

Capacity – MOBCOM

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- ① Wireless Channels (AWGN and LTI)
- ② Fading channels (epsilon, CSIT, freqSelec)
- ③ Multi-user Capacity

① Wireless Channels (AWGN and LTI)

Capacity of wireless channels

Capacity of LTI Gaussian channels

② Fading channels (epsilon, CSIT, freqSelec)

③ Multi-user Capacity

1 Capacity of wireless channels

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- ▶ Fundamental limits : best performance.
- ▶ Tool: information theory.
- ▶ Capacity: max rate s.t. $P_{err} \rightarrow 0$
- ▶ First AWGN, then fading channels (calculate capacity)
- ▶ Derive diversity results
- ▶ Will give insight: allocate W and \bar{P}
- ▶ Inventor of IT: Shannon (1948).
 - > $\exists R > 0$ s.t. $P_{err} \rightarrow 0$ for long T
 - > $C = \max R: P_{err}(R) \rightarrow 0$ as $T \rightarrow \infty$
 - > and $P_{err} \rightarrow 1 \forall R > C$

- ▶ Consider AWGN (Additive White Gaussian Noise)

$$y[m] = x[m] + w[m]$$

- > gives sense to $P_{err} \rightarrow 0$
- > explain capacity

- ▶ 1) $T=1$

- > Since $x[m] = \pm\sqrt{P}$, means

$$P_e = Q\left(\frac{\text{half - distance}}{\sigma_w}\right) = Q\left(\frac{\sqrt{P}}{\sigma_w}\right)$$

- > Not reliable: P_e fails to go 0

1 Early attempts

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- ▶ 2) T large: Consider Repetition Coding:

$$\mathbf{x}_A = \sqrt{P}[1, 1, \dots, 1], \quad \mathbf{x}_B = -\sqrt{P}[1, 1, \dots, 1]$$

- > Let $\mathbf{x} = \mathbf{x}_A$

$$\begin{aligned} \Rightarrow P_e &= Q\left(\sqrt{\frac{\|\mathbf{x}_A - \mathbf{x}_B\|^2}{4\sigma^2}}\right) \\ &= Q\left(\sqrt{\frac{\|\sqrt{P}(2, 2, 2, \dots, 2)\|^2}{4\sigma^2}}\right) \\ &= Q\left(\sqrt{\frac{\|4\sqrt{P}(1, 1, 1, \dots, 1)\|^2}{4\sigma^2}}\right) = Q\left(\sqrt{\frac{P}{\sigma^2} \sum_{i=1}^N 1}\right) = Q\left(\sqrt{\frac{NP}{\sigma^2}}\right) \\ &\approx e^{-\frac{NP}{\sigma^2}} \xrightarrow{N \rightarrow \infty} 0 \end{aligned}$$

- > but

$$R(\text{rate}) = \frac{1}{N}$$

1 Repetition coding *M*-PAM

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► *M*-PAM

$$\mathbf{x} = \sqrt{P}[q, q, \dots, q], \quad q \in \{-(M-1), \dots, -1, 1, 3, \dots, M-3, M-1\}$$

> example: $M = 6$

$$q \in \{-5, -3, -1, 1, 3, 5\}$$

► Say

$$\|\mathbf{x}\| \leq PN, \quad \forall \mathbf{x}$$

$$\Rightarrow \|\sqrt{P}\theta[5, 5, \dots, 5]\|^2 \leq NP$$

$$\|\theta[5, 5, \dots, 5]\|^2 \leq N \Rightarrow \theta = \frac{1}{M-1}$$

half distance between $\sqrt{P}\theta[1, 1, 1, \dots, 1] \rightarrow \sqrt{P}\theta[-1, -1, \dots, -1]$

$$\|\sqrt{P}\frac{1}{M-1}[1, 1, 1, \dots, 1]\|^2 = \frac{PN}{(M-1)^2}$$

1 Repetition coding M -PAM

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- ▶ Probability of error

$$\Rightarrow P_e = Q\left(\frac{\text{half-dist}}{\sigma_w}\right) = Q\left(\sqrt{\frac{PN}{(M-1)^2}}\right)$$

- ▶ Reliable communications if

$$\Rightarrow (M-1)^2 < N \Rightarrow M < \sqrt{N}$$

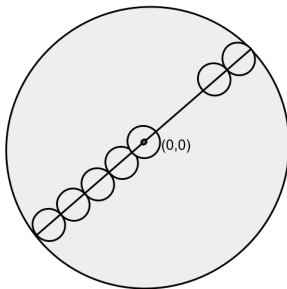
- ▶ Bound on rate

$$\Rightarrow R = \frac{\log M}{N} = \frac{\log \sqrt{N}}{N} \rightarrow 0$$

1 Problem with repetition code:

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- ▶ Repetition code packed every codeword onto single dimension
- ▶ N -dimensional sphere. ($N = 2$ here, but generally very large)



- ▶ But we have N dimensions
 - > Codewords must not be close because noise \rightarrow error.
 - > Must decide on a distance between codewords
- ▶ Try to pack as many such spheres as possible
- ▶ A sphere packing problem in N -dim space

1 Sphere packing

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- ▶ Packing sphere \rightarrow to reach capacity.
- ▶ Sphere packing problem: pack max number of codewords in sphere of certain radius

- ▶ recall:

$$\mathbf{y} = \mathbf{x} + \mathbf{w}, \quad \|\mathbf{x}\|^2 \leq NP$$

- ▶ as $N \rightarrow \infty$

$$\frac{1}{N} \sum_{i=1}^N \|w_i\|^2 \rightarrow \sigma^2 \Rightarrow \|\mathbf{w}\|^2 \rightarrow N\sigma^2 \Rightarrow \|\mathbf{y}\|^2 = \|\mathbf{x} + \mathbf{w}\|^2 \leq NP + N\sigma^2$$

- ▶ thus

$$\|\mathbf{y}\|^2 \leq N(P + \sigma^2) \Rightarrow \mathbf{y} \in \text{Ball}(\text{rad} = \sqrt{N(P + \sigma^2)}), \quad \text{with Prob} \rightarrow 1$$

- ▶ Also given \mathbf{X}_A (transmit codeword) and given $\|\mathbf{w}\| \rightarrow \sqrt{N\sigma^2}$, then
- ▶ \mathbf{y} is on the periphery of a sphere, centered on \mathbf{X}_A , with radius $\sqrt{N\sigma^2}$

1 Sphere packing

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- ▶ Try to pack max number of sphere of radius $\sqrt{N\sigma^2}$ in a bigger sphere of radius $\sqrt{N(P + \sigma^2)}$
- ▶ This max number is

$$\frac{\text{Vol}(\text{Ball}(\text{rad} = \sqrt{N(P + \sigma^2)}))}{\text{Vol}(\text{Ball}(\text{rad} = \sqrt{N\sigma^2}))} \approx \frac{(\sqrt{N(P + \sigma^2)})^N}{(\sqrt{N\sigma^2})^N} = \left(\frac{N(P + \sigma^2)}{N\sigma^2}\right)^{\frac{N}{2}}$$

- ▶ Thus max rate $R = \frac{1}{N} \log\left(\frac{N(P + \sigma^2)}{N\sigma^2}\right)^{\frac{N}{2}} = \frac{1}{2} \log\left(1 + \frac{P}{\sigma^2}\right)$
- ▶ The above is per real space. But we have complex space, thus

$$C = \log\left(1 + \frac{P}{\sigma^2}\right) \text{ bits/complex dimension}$$

- ▶ Cap gives the challenge.
- ▶ Codes : another Story
- ▶ Decoders : another story.
- ▶ People relatively happy with various practical codes
 - > turbo codes (interesting story)
 - > LPPC codes (also interesting story)
- ▶ Commonly used soft decoders (complicated).

1 Capacity-based resource allocation

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- ▶ Capacity analysis helps us allocate resources
- ▶ Total Power \bar{P} (Watts)
- ▶ Total bandwidth W (Hz)
- ▶ Thus $\frac{\bar{P}}{W}$ ($J/s/Hz$) = ($Watts/Hz$)
- ▶ Noise: $\sigma_w^2 = \frac{N_0}{2}$
- ▶ Sample at rate $\frac{1}{W}$ gives

$$y[m] = x[m] + w[x]$$

- > Complex baseband representation (two independent uses of the real channel)
- ▶ $\frac{N_0}{2}$ noise power per real symbol $\Rightarrow N_0$ power of $w[x]$
- ▶ $\frac{\bar{P}}{2W}$ power constraint per real symbol.

1 Capacity-based resource allocation

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$$\blacktriangleright C_{real} = \frac{1}{2} \log\left(1 + \frac{P_{real}}{\sigma_{real}^2}\right) = \frac{1}{2} \log\left(1 + \frac{\bar{P}}{\frac{w}{2}N_0}\right) = \frac{1}{2} \log\left(1 + \frac{\bar{P}}{wN_0}\right)$$

$$\Rightarrow C_{AWGN} = 2C_{real} = \log\left(1 + \frac{\bar{P}}{wN_0}\right)$$

$$\blacktriangleright \text{Consider } SNR = \frac{\bar{P}}{wN_0} \quad (\text{note: SNR wrt complex dimension})$$

$$\Rightarrow C_{complex} = \log(1 + SNR) \quad (\text{bits/complex symbol})$$

$\blacktriangleright C_{AWGN}$: called spectral efficiency.

\blacktriangleright Total capacity

$$C_{AWGN}(\bar{P}, W) = WC_{complex} = W \log(1 + SNR) = W \log\left(1 + \frac{\bar{P}}{WN_0}\right) \text{ bits/sec}$$

1 Capacity-based resource allocation

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► Recall

$$C(\bar{P}, w) = W \log\left(1 + \frac{\bar{P}}{WN_0}\right)$$

- > Choose (invest in) \bar{P} and/or W

► Note:

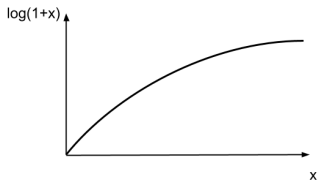
- > \log is concave, i.e., $F''(\text{SNR}) \leq 0$

>

$$\log(1+x) \approx x \log_2 e, \text{ as } x \rightarrow 0, \quad (\text{linear})$$

>

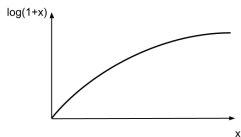
$$\log(1+x) \rightarrow \log(x), \text{ as } x \rightarrow \infty, \quad (\text{logarithmic})$$



1 Power limited regime

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- ▶ low SNR: $\bar{P} \uparrow \Rightarrow C \uparrow$ linearly \Rightarrow invest in \bar{P}
- ▶ high SNR: $\bar{P} \uparrow \Rightarrow C \uparrow$ logarithmically \Rightarrow do NOT invest in \bar{P}



1 Bandwidth limited regime

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- ▶ Fix \bar{P}
- ▶ C increasing function of W
- ▶ if W small \Rightarrow SNR large $\Rightarrow W \uparrow \Rightarrow \log(\cdot) \downarrow$ a little

$\Rightarrow W \log(\cdot) \uparrow$ almost linearly \Rightarrow invest in W (bandwidth limited)

- ▶ if W large \Rightarrow SNR small

$\Rightarrow \log(1+x) \downarrow$ linearly \Rightarrow compensates for $W \uparrow$ linearly

$$W \log\left(1 + \frac{\bar{P}}{N_0 W}\right) \approx W \log_2 e \frac{\bar{P}}{N_0 W} = \log_2 e \frac{\bar{P}}{N_0}$$

- > do not invest in W , but invest in \bar{P} (linear increase) (power limited)

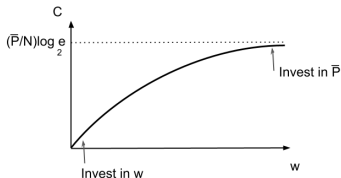
1 Capacity-based resource allocation

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- Minimize $\frac{E_b}{N_0}$: Power efficient region $\bar{P} \rightarrow 0$ (turns out)

$$\begin{aligned}\left(\frac{E_b}{N_0}\right)_{\min} &= \lim_{\bar{P} \rightarrow 0} \frac{\bar{P}}{N_0 C_{AWGN}(\bar{P}, W)} = \frac{\bar{P}}{N_0 W \log_2(1 + \frac{\bar{P}}{N_0 W})} \\ &\approx \frac{\bar{P}}{N_0 W \log_2 e \frac{\bar{P}}{N_0 W}} = \frac{1}{\log_2 e} \\ &\Rightarrow 10 \log_{10} \frac{1}{\log_2 e} = -1.59 \text{ dB}\end{aligned}$$

- > Minimum possible power to send a single bit
- > Orthogonal codes from Hadamard sequences



Capacity of LTI Gaussian channels

1 Capacity of Single-Input Multiple-Output channels

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- ▶ The SIMO channel (L -receive antennas)

$$y_\ell[m] = h_\ell x[m] + w_\ell[m], \quad \ell = 1, \dots, L, \quad w_\ell \sim \mathbb{CN}(0, N_0)$$

- > Channel vector $\mathbf{h} = [h_1, \dots, h_L]^T$ is fixed

- ▶ Capacity achieving policy: receive beamforming

$$\tilde{y}[m] = \frac{\mathbf{h}^\dagger}{\|\mathbf{h}\|} \mathbf{y}[m] = \frac{\|\mathbf{h}^\dagger\|^2}{\|\mathbf{h}\|} x[m] + \underbrace{\frac{\mathbf{h}^\dagger}{\|\mathbf{h}\|} \mathbf{w}[m]}_{\text{AWGN } \sigma^2=1}$$

- > $x[m]$ from a capacity achieving AWGN code
 - > Optimal since $\tilde{y}[m]$ is sufficient statistic

- ▶ Thus

$$SNR = \|\mathbf{h}^\dagger\|^2 \frac{P}{N_0}, \quad E[\|x[m]\|^2] \leq P \quad (\text{Watts/symbol})$$

- ▶ Thus capacity takes form

$$C_{\text{SIMO}} = \log\left(1 + \frac{P}{N_0} \|\mathbf{h}\|^2\right)$$

1 Capacity of Multiple Input Single Output channels

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- ▶ The MISO channel (L -transmit antennas)

$$y[m] = \mathbf{h}^T \mathbf{x}[m] + w[m], \quad w[m] \sim \mathbb{CN}(0, N_0), \quad \mathbf{h} = [h_1, \dots, h_L]^T \text{ fixed}$$

- ▶ Capacity achieving policy: transmit beamforming optimal (proof skipped)

$$\mathbf{x}[m] = \frac{\mathbf{h}^T}{\|\mathbf{h}\|} \tilde{x}[m]$$

> $\tilde{x}[m]$ from opt. AWGN code. $E[\|\tilde{x}[m]\|^2] \leq P$

- ▶ Thus

$$y[m] = \|\mathbf{h}\| \tilde{x}[m] + w[m]$$

- ▶ Thus MISO capacity is

$$C_{tx-beam} = C_{MISO} = \log\left(1 + \frac{P\|\mathbf{h}\|^2}{N_0}\right) \text{ bits/s/Hz}$$

1 Frequency Selective (and parallel) channels

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$$y[m] = \sum_{\ell=0}^{L-1} h_{\ell} x[m - \ell] + w[m], \quad E[|x[m]|^2] \leq P \quad (1)$$

> L taps (convolution)

▶ Solution: Convert to N_c parallel channels (N_c info symbols + L taps)

▶ i th block

$$\tilde{y}_n[i] = \tilde{h}_n \tilde{d}_n[i] + \tilde{w}_n[i] \quad n = 0 \rightarrow N_c - 1 \quad (2)$$

$\tilde{\mathbf{d}}[i] = [\tilde{d}_0[i] \rightarrow \tilde{d}_{N_c-1}[i]]$, DFT of input of i th block, $\|\tilde{\mathbf{d}}[i]\|^2 \leq N_c P$ (Parseval)

$\tilde{\mathbf{w}}[i] = [\tilde{w}_0[i] \rightarrow \tilde{w}_{N_c-1}[i]]$, DFT of noise: $\tilde{\mathbf{w}}[i] \sim \mathbb{CN}(0, N_0 I)$

$\tilde{\mathbf{y}}[i] = [\tilde{y}_0[i] \rightarrow \tilde{y}_{N_c-1}[i]]$, DFT of output of i th block (after some removal)

$\tilde{\mathbf{h}}_{1 \times N_c} = [\tilde{h}_0 \rightarrow \tilde{h}_{N_c-1}]^T = \mathbf{F}_{\text{fourier}} \mathbf{h}$, (N_c - point DFT)

▶ As $N_c \rightarrow \infty \Rightarrow N_c \gg L$

$$\Rightarrow C_{(1)} \approx C_{(2)}$$

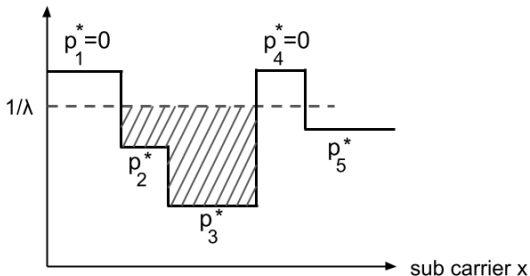
1 Frequency Selective (and parallel) channels

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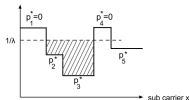
- Power allocation, then use capacity-achieving code over each parallel channel

$$C_{N_c} = \max_{P_0 \rightarrow P_{N_c-1}} \sum_{n=0}^{N_c-1} \log\left(1 + \frac{P_n |\tilde{h}_n|^2}{N_0}\right), \quad s.t. \quad \sum_{n=0}^{N_c-1} P_n = N_c P, \quad P_n \geq 0, \forall n$$

- Optimal $\{P_n\}$ via water-filling (y-axis is $\frac{N_0}{|\tilde{h}_n|^2}$)



1 Freq. Selective/Parallel channels: Lagrangian optimization | 23



► Write Lagrangian $\mathcal{L}(\lambda) = \sum_{n=0}^{N_c-1} \log(1 + \frac{P_n |\tilde{h}_n|^2}{N_0}) - \lambda \sum_{n=0}^{N_c-1} P_n$

► Then evaluate $\frac{\delta \mathcal{L}}{\delta P_n} = \begin{cases} 0 & \text{if } P_n > 0 \\ \leq 0 & \text{if } P_n = 0 \end{cases}$

► Optimal power

$$P_n^* = \left(\frac{1}{\lambda} - \frac{N_0}{|\tilde{h}_n|^2} \right)^+, \quad \text{where } \lambda : \sum_{n=0}^{N_c-1} \overbrace{\left(\frac{1}{\lambda} - \frac{N_0}{|\tilde{h}_n|^2} \right)^+}^{P_n^*} = P N_c$$

► Coding across parallel channels does not improve capacity (turns out)
 > but gives better performance if block length is small

- ① Wireless Channels (AWGN and LTI)
- ② Fading channels (epsilon, CSIT, freqSelec)
Capacity of fading channels
- ③ Multi-user Capacity

Capacities of Fading Channels

2 Capacity of fading channels

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- ▶ Fading channel

$$y[m] = h[m]x[m] + w[m], \quad w[m] \sim \mathbb{CN}(0, N_0), \quad SNR = \frac{P}{N_0}$$

- ▶ h randomly drawn, with $E[|h[m]|^2] = 1$
- ▶ For now only CSIR (No CSIT)
- ▶ Slow fading "channel randomly drawn but here to stay"

$$\Rightarrow y[m] = hx[m] + w[m], \Rightarrow SNR_{Rx} = |h|^2 SNR$$

$$C(h) = \log(1 + SNR|h|^2)$$

- ▶ No lower-limit ($C(h) \rightarrow 0$ maybe) (NO CSIT)
- ▶ Consider outage event

2 Capacity of fading channels

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- ...Consider outage event

$$\text{Error} \iff \text{Capacity}(h) < \text{Rate}$$

- Need another measure: $P(C(h) < R) = P(\text{outage})$

- > applies when R is fixed

$$P_{out}(R) = P(\log(1 + SNR|h|^2) < R) \Rightarrow P(|h|^2 < \frac{2^R - 1}{SNR})$$

- > Note

$$\begin{aligned} P(|h|^2 < \epsilon) &\xrightarrow{\epsilon \rightarrow 0} \epsilon \\ \Rightarrow P_{out}(R) &= P(|h|^2 < \frac{2^R - 1}{SNR}) \xrightarrow{SNR \rightarrow \infty} \frac{2^R - 1}{SNR} \\ &\Rightarrow P_{out}(R) \rightarrow \frac{1}{SNR} \end{aligned}$$

2 ϵ -outage capacity (quasi-static channel)

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- ▶ Another measure of interest - ϵ -outage capacity

$$y = hx + w$$

- ▶ Recall probability of outage

$$P_{out}(R) = P(\log(1 + \rho|h|^2) < R) = P(|h|^2 < \overbrace{\frac{2^R - 1}{\rho}}^x) = \epsilon$$

- ▶ Definition: ϵ -outage capacity

$$C_\epsilon = \max_R : P_{out}(R) \leq \epsilon$$

2 ϵ -outage capacity (quasi-static channel)

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- Evaluate C_ϵ for any SNR

$$\text{Let } x \triangleq \frac{2^R - 1}{\rho} \Rightarrow P(|h|^2 < x) = \epsilon \Rightarrow P(|h|^2 \geq x) = 1 - \epsilon$$

- and denote

$$F(x) \triangleq P(|h|^2 > x) \Rightarrow F(x) = 1 - \epsilon \Rightarrow x = F^{-1}(1 - \epsilon)$$

$$\Rightarrow \frac{2^R - 1}{\rho} = F^{-1}(1 - \epsilon) \Rightarrow 2^R = 1 + \rho F^{-1}(1 - \epsilon)$$

- Thus

$$C_\epsilon = \log(1 + \text{SNR} \cdot F^{-1}(1 - \epsilon))$$

2 ϵ -outage capacity (high SNR)

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- ▶ We have

$$C_\epsilon = \log(1 + SNR \cdot F^{-1}(1 - \epsilon))$$

- ▶ At high SNR

$$C_\epsilon \approx \log(SNR \cdot F^{-1}(1 - \epsilon)) = \log SNR + \log(F^{-1}(1 - \epsilon))$$

- ▶ Thus

$$C_\epsilon = \log SNR - \overbrace{\log\left(\frac{1}{F^{-1}(1 - \epsilon)}\right)}^{\text{constant offset}}$$

2 ϵ -outage capacity (Low SNR)

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- ▶ Low SNR

$$C_\epsilon \approx \log_2 e \text{SNR} \cdot F^{-1}(1 - \epsilon) \approx F^{-1}(1 - \epsilon) C_{\text{AWGN}}$$

- ▶ For Rayleigh fading

$$P(|h|^2 < \epsilon) \approx \epsilon \Rightarrow P(|h|^2 > \epsilon) \approx 1 - \epsilon \Rightarrow F(\epsilon) \approx 1 - \epsilon \Rightarrow F^{-1}(1 - \epsilon) \approx \epsilon$$

$$C_\epsilon \approx \epsilon \text{SNR} \approx \epsilon C_{\text{AWGN}}$$

- ▶ Big problem: Example

$$\epsilon \approx 1\% \rightarrow C_\epsilon \approx \frac{C_{\text{AWGN}}}{100}$$

- ▶ Thus diversity is important

2 Diversity for boosting ϵ -outage capacity

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- Consider SIMO (R_x - diversity) to boost C_ϵ at low SNR

$$P_{out}(R) = P(\log(1 + \rho|\mathbf{h}|^2) < R) \xrightarrow{\rho \rightarrow 0} P(\log_2 e \cdot SNR \cdot |\mathbf{h}|^2 < R)$$

$$\Rightarrow P_{out}(R) \approx P\left(|\mathbf{h}|^2 < \overbrace{\frac{R}{SNR \cdot \log_2 e}}^{\text{small: } R \ll \rho} \right) \stackrel{\text{want}}{=} \epsilon$$

$$P_{out}(R) \approx P(|h_0|^2 < \frac{R}{SNR}, \dots, |h_{L-1}|^2 < \frac{R}{SNR}) = \left(P(|h_0|^2 < \frac{R}{SNR})\right)^L = \epsilon$$

$$\Rightarrow P(|h_0|^2 < \frac{R}{SNR}) = \epsilon^{\frac{1}{L}} =: \delta$$

$$\Rightarrow \frac{R}{SNR} = \delta \quad (\text{since } P(|h_0|^2 < \delta) \approx \delta)$$

$$\frac{R}{SNR} = \epsilon^{\frac{1}{L}} \Rightarrow R = \epsilon^{\frac{1}{L}} SNR \Rightarrow C_\epsilon = \epsilon^{\frac{1}{L}} C_{AWGN}$$

2 Capacity of Time and Frequency selective (parallel) channels | 33

$$y_0[m] = h_0 x_0[m] + w_0[m]$$

$$\vdots$$

$$y_{L-1}[m] = h_{L-1} x_{L-1}[m] + w_{L-1}[m]$$

$$C(\mathbf{h}) = \sum_{\ell=0}^{L-1} \log(1 + |h_\ell|^2 SNR) \rightarrow 0$$

$$P_{out}(R) = P\left(\frac{1}{L} \sum_{\ell=0}^{L-1} \log(1 + |h_\ell|^2 SNR) < R\right)$$

- Fast Fading: $T_{code} \gg T_c \Rightarrow L \rightarrow \infty$

$$\frac{1}{L} \sum_{\ell=0}^{L-1} \log(1 + |h_\ell|^2 SNR) \rightarrow E_h[\log(1 + |h|^2 SNR)]$$

- Fast-fading (ergodic) capacity

$$C_{FF} = E_h\{\log(1 + |h|^2 SNR)\}$$

2 Compare C_{FF} with C_{AWGN}

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- ▶ Employ Jensen's inequality $E\{f(u)\} \leq f(E\{u\})$

$$E_h\{\log(1 + \rho|h|^2)\} = \log(1 + \rho E\{|h|^2\}) \Rightarrow C_{FF} \leq C_{AWGN}$$

- ▶ But generally very close
- ▶ Low SNR:

$$C_{FF} = E\{\log(1 + \rho|h|^2)\} \approx E\{\log_2 e \rho |h|^2\} = \log_2 e \rho \overbrace{E\{|h|^2\}}^1 = C_{AWGN}$$

- ▶ High SNR:

$$C_{FF} \approx E\{\log(\rho|h|^2)\} = \log \rho + E\{\log |h|^2\} \approx C_{AWGN} - \overbrace{0.83}^{2.5\text{dB}}$$

- ▶ TDD reciprocity, FDD feedback (expensive)
- ▶ Slow fading: could do channel inversion, or rate adaptation i.e

$$\log\left(1 + \frac{P_t}{N_0}|h|^2\right) < R_t$$

> \Rightarrow change R_t or P_t (When bad channel)

► Fast Fading: Waterfilling

$$y_0 = h_0 x_0 + w_0$$

$$y_1 = h_1 x_1 + w_1$$

$$\vdots$$

$$y_{L-1} = h_{L-1} x_{L-1} + w_{L-1}$$

$$\Rightarrow \max_{P_0 \rightarrow P_{L-1}} \frac{1}{L} \sum_{\ell=0}^{L-1} \log\left(1 + \frac{P_\ell}{N_0} |h_\ell|^2\right), \quad \sum P_\ell = LP$$

► Seen

$$P_\ell^* = \left(\frac{1}{\lambda} - \frac{N_0}{|h_\ell|^2}\right)^+, \quad \text{for } \lambda : \frac{1}{L} \sum_{\ell=0}^{L-1} \left(\frac{1}{\lambda} - \frac{N_0}{|h_\ell|^2}\right)^+ = P$$

2 Capacity with CSIT - waterfilling

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- Causality an issue, BUT for large L

$$\begin{aligned}\frac{1}{L} \sum_{\ell=0}^{L-1} \left(\frac{1}{\lambda} - \frac{N_0}{|h_\ell|^2} \right)^+ &\rightarrow E \left\{ \left(\frac{1}{\lambda} - \frac{N_0}{|h|^2} \right)^+ \right\} \\ \Rightarrow P_\ell^*(\mathbf{h}) &= \left(\frac{1}{\lambda} - \frac{N_0}{|h_\ell|^2} \right)^+ \text{ for } \lambda : E \left\{ \left(\frac{1}{\lambda} - \frac{N_0}{|h|^2} \right)^+ \right\} = P \\ \Rightarrow C &= E \left\{ \log \left(1 + P^*(h) \frac{|h|^2}{N_0} \right) \right\}\end{aligned}$$

- In the end: For most SNR

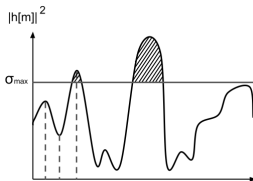
$$C_{AWGN} \approx C_{FF} \approx C_{CSIT}$$

- except (in theory) at low SNR where $C_{CSIT} \gg C_{AWGN}$ WARNING!!

2 Capacity with CSIT - low SNR

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- Send only if $|h[m]|^2 \approx G_{\max}$, and store power while not sending



$$C \approx P \left(|h|^2 \approx G_{\max} \right) \log \left(1 + \frac{G_{\max} \text{SNR}}{P(|h|^2 \approx G_{\max})} \right) \approx G_{\max} \rho \log_2 e \approx G_{\max} C_{\text{AWGN}}$$

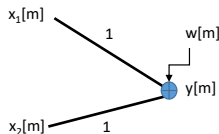
- Would not work in practice!!

- ① Wireless Channels (AWGN and LTI)
- ② Fading channels (epsilon, CSIT, freqSelec)
- ③ Multi-user Capacity

3 Multi-user Capacity – Uplink

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- ▶ Up-link AWGN (multiple-access channel)



$$y[m] = x_1[m] + x_2[m] + w[m]$$

- > $E\{|x_k[m]|^2\} = P_k, k = 1, 2, \quad w[m] \sim \mathbb{CN}(0, N_0)$ iid

- ▶ Capacity region: Best possible R_1, R_2 . First derive outer bounds:

$$R_1 < \log\left(1 + \frac{P_1}{N_0}\right)$$

$$R_2 < \log\left(1 + \frac{P_2}{N_0}\right) \quad \text{single user bounds}$$

$$R_1 + R_2 < \log\left(1 + \frac{P_1 + P_2}{N_0}\right), \quad \text{MISO bound with fixed } h_1 = h_2 = 1$$

- > $E\{|\underline{x}|^2\} = P_1 + P_2$, where $\underline{x} = [x_1, x_2]^T$.

3 Multi-user Capacity – Uplink

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- ▶ Let us achieve these outer bounds.
- ▶ Use capacity-achieving codes for $x_1[m]_{m=1}^{\dots}$, $x_2[m]_{m=1}^{\dots}$
- ▶ First R_x decodes message of user 2 (x_2) by treating x_1 as noise:

$$\Rightarrow R_2 = \log\left(1 + \frac{P_2}{P_1 + N_0}\right)$$

- ▶ Now x_2 removed from y to decode x_1 without interference

$$\Rightarrow R_1 = \log\left(1 + \frac{P_1}{N_0}\right) \text{ naturally optimal}$$

- ▶ but note that

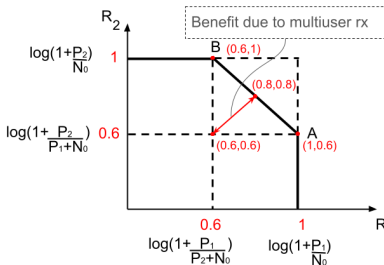
$$R_1 + R_2 = \log\left[1 + \frac{P_1}{N_0}\right] + \log\left[1 + \frac{P_2}{P_1 + N_0}\right] = \log\left[1 + \frac{P_1 + P_2}{N_0}\right]$$

is again optimal

- ▶ but if R_1 is optimal & $R_1 + R_2$ is optimal then R_2 is also optimal.

3 Multi-user Capacity – Uplink

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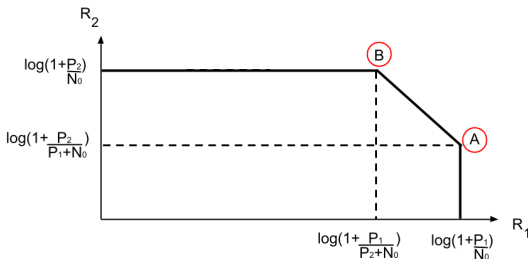


- ▶ To achieve point $A = [\log(1 + \frac{P_1}{N_0}), \log(1 + \frac{P_2}{P_1+N_0})]$
 - > decode x_2 first by treating x_1 as noise $\log(1 + \frac{P_2}{P_1+N_0})$
 - > Remove x_2 and decode x_1 without interference
- ▶ Similarly (reverse order) for point $B = [\log(1 + \frac{P_1}{P_2+N_0}), \log(1 + \frac{P_2}{N_0})]$
 - > decode x_1 first by treating x_2 as noise $\log(1 + \frac{P_1}{P_2+N_0})$
 - > Remove x_1 and decode x_2 without interference
- ▶ Line AB contains all optimal points (optimal sum-rate)
- ▶ Example $P_1 = P_2 = 1$

3 Non-Symmetric Multiple Access

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- ▶ Considering the non-symmetric setting, say $P_1 \gg P_2$.



- ▶ Whom should we decode first?
 - > decode strongest first (small interference from weaker)
 - > then fully decode weaker (point B)
 - > Slightly bother U1, no interference for U2 (Reverse would kill U2)
- ▶ Strong users close to BS transmit fast, with no trouble to rest
 - > No power control. Turns near-far issue into advantage.

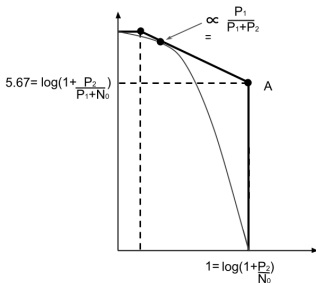
3 Orthogonal Multiple Access

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- Orthogonal Multiple access techniques Optimal? ($\alpha, 1 - \alpha$)

$$R_1 = \alpha W \log\left(1 + \frac{P_1}{\alpha N_0}\right), \quad R_2 = (1 - \alpha) W \log\left(1 + \frac{P_2}{(1 - \alpha) N_0}\right) \text{ bits/s}$$

- \exists only one optimal α : $\alpha = \frac{P_1}{P_1 + P_2}$ that achieves opt sum-capacity
 - > Penalizes weak users ($P_1 \gg P_2 \Rightarrow \alpha \approx 1, \Rightarrow 1 - \alpha \approx 0$)
 - > Logarithmic increase vs. linear decrease
 - > Giving high rate to weak user \Rightarrow entirely sacrifice rate for strong user
 - > Example $P_2 = 0\text{dB}, P_1 = 20\text{dB}$.



3 K-user Uplink Capacity

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- ▶ for $P_i = P_j = P$

$$C_{sum} = \log\left(1 + \frac{kP}{N_0}\right)$$

$$C_{sym} = \frac{1}{k} \log\left(1 + \frac{kP}{N_0}\right) \text{ max rate when all have the same rate}$$

- ▶ achievable with ?? (say $k=2$, $\alpha = \frac{1}{2}$, $\alpha \log(1 + \frac{P}{\alpha N_0})$, $(1 - \alpha) \log(1 + \frac{P}{(1-\alpha)N_0})$) under equal received power
- ▶ OFDM ∇ CDMA, CDMA treats all others as noise

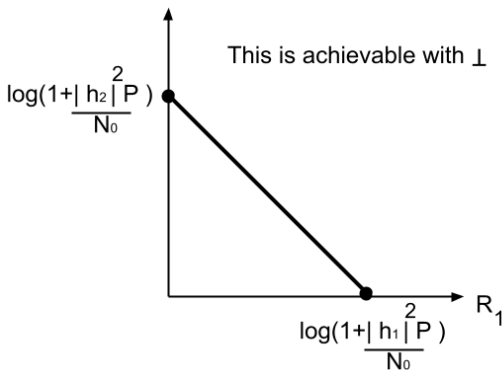
$$C_{sum} = k \log\left(1 + \frac{P}{(k-1)P + N_0}\right) \approx k \frac{P}{(k-1)P + N_0} \log_2 e \approx 1 / \log_2 e \approx 1.44$$

$$\log\left(1 + \frac{kP}{N_0}\right) \uparrow \infty$$

3 Downlink AWGN with fixed fading

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- ▶ $y_k[m] = h_k x[m] + w_k[m] \quad k = 1, 2, \dots, k \text{ fixed}$
- ▶ assume CSTIR $|X[m]|^2 \leq P \quad \frac{\text{Joules}}{\text{symbol}}$
- ▶ Capacity region R_1, R_2
- ▶ $R_k \leq \log(1 + \frac{P|h_k|^2}{N_0})$ all power for user K and rate



$$X[m] = X_1[m] + X_2[m] \quad \text{iid Gaussian codes}$$

$$R_1 = \log\left(1 + \frac{P_1|h_1|^2}{P_2|h_1|^2 + N_0}\right) = \log\left(1 + \frac{(P_1 + P_2)|h_1|^2}{N_0}\right) - \log(1 +)$$

$$R_2 = \log\left(1 + \frac{P_2|h_2|^2}{N_0}\right)$$

- ▶ User 2 can decode anything

$$x = x_1 + x_2$$

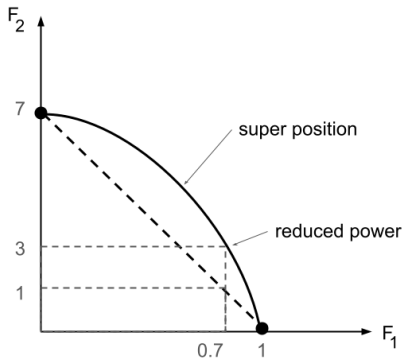
$$R_1 = \log\left(1 + \frac{P_1|h_1|^2}{|h_1|^2 P_2 + N_0}\right) \text{ weaker sees interference}$$

$$R_2 = \log\left(1 + \frac{P_2|h_2|^2}{N_0}\right) \text{ stronger user sees no interference}$$

- ▶ first, decode user 1 data and then Which is generally better than ??
(for all power splits $P_1 + P_2 = P$)

$$R_1 = \alpha \log\left(1 + \frac{P_1|h_1|^2}{\alpha N_0}\right)$$

$$R_2 = (1 - \alpha) \log\left(1 + \frac{P_2|h_2|^2}{(1 - \alpha)N_0}\right)$$



► In General

$$R_k = \log\left(1 + \frac{P_k |h_k|^2}{N_0 + \sum_{j=k+1}^K P_j |h_j|^2}\right) \quad P = \sum_{k=1}^K p_k$$

► sum capacity here (unlike uplink which all user R_x)