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Essential Mathematical Methods for Engineers (MathEng)

The discrete Fourier transform

1. Calculate the DFT, $X(k)$, of the sequence $x(n)$ = ¼, ½, ¼. From the DFT spectrum of $x(n)$, determine the magnitude and phase of the 3-point DFT, $X(k)$, corresponding to the original sequence.

2. A finite-duration sequence of length L is given by:

$$x(n) = \begin{cases} 1, & 0 < n < L-1 \\ 0, & \text{otherwise} \end{cases}.$$

   Determine the DFT of N samples of this sequence where $N > L$. From the result, determine an expression for an N-point DFT of this sequence.

3. A continuous-time signal is known to contain two dominant frequencies at 1605 Hz and 1645 Hz and it is to be sampled at 10 kHz. Given 200 samples, is it possible to resolve the two frequencies by a DFT operation? If they cannot be measured, what frequency values could these two signals have to be altered to, to enable the DFT to adequately resolve the two signals?

4. Work out, by hand, $X(0)$ and $X(1)$, the first two frequency bin outputs for an 8-point DFT, assuming the complex input to be a unit amplitude phasor which rotates 1 cycle during the transform length and has zero starting phase.

5. Given that you are required to analyse a signal comprising the following 8 sample values:

$$j4,\quad -2\sqrt{2} + j2\sqrt{2},\quad -4,\quad -2\sqrt{2} - j2\sqrt{2},\quad -j4,\quad 2\sqrt{2} - j2\sqrt{2},\quad 4,\quad 2\sqrt{2} + j2\sqrt{2}$$

   calculate the first four components of the Fourier transform, $X(0)$, …, $X(3)$, and then sketch the full output for the 8-point DFT analyser.

6. Determine the DFT of the sequence $x(n)$ = 1, 2, 3, 4. Set the two highest frequency components to zero and then perform the IDFT. Compare the resulting approximation to the original sequence and comment on the potential of the DFT for compression.

7. An alternative transform is the discrete Cosine transform (DCT). The DCT is used extensively for image compression. The formulae are:

$$\text{DCT:}\quad S_k = c_k \sum_{n=0}^{N-1} s_n \cos\left[\pi(2n+1)\frac{k}{2N}\right], \qquad \text{IDCT:}\quad s_n = \sum_{k=0}^{N-1} S_k c_k \cos\left[\pi(2n+1)\frac{k}{2N}\right]$$

$$\text{where}\quad c_k = \begin{cases} \sqrt{1/n} & k = 0 \\ \sqrt{2/n} & k > 0 \end{cases}$$

Determine the DCT for the same sequence as in Q6, set the two highest frequency components to zero, and obtain a new approximation for $x(n)$ using the IDCT. Comment on the potential of the DCT for compression and critically compare your results to those obtained in Q6.

8. From the definition of a 4-point DFT, derive, from first principles, the radix-2 DIT algorithm and show the requisite flowchart with the appropriate weights.

9. Derive the FFT DIT algorithm from the definition of the DFT for the specific case where $N = 8$. Indicate the twiddle values as complex number pairs expressed to 6 decimal digits. From the DIT flowchart for the 8-point transform in your lecture notes, extend this into a 16-point transform with W_{16} twiddle values.

10. For the 8-point DIT FFT, follow the input data samples $x(0), \ldots, x(7)$ through all the 2-point butterfly operations on each pass, and obtain the intermediate processed values for each pass in the FFT. Continue to obtain the final complete 8-point transformed output values $X(0)$ to $X(7)$, in terms of the products of twiddle values W_8^0 to W_8^3 with input sample values $x(0), \ldots, x(7)$, and confirm, by comparison with the DFT matrix, that your solution is correct.