Mathematical Methods for Engineers (MathEng)
EXAM
13th February 2018
Duration: 2 hrs, calculators permitted, no documents
This exam paper contains 7 questions and 60 marks.
ATTEMPT ALL QUESTIONS – ANSWER IN ENGLISH

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1. Consider a random variable whose probability density function is given by:

$$f_X(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

A new random variable $Y = y(X)$ is defined by the transformation function:

$$y(x) = -\frac{1}{\lambda}\ln x \qquad (\lambda > 0)$$

(a) Sketch the probability density function of random variable *X*.
(b) Sketch the transformation function *y*(*x*) over the range of the random variable *X*.
(c) Determine an expression for the probability density function of *Y*.
(d) Sketch the probability density function of random variable *Y*.

[x marks]

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2. Using convolution, **sketch** (do not calculate) the response of the system with the illustrated impulse response *h*(*t*) to the input signal *x*(*t*), both illustrated in Figure Q2.

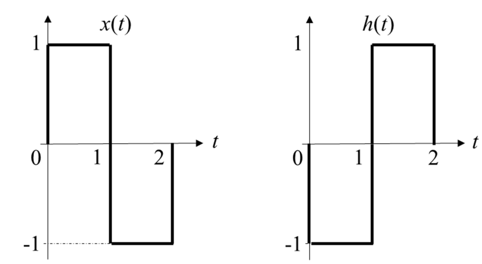

Figure Q2

[8 marks]

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3. Consider the fully rectified sinusoidal signal *x*(*t*) illustrated in Figure Q3.

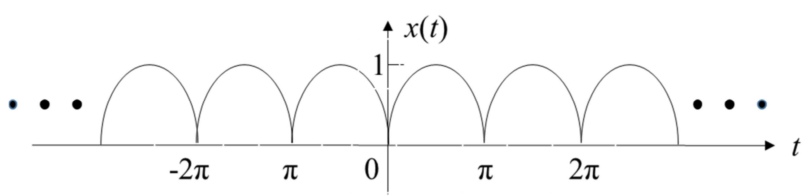

Figure Q3

(a) What is its period, T_0?
(b) What is its fundamental frequency, ω_0?
(c) Determine an expression for the complex Fourier series components given by X_n.
(d) Write down (no calculation needed) an expression for the inverse complex Fourier series.

[6 marks]

4. Sketch the Bode plot (magnitude and phase) of a system with the following transfer function:

$$H(s) = \frac{100}{(s+30)}$$

[8 marks]

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5. A digital filter is defined by difference equation $y[n] = 0.5y[n\text{-}1] + x[n]$. An input signal defined by $x[n] = 0.2^n$, $n \geq 0$ is applied to the filter input.

(a) Sketch a block diagram of the system showing all components of the feedforward and feedback sections and all components of the difference equation.
(b) Determine the z-transform of the input $x[n]$.
(c) Starting with the difference equation $y[n]$, determine an expression for the system transfer function $H[z]$.
(d) By applying a partial fraction expansion, determine the z-transform of the output $Y[z]$.
(e) Using the table of z transforms, determine an expression for the output signal $y[n]$ as a function of n alone.

[12 marks]

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6. (a) Consider the subspace

$$W = \left[\begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix}\right]$$

of the vector space of 2×2 matrices, M_{22}. Determine whether or not $A = \begin{bmatrix} -3 & 3 \\ 6 & -4 \end{bmatrix}$ is an element of W.

(b) Find the *rank* and the *dimension of the nullspace* $N(B)$ of the matrix:

$$B = \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 1 & 3 & 2 & 0 & 4 \\ 1 & 2 & 1 & 1 & 1 \end{bmatrix}$$

(c) Find an orthogonal basis for the column space of the matrix:

$$C = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}$$

[14 marks]

Table of selected Laplace transforms

$$X(s) = \int_{-\infty}^{\infty} x(t) \exp(-st)\, dt$$

N.B.: lower limit is 0 for one-sided Laplace transform

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) \exp(st)\, ds$$

$x(t) \quad (t \geq 0)$	$X(s)$
$\delta(t)$	1
$\delta(t-\alpha)$	$\exp(-\alpha s)$
1 (unit step)	$\frac{1}{s}$
t (unit ramp)	$\frac{1}{s^2}$
$\exp(-\alpha t)$	$\frac{1}{s+\alpha}$
$t\exp(-\alpha t)$	$\frac{1}{(s+\alpha)^2}$
$\sin(\alpha t)$	$\frac{\alpha}{s^2+\alpha^2}$
$\cos(\alpha t)$	$\frac{s}{s^2+\alpha^2}$
$e^{-\alpha t}\sin(\omega t)$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
$e^{-\alpha t}\cos(\omega t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$

Bode plots

Poles or zeros on the real axis:

$$(s+a) = a\left(\frac{s}{a}+1\right) = \frac{1}{\tau}(\tau s + 1)$$

Complex conjugate poles (or zeros):

$$(s^2 + As + B) = (s^2 + 2\zeta\omega_0 s + \omega_0^2) = \omega_0^2((s/\omega_0)^2 + 2\zeta(s/\omega_0) + 1)$$

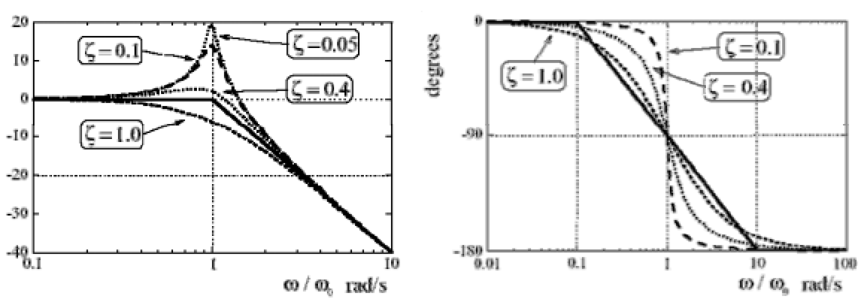

Table of selected z-transforms

$$x_c(t) = \sum_{n=0}^{\infty} x(n\Delta t)\delta(t - n\Delta t)$$

$$X_c(s) = \sum_{n=0}^{\infty} x(n\Delta t)\exp(-n\Delta ts)$$

$$X_c(s) = X(z)|_{z=e^{\Delta ts}}$$

$$X_c(\omega) = X_s(s)|_{s=j\omega}$$

$$X(\omega) = X(z)|_{z=\exp(\Delta tj\omega)}$$

$$X(z) = \sum_{n=0}^{\infty} x(n\Delta t)z^{-n}$$

$x(n)$ $(n \geq 0)$	$X(z)$
$\delta(n)$ unit pulse	1
$\delta(n-m)$	z^{-m}
1 (unit step)	$\dfrac{z}{z-1}$
n (unit ramp)	$\dfrac{z}{(z-1)^2}$
$\exp(-\alpha n)$	$\dfrac{z}{(z-e^{-\alpha})}$
$n\exp(-\alpha n)$	$\dfrac{e^{-\alpha}z}{(z-e^{-\alpha})^2}$
$\sin(\beta n)$	$\dfrac{z\sin(\beta)}{z^2-2z\cos(\beta)+1}$
$\cos(\beta n)$	$\dfrac{z^2-z\cos(\beta)}{z^2-2z\cos(\beta)+1}$
$e^{-\alpha n}\sin(\beta n)$	$\dfrac{ze^{-\alpha}\sin(\beta)}{z^2-2ze^{-\alpha}\cos(\beta)+e^{-2\alpha}}$
$e^{-\alpha n}\cos(\beta n)$	$\dfrac{z^2-ze^{-\alpha}\cos(\beta)}{z^2-2ze^{-\alpha}\cos(\beta)+e^{-2\alpha}}$

Table of selected Fourier transform pairs

Function	$x(t)$	$X(\omega)$
Rectangular function of width τ	$\Pi(t/\tau)$	$\tau\ \mathrm{sinc}(\omega\tau/2)$
Triangular function of width 2τ	$\Lambda(t/\tau)$	$\tau\ \mathrm{sinc}^2(\omega\tau/2)$
Train of impulses every Δt	$\delta_T(t)$	$2\pi/\Delta t\, \Sigma_n \delta(\omega - 2\pi n/\Delta t)$

NB: $\mathrm{sinc}(x) = \sin(\pi x)/\pi x$
NB: $\mathrm{sa}(x) = \sin(x)/x$

Euler's identity

$$\exp(j\theta) = \cos\theta + j\sin\theta$$

$$\cos(n\omega_0 t) = \frac{\exp(jn\omega_0 t) + \exp(-jn\omega_0 t)}{2}$$

$$\sin(n\omega_0 t) = \frac{\exp(jn\omega_0 t) - \exp(-jn\omega_0 t)}{2j}$$

Fourier series and transforms

Trigonometric Fourier series

$$x(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t)$$

$$A_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(n\omega_0\, t)dt$$

$$B_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(n\omega_0\, t)dt$$

Complex Fourier series – periodic and continuous in time, discrete in frequency

$x(t) = \sum_{n=-\infty}^{+\infty} X_n \exp(jn\omega_0 t)$	$X_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\exp(-jn\omega_0 t)\, dt$

Fourier transform – continuous in time, continuous in frequency

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\exp(j\omega t)\, d\omega$	$X(\omega) = \int_{-\infty}^{\infty} x(t)\exp(-j\omega t)\, dt$

Discrete-time Fourier transform – discrete in time, continuous and periodic in frequency

$x(n\Delta t) = \frac{\Delta t}{2\pi}\int_{-\pi/\Delta t}^{\pi/\Delta t} X(\omega)\exp(jn\Delta t\omega)\, d\omega$	$X(\omega) = \sum_{n=-\infty}^{\infty} x(n\Delta t)\exp(-j\omega n\Delta t)$

Discrete Fourier transform – discrete and periodic in time and in frequency

$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\exp\left(\frac{jnk2\pi}{N}\right)$	$X(k) = \sum_{n=0}^{N-1} x(n)\exp\left(\frac{-jnk2\pi}{N}\right)$

Transformation of random variables

$$f_Y(y) = \sum_{i=1}^{N} f_X(x_i) \left|\frac{dx_i}{dy}\right|_{x_i = g_i^{-1}(y)}$$

$$f_{UV}(u, v) = f_{XY}(x, y) \left|\frac{\partial(x, y)}{\partial(u, v)}\right|_{\substack{x = g_1^{-1}(u,v) \\ y = g_2^{-1}(u,v)}}$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

 [END OF PAPER]