

Essential Mathematical Methods for Engineers

Lecture 1b:
Time domain description and convolution

Outline

- time domain description and convolution
 - the impulse response
 - the impulse
 - signal representation
 - system response to an impulse
 - convolution
 - properties of convolution
 - time delay

The impulse response

- time and frequency domain descriptions of the input signals denoted $x(t)$ and $X(s)$ respectively
- identical descriptions of the output: $y(t)$ and $Y(s)$
- what about the system?
 - we've seen a frequency domain description $H(s)$
 - we also have a time domain description $h(t)$

$$h(t) = \mathcal{L}^{-1}[H(s)]$$

this is the impulse response and is an important time domain description of the system

Time domain description and convolution

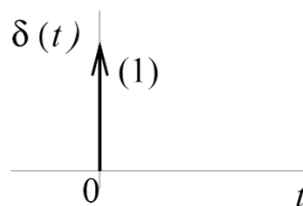
34

The impulse response The impulse

- the unit impulse
 - occurs at time $t = 0$
 - has infinite height
 - elsewhere it is zero
 - the area under the impulse function is equal to 1

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$



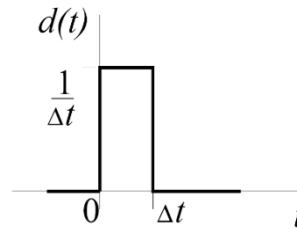
Time domain description and convolution

35

The impulse response

The impulse

- to aid our understanding of the impulse consider a function $d(t)$
 - starts at time $t = 0$
 - lasts for Δt seconds
 - has height $1/\Delta t$
 - an area of 1



- if we reduce the width and increase the height and maintain

$$\int_{-\infty}^{\infty} d(t) dt = 1$$

Δt tends toward zero and $d(t)$ tends toward $\delta(t)$

$$\delta(t) = \lim_{\Delta t \rightarrow 0} d(t)$$

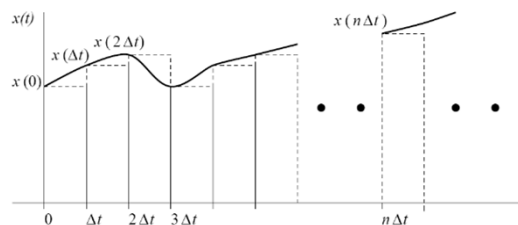
Time domain description and convolution

36

The impulse response

Signal representation

- we've seen how we can represent signals as a sum (or integral) of exponential basis functions
- we can also represent a signal as a sum (or integral) of impulses
- the representation improves as Δt tends toward zero



Time domain description and convolution

37

The impulse response Signal representation

- the pulse at time $t = n\Delta t$ has height $x(n\Delta t)$ so each pulse can be written as:

$$x(n\Delta t)\delta(t - n\Delta t)\Delta t$$

- adding together all of these impulses we have

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(n\Delta t)\delta(t - n\Delta t)\Delta t$$

- and as Δt tends toward zero the
 - pulse becomes an impulse at time $t = n\Delta t$
 - product $n\Delta t$ becomes the continuous time variable τ
 - time step Δt becomes the differential $d\tau$
 - summation becomes an integration

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau$$

where τ is the time at which each impulse occurs

Time domain description and convolution

38

The impulse response System response to an impulse

- we need to apply the Laplace transform

$$L[\delta(t)] = \int_{0^-}^{\infty} \delta(t) \exp(-st) dt$$

but at time $t = 0$ we have $\exp(-st) = 1$ so

$$L[\delta(t)] = \int_{0^-}^{\infty} \delta(t) 1 dt = 1$$

- now to find the transform of the output

$$\begin{aligned} Y(s) &= H(s) 1 \\ &= H(s) \end{aligned}$$

and to find the time domain description

$$\begin{aligned} y(t) &= L^{-1}[Y(s)] \\ &= L^{-1}[H(s)] \\ &= h(t) \end{aligned}$$

Time domain description and convolution

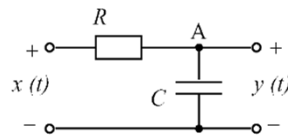
39

The impulse response

System response to an impulse

Example

Calculate the impulse response of the following RC circuit using both a transform domain and time domain approach.



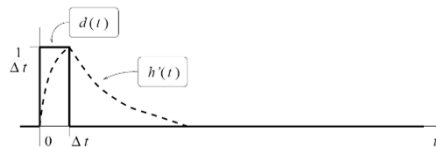
$$H(s) = \frac{1}{1 + RCs}$$

Time domain description and convolution

40

Convolution

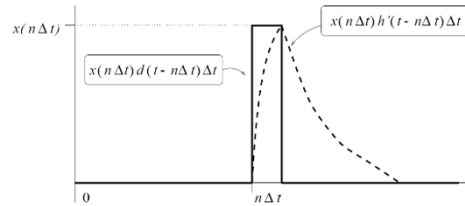
- in order to calculate the response of a system to an input $x(t)$ using the impulse response $h(t)$ recall that
 - the input $x(t)$ can be represented as a summation (integration) of impulses
 - the system is linear and superposition applies
- therefore we can evaluate the response to one of the impulses then integrate the responses to all such impulses to obtain the output $y(t)$
- as Δt tends toward zero $d(t)$ tends toward an impulse and $h'(t)$ tends toward the impulse response



Time domain description and convolution

41

Convolution



- illustrated is the response to a delayed and scaled pulse

$$x(n\Delta t)d(t - n\Delta t)\Delta t$$

- the response is given by

$$x(n\Delta t)h'(t - n\Delta t)\Delta t$$

- using superposition the response to the input

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(n\Delta t)d(t - n\Delta t)\Delta t$$

is given by

$$\hat{y}(t) = \sum_{n=-\infty}^{\infty} x(n\Delta t)h'(t - n\Delta t)\Delta t$$

Time domain description and convolution

42

Convolution

$$\hat{y}(t) = \sum_{n=-\infty}^{\infty} x(n\Delta t)h'(t - n\Delta t)\Delta t$$

- and as $\Delta t \rightarrow 0$ the
 - pulse becomes an impulse at time $n\Delta t$
 - $n\Delta t$ becomes continuous time variable τ
 - $h'(\cdot)$ becomes the impulse response $h(\cdot)$
 - time step Δt becomes the differential $d\tau$
 - the summation becomes an integral

$$y(t) = \int_{-\infty}^{\infty} h(t - \tau)x(\tau)d\tau$$

or alternatively and equivalently

$$y(t) = \int_{-\infty}^{\infty} x(t - \tau)h(\tau)d\tau$$

Time domain description and convolution

43

Convolution

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau$$

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$

- t is the time for which we calculate the output
- τ is the variable used for integration

- notation for convolution:

$$y(t) = x(t) * h(t)$$

- an important relationship between the frequency and time domain
“convolution in the time domain is equivalent to multiplication in the frequency domain”

Time domain description and convolution

44

Convolution

- we can verify the convolution by considering a complex phasor input $x(t) = \exp(st)$ to a system with impulse response $h(t)$

$$\begin{aligned} y(t) &= \int_0^{\infty} x(t-\tau)h(\tau)d\tau \\ &= \int_0^{\infty} \exp(s(t-\tau))h(\tau)d\tau \\ &= \exp(st) \int_0^{\infty} h(\tau) \exp(-s\tau)d\tau \\ &= \exp(st)H(s) \end{aligned}$$

i.e. the response to a complex phasor is a complex phasor scaled by the transfer function $H(s)$

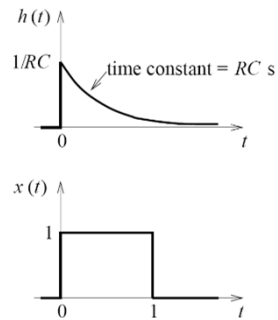
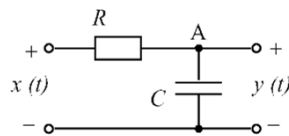
Time domain description and convolution

45

Convolution

Example

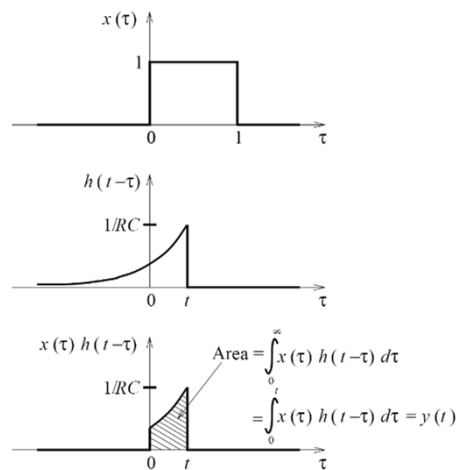
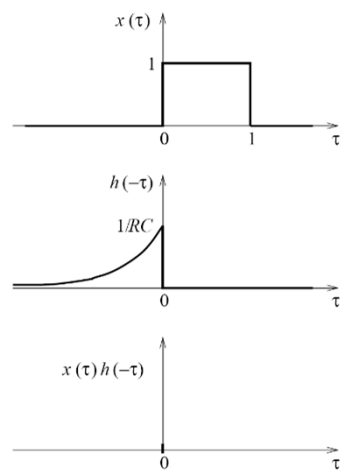
Using convolution, calculate the response of the RC circuit to the rectangular pulse $x(t)$.



Time domain description and convolution

46

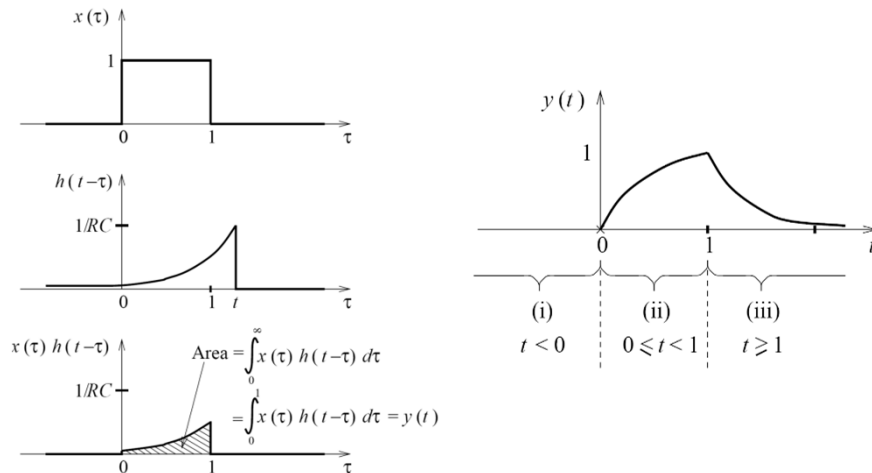
Convolution



Time domain description and convolution

47

Convolution

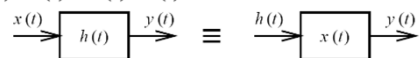


Time domain description and convolution

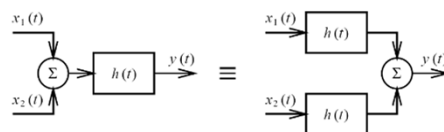
48

Properties of convolution

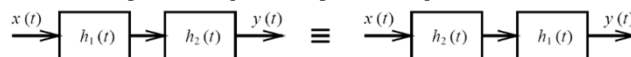
- commutative $x(t) * h(t) = h(t) * x(t)$



- distributive over addition $h(t) * [x_1(t) + x_2(t)] = [h(t) * x_1(t)] + [h(t) * x_2(t)]$



- associative $h_2(t) * [h_1(t) * x(t)] = h_1(t) * [h_2(t) * x(t)]$



- multiplication

$$L[x_1(t) x_2(t)] = \frac{1}{2\pi j} X_1(s) * X_2(s) \quad F[x_1(t) x_2(t)] = \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$

relevant to the study of modulation

*"multiplication in the time domain is equivalent to
convolution in the frequency domain"*

Time domain description and convolution

49

Properties of convolution

Time delay

- in some cases the transform domain is easier to work with than the time domain – e.g. differential equations of *RC* circuits
- for some problems it is the time domain that is better suited, e.g. time delay
- consider a unit impulse $\delta(t)$ applied to a pure time delay of a seconds

$$h(t) = \delta(t - a)$$

- and if we apply $x(t)$ to the same system

$$\begin{aligned} y(t) &= h(t) * x(t) \\ &= \delta(t - a) * x(t) \\ &= x(t - a) \end{aligned}$$

Time domain description and convolution

50

Properties of convolution

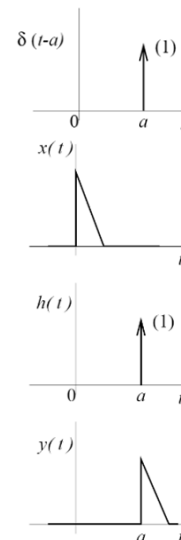
Time delay

- therefore a system with impulse response $\delta(t - a)$ just delays the input by a seconds

- if we have the impulse response we can obtain the transfer function:

$$\begin{aligned} H(s) &= L[\delta(t - a)] \\ &= \exp(-as) \cdot 1 \\ &= \exp(-as) \end{aligned}$$

- we have a complete description of a system from its
 - impulse response
 - transfer function



Time domain description and convolution

51

Summary

You should be:

- familiar with the impulse function and its properties;
- able to work out the impulse response of a simple linear system;
- familiar with the convolution integral and its properties;
- able to evaluate the response of simple linear system to a simple waveform using the integral;
- calculate the response of a simple system to a simple waveform using transform techniques.