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                                               Mathematical Methods for Engineers (MathEng)
                                               EXAM
                                                December 2023
                                                    Duration: 2 hrs, all documents and calculators permitted
                                                    ATTEMPT ALL QUESTIONS - ANSWER IN ENGLISH
                                                1. Using Euler's identity (or any other appropriate method), write down an expression for the
                                                complex Fourier series of the signal x(t):
                                                                                                        x(t) = 3\cos(5t) + 4\sin(10t)
                                                [5 marks]
                                               To find the complex Fourier series of x(t) = 3\cos(5t) + 4\sin(10t),
                                               we use Euler's identity: \cos(\omega t)=rac{e^{j\omega t}+e^{-j\omega t}}{2},\quad \sin(\omega t)=rac{e^{j\omega t}-e^{-j\omega t}}{2i}.
                                               Step 1: Rewrite \cos(5t) and \sin(10t) using Euler's identity
                                                1. 3\cos(5t) \rightarrow 3(\frac{e^{j5t}+e^{-j5t}}{2}) = \frac{3}{2}e^{j5t} + \frac{3}{2}e^{-j5t}
                                                2. 4\sin(10t) \rightarrow 4(\frac{e^{j10t}-e^{-j10t}}{2i}) = \frac{4}{2i}(e^{j10t}-e^{-j10t})
                                                    Recall: \frac{1}{i} = \frac{j}{i^2} = \frac{j}{-1} = -j
                                                    rac{4}{2i}(e^{j10t}-e^{-j10t})=rac{2}{i}(e^{j10t}-e^{-j10t})=rac{2}{j}e^{j10t}-rac{2}{j}e^{-j10t}
                                                                         =rac{2j}{j^2}e^{j10t}-rac{2j}{j^2}e^{-j10t}=rac{2j}{-1}e^{j10t}-rac{2j}{-1}e^{-j10t}=-2je^{j10t}+2je^{-j10t}.
                                                    Thus, x(t) = \frac{3}{2}e^{j5t} + \frac{3}{2}e^{-j5t} - 2je^{j10t} + 2je^{-j10t}.
                                                Step 2: Group the terms
                                               The complex Fourier series representation of x(t) is: x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, where c_k are the complex Fourier coefficients.
                                                Here, x(t) has terms at frequencies \pm 5 and \pm 10. The coefficients c_k are:
                                                 • At k=5: c_5=\frac{3}{2},
                                                 • At k = -5: c_{-5} = \frac{3}{2},
                                                 • At k = 10: c_{10} = -2j
                                                 • At k = -10: c_{-10} = 2j
                                                 • All other c_k = 0.
                                               Final Answer:
                                               The complex Fourier series of x(t) is: \left|x(t)=rac{3}{2}e^{j5t}+rac{3}{2}e^{-j5t}-2je^{j10t}+2je^{-j10t}
ight.
                                                2. Develop an expression for the Fourier Transform of the signal x(t) illustrated in Figure Q2
                                               below:
                                                                                   t(s)
                                                                 0.2
                                                                  Figure Q2
                                               [6 marks]
                                                To develop the Fourier Transform X(f) of the signal x(t) illustrated in the figure, we follow the same steps for a rectangular pulse.
                                                Step 1: Signal Description
                                               The signal x(t) is defined as: x(t) = \begin{cases} 5, & 0 \le t \le 0.2, \\ 0, & \text{otherwise.} \end{cases}
                                                Step 2: Fourier Transform Definition
                                               The Fourier Transform is given by: X(f)=\int_{-\infty}^{\infty}x(t)e^{-j2\pi ft}dt .
                                               Since x(t) is nonzero only in the interval [0,0.2], the limits of integration reduce to [0,0.2]: X(f)=\int_0^{0.2}5e^{-j2\pi ft}dt.
                                                Step 3: Evaluate the Integral
                                                Factor out the constant 5: X(f) = 5 \int_0^{0.2} e^{-j2\pi ft} dt.
                                               The integral of e^{-j2\pi ft} is: \int e^{-j2\pi ft} dt = \frac{e^{-j2\pi ft}}{-i2\pi f}.
                                               Apply the limits of integration: X(f) = 5 \left[ \frac{e^{-j2\pi ft}}{-i2\pi f} \right]_0^{0.2} .
                                               Substitute the limits: X(f) = 5 \cdot \frac{1}{-i2\pi f} \left(e^{-j2\pi f(0.2)} - e^0\right).
                                               Simplify: X(f)=rac{5}{-j2\pi f}ig(e^{-j0.4\pi f}-1ig) .
                                                Step 4: Simplify Further
                                                Using the property e^{-j\theta}-1=-2j\sin\left(\frac{\theta}{2}\right)e^{-j\frac{\theta}{2}} which is derived as follows:
                                                1. Rewrite e^{-j\theta}-1: Expand using Euler's formula: e^{-j\theta}-1=\cos(\theta)-j\sin(\theta)-1=(\cos(\theta)-1)-j\sin(\theta)
                                                2. Factorize Trigonometric Terms: Use the half-angle identities:
                                                      • \cos(\theta) = 1 - 2\sin^2\left(\frac{\theta}{2}\right) \implies \cos(\theta) - 1 = -2\sin^2\left(\frac{\theta}{2}\right)
                                                      • \sin(\theta) = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right).
                                                    Substituting these: e^{-j\theta} - 1 = -2\sin^2\left(\frac{\theta}{2}\right) - j\cdot 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right).
                                                 3. Factor Out Common Terms:
                                                 • Identify Common Factor:
                                                    Both terms contain -2j\sin\left(\frac{\theta}{2}\right) as a common factor: 1. -2\sin^2\left(\frac{\theta}{2}\right): - This can be written as -2j\sin\left(\frac{\theta}{2}\right)\cdot\frac{\sin\left(\frac{\theta}{2}\right)}{i}. 2.
                                                    -j \cdot 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right): - This is already proportional to -2j\sin\left(\frac{\theta}{2}\right).
                                                 • Factorization:
                                                    Factor -2j\sin\left(\frac{\theta}{2}\right) out of the entire expression: e^{-j\theta}-1=-2j\sin\left(\frac{\theta}{2}\right)\cdot\left(\frac{\sin\left(\frac{\theta}{2}\right)}{i}+\cos\left(\frac{\theta}{2}\right)\right).
                                                    Simplify the term : \frac{\sin\left(\frac{\theta}{2}\right)}{j} = -j\sin\left(\frac{\theta}{2}\right). Thus: e^{-j\theta} - 1 = -2j\sin\left(\frac{\theta}{2}\right) \cdot \left(\cos\left(\frac{\theta}{2}\right) - j\sin\left(\frac{\theta}{2}\right)\right).
                                                 • Recognize the Exponential Form: The term \cos\left(\frac{\theta}{2}\right) - j\sin\left(\frac{\theta}{2}\right) is equivalent to e^{-j\frac{\theta}{2}}, using Euler's formula.
                                                5. Simplify: Recognize the term in parentheses as e^{-j\frac{\theta}{2}}: e^{-j\theta}-1=-2j\sin\left(\frac{\theta}{2}\right)e^{-j\frac{\theta}{2}}.
                                                This compactly combines the amplitude term -2j\sin\left(\frac{\theta}{2}\right) and the phase shift e^{-j\frac{\theta}{2}}.
                                               rewrite X(f):
                                               X(f) = \frac{5}{-i2\pi f} \cdot -2j\sin(0.2\pi f)e^{-j0.2\pi f}.
                                               Cancel -j and simplify: X(f) = rac{5 \cdot 2 \sin(0.2\pi f)}{2\pi f} e^{-j0.2\pi f}.
                                               Finally: X(f)=rac{5\sin(0.2\pi f)}{\pi f}e^{-j0.2\pi f}.
                                               Final Expression
                                               X(f)=rac{5\sin(0.2\pi f)}{\pi f}\cdot e^{-j0.2\pi f} where the sinc function is defined as: \mathrm{sinc}(x)=rac{\sin(\pi x)}{\pi x} .
                                                       = 5 \cdot \operatorname{sinc}(0.2f) \cdot e^{-j0.2\pi f}
                                                Interpretation
                                                 • \frac{\sin(0.2\pi f)}{\pi f}: This is the sinc function, representing the frequency-domain shape of the rectangular pulse.
                                                 • e^{-j0.2\pi f}: This is a phase shift due to the non-centered nature of the pulse (starting at t=0).
                                    In [1]:
                                                using FFTW, LinearAlgebra, Plots, LaTeXStrings
                                    In [2]:
                                                 include("modules/operations.jl");
                                    In [3]:
                                                 # Define the unscaled sinc function
                                                 sinc unscaled(x::Real) = x == 0 ? 1.0 : sin(pi * x) / (pi * x)
                                                 # Define the polymorphic sinc function with a normalization option
                                                 sinc(x::Real; normalized::Bool = true) = normalized ? <math>sinc\_unscaled(x / \pi) : sinc\_unscaled(x)
                                                 # Frequency range
                                                 f = range(-40, 40, length=1000)
                                                 # Function components
                                                 A = 5 \cdot * sinc.(0.2 \cdot * f) # Amplitude of the signal
                                                 \phi = e.^{\circ} (-j.* 0.2\pi.* f) # Phase shift
                                                 X = A \cdot * \phi # Combined function
                                                 # Plot with title, labels, and semi-transparent grid
                                                 plot(f, real.(X)
                                                      , label="Real Part", linestyle=:solid, linewidth=2, alpha=0.8, size = (600,400)
                                                      , xlabel="Frequency " * L"f", ylabel="Amplitude"
                                                      , title="Plot of " * L"\mathbf{X}(f) = 5 sinc(0.2 f) e^{-j 0.2 \pi f}"
                                                      , grid=true, gridalpha=0.2 # Enable grid and set transparency
                                                      , framestyle=:box
                                                 # Overlay additional lines
                                                 plot!(f, imag.(X), label="Imaginary Part", linestyle=:dash, linewidth=2, alpha=0.8)
                                                 plot!(f, abs.(X), label="Magnitude", linestyle=:dot, linewidth=2, alpha=0.8)
                                                                          Plot of \mathbf{X}(f) = 5sinc(0.2f)e^{-j0.2\pi f}
                                    Out[3]:
                                                                                                                                Real Part

    Imaginary Part

                                                                                                                               Magnitude
                                                     2
                                               Amplitude
                                                   -2
                                                                                                                         20
                                                        -40
                                                                              -20
                                                                                            Frequency f
                                                3. A linear, time-invariant system has the following transfer function:
                                               H(s) = rac{10(s+100)}{s^2+2s+100}
                                               (a) Derive an expression for H(s) in the usual, normal form.
                                                (b) Determine the frequency-invariant gain {\cal K} and the position of any poles and zeros.
                                               (c) Sketch a Bode plot of the magnitude-frequency response.
                                                (d) Sketch a Bode plot of the phase-frequency response.
                                                [8 marks]
                                                (a) Derive an expression for H(s) in the usual, normal form.
                                                To derive the transfer function H(s) in the usual, normal form, we factorize the numerator and denominator in terms of their natural
                                                frequencies and damping ratios.
                                               The given transfer function is: H(s) = \frac{10(s+100)}{s^2+2s+100}.
                                                Step 1: Denominator Normal Form
                                               The denominator is: s^2 + 2s + 100.
                                                This matches the general form of a second-order system: s^2+2\zeta\omega_ns+\omega_n^2, where \zeta is the damping ratio and \omega_n is the natural
                                               frequency.
                                               Here: \omega_n^2=100 \quad \Rightarrow \quad \omega_n=\sqrt{100}=10, and: 2\zeta\omega_n=2 \quad \Rightarrow \quad \zeta=\frac{2}{2\omega_n}=\frac{2}{20}=0.1.
                                               Thus, the denominator becomes: s^2 + 2s + 100 = (s^2 + 2\zeta\omega_n s + \omega_n^2) = s^2 + 2(0.1)(10)s + 10^2.
                                                Step 2: Numerator Normal Form
                                               The numerator is: 10(s + 100).
                                               Factor out 100 to normalize: 10(s+100)=10\cdot 100\left(\frac{s}{100}+1\right)=1000\left(\frac{s}{100}+1\right).
                                                Step 3: Rewrite in Normal Form
                                                Substitute the factored numerator and denominator into H(s): H(s) = \frac{1000(\frac{s}{100}+1)}{s^2+2(0.1)(10)s+10^2}.
                                               Simplify: H(s) = \frac{1000}{100} \cdot \frac{\left(\frac{s}{100} + 1\right)}{\frac{s^2}{100} + \frac{2(0.1)(10)s}{100} + \frac{10^2}{100}}.
                                               After normalization: H(s)=rac{10\left(rac{s}{100}+1
ight)}{rac{s^2}{100}+rac{2s}{10}+1}.
                                               Alternatively: H(s)=rac{10\left(rac{s}{100}+1
ight)}{rac{s^2}{100}+rac{s}{r}+1}.
                                               This is the normalized form of H(s).
                                                (b): Determine the Frequency-Invariant Gain K and the Positions of Poles and Zeros
                                               1. Transfer Function
                                               The given transfer function is: H(s) = \frac{10(s+100)}{s^2+2s+100}.
                                                2. Frequency-Invariant Gain {\cal K}
                                               The frequency-invariant gain is the gain of the system as s \to 0. This is determined by evaluating the transfer function at s = 0:
                                               K = H(0) = \frac{10(0+100)}{(0)^2+2(0)+100}.
                                               Simplify: K = \frac{10 \cdot 100}{100} = 10.
                                               Thus, the frequency-invariant gain is: K = 10.
                                               3. Poles
                                               The poles are the roots of the denominator s^2+2s+100=0: s^2+2s+100=0.
                                               Solve using the quadratic formula: s=\frac{-b\pm\sqrt{b^2-4ac}}{2a}, where a=1, b=2, and c=100. Substituting:
                                               s = \frac{-2 \pm \sqrt{2^2 - 4(1)(100)}}{2(1)} = \frac{-2 \pm \sqrt{4 - 400}}{2}.
                                               Simplify: s=rac{-2\pm\sqrt{-396}}{2} .
                                               The roots are: s=-1\pm j\sqrt{99}.
                                               Thus, the poles are: s=-1+j\sqrt{99}, \quad s=-1-j\sqrt{99}.
                                                4. Zeros
                                               The zero is the root of the numerator 10(s+100)=0: s+100=0 \quad \Rightarrow \quad s=-100.
                                               Thus, there is one zero at: s=-100.
                                                Final Results:
                                                 • Frequency-Invariant Gain K: K = 10.
                                                 • Poles: s = -1 + j\sqrt{99}, s = -1 - j\sqrt{99}.
                                                 • Zero: s = -100.
                                    In [4]:
                                                 using FFTW, LinearAlgebra
                                                 include("modules/operations.jl");
                                    In [5]:
                                                 # Frequency range (logarithmic scale)
                                                 \omega = 10 .^ range(-1, 3, length=500) # Frequencies from 0.1 to 1000 (log scale)
                                                 # Define the transfer function H(s)
                                                 function H(\omega)
                                                      numerator = 10 \cdot * (j \cdot * \omega \cdot + 100) # Element-wise addition
                                                      denominator = (j \cdot * \omega) \cdot ^2 \cdot + 2 \cdot * (j \cdot * \omega) \cdot + 100 # Element-wise operations
                                                      return numerator ./ denominator # Element-wise division
                                                 end
                                    Out[5]: H (generic function with 1 method)
                                    In [6]:
                                                 using Plots
                                                 using Printf
                                                 using Measures
                                                 # Magnitude response in dB
                                                 magnitude dB = 20 .* log10.(abs.(H.(\omega))) # Broadcasting applied to H, abs, and log10
                                                 # Plot the Bode magnitude plot
                                                 p1 = plot(\omega, magnitude dB)
                                                      , xscale=:log10
                                                      , xlabel="Frequency (rad/s)", ylabel="Magnitude (dB)"
                                                      , title="Bode Magnitude Plot", legend=false, grid=true
                                                      , margin = 5mm
                                                 # Phase response in degrees
                                                 phase_deg = angle.(H.(\omega)) .* (180 / \pi) # Convert phase from radians to degrees
                                                 # Plot the Bode phase plot
                                                 p2 = plot(\omega, phase_deg
                                                      ,xscale=:log10
                                                      ,xlabel="Frequency (rad/s)", ylabel="Phase (degrees)"
                                                      ,title="Bode Phase Plot"
                                                      ,legend=false,grid=true
                                                      ,left margin=10mm, right margin=10mm, top margin=15mm, bottom margin=15mm
                                                 plot(p1, p2, layout = (1, 2), size = (1000, 400))
                                                                     Bode Magnitude Plot
                                                                                                                                                   Bode Phase Plot
                                    Out[6]:
                                                     20
                                                                                                                         egrees)
                                                                                                                               -50
                                                Magnitu
                                                                                                                         Phase (d
                                                                                                                             -100
                                                     -20
                                                                                                                             -150
                                                    -40
                                                                       10<sup>0</sup>
                                                                                                 10<sup>2</sup>
                                                                                                                                                     Frequency (rad/s)
                                                                           Frequency (rad/s)
                                                4. Sketch magnitude and phase responses for a sampled data system with a pair
                                               of complex conjugate zeros and two poles at the origin.
                                                                imag.
                                                                Figure Q4
                                                [4 marks]
                                    In [ ]:
                                               5. A random variable X is uniformly distributed between x=0 and x=1. Via any appropriate method, determine the expected value E[Y] of Y=\exp(X).
                                               [4 marks]
                                               Given Y=\exp(X) and X\sim U(0,1),
                                                1. Expected Value Formula
                                               The expected value of a random variable Y is given by: E[Y] = \int_{-\infty}^{\infty} y f_Y(y) \, dy.
                                               Since X is uniformly distributed, its probability density function (PDF) is: f_X(x) =
                                               For Y=\exp(X), the expected value becomes: E[Y]=\int_0^1 \exp(x) f_X(x)\,dx.
                                               Because f_X(x)=1 for 0\leq x\leq 1, this simplifies to: E[Y]=\int_0^1 \exp(x)\,dx.
                                               2. Solve the Integral
                                               The integral of \exp(x) is: \int \exp(x) \, dx = \exp(x) + C.
                                               Now, evaluate the definite integral: \int_0^1 \exp(x) dx = [\exp(x)]_0^1 = \exp(1) - \exp(0).
                                               Simplify: \int_0^1 \exp(x) dx = e - 1.
                                                3. Final Answer
                                               The expected value is: igl|E[Y]=e-1igr|
                                                6. Identify the pivots and free variables of the following two matrices A and B.
                                                Following the method which we studied in class, find the special solution
                                                corresponding to each free variable and, by combining the special solutions,
                                                describe every solution to Ax = 0 and Bx = 0.
                                                                                         A = egin{bmatrix} 1 & 2 & 2 & 4 & 6 \ 1 & 2 & 3 & 6 & 9 \ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \qquad 	ext{and} \quad B = egin{bmatrix} 2 & 4 & 2 \ 0 & 4 & 4 \ 0 & 8 & 8 \end{bmatrix}.
                                               [7 marks]
                                    In []:
                                                7. For a projection matrix P = A(A^TA)^{-1}A^T , show that P^2 = P and then
                                               explain, in terms of the column space of P, why projections P_b and P(P_b) give
                                               identical results.
                                               [5 marks]
                                               1. Show that P^2 = P
                                                The projection matrix P is defined as: P = A(A^TA)^{-1}A^T, where A is a matrix with linearly independent columns.
                                               Compute P^2:
                                               We want to show: P^2 = P.
                                               Start with P^2: P^2 = P \cdot P = \left(A(A^TA)^{-1}A^T\right) \cdot \left(A(A^TA)^{-1}A^T\right).
                                               Expand the multiplication: P^2 = A(A^TA)^{-1}A^TA(A^TA)^{-1}A^T.
                                               Since A^TA is invertible, A^TA(A^TA)^{-1}=I (identity matrix). So: P^2=A(A^TA)^{-1}(I)A^T=A(A^TA)^{-1}A^T.
                                               This simplifies to: P^2 = P.
                                                2. Projections {\cal P}b and {\cal P}({\cal P}b) Give Identical Results
                                               Interpretation of P:
                                               The projection matrix P projects any vector b onto the column space of A, denoted as Col(A).
                                               Explain Pb:
                                               Pb = P\mathbf{b} = A(A^TA)^{-1}A^T\mathbf{b}. This gives the projection of b onto Col(A).
                                               Explain P(Pb):
                                               P(Pb) = P(Pb). Substitute Pb into P(Pb) : P(Pb) = P \cdot Pb. Since we showed that P^2 = P, this becomes: P(Pb) = Pb.
                                                Why Are Pb and P(Pb) Identical?
                                                 • Pb = P\mathbf{b} is already the projection of \mathbf{b} onto Col(A).
                                                 • Applying P again to Pb does not change it, because projecting a vector already in the subspace \mathrm{Col}(A) onto the same subspace
                                                    leaves it unchanged.
```

• Hence: P(Pb) = Pb.

Conclusion

In []:

3. Column Space Perspective

1. Projection matrix property: $P^2 = P$.

1. The column space of P (and thus Pb) is the **same as** Col(A).

2. Applying P to Pb projects Pb onto $\mathrm{Col}(A)$, but since $Pb \in \mathrm{Col}(A)$, the result is unchanged.

3. **Idempotence**: P an idempotent matrix, which is a key characteristic of projection matrices.

Thus, projections Pb and P(Pb) are identical because projecting a vector already in the column space does nothing.

2. **Projections**: Pb and P(Pb) are identical because Pb lies in the column space, and re-projecting it does not alter it.

In terms of the column space of P:

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