Signal classification

- many ways in which signal may be classified, i.e. periodic, when $x(t) = x(t + T)$, where the smallest value of T defines the period
- we can also classify signals as either energy or power signals
- energy signals
  - non-zero and finite total dissipated energy, E
  - usually exist for a finite interval of time or have most of their energy concentrated in a finite interval of time

$$0 \le E \le \infty, \quad E = \int_{-\infty}^{\infty} x^2(t)dt$$

- power signals
  - non-zero and finite average delivered power, P
  - an example is the unit step function $u(t)$ and a periodic signal of period T such as $x(t)=\sin(2\pi t / T)$

$$0 \le P \le \infty \qquad P = \lim_{T\to\infty} \frac{1}{T}\int_{-T}^{T} x^2(t)dt = \frac{1}{T}\int_{0}^{T} x^2(t)dt$$

Example

Find the energy in the decaying exponential signal $x_1(t)$=5exp(-2t) if $t \ge 0$ and $x_1(t) = 0$ if $t < 0$.

$$\text{Energy} = \int_0^\infty (x(t))^2\, dt = \int_0^\infty 25\,\exp(-4t)\, dt = \frac{-25}{4}\Big[\exp(-4t)\Big]_0^\infty$$

$$= \frac{25}{4}$$

Fourier series
Trigonometric Fourier series

- we can represent any finite power periodic signal $x(t)$ with a period T as a sum of sine and cosine waves:

$$x(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t)$$

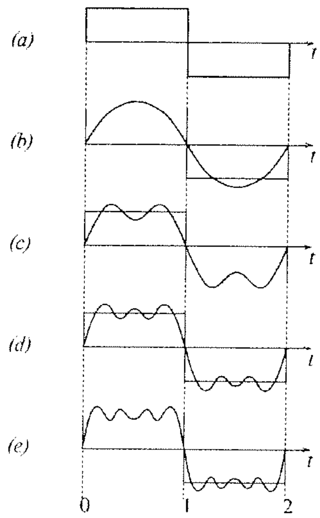
- fundamental frequency:

$$\omega_0 = 2\pi/T \text{ rad/s} \quad \text{or} \quad 1/T \text{ Hz}$$

- harmonics are generally found at $2/T$ Hz, $3/T$ Hz … according to Fourier coefficients:

$$A_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(n\omega_0 t)dt \quad n = 0, 1, 2, \ldots$$

$$B_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(n\omega_0 t)dt \quad n = 1, 2, 3, \ldots$$



Example

Evaluate the Fourier series of the square wave (a)

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The period T of the square wave = 2s $\quad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$

$$A_n = \int_0^2 x(t)\cos(\underbrace{n\pi}_{\omega_0} t)dt$$

$$= \int_0^1 \cos(n\pi t)dt - \int_1^2 \cos(n\pi t)dt$$

$$= 0$$

$$B_n = \int_0^2 x(t)\sin(n\pi t)dt$$

$$= \int_0^1 \sin(n\pi t)dt - \int_1^2 \sin(n\pi t)dt$$

$$= \frac{2}{n\pi}\left(1 - \cos(n\pi)\right)$$

For even values of n the B coeffs are also zero

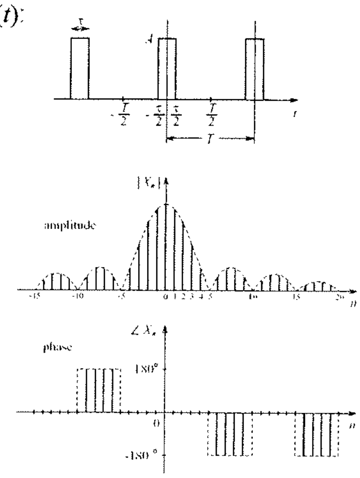
∴ The trigonometric Fourier series representation of the waveform is

$$x(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1-\cos(n\pi)\right)\sin(n\pi t)$$

$$= \frac{4}{\pi}\sin(\pi t) + 0 + \frac{4}{3\pi}\sin(3\pi t) + 0 + \frac{4}{5\pi}\sin(5\pi t) + \text{etc} \ldots$$

Example

Derive an expression for the complex Fourier coefficient, X_n, associated with the periodic signal $x(t)$:



this is n
not ω

$$X_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\exp(-jn\omega_0 t)\,dt$$

$$= \frac{1}{T}\int_{-\tau/2}^{\tau/2} A\exp(-jn\omega_0 t)\,dt$$

$$= \frac{-A}{jn\omega_0 T}\left[\exp\left(\frac{-jn\omega_0\tau}{2}\right) - \exp\left(\frac{jn\omega_0\tau}{2}\right)\right]$$

$$= \frac{A\tau}{T}\,\frac{\sin(n\omega_0\tau/2)}{n\omega_0\tau/2}$$

Orthogonality

- we already looked at the concept of othogonality last time (i.e. QR decomposition and Gram-Schmidt)
- the Fourier series is an orthogonal expansion
- we say two signals $f_1(t)$ and $f_2(t)$ are orthogonal if $\frac{1}{T}\int_{-T/2}^{T/2} f_1(t) f_2(t)\,dt = 0$

and for the complex Fourier series the basis functions are mutually orthogonal:

$$\frac{1}{T}\int_{-T/2}^{T/2} \exp(jn\omega_0 t)\exp^*(jm\omega_0 t) = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

Example

Calculate the power in the simple periodic signal $x(t)$ where:

$$x(t) = a_1 \sin(\omega_0 t) + a_2 \sin(2\omega_0 t)$$

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$$P = \frac{1}{T}\int_0^T x^2(t)\,dt$$

(they're orthogonal!)

$$= \frac{1}{T}\int_0^T a_1^2 \sin^2(\omega_0 t)\,dt + \underbrace{\frac{2}{T}\int_0^T a_1 a_2 \sin(\omega_0 t)\sin(2\omega_0 t)\,dt}_{=0}$$

$$+ \frac{1}{T}\int_0^T a_2^2 \sin^2(2\omega_0 t)\,dt$$

$$= \frac{a_1^2}{2} + \frac{a_2^2}{2}$$

- therefore we have $$X(\omega)=\int_{-\infty}^{\infty}x(t)\exp(-j\omega t)dt$$

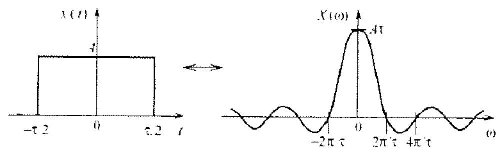
$$x(t)=\lim_{T\to\infty}\sum_{n=-\infty}^{+\infty}X_n'\exp(jn\omega_0 t)\frac{\omega_0}{2\pi}$$

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)\exp(j\omega t)d\omega$$

and we can represent most finite energy signal in this way

Example

Evaluate the Fourier transform of the finite energy signal $x(t)$



$$X(\omega)=\int_{-\infty}^{\infty}x(t)\exp(-j\omega t)\,dt$$

$$=A\int_{-\tau/2}^{\tau/2}\exp(-j\omega t)\,dt = A\left[\frac{-1}{j\omega}\exp(-j\omega t)\right]_{-\tau/2}^{\tau/2}$$

$$=A\left[-\frac{1}{j\omega}\exp\left(-\frac{j\omega\tau}{2}\right)+\frac{1}{j\omega}\exp\left(\frac{j\omega\tau}{2}\right)\right]$$

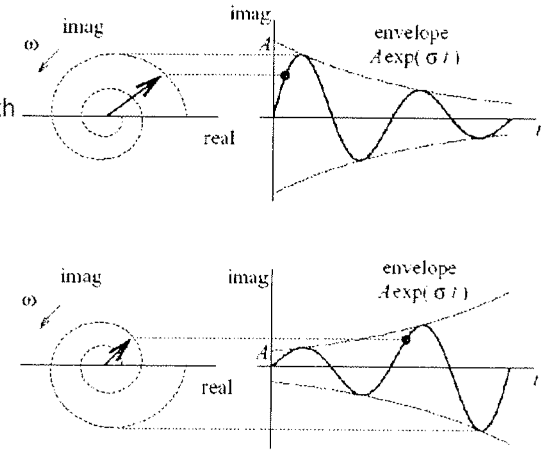
$$=\frac{A\tau\sin(\omega\tau/2)}{\omega\tau/2} = A\tau\,\mathrm{sa}(x)$$

Applicability and physical interpretation

- the basis functions of the Laplace transform are growing or decaying complex phasors

$$A\exp(st) = A\exp(\sigma t)\cos(\omega t) + jA\exp(\sigma t)\sin(\omega t)$$

- the signal $x(t)$ has components with
  - frequency ω
  - magnitude $|X(s)|d\omega/(2\pi)$
  - growth or decay determined by σ
  - phase $\angle X(s)$



Example

Evaluate the Laplace transform of a one-sided signal $x(t) = \exp(-\alpha t)$

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$$X(s) = \int_0^\infty \exp(-\alpha t)\exp(-st)\,dt$$

$$= \int_0^\infty \exp(-(s+\alpha)t)\,dt$$

$$= \frac{-1}{s+\alpha}\left[\exp\left(-(s+\alpha)t\right)\right]_0^\infty$$

then provided that $Re(s) = \sigma > -\alpha$

the function $\exp(-(s+\alpha)t) \to 0$ as $t \to \infty$

$$\therefore \quad X(s) = \frac{1}{s+\alpha}$$

Transform analysis of linear systems
Linear ordinary differential equations

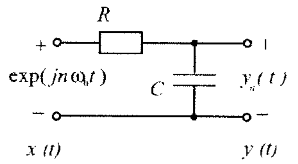
- many linear systems can be modelled with linear ordinary differential equations

$$a_0 y + a_1 \frac{dy}{dt} + \ldots + a_n \frac{d^n y}{dt^n} = b_0 x + b_1 \frac{dx}{dt} + \ldots + b_m \frac{d^m x}{dt^m}$$

where the input, $x(t)$, defines the output, $y(t)$, according to system parameters $a_0 \ldots a_n$ and $b_0 \ldots b_m$

Example

Evaluate the response $y_n(t)$ of the following circuit to n^{th} harmonic, i.e. the complex phasor $\exp(jn\omega_0 t)$



Using Kirchoff's laws (2nd law - algebraic sum of voltage drops around any closed loop = 0)

The system can be described by the differential equation

$$\overset{1}{a_0} y_n(t) + \overset{RC}{a_1} \frac{dy_n}{dt} = \overset{1}{b_0} \exp(jn\omega_0 t)$$

Transient response due to initial condition will have decayed to zero long ago

Assume a solution of the form $y_n(t) = K\exp(jn\omega_0 t)$

$$a_0 K\exp(jn\omega_0 t) + a_1 K(jn\omega_0)\exp(jn\omega_0 t) = b_0 \exp(jn\omega_0 t)$$

$$\therefore \quad K = \frac{b_0}{a_0 + a_1 jn\omega_0}$$

$\therefore$ the response $y_n(t) = \dfrac{b_0}{a_0 + a_1 jn\omega_0} \exp(jn\omega_0 t)$

Laplace transfer function

- defined in the same way as for the Fourier transfer function

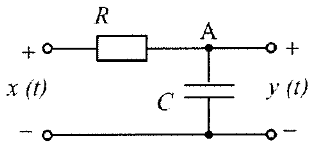
$$H_s = \frac{L[\text{output}]}{L[\text{input}]} = \frac{Y(s)}{X(s)}$$

  and completely specifies system characteristics

- with knowledge of the transfer function we can calculate the response of the system to any input

Example

Evaluate the transfer function of the following circuit:



$$\frac{x(t) - y(t)}{R} = C\frac{dy}{dt}$$

$$\frac{x(t)}{R} = C\frac{dy}{dt} + y(t)$$

& taking laplace transforms

$$\frac{X(s)}{R} = C\{sY(s) - Y(0^-)\} + \frac{Y(s)}{R}$$ & assume $Y(0^-) = 0$

$$\frac{X(s)}{R} = \left\{Cs + \frac{1}{R}\right\} Y(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{1 + RCs}$$

but from $K = \dfrac{1}{1 + jn\omega_0 RC}$

↑ response to the nth harmonic.