Mathematical Methods for Engineers (MathEng)
EXAM
12th February 2010
Duration: 2 hrs, calculators permitted, no documents
ATTEMPT ALL QUESTIONS – ANSWER IN ENGLISH

EURECOM
Sophia Antipolis

There are 10 questions and 80 marks in this exam paper. You should attempt all questions.

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1. Perform an LDU decomposition for the following matrix A and **use it** to solve $Ax = b$ where:

$$A = \begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 8 \\ 21 \end{bmatrix}.$$

What could be the advantage of solving the system in this way rather than by Gaussian elimination?

[5 marks]

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2. Complete the following and explain your answers:
   (a) The eigenvalues of a 2×2 projection matrix are ___ and ___.
   (b) The eigenvalues of a 2×2 reflection matrix are ___ and ___.

[5 marks]

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3. Determine the singular value decomposition (SVD) of the matrix A and orthonormal bases for all four of its subspaces.

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$$

[10 marks]

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4. Determine the energy or power in the following signals (if defined) and hence or otherwise state whether they are energy or power signals.

$$x_1(t) = \sin(2\pi t) \quad \text{and} \quad x_2(t) = \exp(-t)$$

[5 marks]

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5. By graphical time convolution, sketch the system output corresponding to the input signal $x(t)$ and the system impulse response $h(t)$ illustrated in Figure Q6.

[5 marks]

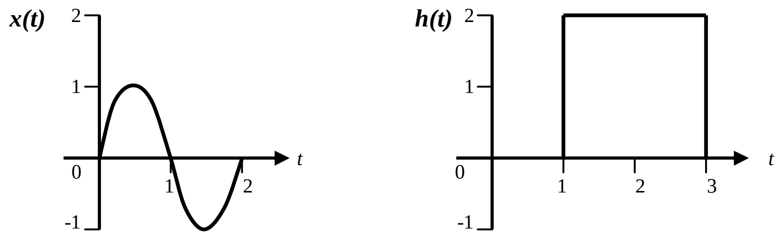

Figure Q6

6. Sketch and describe example impulse responses for systems with pole positions illustrated in the s-planes of Figure Q7 (a) and (b).

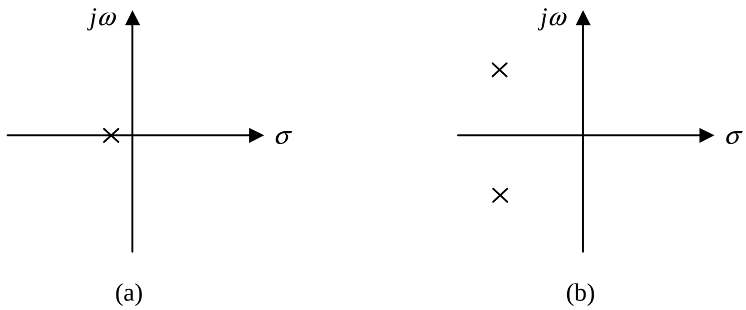

Figure Q7

[5 marks]

7. Sketch Bode plots of the magnitude and phase responses for a systems with transfer function:

$$H(s) = \frac{10(s+100)}{s^2 + 2s + 100}$$

[10 marks]

8. A sequence where $x(n) = 0.2^n$, for $n \geq 0$, is applied to a digital filter with the following difference equation:

$$y(n) = 0.5y(n-1) + x(n)$$

Use transform techniques to develop an expression for the system transfer function $H(z)$ and for the output sequence $y(n)$.

[10 marks]

9. **Using suitable illustrations**, describe the effects of sampling and rectangular windowing on a continuous-time sine wave and its Fourier transform. You should **briefly** discuss the effects of resolution and leakage in your answer.

[10 marks]

10. Two random variables X and Y have the joint probability density function (PDF) given by:

$$f_{XY}(x,y)=\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-m_x}{\sigma_X}\right)^2-2\rho\left(\frac{x-m_x}{\sigma_X}\right)\left(\frac{y-m_Y}{\sigma_Y}\right)+\left(\frac{y-m_Y}{\sigma_Y}\right)^2\right]\right\}$$

(a) Determine the two marginal PDFs $f_X(x)$ and $f_Y(y)$.

Hint: The term within the exponential may be rewritten as:

$$-\frac{1}{2(1-\rho^2)\sigma_Y^2}\left[y-m_Y-\rho\frac{\sigma_Y}{\sigma_X}(x-m_X)\right]^2+\left(\frac{x-m_x}{\sigma_X}\right)^2$$

(b) Show that X and Y are independent when $\rho = 0$.

(c) Show that ρ is the correlation coefficient of X and Y.

[15 marks]

 [END OF PAPER]

Table of selected Laplace transforms

$x(t) \quad (t \geq 0)$	$X(s)$
$\delta(t)$	1
$\delta(t-\alpha)$	$\exp(-\alpha s)$
$1 \quad$ (unit step)	$\frac{1}{s}$
$t \quad$ (unit ramp)	$\frac{1}{s^2}$
$\exp(-\alpha t)$	$\frac{1}{s+\alpha}$
$t\exp(-\alpha t)$	$\frac{1}{(s+\alpha)^2}$
$\sin(\alpha t)$	$\frac{1}{s^2+\alpha^2}$
$\cos(\alpha t)$	$\frac{s}{s^2+\alpha^2}$
$e^{-\alpha t}\sin(\omega t)$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
$e^{-\alpha t}\cos(\omega t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$

Table of selected z-transforms

$x(n) \quad (n \geq 0)$	$X(z)$
$\delta(n)$ unit pulse	1
$\delta(n-m)$	z^{-m}
1 (unit step)	$\dfrac{z}{z-1}$
n (unit ramp)	$\dfrac{z}{(z-1)^2}$
$\exp(-\alpha n)$	$\dfrac{z}{(z-e^{-\alpha})}$
$n\exp(-\alpha n)$	$\dfrac{e^{-\alpha}z}{(z-e^{-\alpha})^2}$
$\sin(\beta n)$	$\dfrac{z\sin(\beta)}{z^2-2z\cos(\beta)+1}$
$\cos(\beta n)$	$\dfrac{z^2-z\cos(\beta)}{z^2-2z\cos(\beta)+1}$
$e^{-\alpha n}\sin(\beta n)$	$\dfrac{ze^{-\alpha}\sin(\beta)}{z^2-2ze^{-\alpha}\cos(\beta)+e^{-2\alpha}}$
$e^{-\alpha n}\cos(\beta n)$	$\dfrac{z^2-ze^{-\alpha}\cos(\beta)}{z^2-2ze^{-\alpha}\cos(\beta)+e^{-2\alpha}}$

Fourier series and transforms

Fourier series – periodic and continuous in time, discrete in frequency	
$x(t) = \sum_{n=-\infty}^{+\infty} X_n \exp\left(jn\omega_0 t\right)$	$X_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \exp(-jn\omega_0 t)\, dt$
Fourier transform – continuous in time, continuous in frequency	
$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \exp\left(j\omega t\right) d\omega$	$X(\omega) = \int_{-\infty}^{\infty} x(t) \exp(-j\omega t)\, dt$
Discrete-time Fourier transform – discrete in time, continuous and periodic in frequency	
$x(n\Delta t) = \frac{\Delta t}{2\pi} \int_{-\pi/\Delta t}^{\pi/\Delta t} X(\omega) \exp\left(jn\Delta t\omega\right) d\omega$	$X(\omega) = \sum_{n=-\infty}^{\infty} x(n\Delta t) \exp(-j\omega n\Delta t)$
Discrete Fourier transform – discrete and periodic in time and in frequency	
$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \exp\left(\frac{jnk2\pi}{N}\right)$	$X(k) = \sum_{n=0}^{N-1} x(n) \exp\left(\frac{-jnk2\pi}{N}\right)$