MathEng2223

March 28, 2025

Mathematical Methods for Engineers (MathEng)

EXAM

December 2023

`Duration: 2 hrs, all documents and calculators permitted`
`ATTEMPT ALL QUESTIONS - ANSWER IN ENGLISH`

1 Using Euler's identity (or any other appropriate method), write down an expression for the complex Fourier series of the signal $x(t)$:

$$x(t) = 3\cos(5t) + 4\sin(10t)$$

[5 marks]

To find the complex Fourier series of $x(t) = 3\cos(5t) + 4\sin(10t)$,

we use Euler's identity: $\cos(\omega t) = \frac{e^{j\omega t}+e^{-j\omega t}}{2}, \quad \sin(\omega t) = \frac{e^{j\omega t}-e^{-j\omega t}}{2j}$.

Step 1: Rewrite $\cos(5t)$ and $\sin(10t)$ using Euler's identity

- $3\cos(5t) \to 3(\frac{e^{j5t}+e^{-j5t}}{2}) = \frac{3}{2}e^{j5t} + \frac{3}{2}e^{-j5t}$
- $4\sin(10t) \to 4(\frac{e^{j10t}-e^{-j10t}}{2j}) = \frac{4}{2j}(e^{j10t} - e^{-j10t})$

  Recall:$\frac{1}{j} = \frac{j}{j^2} = \frac{j}{-1} = -j$

$$\begin{aligned}\frac{4}{2j}(e^{j10t} - e^{-j10t}) &= \frac{2}{j}(e^{j10t} - e^{-j10t}) = \frac{2}{j}e^{j10t} - \frac{2}{j}e^{-j10t} \\ &= \frac{2j}{j^2}e^{j10t} - \frac{2j}{j^2}e^{-j10t} = \frac{2j}{-1}e^{j10t} - \frac{2j}{-1}e^{-j10t} = -2je^{j10t} + 2je^{-j10t}.\end{aligned}$$

Thus, $x(t) = \frac{3}{2}e^{j5t} + \frac{3}{2}e^{-j5t} - 2je^{j10t} + 2je^{-j10t}$.

Step 2: Group the terms The complex Fourier series representation of $x(t)$ is: $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$, where c_k are the complex Fourier coefficients.

Here, $x(t)$ has terms at frequencies ± 5 and ± 10. The coefficients c_k are:

- At $k = 5$: $c_5 = \frac{3}{2}$,
- At $k = -5$: $c_{-5} = \frac{3}{2}$,
- At $k = 10$: $c_{10} = -2j$,
- At $k = -10$: $c_{-10} = 2j$,
- All other $c_k = 0$.

Final Answer: The complex Fourier series of $x(t)$ is:

$$\boxed{x(t) = \frac{3}{2}e^{j5t} + \frac{3}{2}e^{-j5t} - 2je^{j10t} + 2je^{-j10t}}$$

2 Develop an expression for the Fourier Transform of the signal $x(t)$ illustrated in `Figure Q2` below:

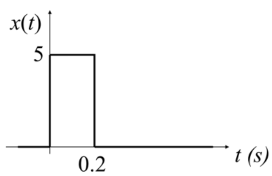


Figure Q2

[6 marks]

To develop the Fourier Transform $X(f)$ of the signal $x(t)$ illustrated in the figure, we follow the same steps for a rectangular pulse.

Step 1: Signal Description The signal $x(t)$ is defined as:

$$x(t) = \begin{cases} 5, & 0 \le t \le 0.2, \\ 0, & \text{otherwise.} \end{cases}$$

Step 2: Fourier Transform Definition The Fourier Transform is given by: $X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$.

Since $x(t)$ is nonzero only in the interval $[0, 0.2]$, the limits of integration reduce to $[0, 0.2]$: $X(f) = \int_0^{0.2} 5e^{-j2\pi ft}dt$.

Step 3: Evaluate the Integral Factor out the constant 5: $X(f) = 5\int_0^{0.2} e^{-j2\pi ft}dt$.

The integral of $e^{-j2\pi ft}$ is: $\int e^{-j2\pi ft}dt = \frac{e^{-j2\pi ft}}{-j2\pi f}$.

Apply the limits of integration: $X(f) = 5\left[\frac{e^{-j2\pi ft}}{-j2\pi f}\right]_0^{0.2}$.

Substitute the limits: $X(f) = 5 \cdot \frac{1}{-j2\pi f}\left(e^{-j2\pi f(0.2)} - e^0\right)$.

Simplify: $X(f) = \frac{5}{-j2\pi f}\left(e^{-j0.4\pi f} - 1\right)$.

Step 4: Simplify Further

Using the property $e^{-j\theta} - 1 = -2j\sin\left(\frac{\theta}{2}\right)e^{-j\frac{\theta}{2}}$ which is derived as follows:

1. **Rewrite $e^{-j\theta} - 1$:** Expand using Euler's formula: $e^{-j\theta} - 1 = \cos(\theta) - j\sin(\theta) - 1 = (\cos(\theta) - 1) - j\sin(\theta)$

2. **Factorize Trigonometric Terms:** Use the half-angle identities:
   - $\cos(\theta) = 1 - 2\sin^2\left(\frac{\theta}{2}\right) \implies \cos(\theta) - 1 = -2\sin^2\left(\frac{\theta}{2}\right)$
   - $\sin(\theta) = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$.

   Substituting these: $e^{-j\theta} - 1 = -2\sin^2\left(\frac{\theta}{2}\right) - j\cdot 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$.

3. **Factor Out Common Terms:**

- **Identify Common Factor:**

  Both terms contain $-2j\sin\left(\frac{\theta}{2}\right)$ as a common factor: 1. $-2\sin^2\left(\frac{\theta}{2}\right)$: - This can be written as $-2j\sin\left(\frac{\theta}{2}\right) \cdot \frac{\sin\left(\frac{\theta}{2}\right)}{j}$. 2. $-j \cdot 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$: - This is already proportional to $-2j\sin\left(\frac{\theta}{2}\right)$.

- **Factorization:**

  Factor $-2j\sin\left(\frac{\theta}{2}\right)$ out of the entire expression: $e^{-j\theta} - 1 = -2j\sin\left(\frac{\theta}{2}\right) \cdot \left(\frac{\sin\left(\frac{\theta}{2}\right)}{j} + \cos\left(\frac{\theta}{2}\right)\right)$.

  Simplify the term : $\frac{\sin\left(\frac{\theta}{2}\right)}{j} = -j\sin\left(\frac{\theta}{2}\right)$. Thus: $e^{-j\theta} - 1 = -2j\sin\left(\frac{\theta}{2}\right) \cdot \left(\cos\left(\frac{\theta}{2}\right) - j\sin\left(\frac{\theta}{2}\right)\right)$.

- **Recognize the Exponential Form:** The term $\cos\left(\frac{\theta}{2}\right) - j\sin\left(\frac{\theta}{2}\right)$ is equivalent to $e^{-j\frac{\theta}{2}}$, using Euler's formula.

5. **Simplify:** Recognize the term in parentheses as $e^{-j\frac{\theta}{2}}$: $e^{-j\theta} - 1 = -2j\sin\left(\frac{\theta}{2}\right)e^{-j\frac{\theta}{2}}$.

This compactly combines the amplitude term $-2j\sin\left(\frac{\theta}{2}\right)$ and the phase shift $e^{-j\frac{\theta}{2}}$.

rewrite $X(f)$:

- $X(f) = \frac{5}{-j2\pi f} \cdot -2j\sin(0.2\pi f)e^{-j0.2\pi f}$.

Cancel $-j$ and simplify: $X(f) = \frac{5\cdot 2\sin(0.2\pi f)}{2\pi f}e^{-j0.2\pi f}$.

Finally: $X(f) = \frac{5\sin(0.2\pi f)}{\pi f}e^{-j0.2\pi f}$.

Final Expression

$$\begin{aligned} X(f) &= \frac{5\sin(0.2\pi f)}{\pi f} \cdot e^{-j0.2\pi f} \\ &= 5 \cdot \mathrm{sinc}(0.2f) \cdot e^{-j0.2\pi f} \end{aligned}$$

where the **sinc function** is defined as: $\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$.

Interpretation

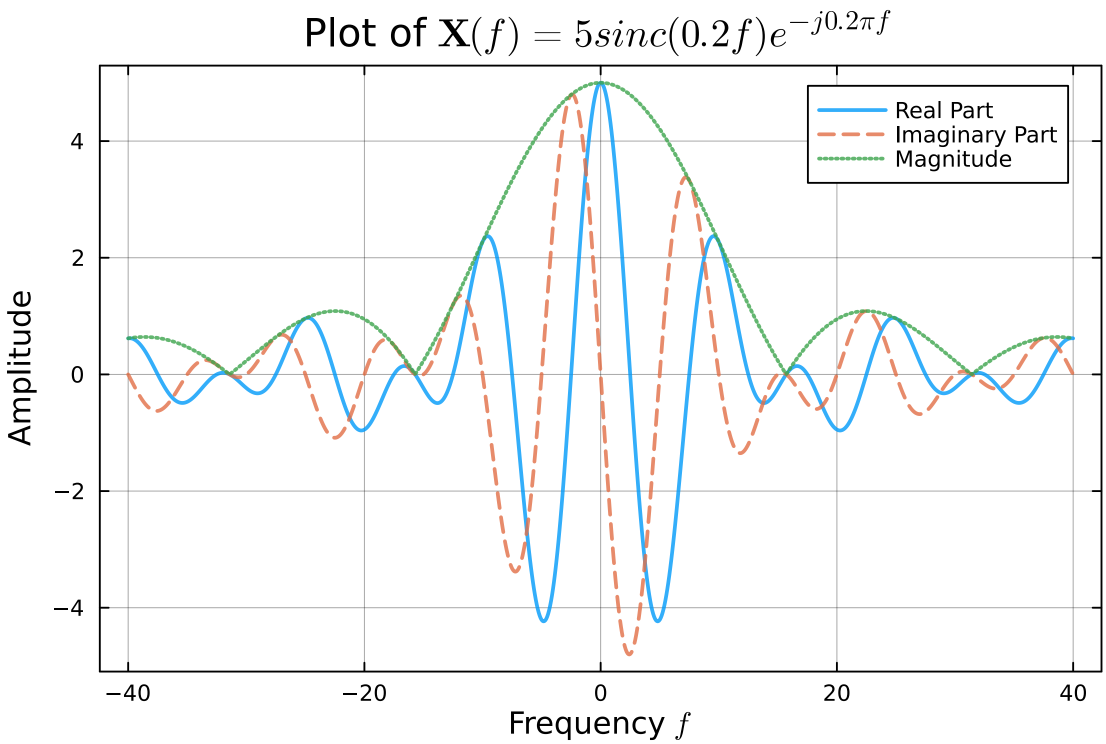
- $\frac{\sin(0.2\pi f)}{\pi f}$**:** This is the sinc function, representing the frequency-domain shape of the rectangular pulse.
- $e^{-j0.2\pi f}$**:** This is a phase shift due to the non-centered nature of the pulse (starting at $t = 0$).

[1]:
```julia
using FFTW, LinearAlgebra, Plots, LaTeXStrings
```

[2]:
```julia
include("../modules/operations.jl");
```

[3]:
```julia
# Define the unscaled sinc function
sinc_unscaled(x::Real) = x == 0 ? 1.0 : sin(pi * x) / (pi * x)

# Define the polymorphic sinc function with a normalization option
sinc(x::Real; normalized::Bool = true) = normalized ? sinc_unscaled(x / π) : sinc_unscaled(x)

# Frequency range
f = range(-40, 40, length=1000)

# Function components
 = 5 .* sinc.(0.2 .* f)  # Amplitude of the signal
  = .^ (-j .* 0.2π .* f)  # Phase shift
 =  .*     # Combined function

# Plot with title, labels, and semi-transparent grid
plot(f, real.()
    , label="Real Part", linestyle=:solid, linewidth=2, alpha=0.8, size = (600,400)
    , xlabel="Frequency " * L"f", ylabel="Amplitude"
    , title="Plot of " * L"\mathbf{X}(f) = 5 sinc(0.2 f) e^{-j 0.2 π f}"
    , grid=true, gridalpha=0.2  # Enable grid and set transparency
    , framestyle=:box
)

# Overlay additional lines
plot!(f, imag.(), label="Imaginary Part", linestyle=:dash, linewidth=2, alpha=0.8)
plot!(f, abs.(), label="Magnitude", linestyle=:dot, linewidth=2, alpha=0.8)
```

[3]:



3 A linear, time-invariant system has the following transfer function:

$H(s) = \frac{10(s+100)}{s^2+2s+100}$

(a) Derive an expression for $H(s)$ in the usual, normal form.

(b) Determine the frequency-invariant gain K and the position of any poles and zeros.

(c) Sketch a Bode plot of the magnitude-frequency response.

(d) Sketch a Bode plot of the phase-frequency response.

[8 marks]

(a) Derive an expression for $H(s)$ in the usual, normal form. To derive the transfer function $H(s)$ in the usual, **normal form**, we factorize the numerator and denominator in terms of their natural frequencies and damping ratios.

The given transfer function is: $H(s) = \frac{10(s+100)}{s^2+2s+100}$.

Step 1: Denominator Normal Form The denominator is: $s^2 + 2s + 100$.

This matches the general form of a second-order system: $s^2 + 2\zeta\omega_n s + \omega_n^2$, where ζ is the damping ratio and ω_n is the natural frequency.

Here: $\omega_n^2 = 100 \quad \Rightarrow \quad \omega_n = \sqrt{100} = 10$, and: $2\zeta\omega_n = 2 \quad \Rightarrow \quad \zeta = \frac{2}{2\omega_n} = \frac{2}{20} = 0.1$.

Thus, the denominator becomes: $s^2 + 2s + 100 = (s^2 + 2\zeta\omega_n s + \omega_n^2) = s^2 + 2(0.1)(10)s + 10^2$.

Step 2: Numerator Normal Form The numerator is: $10(s + 100)$.

Factor out 100 to normalize: $10(s + 100) = 10 \cdot 100 \left(\frac{s}{100} + 1\right) = 1000 \left(\frac{s}{100} + 1\right)$.

Step 3: Rewrite in Normal Form Substitute the factored numerator and denominator into $H(s)$:

$$H(s) = \frac{1000 \left(\frac{s}{100} + 1\right)}{s^2 + 2(0.1)(10)s + 10^2}.$$

Simplify:

$$H(s) = \frac{1000}{100} \cdot \frac{\left(\frac{s}{100} + 1\right)}{\frac{s^2}{100} + \frac{2(0.1)(10)s}{100} + \frac{10^2}{100}}.$$

After normalization:

$$H(s) = \frac{10 \left(\frac{s}{100} + 1\right)}{\frac{s^2}{100} + \frac{2s}{10} + 1}.$$

Alternatively:

$$H(s) = \frac{10 \left(\frac{s}{100} + 1\right)}{\frac{s^2}{100} + \frac{s}{5} + 1}.$$

This is the normalized form of $H(s)$.

(b): Determine the Frequency-Invariant Gain K and the Positions of Poles and Zeros

1. Transfer Function The given transfer function is: $H(s) = \frac{10(s+100)}{s^2+2s+100}$.

2. Frequency-Invariant Gain K The frequency-invariant gain is the gain of the system as $s \to 0$. This is determined by evaluating the transfer function at $s = 0$:

$$K = H(0) = \frac{10(0 + 100)}{(0)^2 + 2(0) + 100}.$$

Simplify: $K = \frac{10 \cdot 100}{100} = 10$.

Thus, the frequency-invariant gain is: $K = 10$.

3. Poles The poles are the roots of the denominator $s^2 + 2s + 100 = 0$: $s^2 + 2s + 100 = 0$.

Solve using the quadratic formula: $s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where $a = 1$, $b = 2$, and $c = 100$. Substituting: $s = \frac{-2 \pm \sqrt{2^2 - 4(1)(100)}}{2(1)} = \frac{-2 \pm \sqrt{4 - 400}}{2}$.

Simplify: $s = \frac{-2 \pm \sqrt{-396}}{2}$.

The roots are: $s = -1 \pm j\sqrt{99}$.

Thus, the poles are: $s = -1 + j\sqrt{99}, \quad s = -1 - j\sqrt{99}$.

4. Zeros The zero is the root of the numerator $10(s + 100) = 0$: $s + 100 = 0 \quad \Rightarrow \quad s = -100$.

Thus, there is one zero at: $s = -100$.

Final Results:

- **Frequency-Invariant Gain K:** $K = 10$.
- **Poles:** $s = -1 + j\sqrt{99}, \quad s = -1 - j\sqrt{99}$.
- **Zero:** $s = -100$.

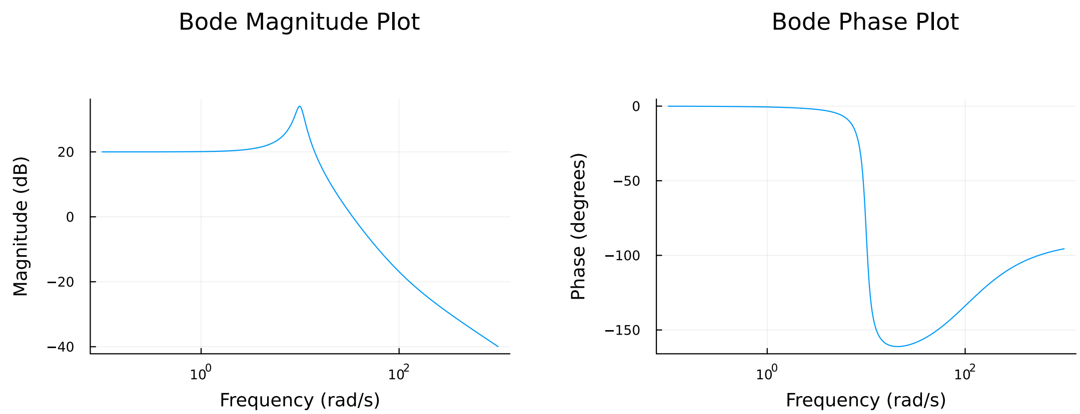
[4]:
```julia
# Frequency range (logarithmic scale)
ω = 10 .^ range(-1, 3, length=500)  # Frequencies from 0.1 to 1000 (log scale)

# Define the transfer function H(s)
function H(ω)
    numerator = 10 .* (j .* ω .+ 100)  # Element-wise addition
    denominator = (j .* ω).^2 .+ 2 .* (j .* ω) .+ 100  # Element-wise operations
    return numerator ./ denominator  # Element-wise division
end
```

[4]: H (generic function with 1 method)

[5]:
```julia
using Plots
using Printf
using Measures

# Magnitude response in dB
magnitude_dB = 20 .* log10.(abs.(H.(ω)))  # Broadcasting applied to H, abs, and
 ↪log10

# Plot the Bode magnitude plot
p1 = plot(ω, magnitude_dB
    , xscale=:log10
    , xlabel="Frequency (rad/s)", ylabel="Magnitude (dB)"
    , title="Bode Magnitude Plot", legend=false, grid=true
    , margin = 5mm
)
```

```
# Phase response in degrees
phase_deg = angle.(H.(ω)) .* (180 / π)  # Convert phase from radians to degrees

# Plot the Bode phase plot
p2 = plot(ω, phase_deg
     ,xscale=:log10
     ,xlabel="Frequency (rad/s)", ylabel="Phase (degrees)"
     ,title="Bode Phase Plot"
     ,legend=false,grid=true
     ,left_margin=10mm, right_margin=10mm, top_margin=15mm, bottom_margin=15mm
)

plot(p1, p2, layout = (1, 2), size = (1000, 400))
```

[5]:



4 Sketch magnitude and phase responses for a sampled data system with a pair of complex conjugate zeros and two poles at the origin.

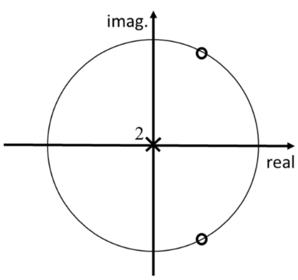


Figure Q4

[4 marks]

The question asks for magnitude and phase response plots for a sampled data system with:

1. **Two complex conjugate zeros** located on the unit circle.
2. **Two poles at the origin**.

This indicates a discrete-time system, and we can describe the transfer function in the z-domain.

Step 1: Transfer Function Representation From the given information:

- **Zeros:** The system has complex conjugate zeros located on the unit circle at $z = e^{j\theta}$ and $z = e^{-j\theta}$. For simplicity, let the zeros be at $z = e^{j\pi/4}$ and $z = e^{-j\pi/4}$.

- **Poles:** Two poles are at the origin ($z = 0$).

The transfer function is:

$$H(z) = K\frac{(z - e^{j\pi/4})(z - e^{-j\pi/4})}{z^2},$$

where K is the gain.

Simplify the numerator using the property of complex conjugates: $(z - e^{j\pi/4})(z - e^{-j\pi/4}) = z^2 - 2z\cos(\pi/4) + 1 = z^2 - \sqrt{2}z + 1$.

Thus, the transfer function becomes: $H(z) = K\frac{z^2-\sqrt{2}z+1}{z^2}$.

Step 2: Frequency Response Substitute $z = e^{j\omega}$ (discrete-time frequency variable):

$H(e^{j\omega}) = K\frac{e^{2j\omega}-\sqrt{2}e^{j\omega}+1}{e^{2j\omega}}$.

Simplify: $H(e^{j\omega}) = K\left(1 - \sqrt{2}e^{-j\omega} + e^{-2j\omega}\right)$.

The magnitude response is: $|H(e^{j\omega})| = |K| \cdot \left|1 - \sqrt{2}e^{-j\omega} + e^{-2j\omega}\right|$.

The phase response is: $\text{Phase}(H(e^{j\omega})) = \arg\left(1 - \sqrt{2}e^{-j\omega} + e^{-2j\omega}\right)$.

Step 3: Magnitude and Phase Plot Characteristics

1. **Magnitude Response:**
   - Peaks occur near the frequencies corresponding to the zeros (here, $\pi/4$ and $-\pi/4$).
   - High attenuation occurs near the poles (low frequencies), since the poles are at the origin.
2. **Phase Response:**
   - The phase changes rapidly near the frequencies of the zeros.
   - At low frequencies, the phase starts at $0°$ due to the dominance of the poles.

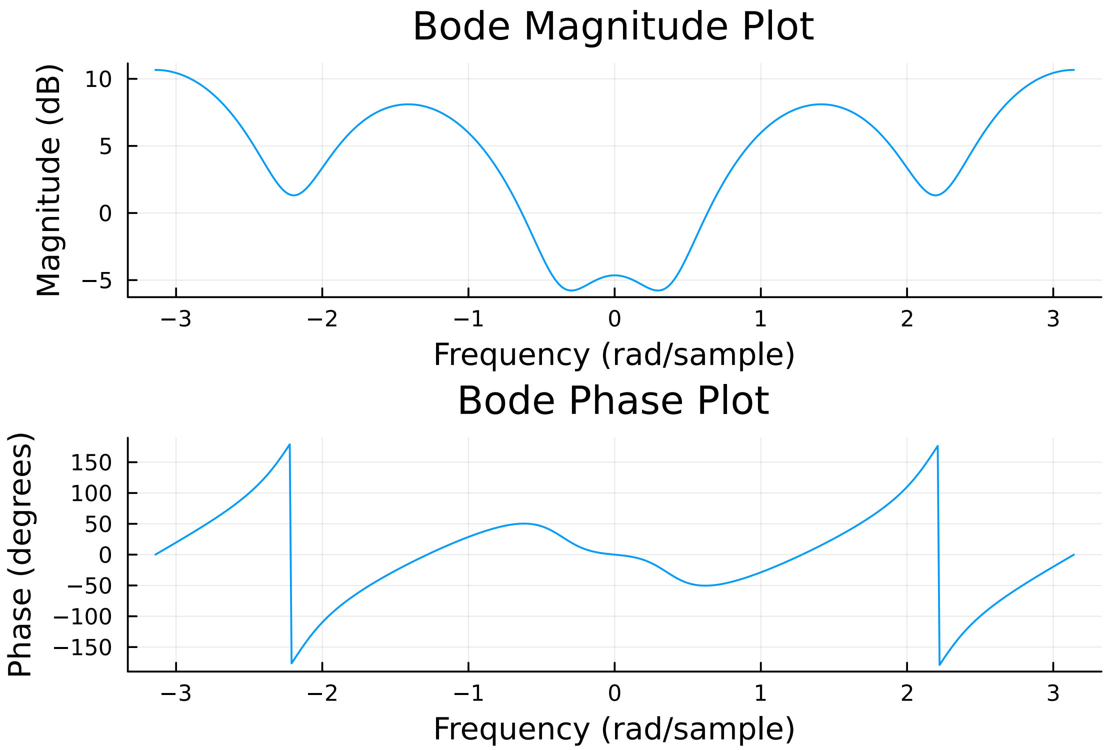
Step 4: Sketch the Plots Here's how the plots would look:

1. **Magnitude Plot:**
   - Start at low values near $\omega = 0$ (due to the poles).
   - Peaks occur near $\omega = \pi/4$ and $\omega = -\pi/4$ (locations of the zeros).
   - Symmetrical around $\omega = 0$.
2. **Phase Plot:**
   - At $\omega = 0$, the phase is $0°$.
   - The phase decreases sharply near $\omega = \pi/4$ and $\omega = -\pi/4$.

[6]:
```
using Plots
using Printf
```

```julia
# Define the transfer function H(e^jw) for the discrete-time system
function H(ω)
    # Complex exponential terms
    z = ^(-j * ω)  # e^(jω)
    z = ^(-2j * ω)  # e^(j2ω)

    # Transfer function
    numerator = z^2 - (2) * z + 1  # (z^2 - sqrt(2)z + 1)
    denominator = z^2  # (z^2)
    return numerator / denominator
end

# Frequency range (from -π to π for discrete systems)
ω = range(-π, π, length=500)

# Magnitude response
magnitude = abs.(H.(ω))
magnitude_dB = 20 .* log10.(magnitude)  # Convert to dB

# Phase response
phase = angle.(H.(ω)) .* (180 / π)  # Convert radians to degrees

# Plot the magnitude response
p1 = plot(
    ω, magnitude_dB,
    xlabel="Frequency (rad/sample)",
    ylabel="Magnitude (dB)",
    title="Bode Magnitude Plot",
    legend=false,
    grid=true
)

# Plot the phase response
p2 = plot(
    ω, phase,
    xlabel="Frequency (rad/sample)",
    ylabel="Phase (degrees)",
    title="Bode Phase Plot",
    legend=false,
    grid=true
)

plot(p1,p2,layout = (2,1))
```

[6]:



5 A random variable X is uniformly distributed between $x = 0$ and $x = 1$. Via any appropriate method, determine the expected value $E[Y]$ of $Y = exp(X)$.

[4 marks]

Given $Y = \exp(X)$ and $X \sim U(0, 1)$,

1. Expected Value Formula The expected value of a random variable Y is given by:

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy.$$

Since X is uniformly distributed, its probability density function (PDF) is:

$$f_X(x) = \begin{cases} 1, & 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

For $Y = \exp(X)$, the expected value becomes: $E[Y] = \int_0^1 \exp(x) f_X(x)\, dx$.

Because $f_X(x) = 1 \text{for} 0 \leq x \leq 1$, this simplifies to: $E[Y] = \int_0^1 \exp(x)\, dx$.

2. Solve the Integral The integral of $\exp(x)$ is: $\int \exp(x)\,dx = \exp(x) + C$.

Now, evaluate the definite integral: $\int_0^1 \exp(x)\,dx = [\exp(x)]_0^1 = \exp(1) - \exp(0)$.

Simplify: $\int_0^1 \exp(x)\,dx = e - 1$.

3. Final Answer The expected value is: $\boxed{E[Y] = e - 1}$

6 Identify the pivots and free variables of the following two matrices A and B. Following the method which we studied in class, find the special solution corresponding to each free variable and, by combining the special solutions, describe every solution to $Ax = 0$ and $Bx = 0$.

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}.$$

[7 marks]

To find the pivots, free variables, and solutions to $Ax = 0$ and $Bx = 0$, we follow these steps:

Step 1: Row Reduction of Matrix A Matrix A is:

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}.$$

Row Reduce A to Row Echelon Form

1. Subtract the first row from the second:

$$\begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}.$$

2. Subtract the third row from the second:

$$\begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Identify Pivots and Free Variables

- **Pivot columns**: The first column (x_1) and third column (x_3).
- **Free variables**: The second column (x_2), fourth column (x_4), and fifth column (x_5).

Step 2: Solve $Ax = 0$ (Homogeneous System) The system is represented as:

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + 4x_4 + 6x_5 &= 0, \\ x_3 + 2x_4 + 3x_5 &= 0. \end{aligned}$$

Back-Substitute:

1. From the second equation: $x_3 = -2x_4 - 3x_5$.
2. Substitute x_3 into the first equation: $x_1 + 2x_2 + 2(-2x_4 - 3x_5) + 4x_4 + 6x_5 = 0$, $x_1 + 2x_2 - 4x_4 - 6x_5 + 4x_4 + 6x_5 = 0$, $x_1 + 2x_2 = 0 \implies x_1 = -2x_2$.

Write the General Solution for $Ax = 0$:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

The **special solutions** correspond to the free variables x_2, x_4, and x_5.

Step 3: Row Reduction of Matrix B Matrix B is:

$$B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}.$$

Row Reduce B to Row Echelon Form

1. Divide the first row by 2:
$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}.$$
2. Divide the second row by 4:
$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 8 & 8 \end{bmatrix}.$$
3. Subtract 8 times the second row from the third:
$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Identify Pivots and Free Variables

- **Pivot columns**: The first column (x_1) and the second column (x_2).
- **Free variable**: The third column (x_3).

Step 4: Solve $Bx = 0$ (Homogeneous System) The system is represented as:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 0, \\ x_2 + x_3 &= 0. \end{aligned}$$

Back-Substitute:

1. From the second equation: $x_2 = -x_3$.
2. Substitute x_2 into the first equation: $x_1 + 2(-x_3) + x_3 = 0$, $x_1 - x_3 = 0 \implies x_1 = x_3$.

Write the General Solution for $Bx = 0$:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

The **special solution** corresponds to the free variable x_3.

Final Answers:

1. For $Ax = 0$:
$$x = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$
2. For $Bx = 0$:
$$x = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

7 For a projection matrix $P = A(A^TA)^{-1}A^T$, show that$P^2 = P$and then explain, in terms of the column space of$P$, why projections $P_b$and$P(P_b)$ give identical results.

[5 marks]

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1. Show that $P^2 = P$ The projection matrix P is defined as: $P = A(A^TA)^{-1}A^T$, where A is a matrix with linearly independent columns.

Compute P^2: We want to show: $P^2 = P$.

Start with P^2: $P^2 = P \cdot P = \left(A(A^TA)^{-1}A^T\right) \cdot \left(A(A^TA)^{-1}A^T\right)$.

Expand the multiplication: $P^2 = A(A^TA)^{-1}A^TA(A^TA)^{-1}A^T$.

Since $A^TA$is invertible,$A^TA(A^TA)^{-1} = I$ (identity matrix). So: $P^2 = A(A^TA)^{-1}(I)A^T = A(A^TA)^{-1}A^T$.

This simplifies to: $P^2 = P$.

2. Projections Pb and $P(Pb)$ Give Identical Results

Interpretation of P: The projection matrix P projects any vector$b$ onto the **column space** of$A$, denoted as $\text{Col}(A)$.

Explain Pb: $Pb = P\mathbf{b} = A(A^TA)^{-1}A^T\mathbf{b}$.

This gives the projection of $\mathbf{b}$ onto $\text{Col}(A)$.

Explain $P(Pb)$: $P(Pb) = P(P\mathbf{b})$.

Substitute Pb into $P(Pb)$: $P(Pb) = P \cdot P\mathbf{b}$.

Since we showed that $P^2 = P$, this becomes: $P(Pb) = P\mathbf{b}$.

Why Are Pb and $P(Pb)$ Identical?

- $Pb = P\mathbf{b}$ is already the projection of $\mathbf{b}$ onto $\text{Col}(A)$.
- Applying P again to Pb does not change it, because projecting a vector already in the subspace $\text{Col}(A)$ onto the same subspace leaves it unchanged.
- Hence: $P(Pb) = Pb$.

3. Column Space Perspective In terms of the column space of P: 1. The column space of P (and thus Pb) is the **same as Col**(A). 2. Applying P to Pb projects Pb onto $\text{Col}(A)$, but since $Pb \in \text{Col}(A)$, the result is unchanged.

Thus, projections Pb and $P(Pb)$ are identical because projecting a vector already in the column space does nothing.

Conclusion

- **Projection matrix property**: $P^2 = P$.
- **Projections**: Pb and $P(Pb)$ are identical because Pb lies in the column space, and re-projecting it does not alter it.
- **Idempotence**: P an idempotent matrix, which is a key characteristic of projection matrices.