Essential Mathematical Methods for Engineers (MathEng)
EXAM
09.30 – 10th February 2014
Duration: 2 hrs, calculators permitted, no documents
This exam paper contains 7 questions and 60 marks.
ATTEMPT ALL QUESTIONS – ANSWER IN ENGLISH

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1. A Gaussian distributed random variable X has mean μ_X and variance σ_X^2. A second random variable Y is defined as a function of X according to:

$$Y = \frac{X - \mu_X}{\sigma_X}$$

(a) Determine an expression for the cumulative density function $F_Y(y)$.
(b) Determine from $F_Y(y)$ an expression for the probability density function $f_Y(y)$ and comment on the result.

[8 marks]

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2. A linear time invariant system has an impulse response $h(t) = e^{-\alpha t}u(t)$, where $\alpha > 0$, and where $u(t)$ is the unit step function. **Sketch and derive an expression** for the output if the system is excited by an input signal $x(t) = u(t)$.

[8 marks]

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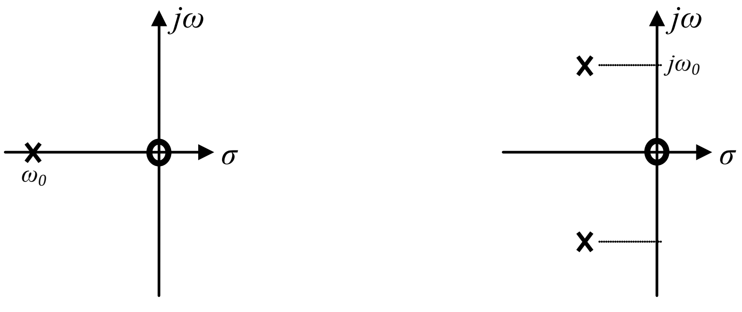
3. Sketch the frequency responses for systems with the following pole/zero positions in the s-plane. State whether each system is a high-pass, low-pass or band-pass filter.



(a) (b)

Figure Q4

[8 marks]

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4. (a) When excited with a unit step function $u[n]$, a linear time invariant, sampled-data system has output $y[n] = 2\left(\frac{1}{3}\right)^n u[n]$. Determine the system impulse response $h[n]$ and then the output when the input $x[n] = \left(\frac{1}{2}\right)^n u[n]$

[12 marks]

5. Determine the DFT of the sequence $x[n]$ = 1, 2, 3, 4. Set the two highest frequency components to zero and then perform the IDFT. Compare the resulting approximation to the original sequence and comment on the potential of the DFT for compression.

[12 marks]

6. Describe the nullspace of *A* and determine an appropriate number of special solutions.

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix}$$

[6 marks]

7. (a) For a projection matrix $P = A(A^T A)^{-1} A^T$, show that $P^2 = P$ and then explain, in terms of the column space of *P*, why projections *Pb* and *P(Pb)* give identical results.

   (b) What can you say about the eigenvalues and eigenvectors of *P*?

   (c) State whether or not *P* has an inverse.

[6 marks]

Table of selected Laplace transforms

$$X(s) = \int_{-\infty}^{\infty} x(t)\exp(-st)\,dt$$

N.B.: lower limit is 0 for one-sided Laplace transform

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\exp(st)\,ds$$

$x(t) \quad (t \geq 0)$	$X(s)$
$\delta(t)$	1
$\delta(t-\alpha)$	$\exp(-\alpha s)$
1 (unit step)	$\frac{1}{s}$
t (unit ramp)	$\frac{1}{s^2}$
$\exp(-\alpha t)$	$\frac{1}{s+\alpha}$
$t\exp(-\alpha t)$	$\frac{1}{(s+\alpha)^2}$
$\sin(\alpha t)$	$\frac{\alpha}{s^2+\alpha^2}$
$\cos(\alpha t)$	$\frac{s}{s^2+\alpha^2}$
$e^{-\alpha t}\sin(\omega t)$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
$e^{-\alpha t}\cos(\omega t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$

Bode plots

Poles or zeros on the real axis:

$$(s+a) = a\left(\frac{s}{a}+1\right) = \frac{1}{\tau}(\tau s+1)$$

Complex conjugate poles (or zeros):

$$(s^2+As+B) = (s^2+2\zeta\omega_0 s+\omega_0^2) = \omega_0^2((s/\omega_0)^2+2\zeta(s/\omega_0)+1)$$

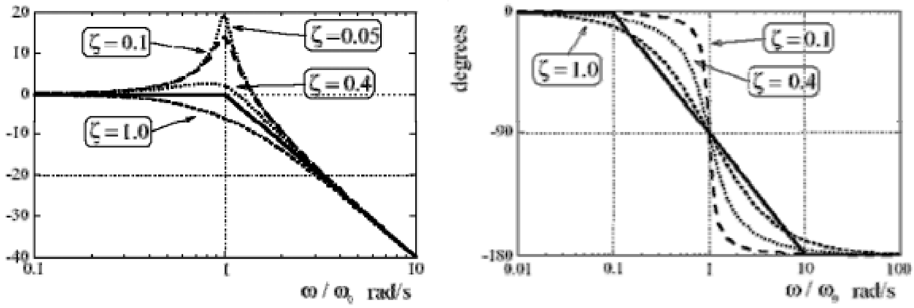

Table of selected z-transforms

$$x_c(t) = \sum_{n=0}^{\infty} x(n\Delta t)\delta(t - n\Delta t)$$

$$X_c(s) = \sum_{n=0}^{\infty} x(n\Delta t)\exp(-n\Delta ts)$$

$$X_c(s) = X(z)|_{z=e^{\Delta ts}}$$

$$X_c(\omega) = X_s(s)|_{s=j\omega}$$

$$X(\omega) = X(z)|_{z=\exp(\Delta tj\omega)}$$

$$X(z) = \sum_{n=0}^{\infty} x(n\Delta t)z^{-n}$$

$x(n)$ $(n \ge 0)$	$X(z)$
$\delta(n)$ unit pulse	1
$\delta(n-m)$	z^{-m}
1 (unit step)	$\dfrac{z}{z-1}$
n (unit ramp)	$\dfrac{z}{(z-1)^2}$
$\exp(-\alpha n)$	$\dfrac{z}{(z-e^{-\alpha})}$
$n\exp(-\alpha n)$	$\dfrac{e^{-\alpha}z}{(z-e^{-\alpha})^2}$
$\sin(\beta n)$	$\dfrac{z\sin(\beta)}{z^2-2z\cos(\beta)+1}$
$\cos(\beta n)$	$\dfrac{z^2-z\cos(\beta)}{z^2-2z\cos(\beta)+1}$
$e^{-\alpha n}\sin(\beta n)$	$\dfrac{ze^{-\alpha}\sin(\beta)}{z^2-2ze^{-\alpha}\cos(\beta)+e^{-2\alpha}}$
$e^{-\alpha n}\cos(\beta n)$	$\dfrac{z^2-ze^{-\alpha}\cos(\beta)}{z^2-2ze^{-\alpha}\cos(\beta)+e^{-2\alpha}}$

Table of selected Fourier transform pairs

Function	$x(t)$	$X(\omega)$
Rectangular function of width τ	$\Pi(t/\tau)$	$\tau\ \mathrm{sinc}(\omega\tau/2)$
Triangular function of width 2τ	$\Lambda(t/\tau)$	$\tau\ \mathrm{sinc}^2(\omega\tau/2)$
Train of impulses every Δt	$\delta_T(t)$	$2\pi/\Delta t\, \Sigma_n \delta(\omega - 2\pi n/\Delta t)$

NB: $\mathrm{sinc}(x) = \sin(\pi x)/\pi x$
NB: $\mathrm{sa}(x) = \sin(x)/x$

Euler's identity

$$\exp(j\theta) = \cos\theta + j\sin\theta$$

$$\cos(n\omega_0 t) = \frac{\exp(jn\omega_0 t) + \exp(-jn\omega_0 t)}{2}$$

$$\sin(n\omega_0 t) = \frac{\exp(jn\omega_0 t) - \exp(-jn\omega_0 t)}{2j}$$

Fourier series and transforms

Trigonometric Fourier series	
$$x(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t)$$ $$A_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(n\omega_0\, t)dt$$ $$B_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(n\omega_0\, t)dt$$	
Complex Fourier series – periodic and continuous in time, discrete in frequency	
$$x(t) = \sum_{n=-\infty}^{+\infty} X_n \exp(jn\omega_0 t)$$	$$X_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\exp(-jn\omega_0 t)\, dt$$
Fourier transform – continuous in time, continuous in frequency	
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\exp(j\omega t)\, d\omega$$	$$X(\omega) = \int_{-\infty}^{\infty} x(t)\exp(-j\omega t)\, dt$$
Discrete-time Fourier transform – discrete in time, continuous and periodic in frequency	
$$x(n\Delta t) = \frac{\Delta t}{2\pi}\int_{-\pi/\Delta t}^{\pi/\Delta t} X(\omega)\exp(jn\Delta t\omega)\, d\omega$$	$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n\Delta t)\exp(-j\omega n\Delta t)$$
Discrete Fourier transform – discrete and periodic in time and in frequency	
$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\exp\left(\frac{jnk2\pi}{N}\right)$$	$$X(k) = \sum_{n=0}^{N-1} x(n)\exp\left(\frac{-jnk2\pi}{N}\right)$$

Transformation of random variables

$$f_Y(y) = \sum_{i=1}^{N} f_X(x_i) \left|\frac{dx_i}{dy}\right|_{x_i = g_i^{-1}(y)}$$

$$f_{UV}(u,v) = f_{XY}(x,y) \left|\frac{\partial(x,y)}{\partial(u,v)}\right|_{\substack{x = g_1^{-1}(u,v) \\ y = g_2^{-1}(u,v)}}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

 [END OF PAPER]