Example

Two random variables X and Y have the joint pdf

$$f_{XY}(x,y) = \begin{cases} Ae^{-(2x+y)}, & x, y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

where A is a constant. Determine A and find the two marginal pdf's.

We evaluate A from $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$

Since $\int_0^{\infty}\int_0^{\infty} e^{-(2x+y)}\,dx\,dy = \frac{1}{2}$, $A = 2$

We find the marginal pdfs as follows

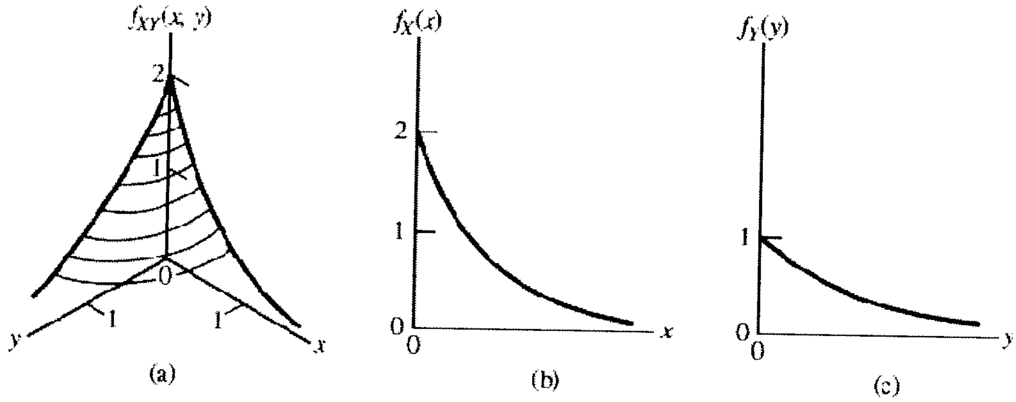
$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \begin{cases} \int_0^{\infty} 2e^{-(2x+y)}\,dy, & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$= \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \begin{cases} e^{-y} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

We see that X & Y are statistically independent since

$$f_{XY}(x,y) = f_X(x)\, f_Y(y)$$



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We find the joint cdf by integrating the joint pdf in both variables:

$$F_{XY}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{XY}(x', y')\,dx'dy' \quad \Big\}\ \text{dummy variables } x', y'$$

$$= \begin{cases} (1-e^{-2x})(1-e^{-y}), & x,y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Note that $F_{XY}(-\infty,-\infty) = 0$ & $F_{XY}(+\infty,+\infty) = 1$ as expected

We can obtain $F_X(x)$ & $F_Y(y)$ from $F_{XY}(x,y)$:

$$F_X(x) = F_{XY}(x, \infty) = \begin{cases} (1-e^{-2x}), & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\& \quad F_Y(y) = F_{XY}(\infty, y) = \begin{cases} (1-e^{-y}), & y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Note that the joint cdf factors into the product of the marginal cdfs as it should for statistically independent random variables.

The conditional pdfs are

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

and $$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \begin{cases} e^{-y}, & y \geq 0 \\ 0, & y < 0 \end{cases}$$

They are equal to the respective marginal pdfs as they should be for independent random variables.

Example

To illustrate the processes of normalisation of joint pdf's, finding marginal from joint pdf's and checking for statistical independence of the corresponding random variables, we consider the joint pdf

$$f_{XY}(x,y) = \begin{cases} \beta xy, & 0 \le x \le y, 0 \le y \le 4 \\ 0, & \text{otherwise} \end{cases}$$

For independence the joint pdf must be the product of the marginal pdf's.

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We find constant β by normalising the volume under the pdf to unity by integrating $f_{XY}(x,y)$ over all x & y:

$$\int_0^4 \int_0^y \beta uv \, du\, dv = 1 \quad \Rightarrow \quad \beta = \frac{1}{32}$$

$\left.\right\}$ u & v dummy variables

Now, integrating over x to obtain $f_Y(y)$

$$f_Y(y) = \int_0^y \frac{xy}{32}\, dx \qquad 0 \le y \le 4$$

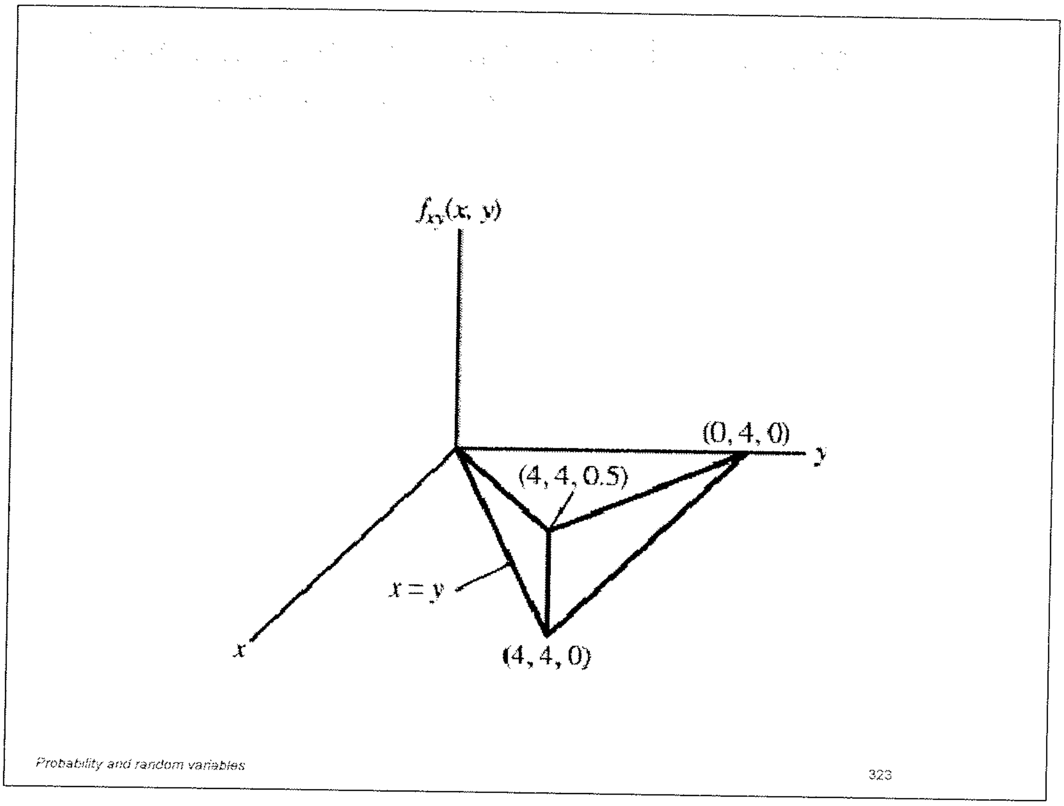
$$= \begin{cases} y^3/64 & 0 \le y \le 4 \\ 0 & \text{otherwise} \end{cases}$$

The pdf on X is similarly obtained as

$$f_X(x) = \int_x^4 \frac{xy}{32}\, dy \qquad 0 \le y \le 4$$

$$= \begin{cases} (x/4)[1-(x/4)^2] & 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

It is clear that the product of the marginal pdfs is not equal to the joint pdf, so the random variables X & Y are not statistically independent.

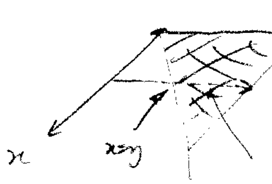


Integrating over x

$$\int_0^y dx$$

integrating from $x = 0$ to $x = y$

Integrating over y



this is y which is $\int_x^4 dy$

the lower limit is $\underline{\underline{x}}$!

Examples

Derive the pdf of random variable Y defined by

$$Y = -\left(\frac{1}{\pi}\right)\Theta + 1$$

where the random variable Θ has a pdf given by

$$f_\Theta(\theta) = \begin{cases} 1/(2\pi), & 0 \le \theta \le 2\pi \\ 0, & \text{otherwise} \end{cases}$$

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$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|_{x=g^{-1}(y)}$$

Since $\frac{dy}{d\theta} = \frac{-1}{\pi}$ the pdf of Y is

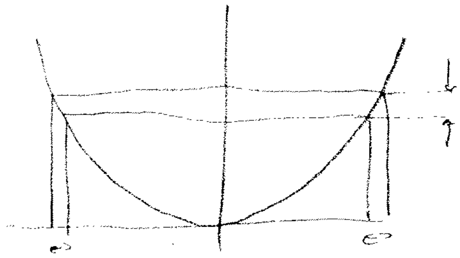
NB the change of limits

$$f_Y(y) = f_\Theta(\theta = -\pi y + \pi)\,|-\pi| = \begin{cases} 1/2, & -1 \leqslant y \leqslant 1 \\ 0, & \text{otherwise} \end{cases}$$

this is $\left|\frac{dg^{-1}(y)}{dy}\right| = \left|\frac{d}{dy}(-\pi y + \pi)\right| = |-\pi|$

Example

Consider the transformation $y = x^2$. If $F_X(x) = 0.5\exp(-|x|)$, find $F_Y(y)$.



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There are two solutions to $x^2 = y$; here are

$x_1 = \sqrt{y}$ for $x \geq 0$ & $x_2 = -\sqrt{y}$ for $x < 0$ $\quad y > 0$

Their derivatives are

$$\frac{dx_1}{dy} = \frac{1}{2\sqrt{y}} \text{ for } x \geq 0 \quad \& \quad \frac{dx_2}{dy} = -\frac{1}{2\sqrt{y}} \text{ for } x < 0 \qquad y > 0$$

Using these results in $f_Y(y) = \sum_{n=1}^{N} f_X(x_i)\left|\frac{dx_i}{dy}\right|_{x_i = g_i^{-1}(y)}$

$$f_Y(y) = \frac{1}{2}e^{-\sqrt{y}}\left|-\frac{1}{2\sqrt{y}}\right| + \frac{1}{2}e^{-\sqrt{y}}\left|\frac{1}{2\sqrt{y}}\right| = \frac{e^{-y}}{2\sqrt{y}}, \quad y > 0$$

Signs are identical due to original pdf $0.5e^{-|x|}$

Since Y cannot be negative, $f_Y(y) = 0$, $y < 0$.

Example

Consider the dart-throwing game discussed in connection with joint cdf's and and pdf's. We assume that the joint pdf in terms of rectangular coordinates for the impact point is:

$$f_{XY}(x, y) = \frac{\exp[-(x^2 + y^2)/2\sigma^2]}{2\pi\sigma^2}$$

where σ^2 is a constant. This is a special case of the joint Gaussian pdf. Instead of rectangular coordinates, we wish to use polar coordinates R and Θ, defined by:

$$R = \sqrt{X^2 + Y^2} \qquad \Theta = \tan^{-1}\left(\frac{Y}{X}\right)$$

See tutorial

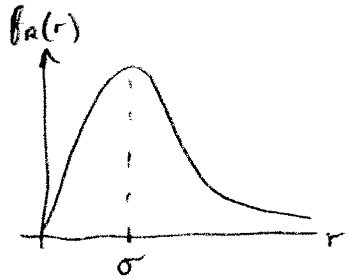
$\therefore\ X = R\cos\Theta = g_1^{-1}(R,\Theta) \quad 0<\Theta<2\pi$

$Y = R\sin\Theta = g_2^{-1}(R,\Theta) \quad 0\le R<\infty$

$R = \sqrt{X^2+Y^2} \qquad \Theta = \tan^{-1}\frac{Y}{X}$

Under this transformation the infinitesimal area $dxdy$ in the xy plane transforms to the area $r\,dr\,d\theta$ in the $r\theta$ plane as determined by the Jacobian, which is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$



$\therefore$ The joint pdf of R & θ is

$$f_{R\Theta}(r,\theta) = \frac{r e^{-r^2/2\sigma^2}}{2\pi\sigma^2} \qquad \begin{matrix} 0<\theta<2\pi \\ 0\le r<\infty \end{matrix}$$

Integrating over θ we obtain the pdf of R alone

$$f_R(r) = \frac{r}{\sigma^2}\exp\left(\frac{-r^2}{2\sigma^2}\right) \qquad 0\le r<\infty$$

which is the Rayleigh pdf. The probability that the dart lands in a ring of radius r from the bull's-eye and having thickness dr is given by $f_R(r)dr$. From the sketch of the Rayleigh pdf we see that the most probable distance for the dart to land from the bull's-eye is $R=\sigma$.

By integrating over r we find that the pdf of Θ is uniform over $(0,2\pi)$.

Example

Suppose the random variable Θ has the pdf

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & |\theta| < \pi \\ 0, & \text{otherwise} \end{cases}$$

Determine $E[\Theta^n]$, referred to as the n^{th} moment of Θ.

The first moment or mean of Θ, $E[\Theta^n]$, is a measure of the location of $f_\Theta(\theta)$ (i.e. the "centre of mass"). Since $f_\Theta(\theta)$ is symmetrically located about $\theta = 0$, it is not surprising that $E[\Theta] = 0$.

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$$E[\Theta^n] = \int_{-\infty}^{\infty} \theta^n f_\Theta(\theta)\,d\theta = \int_{-\pi}^{\pi} \theta^n \frac{d\theta}{2\pi}$$

Since the integrand is odd if n is odd, $E[\Theta^n] = 0$ for n odd

For even n

$$E[\Theta^n] = \frac{1}{\pi}\int_0^{\pi} \theta^n\,d\theta = \frac{1}{\pi}\left[\frac{\theta^{n+1}}{n+1}\right]_0^{\pi} = \frac{\pi^n}{n+1}$$

NB limits

Example

Consider a random variable X that is defined in terms of the uniform random variable Θ considered in the last example by

$$X = \cos\Theta$$

Determine the density function of X, $f_X(x)$ and the first and second moments.

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First, $-1 \leqslant \cos\theta \leqslant 1$ so $f_X(x) = 0$ for $|x| > 1$.

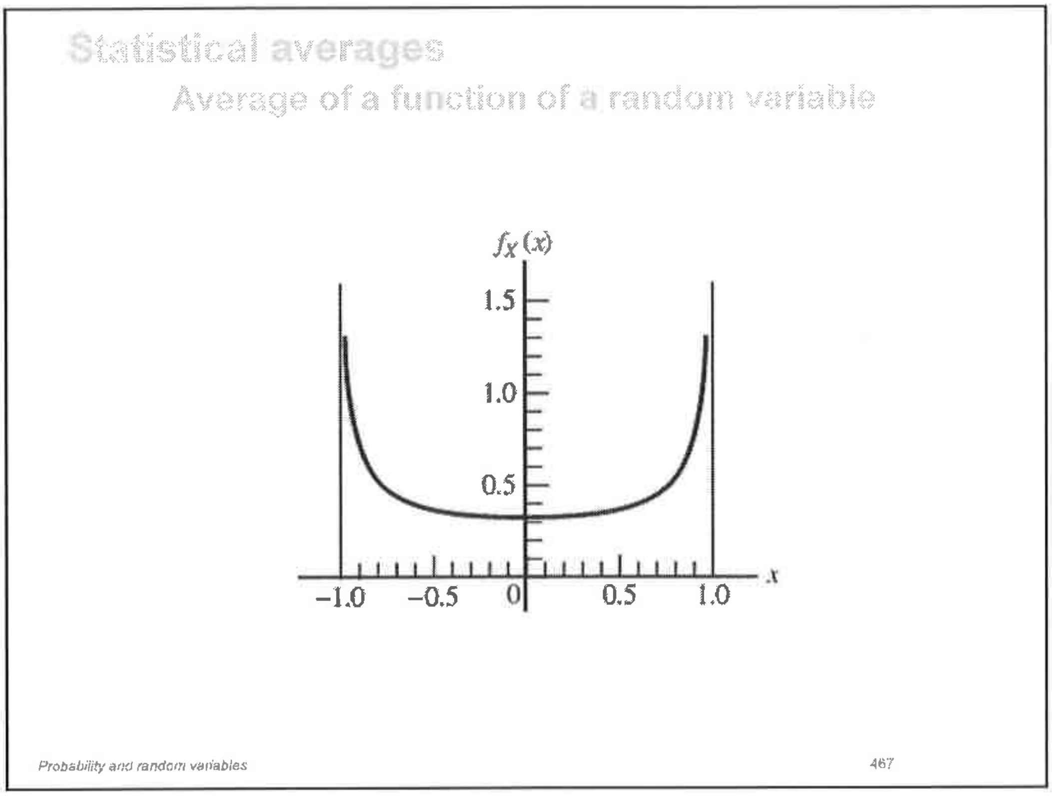
Second, the transformation is not one-to-one, there being two values of Θ for each value of X, since $\cos\theta = \cos(-\theta)$.
But, noting that positive & negative angles have equal probabilities we can write

$$f_X(x) = 2 f_\Theta(\theta)\left|\frac{d\theta}{dx}\right|, \quad |x| < 1$$

Now $\theta = \cos^{-1}x$ & $|d\theta/dx| = (1-x^2)^{-1/2}$, which yields

$$f_X(x) = \begin{cases} 2 \dfrac{1}{2\pi} \dfrac{1}{\sqrt{1-x^2}} & |x| \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{1}{\pi\sqrt{1-x^2}} \\ 0 \end{cases}$$



Now for the 1st & 2nd moments

Using either of the equations we find that

$$\bar{X} = \int_{-1}^{1} \frac{x}{\pi\sqrt{1-x^2}}\,dx = 0$$ because the integrand is odd

$$\overline{X^2} = \int_{-1}^{1} \frac{x^2}{\pi\sqrt{1-x^2}}\,dx = \frac{1}{2}$$ (by a table of integrals)

Alternatively $$\bar{X} = \int_{-\pi}^{\pi} \cos\theta \,\frac{d\theta}{2\pi} = 0$$

$$\overline{X^2} = \int_{-\pi}^{\pi} \cos^2\theta \,\frac{d\theta}{2\pi} = \int_{-\pi}^{\pi} \frac{1}{2}(1+\cos 2\theta)\,\frac{d\theta}{2\pi} = \frac{1}{2}$$

These two methods show the difference between the two approaches for solving the same equation with the two different equations

on slide 453

Example

Consider the joint pdf

$$f_{XY}(x,y) = \begin{cases} Ae^{-(2x+y)}, & x, y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Noting that X and Y are statistically independent, determine the expectation of $g(X, Y) = X\,Y$.

This is the pdf of slide 320

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{XY}(x,y)\, dx\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 2xy\, e^{-(2x+y)} dx\, dy$$

$$= 2\int_0^{\infty} x e^{-2x} dx \int_0^{\infty} y e^{-y} dy = \frac{1}{2}$$

$$\int_0^{\infty} x e^{-2x} dx = \frac{1}{4}$$

$$\int_0^{\infty} y e^{-y} dy = 1$$

remembering that the variables are statistically independent it is no surprise that

$$E[XY] = E[X]E[Y]$$

In fact for statistically independent random variables, it readily follows that

$$E[h(X)\, g(X)] = E[h(X)]\,E[g(Y)]$$

where $h(X)$ & $g(Y)$ are two functions of X & Y, respectively. In the special case where $h(X) = X^m$ & $g(Y) = Y^n$ the average $E[h(X)g(Y)] = E[X^m Y^n]$. These are referred to as the joint moments of order $m+n$ of X & Y.

The joint moments of statistically independent random variables factor.

Example

As a specific example of conditional expectation, consider the firing of projectiles at a target. Projectiles are fired until the target is hit for the first time, after which firing ceases. Assume that the probability of a projectile's hitting the target is p and that the firings are independent of one another. Find the average number of projectiles fired at the target.

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Let N be a random variable denoting the number of projectiles fired at the target. Let the r.v. H be 1 if the first projectile hits the target & 0 if it does not. Using the concept of conditional expectation, we find the average value of N is given by

$$E[N] = E\{E[N|H]\} = p\,E[N|H=1] + (1-p)E[N|H=0]$$

$$= p \times 1 + (1-p)(1 + E[N])$$

where $E[N|H=0] = 1 + E[N]$ because $N \geq 1$ if a miss occurs on the first firing.

Solving the last expression for $E[N]$ gives

$$E[N] = \frac{1}{p}$$

NB we could have evaluated $E[N]$ directly

$$E[N] = 1 \times p + 2 \times (1-p)p + 3 \times (1-p)^2 p + \cdots$$

Example

Determine the variance of the uniform pdf

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

$$E[X] = \int_a^b x \frac{dx}{b-a} = \frac{1}{2}(a+b)$$

$$E[X^2] = \int_a^b x^2 \frac{dx}{b-a} = \frac{1}{3}(b^2+ab+a^2)$$

$$\therefore \sigma_X^2 = \frac{1}{3}(b^2+ab+a^2) - \frac{1}{4}(a^2+2ab+b^2)$$

$$= \frac{1}{12}(a-b)^2$$

Note how the values of a & b influence the variance.

Example

Use a table of Fourier transforms to obtain the characteristic function of the one-sided exponential pdf

$$f_X(x) = \exp(-x)u(x)$$

and determine an expression for its n^{th} moment.

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From a table of transforms

$$e^{-at}u(t) \Leftrightarrow \frac{1}{\alpha + j2\pi f}$$

$$\therefore \quad M_X(jv) = \frac{1}{1-jv}$$

By repeated differentiation or expansion of the characteristic function in a power series in jv it follows from

$$E[X^n] = (-j)^n \left.\frac{\partial^n M_X(jv)}{\partial v^n}\right|_{v=0}$$

that

$$E[X^n] = n!$$

Example

Determine the pdf of Z, the sum of four identically distributed, independent random variables,

$$Z = X_1 + X_2 + X_3 + X_4$$

where the pdf of each X_i is given by

$$f_{X_i}(x_i) = \prod(x_i) = \begin{cases} 1, & |x_i| \le \frac{1}{2} \\ 0, & \text{otherwise}, i = 1,2,3,4 \end{cases}$$

and where $\prod(x_i)$ is the unit rectangular pulse function.

We apply the convolution on the previous slide twice with

$Z_1 = X_1 + X_2$ & $Z_2 = X_3 + X_4$

The pdfs of Z_1 & Z_2 are identical, both being the convolution of a uniform density with itself.

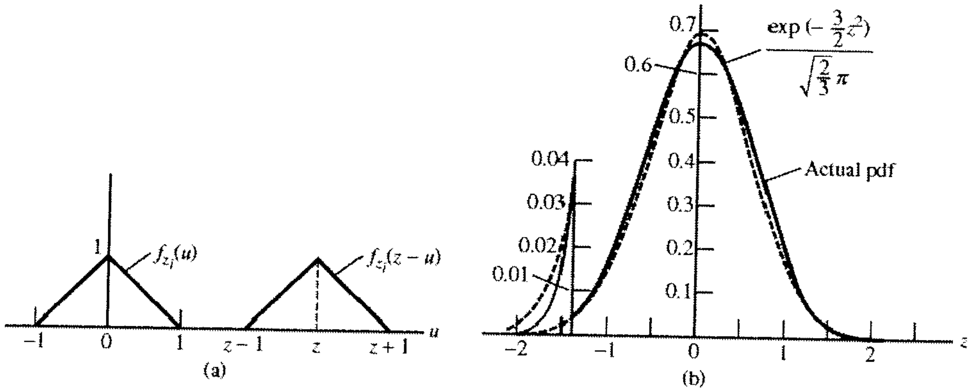
$$f_{Z_i}(z_i) = \Lambda(z_i) = \begin{cases} 1 - |z_i| & |z_i| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Λ is the triangular waveform on the next slide – this is just convolution

where $f_{Z_i}(z_i)$ is the pdf of Z_i, $i = 1,2$. To find $f_Z(z)$ we convolve $f_{Z_i}(z_i)$ with itself. Thus

$$f_Z(z) = \int_{-\infty}^{\infty} f_{Z_i}(z-u) f_{Z_i}(u)\, du$$

The factors in the integrand are sketched over leaf. Clearly $f_Z(z) = 0$ for $z < -2$ or $z > 2$. Since $f_{Z_i}(z_i)$ is even $f_Z(z)$ is also even. Thus we need not consider $f_Z(z)$ for $z < 0$.



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From the sketch it follows that for $1 \le z \le 2$

$$f_Z(z) = \int_{z-1}^{1} (1-u)(1+u-z)\,du = \frac{1}{6}(2-z)^3$$

& for $0 \le z \le 1$ we obtain

$$f_Z(z) = \int_{z-1}^{0} (1-u)(1+u-z)\,du + \int_{0}^{z} (1-u)(1+u-z)\,du + \int_{z}^{1} (1-u)(1-u+z)\,d$$

$$= (1-z) - \frac{1}{3}(1-z)^3 + \frac{1}{6}z^3$$

The graph is illustrated with a plot of $e^{-3/2\,z^2}/(\sqrt{\tfrac{2}{3}\pi})$.

$\Rightarrow$ central limit theorem.

Example

Assuming single births, determine the probability of having 1, 2, 3 and 4 girls in a four-child family when the probability of a female is 0.5.

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$$P_4(2) = \binom{4}{2}\left(\frac{1}{2}\right)^4 = \frac{3}{8}$$

& for 0, 1, 3 & 4 girls it's $\frac{1}{16}$, $\frac{1}{4}$, $\frac{1}{4}$ & $\frac{1}{16}$ respectively.

NB the sum is 1

Example

The probability of error on a single transmission in a digital communication system is $P_E = 10^{-4}$. Using the Poisson approximation to the binomial distribution determine the probability of more than three errors in 1000 transmissions?

$$P(K \leq 3) = \sum_{k=0}^{3} \frac{(\bar{K})^k}{k!} e^{-\bar{K}}$$

where $\bar{K} = (10^{-4})(1000) = 0.1$

hence

$$P(K \leq 3) = e^{-0.1}\left[\frac{(0.1)^0}{0!} + \frac{(0.1)^1}{1!} + \frac{(0.1)^2}{2!} + \frac{(0.1)^3}{3!}\right] \simeq 0.999996$$

$$\therefore \quad P(K > 3) = 1 - P(K \leq 3) \simeq 4 \times 10^{-6}$$

Example

What is the probability of the first error occurring at the 1000th transmission in a digital data transmission system where the probability of error is $p = 10^{-6}$?

$$p(1000) = 10^{-6}(1-10^{-6})^{999}$$

$$= 9.99 \times 10^{-7} \cong 10^{-6}$$