac{2500(j\omega+10)}{j\omega(j\omega+2)((j\omega)^2+30j\omega+2500)}$$

Rearranging in terms for which the Bode plot is known

$$H(\omega) = \frac{2500(j\omega/10+1)10}{j\omega(j\omega/2+1)2((j\omega/50)^2+30j\omega/50^2+1)50^2}$$

$$= \frac{5(j\omega/10+1)}{j\omega(j\omega/2+1)((j\omega/50)^2+30j\omega/50^2+1)}$$

Draw the terms separately as they occur

(i) a constant gain of 5 ie. $20\log_{10}(5) = 14\,dB$

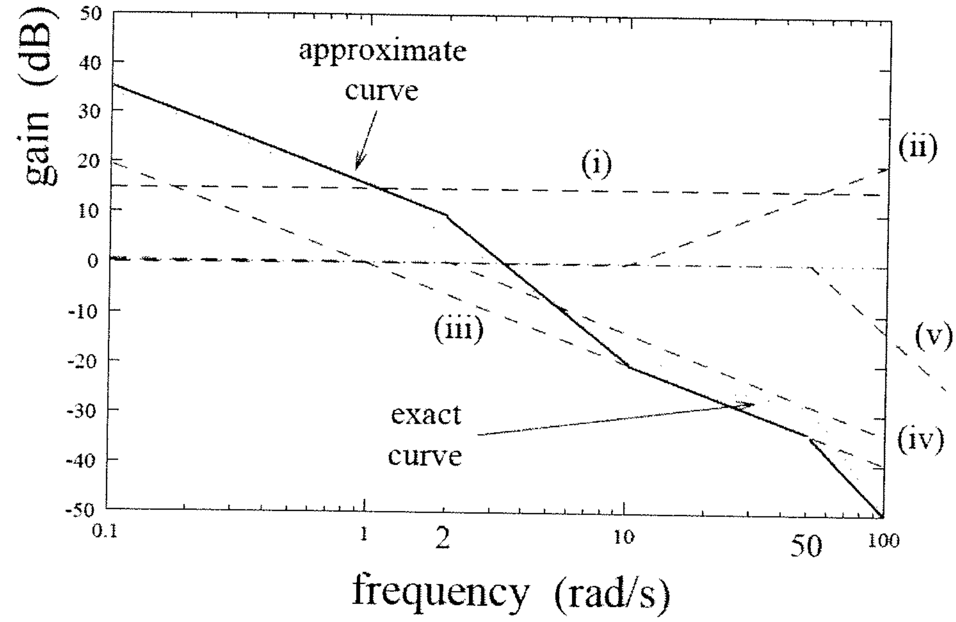
(ii) a zero with a cut in at $\omega = 10$ rad/s

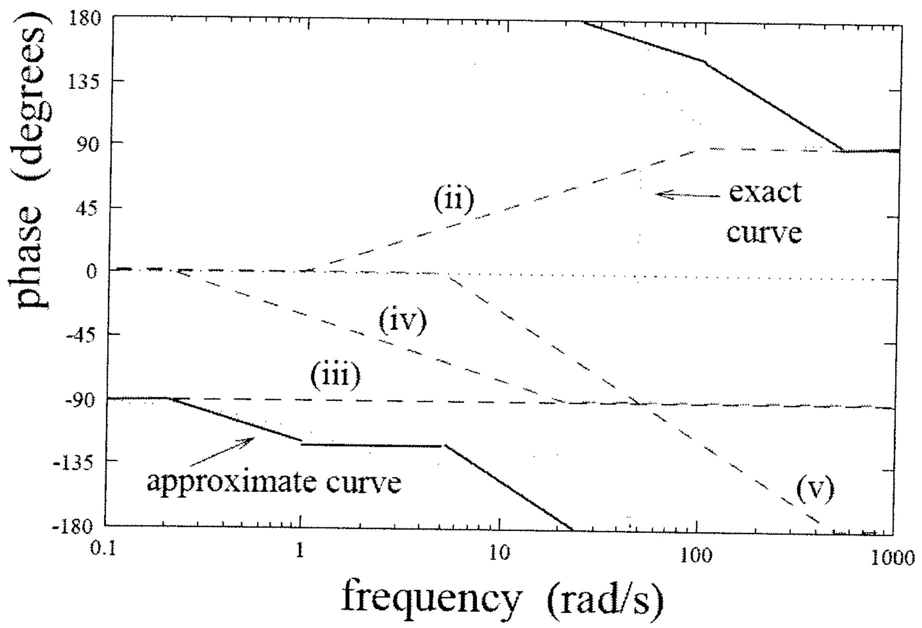
(iii) a pole at the origin

(iv) a pole with a cut off at $\omega = 2$ rad/s

(v) a complex conjugate pair of poles with $\omega_0 = 50$ rad/s and $\zeta = 0.3$

Adding the contributions of the individual terms we obtain:





(6) There are poles at $s = -0.1 \pm j$ thus the system is stable.

The transfer function may be rewritten as

$$H(s) = \frac{1}{s^2 + 0.2s + 1.01} = \frac{1}{(s+0.1)^2 + 1} = \left[\frac{1}{s^2+1}\right]_{s \to s+0.1}$$

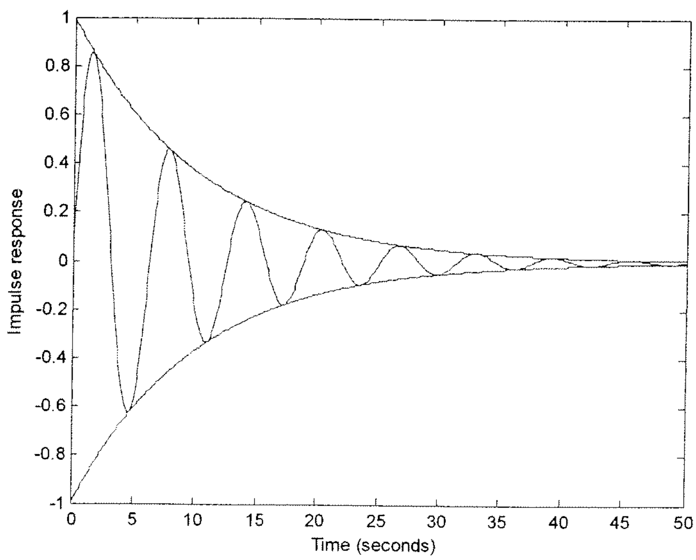
Since $\frac{1}{s^2+1} = \mathcal{L}\{\sin t\}$ from the first shift theorem we have

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+0.2s+1.01}\right\} = e^{-0.1t}\sin t$$

So the impulse response $h(t) = e^{-0.1t}\sin t \quad (t \geq 0)$

The period of oscillation is 2π seconds & the time constant of decay is 10 seconds.

The impulse response & envelope of decay are shown:



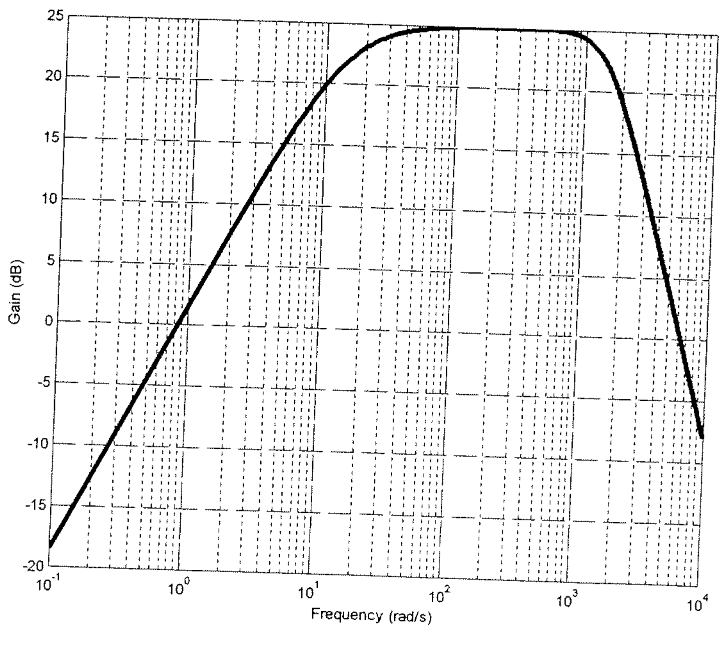
(7) The undamped natural frequency is $\omega_0 = 2$ rad/s & the damping factor is $\zeta = 1/2$

(8) The frequency response is given by

$$H(\omega) = \frac{4 \times 10^7 j\omega}{(j\omega + 15)((j\omega)^2 + 2100 j\omega + 2250000)}$$

$$= \frac{4 \times 10^7 j\omega}{15(j\omega/15 + 1)((j\omega)^2 + 2100 j\omega + 1500^2)}$$

Thus there is a zero at the origin, a real pole at $s = -15$ and a complex conjugate pair of poles with an undamped natural frequency of 1500 rad/s. The filter will be band pass with a lower cut-off frequency of 15 rad/s and a higher cut-off frequency of 1500 rad/s.



The bandwidth is $1500 - 15 = 1485$ rad/s

$= 236$ Hz

(9) From KCL $I(s) = \frac{V(s)}{R} + \frac{V(s)}{1/C(s)}$

therefore $V(s) = \frac{R}{1+sCR} I(s)$ and for a step input

$$V(s) = \frac{R}{1+sCR} \cdot \frac{1}{s}$$

A partial fraction expansion gives

$$V(s) = \frac{-R}{(s+1/CR)} + \frac{R}{s}$$

and taking inverse Laplace transforms gives

$$v(t) = -R\exp(-t/(RC)) + R$$

$$= R(1-\exp(-t/RC))$$

the final value, $v(\infty) = R$ and the rise time is given by

$$0.9v(\infty) = R(1-\exp(-T_r/(RC)))$$

therefore

$$T_r = RC\ln(10) = 1.04\,\mu s$$

The frequency response is given by $H(\omega) = \frac{R}{1+j\omega RC}$

which is a low pass filter with a cut-off frequency of $\omega_c = 1/(RC)$ rad/s which is equivalent to $f_c = 1/(2\pi RC)$ Hz

$$= \frac{\ln(10)}{2\pi T_r}$$

The bandwidth is inversely proportional to the rise time.

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$$E_o(s) = \frac{1}{RCs + 1} E_i(s)$$

i.e. the filter is characterised by

$$G(s) = \frac{1}{RCs + 1}$$

Therefore

$$G(j\omega) = \frac{1}{RCj\omega + 1} = \frac{1 - jRC\omega}{1 + R^2C^2\omega^2} = \frac{1}{1 + R^2C^2\omega^2} - j\frac{RC\omega^2}{1 + R^2C^2\omega^2}$$

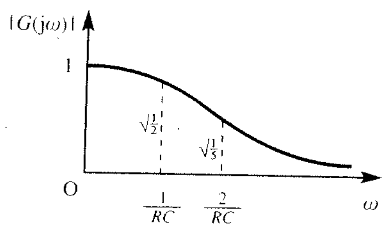
giving the frequency response characteristics

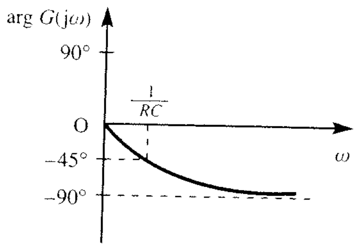
$$|G(j\omega)| = \sqrt{\left[\frac{1}{(1 + R^2C^2\omega^2)^2} + \frac{R^2C^2\omega^2}{(1 + R^2C^2\omega^2)^2}\right]}$$

$$\arg G(j\omega) = -\tan^{-1}(RC\omega)$$

For $\omega = 0$ $|G(j\omega)| = 1$ & $\arg G(j\omega) = 0$

& as $\omega \to \infty$ $|G(j\omega)| \to 0$ & $\arg G(j\omega) \to -\frac{1}{2}\pi$





(11) Expressing $G(s)$ in standard form we obtain

$$G(s) = \frac{10(1+0.2s)}{s(1+0.01s)(1+0.05s)}$$

giving

$$G(j\omega) = \frac{10(1+j0.2\omega)}{j\omega(1+j0.01\omega)(1+j0.05\omega)}$$

& taking logarithms to the base 10 we obtain

$$20\log_{10}|G(j\omega)| = 20\log 10 + 20\log|1+j0.2\omega| - 20\log|j\omega| - 20\log|1+j0.01\omega| - 20\log|1+j0.05\omega|$$

$$\arg G(j\omega) = \arg 10 + \arg(1+j0.2\omega) - \arg j\omega - \arg(1+j0.01\omega) - \arg(1+j0.05\omega)$$

Constructing Bode plots from constituent parts we obtain: