Example

The transfer function of a digital filter is

$$H(z) = \frac{z}{z-a}$$

What is its frequency response?

$$H(\omega) = \frac{\exp(j\omega\Delta t)}{\exp(j\omega\Delta t) - a}$$

The gain & phase response are $|H(\omega)|$ & $\angle H(\omega)$ respectively.

Example

A sequence {1, 2, 3} is applied to a FIR filter with transfer function $1 + 0.5z^{-2} + 0.25z^{-2}$. Use discrete convolution to evaluate the output.

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By inspection the impulse response sequence is $\{1, 0.5, 0.25\}$. This is illustrated at the bottom of the figure overleaf.

Because the sequence is finite there is a simple one-to-one relationship between it and the coefficient of a FIR filter. In this case the three elements of the sequence are the coefficient of a three-tap FIR filter as shown

The output is given by $y(n) = \sum_{m=0}^{2} h(m)\,x(n-m)$

The input sequence is illustrated at the top of the figure. If this sequence is applied to the filter the sample of 1 reaches the filter first, followed by the sample of value 2 and then the sample of value 3. In order to draw the input sequence in a manner which is consistent with the block diagram, we must time reverse the sequence. This is illustrated in the plot of $x(-m)$ against m.

For $n=0$ $\quad y(0) = \sum_{m=0}^{2} h(m)x(-m) = 1$

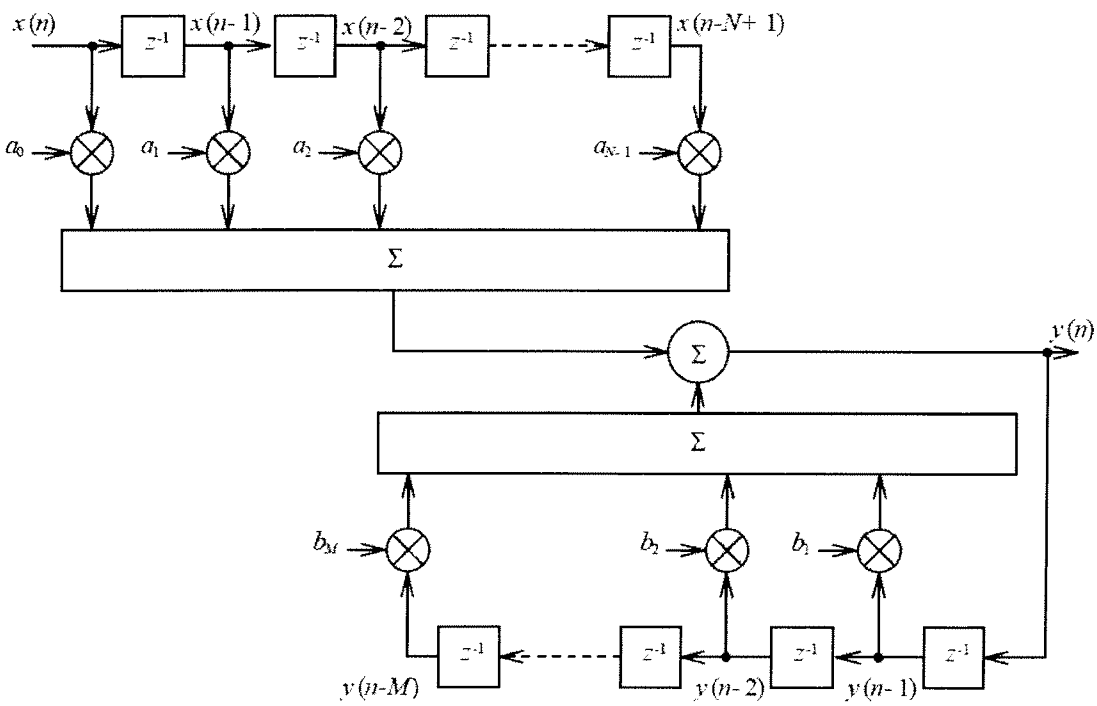
$n=1$ $\quad y(1) = \sum_{m=0}^{2} h(m)x(1-m) = 2.5$

$n=2$ $\quad y(2) = \sum_{m=0}^{2} h(m)x(2-m) = 4.25$

$n=3$ $\quad y(3) = \sum_{m=0}^{2} h(m)x(3-m) = 2$

$n=4$ $\quad y(4) = \sum_{m=0}^{2} h(m)x(4-m) = 0.75$

Thus the complete output sequence is $\{1, 2.5, 4.25, 2, 0.75\}$



Example

A sequence where $x(n) = 0.2^n$, $n \geq 0$ is applied to a digital filter with difference equation $y(n) = 0.5(n\text{-}1) + x(n)$. Use transform techniques to develop an expression for the output $y(n)$.

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The input $x(n)$ has the form $\exp(-\alpha n)$ & from a table of transforms we see that

$$X(z) = \frac{z}{z-0.2}$$

The z-transform of the difference equation is given by

$$Y(z) = 0.5z^{-1}Y(z) + X(z)$$

$$\therefore \quad H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-0.5z^{-1}} = \frac{z}{z-0.5}$$

$\therefore$ The z-transform of the output is given by

$$Y(z) = H(z)X(z) = \frac{z}{(z-0.2)}\frac{z}{(z-0.5)}$$

$$\Rightarrow \quad \frac{Y(z)}{z} = \frac{z}{(z-0.2)(z-0.5)} = \frac{-0.667}{z-0.2} + \frac{1.667}{z-0.5}$$

Thus the output is $y(n) = -0.667(0.2)^n + 1.667(0.5)^n$, $n \geq 0$

Example

What is the inverse z-transform of $X(z) = 1 + 2z^{-1} + 3z^{-2}$?

If we had a sequence of three samples, $x(0) = 1$, $x(1) = 2$ & $x(2) = 3$, the sequence is written as

$$\{x(n)\} = 1, 2, 3$$

The z-transform of the sequence is

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= x(0) z^{-0} + x(1) z^{-1} + x(2) z^{-2}$$

$$= 1 + 2z^{-1} + 3z^{-2}$$

This argument can be operated in reverse. For this z-transform we see by inspection that the sequence of samples is

$$\{x(n)\} = \{1, 2, 3\}$$

— easy since the sequence has a finite number of terms.

Example

The following is a typical example of the type of transform which might be encountered in analysing or designing a sampled data system:

$$X(z) = \frac{3 - \frac{5}{2} z^{-1}}{1 - \frac{3}{2} z^{-1} + \frac{z^{-2}}{2}} = \frac{3z^2 - \frac{5}{2} z}{\left(z^2 - \frac{3}{2} z + \frac{1}{2} \right)}$$

Develop an expression for the nth time sample $x(n)$.

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The transform is not a proper fraction because the highest power in the numerator is equal to the highest power in the denominator. Thus we form a PFE of $X(z)/z$ rather than $X(z)$

$$\frac{X(z)}{z} = \frac{3z - \frac{5}{2}}{z^2 - \frac{3}{2}z + \frac{1}{2}} = \frac{3z - \frac{5}{2}}{(z - \frac{1}{2})(z-1)} = \frac{A}{(z - \frac{1}{2})} + \frac{B}{(z-1)}$$

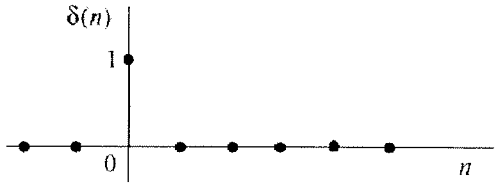
$$= \frac{2}{(z - \frac{1}{2})} + \frac{1}{(z-1)} \qquad \therefore \quad X(z) = \frac{2z}{(z-\frac{1}{2})} + \frac{z}{(z-1)}$$

From tables $\quad \mathcal{Z}^{-1}\left[\frac{z}{(z-1)}\right] = 1 \quad \& \quad \mathcal{Z}^{-1}\left[\frac{z}{(z-\frac{1}{2})}\right] = (1/2)^n$

$$\therefore \quad x(n) = \frac{1}{2^{n-1}} + 1 \qquad n \geq 0$$

Example

Evaluate the z-transform of a unit pulse $\delta(n\Delta t)$: The unit pulse is a discrete sequence with a single sample of value one at time zero.



$$x(n\Delta t) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

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$$X(z) = \sum_{n=0}^{\infty} x(n\Delta t)\, z^{-n}$$

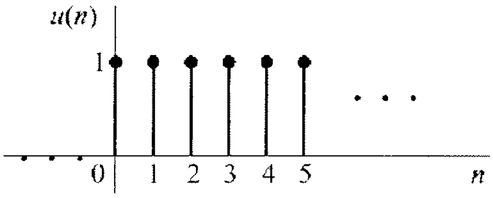
$$= x(0)\, z^{-0}$$

$$= 1$$

The unit impulse can also be called a discrete impulse or just an impulse (which can cause confusion with the analogue impulse).

Example

Evaluate the z-transform of a unit step. A discrete step is a sampled version of the analogue or continuous step.



$$x(n\Delta t) = 1;\ n \geq 0$$
$$= 0;\ n < 0$$

$$X(z) = \sum_{n=0}^{\infty} x(n\Delta t) z^{-n}$$
$$= \sum_{n=0}^{\infty} (1) z^{-n}$$
$$= \sum_{n=0}^{\infty} z^{-n}$$
$$= \sum_{n=0}^{\infty} (z^{-1})^{n}$$
$$= \frac{1}{1-z^{-1}}$$

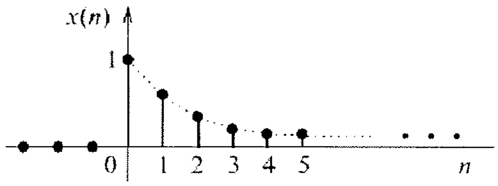
The sum of a geometric series is:

$$\sum_{n=0}^{\infty} c^n = \frac{1}{1-c}$$

provided $|c| < 1$

Example

Evaluate the z-transform of a sampled exponential.



$$x(n\Delta t) = \exp(-\alpha n \Delta t)\ ;\quad n \geq 0$$

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$$X(z) = \sum_{n=0}^{\infty} \exp(-\alpha n \Delta t) z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\exp(-\alpha \Delta t) z^{-1}\right)^n$$

another geometric series

$$= \frac{1}{1 - \exp(-\alpha \Delta t) z^{-1}}$$

$$= \frac{z}{z - \exp(-\alpha \Delta t)}$$

The constant α could be complex with magnitude less than one which would give rise to a discrete decaying sinusoid.

The is is the solution to the example on slide 305

Solution

The input $x(t)$ contains two sine waves. From Figure 3.6, the Fourier transform of a sine wave or cosine wave consists of two impulses. For the 1 kHz component, the impulses are at $\pm 2\pi \times 10^3$ rad/s, each with strength $A\pi$. For the 7.5 kHz component, the impulses are at $\pm 2\pi(7.5 \times 10^3)$ rad/s, each with strength $A\pi/2$. Since we are not asked to consider phase shift in this question, there is no need to distinguish between the transform of a cosine wave and a sine wave – the former, being easier to work with, is adopted here. This Fourier transform is illustrated in Figure 4.27(a). Note: for ease of representation, the frequency axis in Figure 4.27 is labelled in kHz rather than rad/s.

The gain of the RC filter is:

$$|H_{RC}(\omega)| = \frac{1}{|1 + j\omega RC|}$$

This is illustrated approximately in Figure 4.27(b). It is worth noting that for the particular values of R and C, the –3 dB bandwidth of this filter is 5 kHz – half the sampling frequency. At 1 kHz, the gain of this filter is 0.9298 and at 7.5 kHz the gain is 0.3192. While the latter is clearly above the cut-off frequency of the filter, the gain is not negligible. In no sense could it be considered that the low-pass filter has removed the component at 7.5 kHz. The amplitude of the 1 kHz component at the output of the LPF is $0.9298A$ and the amplitude of the 7.5 kHz component is $0.3192 \times A/2 = 0.1596A$. For the 1 kHz component, the Fourier transform consists of impulses at $\pm 2\pi \times 10^3$ rad/s, each with strength $0.9298A\pi$. For the 7.5 kHz component, the Fourier transform consists of impulses at $\pm 2\pi(7.5 \times 10^3)$ rad/s, each with strength $0.1596A\pi$. This Fourier transform is illustrated in Figure 4.27(c). Effectively the Fourier transform of Figure 4.27(c) is obtained by multiplying together the Fourier transform of the input, Figure 4.27(a), and the frequency response of the filter, Figure 4.27(b).

As in section 4.2, the action of the sampler is modelled as multiplication in the time domain with the impulse train of Figure 4.7(a). Hence, using equation (2.15), the Fourier transform of the signal at the output of the sampler is formed by convolving the Fourier transform of Figure 4.27(c) with the Fourier transform of 4.7(b). The 1 kHz component at the output of the LPF is not aliased by the sampling operation and gives rise to impulses at $\pm 2\pi \times 10^3$, $\pm 2\pi(9 \times 10^3)$, $\pm 2\pi(11 \times 10^3)$ etc. rad/s in the Fourier transform of the signal which is present at the output of the sampler. Each impulse has strength $0.9298A\pi/\Delta t$. The 7.5 kHz component at the output of the LPF is aliased by the sampling operation and gives rise to impulses at $\pm 2\pi(2.5 \times 10^3)$, $\pm 2\pi(7.5 \times 10^3)$, $\pm 2\pi(12.5 \times 10^3)$ etc. rad/s, each with strength $0.1596A\pi/\Delta t$. This Fourier transform is illustrated in Figure 4.27(d).

To find the frequency response of the digital filter we require first to calculate the transfer function of the filter. It is convenient to label the signal at the left-most summer of Figure 4.26 as $x_1(n)$. We can then write a difference equation to relate $x(n)$ and $x_1(n)$:

$$x_1(n) = x(n) - 0.81x_1(n-2)$$

Taking z-transforms of both sides yields:

$$X_1(z) = \frac{X(z)}{(1 + 0.81z^{-2})} \tag{4.16}$$

The difference equation which relates the output $y(n)$ to the intermediate signal $x_1(n)$ is

$$y(n) = x_1(n) + 2x_1(n-1) + x_1(n-2)$$

Again taking z-transforms yields:

$$Y(z) = X_1(z)\left(1 + 2z^{-1} + z^{-2}\right) \tag{4.17}$$

Substituting equation (4.16) into (4.17) for $X_1(z)$ leads to the required transfer function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 2z^{-1} + z^{-2}}{1 + 0.81z^{-2}}$$

$$= \frac{(1 + z^{-1})^2}{1 + 0.81z^{-2}}$$

As in section 4.2, the frequency response is obtained by substituting $\exp(j\omega\Delta t)$ for z in the expression for the transfer function. Thus:

$$|H(\omega)| = \frac{|1 + \exp(-j\omega\Delta t)|^2}{|1 + 0.81\exp(-2j\omega\Delta t)|}$$

At 1 kHz, $\omega = 2\pi \times 10^3$, $\omega\Delta t = \pi/5$ and $|H(\omega)| = 2.464$. Because of the properties of digital filters, this will also be the gain at 9, 11, 19, 21 etc. kHz. At 2.5 kHz, $\omega = 2\pi(2.5 \times 10^3)$ rad/s, $\omega\Delta t = \pi/2$ and $|H(\omega)| = 10.526$. Because of the properties of digital filters this will also be the gain at 7.5, 12.5, 17.5 etc. kHz. Although it is not asked for directly in this problem, it is useful to locate the poles and zeros and sketch the frequency response of this digital filter. The result is illustrated in Figure 4.27(d). It is worth noting that this filter is particularly good at amplifying the aliased component at 2.5 kHz.

The frequencies present at the output of the digital filter are 1, 9, 11, 19, 21 etc. kHz – each with an amplitude of

$$2.464\,\frac{0.9298A}{\Delta t} = \frac{2.291A}{\Delta t}$$

and 2.5, 7.5, 12.5, 17.5, 22.5 etc. kHz – each with an amplitude of

$$10.526\,\frac{0.1596A}{\Delta t} = \frac{1.68A}{\Delta t}$$

The Fourier transform of the signal at the output of the digital filter is illustrated in Figure 4.27(f). It has impulses at $\pm 2\pi \times 10^3$, $\pm 2\pi(9 \times 10^3)$, $\pm 2\pi(11 \times 10^3)$ etc. rad/s, each with strength $\pi 2.291A/\Delta t$ and impulses at $\pm 2\pi(2.5 \times 10^3)$, $\pm 2\pi(7.5 \times 10^3)$, $\pm 2\pi(12.5 \times 10^3)$ etc. rad/s, each with strength $\pi 1.68A/\Delta t$. Like any other filter, the Fourier transform of the signal at the output of the digital filter is simply obtained by multiplying the Fourier transform of the input by the frequency response.

The action of the D/A converter is modelled with a ZOH as in section 4.2. From equation (4.2) the frequency response of the ZOH is:

$$|H_{ZOH}(\omega)| = \Delta t\left|\mathrm{sa}\left(\frac{\omega\Delta t}{2}\right)\right|$$

This is illustrated in Figure 4.27(g). As can also be seen from Figure 4.27, the action of the ZOH is to reduce components at 7.5, 9, 11, 12.5 etc. kHz – however it does not completely remove them. In the particular problem, the reconstruction filter is assumed to be an ideal LPF with a cut-off frequency of 5 kHz, Figure 4.27(i). Hence the components at 7.5, 9, 11, 12.5 etc. kHz will be completely removed and it is only necessary to calculate the gain of the ZOH for the 1 and 2.5 kHz components, as these will be the only ones present at the output of the system. At 1 kHz, $\omega\Delta t = \pi/5$ and $|H_{ZOH}(\omega)| = 0.984\Delta t$. The amplitude of the 1 kHz sine wave at the output of the reconstruction filter is thus:

$$0.984\Delta t \,\frac{2.291A}{\Delta t} = 2.254A$$

At 2.5 kHz, $\omega\Delta t = \pi/2$ and $|H_{ZOH}(\omega)| = 0.9\Delta t$. The amplitude of 1 kHz sine wave at the output of the reconstruction filter is thus:

$$0.9\Delta t \,\frac{1.68A}{\Delta t} = 1.512A$$

This is shown in Figure 4.27(j). It is worth noting that the scale factor $1/\Delta t$ on the amplitude of the sampled signals is an arbitrary artefact of the modelling process. It is effectively cancelled by the scale factor Δt present in the gain of the ZOH.

CHAPTER SUMMARY

Sampled data systems offer an alternative to conventional continuous-time analogue processing. The key elements in a sampled data system are: the sample and hold, the quantiser, a DSP unit and a D/A converter. Such systems can be analysed and designed by modelling the sampling process by multiplication with an impulse train and the D/A by a ZOH. Using this model, the effects of aliasing can be examined and the sampling theorem can be verified. Practical systems often require two additional components, namely an anti-aliasing filter and a reconstruction filter.

The z-transform is a modified version of the Laplace transform and provides a convenient mechanism for the design of discrete-time systems. Its application is very similar to the Laplace transform and many of the techniques associated with the Laplace transform have direct counterparts in the z-transform.

Figure 4.28 illustrates the interrelationships between four alternative descriptions of a discrete-time linear system. Such systems can be described by difference rather than differential equations. Taking the z-transform of the difference equation and performing some simple algebraic manipulation yields the system transfer function $H(z)$. If this transfer function is expressed as a ratio of polynomials in z, the roots of the numerator polynomial are the zeros and the roots of the denominator polynomial are the poles. The similarity between the Fourier, Laplace and z-transforms leads to a straightforward way of obtaining the frequency response $H(\omega)$ from the