Mathematical Methods for Engineers (MathEng)

EXAM

December 2023

```
Duration: 2 hrs, all documents and calculators permitted
ATTEMPT ALL QUESTIONS — ANSWER IN ENGLISH
```

1. Using Euler's identity (or any other appropriate method), write down an expression for the complex Fourier series of the signal $x(t)$:

$$x(t) = 3\cos(5t) + 4\sin(10t)$$

[5 marks]

To find the complex Fourier series of $x(t) = 3\cos(5t) + 4\sin(10t)$,

we use Euler's identity: $\cos(\omega t) = \frac{e^{j\omega t}+e^{-j\omega t}}{2}$, $\sin(\omega t) = \frac{e^{j\omega t}-e^{-j\omega t}}{2j}$.

Step 1: Rewrite $\cos(5t)$ and $\sin(10t)$ using Euler's identity

1. $3\cos(5t) \rightarrow 3\left(\frac{e^{j5t}+e^{-j5t}}{2}\right) = \frac{3}{2}e^{j5t} + \frac{3}{2}e^{-j5t}$
2. $4\sin(10t) \rightarrow 4\left(\frac{e^{j10t}-e^{-j10t}}{2j}\right) = \frac{4}{2j}\left(e^{j10t} - e^{-j10t}\right)$

Recall: $\frac{1}{j} = \frac{j}{j^2} = \frac{j}{-1} = -j$

$$\frac{4}{2j}\left(e^{j10t} - e^{-j10t}\right) = \frac{2}{j}\left(e^{j10t} - e^{-j10t}\right) = \frac{2}{j}e^{j10t} - \frac{2}{j}e^{-j10t}$$
$$= \frac{2j}{j^2}e^{j10t} - \frac{2j}{j^2}e^{-j10t} = \frac{2j}{-1}e^{j10t} - \frac{2j}{-1}e^{-j10t} = -2je^{j10t} + 2je^{-j10t}.$$

Thus, $x(t) = \frac{3}{2}e^{j5t} + \frac{3}{2}e^{-j5t} - 2je^{j10t} + 2je^{-j10t}$.

Step 2: Group the terms

The complex Fourier series representation of $x(t)$ is: $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$, where c_k are the complex Fourier coefficients.

Here, $x(t)$ has terms at frequencies ± 5 and ± 10. The coefficients c_k are:

- At $k = 5$: $c_5 = \frac{3}{2}$,
- At $k = -5$: $c_{-5} = \frac{3}{2}$,
- At $k = 10$: $c_{10} = -2j$,
- At $k = -10$: $c_{-10} = 2j$,
- All other $c_k = 0$.

Final Answer:

The complex Fourier series of $x(t)$ is: $\boxed{x(t) = \frac{3}{2}e^{j5t} + \frac{3}{2}e^{-j5t} - 2je^{j10t} + 2je^{-j10t}}$

2. Develop an expression for the Fourier Transform of the signal $x(t)$ illustrated in `Figure Q2` below:

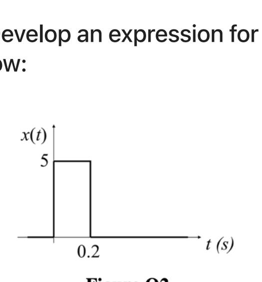


Figure Q2

[6 marks]

To develop the Fourier Transform $X(f)$ of the signal $x(t)$ illustrated in the figure, we follow the same steps for a rectangular pulse.

Step 1: Signal Description

The signal $x(t)$ is defined as: $x(t) = \begin{cases} 5, & 0 \le t \le 0.2, \\ 0, & \text{otherwise.} \end{cases}$

Step 2: Fourier Transform Definition

The Fourier Transform is given by: $X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$.

Since $x(t)$ is nonzero only in the interval $[0, 0.2]$, the limits of integration reduce to $[0, 0.2]$: $X(f) = \int_0^{0.2} 5e^{-j2\pi ft}dt$.

Step 3: Evaluate the Integral

Factor out the constant 5: $X(f) = 5\int_0^{0.2} e^{-j2\pi ft}dt$.

The integral of $e^{-j2\pi ft}$ is: $\int e^{-j2\pi ft}dt = \frac{e^{-j2\pi ft}}{-j2\pi f}$.

Apply the limits of integration: $X(f) = 5\left[\frac{e^{-j2\pi ft}}{-j2\pi f}\right]_0^{0.2}$.

Substitute the limits: $X(f) = 5 \cdot \frac{1}{-j2\pi f}\left(e^{-j2\pi f(0.2)} - e^{0}\right)$.

Simplify: $X(f) = \frac{5}{-j2\pi f}\left(e^{-j0.4\pi f} - 1\right)$.

Step 4: Simplify Further

Using the property $e^{-j\theta} - 1 = -2j\sin\left(\frac{\theta}{2}\right)e^{-j\frac{\theta}{2}}$ which is derived as follows:

1. **Rewrite** $e^{-j\theta} - 1$: Expand using Euler's formula: $e^{-j\theta} - 1 = \cos(\theta) - j\sin(\theta) - 1 = (\cos(\theta) - 1) - j\sin(\theta)$
2. **Factorize Trigonometric Terms:** Use the half-angle identities:
   - $\cos(\theta) = 1 - 2\sin^2\left(\frac{\theta}{2}\right) \implies \cos(\theta) - 1 = -2\sin^2\left(\frac{\theta}{2}\right)$
   - $\sin(\theta) = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$.

   Substituting these: $e^{-j\theta} - 1 = -2\sin^2\left(\frac{\theta}{2}\right) - j\,2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$.
3. **Factor Out Common Terms:**
   - **Identify Common Factor:**

     Both terms contain $-2j\sin\left(\frac{\theta}{2}\right)$ as a common factor: 1. $-2\sin^2\left(\frac{\theta}{2}\right)$: - This can be written as $-2j\sin\left(\frac{\theta}{2}\right) \cdot \frac{\sin\left(\frac{\theta}{2}\right)}{j}$. 2. $-j \cdot 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$: - This is already proportional to $-2j\sin\left(\frac{\theta}{2}\right)$.
   - **Factorization:**

     Factor $-2j\sin\left(\frac{\theta}{2}\right)$ out of the entire expression: $e^{-j\theta} - 1 = -2j\sin\left(\frac{\theta}{2}\right) \cdot \left(\frac{\sin\left(\frac{\theta}{2}\right)}{j} + \cos\left(\frac{\theta}{2}\right)\right)$.

     Simplify the term : $\frac{\sin\left(\frac{\theta}{2}\right)}{j} = -j\sin\left(\frac{\theta}{2}\right)$. Thus: $e^{-j\theta} - 1 = -2j\sin\left(\frac{\theta}{2}\right) \cdot \left(\cos\left(\frac{\theta}{2}\right) - j\sin\left(\frac{\theta}{2}\right)\right)$.
   - **Recognize the Exponential Form:** The term $\cos\left(\frac{\theta}{2}\right) - j\sin\left(\frac{\theta}{2}\right)$ is equivalent to $e^{-j\frac{\theta}{2}}$, using Euler's formula.
5. **Simplify:** Recognize the term in parentheses as $e^{-j\frac{\theta}{2}}$: $e^{-j\theta} - 1 = -2j\sin\left(\frac{\theta}{2}\right)e^{-j\frac{\theta}{2}}$.

This compactly combines the amplitude term $-2j\sin\left(\frac{\theta}{2}\right)$ and the phase shift $e^{-j\frac{\theta}{2}}$.

rewrite $X(f)$:

$X(f) = \frac{5}{-j2\pi f} \cdot -2j\sin(0.2\pi f)e^{-j0.2\pi f}$.

Cancel $-j$ and simplify: $X(f) = \frac{5 \cdot 2\sin(0.2\pi f)}{2\pi f}e^{-j0.2\pi f}$.

Finally: $X(f) = \frac{5\sin(0.2\pi f)}{\pi f}e^{-j0.2\pi f}$.

Final Expression

$$\boxed{\begin{aligned} X(f) &= \frac{5\sin(0.2\pi f)}{\pi f} \cdot e^{-j0.2\pi f} \\ &= 5 \cdot \mathrm{sinc}(0.2f) \cdot e^{-j0.2\pi f} \end{aligned}}$$

where the **sinc function** is defined as: $\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$.

Interpretation

- $\frac{\sin(0.2\pi f)}{\pi f}$: This is the sinc function, representing the frequency-domain shape of the rectangular pulse.
- $e^{-j0.2\pi f}$: This is a phase shift due to the non-centered nature of the pulse (starting at $t = 0$).

In [1]:

```
using FFTW, LinearAlgebra, Plots, LaTeXStrings
```

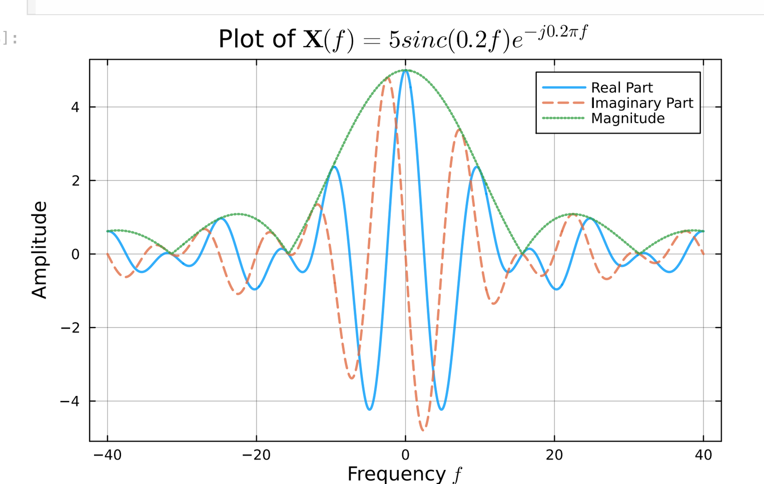
In [2]:

```
include("modules/operations.jl");
```

In [3]:

```
# Define the unscaled sinc function
sinc_unscaled(x::Real) = x == 0 ? 1.0 : sin(pi * x) / (pi * x)

# Define the polymorphic sinc function with a normalization option
sinc(x::Real; normalized::Bool = true) = normalized ? sinc_unscaled(x / π) : sinc_unscaled(x)

# Frequency range
f = range(-40, 40, length=1000)

# Function components
A = 5 .* sinc.(0.2 .* f)  # Amplitude of the signal
ϕ = e.^ (-j .* 0.2π .* f)  # Phase shift
X = A .* ϕ  # Combined function

# Plot with title, labels, and semi-transparent grid
plot(f, real.(X)
    , label="Real Part", linestyle=:solid, linewidth=2, alpha=0.8, size = (600,400)
    , xlabel="Frequency " * L"f", ylabel="Amplitude"
    , title="Plot of " * L"\mathbf{X}(f) = 5 sinc(0.2 f) e^{-j 0.2 π f}"
    , grid=true, gridalpha=0.2  # Enable grid and set transparency
    , framestyle=:box
)

# Overlay additional lines
plot!(f, imag.(X), label="Imaginary Part", linestyle=:dash, linewidth=2, alpha=0.8)
plot!(f, abs.(X), label="Magnitude", linestyle=:dot, linewidth=2, alpha=0.8)
```

Out[3]:



3. A linear, time-invariant system has the following transfer function:

$$H(s) = \frac{10(s+100)}{s^2+3s+100}$$

(a) Derive an expression for $H(s)$ in the usual, normal form.

(b) Determine the frequency-invariant gain K and the position of the poles and zeros.

(c) Sketch a Bode plot of the magnitude frequency response.

(d) Sketch a Bode plot of the phase frequency response.

[8 marks]

(a) Derive an expression for $H(s)$ in the usual, normal form.

To derive the transfer function $H(s)$ in the usual **normal form**, we factorize the numerator and denominator in terms of their natural frequencies and damping ratios.

The given transfer function is: $H(s) = \frac{10(s+100)}{s^2+3s+100}$.

Step 1: Denominator Form

The denominator is: $s^2 + 3s + 100$.

This matches the general form of a second-order system $s^2 + 2\zeta\omega_n s + \omega_n^2$, where ζ is the damping ratio and ω_n is the natural frequency.

Here $\omega_n^2 = 100 \Rightarrow \omega_n = \sqrt{100} = 10$, and $2\zeta\omega_n = 3 \Rightarrow \zeta = \frac{3}{2 \cdot 10} = 0.15$.

Thus, the denominator becomes $s^2 + 3s + 100 = (s^2 + 2(0.15)(10)s + 10^2)$.

Step 2: Numerator Form

The numerator is: $10(s+100)$.

Factor out 100 to normalize: $10(s+100) = 10 \cdot 100\left(\frac{s}{100} + 1\right) = 1000\left(\frac{s}{100} + 1\right)$.

Step 3: Rewrite in Normal Form

[illegible]

This is the normalized form of $H(s)$.

(b): Determine the Frequency-Invariant Gain K and the Positions of Poles and Zeros

1. Transfer Function

The given transfer function is: $H(s) = \frac{10(s+100)}{s^2+3s+100}$.

2. Frequency-Invariant Gain K

The frequency-invariant gain is the gain of the system as $s \to 0$. This is determined by evaluating the transfer function at $s = 0$: $K = H(0) = \frac{10(0+100)}{0^2+3(0)+100}$.

Simplify: $K = \frac{1000}{100} = 10$.

Thus, the frequency invariant gain is $K = 10$.

3. Poles

The poles are the roots of the denominator $s^2 + 3s + 100 = 0$.

[illegible]

4. Zeros

The zero is the root of the numerator $10(s+100) = 0 \Rightarrow s = -100$.

Thus, there is one zero at $s = -100$.

Final Results:

- **Frequency-Invariant Gain** K: $K = 10$.
- **Poles**: [illegible]
- **Zeros**: $s = -100$.

In [4]:

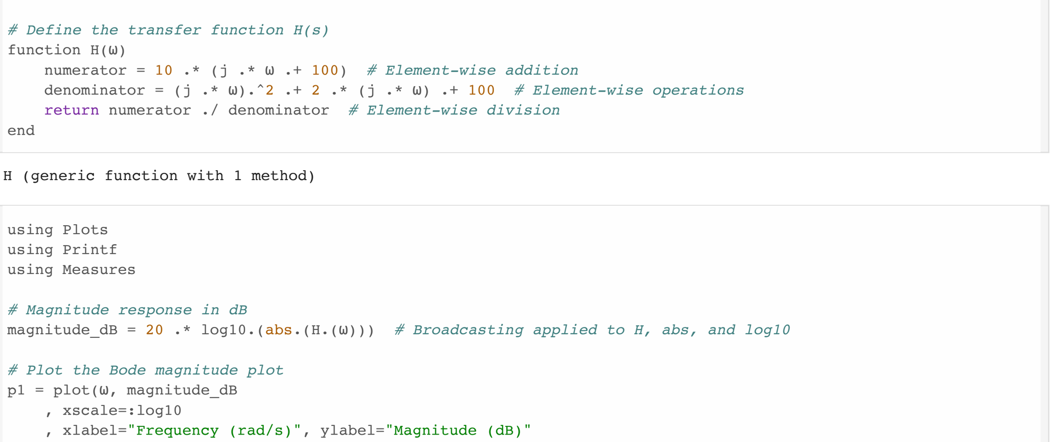
```
[illegible]
```

In [5]:

```
[illegible]
```

In [6]:

```
[illegible]
```



4. Sketch magnitude and phase responses for a sampled data system with a pair of complex conjugate zeros and two poles at the origin.

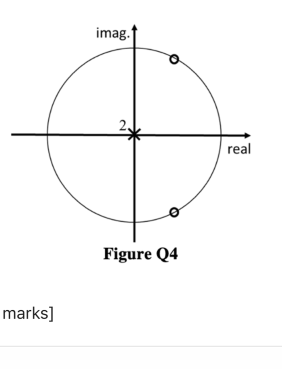


Figure Q4

[4 marks]

5. A random variable X is uniformly distributed between $x = 0$ and $x = 1$. Via any appropriate method, determine the expected value $E[Y]$ of $Y = \exp(X)$.

[4 marks]

Given $Y = \exp(X)$ and $X \sim U(0, 1)$.

1. Expected Value Formula

The expected value of a random variable Y is given by: $E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\,dy$.

Since X is uniformly distributed, its probability density function (PDF) is $f_X(x) = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$

For $Y = \exp(X)$, the expected value becomes: $E[Y] = \int_0^1 \exp(x) f_X(x)\,dx$.

Because $f_X(x) = 1$ for $0 \le x \le 1$, this simplifies to: $E[Y] = \int_0^1 \exp(x)\,dx$.

2. Solve the Integral

The integral of $\exp(x)$ is: $\int \exp(x)\,dx = \exp(x) + C$.

Now, evaluate the definite integral: $\int_0^1 \exp(x)\,dx = [\exp(x)]_0^1 = \exp(1) - \exp(0)$.

Simplify: $\int_0^1 \exp(x)\,dx = e - 1$.

3. Final Answer

The expected value is: $\boxed{E[Y] = e - 1}$.

6. Identify the pivots and free variables of the following two matrices A and B. Following the method which we studied in class, find the special solution corresponding to each free variable, and by combining the special solutions, describe every solution to $Ax = 0$ and $Bx = 0$.

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}.$$

[7 marks]

In [ ]:

7. For a projection matrix $P = A(A^TA)^{-1}A^T$, show that $P^2 = P$ and then explain, in terms of the column space of P, why projections P_b and $P(P_b)$ give identical results.

[5 marks]

---

1. Show that $P^2 = P$

The projection matrix P is defined as: $P = A(A^TA)^{-1}A^T$, where A is a matrix with linearly independent columns.

Compute P^2:

We want to show: $P^2 = P$.

Start with P^2: $P^2 = P \cdot P = \left(A(A^TA)^{-1}A^T\right) \cdot \left(A(A^TA)^{-1}A^T\right)$.

Expand the multiplication: $P^2 = A(A^TA)^{-1}A^TA(A^TA)^{-1}A^T$.

Since A^TA is invertible, $A^TA(A^TA)^{-1} = I$ (identity matrix). So: $P^2 = A(A^TA)^{-1}(I)A^T = A(A^TA)^{-1}A^T$.

This simplifies to: $P^2 = P$.

2. Projections Pb and $P(Pb)$ Give Identical Results

Interpretation of P:

The projection matrix P projects any vector b onto the **column space** of A, denoted as $\mathrm{Col}(A)$.

Explain Pb:

$Pb = P\mathbf{b} = A(A^TA)^{-1}A^T\mathbf{b}$. This gives the projection of $\mathbf{b}$ onto $\mathrm{Col}(A)$.

Explain $P(Pb)$:

$P(Pb) = P(P\mathbf{b})$. Substitute Pb into $P(Pb)$: $P(Pb) = P \cdot P\mathbf{b}$. Since we showed that $P^2 = P$, this becomes: $P(Pb) = P\mathbf{b}$.

Why Are Pb and $P(Pb)$ Identical?

- $Pb = P\mathbf{b}$ is already the projection of $\mathbf{b}$ onto $\mathrm{Col}(A)$.
- Applying P again to Pb does not change it, because projecting a vector already in the subspace $\mathrm{Col}(A)$ onto the same subspace leaves it unchanged.
- Hence: $P(Pb) = Pb$.

3. Column Space Perspective

In terms of the column space of P:

1. The column space of P (and thus Pb) is the **same as** $\mathrm{Col}(A)$.
2. Applying P to Pb projects Pb onto $\mathrm{Col}(A)$, but since $Pb \in \mathrm{Col}(A)$, the result is unchanged.

Thus, projections Pb and $P(Pb)$ are identical because projecting a vector already in the column space does nothing.

Conclusion

1. **Projection matrix property**: $P^2 = P$.
2. **Projections**: Pb and $P(Pb)$ are identical because Pb lies in the column space, and re-projecting it does not alter it.
3. **Idempotence**: P an idempotent matrix, which is a key characteristic of projection matrices.

In [ ]: