<u>Tutorial sheet 3</u>

(1) $f(t) = e^{kt}$

$$\mathcal{L}\{e^{kt}\} = \int_0^\infty e^{-st} e^{kt}\, dt = \lim_{T\to\infty} \int_0^T e^{-(s-k)t}\, dt$$

$$= \lim_{T\to\infty} \frac{-1}{s-k} \left[e^{-(s-k)t} \right]_0^T = \frac{1}{s-k} \left(1 - \lim_{T\to\infty} e^{-(s-k)T} \right)$$

thus $\mathcal{L}\{e^{kt}\} = \dfrac{1}{s-k}$ provided that, for real k,

$\sigma = Re(s) > k$. If k is complex then we require $Re(s) > Re(k)$

Thus we have the Laplace transform pair

$$\left.\begin{aligned} f(t) &= e^{kt} \\ F(s) &= \frac{1}{s-k} \end{aligned}\right\} \quad Re(s) > Re(k)$$

Since $\exp(jat) = \cos at + j \sin at$ we may write

$$f(t) = \sin at = \operatorname{Im} e^{jat} \quad \& \quad g(t) = \cos at = \operatorname{Re} e^{jat}$$

From the result of Q1 with $k = ja$ we have

$$\mathcal{L}\{e^{jat}\} = \frac{1}{s-ja} \qquad Re(s) > 0$$

$$= \frac{s+ja}{s^2+a^2} \qquad Re(s) > 0$$

Thus equating real and imaginary parts and assuming that s is real we have

$$\mathcal{L}\{\sin at\} = \operatorname{Im} \mathcal{L}\{e^{jat}\} = \frac{a}{s^2+a^2}$$

$$\mathcal{L}\{\cos at\} = \operatorname{Re} \mathcal{L}\{e^{jat}\} = \frac{s}{s^2+a^2}$$

which also hold for complex s with $Re(s) > 0$.

(2) Since $\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \int_0^\infty [\alpha f(t) + \beta g(t)] e^{-st} dt$

$$= \int_0^\infty \alpha f(t) e^{-st} dt + \int_0^\infty \beta g(t) e^{-st} dt$$

$$= \alpha \int_0^\infty f(t) e^{-st} dt + \beta \int_0^\infty g(t) e^{-st} dt$$

we have $\underline{\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\}}$

If $f(t)$ & $g(t)$ have abscissae of convergence σ_f & σ_g respectively & $\sigma_1 > \sigma_f$, $\sigma_2 > \sigma_g$ then

$$|f(t)| < M_1 e^{\sigma_1 t}, \quad |g(t)| < M_2 e^{\sigma_2 t}$$

it follows that

$$\begin{aligned} |\alpha f(t) + \beta g(t)| &\leq |\alpha||f(t)| + |\beta||g(t)| \\ &\leq |\alpha| M_1 e^{\sigma_1 t} + |\beta| M_2 e^{\sigma_2 t} \\ &\leq (|\alpha| M_1 + |\beta| M_2) e^{\sigma t} \quad \text{where } \sigma = \max(\sigma_1, \sigma_2) \end{aligned}$$

Exploiting the linearity property we calculate:

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st} t \, dt = \lim_{T \to \infty} \int_0^T e^{-st} t \, dt$$

$$= \lim_{T \to \infty} \left[-\frac{t}{s} e^{-st} - \frac{e^{-st}}{s^2} \right]_0^T = \frac{1}{s^2} - \lim_{T \to \infty} \frac{T e^{-sT}}{s} - \lim_{T \to \infty} \frac{e^{-sT}}{s^2}$$

$$= \frac{1}{s^2} \quad Re(s) > 0$$

& from Q1 we have $\mathcal{L}\{e^{kt}\} = \frac{1}{s-k} \quad Re(s) > Re(k)$

Thus

$$\mathcal{L}\{3t + 2e^{3t}\} = 3\mathcal{L}\{t\} + 2\mathcal{L}\{e^{3t}\}$$

$$= \frac{3}{s^2} + \frac{2}{s-3} \quad Re(s) > \max(0, 3)$$

$$= \frac{3}{s^2} + \frac{2}{s-3} \quad Re(s) > 3$$

(3) Since $\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{at} f(t) e^{-st}\, dt = \int_0^\infty f(t)\, e^{-(s-a)t}\, dt$

and $\mathcal{L}\{f(t)\} = \int_0^\infty f(t) e^{-st}\, dt = F(s) \qquad Re(s) > \sigma_c$

we see that

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a) \qquad Re(s-a) > \sigma_c$$

or $Re(s) > \sigma_c + Re(a)$

$\therefore$ from Q2 we have that $\mathcal{L}\{t\} = \frac{1}{s^2} \qquad Re(s) > 0$

so $\mathcal{L}\{t e^{-2t}\} = F(s+2) = [F(s)]_{s \to s+2} \qquad Re(s) > 0 - 2$

$$= \frac{1}{(s+2)^2} \qquad Re(s) > -2$$

(4) $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$

so that $\frac{d^n F(s)}{ds^n} = \frac{d^n}{ds^n}\int_0^\infty e^{-st}\, dt = \int_0^\infty \frac{\partial^n}{\partial s^n}[e^{-st} f(t)]\, dt$

$= (-1)^n \int_0^\infty e^{-st}\, t^n f(t)\, dt$

$= (-1)^n \mathcal{L}\{t^n f(t)\} \qquad Re(s) > \sigma_c$

For $\mathcal{L}\{t \sin 3t\}$ given that

$\mathcal{L}\{\sin 3t\} = F(s) = \frac{3}{s^2+9} \qquad Re(s) > 0$

we have that

$\mathcal{L}\{t \sin 3t\} = -\frac{dF(s)}{ds} = \frac{6s}{(s^2+9)^2} \qquad Re(s) > 0$

For $\mathcal{L}\{t^2 e^t\}$ given that $\mathcal{L}\{e^t\} = F(s) = \frac{1}{s-1} \qquad Re(s) > 1$

$$\mathcal{L}\{t^2 e^t\} = (-1)^2 \frac{d^2 F(s)}{ds^2} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s-1}\right) = (-1)\frac{d}{ds}\left(\frac{1}{(s-1)^2}\right)$$

$$= \frac{2}{(s-1)^3} \qquad Re(s) > 1$$

but we could have used the first shift theorem.

(5) $\mathcal{L}\left\{\frac{df}{dt}\right\} = \int_0^\infty e^{-st}\frac{df}{dt}\,dt$

& on integrating by parts we have

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = \left[e^{-st}f(t)\right]_0^\infty + s\int_0^\infty e^{-st}f(t)\,dt$$

$$= \quad -f(0) \quad + \quad s\,F(s)$$

so $\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0)$

$$\mathcal{L}\left\{\frac{d^2f}{dt^2}\right\} = \int_0^\infty e^{-st}\frac{d^2f}{dt^2}\,dt = \left[e^{-st}\frac{df}{dt}\right]_0^\infty + s\int_0^\infty e^{-st}\frac{df}{dt}\,dt = -\left[\frac{df}{dt}\right]_{t=0} + s\mathcal{L}\left\{\frac{df}{dt}\right\}$$

$$= -\left[\frac{df}{dt}\right]_{t=0} + s\left[sF(s) - f(0)\right]$$

thus $\mathcal{L}\left\{\frac{d^2f}{dt^2}\right\} = s^2F(s) - sf(0) - \left[\frac{df}{dt}\right]_{t=0} = s^2F(s) - sf(0) - f^{(1)}(0)$

& in general $\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - \sum_{i=1}^{n} s^{n-i} f^{(i-1)}(0)$

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(6) Writing $g(t) = \int_0^t f(\tau)\,d\tau$ we have $\frac{dg}{dt} = f(t)$ $g(0) = 0$

taking Laplace transforms $\mathcal{L}\left\{\frac{dg}{dt}\right\} = \mathcal{L}\{f(t)\}$ which from above

gives $sG(s) = F(s)$ or $\mathcal{L}\{g(t)\} = G(s) = \frac{1}{s}F(s) = \frac{1}{s}\mathcal{L}\{f(t)\}$

which gives the required result.

Starting with $f(t) = t^3 + \sin 2t$

$$F(s) = \mathcal{L}\{t^3\} + \mathcal{L}\{\sin 2t\}$$

$$= \frac{6}{s^4} + \frac{2}{s^2+4}$$

Thus $\mathcal{L}\left\{\int_0^t(\tau^3 + \sin 2\tau)\,d\tau\right\} = \frac{1}{s}F(s) = \frac{6}{s^5} + \frac{2}{s(s^2+4)}$

(7)

(a) Resolving into partial fractions

$$\frac{1}{(s+3)(s-2)} = \frac{-1/5}{s+3} + \frac{1/5}{s-2}$$

Using the result $\mathcal{L}^{-1}\{1/(s+a)\} = e^{-at}$ & the linearity property

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+3)(s-2)}\right\} = -\frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} + \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= -\frac{1}{5}e^{-3t} + \frac{1}{5}e^{2t}$$

(b) Resolving into partial fractions and results in the table of Laplace transforms gives

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2(s^2+9)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/9}{s} + \frac{1/9}{s^2} - \frac{1/9(s+1)}{s^2+9}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1/9}{s} + \frac{1/9}{s^2} - \frac{1/9\, s}{s^2+3^2} - \frac{1/27 \times 3}{s^2+3^2}\right\}$$

$$= \frac{1}{9} + \frac{1}{9}t - \frac{1}{9}\cos 3t - \frac{1}{27}\sin 3t$$

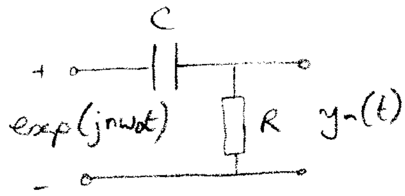
(c) Similarly

$$\mathcal{L}^{-1}\left\{\frac{2}{s^2+6s+13}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{(s+3)^2+4}\right\} = \left[\frac{2}{s^2+2^2}\right]_{s\to s+3}$$

& since $2/(s^2+2^2) = \mathcal{L}\{\sin 2t\}$ the shift theorem gives

$$\mathcal{L}^{-1}\left\{\frac{2}{s^2+6s+13}\right\} = e^{-3t}\sin 2t$$

(8) Using Kirchhoff's current law



$$\text{current} = C\frac{d}{dt}(\exp(jn\omega_0 t) - y) = y/R$$

$$\therefore \; C\exp(jn\omega_0 t)jn\omega_0 - C\frac{dy}{dt} = y/R$$

$$\Rightarrow \quad y + RC\frac{dy}{dt} = RCjn\omega_0\exp(jn\omega_0 t)$$

& assuming a solution of $y_n(t) = K\exp(jn\omega_0 t)$

$$K\exp(jn\omega_0 t) + RCKjn\omega_0\exp(jn\omega_0 t) = RCjn\omega_0\exp(jn\omega_0 t)$$

& solving for K

$$K + RCKjn\omega_0 = RCjn\omega_0 \quad \Rightarrow \quad K = \frac{RCjn\omega_0}{1 + RCjn\omega_0}$$

From $\quad C\frac{d}{dt}(x(t) - y(t)) = \frac{y}{R}$

$$RC\frac{dx}{dt} = RC\frac{dy}{dt} + y$$

Taking Laplace transforms & assuming zero initial conditions

$$RCsX(s) = RCsY(s) + Y(s)$$

$$H(s) = \frac{RCs}{RCs + 1}$$

(9) Applying Kirchhoff's second law to the circuit gives

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = e(t)$$

or, using $i = \frac{dq}{dt}$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = e(t)$$

& substituting the given values for R, C, L & $e(t)$

$$\frac{d^2q}{dt^2} + 160\frac{dq}{dt} + 10^4 q = 20$$

Taking Laplace transforms

$$(s^2 + 160s + 10^4)\,Q(s) = [sq(0) + \dot{q}(0)] + 160q(0) + \frac{20}{s}]$$

where $Q(s)$ is the transform of $q(t)$

We are told that $q(0) = 0$ & $\dot{q}(0) = i(0) = 0$ so the equation is reduced to

$$(s^2 + 160s + 10^4)\,Q(s) = \frac{20}{s}$$

that is

$$Q(s) = \frac{20}{s(s^2 + 160s + 10^4)} = \frac{1/500}{s} - \frac{1}{500}\frac{s + 160}{s^2 + 160s + 10^4}$$

$$= \frac{1}{500}\left[\frac{1}{s} - \frac{(s+80) + 4/3(60)}{(s+80)^2 + (60)^2}\right] = \frac{1}{500}\left[\frac{1}{s} - \left[\frac{s + 4/3 \times 60}{s^2 + 60^2}\right]_{s \to s+80}\right]$$

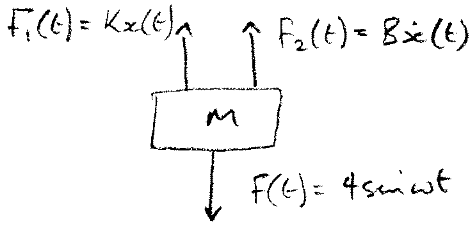
& taking inverse transforms & making use of the shift theorem

$$q(t) = \frac{1}{500}\left(1 - e^{-80t}\cos 60t - \frac{4}{3}e^{-80t}\sin 60t\right)$$

& the resulting current is given by

$$i(t) = \frac{dq}{dt} = \frac{1}{3}e^{-80t}\sin 60t$$

(10) By Newton's law



$$M\ddot{x}(t) = F(t) - F_1(t) - F_2(t)$$

giving

$$\ddot{x}(t) + 6\dot{x}(t) + 25x(t) = 4\sin\omega t$$

Taking Laplace transforms gives

$$(s^2 + 6s + 25)X(s) = [sx(0) + \dot{x}(0)] + 6x(0) + \frac{4\omega}{s^2+\omega^2}$$

$$\Rightarrow \quad X(s) = \frac{4\omega}{(s^2+\omega^2)(s^2+6s+25)}$$

For (a) with $\omega = 2$ $\quad X(s) = \dfrac{8}{(s^2+4)(s^2+6s+25)}$

$$= \frac{4}{195}\,\frac{-4s+14}{s^2+4} + \frac{2}{195}\,\frac{8s+20}{s^2+6s+25}$$

$$= \frac{4}{195}\,\frac{-4s+14}{s^2+4} + \frac{2}{195}\,\frac{8(s+3)-4}{(s+3)^2+16}$$

& taking inverse Laplace transforms gives

$$x(t) = \frac{4}{195}(7\sin 2t - 4\cos 2t) + \frac{2}{195}e^{-3t}(8\cos 4t - \sin 4t)$$

For (b) with $\omega = 5$ $\quad X(s) = \dfrac{20}{(s^2+25)(s^2+6s+25)}$

$$= \frac{-\frac{2}{15}s}{s^2+25} + \frac{1}{15}\,\frac{2(s+3)+6}{(s+3)^2+16}$$

& upon taking inverse Laplace transforms

$$x(t) = -\frac{2}{15}\cos 5t + \frac{1}{15}e^{-3t}\left(2\cos 4t + \frac{3}{2}\sin 4t\right)$$

If the damping term were missing we have $X(s) = \dfrac{20}{(s^2+25)^2}$

& from the derivative of transform property

$$\mathcal{L}\{t\cos 5t\} = -\frac{d}{ds}\mathcal{L}\{\cos 5t\} = -\frac{d}{ds}\left(\frac{s}{s^2+25}\right)$$

that is $\mathcal{L}\{t\cos 5t\} = \dfrac{-1}{s^2+25} + \dfrac{2s^2}{(s^2+25)^2}$

$$= \frac{1}{s^2+25} - \frac{50}{(s^2+25)^2}$$

$$= \frac{1}{5}\mathcal{L}\{\sin 5t\} - \frac{50}{(s^2+25)^2}$$

& by the linearity property

$$\mathcal{L}\{\tfrac{1}{5}\sin 5t - t\cos 5t\} = \frac{50}{(s^2+25)^2}$$

so taking inverse Laplace transforms of $X(s) = \frac{20}{(s^2+25)^2}$

gives $\quad x(t) = \frac{2}{25}(\sin 5t - 5t\cos 5t)$

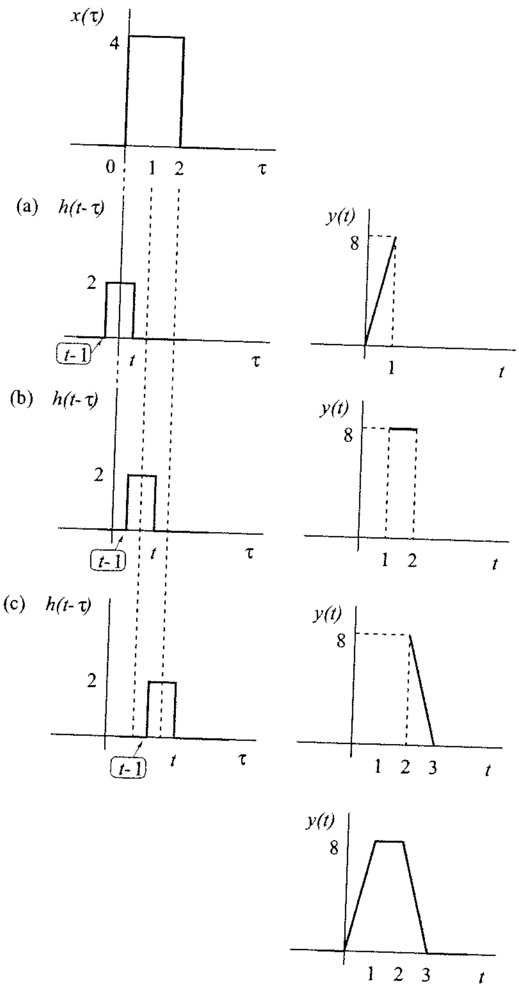
Because of the term $t\cos 5t$ the response is unbounded as $t \to \infty$. This arises because in this case the applied force $F(t) = 4\sin 5t$ is in resonance with the system.

(11) (a) There are three major regions

region 1 : $0 \leq t < 1$ $\quad y(t) = \int_0^t (4)(2)\, dt = 8t$

region 2 : $1 \leq t < 2$ $\quad y(t) = \int_{t-1}^{t} (4)(2)\, dt = 8$

region 3 : $2 \leq t < 3$ $\quad y(t) = \int_{t-1}^{2} (4)(2)\, dt = 8(3-t)$

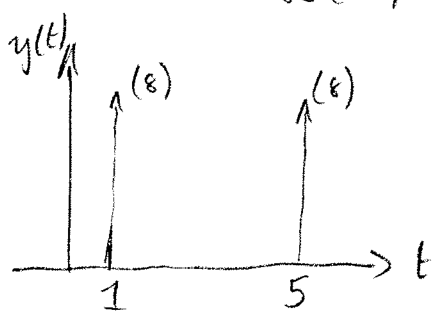


(b) the output $y(t) = h(t) * x(t)$

$$= 4\delta(t-1) * (2\delta(t) + 2\delta(t-4))$$

$$= 8\delta(t-1) * \delta(t) + 8\delta(t-1) * \delta(t-4)$$

$$= 8\delta(t-1) + 8\delta(t-5)$$



(c) There a five major regions

region 1: $0 \le t < 1$ $\quad y(t) = \int_0^t h(\tau)x(t-\tau)\,d\tau$ = shaded area in (a)

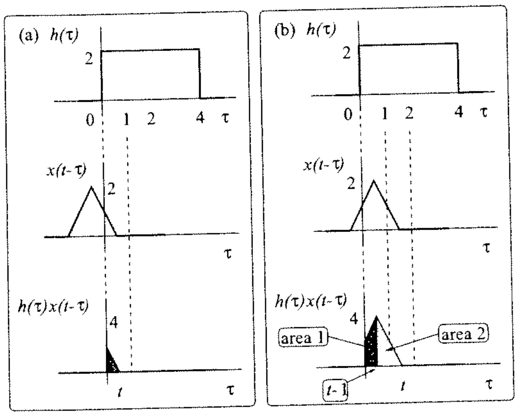
$$= \frac{1}{2}(t)(4t) = 4t^2$$

region 2: $1 \le t < 2$ $\quad y(t) = \int_0^t h(\tau)x(t-\tau)\,d\tau$ = area 1 + area 2 in (b)

$$= \frac{t-1}{2}\left(-4(t-2)+4\right) + \frac{1}{2}(1)(4)$$

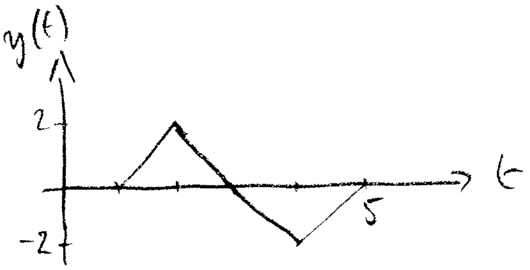
i.e. the area of a trapezium + area of a triangle

$$= 2(t-1)(3-t) + 2$$



The rest of the solution proceeds in the same manner.

(d)



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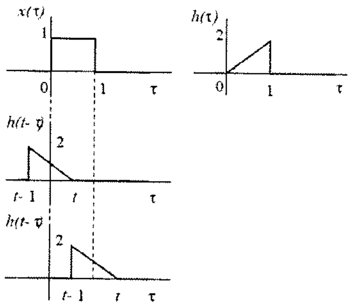
Using equation (2.8)

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Drawing diagrams:



From which, $h(t-\tau) = 2(t-\tau)$ provided $t-1 < \tau < t$ - the equation of a straight line. The convolution in the region $0 \le t < 1$:

$$y(t) = \int_0^t (1)\, 2(t-\tau)\, d\tau$$

$$= 2\int_0^t (t-\tau)d\tau$$

$$= t^2$$

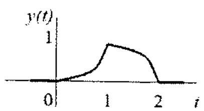
The convolution in the region $1 \le t < 2$:

$$y(t) = \int_{(t-1)}^{1} (1)\, 2(t-\tau)\, d\tau$$

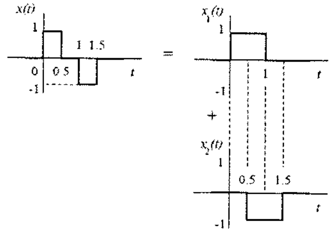
$$= 2\int_{(t-1)}^{1} (t-\tau)d\tau$$

$$= t(2-t)$$

Solution:



New input $x(t)$. The new input can be described as the addition of the old input (which we shall call $x_1(t)$) and a delayed and inverted version of the old input which we shall call $x_2(t)$) as illustrated below:



Thus

$$x(t) = x_1(t) + x_2(t)$$

and

$$y(t) = h(t)*x(t)$$
$$= h(t)*x_1(t) + h(t)*x_2(t)$$
$$= y_1(t) + y_2(t)$$

$y_1(t)$ has already been evaluated and in the first part of the question. $x_2(t)$ is a delayed and invert version of $x_1(t)$

$$x_2(t) = -x_1(t - ½)$$

Thus $y_2(t)$ is a delayed and inverted version of $y_1(t)$

$$y_2(t) = -y_1(t - ½)$$

This gives the result that:

$$y(t) = y_1(t) - y_1(t - ½)$$

Thus for $1/2 \le t < 3/2$

$$y_2(t) = -(t - 1/2)^2$$

and for $3/2 \le t < 5/2$

$$y_2(t) = -(t - 1/2)(2 - (t - 1/2))$$

elsewhere $y_2(t) = 0$ The complete solution for $y(t)$ becomes:

for $0 \le t < ½$.

$$y(t) = t^2$$

for $½ \le t < 1$

$$y(t) = t^2 - (t - ½)^2$$

for $1 \le t < 3/2$

$$y(t) = t(2-t) - (t - ½)^2$$

for $3/2 \le t < 2$

$$y(t) = t(2-t) - (t - ½)(2 - (t - ½))$$

for $2 \le t < 5/2$

$y(t) = -(t - 1/2)(2 - (t - 1/2))$

for $5/2 < t$

$y(t) = 0$