MathEng1920

December 9, 2024

1 Exam

1.1 Mathematical Methods for Engineers (MathEng)

1.2 EXAM

11th February 2020

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Duration: 2 hrs, calculators permitted, no documents
This exam paper contains 7 questions and 40 marks.
ATTEMPT ALL QUESTIONS - ANSWER IN ENGLISH
```

[1]:
```
using FFTW, LinearAlgebra
```

[2]:
```
include("modules/operations.jl");
```

[3]:
```
using Plots
using LaTeXStrings
```

1.2.1 1. Via any appropriate method, and by justifying your response, write an expression for the Fourier transform of $x(t) = \cos w_0 t$.

[5 marks]

The Fourier transform of $x(t) = \cos(\omega_0 t)$ can be derived using the definition of the Fourier transform:

$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$

1.2.2 Step 1: Express $\cos(\omega_0 t)$ in terms of exponentials

We can use Euler's formula to rewrite $\cos(\omega_0 t)$ as a sum of complex exponentials:

$\cos(\omega_0 t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$

1.2.3 Step 2: Apply the Fourier transform to each exponential term

We now take the Fourier transform of each term separately. The Fourier transform of $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$ are well known:

- The Fourier transform of $e^{j\omega_0 t}$ is a delta function shifted to $f = \frac{\omega_0}{2\pi}$:

$\mathcal{F}\{e^{j\omega_0 t}\} = \delta\left(f - \frac{\omega_0}{2\pi}\right)$

- Similarly, the Fourier transform of $e^{-j\omega_0 t}$ is a delta function shifted to $f = -\frac{\omega_0}{2\pi}$:

$\mathcal{F}\{e^{-j\omega_0 t}\} = \delta\left(f + \frac{\omega_0}{2\pi}\right)$

1.2.4 Step 3: Combine the results

Thus, the Fourier transform of $\cos(\omega_0 t)$ becomes:

$$\boxed{X(f) = \frac{1}{2}\left[\delta\left(f - \frac{\omega_0}{2\pi}\right) + \delta\left(f + \frac{\omega_0}{2\pi}\right)\right]}$$

This is the final expression for the Fourier transform of $\cos(\omega_0 t)$. It consists of two delta functions located at $f = \pm\frac{\omega_0}{2\pi}$, indicating that the cosine function contains frequency components at $\pm\frac{\omega_0}{2\pi}$.

1.2.5 Justification

The result makes sense because the cosine function is a combination of two complex exponentials, each contributing a distinct frequency. The delta functions reflect these frequency components in the frequency domain, with equal magnitude contributions at positive and negative frequencies.

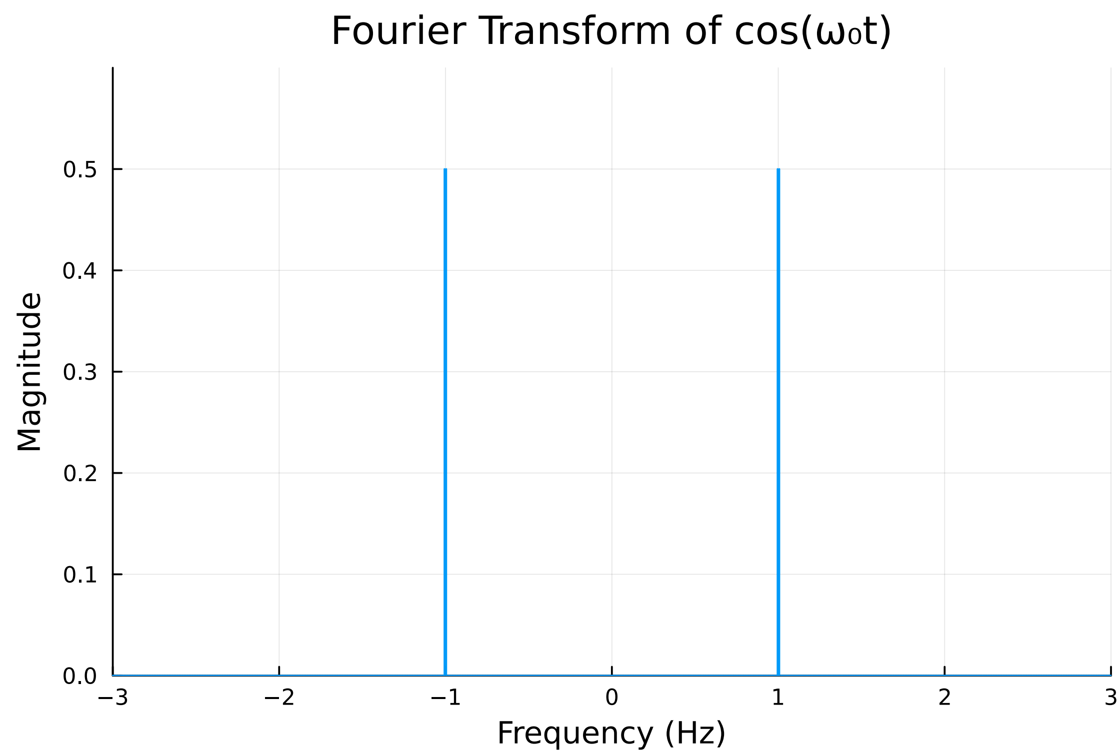
[4]:
```
# Parameters
threshold = 0.01 # The threshold for identifying delta spikes
magnitude = 0.5  # Spike's magnitude
  = 2           # example frequency in rad/s
f =   / 2      # corresponding frequency in Hz

# Frequency range
frequencies = range(-5f , stop=5f , length=1000)
  = zeros(length(frequencies))

# Delta functions at ±f
 [findall(abs.(frequencies .- f ) .< threshold)] .= magnitude
 [findall(abs.(frequencies .+ f ) .< threshold)] .= magnitude

# Plot the Fourier transform with delta spikes
plot(frequencies,
    , st=:stem, label="Fourier Transform"
    , xlabel="Frequency (Hz)", ylabel="Magnitude"
    , title="Fourier Transform of cos( t)"
    , grid = true, legend = false
    , xlims= (-3f , 3f ), ylims = (0, 0.6)
)
```

[4]:



Here is the plot of the Fourier transform of $\cos(\omega_0 t)$. The two delta spikes are located at $\pm f_0$, showing the frequency components at $f_0 = \frac{\omega_0}{2\pi}$. Each spike has a magnitude of 0.5, reflecting the equal contributions of the positive and negative frequency components in the cosine function.

1.2.6 2. Sketch the s-plane pole-zero plot for a system with the following transfer function and then sketch the amplitude frequency response.

$$H(s) = \frac{s^2 + 9}{(s^2 + 0.6s + 4.09)}$$

[5marks]

[5]:
```
# Define poles and zeros
zrs = [j * 3, -j * 3]       # Zeros at ±j3
poles = [-0.3 + 2j, -0.3 - 2j]  # Poles at -0.3 ± j2

# Pole-zero plot
p1 = plot( (zrs),  (zrs)
    , seriestype = :scatter, label="Zeros", legend=:topright
    , xlims = (-5, 5), ylims = (-5, 5)

)
plot!( (poles),  (poles)
    , seriestype = :scatter, label="Poles", marker=:x
```

```
    , xlabel = L"\scr{Re}(s)", ylabel = L"\scr{Im}(s)" , title = "Pole-Zero␣
↪Plot"
)

hline!([0], line=:solid, color=:black, linewidth=0.5)
vline!([0], line=:solid, color=:black, linewidth=0.5)

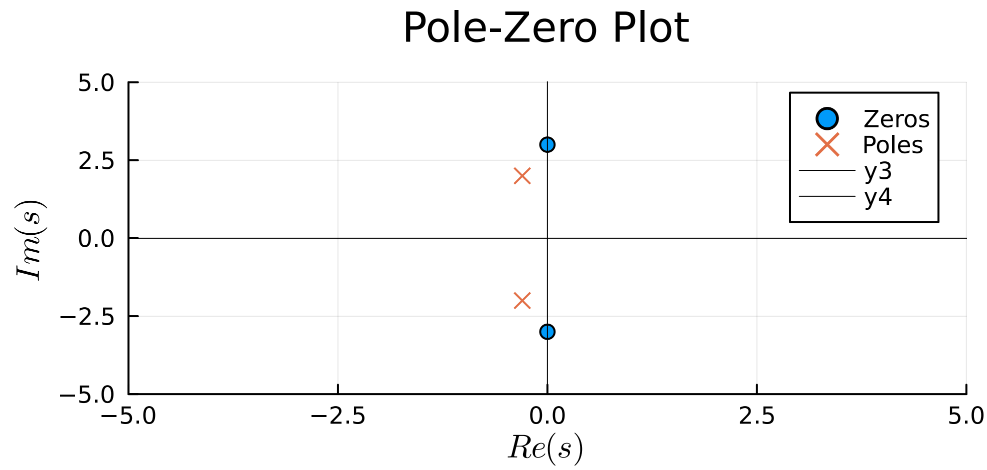
# Define the transfer function H(s)
H(s) = (s^2 + 9) / (s^2 + 0.6 * s + 4.09)

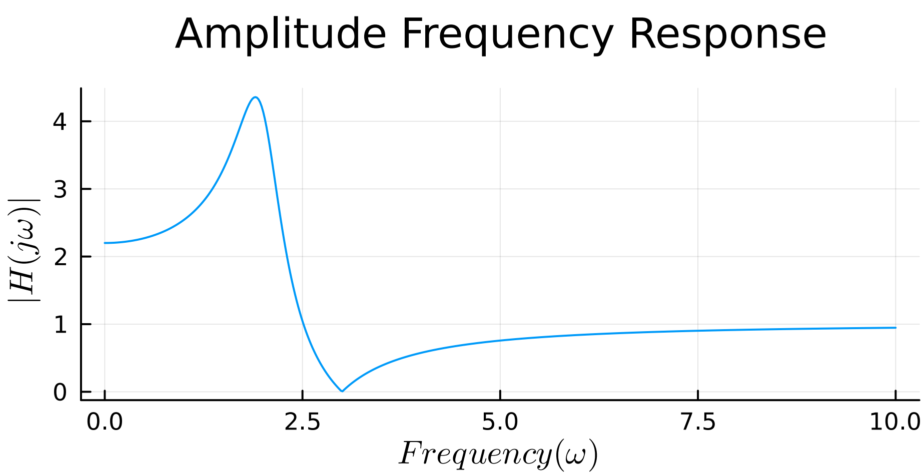
# Frequency range for the amplitude response plot
 = range(0, stop=10, length=500)
H_vals = abs.(H.(j .*  ))

# Plot the amplitude frequency response
p2 = plot( , H_vals
    , xlabel=L"Frequency (\omega)", ylabel=L"|H(j\omega)|"
    , title="Amplitude Frequency Response"
    , legend=false
)

plot(p1, p2
    , layout = (2,1)
    , size = (500,500)
)
```

[5]:


Pole-Zero Plot
5.0
2.5
0.0
−2.5
−5.0
−5.0
−2.5
0.0
2.5
5.0
$Im(s)$
$Re(s)$
Zeros
Poles
y3
y4


Amplitude Frequency Response
4
3
2
1
0
0.0
2.5
5.0
7.5
10.0
$|H(j\omega)|$
$Frequency(\omega)$

```
[ ]:
```