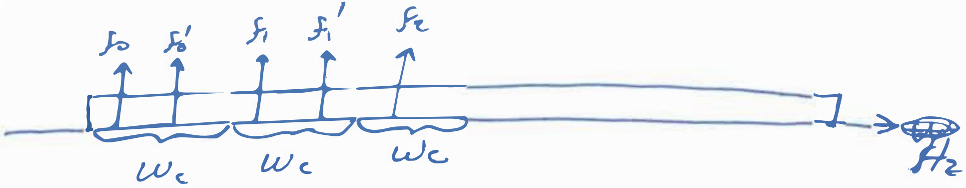
Frequency diversity. + ISI + Single Carrier + DSSS.

when:? $W > W_c$

Recall:



gain diversity on bit b_0 by sending it over different frequencies spaced $\approx W_c$.

$\Rightarrow \frac{W}{W_c} \approx$ diversity.

- Another point of view.

$$W_c \approx \frac{1}{T_d} \quad \& \quad \tau \approx \frac{1}{W} \Rightarrow \frac{W}{W_c} \approx L \quad \left(\begin{matrix}\#\text{ of channel}\\ \text{taps}\end{matrix}\right).$$

Now channel not $y = h\,x + w$ (not flat fading).

but $\underline{y} = \underline{h} \cdot \circledast\, \underline{x} + \underline{w}$ (wideband channel).

$\underline{h} = [h_0 \; h_1 \; \cdots h_{L-1}]_{1 \times L}$

$\underline{x} = [x_0 \; x_1 \; \cdots x_{n-1}]_{1 \times n}$

$\underline{y} = [y_0 \; y_1 \; \cdots y_{L+n-2}]_{1 \times (n+L-1)}$.

$$y[m] = \sum_{\ell=0}^{L-1} h_\ell[m] \cdot x[m-\ell] + w[m] \qquad m = 0 \rightarrow n+L-1.$$

$h_\ell \sim \mathcal{CN}(0, \frac{1}{L})$

but if $T_c \gg T_d \Rightarrow$

$$y[m] = \sum_{\ell=0}^{L-1} h_\ell \cdot x[m-\ell] + w[m]$$

Wideband channel: <u>ISI.</u>

Problem is isi

To increase rate could send. $\underline{x} = \{x_0\ x_1\ x_2 \cdots x_{n-1}\}$

but then $y[m] = \sum_{\ell=0}^{L-1} h_\ell \cdot x[m-\ell]$

e.g

$L=2$ $\{y_0\ y_1\ y_2\ y_3\} = [h_0 x_0 \quad \underbrace{h_0 x_1 + h_1 x_0}_{ISI} \quad \underbrace{h_0 x_2 + h_1 x_1}_{ISI} \quad h_1 x_2\}$

$n=3$ $\underline{x} = \{x_0\ x_1\ x_2\}$

how to detangle signals ??

- 3 main methods.

1). Single Carrier systems with equalization.
   (heavy decoding). (GSM). Complexity ↑, optimal decoding

2) Direct sequence spread spectrum.
   Low rate, ISI ↓ ⇒ (CDMA).

3) Multi career systems IEEE 802.11a (OFDM).
   - Combines advantage from both worlds.
   - encode across diff freqs.
   - diversity ✓.

Single Carrier with ISI equalization

$\underline{y} = \underline{h} * \underline{x} + \underline{w}$. $\quad \underline{h} = [h_0 \rightarrow h_{L-1}] \quad \underline{x} = [x_0 \; x_1 \; \cdots \; x_{n-1}]$.

$$y[m] = \sum_{\ell=0}^{L-1} h_\ell \, x[m-\ell] \qquad m = 0 \rightarrow n+L-2 \qquad T_c \gg T_d \; (\text{e.g. } \ldots)$$

$$h_\ell \sim \mathcal{CN}(0, \tfrac{1}{L}).$$

Task (decode) : ("equalization").

Problem ISI.

To decode, take MISO point of view

$$\underline{y}^T = \{y_0\ y_1 \cdots y_{n+L-2}\} =$$

$$\underline{y} = (h_0\ h_1 \cdots h_{L-1}) \begin{bmatrix} x_0 & x_1 & x_2 & \cdots & x_{n-1} & x_n & \cdot & & & \\ & x_0 & x_1 & x_2 & & \vdots & x_n & \cdot & \cdot & \\ & & x_0 & x_1 & & \vdots & & x_n & \cdot & \\ & 0 & & x_0 & & \vdots & & & x_n & \\ & L \text{ times} & & & \ddots & \vdots & & & & \ddots \\ & 0 & & & & \vdots & & & & \\ & & & & x_0\ x_1 & \cdots & x_{n-L} & & & x_n \end{bmatrix} +$$

$$\underbrace{\qquad}_{L} \qquad \underbrace{\qquad\qquad\qquad\qquad}_{n} \qquad \underbrace{\qquad\qquad}_{L-1} \qquad L \times (n+L$$

e.g. $n=5$ $L=3$

Consider $\{y_0\ y_1\ y_2\ y_3\ y_4\ y_5\ y_6\} = \{h_0\ h_1\ h_2\} \begin{bmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & * & * \\ 0 & x_0 & x_1 & x_2 & x_3 & x_4 & * \\ 0 & 0 & x_0 & x_1 & x_2 & x_3 & x_4 \end{bmatrix}$

Q: guess rank of code.

$\Rightarrow$ this is MISO problem with $L=3$ & code of above structure.

(Q: guess the rate).

From before.

$$P(\underline{x_A} \to \underline{x_B}) \le \prod_{\ell=0}^{L-1} \frac{1}{1 + \frac{e \cdot \lambda_\ell^2}{4}}$$

where λ_ℓ^2 of $\frac{(x_A - x_B)}{\sqrt{e}}$

$\to e^{-L}$ since min rank $\Delta x = L$

since worse is $\begin{pmatrix} x_0 & & 0 & 0 \\ & x_0 & & \\ 0 & & x_0 & \end{pmatrix}$ rank $= L \Rightarrow$ full div.

OFDM

Frequency division:

$$y(t) = x(t) * h(t) \quad (LTI) \ (\text{Continuous time}).$$

$~~y = x * h~~$

$$\cancel{Y(\omega) =} \ FT(y) = FT(x) \ FT(h).$$

$$Y(\omega) = X(\omega) \cdot H(\omega).$$

Review:

$$f(t) \longrightarrow F(\omega)$$

$$f(t-t_0) \longrightarrow F(\omega) \cdot e^{-j\omega \cdot t_0}$$

$$e^{j\omega_0 \cdot t} \longrightarrow 2\pi \, \delta(\omega - \omega_0).$$

$$\Rightarrow \quad V \cdot \frac{1}{2\pi} e^{j\omega_0 t} \xleftarrow{IFT} V \cdot \delta(\omega - \omega_0).$$

- Consider real-time continous signals. (∞ time duration).
- want to send 'v'.
- Modulate v with carrier @ frequency ω_0.

$$x(t) = v \cdot \frac{1}{2\pi} e^{j\omega_0 . t} = FT^{-1}\left(v \cdot \delta(\omega - \omega_0)\right).$$

$$y(t) = h(t) * x(t) = h(t) * v \cdot \frac{1}{2\pi} e^{j2\pi\omega_0} + z$$

$$\Rightarrow Y(\omega) = H(\omega) \cdot X(\omega) + \tilde{z}$$

$$= H(\omega) \cdot v \cdot \delta(\omega - \omega_0) + \tilde{z}$$

$\Rightarrow$ calc $FT(y)$ @ ω_0 (calc $Y(\omega_0)$

$$\Rightarrow Y(\omega_0) = H(\omega_0) \cdot v + \tilde{z} \Rightarrow v \approx \left[\frac{Y(\omega_0)}{H(\omega_0)}\right]$$

Similarly with infinite sequences x (DTFT).

$$x[n] = DTFT^{-1}\left(v \cdot \delta(\omega - \omega_0)\right) = \frac{1}{2\pi} v \cdot e^{jn \cdot \omega_0}$$

$$Y(\omega) = H(\omega)\,\delta(\omega - \omega_0) \cdot v \quad \Rightarrow \quad Y(\omega_0) = H(\omega_0) \cdot v$$

$$\Rightarrow \tilde{v} \approx \left[\frac{Y(\omega_0)}{H(\omega_0)}\right]$$

But note: Finite T & discrete.

$\Rightarrow$ duality now between

$\text{DFT}\big|_{N\text{-point}}$ & $\circledast_{N_c}$ (circular convolution).

$$\underline{x} \circledast_N \underline{h} \longleftrightarrow \left[\text{DFT}_{N_c}(\underline{x})\right]_{k\text{-th elem}} \cdot \left[\text{DFT}_{N_c}(\underline{h})\right]_{k\text{-th element}}.$$

OFDM - need cyclic prefix.

want to send $\underline{\tilde{d}} = [\tilde{d}_0\ \tilde{d}_1\ \dots\ \tilde{d}_{N_c-1}]^T$.

shift to N_c-IDFT $\quad \text{IDFT}(\underline{\tilde{d}}) =: \underline{d} = [d_0 \dots d_{N_c-1}]^T$.

create

$$\underline{x} = [\underbrace{\underset{x[1]}{d_{N_c-L+1}},\ \underset{x[2]}{d_{N_c-L}},\ \dots,\ d_{N_c-2},\ \underset{x[L-1]}{d_{N_c-1}}}_{L-1 \text{ length cyclic prefix.}},\ \underset{x[L]}{d_0}\ \underset{x[L+1]}{d_1}\ d_2\ \dots\ \underset{x[N+L-1]}{d_{N_c-1}}]^T, \ 1 \times (N+L-1).$$

- to be sent over channel of L-taps $\underline{h} = \{h_0\ h_1 \dots h_{L-1}\}$.

$\rightarrow$ send $\underline{x}$ First consider only $N+L-1$ first observ.

$$y[m] = \sum_{\ell=0}^{L-1} h_\ell \cdot x[m-\ell] + w[m] \qquad m = 1, 2, \ldots, N+L-1.$$

& then ignore observations $\{y[1]\ y[2] \ldots, y[L-1]\}$.

$\rightarrow$ left with observations.

$$\underbrace{y[L]\quad y[L+1] \ldots y[N+L-1]}_{N}$$

for these, we have

$$y[m] = \sum_{\ell=0}^{L-1} h_\ell \, d_{[(m-L-\ell) \bmod N_c]} + w[m].$$

Recap.

$\Rightarrow$ for $\quad \underline{y} = [\, y[L] \;\; y[L+1] \; \ldots \; y[L+N_c-1] \,]^T_{1\times N_c}$

$\underline{h} = [\, h_0 \;\; h_1 \ldots h_{L-1} \,] \; 0\;0\;0\;0 \,]_{1\times N_c}$

$\underline{d} := \mathrm{IDFT}(\tilde{\underline{d}}) = [d_0 \ldots d_{N_c-1}]^T.$

$\underline{w} = [\, w[L] \; \ldots \; w[L+N-1] \,]^T_{1\times N}$

$\Rightarrow \quad \underline{y} = \underline{h} \otimes_{N_c} \underline{d} + \underline{w}$.

now mult. duality applies

$\Rightarrow \left[\mathrm{DFT}|_{N_c}(\underline{y}) \right]_n = \left[\mathrm{DFT}|_{N_c}(\underline{h}) \right]_n \cdot \left[\mathrm{DFT}|_{N_c}(\underline{d}) \right]_n + \text{noise}$

but this is actually your data.

recall $\left[\mathrm{DFT}|_{N_c}(\underline{d}) \right]_n := \underbrace{\tilde{d}_n}$ $= \dfrac{1}{\sqrt{N_c}} \displaystyle\sum_{m=0}^{N_c-1} d[m] \, e^{-j2\pi \frac{n \cdot m}{N_c}}, \quad n = 0 \to N_c - 1.$

$\Rightarrow \left[\mathrm{DFT}(\underline{y}) \right]_n = (\mathrm{DFT}(\underline{h}))_n \bullet \underbrace{\tilde{d}_n} + \tilde{w}_n \sim \mathcal{CN}(0,1).$

$\Rightarrow$ can decode 1-by-1.

EQUIVALENT P.O.V.

want to send data symbols $\underline{\tilde{d}} = \begin{bmatrix} \tilde{d}_0 \\ \tilde{d}_1 \\ \vdots \\ \tilde{d}_{N_c-1} \end{bmatrix}$

- get $\underline{d} = IDFT(\underline{\tilde{d}})$ $\quad \underline{d} = U^{-1} \cdot \underline{\tilde{d}}$ $\quad$ where U^{-1} is inverse Fourier Matrix

recall $U = [U_{k,n}] = \frac{1}{\sqrt{N_c}} e^{-j2\pi k \cdot n / N_c}$

$k, n : 0 \to N_c - 1$.

as before

$$\underline{x} = [\underbrace{d_{N_c-L+1}, \dots, d_{N-1}}_{L-1}. \; d_0 \; d_1 \dots d_{N_c-1}]_{1\times(N+L-1)}.$$

$x[1] \qquad x[L] \dots$

$\Rightarrow$

$$\underline{y}' = \underline{h} * \underline{x} + \underline{w}$$

$$\underline{y} = y'[L : L+N_c-1]_{1\times N} = [y[L] \longrightarrow y[L+N-1]]_{1\times N}$$

then have seen that (after extending channel $\underline{h} \to \underline{h} = \{h_0 \dots h_{L-1}, 0 \dots 0\}_{N_c}$)

$$\underline{y} = \underline{h} \circledast_{N_c} \underline{d} + \underline{w}$$

$$\Rightarrow \underline{y} = C . \underline{d} + \underline{w}$$

circulant matrix.

where
$$C = \begin{pmatrix} h_0 & 0 & 0 & 0 & 0 & 0 & h_{L-1} & \cdots & h_2 & h_1 \\ h_1 & h_0 & 0 & 0 & 0 & \cdots & 0 & h_{L-1} & \cdots & h_2 \\ h_2 & h_1 & h_0 & & & & & & h_{L-1} \cdots & h_3 \\ \vdots & & & & & & & & & \vdots \\ h_{L-1} & h_{L-2} & \cdots & h_1 & h_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & h_{L-1} & \cdots & h_2 & h_1 & h_0 & 0 & 0 & 0 & \\ \vdots & & & & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & h_{L-1} & \cdots & h_2 \; h_1 & h_0 \end{pmatrix}_{N \times N}.$$

But circulant matrixes are diagonalizable by Fourier matrices

i.e. $\underline{y} = U^{-1} . \Lambda . U . d + \text{noise}$ i.e. $C = U^{-1} \Lambda U$ $\Lambda = \begin{bmatrix} \lambda_{11} & & \\ & \lambda_{22} & \\ & & \ddots \; \lambda_{NN} \end{bmatrix} = \begin{bmatrix} \tilde{h}_1 & & \\ & \tilde{h}_2 & \\ & & \ddots \; \tilde{h}_N \end{bmatrix}$.

$\Rightarrow$ take DFT of $\underline{y}$

$\Rightarrow U . \underline{y} = U U^{-1} \Lambda U d + U . \underline{w}$ (white) $\Rightarrow U \underline{y} =: \tilde{\underline{y}} = \Lambda . \tilde{\underline{d}}$

$$\Rightarrow \tilde{y}_i = \tilde{h}_i . \tilde{d}_i + \text{white}$$