Mathematical Methods for Engineers (MathEng)
EXAM
13th February 2018
Duration: 2 hrs, calculators permitted, no documents
This exam paper contains 6 questions and 60 marks.
ATTEMPT ALL QUESTIONS – ANSWER IN ENGLISH

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1. Consider a random variable X with the following probability distribution function (PDF):

$$f_X(x) = \begin{cases} mx & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

where m is a constant.

(a) Determine the value of the constant m.
(b) Derive an expression for the cumulative distribution function (CDF) $F_X(x)$.
(c) Sketch the PDF and the CDF.
(d) Derive an expression for the mean μ and the variance σ of the random variable X.

[8 marks]

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2. A linear, time invariant system with impulse response $h(t)$ is fed with an input signal $x(t)$ where:

$$x(t) = \exp(-\alpha t)\, u(t), \alpha > 0, \text{ and} \qquad h(t) = u(t),$$

and where $u(t)$ is the unit step function.

(a) Via graphical methods, **sketch** the output of the system $y(t)$.
(b) Via any appropriate method, **derive** an expression for $y(t)$.

[8 marks]

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3. Consider a system with a transfer function $X(s)$ given by:

$$X(s) = \frac{2s + 4}{s^2 + 4s + 3}$$

By performing a partial fraction expansion, derive an expression for $x(t)$, i.e. the inverse Laplace transform.

[10 marks]

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4. When excited with a unit step function $u[n]$, a linear time invariant, sampled-data system has output $y[n] = 2(1/3)^n u[n]$. Determine the system impulse response $h[n]$ and then the output when the input $x[n] = (1/2)^n u[n]$

[14 marks]

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5. (a) Find an LU decomposition of

$$A = \begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix}$$

(b) Find the inverse of

$$B = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$$

by Gauss-Jordan elimination. **Hint:** $[B \mid I] \rightarrow [I \mid B^{-1}]$

[8 marks]

6. Show that an orthogonal matrix preserves distances and angles. In other words, if $U^T U = I$, then

$$\| U\mathbf{v_1} - U\mathbf{v_2} \| = \|\mathbf{v_1} - \mathbf{v_2}\|$$

and

$$(U\mathbf{v_1}) \cdot (U\mathbf{v_2}) = \mathbf{v_1} \cdot \mathbf{v_2}$$

for all vectors $\mathbf{v_1}, \mathbf{v_2}$.

[12 marks]

Table of selected Laplace transforms

$$X(s) = \int_{-\infty}^{\infty} x(t)\exp(-st)\,dt$$

N.B.: lower limit is 0 for one-sided Laplace transform

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\exp(st)\,ds$$

$x(t)\quad (t \geq 0)$	$X(s)$
$\delta(t)$	1
$\delta(t-\alpha)$	$\exp(-\alpha s)$
1 (unit step)	$\frac{1}{s}$
t (unit ramp)	$\frac{1}{s^2}$
$\exp(-\alpha t)$	$\frac{1}{s+\alpha}$
$t\exp(-\alpha t)$	$\frac{1}{(s+\alpha)^2}$
$\sin(\alpha t)$	$\frac{\alpha}{s^2+\alpha^2}$
$\cos(\alpha t)$	$\frac{s}{s^2+\alpha^2}$
$e^{-\alpha t}\sin(\omega t)$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
$e^{-\alpha t}\cos(\omega t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$

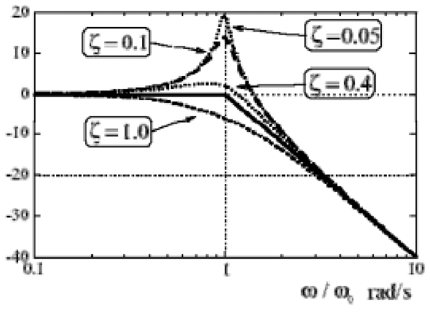
Bode plots

Poles or zeros on the real axis:

$$(s+a) = a\left(\frac{s}{a}+1\right) = \frac{1}{\tau}(\tau s+1)$$

Complex conjugate poles (or zeros):

$$(s^2+As+B) = (s^2+2\zeta\omega_0 s+\omega_0^2) = \omega_0^2((s/\omega_0)^2+2\zeta(s/\omega_0)+1)$$



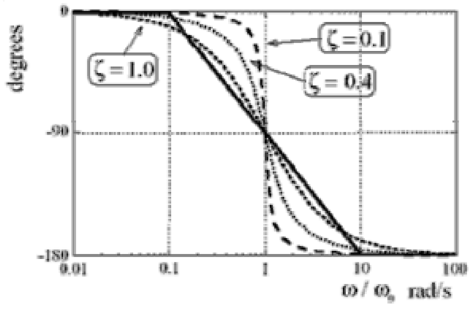


Table of selected z-transforms

$$x_c(t) = \sum_{n=0}^{\infty} x(n\Delta t)\delta(t - n\Delta t)$$

$$X_c(s) = \sum_{n=0}^{\infty} x(n\Delta t)\exp(-n\Delta t s)$$

$$X_c(s) = X(z)|_{z=e^{\Delta t s}}$$

$$X_c(\omega) = X_s(s)|_{s=j\omega}$$

$$X(\omega) = X(z)|_{z=\exp(\Delta t j\omega)}$$

$$X(z) = \sum_{n=0}^{\infty} x(n\Delta t)z^{-n}$$

$x(n)$ $(n \geq 0)$	$X(z)$
$\delta(n)$ unit pulse	1
$\delta(n-m)$	z^{-m}
1 (unit step)	$\dfrac{z}{z-1}$
n (unit ramp)	$\dfrac{z}{(z-1)^2}$
$\exp(-\alpha n)$	$\dfrac{z}{(z-e^{-\alpha})}$
$n\exp(-\alpha n)$	$\dfrac{e^{-\alpha}z}{(z-e^{-\alpha})^2}$
$\sin(\beta n)$	$\dfrac{z\sin(\beta)}{z^2-2z\cos(\beta)+1}$
$\cos(\beta n)$	$\dfrac{z^2-z\cos(\beta)}{z^2-2z\cos(\beta)+1}$
$e^{-\alpha n}\sin(\beta n)$	$\dfrac{ze^{-\alpha}\sin(\beta)}{z^2-2ze^{-\alpha}\cos(\beta)+e^{-2\alpha}}$
$e^{-\alpha n}\cos(\beta n)$	$\dfrac{z^2-ze^{-\alpha}\cos(\beta)}{z^2-2ze^{-\alpha}\cos(\beta)+e^{-2\alpha}}$

Table of selected Fourier transform pairs

Function	$x(t)$	$X(\omega)$
Rectangular function of width τ	$\Pi(t/\tau)$	$\tau\ \mathrm{sinc}(\omega\tau/2)$
Triangular function of width 2τ	$\Lambda(t/\tau)$	$\tau\ \mathrm{sinc}^2(\omega\tau/2)$
Train of impulses every Δt	$\delta_T(t)$	$2\pi/\Delta t\, \Sigma_n \delta(\omega - 2\pi n/\Delta t)$

NB: $\mathrm{sinc}(x) = \sin(\pi x)/\pi x$
NB: $\mathrm{sa}(x) = \sin(x)/x$

Euler's identity

$$\exp(j\theta) = \cos\theta + j\sin\theta$$

$$\cos(n\omega_0 t) = \frac{\exp(jn\omega_0 t) + \exp(-jn\omega_0 t)}{2}$$

$$\sin(n\omega_0 t) = \frac{\exp(jn\omega_0 t) - \exp(-jn\omega_0 t)}{2j}$$

Fourier series and transforms

Trigonometric Fourier series

$$x(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t)$$

$$A_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(n\omega_0\, t)dt$$

$$B_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(n\omega_0\, t)dt$$

Complex Fourier series – periodic and continuous in time, discrete in frequency	
$$x(t) = \sum_{n=-\infty}^{+\infty} X_n \exp(jn\omega_0 t)$$	$$X_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\exp(-jn\omega_0 t)\, dt$$

Fourier transform – continuous in time, continuous in frequency	
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\exp(j\omega t)\, d\omega$$	$$X(\omega) = \int_{-\infty}^{\infty} x(t)\exp(-j\omega t)\, dt$$

Discrete-time Fourier transform – discrete in time, continuous and periodic in frequency	
$$x(n\Delta t) = \frac{\Delta t}{2\pi}\int_{-\pi/\Delta t}^{\pi/\Delta t} X(\omega)\exp(jn\Delta t\omega)\, d\omega$$	$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n\Delta t)\exp(-j\omega n\Delta t)$$

Discrete Fourier transform – discrete and periodic in time and in frequency	
$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\exp\left(\frac{jnk2\pi}{N}\right)$$	$$X(k) = \sum_{n=0}^{N-1} x(n)\exp\left(\frac{-jnk2\pi}{N}\right)$$

Transformation of random variables

$$f_Y(y) = \sum_{i=1}^{N} f_X(x_i) \left|\frac{dx_i}{dy}\right|_{x_i = g_i^{-1}(y)}$$

$$f_{UV}(u,v) = f_{XY}(x,y) \left|\frac{\partial(x,y)}{\partial(u,v)}\right|_{\substack{x = g_1^{-1}(u,v) \\ y = g_2^{-1}(u,v)}}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$