EURECOM
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Essential Mathematical Methods for Engineers (MathEng)

Laplace transforms - time domain description and convolution

1. Determine the Laplace transforms of:

$$f(t) = e^{kt}\,,\ \ f(t) = \sin(at) \text{ and } f(t) = \cos(at)\,.$$

2. Prove the linearity property of the Laplace transform, that is that:

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\}$$

   and subsequently determine the Laplace transform of:

$$f(t) = 3t + 2e^{3t}\,.$$

3. Prove the first shift property of the Laplace transform, that is that for a function $f(t)$ with Laplace transform $F(s)$, with $\mathrm{Re}(s) > \sigma_c$, then:

$$\mathcal{L}\{e^{at}f(t)\} = F(s\text{-}a), \quad \mathrm{Re}(s) > \sigma_c + \mathrm{Re}(a)$$

   and subsequently determine the Laplace transform of:

$$f(t) = te^{-2t}\,.$$

4. Prove the Laplace derivative of transform property (also known as the multiplication-by-t property), that is that for a function $f(t)$ with Laplace transform $F(s)$, with $\mathrm{Re}(s) > \sigma_c$, then the functions $t^n f(t)$ (n = 1,2,…) have Laplace transforms give by:

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n},\ \mathrm{Re(s)} > \sigma_c,$$

   and subsequently determine $\mathcal{L}\{t\sin(3t)\}$ and $\mathcal{L}\{t^2 e^t\}$.

5. Determine the Laplace transforms $\mathcal{L}\{\mathrm{d}f/\mathrm{d}t\}$ and $\mathcal{L}\{\mathrm{d}^2 f/\mathrm{d}t^2\}$.

6. Show that:

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\} = \frac{1}{s}F(s)$$

   and determine

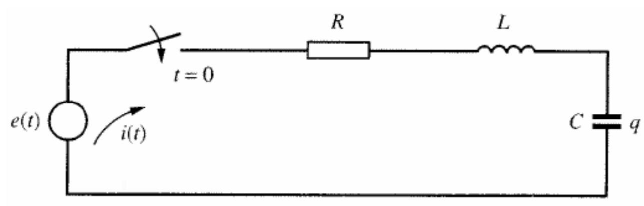
$$\mathcal{L}\left\{\int_0^t (\tau^3 + \sin 2\tau)d\tau\right\}.$$

7. Determine the inverse Laplace transforms:

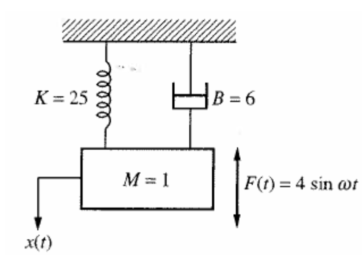
$$\mathcal{L}^{-1}\left\{\frac{1}{(s+3)(s-2)}\right\},\ \mathcal{L}^{-1}\left\{\frac{s+1}{s^2(s^2+9)}\right\}\ \text{and}\ \mathcal{L}^{-1}\left\{\frac{2}{s^2+6s+13}\right\}.$$

8. What would be the response of the system on slide 116 to the phasor $\exp(jn\omega_0 t)$ if the positions of the resistor and capacitor were reversed? Evaluate the new transfer function.

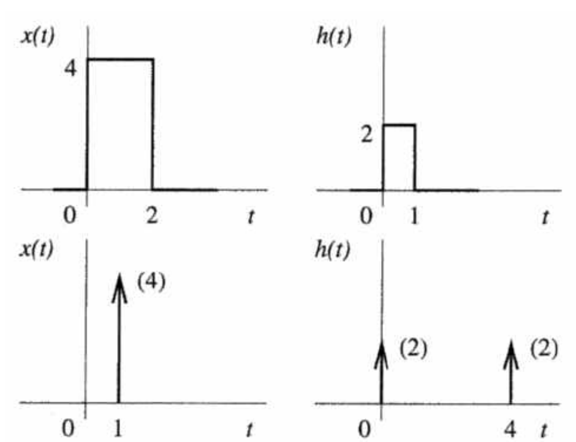
9. An LCR circuit consists of a single resistor R, a capacitor C and an inductor L connected in series together with a voltage source $e(t)$. Prior to closing the switch at time $t = 0$, both the charge on the capacitor and the resulting current in the circuit are zero. Determine the charge $q(t)$ on the capacitor and the resulting current $i(t)$ in the circuit at time t given that $R = 160\ \Omega$, $L = 1$ H, $C = 10^{-4}$ F and $e(t) = 20$ V.

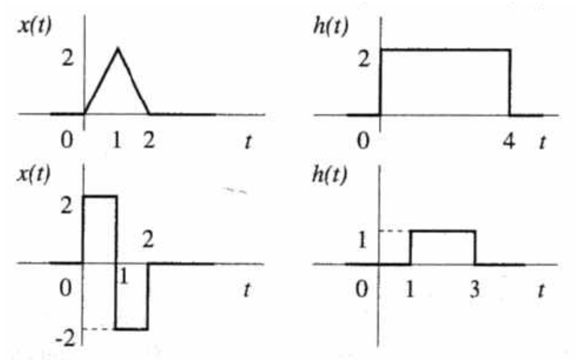


10. The mass of the mass-spring-damper system below is subjected to an externally applied periodic force $F(t) = 4 \sin \omega t$ at time $t = 0$. Determine the resulting displacement $x(t)$ of the mass at time t, given that $x(0) = \dot{x}(0) = 0$ for the two cases (a) $\omega = 2$ and (b) $\omega = 5$. In the case of $\omega = 5$, what would happen to the response if the damper were missing?



11. Convolve the following pairs of input signal and impulse responses. Sketch the outputs and label the horizontal and vertical axis with significant values.





12. In seismic signal processing an explosion is used on the earth's surface to generate an input waveform similar to that shown in the graph below. Given that the impulse response of the earth itself is given by h(t) where

$$h(t) = \begin{cases} 2t & \text{for } 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

use convolution to develop an expression for the output $y(t)$. A problem can arise with the detonator such that the explosion hiccups, resulting in the waveform illustrated in graph (b) below. Derive an expression for the output in this case.