Mathematical Methods for Engineers (MathEng)
EXAM
13th February 2012
Duration: 2 hrs, calculators permitted, no documents
This exam paper contains 8 questions and 80 marks.
ATTEMPT ALL QUESTIONS – ANSWER IN ENGLISH

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1. Determine the Fourier transform $G(f)$ of the square wave $g(t)$ shown in Figure Q1. Then, evaluate and sketch the convolution of $g(t)$ with itself, i.e. $h(t) = g(t) * g(t)$. Determine the Fourier transform of $h(t)$ and explain why it is equal to the square of $G(f)$.

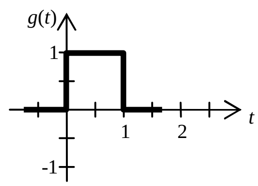

Figure Q1

[12 marks]

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2. Sketch Bode plots of the magnitude and phase responses for a system whose transfer function is given by:

$$H(s) = \frac{2000(s + 0.5)}{s(s + 10)(s + 50)}$$

[16 marks]

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3. A digital filter is described by the following difference equation:

$$y(n) - \tfrac{3}{4}y(n-1) + \tfrac{1}{8}y(n-2) = x(n).$$

Show that the system transfer function is given by:

$$H(z) = \frac{z^2}{(z - 0.5)(z - 0.25)}$$

where $|z| > 0.5$, and determine an expression for the filter impulse response $h(n)$.

[15 marks]

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4. Suppose a pair of random variables is jointly distributed over the unit circle, that is, the joint probability density function $f_{XY}(x, y)$is constant anywhere such that $x^2 + y^2 < 1$:

$$f_{XY}(x, y) = \begin{cases} c, & x^2 + y^2 < 1 \\ 0, & \text{otherwise} \end{cases}$$

Determine c and the marginal probability density functions $f_X(x)$ and $f_Y(y)$.

[12 marks]

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5. (a) A is a 6×7 matrix of rank 4. What are the dimensions of its four subspaces?
(b) Find the bases for the four subspaces associated with the matrix B:

$$B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$$

[6 marks]

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6. Find the eigenvalues and corresponding eigenvectors associated with the matrix A:

$$A = \begin{bmatrix} 0.2 & 0.6 \\ 0.8 & 0.4 \end{bmatrix}$$

[6 marks]

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7. Find an orthonormal basis for the column space of matrix A using the Gram-Schmidt process and thereafter factor it into QR form:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$$

[7 marks]

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8. Under what conditions on b_1, b_2, b_3 and b_4 is the following system solvable? Find an expression for x in that case.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

[6 marks]

Table of selected Laplace transforms

$x(t) \quad (t \geq 0)$	$X(s)$
$\delta(t)$	1
$\delta(t-\alpha)$	$\exp(-\alpha s)$
1 (unit step)	$\dfrac{1}{s}$
t (unit ramp)	$\dfrac{1}{s^2}$
$\exp(-\alpha t)$	$\dfrac{1}{s+\alpha}$
$t\exp(-\alpha t)$	$\dfrac{1}{(s+\alpha)^2}$
$\sin(\alpha t)$	$\dfrac{1}{s^2+\alpha^2}$
$\cos(\alpha t)$	$\dfrac{s}{s^2+\alpha^2}$
$e^{-\alpha t}\sin(\omega t)$	$\dfrac{\omega}{(s+\alpha)^2+\omega^2}$
$e^{-\alpha t}\cos(\omega t)$	$\dfrac{s+\alpha}{(s+\alpha)^2+\omega^2}$

Bode plots

Poles or zeros on the real axis:

$$(s+a) = a\left(\frac{s}{a}+1\right) = \frac{1}{\tau}(\tau s+1)$$

Complex conjugate poles (or zeros):

$$(s^2+As+B) = (s^2+2\zeta\omega_0 s+\omega_0^2) = \omega_0^2((s/\omega_0)^2+2\zeta(s/\omega_0)+1)$$

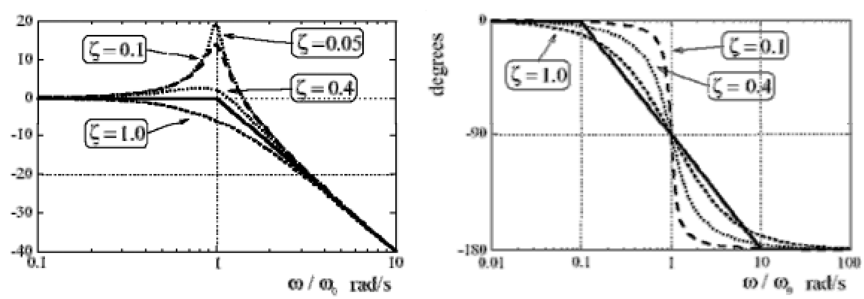

Table of selected z-transforms

$x(n) \quad (n \ge 0)$	$X(z)$
$\delta(n)$ unit pulse	1
$\delta(n-m)$	z^{-m}
1 (unit step)	$\dfrac{z}{z-1}$
n (unit ramp)	$\dfrac{z}{(z-1)^2}$
$\exp(-\alpha n)$	$\dfrac{z}{(z-e^{-\alpha})}$
$n\exp(-\alpha n)$	$\dfrac{e^{-\alpha} z}{(z-e^{-\alpha})^2}$
$\sin(\beta n)$	$\dfrac{z\sin(\beta)}{z^2-2z\cos(\beta)+1}$
$\cos(\beta n)$	$\dfrac{z^2-z\cos(\beta)}{z^2-2z\cos(\beta)+1}$
$e^{-\alpha n}\sin(\beta n)$	$\dfrac{ze^{-\alpha}\sin(\beta)}{z^2-2ze^{-\alpha}\cos(\beta)+e^{-2\alpha}}$
$e^{-\alpha n}\cos(\beta n)$	$\dfrac{z^2-ze^{-\alpha}\cos(\beta)}{z^2-2ze^{-\alpha}\cos(\beta)+e^{-2\alpha}}$

Table of selected Fourier transform pairs

Function	$x(t)$	$X(\omega)$
Rectangular function of width τ	$\Pi(t/\tau)$	$\tau\,\mathrm{sinc}(\omega\tau/2)$
Triangular function of width 2τ	$\Lambda(t/\tau)$	$\tau\,\mathrm{sinc}^2(\omega\tau/2)$
Train of impulses every Δt	$\delta_T(t)$	$2\pi/\Delta t\,\Sigma_n \delta(\omega - 2\pi n/\Delta t)$

NB: $\mathrm{sinc}(x) = \sin(x)/x$

 [END OF PAPER]

Fourier series and transforms

Fourier series – periodic and continuous in time, discrete in frequency	
$x(t) = \sum_{n=-\infty}^{+\infty} X_n \exp(jn\omega_0 t)$	$X_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t)\exp(-jn\omega_0 t)\, dt$
Fourier transform – continuous in time, continuous in frequency	
$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\exp(j\omega t)\, d\omega$	$X(\omega) = \int_{-\infty}^{\infty} x(t)\exp(-j\omega t)\, dt$
Discrete-time Fourier transform – discrete in time, continuous and periodic in frequency	
$x(n\Delta t) = \frac{\Delta t}{2\pi} \int_{-\pi/\Delta t}^{\pi/\Delta t} X(\omega)\exp(jn\Delta t\omega)\, d\omega$	$X(\omega) = \sum_{n=-\infty}^{\infty} x(n\Delta t)\exp(-j\omega n\Delta t)$
Discrete Fourier transform – discrete and periodic in time and in frequency	
$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \exp\left(\frac{jnk2\pi}{N}\right)$	$X(k) = \sum_{n=0}^{N-1} x(n)\exp\left(\frac{-jnk2\pi}{N}\right)$

[END OF PAPER]