Mathematical Methods for Engineers (MathEng)
EXAM
11th February 2020
Duration: 2 hrs, calculators permitted, no documents
This exam paper contains 7 questions and 40 marks.
ATTEMPT ALL QUESTIONS – ANSWER IN ENGLISH

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1. Via any appropriate method, and by justifying your response, write an expression for the Fourier transform of $x(t) = \cos w_0 t$.
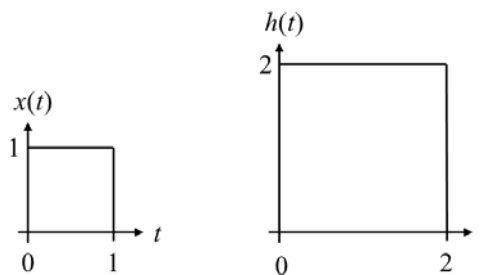
   [5 marks]

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2. Sketch the s-plane pole-zero plot for a system with the following transfer function and then sketch the amplitude frequency response.

$$H(s) = \frac{s^2 + 9}{(s^2 + 0.6s + 4.09)}$$

   [5marks]

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3. A system with the illustrated transfer function $h(t)$ is excited by the illustrated input signal $x(t)$. Using any appropriate method, **sketch** the output $y(t) = x(t) * h(t)$, where * indicates convolution.

   [5 marks]

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4. Using any appropriate technique, derive an expression for the inverse z-transform of:

$$X(z) = \frac{z}{2z^2 - 3z + 1} \quad \text{where} \quad |z| < \frac{1}{2}$$

   [6marks]

5. The joint probability density function for random variables x and y is given by:

$$f_{X,Y}(x,y) = \begin{cases} 8xy & 0 \le x \le 1, 0 \le y \le x \\ 0 & \text{otherwise} \end{cases}.$$

Find:
(a) The marginal probability density function $f_X(x)$;
(b) The marginal probability density function $f_Y(y)$;
(c) The conditional probability density function $f_{X|Y}(x|y)$;
(d) The conditional probability density function $f_{Y|X}(y|x)$.

[8 marks]

6. Determine whether

$$G = \begin{bmatrix} 0 & 0.5 \\ -0.25 & 0.25 \end{bmatrix}$$

is the inverse of any of the following matrices:

$$A = \begin{bmatrix} 4 & -8 \\ 4 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, C = \begin{bmatrix} 2 & -4 \\ 2 & 0 \end{bmatrix}, D = \begin{bmatrix} 2 & -4 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

[4 marks]

7. Use the Gram-Schmidt process to construct an orthogonal set of vectors from the linearly independent set:

$$X_1 = \begin{bmatrix} -4 \\ 3 \\ 6 \end{bmatrix}, X_2 = \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}, X_3 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}.$$

[7 marks]

[TURN OVER]

Table of selected Laplace transforms

$$X(s) = \int_{-\infty}^{\infty} x(t) \exp(-st)\, dt$$

N.B.: lower limit is 0 for one-sided
Laplace transform

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) \exp(st)\, ds$$

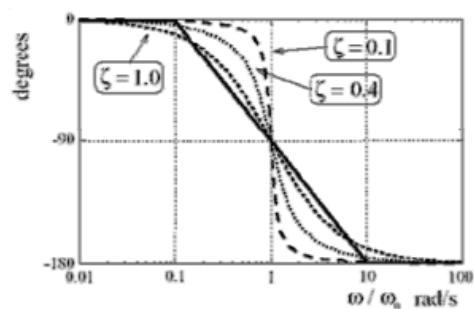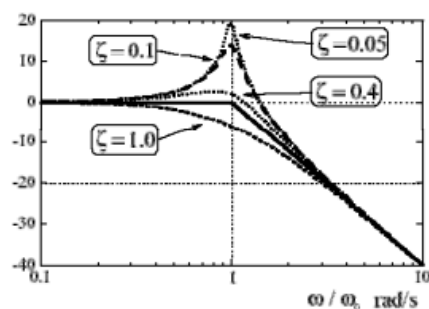| $x(t)$ $(t \geq 0)$	$X(s)$
$\delta(t)$	1
$\delta(t-\alpha)$	$\exp(-\alpha s)$
1 (unit step)	$\dfrac{1}{s}$
t (unit ramp)	$\dfrac{1}{s^2}$
$\exp(-\alpha t)$	$\dfrac{1}{s+\alpha}$
$t\exp(-\alpha t)$	$\dfrac{1}{(s+\alpha)^2}$
$\sin(\alpha t)$	$\dfrac{\alpha}{s^2+\alpha^2}$
$\cos(\alpha t)$	$\dfrac{s}{s^2+\alpha^2}$
$e^{-\alpha t}\sin(\omega t)$	$\dfrac{\omega}{(s+\alpha)^2+\omega^2}$
$e^{-\alpha t}\cos(\omega t)$	$\dfrac{s+\alpha}{(s+\alpha)^2+\omega^2}$

Bode plots

Poles or zeros on the real axis:
$$(s + a) = a\left(\tfrac{s}{a} + 1\right) = \tfrac{1}{\tau}(\tau s + 1)$$

Complex conjugate poles (or zeros):

$$(s^2 + As + B) = (s^2 + 2\zeta\omega_0 s + \omega_0^2) = \omega_0^2((s/\omega_0)^2 + 2\zeta(s/\omega_0) + 1)$$

Table of selected z-transforms

$$x_c(t) = \sum_{n=0}^{\infty} x(n\Delta t)\delta(t - n\Delta t)$$

$$X_c(s) = \sum_{n=0}^{\infty} x(n\Delta t)\exp(-n\Delta ts)$$

$$X_c(s) = X(z)|_{z=e^{\Delta ts}}$$

$$X_c(\omega) = X_s(s)|_{s=j\omega}$$

$$X(\omega) = X(z)|_{z=\exp(\Delta tj\omega)}$$

$$X(z) = \sum_{n=0}^{\infty} x(n\Delta t)z^{-n}$$

$x(n)$ $(n \geq 0)$	$X(z)$
$\delta(n)$ unit pulse	1
$\delta(n - m)$	z^{-m}
1 (unit step)	$\dfrac{z}{z-1}$
n (unit ramp)	$\dfrac{z}{(z-1)^2}$
$\exp(-\alpha n)$	$\dfrac{z}{(z - e^{-\alpha})}$
$n\exp(-\alpha n)$	$\dfrac{e^{-\alpha}z}{(z - e^{-\alpha})^2}$
$\sin(\beta n)$	$\dfrac{z\sin(\beta)}{z^2 - 2z\cos(\beta) + 1}$
$\cos(\beta n)$	$\dfrac{z^2 - z\cos(\beta)}{z^2 - 2z\cos(\beta) + 1}$
$e^{-\alpha n}\sin(\beta n)$	$\dfrac{ze^{-\alpha}\sin(\beta)}{z^2 - 2ze^{-\alpha}\cos(\beta) + e^{-2\alpha}}$
$e^{-\alpha n}\cos(\beta n)$	$\dfrac{z^2 - ze^{-\alpha}\cos(\beta)}{z^2 - 2ze^{-\alpha}\cos(\beta) + e^{-2\alpha}}$

Table of selected Fourier transform pairs

Function	$x(t)$	$X(\omega)$
Rectangular function of width τ	$\Pi(t/\tau)$	$\tau\,\mathrm{sinc}(\omega\tau/2)$
Triangular function of width 2τ	$\Lambda(t/\tau)$	$\tau\,\mathrm{sinc}^2(\omega\tau/2)$
Train of impulses every Δt	$\delta_T(t)$	$2\pi/\Delta t \sum_n \delta(\omega - 2\pi n/\Delta t)$

NB: $\mathrm{sinc}(x) = \sin(\pi x)/\pi x$
NB: $\mathrm{sa}(x) = \sin(x)/x$

Euler's identity

$$\exp(j\theta) = \cos\theta + j\sin\theta$$

$$\cos(n\omega_0 t) = \frac{\exp(jn\omega_0 t) + \exp(-jn\omega_0 t)}{2}$$

$$\sin(n\omega_0 t) = \frac{\exp(jn\omega_0 t) - \exp(-jn\omega_0 t)}{2j}$$

Fourier series and transforms

Trigonometric Fourier series

$$x(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t)$$

$$A_n = \frac{2}{T} \int_{-T/2}^{T/2} x(t)\cos(n\omega_0 t)dt$$

$$B_n = \frac{2}{T} \int_{-T/2}^{T/2} x(t)\sin(n\omega_0 t)dt$$

Complex Fourier series – periodic and continuous in time, discrete in frequency

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n \exp(jn\omega_0 t)$$

$$X_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t)\exp(-jn\omega_0 t)\, dt$$

Fourier transform – continuous in time, continuous in frequency

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\exp(j\omega t)\, d\omega$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\exp(-j\omega t)\, dt$$

Discrete-time Fourier transform – discrete in time, continuous and periodic in frequency

$$x(n\Delta t) = \frac{\Delta t}{2\pi} \int_{-\pi/\Delta t}^{\pi/\Delta t} X(\omega)\exp(jn\Delta t\omega)\, d\omega$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n\Delta t)\exp(-j\omega n\Delta t)$$

Discrete Fourier transform – discrete and periodic in time and in frequency

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \exp\left(\frac{jnk2\pi}{N}\right)$$

$$X(k) = \sum_{n=0}^{N-1} x(n)\exp\left(\frac{-jnk2\pi}{N}\right)$$

[TURN OVER]

Transformation of random variables

$$f_Y(y) = \sum_{i=1}^{N} f_X(x_i) \left|\frac{dx_i}{dy}\right|_{x_i = g_i^{-1}(y)}$$

$$f_{UV}(u,v) = f_{XY}(x,y) \left|\frac{\partial(x,y)}{\partial(u,v)}\right|_{\substack{x = g_1^{-1}(u,v) \\ y = g_2^{-1}(u,v)}}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix}$$

[END OF PAPER]