Essential Mathematical Methods for Engineers

Lecture 2:
Transfer function and system characterisation

Outline

- transfer function, poles and zeros
- transfer function and frequency response
  - frequency response from pole/zero diagram
  - Fourier transform of periodic signals
  - measurement of frequency response
  - Bode plots
  - Fourier and Laplace
- transfer function and impulse response
- time-domain response
  - first order systems
  - second order systems
- rise time and bandwidth

Transfer function, poles and zeros

- for zero initial conditions a linear system can be described by:

$$a_0\, y + a_1 \frac{dy}{dt} + \cdots + a_n \frac{d^n y}{dt^n} = b_0\, x + b_1 \frac{dx}{dt} + \cdots + b_m \frac{d^m x}{dt^m}$$

$$a_0\, Y(s) + a_1\, sY(s) + \cdots + a_n\, s^n Y(s) = b_0\, X(s) + b_1\, sX(s) + \cdots + b_m\, s^m X(s)$$

$$A(s)\, Y(s) = B(s)\, X(s)$$

where $A(s)$ and $B(s)$ are polynomials in s

- therefore $$H(s) = \frac{Y(s)}{X(s)} = \frac{B(s)}{A(s)} = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots + a_m s^n}$$

- poles correspond to values of s for which $H(s)$ is equal to infinity

- zeros correspond to values of s for which $H(s)$ is equal to zero

EURECOM

Transfer function, poles and zeros

Example

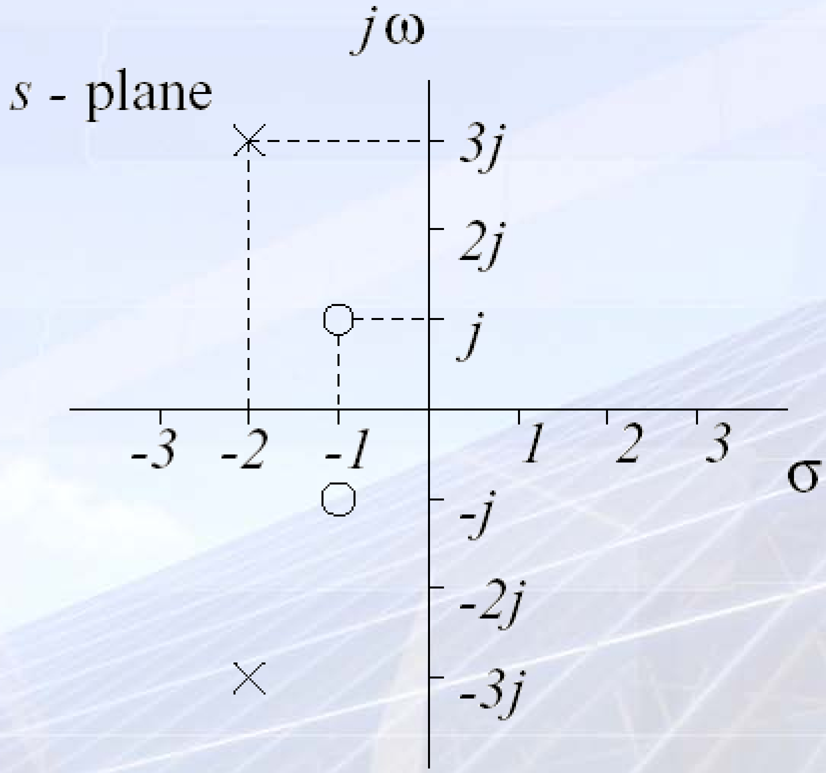
A typical transfer function that might be obtained by applying Kirchhoff's laws to a circuit is:

$$H(s) = \frac{s^2 + 2s + 2}{s^2 + 4s + 13}$$

Determine the positions of its poles and zeros.

Transfer function, poles and zeros

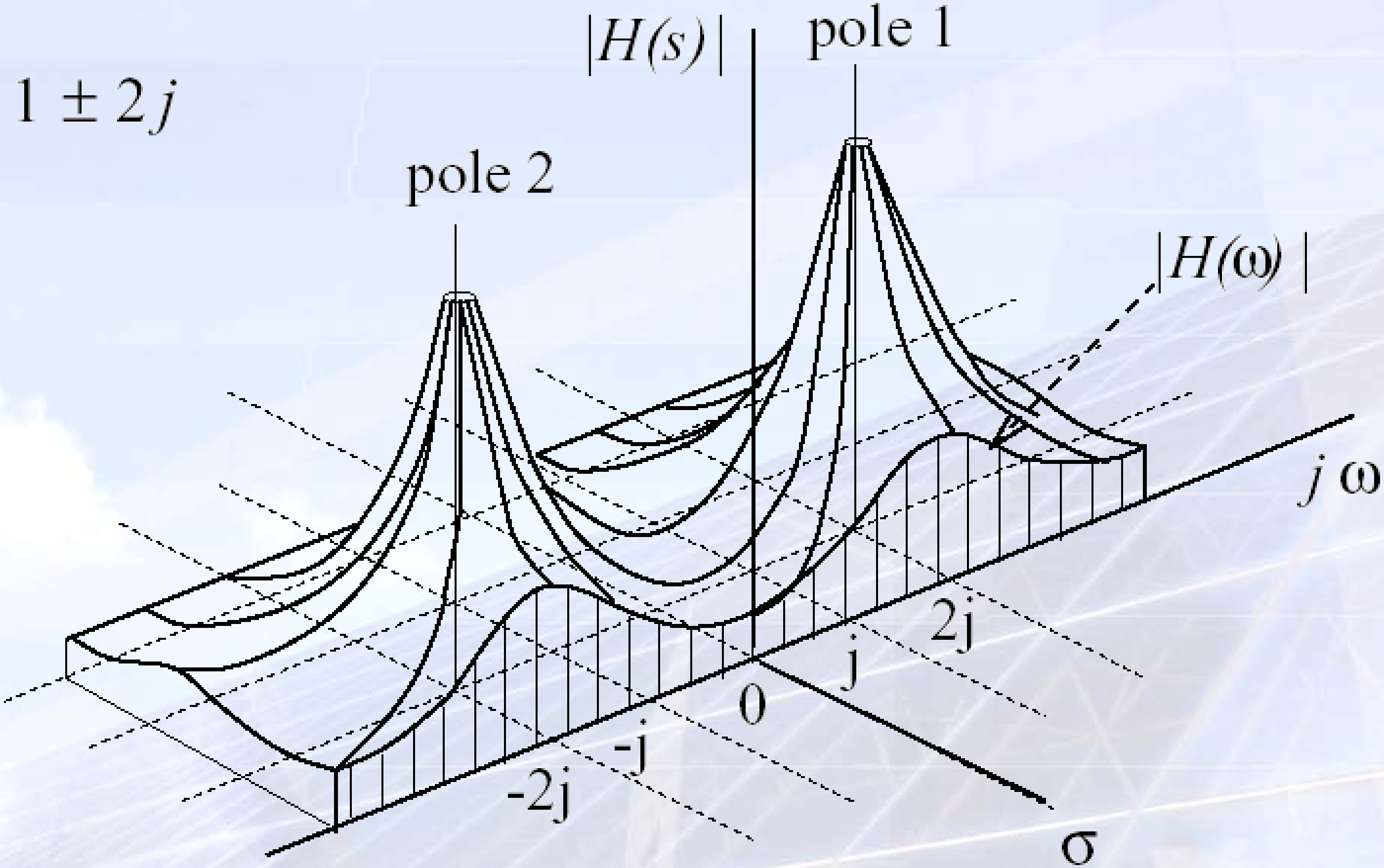
A pole/zero diagram illustrating a pair of complex conjugate poles and zeros:



Transfer function, poles and zeros

A plot of $|H(s)|$ against s for transfer function $H(s) = 1/(s^2 + 2s + 5)$

$$s = -1 \pm 2j$$



Transfer function and frequency response

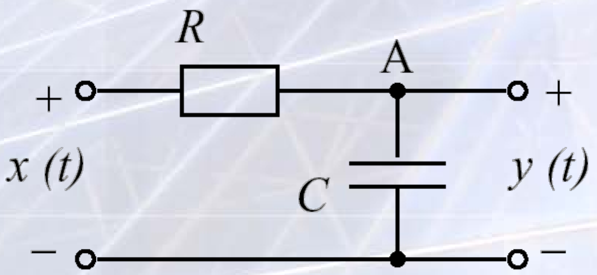
- the frequency response of a system is defined as:

$$H(\omega) = \frac{F[\text{ output }]}{F[\text{ input }]} = \frac{Y(\omega)}{X(\omega)}$$

- given an input $\exp(j\omega t)$ the output is $H(\omega)\exp(j\omega t)$

- for the simple *RC* circuit the frequency response, or Fourier transfer function is given by:

$$H(\omega) = \frac{1}{1 + j\omega RC}$$



Transfer function and frequency response

- note the similarity to the Laplace transform

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{(1 + RCs)} \qquad H(\omega) = \frac{1}{1 + j\omega RC}$$

- for $\sigma = 0$ the Laplace and Fourier basis functions are identical:

$$\exp(st) = \exp(\sigma t + j\omega t)$$

$$\exp(st) = \exp(j\omega t)$$

- we can observe $|H(\omega)|$ in the plot of $|H(s)|$ by setting $\sigma = 0$

- Q: what if the poles moved closer to the axis?

Frequency response from pole/zero diagram

- a transfer function with *m* zeros and *n* poles

$$H(s) = \frac{A(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)}$$

- replacing s with $j\omega$

$$H(\omega) = \frac{A(j\omega - z_1)(j\omega - z_2)\cdots(j\omega - z_m)}{(j\omega - p_1)(j\omega - p_2)\cdots(j\omega - p_n)}$$

- amplitude response

$$|H(\omega)| = \frac{A|j\omega - z_1||j\omega - z_2|\cdots|j\omega - z_m|}{|j\omega - p_1||j\omega - p_2|\cdots|j\omega - p_n|}$$

- phase response

$$\angle H(\omega) = \angle(j\omega - z_1) + \angle(j\omega - z_2) + \cdots + \angle(j\omega - z_m)$$
$$- \angle(j\omega - p_1) - \angle(j\omega - p_2)\cdots - \angle(j\omega - p_n)$$

Transfer function and frequency response

Frequency response from pole/zero diagram

- we can thus obtain the amplitude and phase responses for any ω by evaluating the modulus and argument

- it is useful to visualise the term $(j\omega_0 - p_1)$

- consider a simple example of a single pole system

$$H(s) = \frac{1}{(s - p_1)}$$

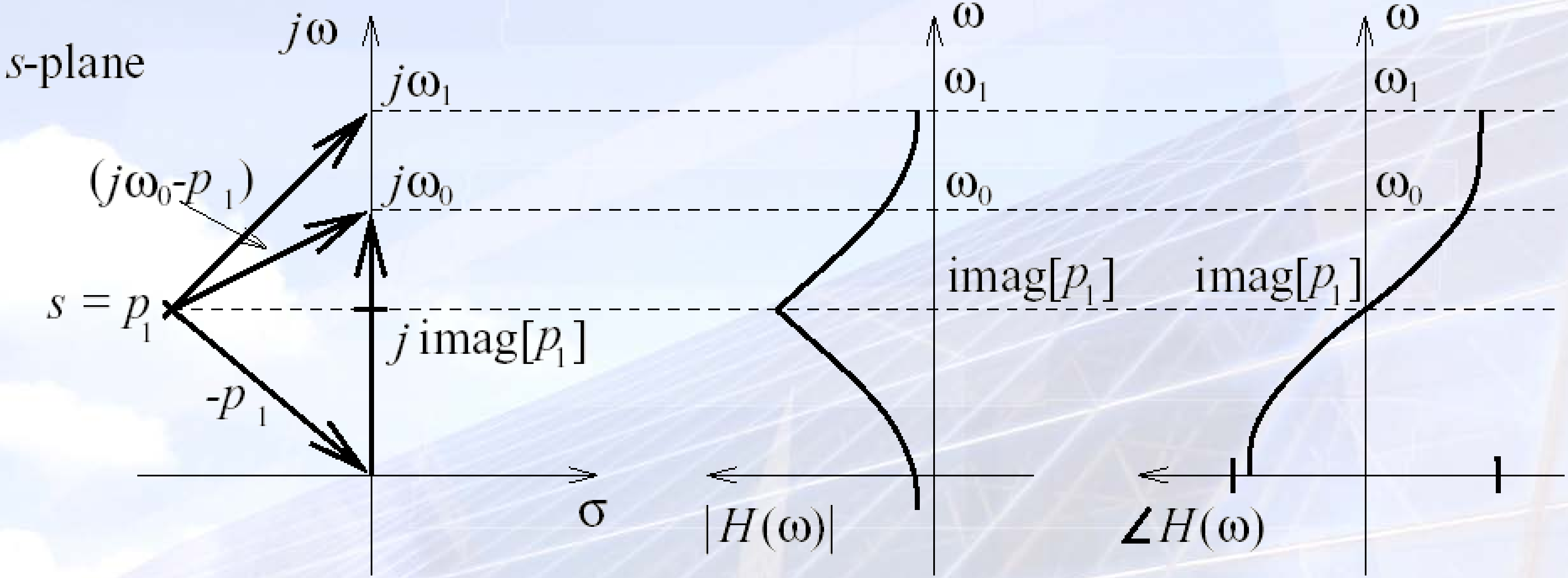
- the frequency response is given by

$$H(\omega) = \frac{1}{(j\omega - p_1)}$$

EURECOM

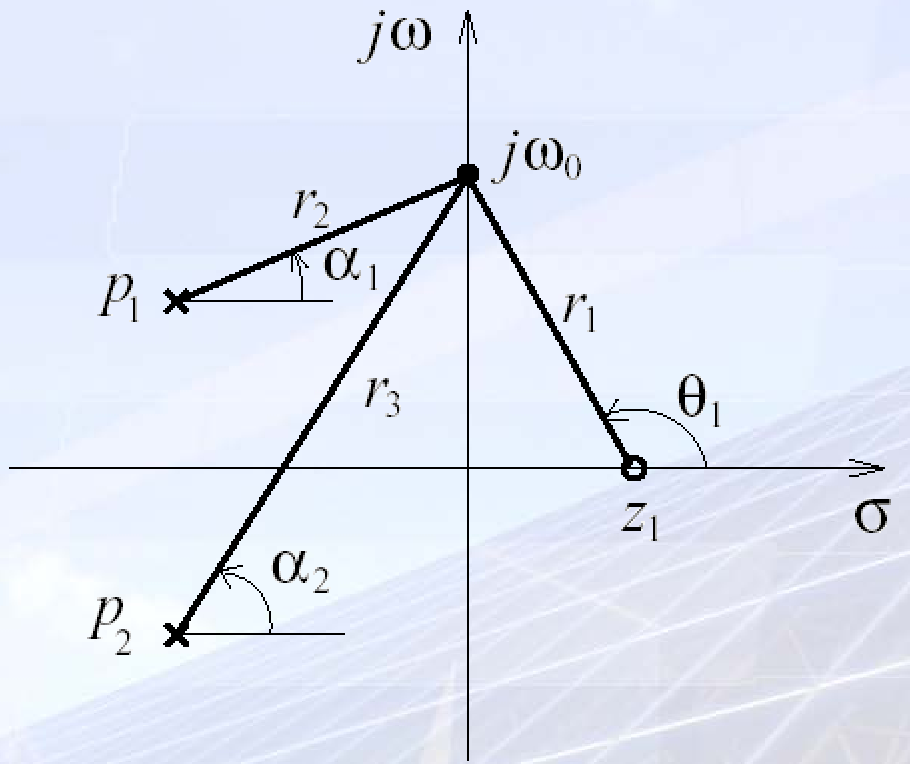
Frequency response from pole/zero diagram

- evaluating the amplitude and phase response through consideration of $(j\omega_0 - p_1)$ as a vector



EURECOM

- evaluating the phase response



$$|H(\omega_0)| = \frac{r_1}{r_2 \, r_3} \qquad \angle H(\omega_0) = \theta_1 - \alpha_1 - \alpha_2$$

Example

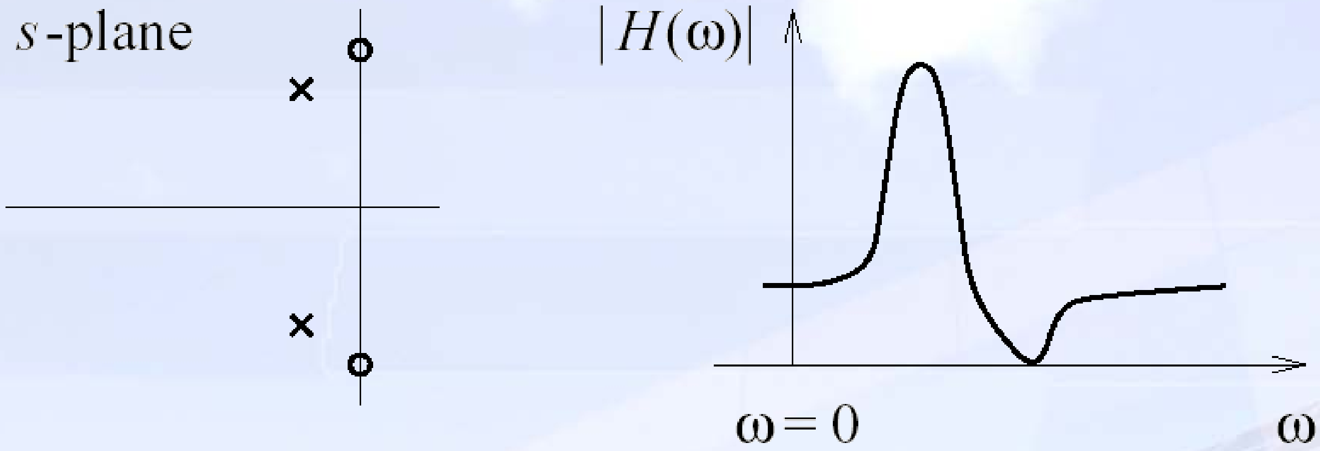
Sketch the amplitude frequency response of each of the following 2 systems:

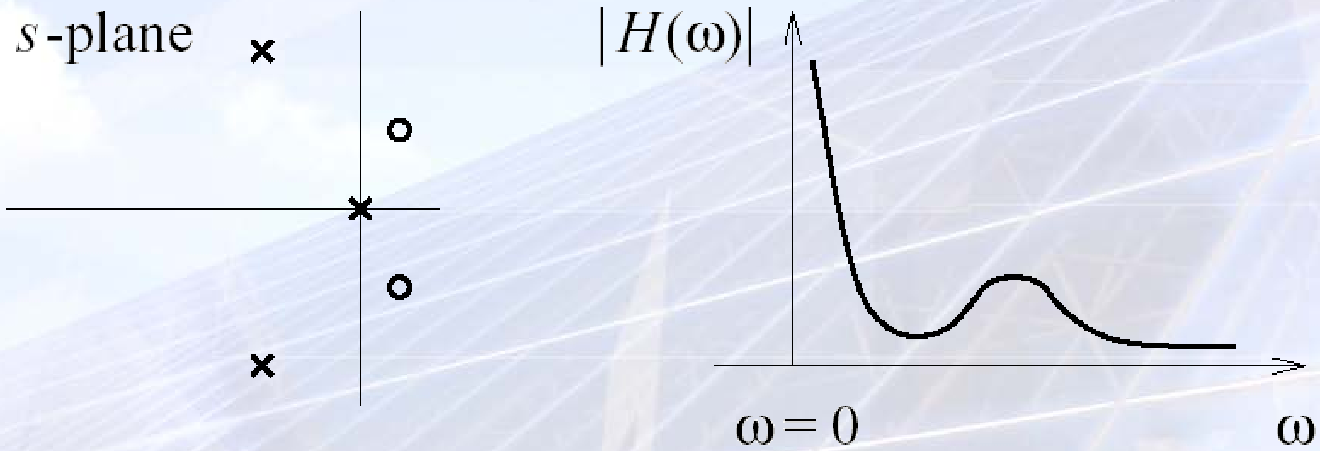
(a) $(s^2 + 9)/(s^2 + 0.6s + 4.09)$

(b) $(s^2 + 0.6s + 4.09)/(s^3 + 0.8s^2 + 9.16s)$

Transfer function and frequency response

Frequency response from pole/zero diagram

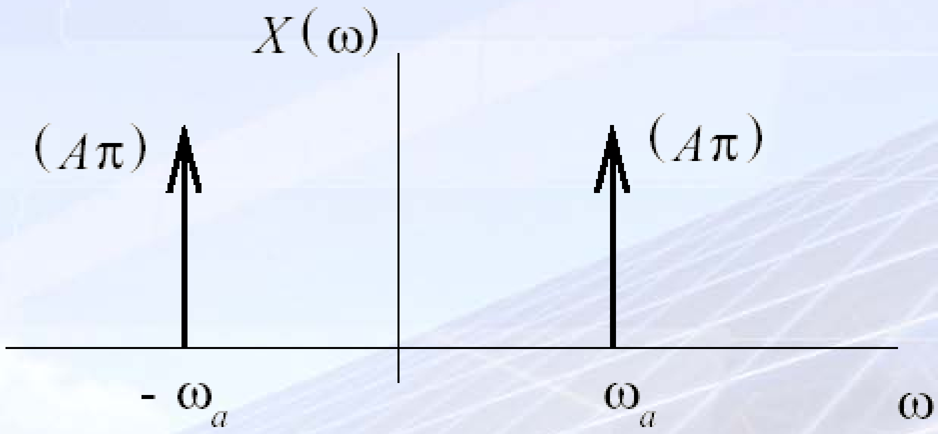




Fourier transform of periodic signals

- a cosine wave $x(t) = A\cos(\omega_a t)$

  can be represented by $X(\omega) = A\pi[\ \delta(\omega - \omega_a) + \delta(\omega + \omega_a)\ ]$



- as justification consider a signal with Fourier transform

$$X(\omega) = \delta(\omega - \omega_a)$$

- its time domain representation is given by

$$x(t) = F^{-1}[\ X(\omega)\ ]$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \exp(j\omega t)\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_a) \exp(j\omega t)\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_a) \exp(j\omega_a t)\, d\omega$$

$$= \frac{1}{2\pi} \exp(j\omega_a t) \int_{-\infty}^{\infty} \delta(\omega - \omega_a)\, d\omega$$

$$= \frac{1}{2\pi} \exp(j\omega_a t)$$

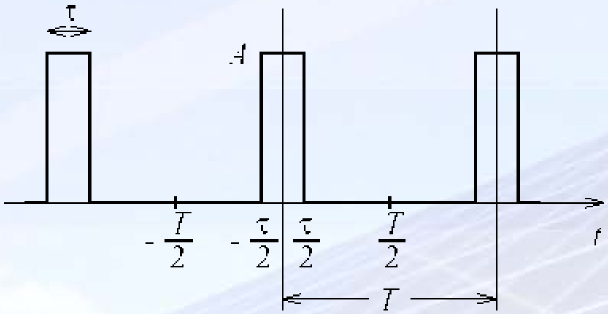
thus $F[\ \exp(j\omega_a t)\ ] = 2\pi\delta(\omega - \omega_a)$

Example

Determine the Fourier transform of a sine wave $A\sin(\omega_a t)$

Example

Derive an expression for the Fourier transform of the following signal:



Note the similarity to the example in the previous lecture

This example shows how the Fourier transform of a periodic signal is easily derived from the Fourier series.

Measurement of frequency response

- we can't generate $\exp(j\omega t)$ in order to measure $H(\omega)\exp(j\omega t)$ and deduce $H(\omega)$

- however we can generate a cosine or sine wave from the addition of two complex phasors

$$\cos(\omega t) = \frac{\exp(j\omega t) + \exp(-j\omega t)}{2} = \Re[\ \exp(j\omega t)\ ]$$

- using superposition the response to a cosine wave is

$$\begin{aligned}
&½\ H(\omega)\exp(j\omega t) + ½\ H(-\omega)\exp(-j\omega t) \\
&= ½\ H(\omega)\exp(j\omega t) + ½\ H^*(\omega)\exp(-j\omega t) \\
&= ½\ H(\omega)\exp(j\omega t) + ½\ (H(\omega)\exp(j\omega t))^* \\
&= \Re[\ H(\omega)\exp(j\omega t)\ ]
\end{aligned}$$

Measurement of frequency response

- but $H(\omega)$ is complex and can be expressed in terms of magnitude and phase, thus

$$
\begin{aligned}
&\Re[\ |H(\omega)| \exp(\ j\angle H(\omega)\ ) \exp(j\omega t)\ ] \\
&= |H(\omega)|\ \Re[\ \exp(\ j\angle H(\omega)\ ) \exp(\ j\omega t\ )\ ] \\
&= |H(\omega)|\ \Re[\ \exp(\ j\omega t + j\angle H(\omega)\ )\ ] \\
&= |H(\omega)| \cos(\omega t + \angle H(\omega)\ )
\end{aligned}
$$

- the output is a cosine wave of identical frequency but amplified by $|H(\omega)|$ and phase shifted by $\angle H(\omega)$

- the complex frequency response is thus given by:

$$H(\omega) = |H(\omega)| \cos(\ \angle H(\omega)\ ) + j|H(\omega)| \sin(\ \angle H(\omega)\ )$$

EURECOM

Bode plots

- a more powerful graphical technique of frequency response analysis

- with any transfer function $H(s)$ there are four general factors
  - a constant gain, K;
  - poles or zeros at the origin, s;
  - poles or zeros on the real axis – terms of the form

$$(s+a)=a\left(\frac{s}{a}+1\right)=\frac{1}{\tau}(\tau s+1)$$

  - complex conjugate poles or zeros, (s^2+As+B)

- the quadratic is usually expressed as

$$(s^2+As+B)=(s^2+2\zeta\omega_0 s+\omega_0^2)$$

where ζ is the damping factor and ω_0 is the natural, undamped frequency

EURECOM

Transfer function and frequency response Bode plots

- scaling by ω_0:

$$(s^2 + As + B) = \omega_0^2(\,(s/\omega_0)^2 + 2\zeta(s/\omega_0) + 1\,)$$

  for which the roots are complex if $\zeta < 1$.

- considering the transfer function

$$H(s) = \frac{K}{s\,(\tau s + 1)\,(\,(s/\omega_0)^2 + 2\zeta(s/\omega_0) + 1\,)}$$

  for which there is a
  - constant gain, K
  - pole at the origin
  - pole on the real axis at $s = -1/\tau$
  - pair of complex poles

Transfer function and frequency response
Bode plots

- to obtain the frequency response

$$H(\omega) = \frac{K}{j\omega(j\omega\tau + 1)\left((j\omega/\omega_0)^2 + 2\zeta(j\omega/\omega_0) + 1\right)}$$

- the logarithmic magnitude:

$$20\log_{10}|H(\omega)| = 20\log_{10}(K) - 20\log_{10}(|j\omega|) - 20\log_{10}(|j\omega\tau + 1|) - 20\log_{10}(|1 + (2\zeta/\omega_0)j\omega + (j\omega/\omega_0)^2|)$$

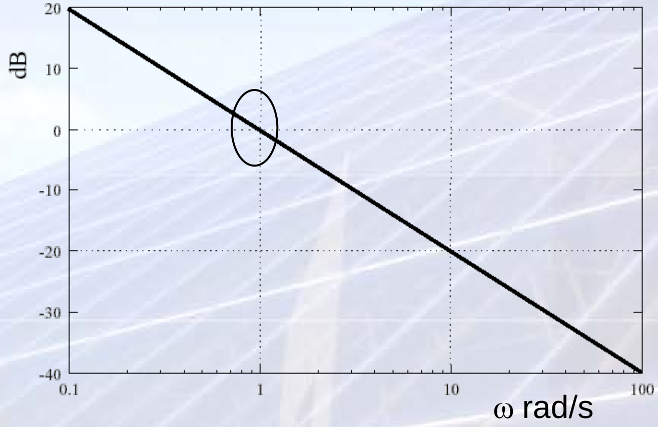
- the phase:

$$\angle H(\omega) = -90^\circ - \tan^{-1}(\omega\tau) - \tan^{-1}\left(\frac{2\zeta\omega_0\omega}{\omega_0^2 - \omega^2}\right)$$

Transfer function and frequency response
Bode plots

- **constant gain term**:
  - a gain of $20\log_{10}(K)$ at all frequencies

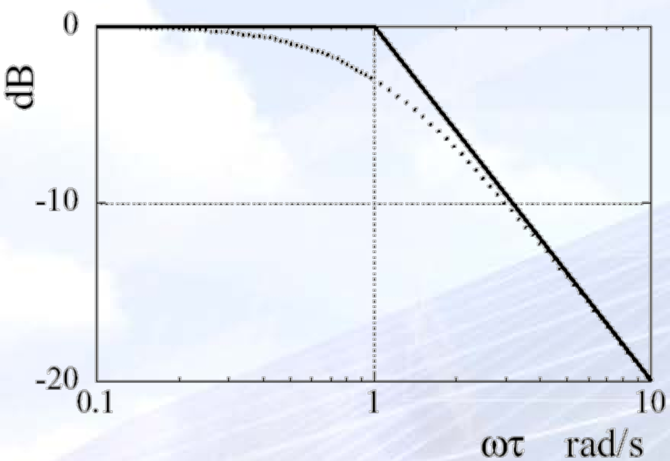
- **pole at origin**:
  - gives rise to a $j\omega$ term in denominator
  - gain of $20\log_{10}(|j\omega|)$ – a straight line with slope -20 dB/decade
  - 0 dB at 1 rad/s
  - phase shift of -90º due to presence of j

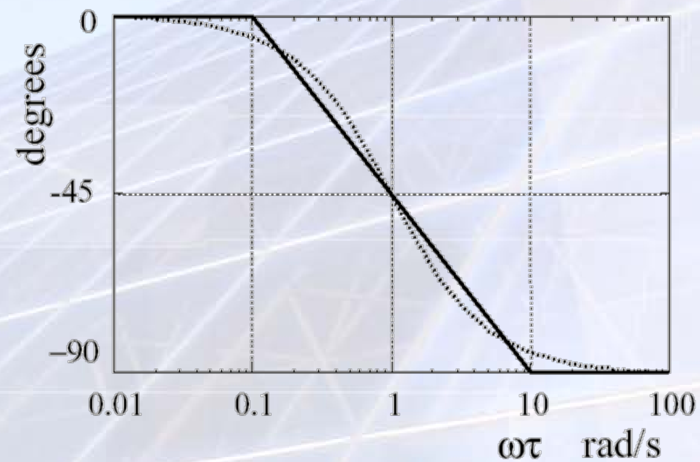


EURECOM

Transfer function and frequency response
Bode plots

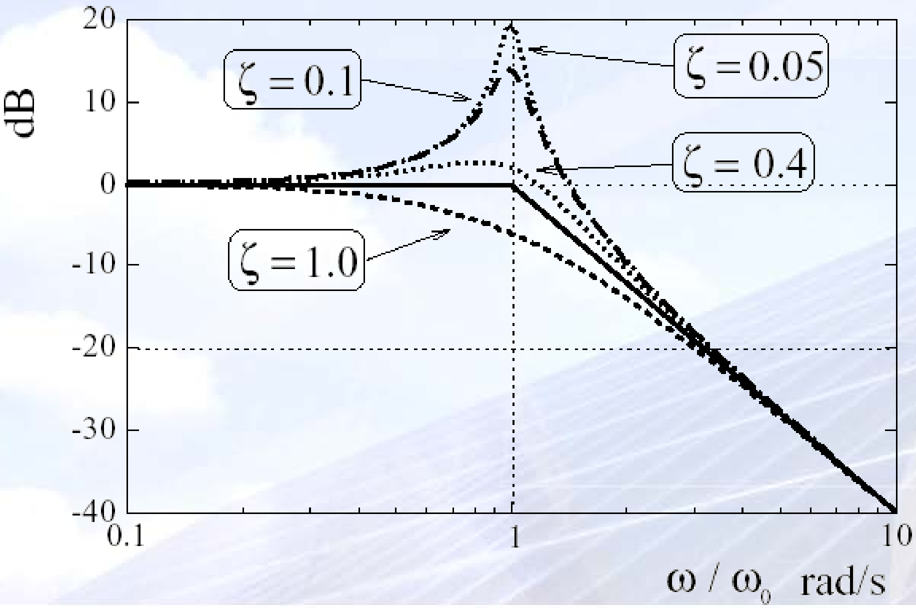
- **pole on real axis**:
  - a pole at $s = -1/\tau$ gives rise to the term $j\omega\tau + 1$ in the denominator
  - at the cut-off frequency $\omega = 1/\tau$, the gain is -3 dB
  - above cut-off the gain has a slope of -20 dB/decade
  - phase of -45º at cut-off,
    - ~0º 1 decade below, ~-90º 1 decade above

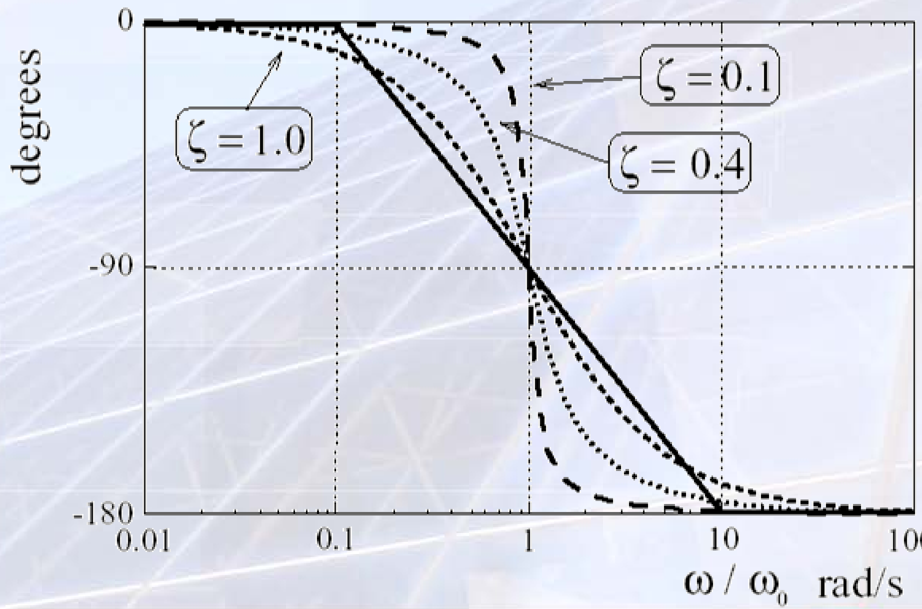




Transfer function and frequency response
Bode plots

- complex conjugate pair of poles
  - gives rise to the term $((j\omega/\omega_0)^2 + 2\zeta(j\omega/\omega_0) + 1)$ in denominator
  - for $\omega < \omega_0$ the gain is ~0 dB
  - above cut-off the gain has a slope of -40 dB/decade
  - at ω_0 the gain is dependent on ζ
  - phase ~-90º at ω_0, ~0º 1 decade below, ~-180º 1 decade above





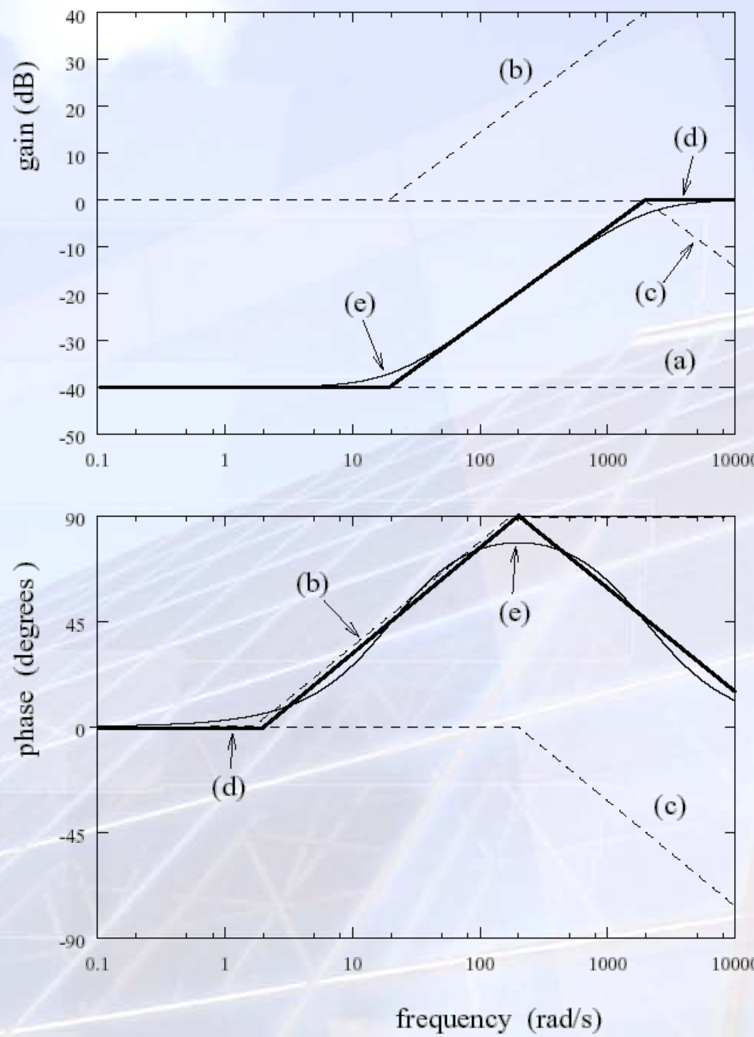
Example

Sketch the Bode plot of a system with the following transfer function:

$$H(s) = \frac{s + 20}{s + 2000}$$

Transfer function and frequency response
Bode plots

$$H(\omega) = \frac{j\omega + 20}{j\omega + 2000} = \frac{j\omega/20 + 1}{(\ j\omega/2000 + 1\ )100}$$



Transfer function and impulse response

- for a system with poles at

$$s = p_1 = \sigma_a + j\omega_a$$

$$s = p_2 = \sigma_a - j\omega_a$$

the transfer function is

$$H(s) = \frac{1}{(s - \sigma_a - j\omega_a)_{p_1} (s - \sigma_a + j\omega_a)_{p_2}}$$

taking a partial fraction expansion

$$H(s) = \frac{A_1}{(\ s - \sigma_a - j\omega_a\ )_{p_1}} + \frac{A_2}{(\ s - \sigma_a + j\omega_a\ )_{p_2}}$$

$$H(s) = \frac{\frac{1}{2j\omega_a}}{(\ s - \sigma_a - j\omega_a\ )_{p_1}} + \frac{\frac{-1}{2j\omega_a}}{(\ s - \sigma_a + j\omega_a\ )_{p_2}}$$

EURECOM

Transfer function and impulse response

- the impulse response is

$$h(t) = \left[\frac{1}{2j\omega_a} \exp(\ (\sigma_a + j\omega_a)\ t\ ) \right]_{p_1} - \left[\frac{1}{2j\omega_a} \exp(\ (\sigma_a - j\omega_a)\ t\ ) \right]_{p_2}$$

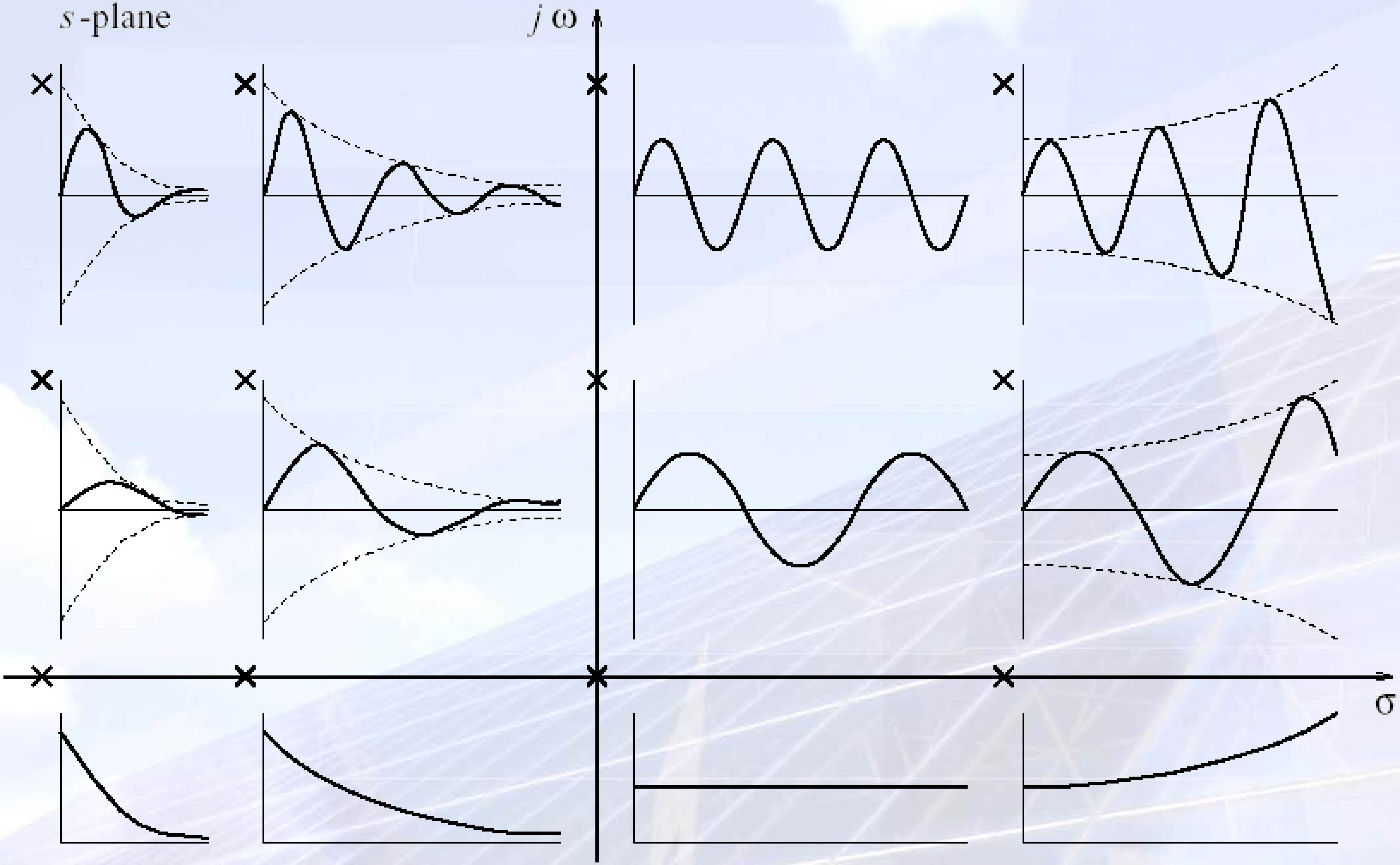
$$= \frac{1}{2j\omega_a} \exp(\ \sigma_a t\ )\ [\ \exp(\ j\omega_a t\ ) - \exp(\ -j\omega_a t\ )\ ]$$

$$= \frac{1}{\omega_a} \exp(\ \sigma_a t\ ) \sin(\ \omega_a t\ )$$

- σ_a controls the decay rate and ω_a the frequency of oscillation

- there is a strong relation between the impulse response and pole location

Transfer function and impulse response



Transfer function and impulse response

- this relationship tells us about the stability of the system
- if a pole has $\sigma > 0$ the output will grow exponentially
  - such a system is said to be unstable
- if a pole has $\sigma < 0$ the response will decay exponentially
  - such a system is said to be stable
- most systems have several poles and zeros

$$H(s) = \frac{A\,(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)}$$

Transfer function and impulse response

- the transfer function may be written as

$$H(s) = \frac{A_1}{(s - p_1)} + \frac{A_2}{(s - p_2)} + \cdots + \frac{A_n}{(s - p_n)}$$

  and taking inverse Laplace transforms

$$h(t) = A_1\, e^{p_1\, t} + A_2\, e^{p_2\, t} + \cdots\ A_n\, e^{p_n\, t} +$$

- the impulse response is the sum of complex phasors such as $e^{p_n t}$
  - these phasors are called the modes of the system
  - all must have $\sigma < 0$ for the system to be stable
  - all poles thus lie in the left-half of the s-plane

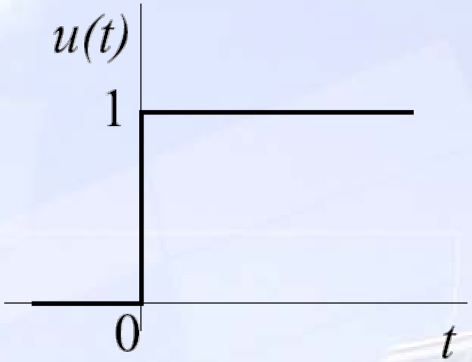
Transfer function and impulse response

Example

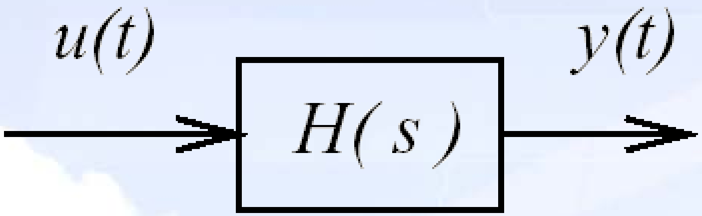
Is the transfer function $H(s) = s/(s^2 + 4s + 68)$ stable or unstable? What is the period of oscillation and the time constant of its impulse response?

Time domain response

- time-domain or transient response
- unit step function $L[u(t)] = \frac{1}{s}$



- when applied to a system



$$Y(s) = H(s)\,\frac{1}{s} \qquad s\,Y(s) = H(s)$$

and taking inverse Laplace transforms of both sides

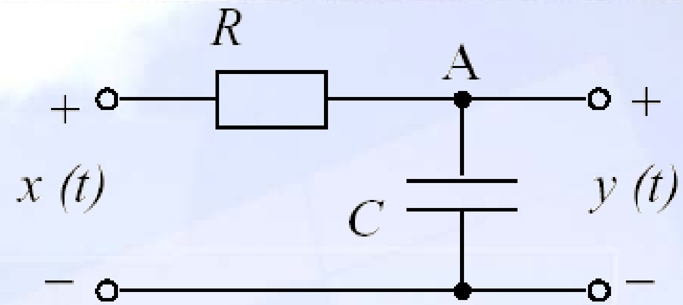
$$\frac{dy}{dt} = h(t)$$

the impulse response is the derivative of the step response

EURECOM

Time domain response

First order systems



- a simple first order, *RC* circuit

  with transfer function $$H(s) = \frac{1}{(1+sRC)}$$

  thus there is one real pole at $s = -1/RC$

- to obtain the step response: $$Y(s) = H(s)\,X(s) = \frac{1}{(1+sRC)}\,\frac{1}{s}$$

$$= \frac{1}{s} - \frac{1}{\left(s + \frac{1}{RC}\right)}$$

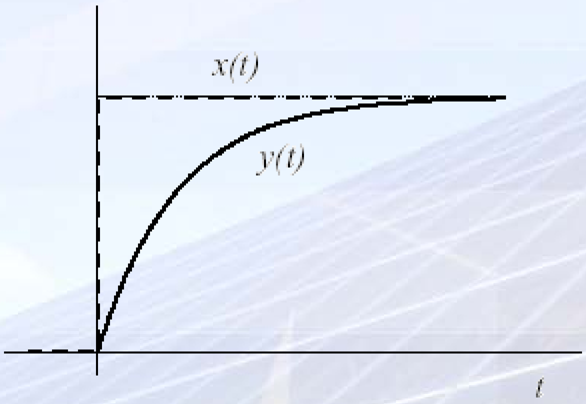
EURECOM

Time domain response
First order systems

- taking inverse Laplace transforms

$$y(t) = 1 - \exp\left(-\frac{t}{RC}\right); \quad t \geq 0.$$

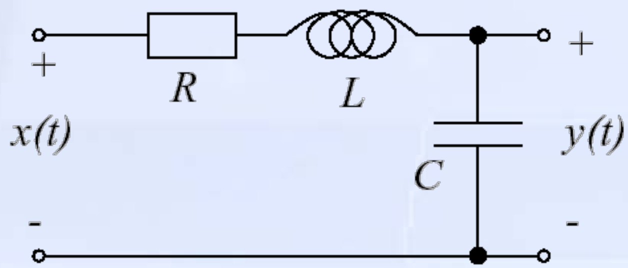
- we obtain the response to the step input



Time domain response

Second order systems

- repeating for an *RLC* circuit



$$Y(s) = \frac{\frac{1}{sC}}{R + sL + \frac{1}{sC}} X(s)$$

therefore

$$H(s) = \frac{1}{s^2 LC + sCR + 1}$$

$$= \frac{\frac{1}{LC}}{s^2 + \frac{R}{L} s + \frac{1}{LC}}$$

EURECOM

Time domain response
Second order systems

- normalising to the form

$$H(s) = \frac{\omega_0^2}{s^2 + 2\zeta\,\omega_0 s + \omega_0^2}$$

we have

$$\omega_0^2 = \frac{1}{LC} \qquad \text{(omega)}$$

$$\zeta = \frac{R}{2\sqrt{\frac{L}{C}}} \qquad \text{(zeta)}$$

- the poles of $H(s)$ are

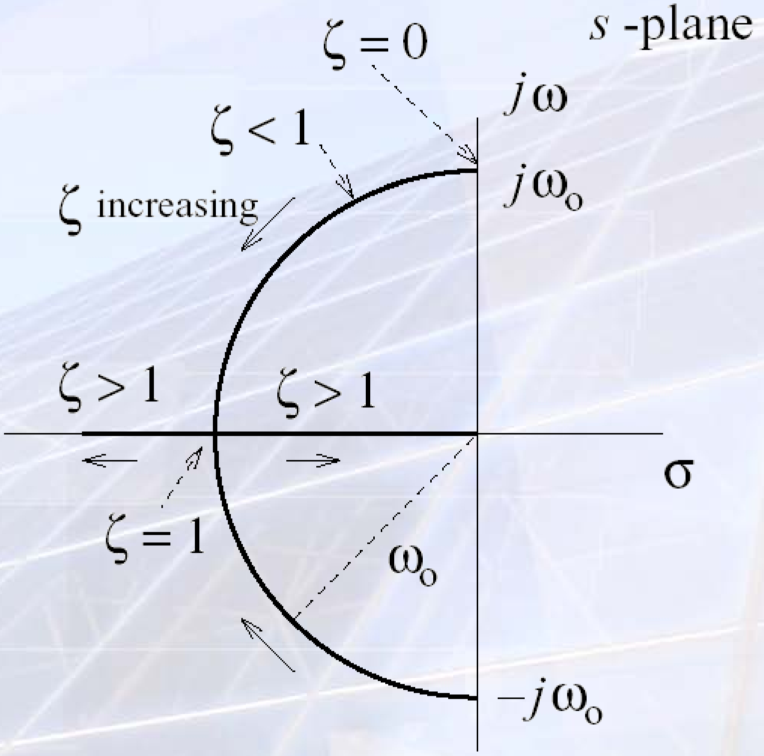
$$s_1 = -\zeta\omega_0 + \omega_0\sqrt{\zeta^2 - 1}$$

$$s_2 = -\zeta\omega_0 - \omega_0\sqrt{\zeta^2 - 1}$$

Time domain response
Second order systems

- three possibilities
  - $\zeta < 1$ => two complex roots (underdamped)
  - $\zeta = 1$ => two real roots at $s = -\omega_0$ (critically damped)
  - $\zeta > 1$ => two real roots (overdamped)

$$s_1 = -\zeta\omega_0 + \omega_0\sqrt{\zeta^2 - 1}$$
$$s_2 = -\zeta\omega_0 - \omega_0\sqrt{\zeta^2 - 1}$$



EURECOM

Time domain response
Second order systems

- what happens to the impulse response as ζ changes

- underdamped ($\zeta < 1$) $$p_1, p_2 = -\zeta\omega_0 \pm j\omega_0 \sqrt{1-\zeta^2}$$

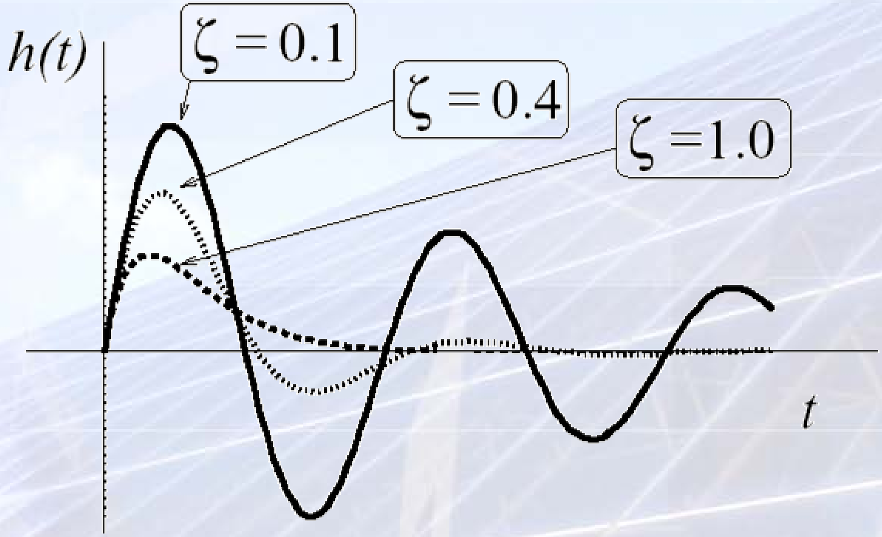
  – the impulse response is given by

$$h(t) = \frac{\omega_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_0 t} \sin(\sqrt{1-\zeta^2}\,\omega_0\, t)$$

- regarding the poles
  – the real part determines the time constant associated with the exponential decay
  – the imaginary part determines the frequency of oscillation

Time domain response
Second order systems

- critically damped ($\zeta = 1$)
  - two identical real poles $p_1 = p_2 = -\omega_0$
    - for real poles there is no oscillation
  - the impulse response is $h(t) = \omega_0 \; t \; e^{-\omega_0 t}$



Time domain response
Second order systems

- overdamped ($\zeta > 1$)
  - equivalent to two first order systems with two different real poles

$$p_1, p_2 = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$$

  - the impulse response is

$$h(t) = \frac{\omega_0}{\sqrt{\zeta^2 - 1}} e^{-\zeta\omega_0 t} \sinh\left(\sqrt{\zeta^2 - 1}\,\omega_0 t\right)$$

  - again there is no oscillation due to real poles

Time domain response
Second order systems

- for the step response

$$Y(s) = H(s)\; L[u(t)]$$

$$= \frac{\omega_0}{(\; s^2 + 2\zeta\,\omega_0 s + \omega_0^2 \;)}\;\frac{1}{s}$$

and again taking inverse Laplace transforms

$$y(t) = L^{-1}[Y(s)]$$

$$= 1 - \frac{\exp(\,-\zeta\,\omega_0 t\,)}{\sqrt{1-\zeta^2}}\,\sin\!\left(\sqrt{1-\zeta^2}\;\omega_0\; t + \theta\right)$$

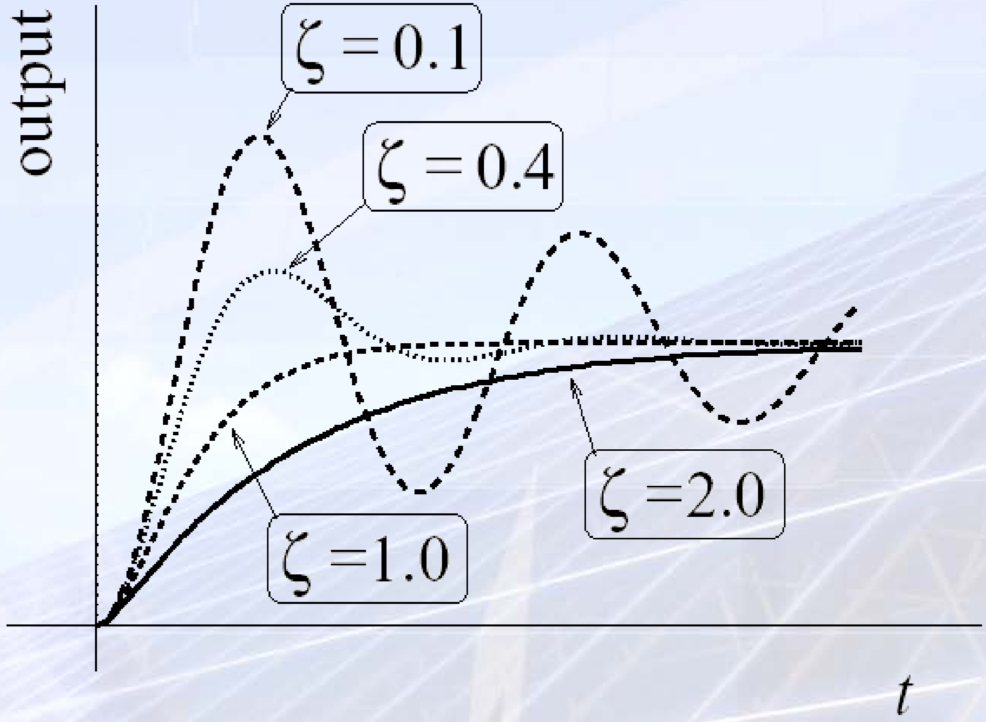
where

$$\theta = \tan^{-1}\!\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

Time domain response
Second order systems

- the last result applies to the underdamped case

- for other damping factor values



EURECOM

Example

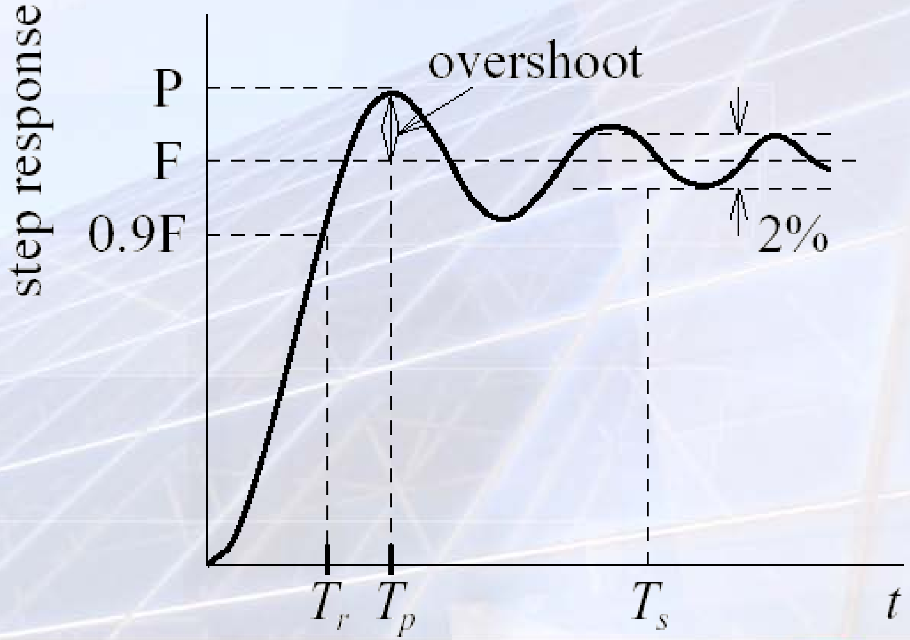
What is the damping factor and undamped natural frequency of a system with transfer function:

$$H(s) = 7s/(12s^2 + 118.8s + 2700)?$$

Rise time and bandwidth

- the rise time measures the speed of response
  - rise time, T_r: time taken to reach 90% of final value, F
  - peak time, T_p: time to first peak, P, for $\zeta < 1$
  - settling time, T_s: time to get to within 2% of F
  - percentage overshoot

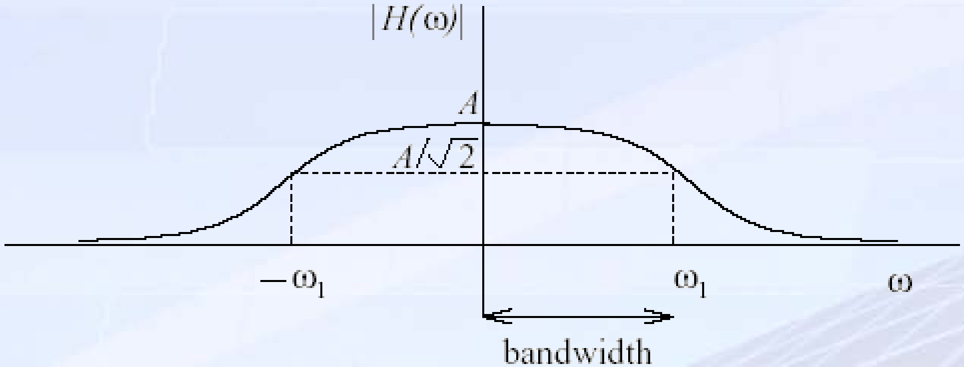
$$100 \frac{(P - F)}{F}$$

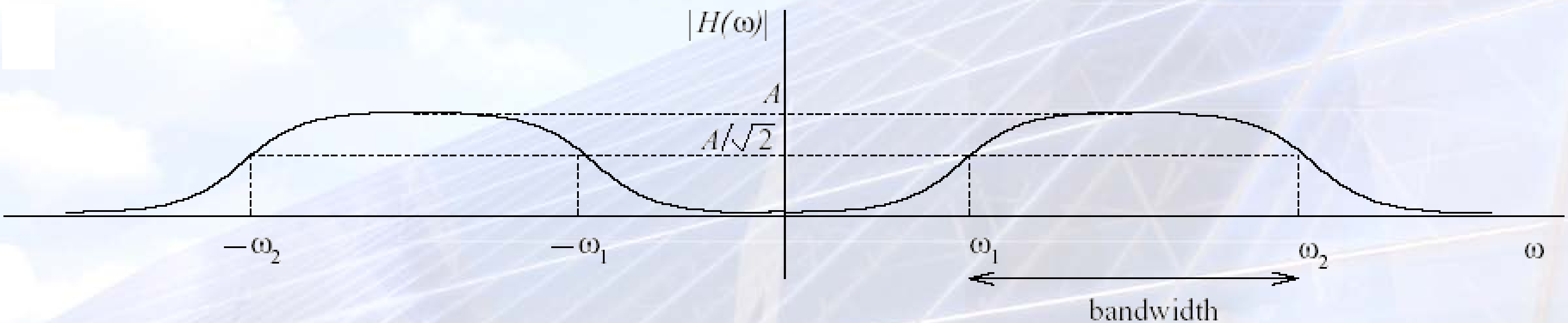


EURECOM

Rise time and bandwidth

- the half power bandwidth ω_B is the interval of frequencies where the gain varies by less than 3 dB





Rise time and bandwidth

- the step response of a first order system is given by

$$y(t) = 1 - \exp\left(-\frac{t}{RC}\right)$$

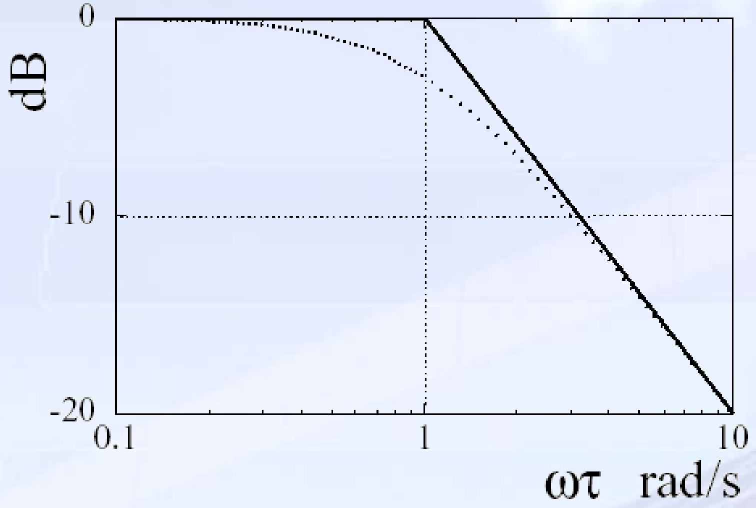
- since the final value $F = y(\infty) = 1$, T_r can be found from

$$0.9F = 1 - \exp\left(-\frac{T_r}{RC}\right)$$

- thus $T_r = RC \ln(10)$ s

Rise time and bandwidth

- the frequency response for a first order system:



- the bandwidth is $1/\tau = 1/RC$ rad/s for the simple *RC* circuit
- thus the rise time is the reciprocal of the bandwidth
- a faster response would require more bandwidth

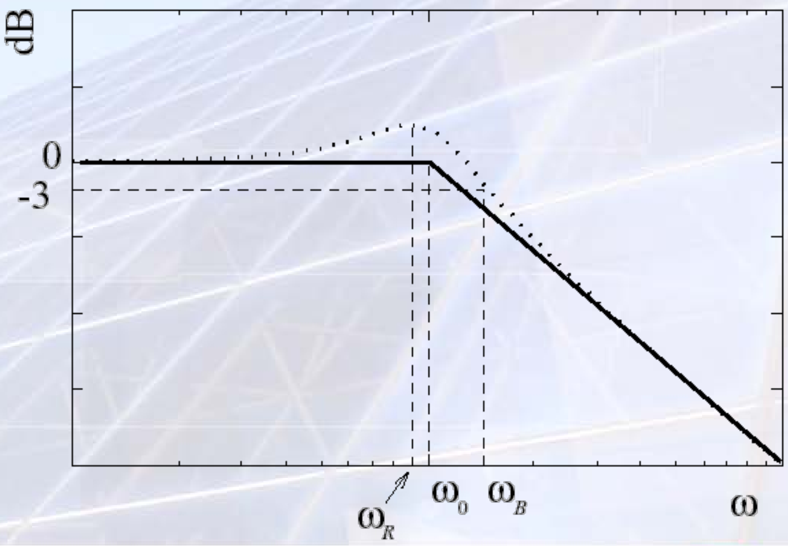
Rise time and bandwidth

- for an underdamped second order system

$$T_p = \frac{\pi}{\omega_0 \sqrt{1-\zeta^2}} \text{ s}$$

  and the percentage overshoot, $\mathrm{PO} = 100 \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)$

- a typical response:
  - the peak is associated with ω_0 and ω_R – the resonant frequency
  - the bandwidth depends on ζ



EURECOM

Rise time and bandwidth

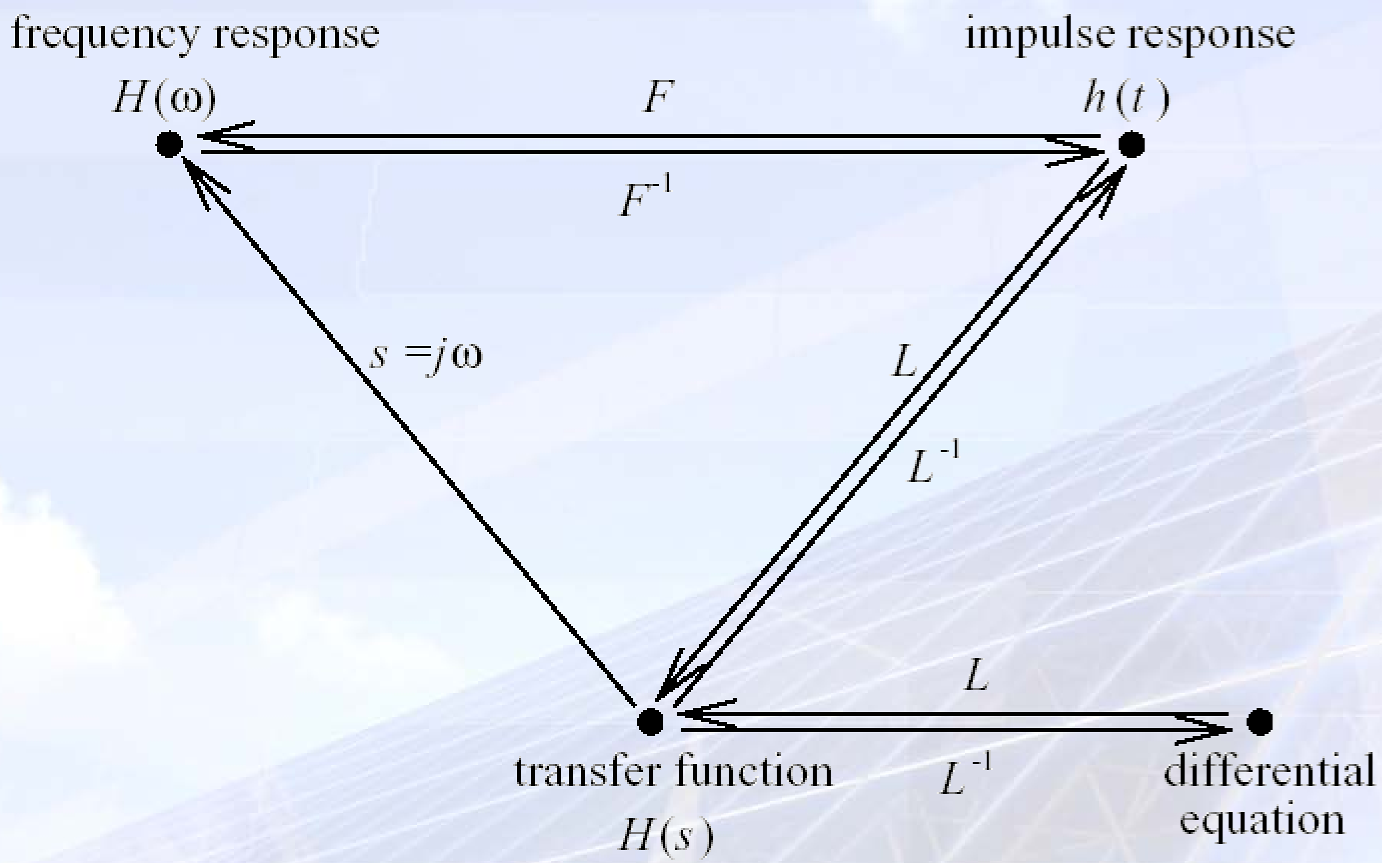
- two important trade-offs
  - case (i): ζ constant
    - if ω_B is increased then so is ω_0 which reduces T_p and in turn T_r
    - thus the rise time is inversely proportional to bandwidth
    - a faster response requires more bandwidth
  - case (ii): ω_B constant
    - can be achieved by holding ω_0 constant
    - T_p can be reduced by reducing ζ which would also reduce T_r
    - thus a faster response can be achieved just by reducing ζ
    - greater overshoot in the time domain
    - higher resonant peak in the frequency domain

Rise time and bandwidth

Example

A low pass, second order system has a peak time of 2 s and an overshoot of 10%. Estimate its bandwidth.

Summary



frequency response
$H(\omega)$
F
impulse response
$h(t)$
F^{-1}
$s = j\omega$
L
L^{-1}
L
transfer function
$H(s)$
L^{-1}
differential
equation

Summary

- you should be able to:
  - calculate the poles and zeros of first and second order systems and plot them in the s-plane;
  - sketch the frequency response of a system from its pole/zero map;
  - draw the Bode plots of first, second and third order systems;
  - sketch the impulse response of first and second order systems from their transfer function;
  - identify if a system is stable or unstable;
  - sketch the step response of first and second order systems from their transfer function;
  - describe how the rise time of a second order system is related to both overshoot and bandwidth.