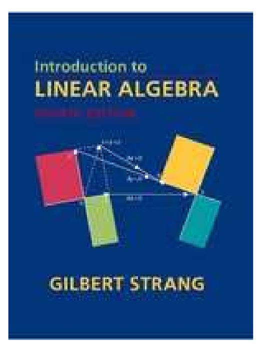
Essential Mathematical Methods for Engineers

Lecture 6:
Linear algebra 1

Linear Algebra

- this lecture is based upon
  - "***Introduction to Linear Algebra***," Strang, Wellesley Cambridge Press 2009
  - several copies in the library



Outline

- solving linear equations
  - row and column pictures
  - elimination
  - the inverse matrix
  - factorisation
- vector spaces and subspaces
  - column space
  - nullspace
  - rank and row reduced form
  - special solutions
  - complete solution
  - independence, basis and dimension
  - the four fundamental subspaces
  - the first fundamental theorem of linear algebra

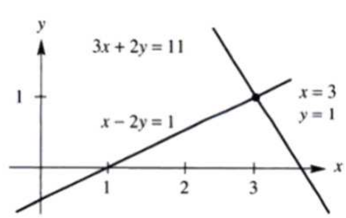
Linear algebra 3

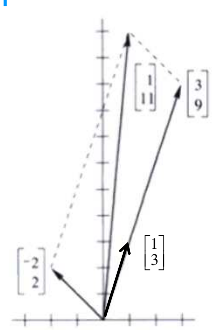
Solving linear equations
Vectors and linear equations

$$\begin{aligned} x-2y&=1 \\ 3x+2y&=11 \end{aligned}$$

$$x\begin{bmatrix}1\\3\end{bmatrix}+y\begin{bmatrix}-2\\2\end{bmatrix}=\begin{bmatrix}1\\11\end{bmatrix}$$

- row picture
- column picture





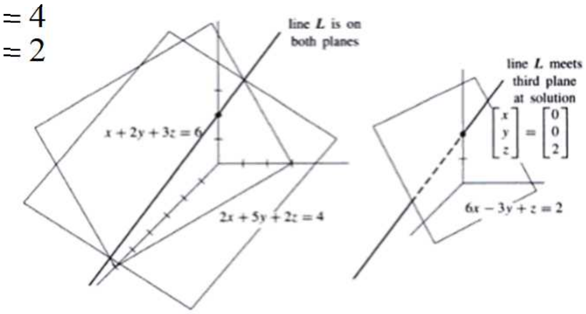
$$\begin{bmatrix}1&-2\\3&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\11\end{bmatrix}$$

- matrix equation

Linear algebra 4

Three equations in three unknowns

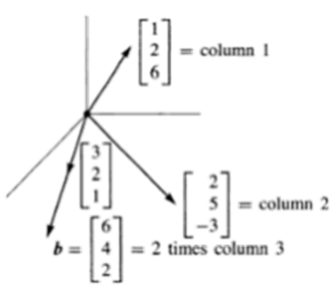
$$
\begin{aligned}
x+2y+3z&=6\\
2x+5y+2z&=4\\
6x-3y+z&=2
\end{aligned}
$$



the ***row picture*** shows three planes meeting at a single point

Three equations in three unknowns

the ***column picture*** combines three columns to produce the fourth



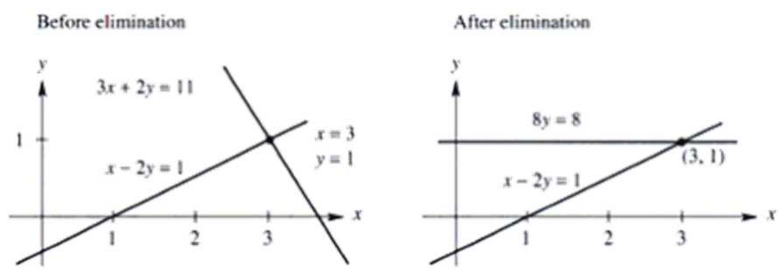
$$
x\begin{bmatrix}1\\2\\6\end{bmatrix}+y\begin{bmatrix}2\\5\\-3\end{bmatrix}+z\begin{bmatrix}3\\2\\1\end{bmatrix}=\begin{bmatrix}6\\4\\2\end{bmatrix}
$$

whatever the approach, the solution is $(x, y, z)=(0,0,2)$

Elimination

$$\begin{aligned} x-2y&=1 \\ 3x+2y&=11 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x-2y&=1 \\ 8y&=8 \end{aligned}$$

- elimiation produces an ***upper triangular system***
  - with the same solution
  - solved by back substitution



Elimination

$$\begin{aligned} x-2y&=1 \\ 3x+2y&=11 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x-2y&=1 \\ 8y&=8 \end{aligned}$$

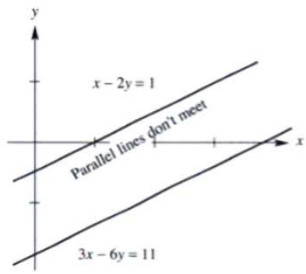
- concept of pivots
  - to solve n equations we need n pivots
- three cases where n pivots *might* not be possible
  - no solution
  - infinitely many solutions
  - temporary failure – solved with row exchange

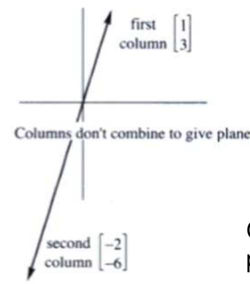
Elimination

- case 1: no solution

$$\begin{aligned} x-2y&=1 \\ 3x-6y&=11 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x-2y&=1 \\ 0y&=8 \end{aligned}$$

Row picture





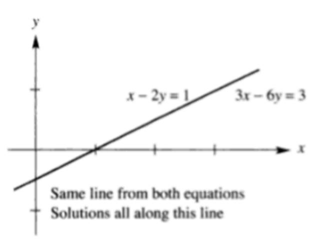
Column picture

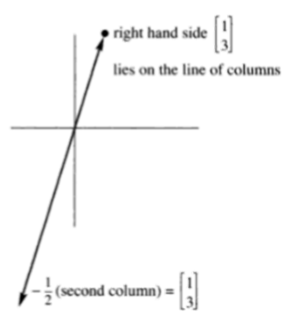
Elimination

- case 2: infinitely many solutions

$$\begin{aligned} x-2y&=1 \\ 3x-6y&=3 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x-2y&=1 \\ 0y&=0 \end{aligned}$$

Row picture





Column picture

Elimination

- case 3: temporary failure

$$\begin{aligned} 0x + 2y &= 4 \\ 3x - 2y &= 5 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 3x - 2y &= 5 \\ 2y &= 4 \end{aligned}$$

by interchanging rows we obtain an upper triangular system with 2 pivots and one unique solution

- case 1 and 2 are singular – no second pivot
  - no solution, or infinitely many solutions
- case 3 is nonsingular – two unique pivots
  - one unique solution

Elimination

Example: you should have no difficulty extending this to higher order systems, e.g.:

$$\begin{aligned} 2x + 4y - 2z &= 2 \\ 4x + 9y - 3z &= 8 \\ -2x - 3y + 7z &= 10 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 2x + 4y - 2z &= 2 \\ 1y + 1z &= 4 \\ 4z &= 8 \end{aligned}$$

Elimination using matrices

$$Ax = b$$

$$\begin{aligned} 2x+4y-2z&=2 \\ 4x+9y-3z&=8 \\ -2x-3y+7z&=10 \end{aligned} \qquad \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

has solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

- Ax is a combination of the columns of A

$$Ax = (-1)\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 9 \\ -3 \end{bmatrix} + 2\begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

Elimination using matrices

- we can perform elimination using matrices
- the matrix which performs the first step of the elimination is

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  and correspondingly

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ -2 & -3 & 7 \end{bmatrix}$$

  so we have that $EAx=Eb=[2\ 4\ 10]^T$

- note the 2D associative law: $A(BC)=(AB)C$
  - this is not commutative: $AB \neq BA$

Row exchange

- P matrices exchange or permute rows
  - i.e. when there is a zero in the pivot position

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 4 & 1 \\ 0 & 0 & 3 \\ 0 & 6 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 1 \\ 0 & 6 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

needed
for case 3

Linear algebra 15

The augmented matrix

- E and P matrices can be applied to A and b separately
  - also to the augmented matrix A'

$$A' = [A \;\; b] = \begin{bmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{bmatrix}$$

  - and thus

$$EA' = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{bmatrix}$$

- each row of E acts on A' to give a row of EA'
- E acts on each column of A' to give a column of EA'

Linear algebra 16

Rows and Columns of AB

- column picture
  - A multiplies each column of B and gives a column of AB
  - each column of AB is a combination of the columns of A
- row picture
  - each row of A multiplies the whole of matrix B to give a row of AB
  - each row of AB is a combination of the rows of B
- note the row-column picture AND the column-row picture

Linear algebra 17

Laws for matrix operations

Addition

- commutative

$$A + B = B + A$$

- distributive

$$c(A + B) = cA + cB$$

- associative

$$A + (B + C) = (A + B) + C$$

Multiplication

- commutative

$$AB \neq BA$$

- distributive

$$C(A + B) = CA + CB$$
$$(A + B)C = AC + BC$$

- associative

$$A(BC) = (AB)C$$

Matrix powers

$$A^p = AAA\cdots A \quad \text{(p factors)} \qquad (A^p)(A^q) = A^{p+q} \qquad (A^p)^q = A^{pq}$$

normal rules apply when p or q are negative A^{-1} is the inverse

Linear algebra 18

Inverse matrix

$$A^{-1}Ax = A^{-1}b \quad \text{gives} \quad x = A^{-1}b$$

- but it's not needed to solve $Ax=b$
  - we can just use elimination
  - still of interest and a fundamental property of matrices
- inverse exists if there is a matrix A^{-1} such that

$$A^{-1}A = I \quad \text{or} \quad AA^{-1} = I$$

or if there are n pivots

Inverse matrix

- the inverse is unique since if $BA=I$ and $AC=I$
  $$B(AC) = (BA)C \quad \text{gives} \quad BI = IC \quad \text{or} \quad B = C$$
- if A is invertible then the only solution to $Ax=b$ is $x=A^{-1}b$
  $$\text{Multiply} \quad Ax = b \quad \text{by} \quad A^{-1} \quad \text{then} \quad x = A^{-1}Ax = A^{-1}b$$
- if there is a non-zero vector x such that $Ax=0$ then A is not invertible – if A is invertible the only solution to $Ax=0$ is $x=0$
- a 2×2 matrix is invertible only if $ad\text{-}bc \neq 0$
  $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
- for diagonal matrices
  $$\text{If} \quad A = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} \quad \text{then} \quad A^{-1} = \begin{bmatrix} 1/d_1 & & \\ & \ddots & \\ & & 1/d_n \end{bmatrix}$$

Inverse of a product

- if A and B are invertible then

$$(AB)^{-1} = B^{-1}A^{-1}$$

since

$$B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I$$

- similarly

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

Inverse by Gauss-Jordan Elimination

- we try to solve $AA^{-1}=I$ one column at a time

$$AA^{-1} = A[x_1 \ \ x_2 \ \ x_3] = [e_1 \ \ e_2 \ \ e_3] = I$$

- we have only to solve $Ax_n=e_n$ to obtain A^{-1} – we solve 3 systems of equations
- using an augmented matrix $[A\ I]$ we can determine $[I\ A^{-1}]$ and solve all n equations together
  - upper triangular form through Gaussian elimination

$$\begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & -1 & 1/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{bmatrix}$$

which we could solve by back substitution

Inverse by Gauss-Jordan Elimination

- Jordan continued to reduced echelon form

$$\left[\begin{array}{cccccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & -1 & 1/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cccccc} 2 & 0 & 0 & 3/2 & 1 & 1/2 \\ 0 & 3/2 & 0 & 3/4 & 3/2 & 3/4 \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{array}\right]$$

and dividing rows by their pivots

$$\left[\begin{array}{cccccc} 2 & 0 & 0 & 3/2 & 1 & 1/2 \\ 0 & 3/2 & 0 & 3/4 & 3/2 & 3/4 \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cccccc} 1 & 0 & 0 & 3/4 & 1/2 & 1/4 \\ 0 & 1 & 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & 1 & 1/4 & 1/2 & 3/4 \end{array}\right]$$

so that the inverse is

$$A^{-1} = \left[\begin{array}{ccc} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{array}\right]$$

Linear algebra 23

Singluar versus invertible

- an $n \times n$ matrix must have n pivots in order to be invertible
  - we can solve all the equations $Ax_i=e_i$ and the columns x_i give A^{-1}
  - then $AA^{-1}=I$
- elimination is really a long sequence of manipulations

$$(D^{-1} \cdots E \cdots P \cdots E)A = I$$

where D^{-1} divides by the pivots
  - this gives a left inverse such that $A^{-1}A=I$ - the same solution
- if there are not n pivots then A^{-1} does not exist

Linear algebra 24

LU factorisation

- a factorisation of A into the product of two triangular matrices

$$E_{21}A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} = U$$

$$E_{21}^{-1}U = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = A$$

  or $LU=A$

- the L matrix includes the inverses of all the E matrices

- combining all the elimination matrices

$$A = \left(E^{-1} \cdots P^{-1} \cdots E^{-1}\right)U$$

LU factorisation

$$\left(E_{32}E_{31}E_{21}\right)A = U \quad \text{becomes} \quad A = \left(E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}\right)U \quad \text{which is} \quad A = LU$$

- some points
  - every inverse matrix E^{-1} is lower triangular
    - its off diagonal entry ℓ_{ij} undoes the subtraction in E with $-\ell_{ij}$
    - the diagonals of E and E^{-1} contain 1
  - the product of the E's is still lower triangular – this is L
  - each multiplier ℓ_{ij} goes directly into its i,j position unchanged in the product L
  - when a row of A starts with zeros, so does that row in L
  - when a column of A starts with zeros, so does that column of U

LU factorisation

- U has the pivots on its diagonal whereas L has 1s
- if we write U as $$\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} 1 & u_{12}/d_1 & u_{13}/d_1 & \vdots \\ & 1 & u_{23}/d_2 & \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix}$$

  we can now write $A=LDU$
  - the new upper triangular matrix is also referred to as U
  - in this form we assume that each row of U has been divided by its pivot, e.g.:

$$\begin{bmatrix} 2 & 8 \\ 6 & 29 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 8 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & \\ & 5 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

One square system = two triangular systems

- the LU decomposition is important in solving $Ax=b$
  - rewrite as $LUx=b$
  - we apply the forward elimination steps of L to b
    - effectively solve $Lc=b$
  - then solve $Ux=c$ by back substitution
- but what have we achieved?

One square system = two triangular systems

Example

Solve the following using an LU decomposition:

$$\begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 8 \\ 21 \end{bmatrix}$$

Transposes and permutations

- simple exchange of rows and columns: $(A^T)_{ij}=A_{ji}$
- rules
  - under addition: $(A+B)^T = A^T+B^T$
  - under multiplication: $(AB)^T = B^T A^T$
  - inverses: $(A^{-1})^T=(A^T)^{-1}$

  Ax combines the columns of A while $x^T A^T$ combines the rows of A^T – the same combinations of the same vectors
    - the transpose of the column Ax is the row $x^T A^T$, thus

$$(Ax)^T = x^T A^T$$

Transposes and permutations

- but what about $(AB)^T$?
  - assume that $B=[x_1\ x_2]$ has two columns
  - the columns of AB are Ax_1 and Ax_2
  - their transposes are $x_1^T A^T$ and $x_2^T A^T$
  - these are the rows of $B^T A^T$

$$\text{Transposing} \quad AB = \begin{bmatrix} Ax_1 & Ax_2 & \cdots \end{bmatrix}$$

$$\text{gives} \quad \begin{bmatrix} x_1^T A^T \\ x_2^T A^T \\ \vdots \end{bmatrix} \quad \text{which is} \quad B^T A^T$$

Linear algebra 31

Transposes and permutations

- this rule extents to more than two matrices
  - $(ABC)^T = C^T B^T A^T$
  - if $A=LDU$ then $A^T=U^T D L^T$ – the pivot matrix $D=D^T$

- if we apply this rule to both sides of $A^{-1}A=I$ we get

$$A^T (A^{-1})^T = I$$

and similarly if we do the same with $AA^{-1}=I$ we get

$$(A^{-1})^T A^T = I$$

A^T is invertible exactly when A is invertible and we can swap the order of transposing and inverting

Linear algebra 32

Symmetric matrices

- any matrix where $A^T = A$
  - their inverses are also symmetric
- for any non-symmetric matrix R, both $R^T R$ and RR^T are symmetric – easy to prove
- we can apply elimination to a symmetric matrix
  - the smaller matrices remain symmetric as elimination proceeds

$$\begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

  - and U is the transpose of L and we have that

$$A = LDL^T \qquad \left(LDL^T\right)^T = \left(L^T\right)^T D^T L^T = LDL^T$$

Permutation matrices

- row exchanges, e.g. for a 3x3 matrix

$$I = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \quad P_{21} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} \quad P_{32}P_{21} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}$$

$$P_{31} = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \quad P_{32} = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} \quad P_{21}P_{32} = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}$$

- P^{-1} is also a permutation matrix
  - the four left matrices above are their own inverse
  - the two right matrices are inverses of each other
    - note that the order is reversed!!!
- $P^{-1} = P^T$
  - the four left matrices above are their own transposes
  - the two right matrices are transposes of each other

LU factorization with row exchange

- row exchanges are sometimes required in order to produce pivots

$$A = \left(E^{-1}\cdots P^{-1}\cdots E^{-1}\cdots P^{-1}\cdots\right)U$$

- we can combine all permutations into a single P
  - but where do we put it?

- two solutions
  - move row exchanges to the left side
    - $PA=LU$
  - perform row exchanges after elimination
    - $A=L_1P_1U_1$
    - P_1 puts rows into the right order in U_1

LU factorization with row exchange

- e.g.:

$$\underset{A}{\begin{bmatrix}0&1&1\\1&2&1\\2&7&9\end{bmatrix}} \Rightarrow \underset{PA}{\begin{bmatrix}1&2&1\\0&1&1\\2&7&9\end{bmatrix}} \Rightarrow \underset{\ell_{31}=2}{\begin{bmatrix}1&2&1\\0&1&1\\0&3&7\end{bmatrix}} \Rightarrow \underset{\ell_{32}=3}{\begin{bmatrix}1&2&1\\0&1&1\\0&0&4\end{bmatrix}}$$

the matix is now in good order and it factorises as follows

$$PA = \begin{bmatrix}1&0&0\\0&1&0\\2&3&1\end{bmatrix}\begin{bmatrix}1&2&1\\0&1&1\\0&0&4\end{bmatrix} = LU$$

so long as A is invertible

Vector spaces and subspaces

- columns of Ax and AB are linear combinations of n vectors
  - the columns of A
  - vector spaces and subspaces
- most simple examples are R^1, R^2, R^3, etc
  - R^n : n-dimensional space
  - all column vectors v with n components
- rules
  - commutative law: $v + w = w + v$
  - distributive law: $c(v + w) = cv + cw$
  - zero vector: $0 + v = v$

Vector spaces and subspaces

- some other vector spaces
  - M : all real 2 by 2 matrices
  - F : all real functions of $f(x)$
  - Z : consists only of a zero vector
- in all cases additions and multplications result in new vectors that stay in the vector space

Subspaces

- a subspace of a vector space is a set of vectors that satisfies two requirements
- if v and w are vectors in the subspace and c is any scalar, then
  - $v + w$ is in the subspace
  - cv is in the subspace
- all linear combinations stay in the subspace
- every subspace contains the zero vector

Linear algebra 39

Subspaces

- planes that don't contain the origin are not subspaces
  - i.e. for v on such a plane, $-v$ & $0v$ are not on the plane
- keeping only parts of the subspace violate the conditions
  - e.g. keeping only vectors (x,y) where x & $y > 0$
  - e.g. even if we include (x,y) where x & $y < 0$
- a subspace containing v and w contain all linear combinations of $cv + dw$

Linear algebra 40

The column space of A

- we are interested in subspaces associated with $Ax = b$
- when A is not invertible the system is solvable for some b but not for others
  - it is solvable only when b is in the column space of A
- the column space of A contains all linear combinations of its columns – all possible combinations Ax
- the columns of an $m \times n$ matrix A have m components
  - the set of all column combinations Ax are a subspace
  - the column space is a subspace of R^m

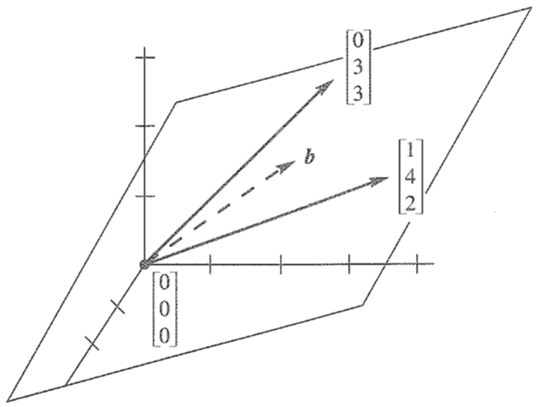
The column space of A

Example

$$Ax = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b \quad \text{which is} \quad x_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} = b$$

- the column combinations fill up the column space which is in R^3
  - the column space is actually a subspace of R^3
- if the right side b lies on that plane then it is one of those combinations and the corresponding (x_1, x_2) is a solution
- if the right side is not on the plane then there is no solution to what are 3 equations in 2 unknowns
- the column space is denoted by $C(A)$

The column space of A



$$A = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix}$$

$$b = x_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$$

The column space of A

Example

Describe the column spaces of the following matrices

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

but note that there will be more solutions to B than I and A… why?

The nullspace of A

- the nullspace describes the solutions to $Ax = 0$
  - one obvious solution, $x = 0$
  - for invertible matrices it's the only solution
- an $m \times n$ matrix has a nullspace $N(A)$ in R^n
  - specifically the nullspace is a subspace of R^n
- if the right side b is not 0 then the solutions of $Ax = b$ do not form a subspace: $x = 0$ is only a solution if $b = 0$
  - when the set of solutions does not include $x = 0$ it cannot be a subspace

Linear algebra 45

The nullspace of A

Example

The equation $x + 2y + 3z = 0$ comes from the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

The equation produces a plane through the origin. This is the nullspace, a subspace of R^3.

The solutions to $x + 2y + 3z = 6$ also form a plane, but not a subspace.

Linear algebra 46

The nullspace of A

Example

Describe the nullspace of

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

The nullspace $N(A)$ contains all multiples of

$$s = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

so the nullspace is a line.

Do the same for

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \quad B = \begin{bmatrix} A \\ 2A \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} A & 2A \end{bmatrix}$$

The nullspace of A

$$C = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix} \quad \text{becomes} \quad U = \left[\begin{array}{cc:cc} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{array}\right]$$

$$\begin{array}{cc:cc} \uparrow & \uparrow & \uparrow & \uparrow \\ \multicolumn{2}{c:}{\text{pivot columns}} & \multicolumn{2}{c}{\text{free columns}} \end{array}$$

$$U = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix} \quad \text{becomes} \quad R = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ \multicolumn{2}{c}{\text{pivot columns contain } I} \end{array}$$

$$s_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad s_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \begin{array}{l} \leftarrow \text{pivot} \\ \leftarrow \text{variables} \\ \leftarrow \text{free} \\ \leftarrow \text{variables} \end{array}$$

- produce zeros above the pivots by eliminating upwards
- produce ones in the pivots by dividing the whole row by its pivot
- the r.h.s is not changed – it's all zero!
  - nullspace stays the same

The nullspace of A

- to describe the nullspace we determine the special solutions to $Ax = 0$
  - $N(A)$ contains all combinations of the special solutions
- when the only solution to $Ax = 0$ is $x = 0$, the nullspace contains only that special vector $x = 0$
  - the zero or trivial combination
  - the nullspace is Z
  - this tells us that the columns of A are independent
  - no combination of columns gives us the zero vector except the zero combination

Linear algebra 49

Solving Ax = 0 by elimination

- involves:
  - forward elimination from $A \to U \to R$
  - backward substitution in $Ux = 0$ or $Rx = 0$ to find x
- there may not be n pivots, e.g.

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 8 & 10 \\ 3 & 3 & 10 & 13 \end{bmatrix} \quad \text{becomes} \quad U = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

there are four unknowns and only two pivots so there are infinitely many solutions – but how to describe them?
  - pivot variables and free variables

Linear algebra 50

Solving Ax = 0 by elimination

$$x = x_2\begin{bmatrix}-1\\1\\0\\0\end{bmatrix} + x_4\begin{bmatrix}-1\\0\\-1\\1\end{bmatrix} = \begin{bmatrix}-x_2-x_4\\x_2\\-x_4\\x_4\end{bmatrix}$$

special special complete

- P : the pivot variables are x_1 and x_3
- F : the free variables are x_2 and x_4
- these solve $Ux=0$ and therefore $Ax=0$
- every solution is
  - a combination of the special solution
  - in the nullspace $N(A)$
- the combinations fill out the nullspace

Echelon matrices

$$U = \begin{bmatrix} x & x & x & x & x & x & x \\ 0 & x & x & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3 pivot variables x_1, x_2, x_6

4 free variables x_3, x_4, x_5, x_7

four special solutions in $N(U)$

- what are the column and nullspaces?
- if A has more columns than rows $(n > m)$
  - there at most m pivots
  - there is at least 1 free variable
  - $Ax = 0$ has at least one special solution – not $x = 0$
- the number of free variables dictates the dimension of the nullspace – a subspace

The reduced Echelon matrix, *R*

$$U = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{becomes} \quad R = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- if A is invertible then $R = I$
- zeros in R make it easy to find the special solutions
  - the nullspace $N(A) = N(U) = N(R)$

$$x = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \quad \text{as before}$$

Linear algebra 53

The rank and row reduced echelon form

- R can be obtained from an elimination matrix: $EA = R$
  - square matrix E is the product of the elementary matrices, E_{ij}, P_{ij} and D^{-1}
  - E is obtained from row reduction on $[\,A \;\; I\,]$ since

$$E[A \;\; I] = [R \;\; E]$$

  - which is Gauss-Jordan elimination
- when A is square and invertible, $EA = R = I$
  - E is then A^{-1}
- here we consider all (rectangular) matrices
  - E will obtain R, but R will not necessarily equal I
  - it shows us the pivot columns and special solutions

Linear algebra 54

The rank of a matrix

- reflects the true size of the linear system
- the rank is equal to the number of pivots, r
  - $r \le m$ and $r \le n$
  - when $r = m$ the matrix is of full row rank
    - no zero rows in R
  - when $r = n$ the matrix is of full column rank
    - no free variables
- a square, invertible matrix has $r = m = n$ and $R = I$
- at higher levels
  - the matrix has r independent rows and columns
  - r is the dimension of the row and column space

The pivot columns

- pivots in R are all 1
  - respective columns form an I matrix which is $r \times r$

$$A = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 1 & 3 & 1 & 6 & -4 \end{bmatrix} \quad \text{becomes} \quad R = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- for this example

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \quad \text{and} \quad E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

the r pivot columns of A are the first r columns of E^{-1}

- $A=E^{-1}R$: each column of A is E^{-1} times a column of R
- 1's in the pivots of R pick out first r columns of E^{-1}

The pivot columns

- pivot columns are **not** combinations of earlier columns
  - clearly true for R and also true for A
  - since $Ax = 0$ exactly when $Rx = 0$
    - solutions do not change during elimination
- free columns **are** combinations of earlier columns
  - the combinations are given exactly by the special solutions

Special solutions

- special solutions have one free variable equal to 1
  - all others are zero
- special solutions can be read off directly from R to give the nullspace matrix N

$$Rx = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad N = \begin{array}{cl} \begin{bmatrix} -3 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & -4 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{array}{l} \text{not free} \\ \text{free} \\ \text{not free} \\ \text{free} \\ \text{free} \end{array} \end{array}$$

which gives the full solution to $Ax = 0$ (and $Rx = 0$)

- note once again the presence of the I matrix

Special solutions

- there is a special solution for each free variable
  - with r pivot columns there are $n - r$ free variables
  - free solutions are independent
- for $Ax = 0$
  - n unknowns
  - r independent equations
  - $n - r$ independent solutions
- in the case where the first r columns are pivot columns

$$\begin{array}{cc} & \begin{array}{cc} & \end{array} \\ R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} & \begin{array}{l} r \text{ pivot rows} \\ m\text{-}r \text{ zero rows} \end{array} \end{array} \qquad N = \begin{bmatrix} -F \\ I \end{bmatrix} \begin{array}{l} r \text{ pivot variables} \\ n\text{-}r \text{ free variables} \end{array}$$

r pivot columns $\quad n\text{-}r$ free columns

Linear algebra 59

Special solutions

- and note that $RN = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -F \\ I \end{bmatrix} = I(-F) + FI = 0$
- the columns of N solve $Rx = Ax = 0$
- pivot variables come by changing signs (F to $-F$) in the free columns of R

$$I \begin{bmatrix} \text{pivot} \\ \text{variables} \end{bmatrix} = -F \begin{bmatrix} \text{free} \\ \text{variables} \end{bmatrix}$$

- in each special solution the free variables are a column of I
- then the pivot variables are a column of $-F$
- they give the nullspace matrix N

Linear algebra 60

Special solutions

- this holds irrespective of the order of the pivot and free columns
- also note that no matter what method we use to reduce A we always obtain the same R

Example: The special solutions of $x_1 + 2x_2 + 3x_3 = 0$ are

$$N = \begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the coefficient matrix is $[1\ 2\ 3]=[\ I\ \ F\ ]$. The rank is 1 so there are $n - r = 2$ special solutions in N. Their first components are $-F = [-2\ -3]$. The other free variables come from I.

The complete solution to Ax = b

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix} \quad \text{has the augmented matrix} \quad \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 1 & 3 & 1 & 6 & 7 \end{bmatrix} = [A \ \ b]$$

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix} \quad \text{has the augmented matrix} \quad \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = [R \ \ d]$$

- $x_2 = x_4 = 0$ are free variables, $x_1 = 1$ and $x_3 = 6$ are pivots variables taken from d, giving the particular solution $x_p = [\ 1\ 0\ 6\ 0\ ]^{\mathrm{T}}$

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$$

Pivot variables 1,6
Free variables 0, 0

The complete solution to Ax = b

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$$

Pivot variables 1,6
Free variables 0, 0

- $x_2 = x_4 = 0$ are free variables, $x_1 = 1$ and $x_3 = 6$ are pivots variables taken from d, giving $x_p = (\ 1, 0, 6, 0\ )$
  - after row reduction we solve $Rx = d$
  - the particular solution solves $Ax_p = b$
  - the $n - r$ special solutions solve $Ax_n = 0$

The complete solution to Ax = b

- given a square invertible matrix, what are x_p and x_n?
  - the particular solution is $A^{-1}b$
  - there are no special solutions – no free variables
    - the null space contains only the zero vector
  - the complete solution is $x = x_p + x_n = A^{-1}b + 0$
  - solution $A^{-1}b$ appears in the extra column
    - the reduced form of A is $R = I$
  - in this case $[\ A \quad b\ ]$ is reduced to $[\ I \quad A^{-1}b]$
  - $Ax = b$ is reduced all the way to $x = A^{-1}b$

The complete solution to Ax = b

Example: Find the condition on (b_1, b_2, b_3) for $Ax = b$ to be solveable if:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

This condition puts b in the column space of A. Find the complete $x = x_p + x_n$.

A has full column rank $r = n$

– here $R = \begin{bmatrix} n \text{ by } n \text{ identity matrix} \\ m-n \text{ rows of zeros} \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}$

there are no free variables – F is empty

The complete solution to Ax = b

- for every matrix A of full column rank $(m \geq n = r)$
  - all columns of A are pivot columns
  - there are no free variables or special solutions
  - $N(A)$ contains only $x = 0$
  - if $Ax = b$ has a solution it is unique
- we will also see that
  - A has independent columns
  - $A^T A$ is invertible
  - there will be $m - n$ zero rows in R
  - there will be $m - n$ conditions on b in order to have $0 = 0$ in those rows

The complete solution to Ax = b

- for every matrix A of full row rank ($r = m \le n$)
  - all rows of A have pivots and R has no zero rows
  - $Ax = b$ has a solution for any right side b
  - the column space is the whole space R^m
  - there are $n - r = n - m$ special solutions in the nullspace of A
  - the rows are linearly independent

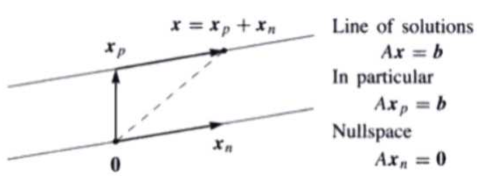
The complete solution to Ax = b

- **Example:** The following system has $n = 3$ unknowns but only two equations. The rank is $r = m = 2$.

$$\begin{aligned} x + y + z &= 3 \\ x + 2y - z &= 4 \end{aligned}$$

Show that the complete solution is given by

$$x = x_p + x_n = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$



The complete solution to Ax = b

- four possibilities depending on the rank
  - $r = m$ and $r = n$: square and invertible, 1 solution
  - $r = m$ and $r < n$: short and wide, infinte solutions
  - $r < m$ and $r = n$: tall and thin, 0 or 1 solution
  - $r < m$ and $r < n$: unknown shape, 0 or infinite solutions

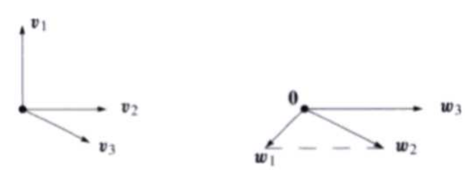
- R will fall into the same category as A

- with the pivot columns first

$$R = [I] \quad [I \;\; F] \quad \begin{bmatrix} I \\ 0 \end{bmatrix} \quad \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

Linear independence

- the columns of A are linearly independent when the only solution to $Ax = 0$ is $x = 0$
  - no other combination Ax gives the zero vector
  - $N(A)$ contains only the zero vector
- the sequence of vectors $v_1, v_2, \ldots, v_n$ is linearly independent if the only combination that gives the zero vector is $0v_1 + 0v_2 + \ldots + 0v_n$



Linear independence

Example: Determine whether the columns of A are independent or dependent if we know that:

$$Ax = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} = 0$$

Linear independence

- the columns of A are independent exactly when $r = n$
  - there are n pivots
  - only $x = 0$ is in the nullspace
- for square matrices independent columns imply independent rows
- any set of n vectors in R^m must be linearly dependent for $n > m$, then $Ax = 0$ will have a nonzero solution

Vectors that span a subspace

- combinations of the columns, Ax, span the column space
  - the column space is spanned by the columns
  - $C(A)$ is a subspace of R^m
- a set of vectors spans a space if their linear combinations fill the space
- combinations of the rows of A span the row space
  - $C(A^T)$ is a subspace of R^n

Vectors that span a subspace

Example: the column space of A is spanned by the two columns of A – a plane in R^3. The row space is spanned by the three rows of A – all of R^2.

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 7 \\ 3 & 5 \end{bmatrix} \qquad m = 3 \quad \text{and} \quad n = 2$$

A basis for a vector space

- two properties
  - the vectors are linearly independent
  - the vectors span the space
- every vector in the vector space is a combination of the basis vectors
- the standard bases for R^n come from the n by n identity matrix I – but they are not the only possibilities
  - the columns of any invertible n by n matrix give a valid basis

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

A basis for a vector space

- when A is invertible its columns are independent
  - only solution to $Ax = 0$ is $x = 0$
  - columns span the whole space R^n because every vector b is a combination of the columns
  - $Ax = b$ can always be solved by $x = A^{-1}b$
- the vectors $v_1, \ldots, v_n$ are a basis for R^n exctly when they are the columns of an n by n invertible matrix
  - R^n has infinitely many bases
- the pivot columns of A are a basis for its column space
- the pivot rows of A are a basis for its row space

A basis for a vector space

Example: Reduce A to R and then find and compare bases for their column and row spaces.

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$$

Example: Find a basis for the column space of

$$R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example: How would you find a basis for the space spanned by five vectors in R^7?

N.B. the differences between the column spaces of A and R !!!

Dimension of a vector space

- all the different bases for the same space have the same number of vectors – this is the dimension of the space
- if $v_1,\ldots,v_m$ and $w_1,\ldots,w_n$ are two bases for the same space, then $m = n$
  - w_1 must be a combination of the v's
  - if $w_1 = a_{11}v_1+\ldots+a_{m1}v_m$ – the first column of a matrix multiplication

$$W = \begin{bmatrix} w_1 w_2 \cdots w_n \end{bmatrix} = \begin{bmatrix} v_1 v_2 \cdots v_m \end{bmatrix} \begin{bmatrix} a_{11} & & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{bmatrix} = VA$$

  - we don't know each a_{ij} but we know that A is m by n

Dimension of a vector space

- A has a row for every v and a column for every w
  - if we assume that $n > m$ it is a short, wide $m \times n$ matrix
  - there is a nonzero solution to $Ax = 0$
  - then $VAx = 0$ and $Wx = 0$
  - a combination of the w's gives zero
  - columns of W cannot be a basis
  - same outcome if we consider $m > n$,
  - therefore m must equal n
- the dimension of a vector space is the number of vectors in every basis
- the dimension of the column space is the rank of the matrix

Bases for matrix spaces and function spaces

- not just column vectors – matrices and functions too!
- all 2 by 2 matrices might have bases

$$A_1, A_2, A_3, A_4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

bases easily found for e.g. all diagonal, all upper triangular matrices, or all symmetric matrices etc.

Dimensions of the four subspaces

- the rank of a matrix defines the dimension of all four subspaces
  - two subspaces from A, two from A^T

- row space $C(A^T)$ a subspace of R^n
  - also the column space of A^T
- column space $C(A)$ a subspace of R^m
- nullspace $N(A)$ a subspace of R^n
- left nullspace $N(A^T)$ a subspace of R^m
  - nullspace of A^T – obtained by solving $A^Ty = 0$
    - an $n \times m$ system
    - vectors y go on the left when written as $y^TA = 0^T$

- all of these spaces are connected

Linear algebra 81

Dimensions of the four subspaces

- we will learn about two fundamental theorems of linear algebra

- Fundamental Theorem 1
  - the column space and the row space have the same dimension r – the rank of the matrix A
  - the two nullspaces have dimensions n-r and m-r
    - they make up the full dimensions n and m

- Fundamental Theorem 2
  - we will see how these four subspaces fit together

Linear algebra 82

Dimensions of the four subspaces

- we can reduce A to its row echelon form R
  - the dimensions of the four subspaces are the same for both A and R

Dimensions of the four subspaces

Example

Take a matrix R, where $m = 3$, $n = 5$ and $r = 2$

$$R = \begin{bmatrix} 1 & 3 & 5 & 0 & 9 \\ 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} \text{pivot rows 1 and 2} \\ \\ \text{pivot columns 1 and 4} \end{array}$$

- the rank of R is $r = 2$
- the ***row space*** of R has dimension r
  - pivot rows 1 and 2 span the row space, they are independent and form a basis

Dimensions of the four subspaces

…example (cont.)

- the ***column space*** of R has dimension r
  - pivot columns 1 and 4 span the column space, they are independent and form a basis
    - $c_2 = 3c_1$: special solution is $s_2 = (-3, 1, 0, 0, 0)^T$
    - $c_3 = 5c_1$: special solution is $s_3 = (-5, 0, 1, 0, 0)^T$
    - $c_5 = 9c_1 + 8c_4$: special solution is $s_5 = (-9, 0, 0, -8, 1)^T$

Linear algebra 85

Dimensions of the four subspaces

…example (cont.)

- the ***nullspace*** has dimension $n - r = 5 - 2 = 3$
  - there are no pivots in columns 2, 3 and 5
  - these three free variables lead to the three special solutions to $Rx = 0$ which are independent and form a basis
  - we have the same special solutions as before
  - $Rx = 0$ has the complete solution $x = x_2s_2 + x_3s_3 + x_5s_5$
  - the pivot variables x_1 and x_4 are totally determined by $Rx = 0$

Linear algebra 86

Dimensions of the four subspaces

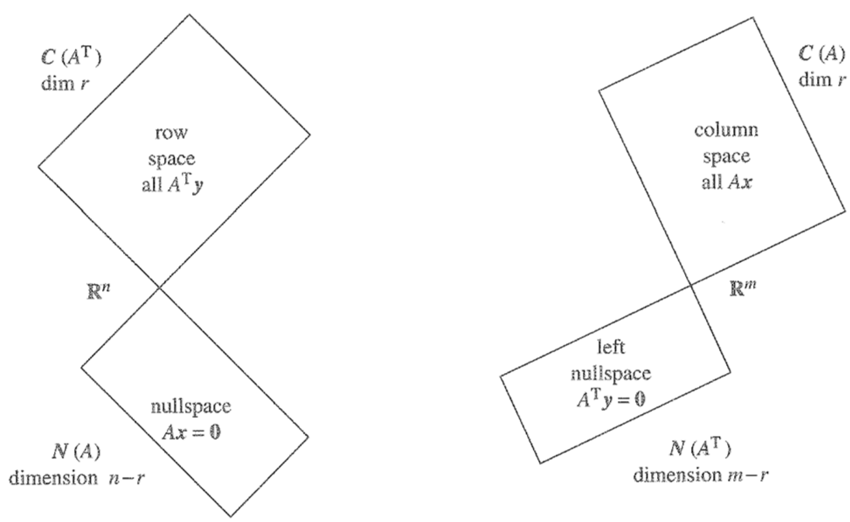
…example (cont.)

- the nullspace of R^T has dimension $m - r = 3 - 2 = 1$
- $R^T y = 0$ looks for combinations of the columns of R^T that produce 0:
  - $y_1(1,3,5,0,9) + y_2(0,0,0,1,8) + y_3(0,0,0,0,0) = (0,0,0,0,0)$
- the nullspace thus contains all vectors in $(0,0,y_3)$
  - it is the line of all multiples of the basis vector $(0,0,1)$

- in R^n the row space and nullspace have dimensions r and $n - r$ (the two add to n)
- in R^m the column space and the left nullspace have dimension r and $m - r$ (the two add to m)

Linear algebra 87

Dimensions of the four subspaces



Linear algebra 88

Dimensions of the four subspaces

- A and R have the same subspace dimensions
  - they are connected according to $EA = R$ and $A = E^{-1}R$
- A has the same row space, dimension and basis as R
  - every row of A is a combination of the rows of R
  - every row of R is a combination of the rows of A
- the column spaces of A and R have the same dimension
  - but the column spaces are NOT the same!
  - $Ax = 0$ when $Rx = 0$ – same combination of columns
  - r pivot columns of A are the basis for its column space
  - r pivot columns of R are the basis for its column space

Dimensions of the four subspaces

- A has the same nullspace, dimension and basis as R
  - elimination doesn't change the solutions
  - the special solutions are a basis for this nullspace
  - there are $n - r$ free variables, so the dimension is $n - r$
- the left nullspace of A has dimension $m - r$
  - if we know the dimensions of A then we know them for A^T
  - A^T is $n \times m$ so the "whole space" is now R^m
  - $r + (m - r) = m$

1st Fundamental Theorem of Linear Algebra

the column space and row space both have dimension r

$$C(A) = C(A^T) = r$$

the nullspaces have dimensions $n - r$ and $m - r$

$$N(A) = n - r \quad \text{and} \quad N(A^T) = m - r$$

Linear algebra 91

1st Fundamental Theorem of Linear Algebra

Example

Describe the spaces of

$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

Linear algebra 92