

Mathematical Methods for Engineers (MathEng)

EXAM

13th February 2017

Duration: 2 hrs, calculators permitted, no documents

This exam paper contains 7 questions and 60 marks.

ATTEMPT ALL QUESTIONS – ANSWER IN ENGLISH



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1. State with appropriate mathematical justification whether $x_1[n]=u[n]$ (the unit step function) and $x_2[n]=2e^{j3n}$ are energy signals or power signals.

[6 marks]

2. Determine the Fourier and Laplace transforms of the waveform $x(t)$ illustrated in Figure Q1 and sketch a plot of the magnitude of the Fourier transform.

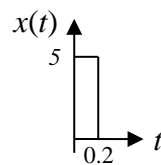


Figure Q2

[8 marks]

3. Sketch a pole/zero map and magnitude frequency response for the following transfer function:

$$H(z) = \frac{z^2 + 2z + 1}{z^3}$$

Determine the gain at zero frequency, one-quarter of the sampling frequency and at half the sampling frequency.

[6 marks]

4. With the aid of appropriate frequency domain diagrams, explain the problem of aliasing and the sampling conditions under which aliasing is avoided.

[8 marks]

5. Find the discrete Fourier transform $X[k]$ of sequence $x[n]=0, 1, 2, 3$.

[6 marks]

6. The joint probability density function of two continuous random variables is given by

$$f_{XY}(x, y) = \begin{cases} cxy, & 0 \leq x \leq 4, 1 \leq y \leq 5 \\ 0, & \text{otherwise} \end{cases}$$

- (a) Determine the value of the constant, c .
(b) Determine the probability (cumulative) distribution function of X , i.e. $F_X(x)$

[12 marks]

7. (a) Find an orthonormal basis for the span of the following vectors

$$\begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix}$$

- (b) Show that the orthogonal projection of u onto v is the same as the orthogonal projection of u onto av for any $a \neq 0$.
(c) Show that, if two matrices Q_1 and Q_2 are orthogonal, then their product Q_1Q_2 is also orthogonal.

[14 marks]

Table of selected Laplace transforms

$$X(s) = \int_{-\infty}^{\infty} x(t) \exp(-st) dt$$

N.B.: lower limit is 0 for one-sided Laplace transform

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) \exp(st) ds$$

$x(t) \quad (t \geq 0)$	$X(s)$
$\delta(t)$	1
$\delta(t - \alpha)$	$\exp(-\alpha s)$
1 (unit step)	$\frac{1}{s}$
t (unit ramp)	$\frac{1}{s^2}$
$\exp(-\alpha t)$	$\frac{1}{s + \alpha}$
$t \exp(-\alpha t)$	$\frac{1}{(s + \alpha)^2}$
$\sin(\alpha t)$	$\frac{\alpha}{s^2 + \alpha^2}$
$\cos(\alpha t)$	$\frac{s}{s^2 + \alpha^2}$
$e^{-\alpha t} \sin(\omega t)$	$\frac{\omega}{(s + \alpha)^2 + \omega^2}$
$e^{-\alpha t} \cos(\omega t)$	$\frac{s + \alpha}{(s + \alpha)^2 + \omega^2}$

Bode plots

Poles or zeros on the real axis:

$$(s + a) = a \left(\frac{s}{a} + 1 \right) = \frac{1}{\tau} (\tau s + 1)$$

Complex conjugate poles (or zeros):

$$(s^2 + As + B) = (s^2 + 2\zeta\omega_0 s + \omega_0^2) = \omega_0^2 ((s/\omega_0)^2 + 2\zeta(s/\omega_0) + 1)$$

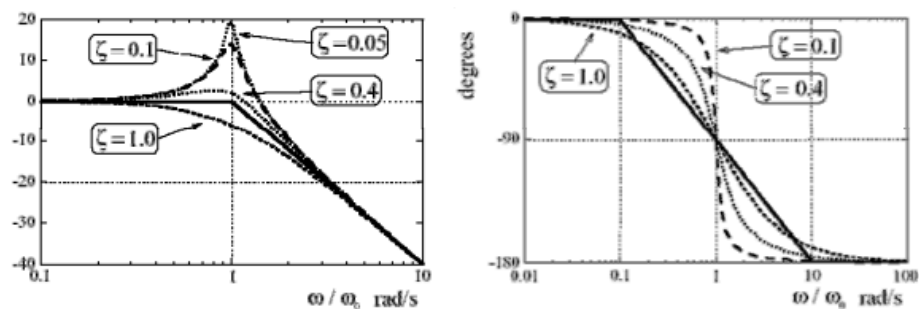


Table of selected z-transforms

$$x_c(t) = \sum_{n=0}^{\infty} x(n\Delta t)\delta(t - n\Delta t)$$

$$X_c(s) = \sum_{n=0}^{\infty} x(n\Delta t)\exp(-n\Delta ts)$$

$$X_c(s) = X(z)|_{z=e^{\Delta ts}}$$

$$X_c(\omega) = X_s(s)|_{s=j\omega}$$

$$X(\omega) = X(z)|_{z=\exp(\Delta t j\omega)}$$

$$X(z) = \sum_{n=0}^{\infty} x(n\Delta t)z^{-n}$$

$x(n) \ (n \geq 0)$	$X(z)$
$\delta(n)$ unit pulse	1
$\delta(n-m)$	z^{-m}
1 (unit step)	$\frac{z}{z-1}$
n (unit ramp)	$\frac{z}{(z-1)^2}$
$\exp(-\alpha n)$	$\frac{z}{(z-e^{-\alpha})}$
$n \exp(-\alpha n)$	$\frac{e^{-\alpha} z}{(z-e^{-\alpha})^2}$
$\sin(\beta n)$	$\frac{z \sin(\beta)}{z^2 - 2z \cos(\beta) + 1}$
$\cos(\beta n)$	$\frac{z^2 - z \cos(\beta)}{z^2 - 2z \cos(\beta) + 1}$
$e^{-\alpha n} \sin(\beta n)$	$\frac{ze^{-\alpha} \sin(\beta)}{z^2 - 2ze^{-\alpha} \cos(\beta) + e^{-2\alpha}}$
$e^{-\alpha n} \cos(\beta n)$	$\frac{z^2 - ze^{-\alpha} \cos(\beta)}{z^2 - 2ze^{-\alpha} \cos(\beta) + e^{-2\alpha}}$

Table of selected Fourier transform pairs

Function	$x(t)$	$X(\omega)$
Rectangular function of width τ	$\Pi(t/\tau)$	$\tau \operatorname{sinc}(\omega\tau/2)$
Triangular function of width 2τ	$\Lambda(t/\tau)$	$\tau \operatorname{sinc}^2(\omega\tau/2)$
Train of impulses every Δt	$\delta_T(t)$	$2\pi/\Delta t \sum_n \delta(\omega - 2\pi n/\Delta t)$

NB: $\operatorname{sinc}(x) = \sin(\pi x)/\pi x$

NB: $\operatorname{sa}(x) = \sin(x)/x$

Euler's identity

$$\exp(j\theta) = \cos \theta + j \sin \theta$$

$$\cos(n\omega_0 t) = \frac{\exp(jn\omega_0 t) + \exp(-jn\omega_0 t)}{2}$$

$$\sin(n\omega_0 t) = \frac{\exp(jn\omega_0 t) - \exp(-jn\omega_0 t)}{2j}$$

Fourier series and transforms
Trigonometric Fourier series

$$x(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t)$$

$$A_n = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \cos(n\omega_0 t) dt$$

$$B_n = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \sin(n\omega_0 t) dt$$

Complex Fourier series – periodic and continuous in time, discrete in frequency

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n \exp(jn\omega_0 t)$$

$$X_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \exp(-jn\omega_0 t) dt$$

Fourier transform – continuous in time, continuous in frequency

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \exp(j\omega t) d\omega$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \exp(-j\omega t) dt$$

Discrete-time Fourier transform – discrete in time, continuous and periodic in frequency

$$x(n\Delta t) = \frac{\Delta t}{2\pi} \int_{-\pi/\Delta t}^{\pi/\Delta t} X(\omega) \exp(jn\Delta t\omega) d\omega$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n\Delta t) \exp(-j\omega n\Delta t)$$

Discrete Fourier transform – discrete and periodic in time and in frequency

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \exp\left(\frac{jnk2\pi}{N}\right)$$

$$X(k) = \sum_{n=0}^{N-1} x(n) \exp\left(\frac{-jnk2\pi}{N}\right)$$

Transformation of random variables

$$f_Y(y) = \sum_{i=1}^N f_X(x_i) \left| \frac{dx_i}{dy} \right|_{x_i=g_i^{-1}(y)}$$

$$f_{UV}(u, v) = f_{XY}(x, y) \left| \frac{\partial(x, y)}{\partial(u, v)} \right|_{\substack{x=g_1^{-1}(u, v) \\ y=g_2^{-1}(u, v)}}$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$