Column spaces

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The column space of I is all of R^2
- the x, y plane

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

The column space of A is only a line in R^2 since column 2 is twice column 1
- there's really only one equation

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

Again the column space is all of R^2 since columns 1 & 3 are independent.

NB. B will have more solutions than I or A since more than $\underline{1}$ x will solve $Bx = b$ on account of the dependence between columns 1 & 2.

Special Solution

$$x_1 + 2x_2 + 3x_3 = 0$$

This is:

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

So $A = R = \left[\begin{array}{c|cc} 1 & 2 & 3 \end{array}\right]$

Pivot | free

and from $RN = 0$

$$I(-F) = -F(I)$$

$$N = \left[\begin{array}{c} -F \\ \hline I \end{array}\right] = \begin{array}{l} \left[\begin{array}{cc} -2 & -3 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] \end{array} \begin{array}{l} \} -F \\ \} I \end{array}$$

Nullspace Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

Note that:

$$R = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

And therefore all multiples of the vector

$$s = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

are in the nullspace of A. The nullspace is therefore a line.

For $A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$ the nullspace contains only Z since A is of full rank.

For $B = \begin{bmatrix} A \\ 2A \end{bmatrix}$ the nullspace is still Z since B is of full column rank.

The nullspace of C is bigger since it is not of full column rank. The nullspace in this case is a plane in R^4 - see solution to determine the nullspace basis in lecture notes.

Complete solution to $Ax=b$

$$A' = \begin{bmatrix} 1 & 1 & b_1 \\ 1 & 2 & b_2 \\ -2 & -3 & b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & b_1 \\ 0 & 1 & b_2-b_1 \\ 0 & -1 & b_3+2b_1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2b_1-b_2 \\ 0 & 1 & b_2-b_1 \\ 0 & 0 & b_3+\cancel{2}b_1+b_2 \end{bmatrix}$$

$\therefore$ for the system to be consistent:

$$b_3 + \cancel{2}b_1 + b_2 = 0$$

in order that b is in A's column space.

Note that, in this particular case, there are no free variables – A is of full column rank – and thus there are no special solutions.

The particular solution is given by

$$x_p = \begin{bmatrix} 2b_1-b_2 \\ b_2-b_1 \end{bmatrix}$$

which is at the top of the augmented matrix

Also, note that if $b_3+2b_1+b_2 \neq 0$, then there's no solution.

Complete solution to $Ax=b$ – 2

$$x + y + z = 3$$
$$x + 2y - z = 4$$

N.B. it's a 2×3 matrix of rank 2 – it is column rank deficient.

gives an augmented matrix

$$A' = [A \;\; b] = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & -1 & 4 \end{bmatrix}$$

$$[U \;\; d] = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

$$[R \;\; d] = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

pivot variables — free variable — this is d

$\therefore$ the particular solution is given by

$$x_p = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

checking:

$$Ax_p = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \checkmark$$

$Ax = Ax_p + Ax_n$
$= b + 0$
the null component has no effect on the result.

The special solution is given by

$$x_n = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

checking:

$$Ax_n = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \checkmark$$

And thus the full solution is given by:

$$x = x_p + \alpha x_n = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

Linear independence

If we know that

$$Ax = 0$$

for some $x \neq 0$ then there is some linear combination of the columns which gives 0. Then the columns of A cannot all be independent.

Elimination would reveal the system's true dimension

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \Rightarrow U = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$r = 2$$

Column 3 is a linear combination of columns 1 & 2

Vector space bases

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \quad \text{gives} \quad R = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

the column space of R is spanned by

$$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The corresponding column space of A is spanned by

$$v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$ THEY ARE NOT THE SAME

But, the row spaces of both A & R are spanned by $u = [1 \ 2]$

– reduction / elimination operation are applied to rows!

$$R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

↑ ↑ pivots

$u_1 = [1\ 2\ 0\ 3] \quad u_2 = [0\ 0\ 1\ 4]$

The general process to identify bases involves reducing A to R, identifying the pivot columns in R and by picking off the corresponding columns in A. They form a column basis. The basis for A's row space can be picked straight from R, since it is the same for A.

Column & row spaces

$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$m = 1,\ n = 3,\ r = 1$

- the row space is a line in R^3
- the nullspace is a plane in R^3
  two special solutions: $s_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ $s_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$
  since its dimension $= n - r = 2$
- the column space is in R^1 - all of R^1 since the rank of A is 1
- the left nullspace is empty since $m - r = 0$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$m = 2,\ n = 3,\ r = 1$

- the row space is the same line in R^3
- the nullspace is the same
- the column space is now spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ which is the column of A corresponding to the pivot column of R - it is <u>NOT</u> $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- the dimension of the left nullspace is now $m - r = 1$
  solutions are of the form $A'y = 0$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and thus any multiple of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a solution