MathEng2223

December 9, 2024

Mathematical Methods for Engineers (MathEng)

EXAM

December 2023

```
Duration: 2 hrs, all documents and calculators permitted
ATTEMPT ALL QUESTIONS - ANSWER IN ENGLISH
```

1 Using Euler's identity (or any other appropriate method), write down an expression for the complex Fourier series of the signal $x(t)$:

$$x(t) = 3\cos(5t) + 4\sin(10t)$$

[5 marks]

To find the complex Fourier series of $x(t) = 3\cos(5t) + 4\sin(10t)$,

we use Euler's identity: $\cos(\omega t) = \frac{e^{j\omega t}+e^{-j\omega t}}{2}, \quad \sin(\omega t) = \frac{e^{j\omega t}-e^{-j\omega t}}{2j}$.

Step 1: Rewrite $\cos(5t)$ and $\sin(10t)$ using Euler's identity

- $3\cos(5t) \rightarrow 3(\frac{e^{j5t}+e^{-j5t}}{2}) = \frac{3}{2}e^{j5t} + \frac{3}{2}e^{-j5t}$
- $4\sin(10t) \rightarrow 4(\frac{e^{j10t}-e^{-j10t}}{2j}) = \frac{4}{2j}(e^{j10t} - e^{-j10t})$

  Recall:$\frac{1}{j} = \frac{j}{j^2} = \frac{j}{-1} = -j$

$= \backslash frac2j(e^{j10t} - e^{-j10t}) = \backslash frac2je^{j10t} - \backslash frac2je^{-j10t}\backslash nobreak\backslash break = \backslash frac2jj^2e^{j10t} - \backslash frac2jj^2e^{-j10t} = \backslash fra$

Thus, $x(t) = \frac{3}{2}e^{j5t} + \frac{3}{2}e^{-j5t} - 2je^{j10t} + 2je^{-j10t}$.

Step 2: Group the terms The complex Fourier series representation of $x(t)$ is: \$ x(t) = $\_\{k = -\infty\}\hat{}\infty c\_ke\hat{}\{jk\omega\_0t\}, \$ wherec_k$ are the complex Fourier coefficients.

Here, $x(t)$ has terms at frequencies ± 5 and ± 10. The coefficients c_k are:

- At $k = 5$: $c_5 = \frac{3}{2}$,

- At $k = -5$: $c_{-5} = \frac{3}{2}$,
- At $k = 10$: $c_{10} = -2j$,
- At $k = -10$: $c_{-10} = 2j$,
- All other $c_k = 0$.

Final Answer: The complex Fourier series of $x(t)$ is:

$$\boxed{x(t) = \frac{3}{2}e^{j5t} + \frac{3}{2}e^{-j5t} - 2je^{j10t} + 2je^{-j10t}}$$

2 Develop an expression for the Fourier Transform of the signal $x(t)$ illustrated in `Figure Q2` below:

[6 marks]

To develop the Fourier Transform $ X(f) $ of the signal $ x(t) $ illustrated in the figure, we follow the same steps for a rectangular pulse.

Step 1: Signal Description The signal $ x(t) $ is defined as: $ x(t) =

$$\begin{cases} 5, & 0 \leq t \leq 0.2, \\ 0, & \text{otherwise.} \end{cases}$$

$

Step 2: Fourier Transform Definition The Fourier Transform is given by: $ X(f) = _{-∞}^∞x(t) e^{-j 2 f t} dt. $

Since $ x(t) $ is nonzero only in the interval $ [0, 0.2] $, the limits of integration reduce to $ [0, 0.2] : X(f) = _0^{0.2} 5 e^{-j 2 f t} dt. $

Step 3: Evaluate the Integral Factor out the constant $ 5 : X(f) = 5 _0^{0.2} e^{-j 2 f t} dt. $

The integral of $ e^{-j 2 f t} $ is: $ e^{-j 2 f t} dt = e$^{-j2\pi ft} \frac{}{-j2\pi f.\$}$

Apply the limits of integration: $ X(f) = 5 $\left[\frac{e^{-j2\pi ft}}{-j2\pi f}\right]$ _0^{0.2}.$

Substitute the limits: $ X(f) = 5 1$\frac{}{-j2\pi f(e\hat{}\{-j2\pi f(0.2)\} - e\hat{}\{0\}).\$}$

Simplify: $ X(f) = 5$\frac{}{-j2\pi f(e\hat{}\{-j0.4\pi f\} - 1).\$}$

Step 4: Simplify Further

Using the property $e^{-j\theta} - 1 = -2j\sin\left(\frac{\theta}{2}\right)e^{-j\frac{\theta}{2}}$ **which is derived as follows:**

1. **Rewrite $ e^{-j } - 1 : $**ExpandusingEuler'sformula* : e^{-j } - 1 = $\cos(\theta) - j\sin(\theta) - 1 = (\cos(\theta) - 1) - j\sin(\theta)$$**Factorize Trigonometric Terms:***Usethehalf − angleidentities :*

2. • $ $\cos(\theta) = 1 - 2\sin\hat{}2\left(\frac{\theta}{2}\right) \implies \cos(\theta) - 1 = -2\sin\hat{}2\left(\frac{\theta}{2}\right)$ $$\sin(\theta) = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$ $.

Substituting these: $ e^{-j } - 1 = $\{-2\sin\hat{}2\left(\frac{\theta}{2}\right)\} - j\{\cdot 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\}.\$$**Factor Out Common Terms:**

3. **Identify Common Factor:**

   Both terms contain $ -2jsin $\left(\frac{\theta}{2}\right) \$ as a common factor : 1.\$ - 2\sin\hat{}2\left(\frac{\theta}{2}\right)\$: -This can be written as\$ - 2j\sin\left(\frac{\theta}{2}\right) \cdot \frac{\sin\left(\frac{\theta}{2}\right)}{j}\$.2.\$ - j \cdot 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\$: -This is already proportional to\$ - 2j\sin\left(\frac{\theta}{2}\right)\$$.**Factorization:**

Factor $ -2jsin $\left(\frac{\theta}{2}\right) \$ out of the entire expression : \$e\hat{}\{-j\theta\} - 1 = -2j\sin\left(\frac{\theta}{2}\right) \cdot \left(\frac{\sin\left(\frac{\theta}{2}\right)}{j} + \cos\left(\frac{\theta}{2}\right)\right).\$$

Simplify the term : $ sin $\left(\frac{\theta}{2}\right)\frac{}{j = -j\sin\left(\frac{\theta}{2}\right).\$Thus: e^{-j\theta} - 1 = -2j\sin\left(\frac{\theta}{2}\right)\cdot\left(\cos\left(\frac{\theta}{2}\right) - j\sin\left(\frac{\theta}{2}\right)\right).}$

- **Recognize the Exponential Form:** The term $ $\cos\left(\frac{\theta}{2}\right) - j\sin\left(\frac{\theta}{2}\right) \$ is equivalent to \$e\hat{}\{-j\frac{\theta}{2}\}\$, using Euler's formula.$

5. **Simplify:** Recognize the term in parentheses as $ e^{-j $_{2\}:}$ e^{-j } - 1 = -2j sin $\left(\frac{\theta}{2}\right) e\hat{}\{-j\frac{\theta}{2}\}.\$$

   This compactly combines the amplitude term $ -2j sin $\left(\frac{\theta}{2}\right) \$ and the phase shift \$e\hat{}\{-j\frac{\theta}{2}\}\$.$

rewrite $ X(f) : X(f) = 5$\frac{}{-j2\pi f \cdot -2j\sin(0.2\pi f)e\hat{}\{-j0.2\pi f\}.\$}$

Cancel $ -j $ and simplify: $ X(f) = 5 2 $\sin(0.2\pi f)\frac{}{2\pi f e\hat{}\{-j0.2\pi f\}.\$}$

Finally: $ X(f) = 5 $\sin(0.2\pi f)\frac{}{\pi f e\hat{}\{-j0.2\pi f\}.\$}$

Final Expression $ □ $ where the **sinc function** is defined as: $ sinc(x) = $\sin(\pi x)\frac{}{\pi x.\$}$

Interpretation

- $\frac{\sin(0.2\pi f)}{\pi f}$**:** This is the sinc function, representing the frequency-domain shape of the rectangular pulse.
- $e^{-j0.2\pi f}$**:** This is a phase shift due to the non-centered nature of the pulse (starting at $ t = 0 $).

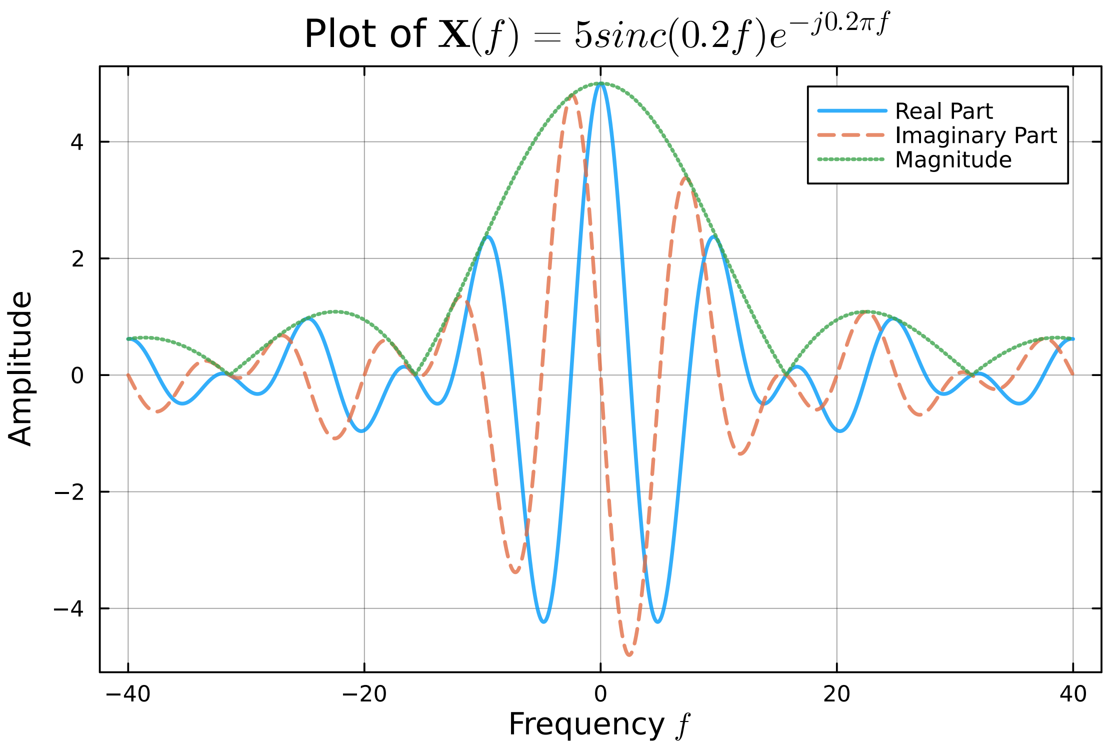
[1]:
```julia
using FFTW, LinearAlgebra, Plots, LaTeXStrings
```

[2]:
```julia
include("modules/operations.jl");
```

[3]:
```julia
# Define the unscaled sinc function
sinc_unscaled(x::Real) = x == 0 ? 1.0 : sin(pi * x) / (pi * x)

# Define the polymorphic sinc function with a normalization option
sinc(x::Real; normalized::Bool = true) = normalized ? sinc_unscaled(x / ) :
    sinc_unscaled(x)

# Frequency range
f = range(-40, 40, length=1000)

# Function components
```

```
 = 5 .* sinc.(0.2 .* f)  # Amplitude of the signal
 =  .^ (-j .* 0.2  .* f)  # Phase shift
 =   .*    # Combined function

# Plot with title, labels, and semi-transparent grid
plot(f, real.( )
    , label="Real Part", linestyle=:solid, linewidth=2, alpha=0.8, size =␣
 ↪(600,400)
    , xlabel="Frequency " * L"f", ylabel="Amplitude"
    , title="Plot of " * L"\mathbf{X}(f) = 5 sinc(0.2 f) e^{-j 0.2   f}"
    , grid=true, gridalpha=0.2  # Enable grid and set transparency
    , framestyle=:box
)

# Overlay additional lines
plot!(f, imag.( ), label="Imaginary Part", linestyle=:dash, linewidth=2,␣
 ↪alpha=0.8)
plot!(f, abs.( ), label="Magnitude", linestyle=:dot, linewidth=2, alpha=0.8)
```

[3]:



3 A linear, time-invariant system has the following transfer function:

\$ H(s) = 10(s + 100)$_{\overline{s^2+2s+100\$}}$

(a) Derive an expression for \$ H(s) \$ in the usual, normal form.

(b) Determine the frequency-invariant gain \$ K \$ and the position of any poles and zeros.

(c) Sketch a Bode plot of the magnitude-frequency response.

(d) Sketch a Bode plot of the phase-frequency response.

[8 marks]

(a) Derive an expression for \$ H(s) \$ in the usual, normal form. To derive the transfer function \$ H(s) \$ in the usual, **normal form**, we factorize the numerator and denominator in terms of their natural frequencies and damping ratios.

The given transfer function is: \$ H(s) = 10(s + 100)$\frac{}{s^2+2s+100.\$}$

Step 1: Denominator Normal Form The denominator is: \$ s^2 + 2s + 100. \$

This matches the general form of a second-order system: \$ s^2 + 2 _n s + _n^2, \$ where \$ \$ is the damping ratio and \$ _n \$ is the natural frequency.

Here: \$ _n^2 = 100 _n = $\sqrt{100} = 10, \$and : \$2\zeta\omega\_n = 2 \quad \Rightarrow \quad \zeta = \frac{2}{2\omega_n} = \frac{2}{20} = 0.1.\$$

Thus, the denominator becomes: \$ s^2 + 2s + 100 = (s^2 + 2 _n s + _n^2) = s^2 + 2(0.1)(10)s + 10^2. \$

Step 2: Numerator Normal Form The numerator is: \$ 10(s + 100). \$

Factor out \$ 100 \$ to normalize: \$ 10(s + 100) = 10 100 $\left(\frac{s}{100} + 1\right) = 1000\left(\frac{s}{100} + 1\right).\$$

Step 3: Rewrite in Normal Form Substitute the factored numerator and denominator into \$ H(s) : H(s) = 1000 $\left(\frac{s}{100} + 1\right)\frac{}{s^2+2(0.1)(10)s+10^2.\$}$

Simplify: \$ H(s) = 1000$\frac{}{100\cdot\frac{\left(\frac{s}{100}+1\right)}{\frac{s^2}{100}+\frac{2(0.1)(10)s}{100}+\frac{10^2}{100}}.\$}$

After normalization: \$ H(s) = 10 $\left(\frac{s}{100} + 1\right)\frac{}{\frac{s^2}{100}+\frac{2s}{10}+1.\$}$

Alternatively: \$ H(s) = 10 $\left(\frac{s}{100} + 1\right)\frac{}{\frac{s^2}{100}+\frac{s}{5}+1.\$}$

This is the normalized form of \$ H(s) \$.

(b): Determine the Frequency-Invariant Gain \$ K \$ and the Positions of Poles and Zeros

1. Transfer Function The given transfer function is: \$ H(s) = 10(s + 100)$\frac{}{s^2+2s+100.\$}$

2. Frequency-Invariant Gain $ K $ The frequency-invariant gain is the gain of the system as $ s →0 $. This is determined by evaluating the transfer function at $ s = 0 : K = H(0) = 10(0 + 100)$_{(0)^2+2(0)+100.$}$

Simplify: $ K = 10 100$_{100=10.$}$

Thus, the frequency-invariant gain is: $ K = 10. $

3. Poles The poles are the roots of the denominator $ s^2 + 2s + 100 = 0 : s^2 + 2s + 100 = 0. $

Solve using the quadratic formula: $ s = -b ±$\sqrt{b^2 - 4ac}_{2a,\$where\$a=1\$,\$b=2\$,and\$c=100.Substituting:}$ s = -2 ±$\sqrt{2^2 - 4(1)(100)}_{2(1)=\frac{-2\pm\sqrt{4-400}}{2}.\$}$

Simplify: $ s = -2 ±$\sqrt{-396}_{2.\$}$

The roots are: $ s = -1 ±j$\sqrt{99}.\$$

Thus, the poles are: $ s = -1 + j$\sqrt{99}, \quad s = -1 - j\sqrt{99}.\$$

4. Zeros The zero is the root of the numerator $ 10(s + 100) = 0 : s + 100 = 0 s = -100. $

Thus, there is one zero at: $ s = -100. $

Final Results:

- **Frequency-Invariant Gain $ K : ** K = 10. $

- **Poles:** $ s = -1 + j$\sqrt{99}, \quad s = -1 - j\sqrt{99}.\$$**Zero:**$$s = -100.\$$

[4]:
```
using FFTW, LinearAlgebra
include("modules/operations.jl");
```

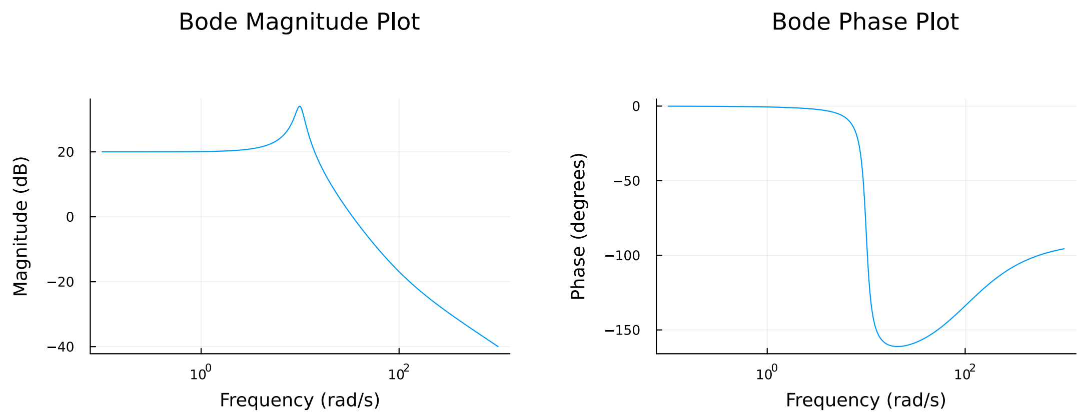
[5]:
```
# Frequency range (logarithmic scale)
 = 10 .^ range(-1, 3, length=500)  # Frequencies from 0.1 to 1000 (log scale)

# Define the transfer function H(s)
function H( )
    numerator = 10 .* (j .*   .+ 100)  # Element-wise addition
    denominator = (j .*  ).^2 .+ 2 .* (j .*  ) .+ 100  # Element-wise operations
    return numerator ./ denominator  # Element-wise division
end
```

[5]: H (generic function with 1 method)

[6]:
```
using Plots
using Printf
using Measures

# Magnitude response in dB
```

```
magnitude_dB = 20 .* log10.(abs.(H.( )))  # Broadcasting applied to H, abs, and
 ↪log10

# Plot the Bode magnitude plot
p1 = plot( , magnitude_dB
    , xscale=:log10
    , xlabel="Frequency (rad/s)", ylabel="Magnitude (dB)"
    , title="Bode Magnitude Plot", legend=false, grid=true
    , margin = 5mm
)

# Phase response in degrees
phase_deg = angle.(H.( )) .* (180 /  )  # Convert phase from radians to degrees

# Plot the Bode phase plot
p2 = plot( , phase_deg
    ,xscale=:log10
    ,xlabel="Frequency (rad/s)", ylabel="Phase (degrees)"
    ,title="Bode Phase Plot"
    ,legend=false,grid=true
    ,left_margin=10mm, right_margin=10mm, top_margin=15mm, bottom_margin=15mm
)

plot(p1, p2, layout = (1, 2), size = (1000, 400))
```

[6]:



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3.1 4. Sketch magnitude and phase responses for a sampled data system with a pair of complex conjugate zeros and two poles at the origin.

[4 marks]

[ ]:

4 A random variable X is uniformly distributed between $x = 0$ and $x = 1$. Via any appropriate method, determine the expected value $E[Y]$ of $Y = exp(X)$.

[4 marks]

Given \$ Y = $\exp(X)$\$*and*\$$X \sim U(0,1)$\$,

1. Expected Value Formula The expected value of a random variable \$ Y \$ is given by: \$ E[Y] = _{-∞}^∞y f_Y(y) , dy. \$

Since \$ X \$ is uniformly distributed, its probability density function (PDF) is: \$ f_X(x) =

$$\begin{cases} 1, & 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

\$

For \$ Y = $\exp(X), theexpectedvaluebecomes:$ E[Y] = _0^1 $\exp(x) f\_X(x), dx.$\$

Because \$ f_X(x) = 1 \$ for \$ 0 x 1 , $thissimplifiesto:$ E[Y] = _0^1 $\exp(x), dx.$\$

2. Solve the Integral The integral of \$ $\exp(x)$\$$is:$ \$ $\int \exp(x), dx = \exp(x) + C.$\$

Now, evaluate the definite integral: \$ _0^1 $\exp(x), dx = [\exp(x)]\_0\hat{}1 = \exp(1) - \exp(0).$\$

Simplify: \$ _0^1 $\exp(x), dx = e - 1.$\$

3. Final Answer The expected value is: \$ $\boxed{E[Y] = e - 1}$ \$

5 Identify the pivots and free variables of the following two matrices A and B. Following the method which we studied in class, find the special solution corresponding to each free variable and, by combining the special solutions, describe every solution to $Ax = 0$ and $Bx = 0$.

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \qquad \text{and} \quad B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}.$$

[7 marks]

[ ]:

6 For a projection matrix $P = A(A^TA)^{-1}A^T$, show that $P^2 = P$ and then explain, in terms of the column space of P, why projections P_b and $P(P_b)$ give identical results.

[5 marks]

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1. Show that $ P^2 = P $ The projection matrix $ P $ is defined as: $ P = A (A^T A)^{-1} A^T, $ where $ A $ is a matrix with linearly independent columns.

∗∗Compute $ P^2 : ∗∗ *Wewanttoshow* : P^2 = P. $

Start with $ P^2 : P^2 = P P = $(A(A\hat{}TA)\hat{}\{-1\}A\hat{}T) \cdot (A(A\hat{}TA)\hat{}\{-1\}A\hat{}T)$.$

Expand the multiplication: $ P^2 = A (A^T A)^{-1} A^T A (A^T A)^{-1} A^T. $

Since $ A^T A $ is invertible, $ A^T A (A^T A)^{-1} = I $ (identity matrix). So: $ P^2 = A (A^T A)^{-1} (I) A^T = A (A^T A)^{-1} A^T. $

This simplifies to: $ P^2 = P. $

2. Projections $ Pb $ and $ P(Pb) $ Give Identical Results

Interpretation of $ P $: The projection matrix $ P $ projects any vector $ b $ onto the **column space** of $ A $, denoted as $ Col(A) $.

∗∗Explain $ Pb : ∗∗ Pb = P b = A (A^T A)^{-1} A^T b. $ This gives the projection of $ b $ onto $ Col(A) $.

∗∗Explain $ P(Pb) : ∗∗ P(Pb) = P(P b). $ Substitute $ Pb $ into $ P(Pb) : P(Pb) = P P b. $ Since we showed that $ P^2 = P , *thisbecomes* : P(Pb) = P b. $

Why Are $ Pb $ and $ P(Pb) $ Identical?

- $ Pb = P b $ is already the projection of $ b $ onto $ Col(A) $.
- Applying $ P $ again to $ Pb $ does not change it, because projecting a vector already in the subspace $ Col(A) $ onto the same subspace leaves it unchanged.
- Hence: $ P(Pb) = Pb. $

3. Column Space Perspective In terms of the column space of $ P $: 1. The column space of $ P $ (and thus $ Pb $) is the **same as $ Col(A) $**. 2. Applying $ P $ to $ Pb $ projects $ Pb $ onto $ Col(A) $, but since $ Pb Col(A) $, the result is unchanged.

Thus, projections $ Pb $ and $ P(Pb) $ are identical because projecting a vector already in the column space does nothing.

Conclusion

- **Projection matrix property**: $ P^2 = P $.
- **Projections**: $ Pb $ and $ P(Pb) $ are identical because $ Pb $ lies in the column space, and re-projecting it does not alter it.
- **Idempotence**: P an idempotent matrix, which is a key characteristic of projection matrices.

[ ]: