Mathematical Methods for Engineers (MathEng)
EXAM
4th February 2015
Duration: 2 hrs, calculators permitted, no documents
This exam paper contains 5 questions and 50 marks.
ATTEMPT ALL QUESTIONS – ANSWER IN ENGLISH

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1. (a) Determine an expression for the trigonometric Fourier series for the period signal in Figure Q1.
   (b) Hence or otherwise, determine an expression for the complex Fourier series of the same signal.

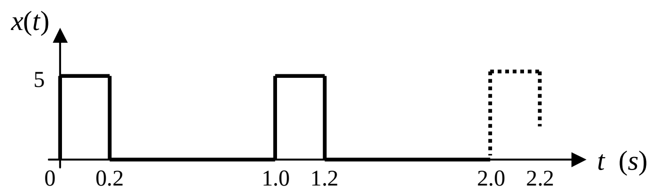

Figure Q1

[9 marks]

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2. A digital filter has the following difference equation:

$$y(n) = 1.6y(n-1) - 0.8y(n-2) + x(n)$$

(a) Assuming zero initial conditions and without using transform methods, determine the first four terms of the unit pulse response $\{h(n)\}$.
(b) Determine the transfer function $H(z)$ and state clear the position of any poles and/or zeros.
(c) Via partial fraction expansions and by using the look-up table of z-transforms, determine an expression for $h(n)$.
(d) Use discrete convolution to calculate the first eight samples of the output when the following sequence is applied to the filter:

$$\{0, 0.25, 0.5, 0.75, 1.0, 0, 0, 0, \dots, 0\}.$$

[11 marks]

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3. (a) Plot separate amplitude and phase responses for two different, analogue filters whose pole/zero plots are illustrated in Figure Q3 (a) and (b).
   (b) State clearly whether each filter is a high-pass, low-pass, band-pass or band-stop.
   (c) Suppose that the analogue filters are to be replaced by digital filters. Justifying your answer and stating clearly any assumptions you make, sketch approximate z-plane pole/zero plots for digital filters having frequency responses similar to those illustrated in Figure Q3.

[8 marks]

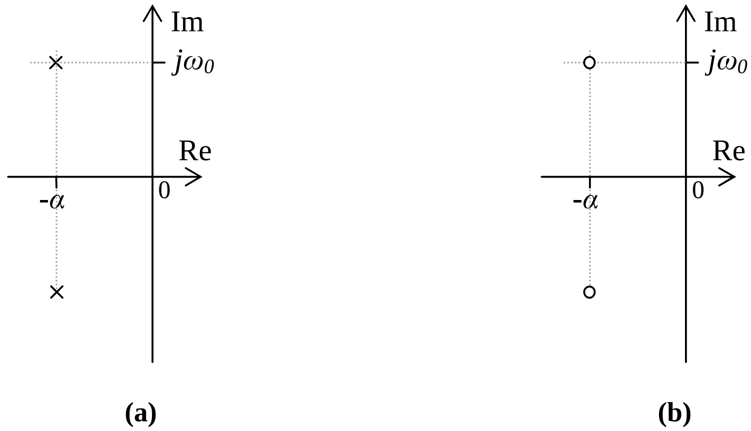

Figure Q3

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4. The joint probability density function of two random variables is given by:

$$f_{XY}(x,y) = C(1 - x - y)$$

where $0 \le x \le 1 - y$ and where $0 \le y \le 1$. Determine C, the two marginal probability density functions and then state whether or not X and Y are independent.

[12 marks]

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5. A system of linear equations is given by $Ax=b$:

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix}$$

(a) Write down the augmented matrix and then its row reduced echelon form.
(b) Identify the pivot and free variables and the rank of matrix A.
(c) Determine the special solutions and, **while ensure that the *I* matrix appears in the rows of *N* which correspond to the free variables**, give the nullspace matrix N.
(d) Determine the particular solution x_p.
(e) Write the full solution to $Ax=b$ in the form $x = x_p + x_n$ where there is an x_n for each special solution.

[10 marks]

Table of selected Laplace transforms

$$X(s) = \int_{-\infty}^{\infty} x(t)\exp(-st)\,dt$$

N.B.: lower limit is 0 for one-sided Laplace transform

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\exp(st)\,ds$$

$x(t) \quad (t \geq 0)$	$X(s)$
$\delta(t)$	1
$\delta(t-\alpha)$	$\exp(-\alpha s)$
1 (unit step)	$\frac{1}{s}$
t (unit ramp)	$\frac{1}{s^2}$
$\exp(-\alpha t)$	$\frac{1}{s+\alpha}$
$t\exp(-\alpha t)$	$\frac{1}{(s+\alpha)^2}$
$\sin(\alpha t)$	$\frac{\alpha}{s^2+\alpha^2}$
$\cos(\alpha t)$	$\frac{s}{s^2+\alpha^2}$
$e^{-\alpha t}\sin(\omega t)$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
$e^{-\alpha t}\cos(\omega t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$

Bode plots

Poles or zeros on the real axis:

$$(s+a) = a\left(\frac{s}{a}+1\right) = \frac{1}{\tau}(\tau s+1)$$

Complex conjugate poles (or zeros):

$$(s^2+As+B) = (s^2+2\zeta\omega_0 s+\omega_0^2) = \omega_0^2((s/\omega_0)^2+2\zeta(s/\omega_0)+1)$$

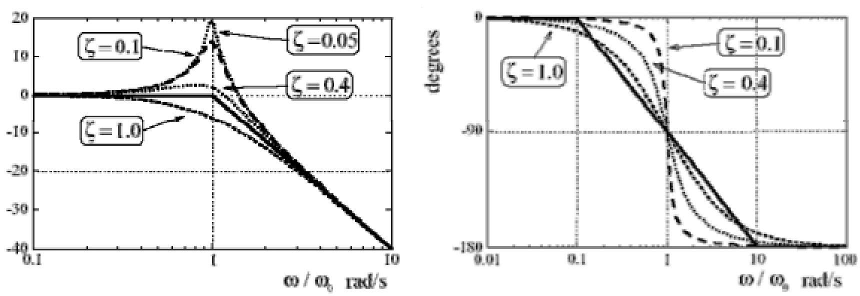

Table of selected z-transforms

$$x_c(t) = \sum_{n=0}^{\infty} x(n\Delta t)\delta(t - n\Delta t)$$

$$X_c(s) = \sum_{n=0}^{\infty} x(n\Delta t)\exp(-n\Delta ts)$$

$$X_c(s) = X(z)|_{z=e^{\Delta ts}}$$

$$X_c(\omega) = X_s(s)|_{s=j\omega}$$

$$X(\omega) = X(z)|_{z=\exp(\Delta tj\omega)}$$

$$X(z) = \sum_{n=0}^{\infty} x(n\Delta t)z^{-n}$$

$x(n) \quad (n \geq 0)$	$X(z)$
$\delta(n)$ unit pulse	1
$\delta(n-m)$	z^{-m}
1 (unit step)	$\dfrac{z}{z-1}$
n (unit ramp)	$\dfrac{z}{(z-1)^2}$
$\exp(-\alpha n)$	$\dfrac{z}{(z-e^{-\alpha})}$
$n\exp(-\alpha n)$	$\dfrac{e^{-\alpha}z}{(z-e^{-\alpha})^2}$
$\sin(\beta n)$	$\dfrac{z\sin(\beta)}{z^2-2z\cos(\beta)+1}$
$\cos(\beta n)$	$\dfrac{z^2-z\cos(\beta)}{z^2-2z\cos(\beta)+1}$
$e^{-\alpha n}\sin(\beta n)$	$\dfrac{ze^{-\alpha}\sin(\beta)}{z^2-2ze^{-\alpha}\cos(\beta)+e^{-2\alpha}}$
$e^{-\alpha n}\cos(\beta n)$	$\dfrac{z^2-ze^{-\alpha}\cos(\beta)}{z^2-2ze^{-\alpha}\cos(\beta)+e^{-2\alpha}}$

Table of selected Fourier transform pairs

Function	$x(t)$	$X(\omega)$
Rectangular function of width τ	$\Pi(t/\tau)$	$\tau\ \mathrm{sinc}(\omega\tau/2)$
Triangular function of width 2τ	$\Lambda(t/\tau)$	$\tau\ \mathrm{sinc}^2(\omega\tau/2)$
Train of impulses every Δt	$\delta_T(t)$	$2\pi/\Delta t\, \Sigma_n \delta(\omega - 2\pi n/\Delta t)$

NB: $\mathrm{sinc}(x) = \sin(\pi x)/\pi x$
NB: $\mathrm{sa}(x) = \sin(x)/x$

Euler’s identity

$$\exp(j\theta) = \cos\theta + j\sin\theta$$

$$\cos(n\omega_0 t) = \frac{\exp(jn\omega_0 t) + \exp(-jn\omega_0 t)}{2}$$

$$\sin(n\omega_0 t) = \frac{\exp(jn\omega_0 t) - \exp(-jn\omega_0 t)}{2j}$$

Fourier series and transforms

Trigonometric Fourier series	
$$x(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t)$$ $$A_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(n\omega_0\, t)dt$$ $$B_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(n\omega_0\, t)dt$$	
Complex Fourier series – periodic and continuous in time, discrete in frequency	
$$x(t) = \sum_{n=-\infty}^{+\infty} X_n \exp(jn\omega_0 t)$$	$$X_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\exp(-jn\omega_0 t)\, dt$$
Fourier transform – continuous in time, continuous in frequency	
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\exp(j\omega t)\, d\omega$$	$$X(\omega) = \int_{-\infty}^{\infty} x(t)\exp(-j\omega t)\, dt$$
Discrete-time Fourier transform – discrete in time, continuous and periodic in frequency	
$$x(n\Delta t) = \frac{\Delta t}{2\pi}\int_{-\pi/\Delta t}^{\pi/\Delta t} X(\omega)\exp(jn\Delta t\omega)\, d\omega$$	$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n\Delta t)\exp(-j\omega n\Delta t)$$
Discrete Fourier transform – discrete and periodic in time and in frequency	
$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\exp\left(\frac{jnk2\pi}{N}\right)$$	$$X(k) = \sum_{n=0}^{N-1} x(n)\exp\left(\frac{-jnk2\pi}{N}\right)$$

Transformation of random variables

$$f_Y(y) = \sum_{i=1}^{N} f_X(x_i) \left|\frac{dx_i}{dy}\right|_{x_i = g_i^{-1}(y)}$$

$$f_{UV}(u,v) = f_{XY}(x,y) \left|\frac{\partial(x,y)}{\partial(u,v)}\right|_{\substack{x = g_1^{-1}(u,v) \\ y = g_2^{-1}(u,v)}}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

 [END OF PAPER]