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Essential Mathematical Methods for Engineers (MathEng)

Linear algebra B

Tutorial Questions
(from course textbook "Introduction to linear algerbra", by G. Strang)

To save time, it is suggested that you use Matlab as required, e.g. for matrix inverses.

1. (a) If $Ax = b$ has a solution and $A^T y = 0$, show that y is perpendicular to b.
   (b) If $Ax = b$ has no solution and $A^T y = 0$, explain why y is not perpendicular to b.

2. Suppose A is a symmetric matrix.
   (a) Why is its column space perpendicular to its nullspace?
   (b) If $Ax = 0$ and $Az = 5z$, why is x perpendicular to z?

3. If S is the subspace of R^3 containing only the zero vector, what is $S^\perp$? If S is spanned by (1, 1, 1), what is $S^\perp$? If S is spanned by (2, 0, 0) and (0, 0, 3), what is $S^\perp$?

4. Suppose the columns of A are unit vectors, all mutually perpendicular. What is $A^T A$?

5. Find a matrix with $v = (1, 2, 3)$ in the row space and the column space. Find another matrix with v in the nullspace and column space. Which pairs of subspaces can v not be in?

6. Project the vector $b = (1, 2, 2)$ onto the line through $a = (1, 1, 1)$ and check that e is perpendicular to a. Find the corresponding projection matrix P and verify that $P^2 = P$. Mutiply Pb in each case to compute the projection p.

7. Project b onto the column space of A by solving $A^T A\hat{x} = A^T b$ and $p = A\hat{x}$ for:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

   Find $e = b - p$. It should be perpendicular to the columns of A.

8. If A is doubled, then $P = 2A(4A^T A)^{-1} 2A^T$. This is the same as $A(A^T A)^{-1} A^T$. The column space of $2A$ is the same as ______. Is $\hat{x}$ the same for A and $2A$?

9. Multiply the matrix $P = A(A^T A)^{-1} A^T$ by itself and cancel to prove that $P^2 = P$. Explain why $P(Pb)$ always equals Pb. The vector Pb is in the column space so its projection is _____.

10. Find the height C of the best horizontal line to fit $b = (0, 8, 8, 20)$ at times $t = (0, 1, 2, 3)$. An exact fit would solve the unsolveable equations $C = 0$, $C = 8$, $C = 8$, $C = 20$. Find the 4 by 1 matrix A in these equations and solve $A^T A\hat{x} = A^T b$.

11. Find the closest line $b = Dt$ through the origin to the same points as in Q13.

12. Suppose the measurements at t = -1, 1, 2 are b = (5, 13, 17). Compute $\hat{x}$, the closest line and e. The error e = 0 because this b is _____. Which of the four subspaces contains the error vector e? Which contains p? Which contains $\hat{x}$? What is the nullspace of A?

13. (a) Find orthonormal vectors q_1 and q_2 in the plane of a = (1, 3, 4, 5, 7) and b = (-6, 6, 8, 0, 8).
    (b) Which vector in this plane is closest to (1, 0, 0, 0, 0)?

14. What multiple of a = (4, 5, 2, 2) is closest to b = (1, 2, 0, 0)? Find orthonormal vectors q_1 and q_2 in the plane of a and b.

15. Find q_1, q_2, q_3 (orthonormal) as combinations of a, b, c (independent columns). Then write A as QR.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}$$

16. From an example in class we saw that

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 0.70 & 0.45 \\ 0.30 & 0.55 \end{bmatrix} \quad \text{and} \quad A^\infty = \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$$

The matrix A^2 is halfway between A and A^∞. Explain why $A^2 = \frac{1}{2}(A + A^\infty)$ from the eigenvalues and eigenvectors of these three matrices.
(a) Show from A how a row exchange can produce different eigenvalues.
(b) Why is a zero eigenvalue not changed by the steps of elimination?

17. Compute the eigenvalues and eigenvectors of A and A^{-1}:

$$A = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} -3/4 & 1/2 \\ 1/2 & 0 \end{bmatrix}$$

A^{-1} has the _____ eigenvectors as A. When A has the eigenvalues λ_1 and λ_2, its inverse has eigenvalues _____.

18. Find the eigenvalues and eigenvectors for the projection matrices P and P^{100}:

$$P = \begin{bmatrix} .2 & .4 & 0 \\ .4 & .8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If two eigenvectors share the same λ, so do all the linear combinations. Find an eigenvector of P with no zero components.

19. The sum of the diagonal entries (the trace) equals the sum of the eigenvalues.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad \det(A - \lambda I) = \lambda^2 - (a+d)\lambda + ad - bc = 0$$

If A has λ_1 = 3 and λ_2 = 4 then $\det(A - \lambda I)$ = _____. The quadratic formula gives the eigenvalues λ = $(a + d + \sqrt{\quad})/2$ and λ = _____. Their sum is _____.

20. If A has $\lambda_1 = 4$ and $\lambda_2 = 5$ then $\det(A - \lambda I) = (\lambda - 4)(\lambda - 5) = \lambda^2 - 9\lambda + 20$. Find three matrices that have trace $a + d = 9$ and determinant 20 and $\lambda = 4, 5$.

21. Choose a, b, c so that $\det(A - \lambda I) = 9\lambda - \lambda^3$. Then the eigenvalues are -3, 0, 3:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{bmatrix}$$

22. Factor these two matrices into $A = S\Lambda S^{-1}$:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

23. If A has $\lambda_1 = 2$ with eigenvector $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\lambda_2 = 5$ with eigenvector $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, use $S\Lambda S^{-1}$ to find A. No other matrix has the same λ's and x's.

24. Complete these matrices so that $\det A = 25$. Then check that $\lambda = 5$ is repeated – the determinant of $A - \lambda I$ is $(\lambda - 5)^2$. Find an eigenvector with $Ax = 5x$. These matrices will not be diagonaliseable because there is no second line of eigenvectors.

$$A = \begin{bmatrix} 8 & \\ & 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 9 & 4 \\ & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 10 & 5 \\ -5 & \end{bmatrix}$$

25. Find an orthogonal matrix Q that diagonalises this symmetric matrix:

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}$$

46. (a) Suppose that, for a symmetric matrix, $Ax = \lambda x$ and $Ay = 0y$ and $\lambda \neq 0$. Then y is in the nullspace and x is in the column space. Why are these subspaces orthogonal?

(b) If the second eigenvalue is now a nonzero number β, apply this argument to $A - \beta I$. **Hint:** note that one of the eigenvalues (of $A - \beta I$) moves to zero and the eigenvectors stay the same – so they are perpendicular.

47. Which of these classes of matrices do A and B belong to: invertible, orthogonal, projection, permutation, diagonaliseable, Markov?

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Which of these factorisations are possible for A and B: LU, QR, $S\Lambda S^{-1}$, $Q\Lambda Q^T$.

48. What number b in $\begin{bmatrix} 2 & b \\ 1 & 0 \end{bmatrix}$ makes $A = Q\Lambda Q^T$ possible? What number makes $A = Q\Lambda Q^T$ impossible?

49. Compute A^TA and its eigenvalues σ_1^2, 0 and unit eigenvectors v_1 and v_2 for:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$$

Then compute AA^T and its eigenvalues σ_1^2, 0 and unit eigenvectors u_1 and u_2 and verify that $Av_1 = \sigma_1 u_1$. Write out the SVD and find orthogonal bases for the four fundamental subspaces of this A.