Mathematical Methods for Engineers (MathEng)
EXAM
4th February 2013
Duration: 2 hrs, calculators permitted, no documents
This exam paper contains 7 questions and 80 marks.
ATTEMPT ALL QUESTIONS – ANSWER IN ENGLISH

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1. Determine the Fourier and Laplace transforms of the waveform $x(t)$ illustrated in Figure Q1. Then write the Fourier transform in the form of a sinc( ) function and sketch plots of the magnitude and phase spectra.

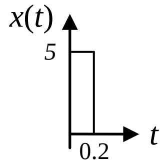

Figure Q1

[20 marks]

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2. For a system with the following difference equation:

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n)$$

and assuming zero initial conditions, determine expressions for:

(a) the system transfer function $H(z)$;
(b) the impulse response $h(n)$;
(c) the frequency response $H(\omega)$.

Make sketches of the poles and zeros in the z-plane and the magnitude frequency response and then state whether the system is low-pass or high-pass.

[15 marks]

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3. A random variable X is uniformly distributed between $x = 0$ and $x = 1$. Via any appropriate method, determine the expected value of $Y = \exp(X)$.

[5 marks]

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4. A random variable Z has a probability density function given by:

$$f_Z(z) = a\,\exp(-b|z|)$$

where both a and b are constants. Determine the value of a and then the characteristic function of Z.

[15 marks]

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5. (a) Identify and **briefly** describe the four fundamental subspaces of a matrix and then discuss the link between the rank of the matrix, the dimensionality of each subspace and any orthogonality between them. (No proof is required.)

   (b) Determine the rank of the following matrix A and the dimensions of its four fundamental subspaces:

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \\ 0 & -1 \end{bmatrix}$$

[7 marks]

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6. (a) Consider the **full-size** SVD of any rank-r $m \times n$ matrix $A=U\Sigma V^T$ where both U and V are square matrices. Describe the link between the four fundamental subspaces of A and the columns of matrices U and V.

   (b) Compute matrices U and V for the matrix A in question 5. According to your previous description, identify matrices U_1, U_2, V_1 and V_2 (when they exist) so as to split the columns of U and V into $U=[U_1\ U_2]$ and $V=[V_1\ V_2]$. This is the **reduced-size** SVD of $A=U_1\Sigma_1 V_1^T$, where:

$$\Sigma_1 = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{7} \end{bmatrix}$$

   (c) The matrix P which projects any vector x in R^m onto the column space of an $m \times n$ matrix A is given by $P = A(A^T A)^{-1} A^T$. Fully justifying your answer, show that in the case of the matrix A in question 5, this expression simplifies to $P = U_1 {U_1}^T$. (You are **not** required to compute P.)

[13 marks]

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7. A signal vector x is corrupted by noise n to produce the noisy vector $z = x + n$ in R^3, given by:

$$z = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Given that the signal vector x lies in the subspace spanned by the columns of the matrix A in question 5 and that the noise vector n lies in the orthogonal complement of this subspace, recover x from z. Give a basis for the noise subspace.

N.B. Your ability to answer this question requires some of the solutions to question 6. If you are unable to answer question 6 you should explain the method used to answer question 7.

[5 marks]

Table of selected Laplace transforms

$$X(s) = \int_{-\infty}^{\infty} x(t)\exp(-st)\,dt$$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\exp(st)\,ds$$

$x(t) \quad (t \geq 0)$	$X(s)$
$\delta(t)$	1
$\delta(t-\alpha)$	$\exp(-\alpha s)$
1 (unit step)	$\frac{1}{s}$
t (unit ramp)	$\frac{1}{s^2}$
$\exp(-\alpha t)$	$\frac{1}{s+\alpha}$
$t\exp(-\alpha t)$	$\frac{1}{(s+\alpha)^2}$
$\sin(\alpha t)$	$\frac{1}{s^2+\alpha^2}$
$\cos(\alpha t)$	$\frac{s}{s^2+\alpha^2}$
$e^{-\alpha t}\sin(\omega t)$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
$e^{-\alpha t}\cos(\omega t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$

Bode plots

Poles or zeros on the real axis:

$$(s+a) = a\left(\frac{s}{a}+1\right) = \frac{1}{\tau}(\tau s+1)$$

Complex conjugate poles (or zeros):

$$(s^2+As+B) = (s^2+2\zeta\omega_0 s+\omega_0^2) = \omega_0^2((s/\omega_0)^2+2\zeta(s/\omega_0)+1)$$

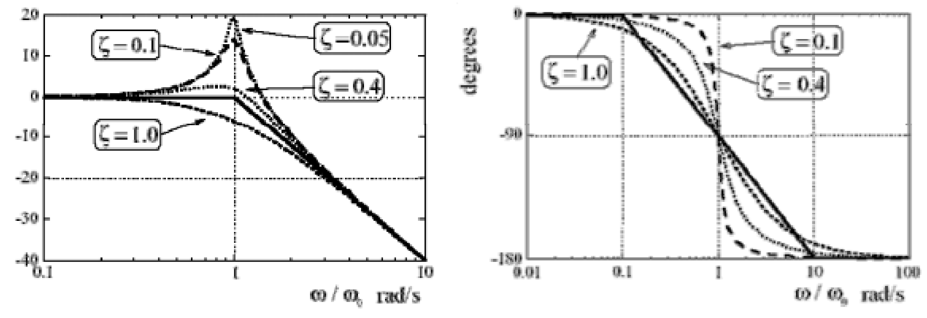

Table of selected z-transforms

$$x_c(t) = \sum_{n=0}^{\infty} x(n\Delta t)\delta(t - n\Delta t)$$

$$X_c(s) = \sum_{n=0}^{\infty} x(n\Delta t)\exp(-n\Delta ts)$$

$$X_c(s) = X(z)|_{z=e^{\Delta ts}}$$

$$X_c(\omega) = X_s(s)|_{s=j\omega}$$

$$X(\omega) = X(z)|_{z=\exp(\Delta tj\omega)}$$

$$X(z) = \sum_{n=0}^{\infty} x(n\Delta t)z^{-n}$$

$x(n)$ $(n \geq 0)$	$X(z)$
$\delta(n)$ unit pulse	1
$\delta(n-m)$	z^{-m}
1 (unit step)	$\frac{z}{z-1}$
n (unit ramp)	$\frac{z}{(z-1)^2}$
$\exp(-\alpha n)$	$\frac{z}{(z-e^{-\alpha})}$
$n\exp(-\alpha n)$	$\frac{e^{-\alpha}z}{(z-e^{-\alpha})^2}$
$\sin(\beta n)$	$\frac{z\sin(\beta)}{z^2-2z\cos(\beta)+1}$
$\cos(\beta n)$	$\frac{z^2-z\cos(\beta)}{z^2-2z\cos(\beta)+1}$
$e^{-\alpha n}\sin(\beta n)$	$\frac{ze^{-\alpha}\sin(\beta)}{z^2-2ze^{-\alpha}\cos(\beta)+e^{-2\alpha}}$
$e^{-\alpha n}\cos(\beta n)$	$\frac{z^2-ze^{-\alpha}\cos(\beta)}{z^2-2ze^{-\alpha}\cos(\beta)+e^{-2\alpha}}$

Table of selected Fourier transform pairs

Function	$x(t)$	$X(\omega)$
Rectangular function of width τ	$\Pi(t/\tau)$	$\tau\ \mathrm{sinc}(\omega\tau/2)$
Triangular function of width 2τ	$\Lambda(t/\tau)$	$\tau\ \mathrm{sinc}^2(\omega\tau/2)$
Train of impulses every Δt	$\delta_T(t)$	$2\pi/\Delta t\,\Sigma_n \delta(\omega - 2\pi n/\Delta t)$

NB: $\mathrm{sinc}(x) = \sin(x)/x$

Fourier series and transforms

Trigonometric Fourier series

$$x(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t)$$

$$A_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(n\omega_0\, t)dt$$

$$B_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(n\omega_0\, t)dt$$

Complex Fourier series – periodic and continuous in time, discrete in frequency

$x(t) = \sum_{n=-\infty}^{+\infty} X_n \exp(jn\omega_0 t)$	$X_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\exp(-jn\omega_0 t)\, dt$

Fourier transform – continuous in time, continuous in frequency

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\exp(j\omega t)\, d\omega$	$X(\omega) = \int_{-\infty}^{\infty} x(t)\exp(-j\omega t)\, dt$

Discrete-time Fourier transform – discrete in time, continuous and periodic in frequency

$x(n\Delta t) = \frac{\Delta t}{2\pi}\int_{-\pi/\Delta t}^{\pi/\Delta t} X(\omega)\exp(jn\Delta t\omega)\, d\omega$	$X(\omega) = \sum_{n=-\infty}^{\infty} x(n\Delta t)\exp(-j\omega n\Delta t)$

Discrete Fourier transform – discrete and periodic in time and in frequency

$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\exp\left(\frac{jnk2\pi}{N}\right)$	$X(k) = \sum_{n=0}^{N-1} x(n)\exp\left(\frac{-jnk2\pi}{N}\right)$

Transformation of random variables

$$f_Y(y) = \sum_{i=1}^{N} f_X(x_i) \left|\frac{dx_i}{dy}\right|_{x_i = g_i^{-1}(y)}$$

$$f_{UV}(u,v) = f_{XY}(x,y) \left|\frac{\partial(x,y)}{\partial(u,v)}\right|_{\substack{x = g_1^{-1}(u,v) \\ y = g_2^{-1}(u,v)}}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$