Mathematical Methods for Engineers (MathEng)
EXAM Nick Evans
17th February 2008
Duration: 2 hrs, calculators permitted, no documents
Total: 60 marks
ATTEMPT ALL QUESTIONS – ANSWER IN ENGLISH

INSTUCTIONS: There are 8 questions in this exam paper. The first 7 questions total 60 marks and if you answer these questions correctly you will score 20/20. Optionally you may also attempt question 8 for which you can gain extra marks upto a maximum of 20/20.

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1. (a) Identify the four subspaces associated with a matrix and describe how each of their dimensions is related to the rank of the matrix. Describe the orthogonality between the four subspaces in your answer.
   (b) Giving the dimension of each subspace in your answer, find a basis for each of the four subspaces associated with the following matrices and comment on the results:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{bmatrix}$$

   (c) Assuming that you were given the singular value decomposition (SVD) of the matrices how would you be able to answer part (b)? (You do **not** need to prove this nor calculate the SVD in your answer.)

[10 marks]

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2. When a matrix A is square, symmetric and positive-definite, the solution to $Ax = b$ minimises a scalar, quadratic function of a vector with the form $f(x) = \frac{1}{2}x^T Ax - b^T x + c$ where x and b are vectors and c is a scalar constant.

   Make two copies of the contour diagram in Figure Q2 and, starting in both cases from the point $x_{(0)}$, plot two search iterations/directions to illustrate and help describe the different behaviour of (i) the method of steepest descent and (ii) conjugate gradient algorithms for a simple 2-dimensional problem of this sort, where $x_{(i)} = [x_1 \quad x_2]^T$. (The contours of the quadratic form illustrate points with constant $f(x)$ and the centre of the smallest elipse is the minimum.)

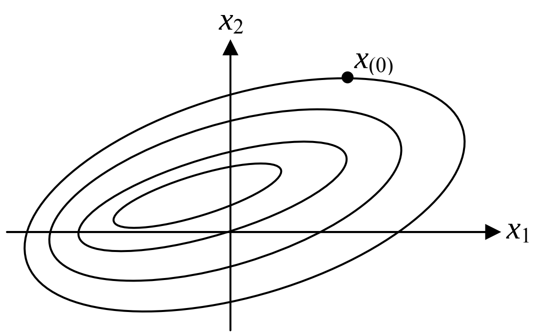

Figure Q2

[5 marks]

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3. The function $f(x,y) = 2x^2 + 3y^2$ may be solved, subject to the constraint $2x + y = 1$, simply by eliminating a variable. Show how the solution may be also be obtained using Lagrange multipliers.

[8 marks]

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4. Sketch the bode plots (magnitude and phase) for the transfer function:

$$H(s) = \frac{20s(s+100)}{(s+2)(s+10)}$$

[10 marks]

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5. (a) Find the discrete Fourier transform (DFT) of the 3-point signal $f(n)$ illustrated in solid black lines in Figure Q5(a) and plot the magnitude and phase spectrums.

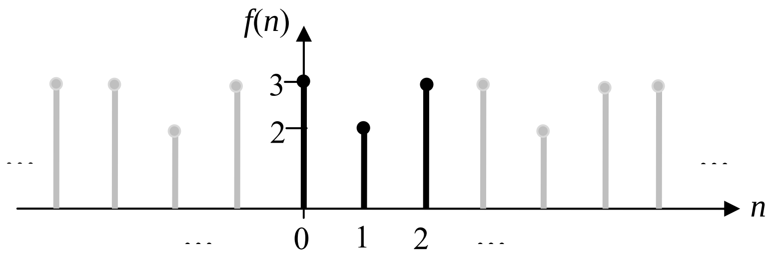

Figure Q5(a)

(b) Repeat part (a) by padding three zeros to $f(n)$ as illustrated in Figure Q5(b). Compare and comment on the result.

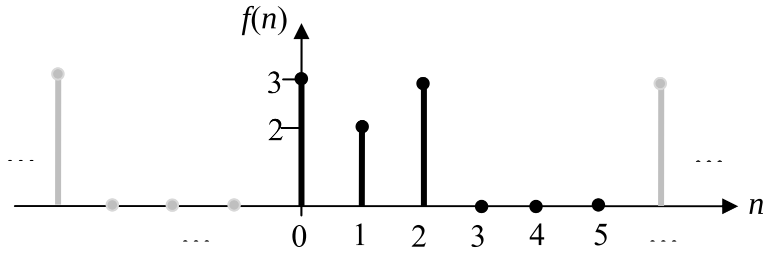

Figure Q5(b)

[15 marks]

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6. Figure Q6 illustrates three bivariate Gaussian probability density functions (PDFs) and their corresponding contour plots (left and right columns respectively) for random variables *X* and *Y*. Describe the differences between the PDFs in terms of the mean and variance of each component and their correlation.

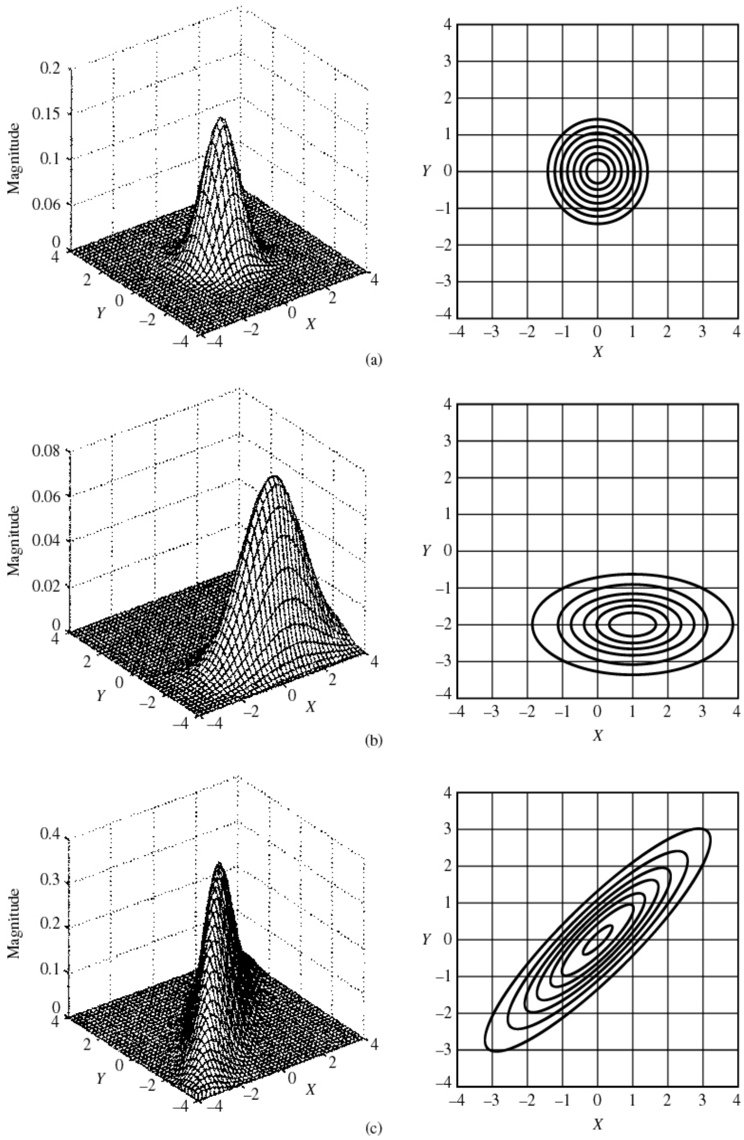

Figure Q6

[6 marks]

7. A zero-mean stationary white noise signal $x(n)$ is applied to a finite impulse response (FIR) filter with impulse response sequence {0.5, 0.75} as illustrated in Figure Q7. Derive an expression for the power spectral density (PSD) of the signal at the output of the filter.

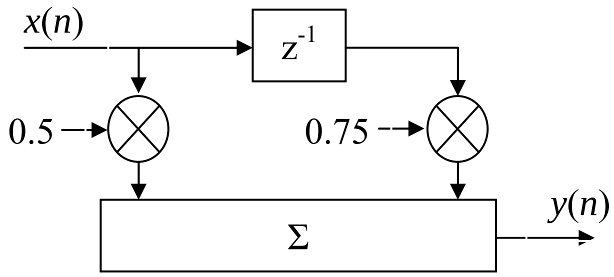

Figure Q7

[6 marks]

This question is optional as per the instructions on page 1.

8. A digital audio recording is contaminated with noise at 2 kHz. You are tasked with designing a second-order notch filter that (i) removes frequencies at 50 Hz (gain = 0) and (ii) has a rapid recovery to pass frequencies both sides of 50 Hz unattenuated (gain ≈ 1). The highest frequency to be processed is 4 kHz. Show how such a filter may be realised with a transfer function of the form:

$$H(z) = K\frac{(z^2+1)}{(z^2+a^2)}$$

where a is a real number. Stating any assumptions and/or design choices that you make:

(a) determine an approximate suitable value for a and sketch a plot of the z-plane showing any poles and zeros of your design;
(b) determine an expression in a for the value of K;
(c) derive an expression for the magnitude response as a function of a, and
(d) give a rough plot of the magnitude response for different values of a.

[extra marks]

Table of selected Laplace transforms

$x(t) \quad (t \geq 0)$	$X(s)$
$\delta(t)$	1
$\delta(t-\alpha)$	$\exp(-\alpha s)$
$1 \quad$ (unit step)	$\frac{1}{s}$
$t \quad$ (unit ramp)	$\frac{1}{s^2}$
$\exp(-\alpha t)$	$\frac{1}{s+\alpha}$
$t\exp(-\alpha t)$	$\frac{1}{(s+\alpha)^2}$
$\sin(\alpha t)$	$\frac{1}{s^2+\alpha^2}$
$\cos(\alpha t)$	$\frac{s}{s^2+\alpha^2}$
$e^{-\alpha t}\sin(\omega t)$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
$e^{-\alpha t}\cos(\omega t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$

[TURN OVER]

Table of selected z-transforms

$x(n) \quad (n \geq 0)$	$X(z)$
$\delta(n)$ unit pulse	1
$\delta(n-m)$	z^{-m}
1 (unit step)	$\dfrac{z}{z-1}$
n (unit ramp)	$\dfrac{z}{(z-1)^2}$
$\exp(-\alpha n)$	$\dfrac{z}{(z-e^{-\alpha})}$
$n\exp(-\alpha n)$	$\dfrac{e^{-\alpha}z}{(z-e^{-\alpha})^2}$
$\sin(\beta n)$	$\dfrac{z\sin(\beta)}{z^2-2z\cos(\beta)+1}$
$\cos(\beta n)$	$\dfrac{z^2-z\cos(\beta)}{z^2-2z\cos(\beta)+1}$
$e^{-\alpha n}\sin(\beta n)$	$\dfrac{ze^{-\alpha}\sin(\beta)}{z^2-2ze^{-\alpha}\cos(\beta)+e^{-2\alpha}}$
$e^{-\alpha n}\cos(\beta n)$	$\dfrac{z^2-ze^{-\alpha}\cos(\beta)}{z^2-2ze^{-\alpha}\cos(\beta)+e^{-2\alpha}}$

Fourier series and transforms

Fourier series – periodic and continuous in time, discrete in frequency	
$x(t) = \sum_{n=-\infty}^{+\infty} X_n \exp\,(jn\omega_0 t)$	$X_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\exp(-jn\omega_0 t)\,dt$
Fourier transform – continuous in time, continuous in frequency	
$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\exp\,(j\omega t)\,d\omega$	$X(\omega) = \int_{-\infty}^{\infty} x(t)\exp(-j\omega t)\,dt$
Discrete-time Fourier transform – discrete in time, continous and periodic in frequency	
$x(n\Delta t) = \frac{\Delta t}{2\pi}\int_{-\pi/\Delta t}^{\pi/\Delta t} X(\omega)\exp\,(jn\Delta t\omega)\,d\omega$	$X(\omega) = \sum_{n=-\infty}^{\infty} x(n\Delta t)\exp(-j\omega n\Delta t)$
Discrete Fourier transform – discrete and periodic in time and in frequency	
$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\exp\left(\frac{jnk2\pi}{N}\right)$	$X(k) = \sum_{n=0}^{N-1} x(n)\exp\left(\frac{-jnk2\pi}{N}\right)$