Projection

$$a = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$p = a\hat{x} = a\frac{a^T b}{a^T a} = Pb$$

$$\therefore \quad P = \frac{aa^T}{a^T a}$$

$$= \frac{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \end{bmatrix}}{\begin{bmatrix} 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}} = \frac{\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix}}{9}$$

& note that the rank of P is 1
– projection onto a line

$$b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$p = Pb = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 5 \\ 10 \\ 10 \end{bmatrix}$$

$$p' = Pp = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix} \times \frac{1}{9}\begin{bmatrix} 5 \\ 10 \\ 10 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 5 \\ 10 \\ 10 \end{bmatrix}$$

– if we project again, nothing happens since p is already on the line

Projections onto a subspace

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

Starting with $A^TA\hat{x} = A^Tb$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \qquad A^Tb = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

Now solve by elimination for $\hat{x}$

$$\begin{bmatrix} A & b \end{bmatrix} = \left[\begin{array}{cc|c} 3 & 3 & 6 \\ 3 & 5 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} 3 & 3 & 6 \\ 0 & 2 & -6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix} \qquad \hat{x} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

Really: A^TA A^Tb

Now, $p = A\hat{x} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$

$e = b - p$

$= \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

For $P = A(A^TA)^{-1}A^T$

$$\text{inv}(A^TA) = \begin{bmatrix} 5/6 & -3/6 \\ -3/6 & +3/6 \end{bmatrix}$$

& so $P = \frac{1}{6}\begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}$

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check: e should be perpendicular to A

$$e^Ta_1 = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0 \checkmark \qquad e^Ta_2 = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = 0 \checkmark$$

check: $p = Pb = \frac{1}{6}\begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}\begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \checkmark$

can also check that $P^2 = P$

Least squares approximation

$$b = C + Dt$$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \quad x = \begin{bmatrix} C \\ D \end{bmatrix} \quad b = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

NB. Same problem as previous example

NB $[A \ b]$ reduces to $[R \ d] = \left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$ ← inconsistent

Solve in the same way from $A^T A \hat{x} = A^T b$

$$\hat{x} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

Gram-Schmidt

$$a = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad b = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \qquad c = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}$$

$$A = a$$

$$B = b - \frac{A^T b}{A^T A} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} - \frac{\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}}{\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

and checking: $A^T B = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = 0$ ✓

$$C = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B$$

$$= \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} - \frac{\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}}{\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and checking $A^T C = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$ ✓

$B^T C = \begin{bmatrix} 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$ ✓

$$\therefore \quad A = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

but this basis is only orthogonal - not orthonormal

$$\|A^2\| = A^T A = 2 \qquad \|B\|^2 = B^T B = 6 \qquad \|C\|^2 = C^T C = 3$$

$$q_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad q_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \qquad q_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

& now $q_i^T q_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

this basis is orthonormal

Eigenvalue problem

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

$$Ax = \lambda x$$
$$\lambda x - Ax = 0$$
$$(\lambda I - A)x = 0$$
$$|\lambda I - A| = 0$$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \right| = \left| \begin{bmatrix} \lambda+2 & -1 \\ -1 & \lambda+2 \end{bmatrix} \right|$$
$$= (\lambda+2)(\lambda+2) - 1$$
$$= (\lambda+1)(\lambda+3)$$

& so either $\lambda = -1$ or $\lambda = -3$

when $\lambda = -1$ $\left(\underbrace{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}_{(\lambda I)} - \underbrace{\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}}_{(A)} \right) x = 0$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} x = 0 \quad \Rightarrow \quad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

when $\lambda = -3$ $\left(\begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} - \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \right) x = 0$

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} x = 0 \quad \Rightarrow \quad x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Eigenvalues 2

$$A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

gives $c(\lambda) = \begin{vmatrix} \lambda-1 & -1 & 2 \\ 1 & \lambda-2 & -1 \\ 0 & -1 & \lambda+1 \end{vmatrix} = 0$

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expanding along first column

$$(\lambda-1)\begin{vmatrix} \lambda-2 & -1 \\ -1 & \lambda+1 \end{vmatrix} - \begin{vmatrix} -1 & 2 \\ -1 & \lambda+1 \end{vmatrix}$$

$$= (\lambda-1)\left[(\lambda-2)(\lambda+1) - 1\right] - \left[2 - (\lambda+1)\right]$$

$$= \lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

we could solve this to get λ_1, λ_2 or λ_3

or factorize the determinant

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix}$$

$\div$ c3 by $-1-\lambda$

$$\Rightarrow -(1+\lambda)\begin{vmatrix} 1-\lambda & 1 & 1 \\ -1 & 2-\lambda & 0 \\ 0 & 1 & 1 \end{vmatrix}$$

subtract row 3 from row 1

$$-(1+\lambda)\begin{vmatrix} 1-\lambda & 0 & 0 \\ -1 & 2-\lambda & 0 \\ 0 & 1 & 1 \end{vmatrix} = -1(1+\lambda)(1-\lambda)(2-\lambda)$$

$\therefore \lambda = -1, 1$ or 2

$\underline{\lambda_1 = 2 \quad \lambda_2 = 1 \quad \lambda_3 = -1}$

Eigenvalues . 3

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$|\lambda I - A| = 0 = \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} \lambda - 1 & \\ & \lambda - 1 \end{vmatrix}$$

$$= (\lambda - 1)^2$$

and eigenvalues are 1 and 1 - repeated values!

Eigenvalues then come from

$$(\lambda I - A)x = 0$$

$$\left(\begin{bmatrix} 1 & \\ & 1 \end{bmatrix} - \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \right) x = 0 \quad \Rightarrow \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x = 0$$

This can be satisfied by any value of x.

We take eigenvector $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

& any combination will also be an eigenvector.

Geometrically, any vector is mapped onto itself

- A is actually I!

Eigenvalues 4

$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$|\lambda I - B| = 0 = \left| \begin{bmatrix} \lambda & \\ & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 1 \end{vmatrix}$

$= (\lambda - 1)^2$

again, we have repeated eigenvalues

To obtain the eigenvectors:

$(\lambda I - B) x = 0$

$\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} x = 0$

The solution $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ must have $x_2 = 0$

and the eigenvector is therefore $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

+ any multiple.

SVD

$$A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$$

Start by determining V

$$A^TA = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

So now determine the eigenvalues of A^TA

$$(A^TA)x = \lambda x$$

$$(\lambda I - A^TA)x = 0 \qquad x \neq 0$$

$$\therefore \det(\lambda I - A^TA) = 0$$

$$\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}\right) = \det\left(\begin{bmatrix} \lambda-5 & -3 \\ -3 & \lambda-5 \end{bmatrix}\right)$$

$$= (\lambda-5)(\lambda-5) - 9 = (\lambda-2)(\lambda-8)$$

Letting $\lambda = 8$

$$\begin{bmatrix} +3 & -3 \\ -3 & +3 \end{bmatrix}\begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad x = \begin{bmatrix} +1 \\ +1 \end{bmatrix} \qquad \& \qquad v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} +1 \\ +1 \end{bmatrix}$$

Letting $\lambda = 2$

$$\begin{bmatrix} -3 & -3 \\ -3 & -3 \end{bmatrix}\begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad x = \begin{bmatrix} -1 \\ +1 \end{bmatrix} \qquad \& \qquad v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ +1 \end{bmatrix}$$

& note that v_1 & v_2 are perpendicular

(A^TA is symmetric)

$$Av_1 = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} +1/\sqrt{2} \\ +1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix}$$

$$= 2\sqrt{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \sigma_1 u_1$$

$$Av_2 = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -1/\sqrt{2} \\ +1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}$$

$$= \sqrt{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \sigma_2 u_2$$

u_1 & u_2 are perpendicular

$$\therefore A = U\Sigma V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2\sqrt{2} & \\ & \sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$