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Essential Mathematical Methods for Engineers (MathEng)

Probability and random variables

1. Show that if two events are mutually exclusive and statistically independent, the probability of one, or both, is zero.

2. Given a binary communication channel where A = input and B = output, let $P(A) = 0.4$ and, $P(B|A) = 0.9$, and $P(\overline{B}\,|\,\overline{A}) = 0.6$. Find $P(A|B)$ and $P(A\,|\,\overline{B})$.

3. Given the table of joint probabilities

	B_1	B_2	B_3	$P(A_i)$
A_1	0.05		0.45	0.55
A_2		0.15	0.10	
A_3	0.05	0.05		0.15
$P(B_j)$				1.0

   a. find the omitted probabilities, and
   b. find the probabilities $P(A_3|B_3)$, $P(B_2|A_1)$, and $P(B_3|A_2)$.

4. A certain continuous random variable has the cumulative distribution function:

$$F_X(x) = \begin{cases} 0, & x < 0 \\ Ax^3, & 0 \le x \le 10 \\ B, & x > 10 \end{cases}$$

   a. Find the proper values for A and B.
   b. Obtain and plot the pdf $f_X(x)$.
   c. Compute $P(X > 7)$.
   d. Compute $P(3 \le X < 7)$.

5. Proving your answers, test X and Y for independence if $f_{XY}(x,y) = A\exp(-|x|-2|y|)$.

6. Suppose a pair of random variables is jointly distributed over the unit circle. That is, the joint pdf $f_{XY}(x,y)$ is constant anywhere such that $x^2 + y^2 < 1$:

$$f_{XY}(x,y) = \begin{cases} c, & x^2 + y^2 < 1 \\ 0, & \text{otherwise} \end{cases}.$$

   Determine c and the marginal pdfs, $f_X(x)$ and $f_Y(y)$.

7. The joint pdf of the random variables X and Y is:

$$f_{XY}(x, y) = Axy\,\exp[-(x+y)], \quad x \geq 0 \text{ and } y \geq 0$$

   a. Find the constant A.
   b. Find the marginal pdf's of X and Y, $F_X(x)$ and $F_Y(y)$.
   c. Are X and Y statistically independent? Justify your answer.

8. A nonlinear system has input X and output Y. The pdf of the input is a zero mean Gaussian. Determine the pdf of the output, assuming that the nonlinear system has the following input/output relationship:

$$Y = \begin{cases} aX, & X \geq 0 \\ 0, & X < 0 \end{cases}.$$

9. Let $f_X(x) = A\,\exp(-b|x|)$ for all x.
   a. Find the relationship between A and b such that this function is a pdf.
   b. Calculate $E[X]$.
   c. Find $E[X^2]$ and the variance?

   [Hint: $\int_0^\infty x^2 e^{-bx} dx = 2/b^3$]

10. A random variable has pdf where

$$f_X(x) = \frac{1}{2}\delta(x-4) + \frac{1}{8}[u(x-3) - u(x-7)]$$

   where $u(x)$ is the unit step. Determine the mean and the variance of the random variable thus defined. [Hint: $\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$.]

11. Two Gaussian random variables X and Y, with zero mean and variance σ^2, between which there is a correlation coefficient ρ, have a joint probability density given by:

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left[-\frac{x^2 - 2\rho xy + y^2}{2\sigma^2(1-\rho^2)}\right]$$

   Verify that the symbol ρ in the expression for $f(x,y)$ is the correlation coefficient. That is, evaluate $E[XY]/\sigma^2$.

12. A random variable X has the probability density function

$$f_X(x) = \begin{cases} ae^{-ax}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

where a is an arbitrary positive constant.

   a. Determine the characteristic function $M_X(jv)$.
   b. Use the characteristic function to determine $E[X]$ and $E[X^2]$.
   c. Compute σ_X^2 .

13. Assume that two random variables X and Y are jointly Gaussian.
   a. Obtain the marginal pdfs $f_X(x)$ and $f_Y(y)$.
   b. Use the marginal pdfs to compute $E[X]$, $E[Y]$, var{X}, and var{Y}.

14. A stationary mobile unit receives multiple independent signal components from its transmitter. Each component undergoes scattering, reflection and diffraction caused by buildings and other artificial and natural structures. Show that:

   a. the envelope of the received signal is Rayleigh distributed, and that
   b. the power of the received signal has an exponential distribution.

   If the average power being received is 100 μW what is the probability that the received power will be less that 50 μW?