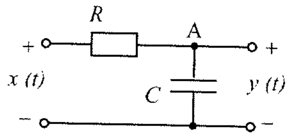
System response to an impulse

Example

Calculate the impulse response of the following RC circuit using both a transform domain and time domain approach.



Thus the transform domain approach is the most straightforward but the time domain approach does provide additional insights....

The impulse is a 'kick' to the system which provides energy to observe the natural response.

Transform domain $\quad H(s) = \dfrac{1}{1 + RCs}$

$$h(t) = \mathcal{L}^{-1}\left(\frac{1}{1+RCs}\right) = \frac{1}{RC}\exp\left(\frac{-1}{RC}t\right) \quad t \geq 0$$

from table of transforms

$$\exp(-\alpha t) = \frac{1}{s+\alpha}$$

$$\frac{1}{\alpha}\exp(-\alpha t) = \frac{1}{\alpha}\left(\frac{1}{s/\alpha + 1}\right)$$

Time domain

$x(t) = \delta(t)$ only time at which input $\neq 0$

at $t=0$ current through R

$$i(t) = \frac{x(t) - y(t)}{R}$$

assuming $y(t)$ is initially equal to zero ie $y(t) = 0$ at $t=0$

$$i(t) = \frac{\delta(t)}{R}$$

which only exists at $t = 0$

The relationship between current & voltage on the capacitor yields $i(t) = C\frac{dy}{dt}$ $\left(i(t) = \frac{\delta(t)}{R} \right)$

$$\therefore dy = \frac{1}{RC}\delta(t)\,dt$$

& since the impulse only exists at time $t=0$ it is sufficient to integrate between $t=0^-$ & $t=0^+$ to evaluate the voltage left on the capacitor at $t=0^+$ by the current impulse

$$y(0^+) = \int_{0^-}^{0^+} dy$$

$$= \frac{1}{RC}\int_{0^-}^{0^+} \delta(t)\,dt = \frac{1}{RC}$$

<u>After</u> $t=0$ the input is zero, i.e. $x(t)=0$

Applying Kirchhoff's current law again

$$-\frac{y(t)}{R} = C\frac{dy}{dt}. \qquad \text{ie } x(t)=0$$

$$\frac{dy}{dt} + \frac{1}{RC}y(t) = 0$$

& with initial condition $y(0^+) = 1/RC$

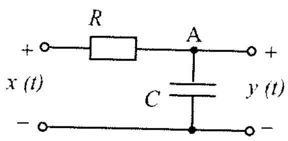
the solution is

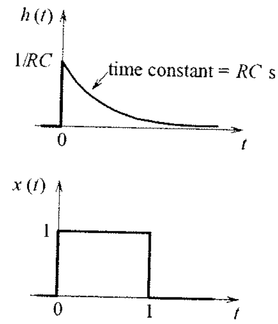
$$y(t) = \frac{1}{RC}\exp\left(-\frac{t}{RC}\right)$$

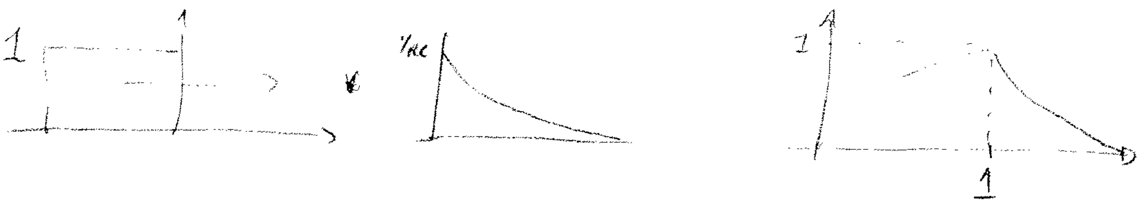
which is the impulse response.

Example

Using convolution, calculate the response of the RC circuit to the rectangular pulse $x(t)$.







Solution over page

EXAMPLE 2.2

Filter output using convolution

Recall the simple RC circuit of Figure 1.12. In Example 2.1 its impulse response was calculated using both the time-domain and transform-domain approaches. This impulse response is illustrated in Figure 2.5(a). Using convolution, calculate the response of this circuit to the rectangular pulse $x(t)$ shown in Figure 2.5(b).

Solution

The problem is to calculate the output $y(t)$ (using convolution) at every value of time t. The convolution integral is usually performed in pieces. The way to identify which bits are involved is to draw diagrams of the signals involved. This subdivides the problem into smaller problems which when solved describe the output at different regions on the time axis. For this particular example consider three particular regions of the time axis.

The first key step in convolution is to draw diagrams of the signals involved.

Region (i): What is the output at time zero? Start by mechanically applying the convolution integral which states:

$$y(t) = \int_{0}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Replacing t with 0 gives an expression for the output at time zero:

$$y(0) = \int_{0}^{\infty} x(\tau)\, h(0-\tau)\, d\tau$$

$$= \int_{0}^{\infty} x(\tau)\, h(-\tau)\, d\tau$$

In words this means: multiply $x(\tau)$ by $h(-\tau)$ at every value of τ to get the product $x(\tau)\, h(-\tau)$, and then integrate the product from $\tau = 0$ to infinity. Figure 2.6 shows $x(\tau)$ drawn as a function of τ. Below it $h(-\tau)$ is drawn as a function of τ. $h(-\tau)$ is the same shape as $h(t)$ but it is the opposite way round, i.e. *time-reversed.* To calculate the product, multiply the two graphs together at every value of τ. For every value of τ less than zero, $x(\tau) = 0$. Thus the product will be zero when τ is less than zero. For every value of τ greater than zero, $h(-\tau) = 0$. Thus the product

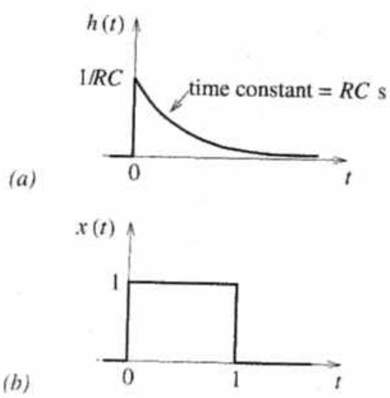


Figure 2.5
Convolution example: (a) impulse response $h(t)$ and (b) input $x(t)$.

will be zero when τ is greater than zero. The product will be zero at every value of τ. The integral of the product is zero:

$$y(0) = 0$$

This result might have been expected since the input $x(t)$ does not start until $t = 0$.

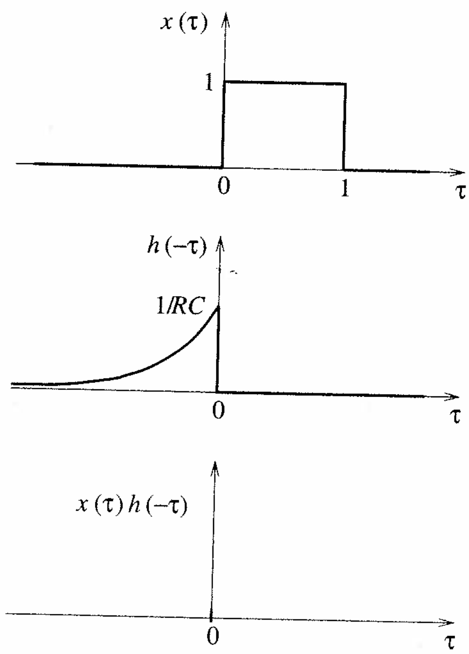
Region (ii): What is the output when t is between 0 and 1 second (i.e. $0 \le t < 1$)? Again, start by mechanically applying the convolution integral:

$$y(t) = \int_{0}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Figure 2.7 shows $x(\tau)$ drawn as a function of τ. Below it $h(t-\tau)$ is drawn as a function of τ. Comparing Figure 2.7 with 2.6 it can be seen that $h(t-\tau)$ is obtained by *sliding* $h(-\tau)$ to the right by t seconds. Recall that t is constrained to lie between 0 and 1. Hence the vertical edge of $h(t-\tau)$ must lie somewhere between the rising edge and falling edge of $x(\tau)$. Again applying the convolution integral mechanically, multiply the two waveforms together at every value of τ to obtain the product waveform $x(\tau)h(t-\tau)$. Note that for values of τ less than zero, $x(\tau) = 0$ and hence the product will be zero for these values of τ. Further, for values of τ greater than t, $h(t-\tau)$ is zero and hence the product will also be zero when τ is greater than t. Finally, integrate the product between zero and infinity. However it is clear that the product is zero for values of τ less than zero and greater than t. Thus the integration is simply the area under the product curve between 0 and t:

$$y(t) = \int_{0}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

Figure 2.6
Convolution example: time reversal.



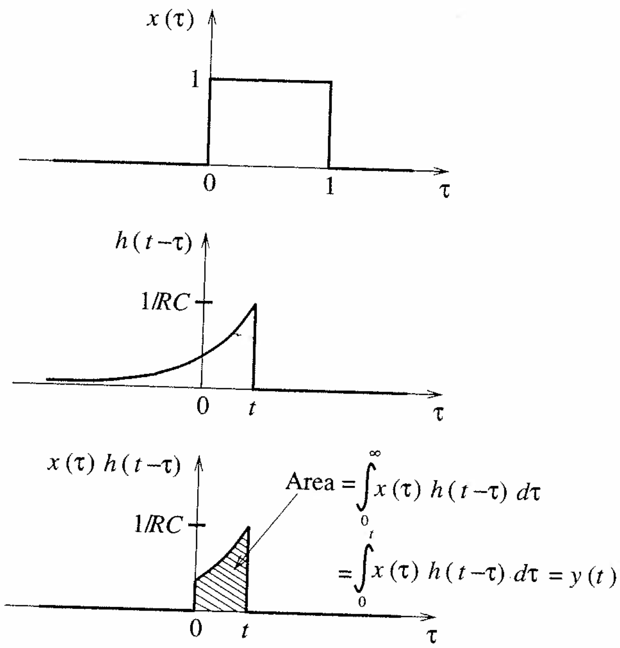


Figure 2.7
Convolution example: slide into second region.

Region (iii): Figure 2.8 illustrates the waveforms when t is greater than one. In this case the vertical edge of $h(t-\tau)$ has *slid* further to the right so that it is beyond the falling edge of $x(\tau)$. Multiply the waveforms together as before. Note that $x(\tau)$ is zero for all values of τ greater than one and hence the product will also be zero for these values of τ. Integrate the product; the integral is simply the area under the product curve between zero and one, i.e.:

$$y(t) = \int_0^1 x(\tau)\, h(t-\tau)\, d\tau$$

Thus it has been shown that the integration can be performed by separating the time axis (t) into three separate regions and simplifying the expression within each region. Thus it might be expected that three separate equations are needed to describe the output $y(t)$ – one for each region. We will now develop expressions for the output $y(t)$. Having identified the regions, it now remains to develop expressions for the output within each of these regions:

The second step in convolution is to divide the time axis into regions.

Region (i): $t < 0$ – from the graphical arguments the answer is simply:

$$y(t) = 0$$

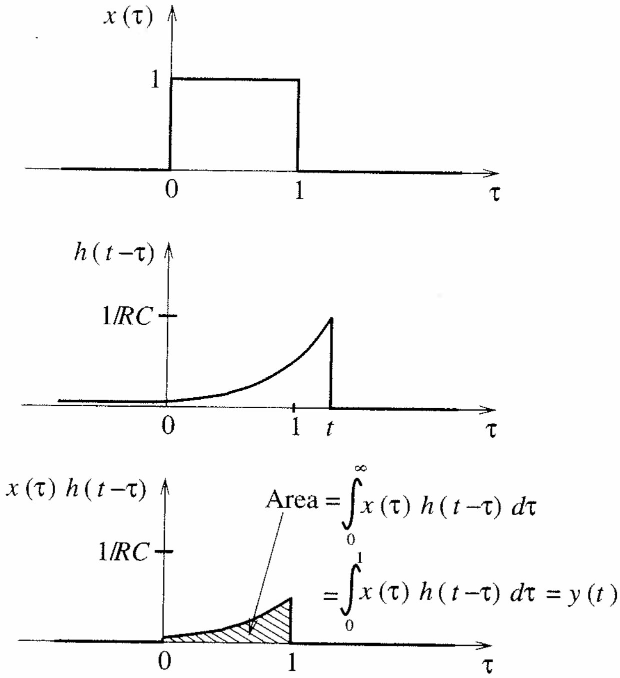
Region (ii): $0 \le t < 1$ – start with the result obtained from the graphical arguments:

The final step in convolution is to perform the integrations in each region.

$$y(t) = \int_0^t x(\tau)\, h(t-\tau)\, d\tau$$

Within this region $x(\tau)$ is 1. An expression for the impulse response $h(t)$ was developed in Example 2.1. An expression for $h(t-\tau)$ is obtained by simply substituting $t-\tau$ for t:

Figure 2.8
Convolution example: slide into third region.



$$y(t) = \int_0^t 1 \frac{1}{RC} \exp\left(-\frac{t-\tau}{RC}\right) d\tau$$

It is important at this point to remember that integration is with respect to τ and not t. For the purposes of this integration, t behaves like a constant and can be taken outside the integral sign:

$$y(t) = \frac{1}{RC} \exp\left(-\frac{t}{RC}\right) \int_0^t \exp\left(\frac{\tau}{RC}\right) d\tau$$

The integration is now fairly straightforward:

$$y(t) = \frac{1}{RC} \exp\left(-\frac{t}{RC}\right) \left[RC \exp\left(\frac{\tau}{RC}\right) \right]_0^t$$

$$= 1 - \exp\left(-\frac{t}{RC}\right)$$

Region (iii): $t \geq 1$ – from the graphical arguments:

$$y(t) = \int_0^1 x(\tau)\ h(t-\tau)\ d\tau$$

Within this region $x(\tau)$ is 1. Thus:

$$y(t) = \int_0^1 1 \frac{1}{RC} \exp\left(-\frac{t-\tau}{RC}\right) d\tau$$

$$= \exp\left(-\frac{t}{RC}\right)\left[\exp\left(\frac{1}{RC}\right) - 1\right]$$

The three separate portions of the solution are illustrated in Figure 2.9. As a final check, it is worth verifying that the expression for each region should coincide at the boundaries. Thus for $t = 1$, both the result for region (ii) and the result for region (iii) give: $y(1) = 1 - \exp(-1/(RC))$.

Self Assessment Question 2.4: Rework Example 2.2 for an input $x(t)$ where: $x(t) = 1$ when $1 \leq t < 2$ and $x(t) = 0$ elsewhere.

2.3. Properties of convolution

Linear systems exhibit many properties. Since the convolution operation describes a linear system in the time-domain, the properties of linear systems can be used to simplify what might seem like daunting convolution expressions.

- Convolution is commutative:

$$x(t) * h(t) = h(t) * x(t) \tag{2.11}$$

If a signal $x(t)$ is applied to a system with impulse response $h(t)$ as in Figure 2.10, we get the same result as that from applying a signal $h(t)$ to a system with impulse response $x(t)$.

- Convolution is distributive over addition:

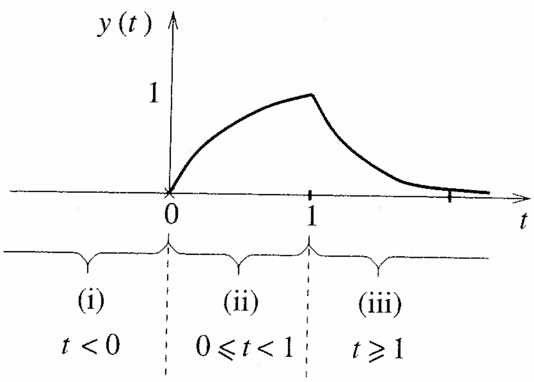


Figure 2.9
Convolution example: complete solution.

Figure 2.10
Cumulative property of convolution.