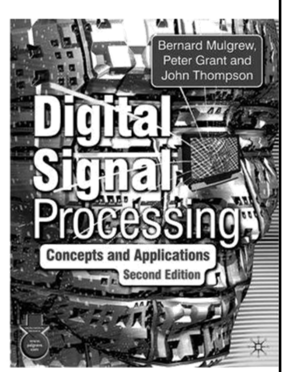
Essential Mathematical Methods for Engineers

Lecture 1a:
Signal representation and system response

Outline

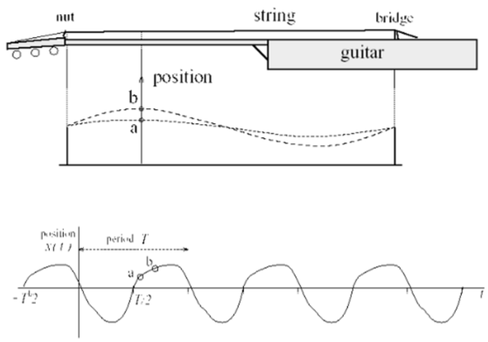
- signal representation and system response
  - signals and systems
  - signal classification
  - Fourier series
  - the Fourier transform
  - Laplace transform
  - transform analysis of linear systems
  - transfer function

course text:
Digital Signal Processing: Concepts and Applications
Mulgrew, Grant and Thompson
Palgrave Macmillan



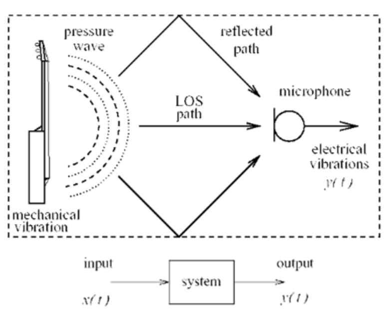
Signals and systems

- a familiar, simple example of a signal
- motion of a plucked guitar string alters pressure of surrounding air
- air pressure oscillates in sympathy
- pressure radiates to microphone and converted to electrical voltage



Signals and systems

- pressure waves radiate in all directions
- difference in path lengths can cause interference
- a simple example of a system
  – the input is the position signal, $x(t)$
  – the output is the electrical signal, $y(t)$



Signal classification

- many ways in which signal may be classified, i.e. periodic, when $x(t) = x(t + T)$, where the smallest value of T defines the period
- we can also classify signals as either energy or power signals
- energy signals
  - non-zero and finite total dissipated energy, E
  - usually exist for a finite interval of time or have most of their energy concentrated in a finite interval of time

finite energy = zero power
finite power = infinite energy

$$0 \le E \le \infty, \quad E = \int_{-\infty}^{\infty} x^2(t)dt$$

- power signals
  - non-zero and finite average delivered power, P, i.e. $0 < P < \infty$
  - an example is the unit step function $u(t)$ and a periodic signal of period T such as $x(t)=\sin(2\pi t / T)$

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} x^2(t)dt \quad P = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x^2(t)dt \quad P = \frac{1}{T}\int_{0}^{T} x^2(t)dt$$

Example

Find the energy in the decaying exponential signal $x_1(t)=5\exp(-2t)$ if $t \ge 0$ and $x_1(t) = 0$ if $t < 0$.

Fourier series
Trigonometric Fourier series

- we can represent any finite power periodic signal $x(t)$ with a period T as a sum of sine and cosine waves:

$$x(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t)$$

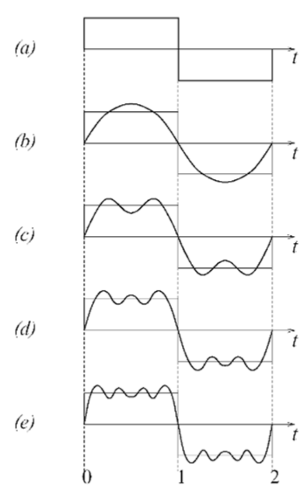
- fundamental frequency:

$\omega_0 = 2\pi/T$ rad/s or $1/T$ Hz

- harmonics are generally found at $2/T$ Hz, $3/T$ Hz … according to Fourier coefficients:

$$A_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(n\omega_0 t)dt \quad n = 0, 1, 2, \ldots$$

$$B_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(n\omega_0 t)dt \quad n = 1, 2, 3, \ldots$$



Example

Evaluate the Fourier series of the square wave (a)

Fourier series
Complex phasors

- sine and cosine waves may be described using complex phasors
- the complex phasor can be split into real and imaginary terms:

$$A\exp(j\omega_0 t) = A\cos(\omega_0 t) + jA\sin(\omega_0 t)$$

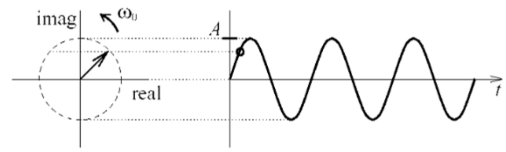
the complex phasor may be interpreted as a vector of length A rotating anticlockwise at w_0 rad/s

Euler's formula
$e^{j\theta} = \cos\theta + j\sin\theta$

- we may thus write

$$A\cos(n\omega_0 t) = \Re\{A\exp(j\omega_0 t)\}$$
$$A\sin(n\omega_0 t) = \Im\{A\exp(j\omega_0 t)\}$$

$$\cos(n\omega_0 t) = \frac{\exp(jn\omega_0 t) + \exp(-jn\omega_0 t)}{2}$$
$$\sin(n\omega_0 t) = \frac{\exp(jn\omega_0 t) - \exp(-jn\omega_0 t)}{2j}$$



Fourier series
Complex Fourier series

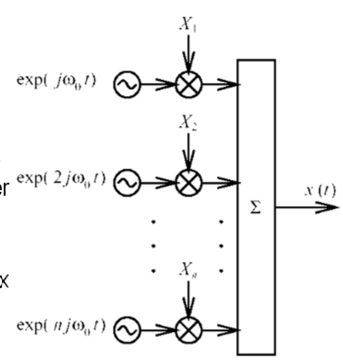
- substituting the last two equations on the last slide into those for the trigonometric Fourier series gives us the complex Fourier series:

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n \exp(jn\omega_0 t)$$

which we can interpret as a bank of phasor generators with increasing frequencies and amplitudes X_n

$$X_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\exp(-jn\omega_0 t)dt$$

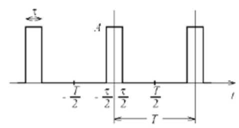
- one difference between the trigonometric and the complex Fourier series – the trigonometric Fourier series has three equations whereas the complex Fourier series has only two
  - but X_ns in the complex form are often complex

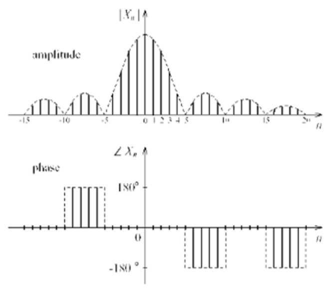


Fourier series
Complex Fourier series

Example

Derive an expression for the complex Fourier coefficient, X_n, associated with the periodic signal $x(t)$:





Example magnitude and phase spectra for $T = 5\tau$

Fourier series
Relationship between Fourier series

- there is a simple relationship between the trigonometric and complex Fourier series coefficients

$$X_0 = \frac{A_0}{2}, \qquad X_n = \frac{A_n - jB_n}{2} \ (n > 0), \qquad X_n = \frac{A_{-n} + jB_{-n}}{2} \ (n < 0)$$

  thus the complex Fourier series of a real signal exhibits complex conjugate (Hermitian) symmetry: $X_{-n} = X_n^*$ or $|X_{-n}| = |X_n|$

  and the phase is asymmetrical: $\angle X_{-n} = -\angle X_n$

- we thus need never calculate X_n for negative n and we can easily move between the two representations

Fourier series
Orthogonality

- the Fourier series is an orthogonal expansion
- we say two signals $f_1(t)$ and $f_2(t)$ are orthogonal if $\frac{1}{T}\int_{-T/2}^{T/2} f_1(t) f_2^*(t)\,dt = 0$

and for the complex Fourier series the basis functions are mutually orthogonal:

$$\frac{1}{T}\int_{-T/2}^{T/2} \exp(jn\omega_0 t)\exp^*(jm\omega_0 t) = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

Example

Note the benefit of othogonality in calculating the power in the simple periodic signal $x(t)$ where:

$$x(t) = a_1 \sin(\omega_0 t) + a_2 \sin(2\omega_0 t)$$

Fourier series
Parseval's theorem for periodic signals

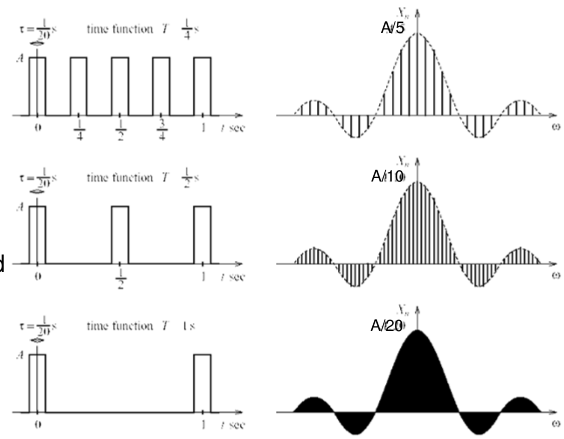
- a consequence of orthogonality
- the power in a signal may be calculated from either the trigonometric or complex Fourier coefficients:

$$P = \frac{A_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\left(A_n^2 + B_n^2\right)$$

$$P = \sum_{n=-\infty}^{\infty} \left|X_n\right|^2$$

The Fourier transform

- applicable to non-periodic signals
- consider what happens when the period of a periodic signal increases
- as the period is doubled the frequency spacing and magnitude halves
- the shape doesn't change
- NB – the horizontal axis is now ω – not n – the frequency of the n^{th} harmonic is now $n\omega_0$



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The Fourier transform

- perhaps we can determine the Fourier representation of a single pulse by letting T get very large?
  - but if it gets too large the representation will disappear
  - Fourier coefficients are calculated from $X_n = \frac{1}{T}\int_{-\tau/2}^{\tau/2} x(t)\exp(-jn\omega_0 t)dt$
  - as $T \to \infty$ the $X_n \to 0$
- we could avoid this problem by defining the Fourier coefficients

$$\begin{aligned} X_n^{'} &= T\,X_n \\ &= \int_{-T/2}^{T/2} x(t)\exp(-jn\omega_0 t)dt \end{aligned}$$

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The Fourier transform

- the Fourier series is then given by

$$x(t) = \sum_{n=-\infty}^{+\infty} \frac{X_n^{'}}{T} \exp(jn\omega_0 t)$$
$$= \sum_{n=-\infty}^{+\infty} X_n^{'} \exp(jn\omega_0 t)\frac{\omega_0}{2\pi}$$

now as $T \to \infty$

- the spectral lines get closer together and the separation between ω_0 becomes the differential $d\omega$
- the harmonic frequency $n\omega_0$ becomes the continuous frequency variable ω
- the discrete spectrum X_n' becomes a continuous spectrum $X(\omega)$
- the summation of all discrete frequency components becomes an integration over all possible frequencies:

$$X(\omega) = \lim_{T\to\infty} X_n^{'}$$
$$= \lim_{T\to\infty} \int_{-T/2}^{T/2} x(t)\exp(-jn\omega_0 t)dt$$

The Fourier transform

- therefore we have $$X(\omega) = \int_{-\infty}^{\infty} x(t)\exp(-j\omega t)dt$$

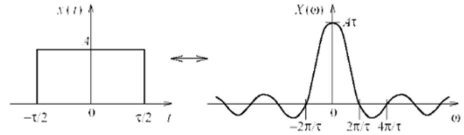
$$x(t) = \lim_{T\to\infty} \sum_{n=-\infty}^{+\infty} X_n^{'} \exp(jn\omega_0 t)\frac{\omega_0}{2\pi}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\exp(j\omega t)d\omega$$

and we can represent most finite energy signals in this way

Example

Evaluate the Fourier transform of the finite energy signal $x(t)$



The Fourier transform
Sinc and sampling functions

- this last result occurs frequently so we assign it a convenient abbreviation

$$\mathrm{sa}(x) = \frac{\sin(x)}{x}$$

which is known as the sampling function

- an alternative better suited to Hz than rad/s

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

The Fourier transform
Physical interpretation and Parseval's theorum

- we have seen how we can represent an aperiodic signal as a sum of cosine waves **at all possible frequencies**
  - the signal is not periodic thus it cannot have harmonics
- the signal $x(t)$ has a component with
  - small frequency band ω to $\omega + d\omega$ rad/s
  - magnitude $|X(\omega)|d\omega/(2\pi)$
  - phase $\angle X(\omega)$

$$\frac{|X(\omega)|d\omega}{2\pi}\cos(\omega t + \angle X(\omega))$$

- Parseval's theorem for finite energy signals $E = \int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2 d\omega$
- $|X(\omega)|^2/(2\pi)$
  - defines how the energy is distributed in frequency
  - is known as the energy spectral density

The Laplace transform

- the Fourier transform only exists for finite energy signals – for $u(t)$ (a power signal):

$$U(0) = \int_{-\infty}^{\infty} u(t)\exp(-j0t)dt$$
$$= \int_{0}^{\infty} dt$$
$$= \infty$$

- the solution is to multiply $x(t)$ by a convergence factor $\exp(-\sigma t)$

$$x_\sigma(t) = \exp(-\sigma t)x(t)$$

so that

$$X_\sigma(\omega) = \int_{-\infty}^{\infty} x_\sigma(t)\exp(-j\omega t)dt$$
$$= \int_{-\infty}^{\infty} x(t)\exp(-(\sigma + j\omega)t)dt$$

which we rewrite using $s = \sigma + j\omega$ to give us the two-sided or bilateral Laplace transform

$$X(s) = \int_{-\infty}^{\infty} x(t)\exp(-st)dt$$

The Laplace transform

- the inverse Laplace transform

$$x(t) = \exp(\sigma t)x_\sigma(t)$$
$$= \exp(\sigma t)\frac{1}{2\pi}\int_{-\infty}^{\infty} X_\sigma(\omega)\exp(j\omega t)d\omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_\sigma(\omega)\exp((\sigma + j\omega)t)d\omega$$

therefore

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)\exp(st)ds$$

$s = \sigma + j\omega$

$\frac{ds}{d\omega} = j$

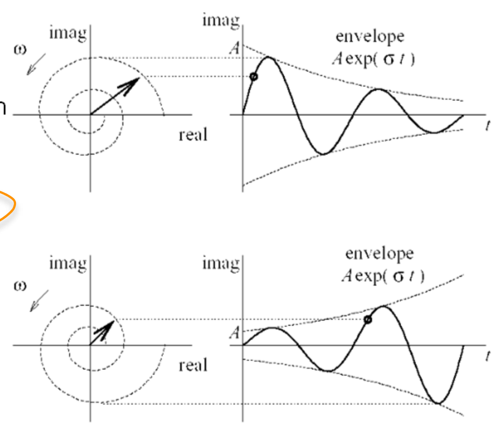
- if we assume causality we have the one-sided Laplace transform

$$X(s) = \int_{0^-}^{\infty} x(t)\exp(-st)dt$$

The Laplace transform
Applicability and physical interpretation

- the basis functions of the Laplace transform are growing or decaying complex phasors

$$A\exp(st) = A\exp(\sigma t)\cos(\omega t) + jA\exp(\sigma t)\sin(\omega t)$$



- the signal $x(t)$ has components with
  - frequency ω
  - magnitude $|X(s)|d\omega/(2\pi)$
  - growth or decay determined by σ [constant]
  - phase $\angle X(s)$

$$\frac{|X(s)|d\omega}{2\pi}\exp(\sigma t)\cos(\omega t + \angle X(s))$$

Example

Evaluate the Laplace transform of a one-sided signal $x(t) = \exp(-\alpha t)$

Transform analysis of linear systems

- the transforms all represent signals as weighted sums (integrals) of exponential orthogonal basis functions
  - e.g. for the complex Fourier series we have weights X_n and mutually orthogonal basis functions $\exp(jn\omega_0 t)$
- this representation is fundamental to the analysis of linear systems and to evaluating the response of such systems to a wide range of inputs
  - superposition: a system with a number of inputs has an output equal to the sum of the output of each input

Transform analysis of linear systems
Linear ordinary differential equations

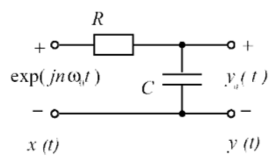
- many linear systems can be modelled with linear ordinary differential equations

$$a_0 y + a_1 \frac{dy}{dt} + \ldots + a_n \frac{d^n y}{dt^n} = b_0 x + b_1 \frac{dx}{dt} + \ldots + b_m \frac{d^m x}{dt^m}$$

where the input, $x(t)$, defines the output, $y(t)$, according to system parameters $a_0 \ldots a_n$ and $b_0 \ldots b_m$

Example

Evaluate the response $y_n(t)$ of the following circuit to the n^{th} harmonic, i.e. the complex phasor $\exp(jn\omega_0 t)$



$$a_0 y_n(t) + a_1 \frac{dy_n}{dt} = b_0 \exp(jn\omega_0 t)$$

$$y_n(t) + RC \frac{dy_n}{dt} = \exp(jn\omega_0 t)$$

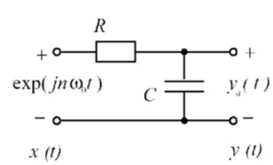
Transform analysis of linear systems
Linear ordinary differential equations

- the response of the circuit to the n^{th} basis function is characterised by the system transfer function H_n

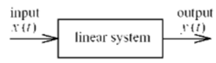
$$y_n(t) = H_n \exp(jn\omega_0 t)$$

where for the given system

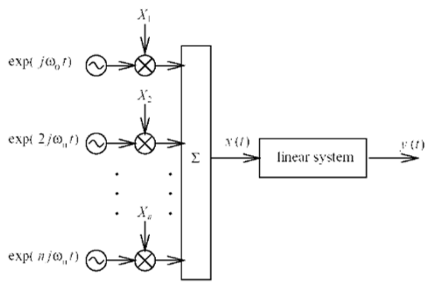
$$\begin{aligned} H_n &= \frac{b_0}{a_0 + (jn\omega_0)a_1} \\ &= \frac{1}{1 + (jn\omega_0)RC} \end{aligned}$$

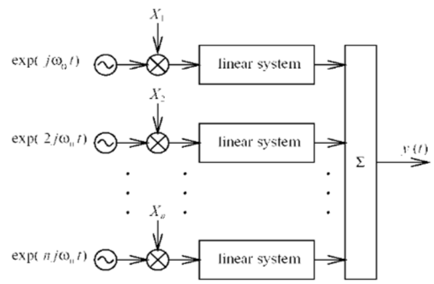


Transform analysis of linear systems
Response of a linear system to a periodic input

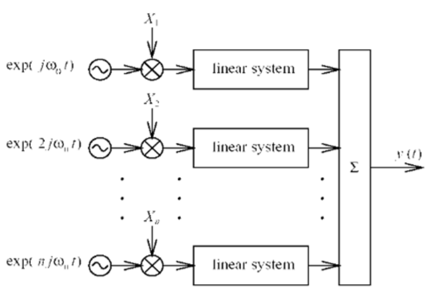


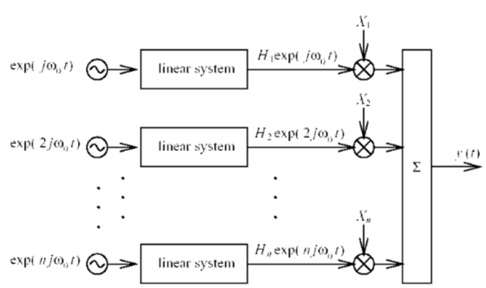
- what if we want to calculate more generally the output $y(t)$ to any periodic input $x(t)$?
- according to the principle of superposition:





Transform analysis of linear systems
Response of a linear system to a periodic input

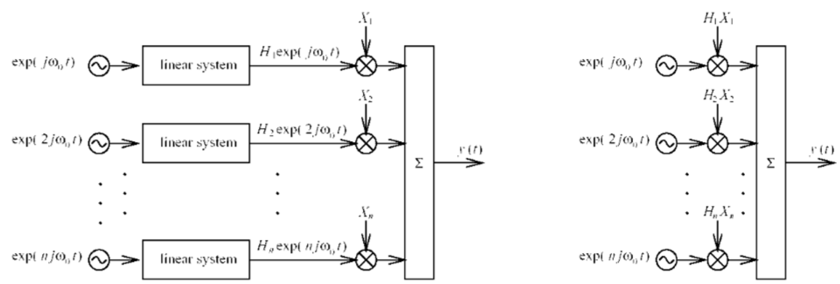




- the system is linear so we can perform the multiplication by X_n at the input or equivalently at the output
- H_n characterises the system response to the n^{th} complex phasor

Transform analysis of linear systems
Response of a linear system to a periodic input

- the diagrams are equivalent in structure
- in the second diagram H_n and X_n have been bought together
  - these are the complex Fourier coefficients of the output, $y(t)$
  - this is the Fourier series representation, Y_n



Transform analysis of linear systems
Response of a linear system to a periodic input

- there is thus a simple relationship between the complex Fourier coefficients of the input, X_n and those of the output, Y_n

$$Y_n = H_n X_n$$

- since each basis function is scaled by X_n to form the input $x(t)$
- the response of the system to each Fourier component $X_n \exp(jn\omega_0 t)$ is given by

$$X_n H_n \exp(jn\omega_0 t)$$

- using superposition we thus have

$$y(t) = \sum_{n=-\infty}^{\infty} H_n X_n \exp(jn\omega_0 t)$$

and $Y_n = H_n X_n$ is a complex Fourier coefficient of the output

Transform analysis of linear systems
General approach

- to evaluate the response $y(t)$ of a linear system to an input $x(t)$
  - represent the input signal as a weighted sum of exponential basis functions
  - obtain the appropriate linear differential equation which characterises the system
  - obtain the response of the system to each basis function
  - apply principles of superposition to determine the output

- we have considered periodic input signals and hence steady-state responses
  - the Laplace transform applies equally to steady-state and transient responses to non-periodic signals

Laplace transfer function

- defined in the same way as for the Fourier transfer function

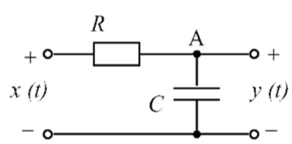
$$H_s = \frac{L[\text{output}]}{L[\text{input}]} = \frac{Y(s)}{X(s)}$$

and completely specifies system characteristics

- with knowledge of the transfer function we can calculate the response of the system to any input

Example

Evaluate the transfer function of the following circuit:



$$\frac{x(t) - y(t)}{R} = C\frac{dy}{dt}$$

Summary

You should be able to:

- recognise both signals and systems;
- evaluate the complex Fourier series of simple waveforms and know what the complex weights signify;
- evaluate the Fourier and Laplace transforms of simple waveforms and know what the transforms signify;
- understand the role of these transforms in evaluating the response of a linear system to a particular signal;
- calculate the response of a simple system to a simple waveform using transform techniques.