# Tutorial sheet 5.

(1) $$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n\Delta t} = \frac{1}{4} + \frac{1}{2} e^{-j\omega\Delta t} + \frac{1}{4} e^{-j2\omega\Delta t}$$

$$= \left\{ 1/2 + 1/4\, e^{j\omega\Delta t} + 1/4\, e^{-j\omega\Delta t} \right\} e^{-j\omega\Delta t}$$

$$= \frac{1}{2}\left\{ 1 + \cos\omega\Delta t \right\} e^{-j\omega\Delta t}$$

replacing $\omega\Delta t$ with $\frac{2\pi k}{3}$

$$X(k) = \frac{1}{2}\left\{ 1 + \cos\left(\frac{2\pi k}{3}\right)\right\} e^{-j\frac{2\pi k}{3}}$$

$$\therefore \quad X(0) = 1; \quad X(1) = \frac{1}{4} e^{-j2\pi/3}; \quad X(2) = \frac{1}{4} e^{-j4\pi/3}$$

(2) $$X(\omega) = \sum_{n=0}^{N} x(n) e^{-jn\omega\Delta t} = \sum_{n=0}^{L-1} e^{-jn\omega\Delta t} = \frac{1 - e^{-j\omega\Delta t L}}{1 - e^{-j\omega\Delta t}}$$

$$= \frac{e^{-j\omega\Delta t L/2}}{e^{-j\omega\Delta t/2}} \frac{\left\{ e^{j\omega\Delta t L/2} - e^{-j\omega\Delta t L/2}\right\}}{\left\{ e^{j\omega\Delta t/2} - e^{-j\omega\Delta t/2}\right\}} = e^{-j\omega\Delta t(L-1)/2} \frac{\sin\left(\frac{\omega\Delta t L}{2}\right)}{\sin\left(\frac{\omega\Delta t}{2}\right)}$$

& by making the substitution $\frac{2\pi k}{N} = \omega\Delta t$ we obtain

$$X(k) = e^{-j\frac{\pi k(L-1)}{N}} \frac{\sin\left(\frac{\pi k L}{N}\right)}{\sin\left(\frac{\pi k}{N}\right)}$$

or from noting

$$X(k) = \sum_{n=0}^{L-1} e^{-j\frac{2\pi n k}{N}}$$

(3.) The frequency resolution of the DFT is $\frac{f_s}{N}$

$$= \frac{10 \times 10^3}{200} = 50\,Hz$$

But the difference between the two dominant frequencies ($\Delta f$) is 40 Hz so they cannot be resolved. For resolution $\Delta f$ must be a multiple of 50 Hz i.e. 1600 Hz & 1650 Hz would be resolvable as separate DFT output signals.

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For an 8-point DFT the coefficient matrix is:-

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ 1 & W_8^2 & W_8^4 & W_8^6 & W_8^0 & W_8^2 & W_8^4 & W_8^6 \\ 1 & W_8^3 & W_8^6 & W_8^1 & W_8^4 & W_8^7 & W_8^2 & W_8^5 \\ 1 & . & . & . & . & . & . & . \\ 1 & W_8^7 & W_8^6 & W_8^5 & W_8^4 & W_8^3 & W_8^2 & W_8^1 \end{bmatrix}$$

Fourier coefficients are obtained from

$$W_N^{nk} = e^{-j\frac{n\pi\, nk}{N}} \quad \text{where} \quad N = 8$$

$k$ = component no. $(0-7)$

$n$ = weight no.

For $k = 0$, $e^{-j0} = 1$ irrespective of $n$ value.

For $k = 1$, coefficients are:

$$e^{\frac{-j\,0}{8}},\ e^{-j\frac{2\pi}{8}},\ e^{-j\frac{2\pi\, 2}{8}},\ e^{-j\frac{2\pi\, 3}{8}},\ e^{-j\frac{2\pi\, 4}{8}},\ e^{-j\frac{2\pi\, 5}{8}},\ etc.$$

$$= 1,\ e^{-j\frac{\pi}{4}},\ e^{-j\frac{\pi}{2}},\ e^{-j\frac{3\pi}{4}},\ e^{-j\pi},\ e^{-j\frac{5\pi}{4}},\ etc.$$

$$= 1,\ .7-j.7,\ -j,\ -.7-j.7,\ -1,\ +.7+j.7,\ j,\ etc.$$

The full matrix is then:-

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & .7-j.7 & -j & -.7-j.7 & -1 & -.7+j.7 & j & .7+j.7 \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & -.7-j.7 & j & .7-j.7 & -1 & .7+j.7 & -j & -.7+j.7 \\ 1 & . & . & . & . & . & . & . \\ 1 & .7+j.7 & j & -.7+j.7 & -1 & -.7-j.7 & -j & .7-j.7 \end{bmatrix}$$

The samples of $\cos\omega t$ and $\sin\omega t$ are:-

| | $x(0)$ | $x(1)$ | $x(2)$ | $x(3)$ | $x(4)$ | $x(5)$ | $x(6)$ | $x(7)$ |
|---|---|---|---|---|---|---|---|---|
| cos | 1.0 | .707 | 0 | −.707 | −1.0 | −.707 | 0 | .707 |
| sin | 0 | .707 | 1.0 | .707 | 0 | −.707 | −1 | −.707 |

The Fourier transformed output values of the above input data samples are then given as:-

$$X(0) = (1+j0)\,[\,(1+j0) + (.707 + j\,.707) + (\ \ ) + \cdots + (\ \ )\ \text{etc}]$$

$$= 0 + j0$$

$$X(1) = \sum_0^7 (a+jb)(a-jb) = 8\,(a^2+b^2) = 8$$

Note in this **special case** (zero starting phase and one cycle per block length) the sampled phasor in the 2nd row of the $8\times 8$ DFT matrix is the complex conjugate of the input sampled phasor. This simplifies the calculation of $X(1)$ to the expression above.

$$X(2) = 0 + j\,0$$

etc.

(5)

Now signal samples are given in question and we apply the DFT matrix again from problem 9.4.

For $X(0)$ we simply sum the 8 sample values.

$j4$, $-2\sqrt{2} + j2\sqrt{2}$, $-4$, $-2\sqrt{2} - j2\sqrt{2}$, etc.

to get the answer = 0

as here all the DFT coefficients are +1.

For $X(1)$ we now multiply the signal values with the second row of the DFT matrix:

| SIGNAL VALUES | | SECOND DFT ROW | | |
|---|---|---|---|---|
| $j4$ | $\times$ | $1$ | $=$ | $j4$ |
| $-2\sqrt{2}+j2\sqrt{2}$ | $\times$ | $.7-j7$ | $=$ | $-2+2+2j+2j$ |
| $-4$ | $\times$ | $-j$ | $=$ | $j4$ |
| $-2\sqrt{2}-j2\sqrt{2}$ | $\times$ | $-.7-j.7$ | $=$ | $+2-2+2j+2j$ |
| $-j4$ | $\times$ | $-1$ | $=$ | $j4$ |
| $2\sqrt{2}-j2\sqrt{2}$ | $\times$ | $-.7+j.7$ | $=$ | $-2+2+2j+2j$ |
| $4$ | $\times$ | $j$ | $=$ | $j4$ |
| $2\sqrt{2}+2\sqrt{2}$ | $\times$ | $.7+j.7$ | $=$ | $+2-2+2j+2j$ |

Total for $X(1) = 8 \times 4j = 32j$

For $X(2)$ we now multiply the signal values with the third row of the DFT matrix:

| SIGNAL VALUES | | THIRD DFT ROW | | |
|---|---|---|---|---|
| $j4$ | $\times$ | $1$ | $=$ | $j4$ |
| $-2\sqrt{2}+j2\sqrt{2}$ | $\times$ | $-j$ | $=$ | $+2\sqrt{2}+j2\sqrt{2}$ |
| $-4$ | $\times$ | $-1$ | $=$ | $4$ |
| $-2\sqrt{2}-j2\sqrt{2}$ | $\times$ | $+j$ | $=$ | $+2\sqrt{2}-j2\sqrt{2}$ |
| $-j4$ | $\times$ | $1$ | $=$ | $-j4$ |
| $+2\sqrt{2}-j2\sqrt{2}$ | $\times$ | $-j$ | $=$ | $-2\sqrt{2}-j2\sqrt{2}$ |
| $4$ | $\times$ | $-1$ | $=$ | $-4$ |
| $+2\sqrt{2}+j2\sqrt{2}$ | $\times$ | $+j$ | $=$ | $-2\sqrt{2}+j2\sqrt{2}$ |

TOTAL SUMS TO ZERO for $X(2)$

Similarly for 3rd component sum = 0 and $X(3) = 0$.

Full DFT output is thus 0, $32j$, 0, 0, etc. As the input signal is complex the X(7) value will be zero and *not* $32j$.

⑥

$x_n = \{1, 2, 3, 4\}$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}nk}$$

$$X(0) = \sum_{n=0}^{3} x(n) e^{-j\frac{2\pi}{4}n.0} = \sum_{n=0}^{3} x(n) = 1+2+3+4 = \underline{10}$$

$$X(1) = \sum_{n=0}^{3} x(n) e^{-j\frac{2\pi}{4}n.1} = \sum_{n=0}^{3} x(n) e^{-j\frac{\pi}{2}} = 1.e^{-j0} + 2e^{-j\pi/2} + 3e^{-j\pi} + 4e^{-j3\pi/2}$$

$$= 1 - j2 - 3 + j4 = \underline{-2+j2}$$

$$X(2) = \sum_{n=0}^{3} x(n) e^{-j\frac{2\pi}{4}n.2} = \sum_{n=0}^{3} x(n) e^{-j\pi n} = 1e^{-j0} + 2e^{-j\pi} + 3e^{-j2\pi} + 4e^{-j3\pi}$$

$$= 1 - 2 + 3 - 4 = \underline{-2}$$

$$X(3) = \sum_{n=0}^{3} x(n) e^{-j\frac{2\pi}{4}n.3} = \sum_{n=0}^{3} x(n) e^{-j\frac{3\pi n}{2}} = 1e^{-j0} + 2e^{-j\frac{3}{2}\pi} + 3e^{-j3\pi} + 4e^{-j\frac{9}{2}\pi}$$

$$= 1 + j2 - 3 - j4 = \underline{-2-j2}$$

Thus the DFT of the sequence is $X_k = \underline{\{10, -2+j2, -2, -2-j2\}}$

Setting the highest two frequency components to zero we obtain $\hat{X}_k = \underline{\{10, -2+j2, 0, 0\}}$

Now using the IDFT

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}nk}$$

$$x(0) = \frac{1}{4} \sum_{k=0}^{3} X(k) e^{j\frac{2\pi}{4}.0.k} = \frac{1}{4} \sum_{k=0}^{1} X(k) = \frac{1}{4}(10 - 2 + 2j) = \underline{2 + j0.5}$$

$$x(1) = \frac{1}{4} \sum_{k=0}^{3} X(k) e^{j\frac{2\pi}{4}.1.k} = \frac{1}{4} \sum_{k=0}^{1} X(k) e^{j\pi/2 k} = \frac{1}{4}\left(10 + (-2+j2)e^{j\pi/2}\right)$$

$$= \frac{1}{4}(10 - j2 - 2) = \underline{2 - j0.5}$$

$$x(2) = \frac{1}{4}\sum_{k=0}^{3} X(k) e^{j\frac{2\pi}{4}.2.k} = \frac{1}{4}\sum_{k=0}^{1} X(k)\, e^{j\pi k} = \frac{1}{4}\left(10 + (-2+j2)\, e^{j\pi}\right)$$

$$= \frac{1}{4}\left(10 + 2 - j2\right) = \underline{3 - j0.5}$$

$$x(3) = \frac{1}{4}\sum_{k=0}^{3} X(k)\, e^{j\frac{2\pi}{4}.3.k} = \frac{1}{4}\sum_{k=0}^{1} X(k)\, e^{j\frac{3\pi}{2}k} = \frac{1}{4}\left(10 + (-2+j2)\, e^{j\frac{3\pi}{2}}\right)$$

$$= \frac{1}{4}\left(10 + j2 + 2\right) = \underline{3 + j0.5}$$

Thus the IDFT of the sequence $\hat{X}_k$ is given by

$$\hat{x}_n = \{\underline{2 + j0.5,\ 2 - j0.5,\ 3 - j0.5,\ 3 + j0.5}\}$$

& taking the magnitude

$$|x_n| = \{\underline{2.1,\ 2.1,\ 3.0,\ 3.0}\}$$

(7) $x_n = \{1, 2, 3, 4\}$

$$X(k) = c(k)\sum_{n=0}^{N-1} x(n)\cos\left[\pi(2n+1)\frac{k}{2N}\right]$$

$$X(0) = \sqrt{\frac{1}{4}}\left[1\cos(0) + 2\cos(0) + 3\cos(0) + 4\cos(0)\right]$$

$$= \underline{5}$$

$$X(1) = \sqrt{\frac{2}{4}}\left[1\cos(\pi/8) + 2\cos(3\pi/8) + 3\cos(5\pi/8) + 4\cos\left(\frac{7\pi}{8}\right)\right]$$

$$= \underline{-2.23}$$

$$X(2) = \sqrt{\frac{2}{4}}\left[1\cos(2\pi/8) + 2\cos(6\pi/8) + 3\cos\left(\frac{10\pi}{8}\right) + 4\cos(14\pi/8)\right]$$

$$= 0$$

$$X(3) = \sqrt{\frac{2}{4}}\left[1\cos(3\pi/8) + 2\cos(9\pi/8) + 3\cos(15\pi/8) + 4\cos(21\pi/8)\right]$$

$$= -0.159$$

Thus the DCT of the sequence is $X_k = \{5, -2.23, 0, -0.159\}$

Setting the highest frequency components to zero we obtain $\hat{X}_k = \{5, -2.23, 0, 0\}$

Now using the IDCT

$$x(n) = \sum_{k=0}^{N-1} X(k)\,c(k)\cos\left[\pi(2n+1)\frac{k}{2N}\right]$$

$$x(0) = 5\sqrt{\frac{1}{4}}\cos(0) - 2.23\sqrt{\frac{2}{4}}\cos(\pi/8) = 1.0$$

$$x(1) = 5\sqrt{\frac{1}{4}}\cos(0) - 2.23\sqrt{\frac{2}{4}}\cos(3\pi/8) = 1.9$$

$$x(2) = 5\sqrt{\frac{1}{4}}\cos(0) - 2.23\sqrt{\frac{2}{4}}\cos\left(\frac{5\pi}{8}\right) = 3.1$$

$$x(3) = 5\sqrt{\frac{1}{4}}\cos(0) - 2.23\sqrt{\frac{2}{4}}\cos\left(\frac{7\pi}{8}\right) = 4.0$$

(8.) The four-point DFT equation is given by:

$$X(k) = \sum_{n=0}^{3} x(n) W_4^{nk}$$

which may be split into even & odd sums as:

$$X(k) = \underbrace{\sum_{m=0}^{1} x(2m) W_4^{2mk}}_{\text{even values of } n} + \underbrace{\sum_{m=0}^{1} x(2m+1) W_4^{(2m+1)k}}_{\text{odd values of } n}$$

Removing the constant twiddle factor we obtain

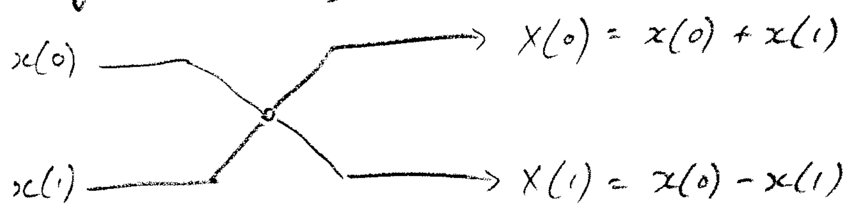
$$X(k) = \sum_{m=0}^{1} x(2m) W_4^{2mk} + W_4^{k} \sum_{m=0}^{1} x(2m+1) W_2^{mk} \qquad (1)$$

$$= X_1(k) + W_4^{k} X_2(k)$$

Now we need to express how these two DFTs are combined. Using a 2-point butterfly operation:

$$\begin{bmatrix} X(0) \\ X(1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \end{bmatrix}$$

or diagramatically

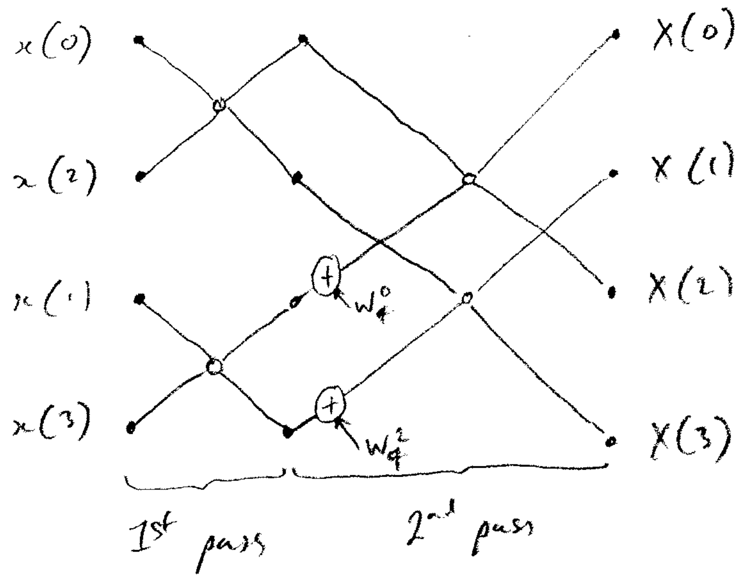



Now if we consider the FFT bin $X(k+2)$, we obtain:

$$X(k+2) = X_1(k+2) + W_4^{k} X_2(k+2)$$
$$= X_1(k) - W_4^{k} X_2(k) \qquad (2)$$

which follows since $X_1(k)$ and $X_2(k)$ are 2-point DFTs with period 2 & $W_4^k = -W_4^{(k+2)}$. Considering equations (1) & (2) we can see that the outputs $X(k)$ & $X(k+2)$ are formed by a 2-point butterfly operation on the inputs $X_1(k)$ & $W_4^k X_2(k)$. Therefore we can form all four FFT outputs using two sets of 2-point butterfly operations for index values $k=0$ & $1$.

The flowchart of the 4-point FFT structure is as shown below:



(9) The eight-point DFT is given by:

$$X(k) = \sum_{n=0}^{7} x(n) W_8^{nk}$$

which may be written as:

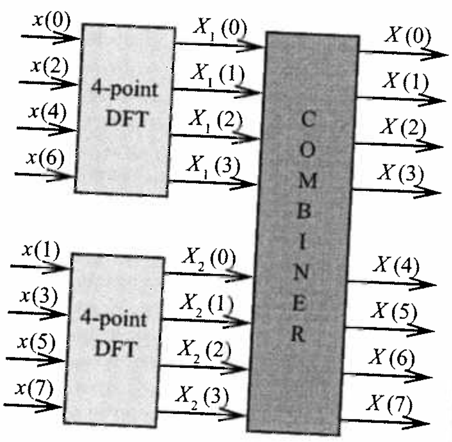
$$X(k) = \sum_{m=0}^{3} x(2m) W_8^{2mk} + \sum_{m=0}^{3} x(2m+1) W_8^{(2m+1)k}$$

and simplified to give:

$$X(k) = \sum_{m=0}^{3} x(2m) W_4^{mk} + W_8^{k} \sum_{0}^{3} x(2m+1) W_4^{mk} \qquad (1)$$

$$= X_1(k) + W_8^{k} X_2(k)$$

$X(k)$ is the sum of two 4-point FFTs, $X_1(k)$ & $X_2(k)$, whose structure was derived in Q8.




Now considering the combination:

$$X(k+4) = X_1(k+4) + W_8^{(k+4)} X_2(k+4) \qquad (2)$$

$$= X_1(k) - W_8^{k} X_2(k)$$

which holds because $X_1$ & $X_2$ have period 4 (being 4-point FFTs). Similarly $W_8^{(k+4)} = -W_8^{k}$.

From equations (1) & (2) we see that the outputs $X(k)$ & $X(k+4)$ are formed by the two point butterfly operation on the inputs $X_1(k)$ & $W_8^{k} X_2(k)$.

Therefore we can form all eight FFT outputs using four sets of 2-point butterfly operations for index values $k = 0, 1, 2$ & $3$.

The flowchart of the 8-point FFT is as in your lecture notes.

The 16-point FFT can be split into two 8-point FFTs which process even ($X_e(k)$) and odd samples ($X_o(k)$) of $x(n)$. Each 8-point FFT structure is as derived above. The two sets of 8-point FFT outputs are then combined to form the 16-point FFT in a final combiner stage.

We can write two equations for the 16-point FFT $X(k)$ & $X(k+8)$ in terms of the even 8-point FFT $X_e(k)$ and the odd 8-point FFT $X_o(k)$:

$$X(k) = X_e(k) + W_{16}^{k} X_o(k)$$
$$X(k+8) = X_e(k) - W_{16}^{k} X_o(k)$$

This is the final combiner stage of the 16-point FFT consisting of 8 2-point butterfly operations. The butterfly inputs are $X_e(k)$ & $W_{16}^{k} X_o(k)$ for indices $k = 0 \ldots 7$.

(10) The intermediate outputs from the first pass DIT FFT are given as:

$$x(0) + x(4)$$
$$x(0) - x(4)$$
$$x(2) + x(6)$$
$$x(2) - x(6)$$
$$x(1) + x(5)$$
$$x(1) - x(5)$$
$$x(3) + x(7)$$
$$x(3) - x(7)$$

The intermediate outputs on the second pass are then:

$$x(0) + x(4) + W^0[x(2) + x(6)]$$
$$x(0) - x(4) + W^2[x(2) - x(6)]$$
$$x(0) + x(4) - W^0[x(2) + x(6)]$$
$$x(0) - x(4) - W^2[x(2) - x(6)]$$
$$x(1) + x(5) + W^0[x(3) + x(7)]$$
$$\vdots \qquad\qquad \vdots$$

The third pass outputs then yield the full 8-point transform output values.