Essential Mathematical Methods for Engineers

Lecture 7:
Linear algebra 2

Outline

- orthogonality
  - projections
  - least squares approximation
  - Gram-Scmidt
- eigenvales and eigenvectors
  - diagonalisation
  - symmetric matrices
  - singular value decomposition (SVD)

Orthogonality

- the row space is perpendicular to the nullspace
  - every row of A is perpendicular to every solution of $Ax = 0$
- the column space is perpendicular to the left nullspace
  - every column of A is perpendicular to every solution of $A^T y = 0$
- two spaces are perpendicular when
  - $v.w = 0$ or $v^T w = 0$ for all v in V and w in W
- about the left nullspace
  - it is never reached by Ax
  - when b is outside the column space we cannot solve $Ax = b$
  - the left nullspace contains the error in the least squares solution

2nd Fundamental Theorem of Linear Algebra

- the fundamental subspaces are not only orthogonal
  - they are orthogonal complements

the nullspace is the orthogonal complement
of the row space (in R^n)

the left nullspace is the orthogonal complement
of the column space (in R^m)

2nd Fundamental Theorem of Linear Algebra

- every vector x can be split into
  - row space component x_r
  - nullspace component x_n

  so we really have $A(x_r + x_n)$
  - $Ax_r = Ax = b$
  - $Ax_n = 0$
- every vector x goes to the column space
  - every vector b in the column space comes from one and only one vector in the row space
  - there is an invertible matrix hiding in A
    - the pseudoinverse

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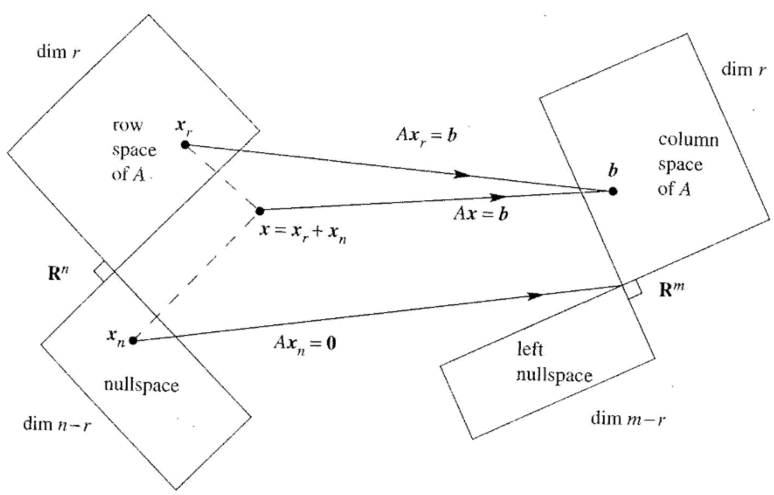
2nd Fundamental Theorem of Linear Algebra



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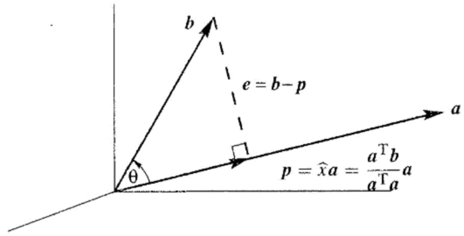
Projections

- we are given
  - a point $b = (b_1,\ldots,b_m)$ in m-dimensional space
  - a line through the origin $a = (a_1,\ldots,a_m)$
- we want the point p on a that is closest to b
  - the line connecting p and b is perpendicular to a



subspaces & column spaces

Projections

- p is a multiple of a : $p = \hat{x}a$
- the dotted line e is: $b - p = b - \hat{x}a$
- noting perpendicularity

$$a \cdot (b - \hat{x}a) = 0 \quad \text{or} \quad a \cdot b - \hat{x}a \cdot a = 0 \quad \text{or} \quad \hat{x} = \frac{a \cdot b}{a \cdot a} = \frac{a^T b}{a^T a}$$

- the projection of b onto the line through a is the vector

$$p = \hat{x}a = \frac{a^T b}{a^T a} a$$

- the length of the line is $||p|| = ||b|| \cos \theta$

Projections

- the projection matrix is given by

$$p = a\hat{x} = a\frac{a^T b}{a^T a} = Pb \quad \text{thus} \quad P = \frac{aa^T}{a^T a}$$

- what is the rank of P?
- two special cases
  - if $b = a$ then $\hat{x} = 1$ because a is projected onto itself
  - if b is perpendicular to a then $a^T b = 0$

Example

Find the projection matrix onto the line through $a = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Projections

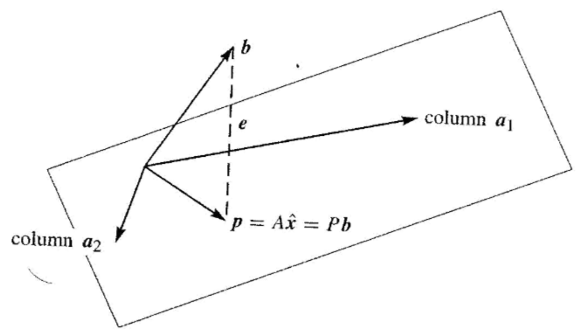
Example

Project $b = [1\ 1\ 1]^T$ onto $a = [1\ 2\ 2]^T$ to find p

What happens if we project this new vector again?

Projections onto a subspace

- with n linearly independent vectors in R^m find the combination that is closest to a given vector b
  - each b in R^m is projected onto a's subspace



Projections onto a subspace

- e goes from b to the nearest point $A\hat{x}$ in the subspace
  - the error vector $e = b - A\hat{x}$ is perpendicular to the subspace
  - it makes a right angle with all the vectors $a_1, \ldots, a_n$

$$\begin{matrix} a_1^T(b - A\hat{x}) = 0 \\ \vdots \\ a_n^T(b - A\hat{x}) = 0 \end{matrix} \quad \text{or} \quad \begin{bmatrix} - a_1^T - \\ \vdots \\ - a_n^T - \end{bmatrix} \begin{bmatrix} b - A\hat{x} \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$A^T(b - A\hat{x}) = 0 \quad \text{or} \quad A^T A\hat{x} = A^T b$$

  - A^TA is symmetric and it's invertible if the a's are independent

$$\hat{x} = (A^T A)^{-1} A^T b$$

  - the projection of b onto the subspace is the vector

$$p = A\hat{x} = A(A^T A)^{-1} A^T b$$

  - and the $n \times n$ projection matrix that produces $p = Pb$ is

$$P = A(A^T A)^{-1} A^T$$

Projections onto a subspace

- the key equation $A^T(b - A\hat{x}) = 0$ is given by linear algebra
  - the subspace is the column space of A
  - $e = b - A\hat{x}$ is perpendicular to the column space
  - $b - A\hat{x}$ is in the left nullspace, therefore $A^T(b - A\hat{x}) = 0$

- thus the left nullspace is important in projections and contains the error vector $e = b - A\hat{x}$
  - b is split into the projection p and the error $e = b - p$

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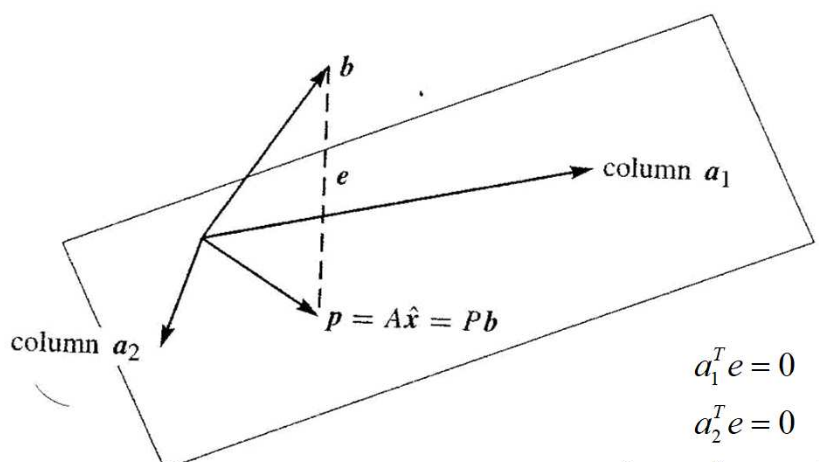
Projections onto a subspace



$$a_1^T e = 0$$
$$a_2^T e = 0$$
$$A^T e = A^T(b - A\hat{x}) = 0$$

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Projections onto a subspace

Example

Find $\hat{x}$, p and P when

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

Question

What is the effect of multiplying a projected vector by the same P again?

Least squares approximation

- sometimes, for example when we have too many equations, there is no solution to $Ax = b$
  - e.g. when $m > n$, the n columns span a small part of the m-dimensional space
  - unless all measurements are perfect $e = b - Ax \neq 0$
    - when $e = 0$, x is an exact solution to $Ax = b$
    - when $||e||$ is as small as possible $\hat{x}$ is a least squares solution
- when there is no solution, we instead solve $A^T A\hat{x} = A^T \mathrm{b}$
  - every vector in b has a part in the column space (p) and another perpendicular part in the left nullspace (e) – so remove e

$$Ax = b \;= p + e \quad \text{is impossible;} \quad Ax = p \quad \text{is solvable}$$

Least squares approximation

- the solution $\hat{x}$ to $Ax = p$ makes the error as small as possible because for any x

$$\| Ax - b \|^2 = \| Ax - p \|^2 + \| e \|^2$$

  this is simply $a^2 + b^2 = c^2$ for a right angle triangle

- vector $Ax - p$ in the column space is perpendicular to e in the left nullspace

- $Ax - p$ is reduced to zero by choosing x to be $\hat{x}$
  - this leaves the smallest possible error, i.e. e

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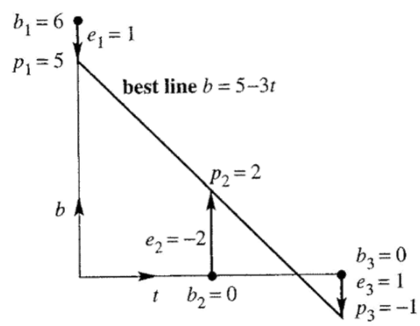
Least squares approximation

Example

Find a line $b = C+Dt$ through the three points (0,6), (1,0) and (2,0).



- the solution is $C = 5$ and $D = -3$, i.e. $b = 5 - 3t$ is the best line
  - note the vertical error distances e_1, e_2, e_3 chosen to minimise the total error $E = e_1^2 + e_2^2 + e_3^2$

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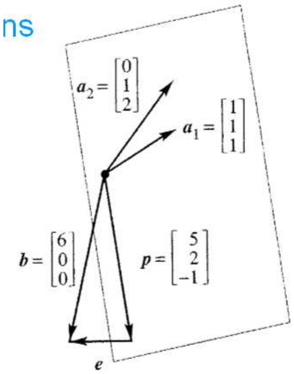
Least squares approximation

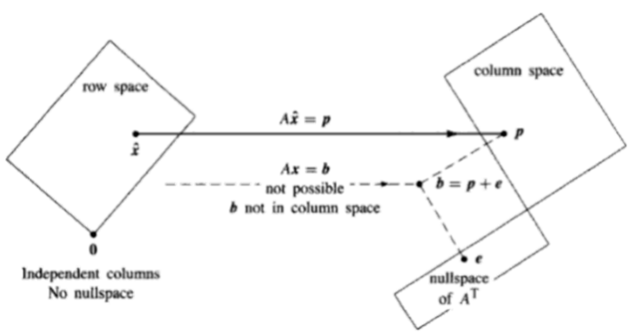
- and in 3 dimensions



$$A^T A\hat{x} = A^T b$$

- b is not in the column space of A – hence there's no solution
  - the smallest error is $e = b - A\hat{x}$ which is perpendicular to the plane

Least squares approximation



- here we are splitting up $b = p + e$
  - instead of solving $Ax = b$ we solve $A\hat{x} = p$
  - error $e = b - p$

Fitting a straight line

- we wish to fit a line to some points (t_i,b_i)

$$Ax=b \quad \text{is} \quad \begin{array}{c} C+Dt_1=b_1 \\ C+Dt_2=b_2 \\ \vdots \\ C+Dt_m=b_m \end{array} \quad \text{and} \quad A=\begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix}$$

  the column space is so thin that b most probably lies outside
  - components of e are the vertical distances $e_1,\ldots,e_m$ to the closest line which has heights $p_1,\ldots,p_m$
- the equations $A^T A\hat{x}=A^T b$ give $\hat{x}=(C,D)$
  - the errors are $e_i=b_i-C-Dt_i$

Fitting a straight line

- $b = C + Dt$ exactly fits the data points if

$$\begin{array}{c} C+Dt_1=b_1 \\ \vdots \\ C+Dt_m=b_m \end{array} \quad \text{or} \quad \begin{bmatrix} 1 & t_1 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix}=\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

- the columns are independent so we multiply by A^T to get

$$A^T A\hat{x}=A^T b$$

$$A^T A=\begin{bmatrix} 1 & \cdots & 1 \\ t_1 & \cdots & t_m \end{bmatrix}\begin{bmatrix} 1 & t_1 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix}=\begin{bmatrix} m & \Sigma t_i \\ \Sigma t_i & \Sigma t_i^2 \end{bmatrix}$$

$$A^T b=\begin{bmatrix} 1 & \cdots & 1 \\ t_1 & \cdots & t_m \end{bmatrix}\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}=\begin{bmatrix} \Sigma b_i \\ \Sigma t_i b_i \end{bmatrix}$$

Fitting a straight line

- the line $C + Dt$ which minimises $e_1{}^2 + \ldots + e_m{}^2$ is determined by

$$A^T A\hat{x} = A^T b$$

$$\begin{bmatrix} m & \Sigma t_i \\ \Sigma t_i & \Sigma t_i^2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} \Sigma b_i \\ \Sigma t_i b_i \end{bmatrix}$$

- the vertical errors at the m points on the line are the components of $e = b - p = b - A\hat{x}$ which is perpendicular to the columns of A

- it is in the nullspace of A^T and the best $\hat{x} = (C, D)$ minimises the total error E, the sum of the squares

$$E(x) = \| Ax - b \|^2 = (C + Dt_1 - b_1)^2 + \cdots + (C + Dt_m - b_m)^2$$

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Fitting a parabola

- problems with parabolas $b = C + Dt + Et^2$ are still problems in **linear** algebra

- e.g. fitting $b_1, \ldots, b_m$ at times $t_1, \ldots, t_m$ to such a parabola

$$\begin{matrix} C + Dt_1 + Et_1^2 = b_1 \\ \vdots \\ C + Dt_m + Et_1^2 = b_m \end{matrix} \quad \text{has the } m \text{ by 3 matrix} \quad A = \begin{bmatrix} 1 & t_1 & t_1^2 \\ \vdots & \vdots & \vdots \\ 1 & t_m & t_m^2 \end{bmatrix}$$

the best parabola chooses $\hat{x} = (C, D, E)$ to satisfy

$$A^T A\hat{x} = A^T b$$

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Orthogonal bases and Gram-Schmidt

- orthogonal vectors are good
  - can make A^TA diagonal

  we want to be able to choose orthogonal vectors

- vectors $q_1,\ldots,q_n$ are orthogonal when their dot products are zero, i.e. $q_i \cdot q_j$ or $q_i^T q_j = 0$ for $i \neq j$

- they are orthonormal if for each i, $||q_i|| = 1$

- the matrix of orthonormal vectors is written Q
  - Q is easy to work with since $Q^TQ = I$ and $Q^T = Q^{-1}$
  - if its only orthogonal then it's still diagonal

Orthogonal bases and Gram-Schmidt

- some special orthogonal matrices
  - rotation matrices

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{and} \quad Q^T = Q^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

  - permutation matrices

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ z \\ x \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

  and their inverse/transpose

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} y \\ z \\ x \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Orthogonal bases and Gram-Schmidt

- reflection matrices
  - if u is a unit vector and $Q = I - 2uu^T$, $Q^T = Q^{-1} = Q$, then

$$Q^T = I - 2uu^T = Q \quad \text{and} \quad Q^T Q = I - 4uu^T + 4uu^T uu^T = I$$

- note that if a vector is on the mirror line then it doesn't change under multiplication by Q

- rotation, permutation and reflection all preserve length

$$\| Qx \| = \| x \| \quad \text{for any } x$$

- dot products are also preserved

$$(Qx)^T (Qy) = x^T Q^T Qy = x^T y$$

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Orthogonal bases and Gram-Schmidt

- Q matrices are important for numerical computation because

$$\| Qx \|^2 = \| x \|^2$$

$$\text{because} \quad (Qx)^T (Qx) = x^T Q^T Qx = x^T Ix = x^T x$$

numbers can never grow too large when vectors of fixed length are used – computer codes which use Q matrices are numerically stable

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Projections using orthogonal bases

- when we have Q instead of A projection formulae become

$$A^T A\hat{x} = A^T b \qquad \hat{x} = Q^T b$$

$$p = A\hat{x} \qquad p = Q\hat{x}$$

$$P = A(A^T A)^{-1} A^T \qquad P = QQ^T$$

  and there are no matrices to invert

- there are n separate 1-dimensional projections and there is no coupling ($A^T A$) from A matrices

$$p = Q\hat{x} = QQ^T b = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} \begin{bmatrix} q_1^T b \\ \vdots \\ q_n^T b \end{bmatrix} = q_1(q_1^T b) + \cdots + q_n(q_n^T b)$$

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Projections using orthogonal bases

- when Q is square the subspace is the whole space
  - $Q^T = Q^{-1}$ and $\hat{x} = Q^T b$ is equal to $x = Q^{-1} b$
  - b's projection onto the whole space is b itself
  - $P = QQ^T = I$ and the solution is exact!

- when $p = b$ the formulae decomposes b into its constituent 1-dimensional projections

$$b = q_1(q_1^T b) + \cdots + q_n(q_n^T b)$$

  the vector is decomposed into perpendicular components
  - Fourier!

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Projections using orthogonal bases

Example

The orthogonal matrix has orthonormal columns q^1, q^2 and q^3

$$Q = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \quad \text{has first column} \quad q_1 = \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$$

the separate projections of $b = (0, 0, 1)$ onto the columns is

$$q_1(q_1^T b) = \frac{2}{3} q_1 \quad \text{and} \quad q_2(q_2^T b) = \frac{2}{3} q_2 \quad \text{and} \quad q_3(q_3^T b) = -\frac{1}{3} q_3$$

their sum is the projection of b onto the whole space, b

$$\frac{2}{3} q_1 + \frac{2}{3} q_2 - \frac{1}{3} q_3 = \frac{1}{9}\begin{bmatrix} -\ 2 + 4 - 2 \\ 4 - 2 - 2 \\ 4 + 4 + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = b$$

The Gram-Schmidt Process

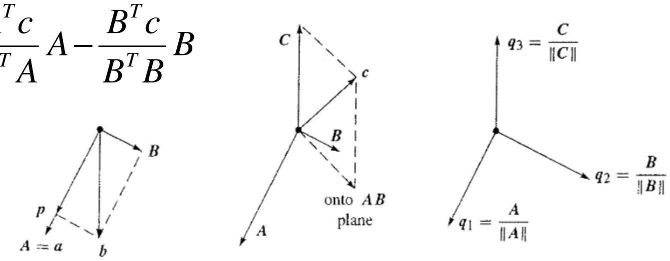
- a means of obtaining orthonormal vectors
- we start with (e.g. 3) vectors a, b, c and want to determine a number of orthonormal bases, say A, B, C
- begin by choosing $A = a$ as the first direction
- the next direction B must be perpendicular to A
  - start with b and remove its projection onto A

$$B = b - \frac{A^T b}{A^T A} A$$

The Gram-Schmidt Process

- by taking dot products we can verify that A and B are perpendicular, $A^T B = 0$
- B is the error vector e, that is perpendicular to A
- the third direction starts with c

$$C = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B$$



The Gram-Schmidt Process

Example

Calculate an orthonormal basis for the three vectors

$$a = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \quad \text{and} \quad c = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}$$

The factorisation $A = QR$

- there is a matrix connecting A and Q
- we can see it easily already

$$\begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} q_1^T a & q_1^T b & q_1^T c \\ & q_2^T b & q_2^T c \\ & & q_3^T c \end{bmatrix} \quad \text{or} \quad A = QR$$

- e.g. from the last example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & -3 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{18} \\ 0 & \sqrt{6} & -\sqrt{6} \\ 0 & 0 & \sqrt{3} \end{bmatrix} = QR$$

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The factorisation $A = QR$

- the square matrix R is upper triangular with positive diagonal
- it's useful for least squares

$$A^T A = R^T Q^T QR = R^T R$$

- the least squares equation simplifies to

$$R^T R\hat{x} = R^T Q^T b \quad \text{or} \quad R\hat{x} = Q^T b$$

- instead of solving $Ax = b$, which is impossible, we solve $R\hat{x} = Q^T b$ by back substitution – which is very fast

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The factorisation $A = QR$

Example

Determine the QR decomposition of

$$A = \begin{bmatrix} 12 & -51 & 4 \\ 6 & 167 & -68 \\ -4 & 24 & -41 \end{bmatrix}$$

Eigenvalues and Eigenvectors

- take a matrix and look at its powers

$$\begin{array}{cccc} A & A^2 & A^3 & A^{100} \\ \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} & \begin{bmatrix} 0.7 & 0.45 \\ 0.3 & 0.55 \end{bmatrix} & \begin{bmatrix} 0.650 & 0.525 \\ 0.350 & 0.475 \end{bmatrix} & \begin{bmatrix} 0.6000 & 0.6000 \\ 0.4000 & 0.4000 \end{bmatrix} \end{array}$$

- A^{100} was found by looking at eigenvalues
- certain special vectors x do not change direction when multiplied by A – they are the eigenvectors of A
- they can, however, change their length and do so according to the corresponding eigenvalue, λ

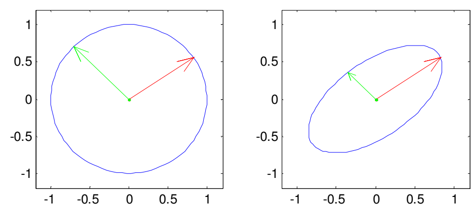
Eigenvalues and Eigenvectors

- the basic equation

$$Ax = \lambda x$$

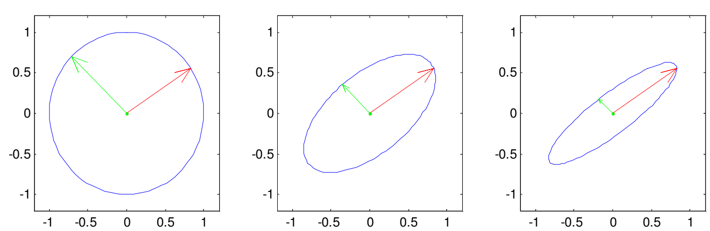
- example

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \qquad S = \begin{bmatrix} 0.6 & 1 \\ 0.4 & -1 \end{bmatrix} \qquad V = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$



Eigenvalues and Eigenvectors

- what happens if we multiply by A again?



$$A^{99}\begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} \quad \text{is really} \quad \overset{(0.8)}{\cancel{x_1}} + (0.2)\left(\frac{1}{2}\right)^{99} x_2 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} + \begin{bmatrix} \text{very} \\ \text{small} \\ \text{number} \end{bmatrix}$$

- any vector is a combination of the eigenvectors

Eigenvalues and Eigenvectors

- eigenvector x_1 is a steady-state that doesn't change
  - $\lambda_1 = 1$
- eigenvector x_2 is a decaying mode
  - $\lambda_1 = 0.5$
- the higher the power of A the closer its columns approach the steady state
- A is a Markov matrix
  - largest eigenvalue of 1
  - [0.6 0.4]' is the steady state

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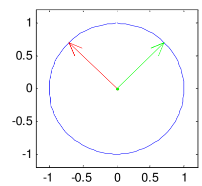
Eigenvalues and Eigenvectors

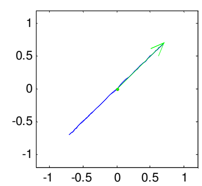
Example

The projection matrix P has eigenvalues of 0 and 1. Its eigenvectors are x_1=(1,1) and x_2=(1,-1).

$$P = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

- each column of P adds up to 1, so $\lambda = 1$ is an eigenvalue
- P is singular, so $\lambda = 0$ is an eigenvalue
- P is symmetric, so x_1 and x_2 are perpendicular





What can you say about the nullspace and the column space?

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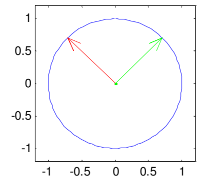
Eigenvalues and Eigenvectors

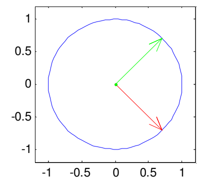
$$R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Example

The reflection matrix R has eigenvalues of 1 and -1.

- eigenvector x_1=(1,1) is unchanged
- eigenvector x_2=(-1,1) is reflected
- the eigenvectors are the same as for P





The eigenvalue problem

- refers to the seeking of non-trivial solutions to

$$Ax = \lambda x$$
$$(\lambda I - A)x = 0$$

values of the scalar λ are the eigenvalues and the corresponding values of $x \neq 0$ are the eigenvectors which make up the nullspace of A - λI

- a non-trivial solution exists if A - λI is not invertible, or if $\det(A - \lambda I) = 0$
- for a matrix A of size $n \times n$, there will be n eigenvalues
- each eigenvalue leads to a corresponding eigenvector

The eigenvalue problem

Example

Calculate the eigenvalues and eigenvectors of:

$$A=\begin{bmatrix}-2 & 1\\ 1 & -2\end{bmatrix} \qquad B=\begin{bmatrix}1 & 1 & -2\\ -1 & 2 & 1\\ 0 & 1 & -1\end{bmatrix}$$

Example

Calculate the eigenvalues and eigenvectors of:

$$A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \text{ and } B=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$$

where we have repeated eigenvalues

Properties of eigenvalues for $n \times n$ matrices

- Property 1
  The sum of the eigenvalues:
  $$\sum_{i=1}^{n}\lambda_i = \operatorname{trace} A = \sum_{i=1}^{n} a_{ii}$$
- Property 2
  The product of the eigenvalues:
  $$\prod_{i=1}^{n}\lambda_i = |A|$$
- Property 3
  The eigenvalues of the inverse, if it exists, are:
  $$\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \cdots, \frac{1}{\lambda_n}$$
- Property 4
  The eigenvalues of the transpose are:
  $$\lambda_1, \lambda_2, \cdots, \lambda_n$$
- Property 5
  If k is a scalar then the eigenvalues of kA are:
  $$k\lambda_1, k\lambda_2, \cdots, k\lambda_n$$
- Property 6
  If k is a scalar then the eigenvalues of $A + kI$ are:
  $$\lambda_1 \pm k, \lambda_2 \pm k, \cdots, \lambda_n \pm k$$
- Property 6
  If k is a positive integer then the eigenvalues of A^k are:
  $$\lambda_1^k, \lambda_2^k, \cdots, \lambda_n^k$$

Diagonalisation

- when x is an eigenvector multiplication by A is equivalent to multiplication by a single number: $Ax = \lambda x$
- it is like having a diagonal matrix
  - the 100th power of a diagonal matrix is easy to compute
- here we show that A turns into a diagonal matrix when the eigenvalues are properly used

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Diagonalisation

- suppose $n \times n$ matrix A has n linearly independent eigenvectors
- if we put them into the columns of an eigenvector matrix, S, then

$$S^{-1}AS = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Proof:

$$AS = A\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \cdots & \lambda_n x_n \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 x_1 & \cdots & \lambda_n x_n \end{bmatrix} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = S\Lambda$$

$$AS = S\Lambda \quad \text{is} \quad S^{-1}AS = \Lambda \quad \text{or} \quad A = S\Lambda S^{-1}$$

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Diagonalisation

- some points:
  - S has an inverse because we assumed the eigenvectors were linearly independent
    - without linearly independent eigenvectors we can't diagonalise the matrix
  - matrices A and Λ have the same eigenvalues but different eigenvectors
    - the eigenvectors in S diagonalise A
    - the eigenvectors of Λ are NOT the same as those of A and are the columns of I
      - diagonalisation aligns the new eigenvectors with the coordinate axes

Diagonalisation

An example for the projection matrix P which has eigenvalues of 1 and 0 and corresponding eigenvectors (1,1) and (-1,1)

$$\underset{S^{-1}}{\begin{bmatrix} .5 & .5 \\ -.5 & .5 \end{bmatrix}} \underset{P}{\begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}} \underset{S}{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}} = \underset{\Lambda}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}$$

Note that

- $P^2 = P$
- $\Lambda^2 = \Lambda$
- the column space has swung around from (1,1) to (1,0)
- the nullspace has swung around from (-1,1) to (0,1)
- diagonalisation lines up the **new** eigenvectors with the co-ordinate axes

Diagonalisation

- suppose all eigenvalues of A are different
  - then the eigenvectors are independent
  - then A is diagonalisable

proofs in course text

- the eigenvector matrix is not unique
  - multiplying its columns by any non-zero constant gives the same Λ
- to diagonalise A we must use an eigenvector matrix
- matrices with repeated eigenvalues (too few) are not diagonalisable

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Diagonalisation

- our Markov matrix A can be diagonalised as follows

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.6 & 1 \\ 0.4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0.4 & -0.6 \end{bmatrix} = S\Lambda S^{-1}$$

- note also that, along the same lines as before

$$A^k = S\Lambda^k S^{-1} = \begin{bmatrix} 0.6 & 1 \\ 0.4 & -1 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & 0.5^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0.4 & -0.6 \end{bmatrix}$$

$$A^\infty = \begin{bmatrix} 0.6 & 1 \\ 0.4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0.4 & -0.6 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$$

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Diagonalisation

- what's the point?
- we've already seen their use in calculating matrix powers
- there are more applications
  - Fibonacci sequences
  - Markov processes
  - differential equations
  - exponential of a matrix
  - quadratic forms

Linear algebra 53

Symmetric matrices

- $A = S\Lambda S^{-1}$ has particular properties when A is symmetric
- the ***spectral theorem*** tells us that for a symmetric matrix
  - the eigenvalues are real
  - the eigenvectors can be chosen to be orthonormal
- we denote such a diagonalisation by $A = Q\Lambda Q^{-1} = Q\Lambda Q^{\mathrm{T}}$
- to see that $Q\Lambda Q^{\mathrm{T}}$ is symmetric take its transpose

$$\left(Q\Lambda Q^T\right)^T = \left(Q^T\right)^T \Lambda^T Q^T = Q\Lambda Q^T$$

Linear algebra 54

Singular value decomposition (SVD)

- A is an $m \times n$ matrix, square or rectangular
  - row space is r-dimensional in R^n
  - column space is r-dimensional in R^m
  - we will choose orthonormal bases for these spaces
    - row space basis $v_1,\ldots,v_r$
    - column space basis $u_1,\ldots,u_r$
- e.g. for an invertible 2×2 matrix where $m=n=r=2$
  - row space is the plane R^2
  - we want v_1 and v_2 to be perpendicular unit vectors
    - an orthogonal basis
  - also want Av_1 and Av_2 to be perpendicular
  - unit vectors $u_1=Av_1/\|Av_1\|$ and $u_2=Av_2/\|Av_2\|$ will be orthogonal

Linear algebra 55

Singular value decomposition (SVD)

- for the matrix

$$A=\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$$

note that
  - no single orthogonal basis Q will make $Q^{-1}AQ$ diagonal
  - we cannot use the eigenvectors of A to form the basis as they aren't orthonormal
  - A is not symmetric and we need two different orthogonal matrices to diagonalise it

Linear algebra 56

Singular value decomposition (SVD)

- inputs v_1 and v_2 give outputs Av_1 and Av_2
  - we want them to line up with u_1 and u_2
- the basis vectors have to give $Av_1=\sigma_1 u_1$ and $Av_2=\sigma_2 u_2$
  - σ_1 and σ_2 are just the lengths $\|Av_1\|$ and $\|Av_2\|$
- with v_1 and v_2 as columns

$$A\begin{bmatrix} \vdots & \vdots \\ v_1 & v_2 \\ \vdots & \vdots \end{bmatrix} = \begin{bmatrix} \vdots & \vdots \\ \sigma_1 u_1 & \sigma_2 u_2 \\ \vdots & \vdots \end{bmatrix} = \begin{bmatrix} \vdots & \vdots \\ u_1 & u_2 \\ \vdots & \vdots \end{bmatrix}\begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix}$$

$$AV = U\Sigma$$

Linear algebra 57

Singular value decomposition (SVD)

$$A\begin{bmatrix} \vdots & \vdots \\ v_1 & v_2 \\ \vdots & \vdots \end{bmatrix} = \begin{bmatrix} \vdots & \vdots \\ \sigma_1 u_1 & \sigma_2 u_2 \\ \vdots & \vdots \end{bmatrix} = \begin{bmatrix} \vdots & \vdots \\ u_1 & u_2 \\ \vdots & \vdots \end{bmatrix}\begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix}$$

$$AV = U\Sigma$$

- Σ is diagonal and like the matrix Λ
  - Λ contained the eigenvalues
  - Σ contains the singular values σ_1 and σ_2
- when U and V both equal S we have $AS = S\Lambda$ which gives $S^{-1}AS = \Lambda$ - it is diagonalised
  - but the vectors in S are not generally orthogonal

Linear algebra 58

Singular value decomposition (SVD)

- we require U and V to be orthogonal
  - basis vectors in their columns must be orthonormal

$$V^T V = \begin{bmatrix} \cdots & v_1^T & \cdots \\ \cdots & v_2^T & \cdots \end{bmatrix} \begin{bmatrix} \vdots & \vdots \\ v_1 & v_2 \\ \vdots & \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

  - thus $V^T V = I$ which means $V^T = V^{-1}$
    - similarly $U^T U = I$ and $U^T = U^{-1}$

- this is the SVD

$$AV = U\Sigma \quad \text{and then} \quad A = U\Sigma V^{-1} = U\Sigma V^T$$

where U and V are orthogonal

Linear algebra 59

Singular value decomposition (SVD)

- to see V by itself multiply A^T by A

$$\begin{aligned} A^T A &= \left(U\Sigma V^T\right)^T (U\Sigma V^T) = V\Sigma^T U^T U\Sigma V^T \\ &= V\Sigma^T \Sigma V^T \\ &= V \begin{bmatrix} \sigma_1^2 & \\ & \sigma_2^2 \end{bmatrix} V^T \end{aligned}$$

which is an ordinary factorisation exactly like $A = Q\Lambda Q^T$
  - except that A is really $A^T A$
  - the columns of V are now the eigenvectors of $A^T A$!!!

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Singular value decomposition (SVD)

Example

Show that the SVD of

$$A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$$

is given by

$$A = U\Sigma V^T$$

$$\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & \\ & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

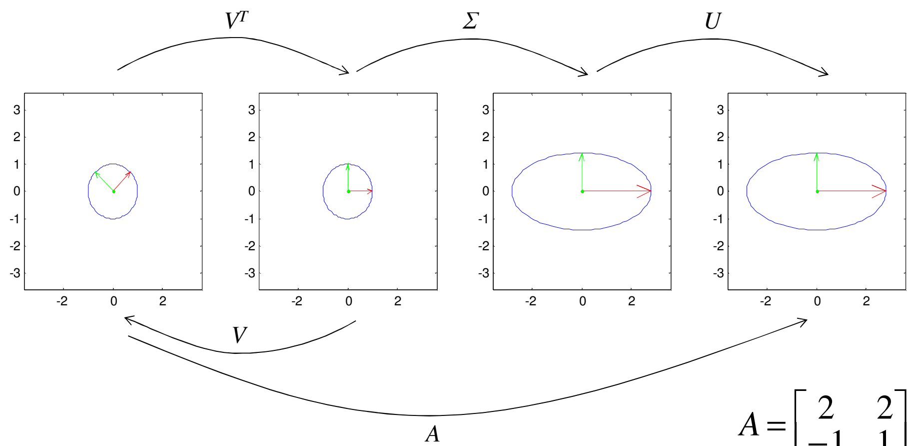
Singular value decomposition (SVD)

- we can also calculate the u's first, then the v's

$$AA^T = (U\Sigma V^T)\left(U\Sigma V^T\right)^T = U\Sigma V^T V\Sigma^T U^T$$
$$= U\Sigma^T \Sigma U^T$$
$$= U\begin{bmatrix} \sigma_1^2 & \\ & \sigma_2^2 \end{bmatrix} U^T$$

- we have an ordinary factorisation of AA^T
  - columns of U are its eigenvectors
- we can show that it gives the same result as in the previous example

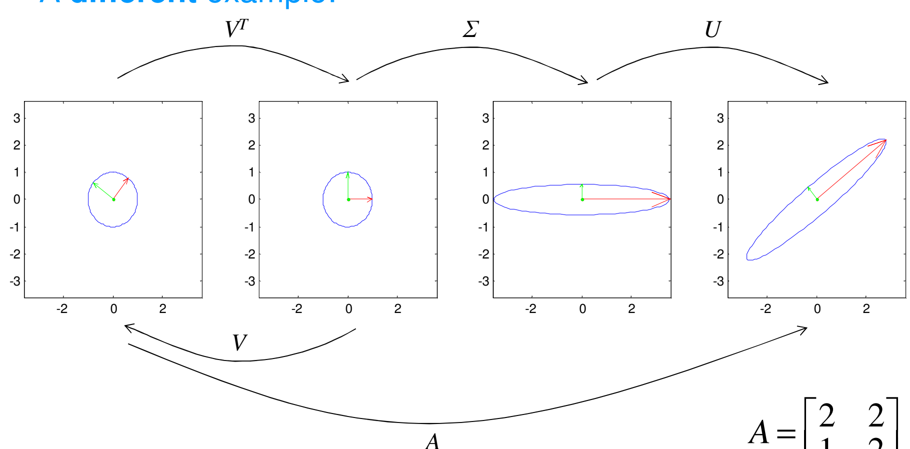
Singular value decomposition (SVD)



$$A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$$

Singular value decomposition (SVD)

A **different** example:



$$A = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}$$

Singular value decomposition (SVD)

Example

Find the SVD of the following singular matrix:

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$$

Remember that it is **singular**!

$$A = U\Sigma V^T$$

$$\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

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Singular value decomposition (SVD)

- U and V contain orthonormal bases for all four subspaces
  - first r columns of V : row space of A
  - last n-r columns of V : nullspace of A
  - first r columns of U : column space of A
  - last m-r columns of U : nullspace of A^T
- some points:
  - the v's are eigenvectors of A^TA
  - the u's are eigenvectors of AA^T
  - A^TA and AA^T have the same eigenvalues
  - Av_i has to fall in the direction of u_i, $Av_i=\sigma_i u_i$

Linear algebra 66

Singular value decomposition (SVD)

- starting from $A^T A v_i = \sigma_i^2 v_i$ and multiplying
  - first by v_i^T

    $v_i^T A^T A v_i = \sigma_i^2 v_i^T v_i$ gives $\|Av_i\|^2 = \sigma_i^2$ so that $\|Av_i\| = \sigma_i$

    were we used $(v_i^T A^T)(Av_i) = ||Av_i||^2$
  - then by A

$AA^T A v_i = \sigma_i^2 A v_i$ gives $u_i = Av_i / \sigma_i$ as a unit eigenvector of AA^T

- this gives us the proof that $Av_i = \sigma_i u_i$
  - A is diagonalised by the two bases

$$A = U \Sigma V^T$$

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Summary

- You should be able to
  - work comfortably with matrices
  - calculate matrix determinants and inverses
  - solve linear equations via matrix methods
  - understand the concept of a matrix rank
  - understand the principles of orthogonality and the 4 subspaces
  - determine eigenvalues and eigenvectors
  - determine matrix decompositions

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