Essential Mathematical Methods for Engineers

Lecture 4:
The discrete Fourier transform

Outline

- development of the DFT
  - the continuous Fourier transform
  - Fourier transform of a finite length data record
  - definition of the DFT
  - properties of the DFT

- computation of the discrete Fourier transform
  - DFT matrix coefficient values
  - matrix formulation of the DFT
  - analogies for the DFT

- resolution and window responses
  - resolution
  - leakage effects
  - the rectangular window
  - Hanning window
  - Hamming window

Development of the discrete Fourier transform
The continuous Fourier transform

- an analogue deterministic signal has Fourier transform:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \exp(- j\omega t) \, dt$$

- sampling $x(t)$ every Δt seconds we have

$$x_c(t) = \sum_{n=-\infty}^{\infty} x(n\Delta t) \, \delta(t - n\Delta t)$$

- the Fourier transform of the sampled signal is given by

$$X_c(\omega) = \int_{-\infty}^{\infty} x_c(t) \exp(- j\omega t) \, dt$$

$$X_c(\omega) = \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} x(n\Delta t) \, \delta(t - n\Delta t) \right] \exp(- j\omega t) \, dt$$

Development of the discrete Fourier transform
The continuous Fourier transform

- by changing the order of the integration and summation:

$$X_c(\omega) = \sum_{n=-\infty}^{\infty} x(n\Delta t) \left[\int_{-\infty}^{\infty} \delta(t - n\Delta t) \exp(-j\omega t)\, dt \right]$$

- according to the properties of the impulse:

$$X_c(\omega) = \sum_{n=-\infty}^{\infty} x(n\Delta t)\ \exp(-j\omega n\Delta t)$$

which is known as the discrete-time Fourier transform (DTFT)

  – we still need an infinite number of time samples

Fourier transform of a finite length data record

- assuming that we have only a finite number of samples

$$\hat{X}_c(\omega) = \sum_{n=0}^{N-1} x(n\Delta t)\ \exp(-j\omega n\Delta t)$$

- this is equivalent to multiplying $x(n\Delta t)$ by a rectangular window $w_T(t)$ of width $T = N\Delta t$ seconds:

$$\hat{x}_c(t) = \sum_{n=0}^{N-1} x(n\Delta t)\ \delta(t - n\Delta t) = x_c(t)\ w_T(t)$$

- this is equivalent to convolution in the frequency domain:

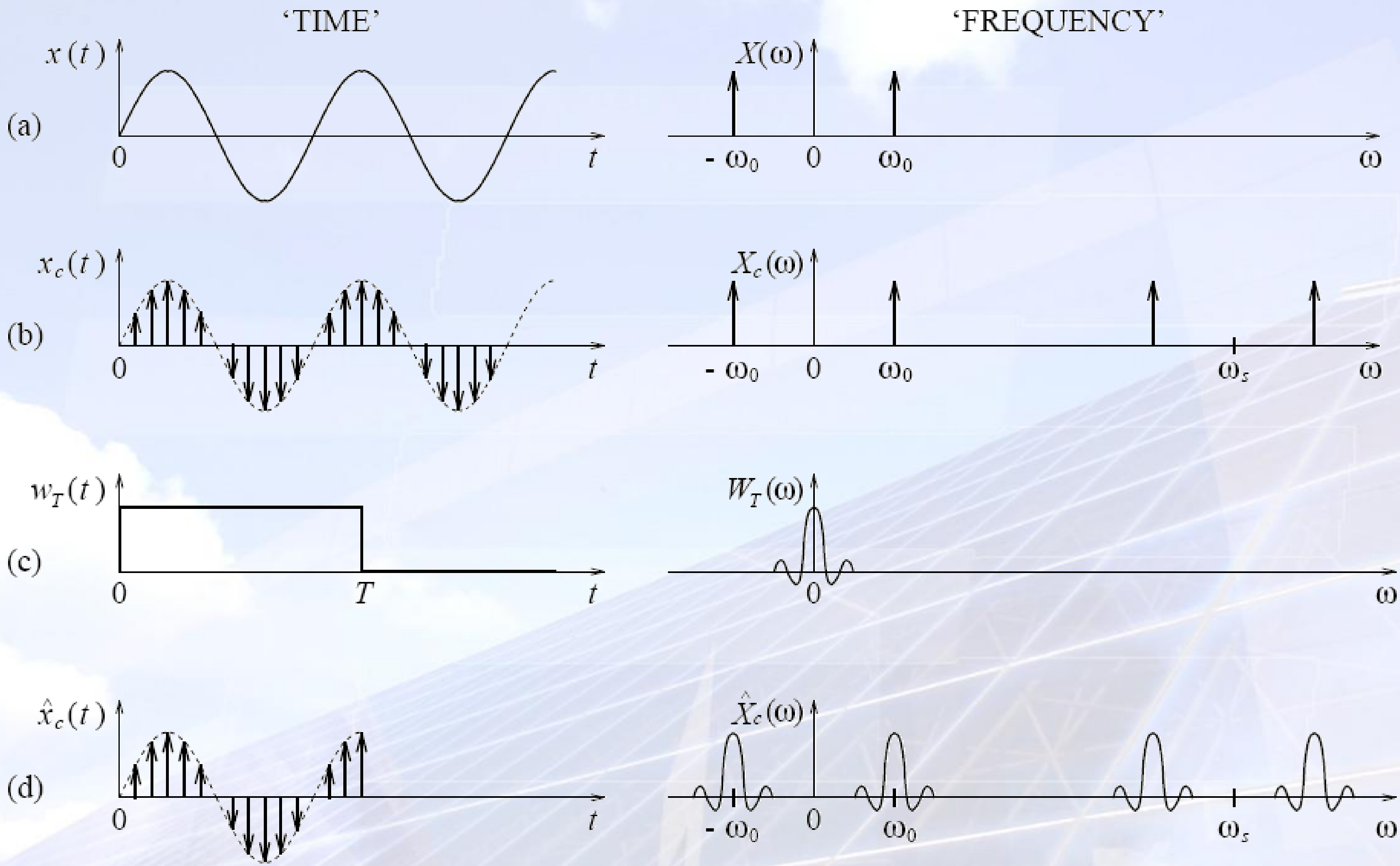
$$\hat{X}_c(\omega) = \frac{1}{2\pi} X_c(\omega) * W_T(\omega)$$

Example

Illustrate the effects of sampling and rectangular windowing on a continuous-time sine wave and its Fourier transform.

NB the difference between $X_C(\omega)$ and $\hat{X}_C(\omega)$

Development of the discrete Fourier transform
Fourier transform of a finite length data record



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Development of the discrete Fourier transform

Fourier transform of a finite length data record

- the Fourier transform of any sampled data signal is a continuous function of frequency

- working with actual data we calculate it numerically at specific values of ω

- in practice it is calculated at N equally spaced frequencies between 0 and the sampling frequency, ω_S

$$\Delta\omega = 2\pi/(\Delta t\ N)$$

$$\Delta f = 1/(\Delta t\ N)$$

$$(T = N\ \Delta t)$$

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Development of the discrete Fourier transform

Fourier transform of a finite length data record

- $\Delta\omega$ can be arbitrarily decreased by increasing the transform length N

sampling in the frequency domain

- thus from $\hat{X}_c(\omega) = \sum_{n=0}^{N-1} x(n\Delta t)\ \exp(-j\omega n\Delta t)$

we have $\hat{X}_c(k\,\Delta\omega) = \sum_{n=0}^{N-1} x(n\Delta t)\ \exp(-jk\Delta\omega n\Delta t)$

$$\hat{X}_c(k) = \sum_{n=0}^{N-1} x(n)\ \exp\left(\frac{-jnk2\pi}{N}\right)$$

- this is the discrete Fourier transform (DFT)
  - usually we write $\hat{X}_c(k)$ simply as $X(k)$

Definition of the DFT

- the DFT is defined as:

$$X(k) = \sum_{n=0}^{N-1} x(n) \exp\left(\frac{-jnk2\pi}{N} \right)$$

where $0 \leq n < N-1$ and $0 \leq k < N-1$.

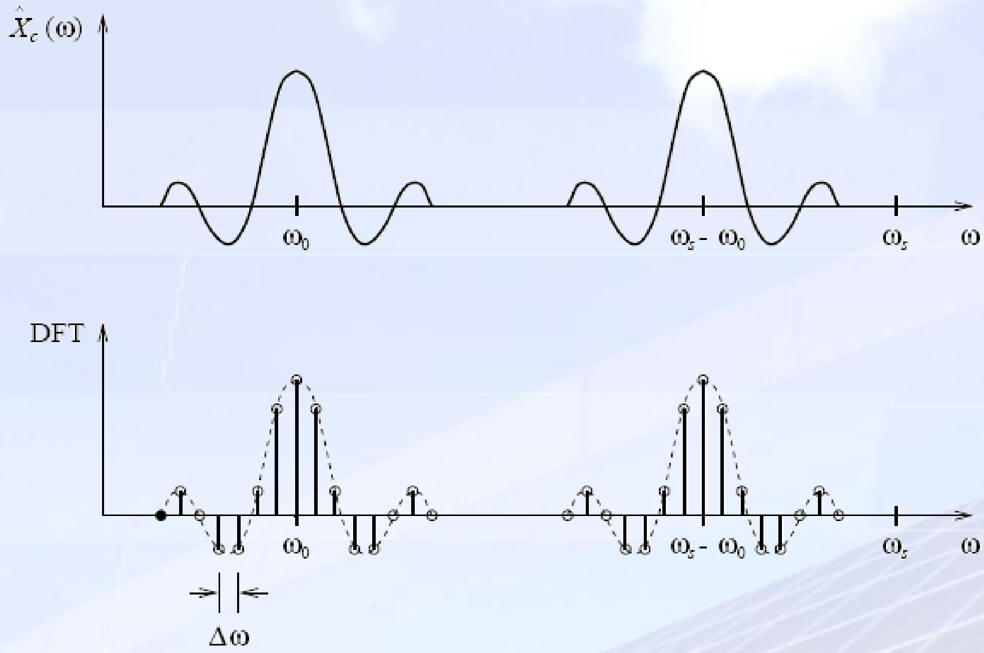
- thus we calculate N equally spaced values of $X_C(\omega)$ from N consecutive samples of the analogue signal
  - k is often referred to as the 'bin' number and refers to a frequency denoted ω_k

- the inverse DFT is given by

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \exp\left(\frac{jnk2\pi}{N} \right)$$

where again $0 \leq n < N-1$ and $0 \leq k < N-1$.

Development of the discrete Fourier transform
Definition of the DFT



sampling in the frequency domain

- observing the DFT of the previous example
  - the energy is 'smeared' around ω_0 even though there is only one frequency present in $x(t)$

Properties of the DFT

- the symmetry property implies that the magnitude DFT of a real data sequence is symmetrical about the DC or 0 Hz component

- for complex input data the complex conjugate symmetry property applies and the phase response is skew symmetric about 0 Hz – the phase at $+\omega$ is equal to the negative of the phase at $-\omega$

$$X(-k) = \sum_{n=0}^{N-1} x(n) \ \exp\left(\frac{-jn - k2\pi}{N} \right)$$

$$= \sum_{n=0}^{N-1} x(n) \ \exp\left(\frac{jnk2\pi}{N} \right)$$

$$= X^*(k)$$

Development of the discrete Fourier transform Properties of the DFT

- this symmetry for real signals gives rise to the aliased repeat at ω_s
- the modulus of the DFT components from zero up to $\omega_s / 2$ thus repeats from $\omega_s / 2$ to ω_s in a mirror image fashion
  - this can be shown by substituting $k = N - r$ where $r < N / 2$

$$X(N-r) = \sum_{n=0}^{N-1} x(n) \ \exp\left(\frac{-jn(N-r)2\pi}{N} \right)$$

$$= \sum_{n=0}^{N-1} x(n) \ \exp\left(\frac{jnr2\pi}{N} \right) \exp(-jn2\pi)$$

$$= \sum_{n=0}^{N-1} x(n) \ \exp\left(\frac{jnr2\pi}{N} \right)$$

$$X(N-r) = X^*(r)$$

Development of the discrete Fourier transform
Properties of the DFT

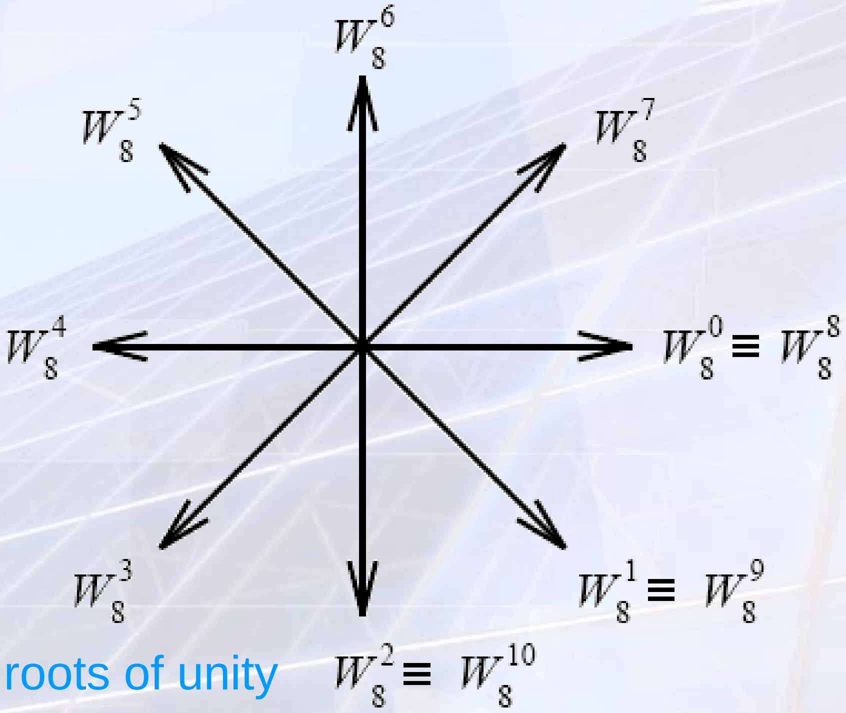
- we can also show that $X(N+k) = X(k)$
- the shift property implies that the magnitude DFT of a sampled signal record is fixed or constant, irrespective of the precise timing of the signal record within the DFT block
- the timing information is contained in the DFT output phase components

Computation of the DFT

- for convenience we can write $W_N^k = \exp(-j2\pi k/N)$

  where W_N^1 is the fundamental N^{th} root of unity (representing the first of the N, N^{th} roots)

- these are complex numbers which, when raised to the power N, all equal 1



the eight, eighth roots of unity

Matrix formulation

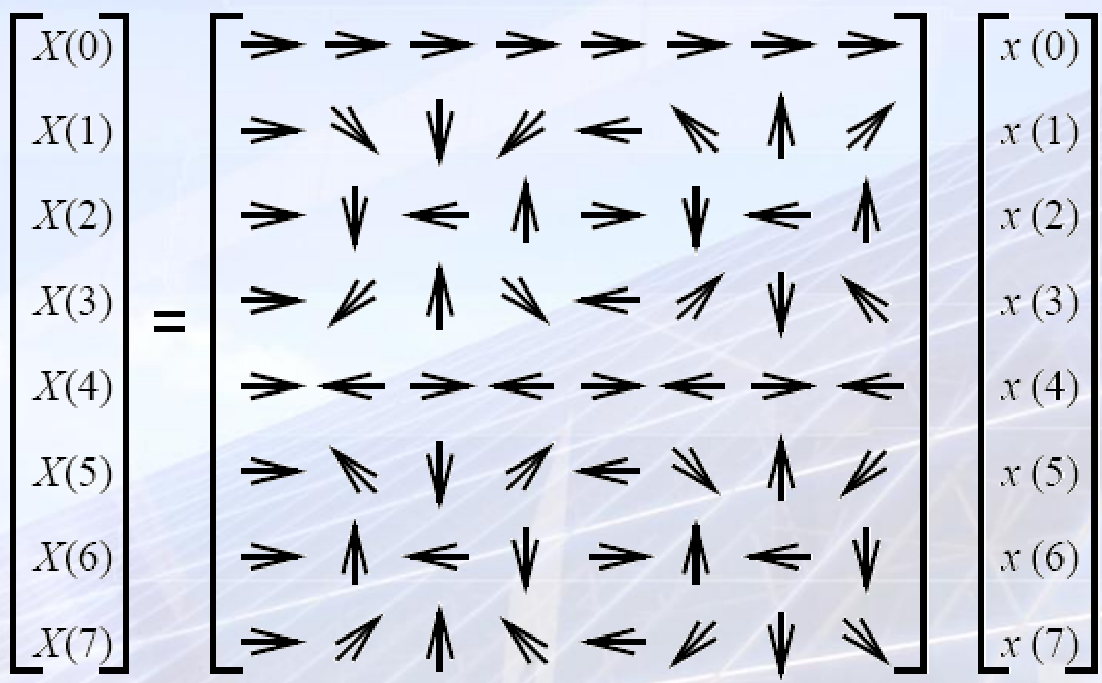
$$
\begin{array}{lcccccccc}
X(0) & = & x(0)\,W_N^0 & + & x(1)\,W_N^0 & + & . & . & + & x(N-1)\,W_N^0 \\
X(1) & = & x(0)\,W_N^0 & + & x(1)\,W_N^1 & + & . & . & + & x(N-1)\,W_N^{N-1} \\
X(2) & = & x(0)\,W_N^0 & + & x(1)\,W_N^2 & + & . & . & + & x(N-1)\,W_N^{N-2} \\
. & & . & & . & & & & & . \\
. & & . & & . & & & & & . \\
X(N-1) & = & x(0)\,W_N^0 & + & x(1)\,W_N^{N-1} & + & . & . & + & x(N-1)\,W_N^1
\end{array}
$$

$$
\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ . \\ . \\ X(N-1) \end{bmatrix}
=
\begin{bmatrix}
1 & 1 & 1 & . & . & 1 \\
1 & W_N^1 & W_N^2 & . & . & W_N^{N-1} \\
1 & W_N^2 & W_N^4 & . & . & W_N^{N-2} \\
. & . & . & & & . \\
. & . & . & & & . \\
1 & W_N^{N-1} & W_N^{N-2} & . & . & W_N^1
\end{bmatrix}
\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(N-1) \end{bmatrix}
$$

Computation of the DFT
Matrix formulation

Example

Explain the operation of the 8-point DFT by a complex input phasor with an integer number of cycles within the 8-sample input data sequence.



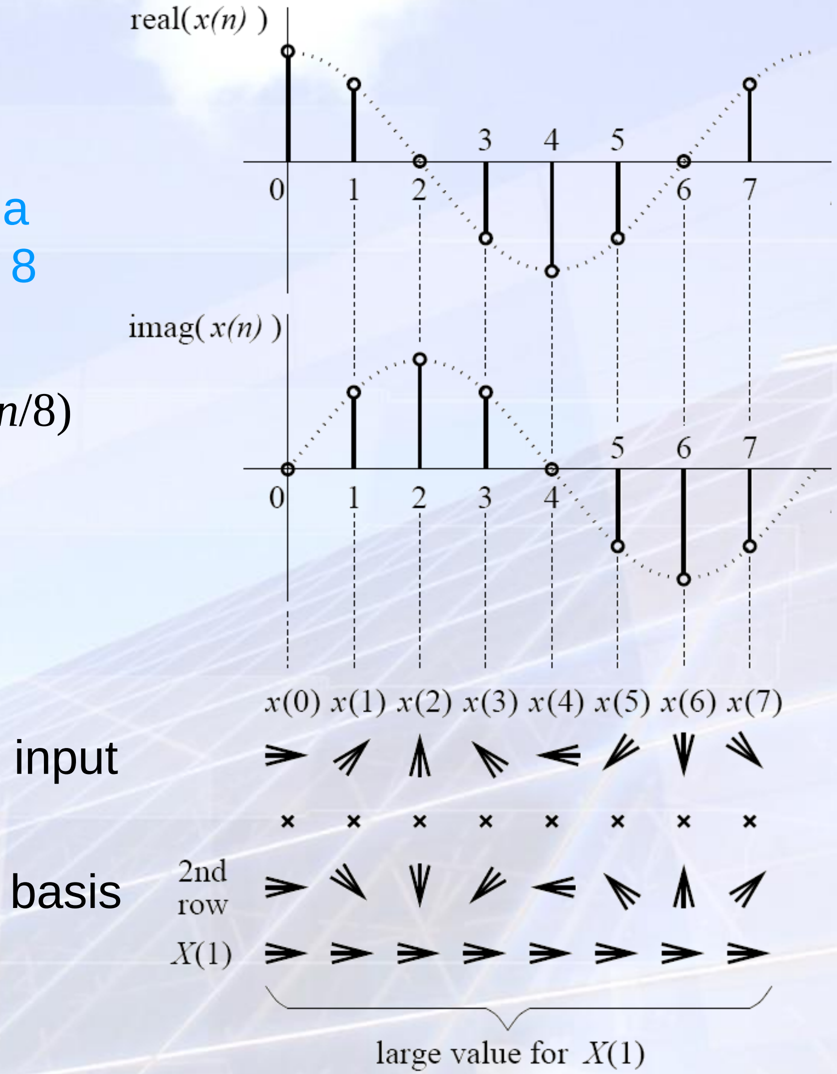
Computation of the DFT
Matrix formulation

e.g. for a single input signal consisting of a sampled complex phasor with a period of 8 samples:

$$x(n) = \exp(j2\pi n/8) = \cos(2\pi n/8) + j\sin(2\pi n/8)$$

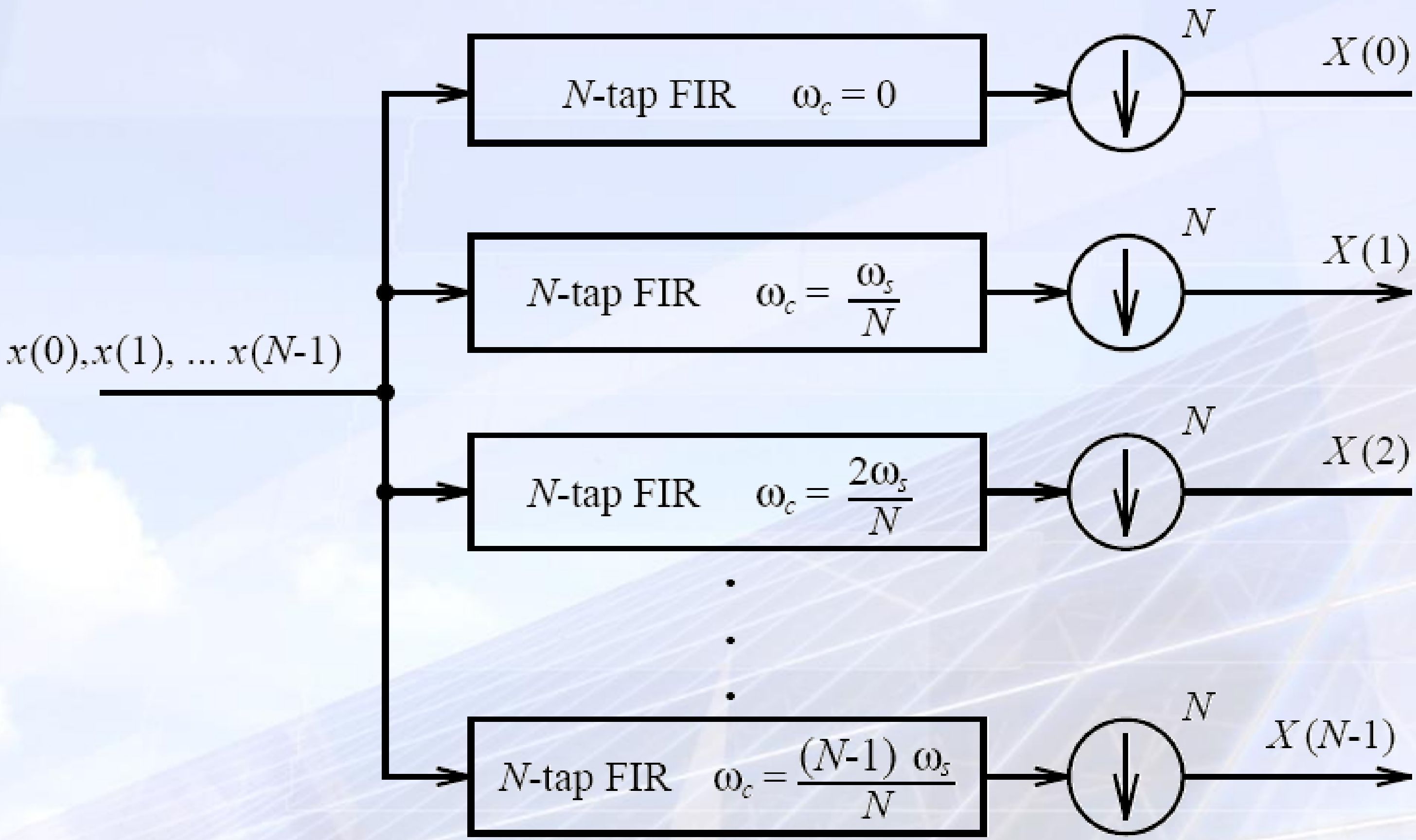
Questions:

- what is the output for $X(1)$?
- what is the output for $X(2)$?
- what is the output for $\theta \neq 0$?



Analogies for the DFT

- the DFT may be viewed as a filterbank



Computation of the DFT
Analogies for the DFT

Resolution and window responses
Resolution

- when processing N input samples, the transformed output has a $\operatorname{sinc} x$ shape

- if the input signal comprises a sinusoid with N complete cycles within the N samples then the output occurs on bin $k = n$ only
  - a $\operatorname{sinc} x$ shape which peaks on bin n and zeros elsewhere

- the resolution is the ability to distinguish between two or more closely spaced sinusoidal inputs without error
  - the smallest resolution, $\Delta\omega$, is equal to the filter spacing ω_s / N
    - related to the sample rate and transform size

- the resolution specifies the minimum distinguishable frequency gap
  - two sinusoids appearing in two neighbouring, different bins

Leakage effects

- the continuous Fourier transform is valid when the integration is performed over an interval $-\infty$ to $+\infty$ or an integer number of cycles of the waveform

- when the DFT is applied over a non-integer number of cycles then leakage is the result

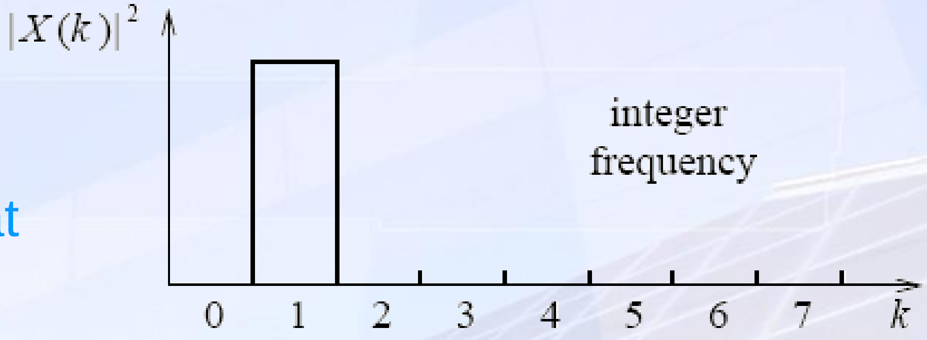
- a series of N samples corresponds to an infinite series multiplied by a rectangular window of width $T = N\Delta t$ seconds

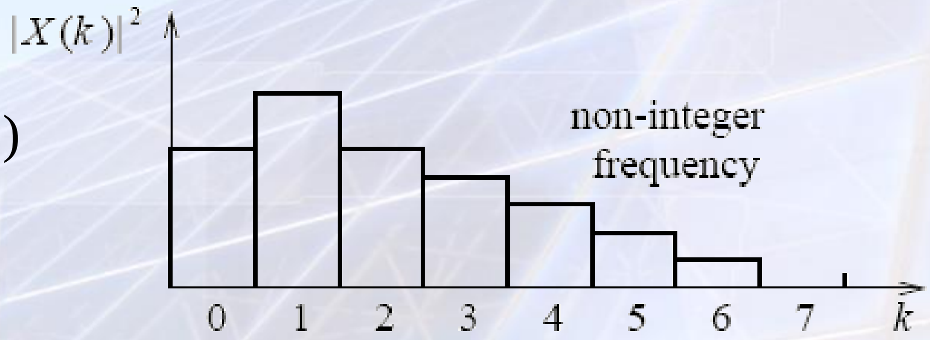
- this is equivalent to convolving the FT of the continuous signal with the function $T\,\mathrm{sa}(\omega T/2)$
  - we've already seen how this smears $X(\omega)$ on a frequency scale of approximately $2/T$ Hz

- tapered windows are applied to weight the input data samples to control leakage and to minimise its effect on reducing the dynamic range capability of the transform

Resolution and window responses
Leakage effects

- the effect is worst when there are $n + \frac{1}{2}$ cycles in the input data block length
- the resultant output $X(n)$ is no longer stationary and is rotating at ½ a cycle per block length
  - $X(n)$ is diminished in amplitude
  - 'missing' energy spreads to neighbouring bins
  - in the worst case $X(n) = X(n+1)$





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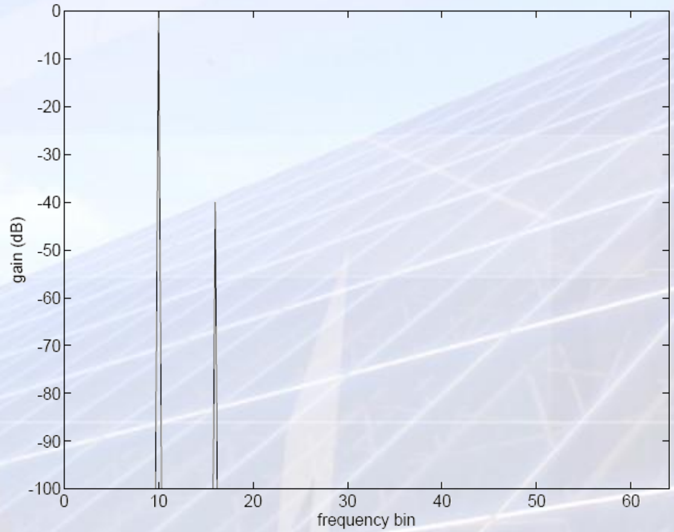
The rectangular window

- every DFT process is windowed
  - this is how we achieve a finite duration time-series

- the following illustrates 2 sine waves with a 128-point DFT
  1. amplitude 1, frequency of 10 cycles / block length ($10\omega_s/N$)
  2. relative amplitude 0.1, frequency of 16 cycles / block length

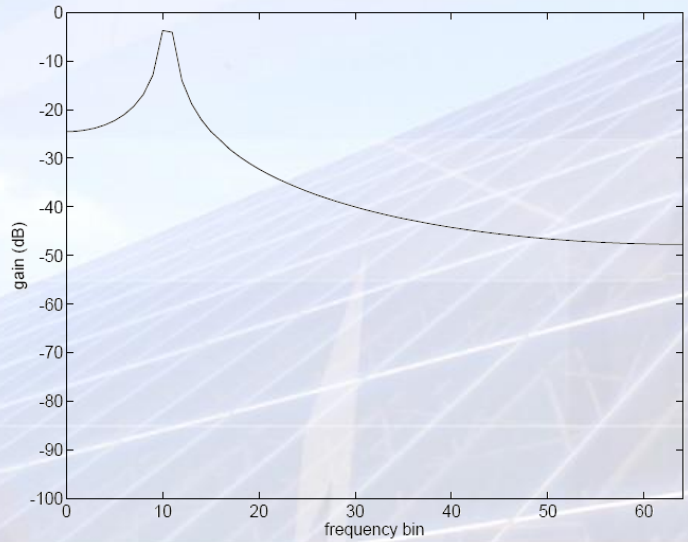
- the DFT samples lie at the zeros on the sidelobes of the $\operatorname{sinc} x$ responses



Resolution and window responses
The rectangular window

- if we increase the frequency of the larger sine wave by half a resolution cell in the DFT to $10.5\omega_s/N$ it now falls half way between bins 10 and 11

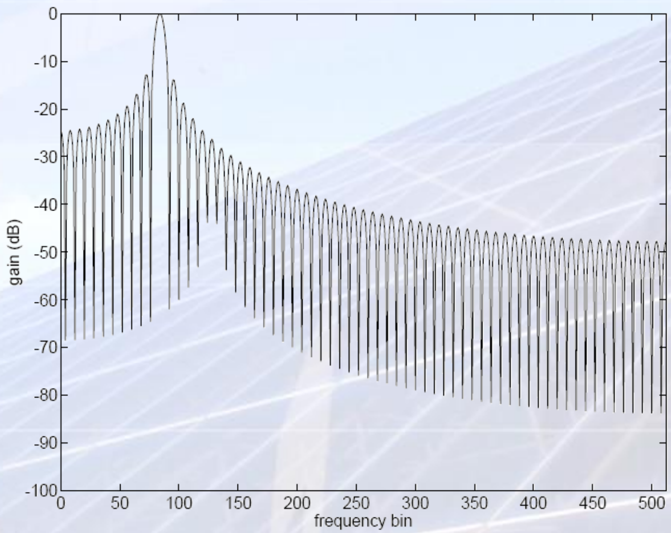
- effects:
  - the peak magnitude is reduced
  - leakage spread right across the spectrum and masks the smaller signal
  - the dynamic range is greatly reduced



Resolution and window responses
The rectangular window

- the leakage follows the envelope of the sinc function
  - the output sampling grid in the 128-point DFT is no longer sampling at the zeros of the $\operatorname{sinc} x$ function

- if we extend the data record from 128 to 1024 samples by adding 896 zero elements we introduce an interpolation effect which allows plotting of the fine detailed structure of the sidelobes in these $\operatorname{sinc} x$ responses

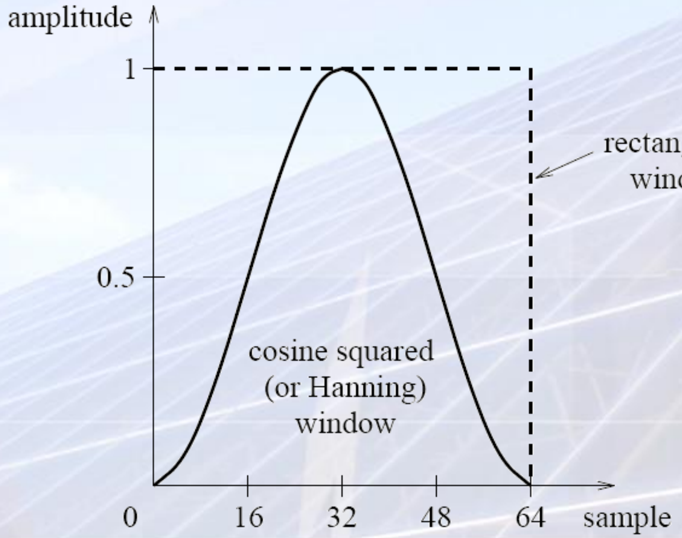


Resolution and window responses
Hanning window

- a tapered window may be used to reduce leakage
- the Hanning window is a 'raised cosine' window

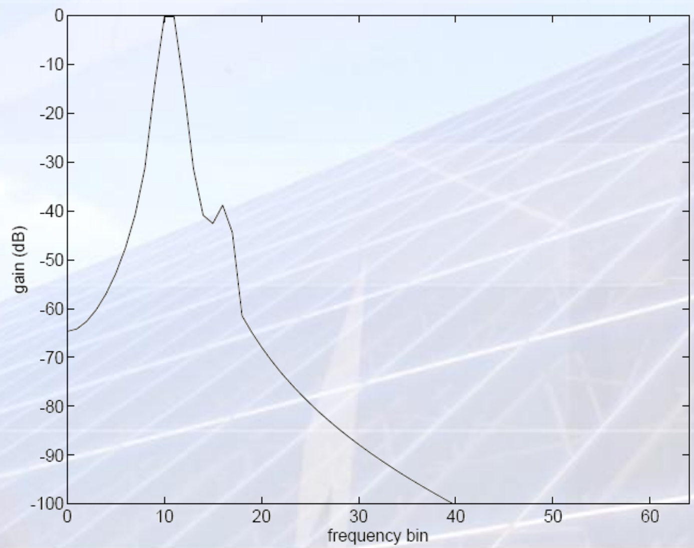
$$w_T(n) = 0.5 - 0.5\cos\left(\frac{2\pi n}{N}\right)$$

and is used to weight the input sample sequence $x(n)$ prior to processing in the DFT, i.e. $x'(n) = x(n) \times w_n$



Resolution and window responses
Hanning window

- the plot below shows the 128-point DFT with the Hanning window
  - signal energy is reduced (it is normalised below to 0 dB peak)
  - the leakage is reduced
  - the smaller signal is now visible
  - the base width of the responses is greater thus the resolution is comparatively degraded
    - more difficult to distinguish between closely spaced sinusoidal inputs



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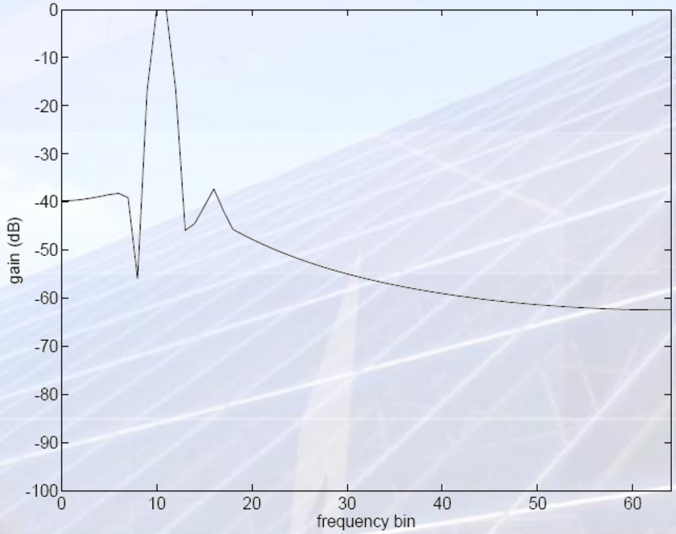
Hamming window

- the Hamming window is a slight modification of the Hanning window to add an 8% magnitude value at the window edge

$$w_T(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{N}\right)$$

- the Hamming window has a maximum sidelobe level of -43 dB (c.f. -32 dB for the Hanning window) but the roll-off is less rapid
  - the roll-off 'bottoms out' at about -50dB
  - with better resolution the smaller signal is more visible



Summary

You should:

- understand how the DFT arises;
- be able to apply the DFT to a given signal;
- recognise the effect of resolution and windowing functions upon the DFT;