0 Outline

1. Wireless Channels (AWGN and LTI)
2. Fading channels (epsilon, CSIT, freqSelec)
3. Multi-user Capacity

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   - Capacity of wireless channels
   - Capacity of LTI Gaussian channels
2. Fading channels (epsilon, CSIT, freqSelec)
3. Multi-user Capacity

1 Capacity of wireless channels

- Fundamental limits : best performance.
- Tool: information theory.
- Capacity: max rate s.t. $P_{err} \to 0$
- First AWGN, then fading channels (calculate capacity)
- Derive diversity results
- Will give insight: allocate W and $\bar{P}$

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- Inventor of IT: Shannon (1948).
  - $\exists R > 0$ s.t. $P_{err} \to 0$ for long T
  - $C = \max R$: $P_{err}(R) \to 0$ as $T \to \infty$
  - and $P_{err} \to 1$ $\forall R > C$

1 The AWGN channel

- Consider AWGN (Additive White Gaussian Noise)

$$y[m] = x[m] + w[m]$$

  - gives sense to $P_{err} \to 0$
  - explain capacity

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  - explain capacity
- 1) T=1
  - Since $x[m] = \pm\sqrt{P}$, means

$$P_e = Q(\frac{half - distance}{\sigma_w}) = Q(\frac{\sqrt{P}}{\sigma_w})$$

  - Not reliable: P_e fails to go 0

1 Early attempts

- 2) T large: Consider Repetition Coding:

$$\mathbf{x}_A = \sqrt{P}[1, 1, ..., 1], \quad \mathbf{x}_B = -\sqrt{P}[1, 1, ..., 1]$$

  - Let $\mathbf{x} = \mathbf{x}_A$

- 2) T large: Consider Repetition Coding:

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> Let $\mathbf{x} = \mathbf{x}_A$

$$\begin{aligned}
\Rightarrow P_e &= Q\left(\sqrt{\frac{||\mathbf{x}_A - \mathbf{x}_B||^2}{4\sigma^2}}\right) \\
&= Q\left(\sqrt{\frac{||\sqrt{P}(2,2,2,...,2)||^2}{4\sigma^2}}\right) \\
&= Q\left(\sqrt{\frac{||4\sqrt{P}(1,1,1,...,1)||^2}{4\sigma^2}}\right) = Q\left(\sqrt{\frac{P}{\sigma^2}\sum_{i=1}^{N} 1}\right) = Q\left(\sqrt{\frac{NP}{\sigma^2}}\right) \\
&\approx e^{-\frac{NP}{\sigma^2}} \xrightarrow{N\to\infty} 0
\end{aligned}$$

> but

$$R(rate) = \frac{1}{N}$$

1 Repetition coding M-PAM

- M-PAM

$$\mathbf{x} = \sqrt{P}[q, q, ..., q], \quad q \in \{-(M-1), ..., -1, 1, 3, ..., M-3, M-1\}$$

  - example: $M = 6$

$$q \in \{-5, -3, -1, 1, 3, 5\}$$

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- Say

$$||\mathbf{x}|| \leq PN, \ \forall \mathbf{x}$$

$$\Rightarrow ||\sqrt{P}\theta[5, 5, .., 5]||^2 \leq NP$$

$$||\theta[5, 5, ..., 5]||^2 \leq N \Rightarrow \theta = \frac{1}{M-1}$$

half distance between $\sqrt{P}\theta[1, 1, 1, ..., 1] \rightarrow \sqrt{P}\theta[-1, -1, ..., -1]$

$$||\sqrt{P}\frac{1}{M-1}[1, 1, 1, ..., 1]||^2 = \frac{PN}{(M-1)^2}$$

- Probability of error

$$\Rightarrow P_e = Q(\frac{half - dist}{\sigma_w}) = Q(\sqrt{\frac{PN}{(M-1)^2}})$$

- Reliable communications if

$$\Rightarrow (M-1)^2 < N \Rightarrow M < \sqrt{N}$$

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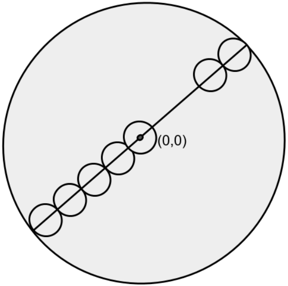
- Bound on rate

$$\Rightarrow R = \frac{\log M}{N} = \frac{\log \sqrt{N}}{N} \rightarrow 0$$

1 Problem with repetition code:

- Repetition code packed every codeword onto single dimension
- N-dimensional sphere. ($N = 2$ here, but generally very large)



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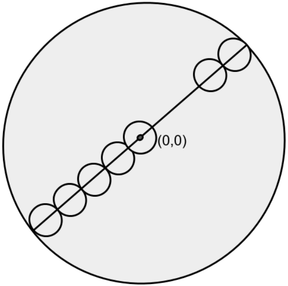

- But we have N dimensions
  - Codewords must not be close because noise $\rightarrow$ error.
  - Must decide on a distance between codewords
- Try to pack as many such spheres as possible
- A sphere packing problem in N-dim space

- Packing sphere $\rightarrow$ to reach capacity.
- Sphere packing problem: pack max number of codewords in sphere of certain radius
- recall:

$$\mathbf{y} = \mathbf{x} + \mathbf{w}, \quad ||\mathbf{x}||^2 \leq NP$$

- as $N \rightarrow \infty$

$$\frac{1}{N}\sum_{i=1}^{N}||w_i||^2 \rightarrow \sigma^2 \Rightarrow ||\mathbf{w}||^2 \rightarrow N\sigma^2 \Rightarrow ||\mathbf{y}||^2 = ||\mathbf{x}+\mathbf{w}||^2 \leq NP + N\sigma^2$$

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- thus

$$||\mathbf{y}||^2 \leq N(P+\sigma^2) \Rightarrow \mathbf{y} \in Ball(rad = \sqrt{N(P+\sigma^2)}), \quad \text{with Prob} \rightarrow 1$$

- Also given $\mathbf{X}_A$ (transmit codeword) and given $||\mathbf{w}|| \rightarrow \sqrt{N\sigma^2}$, then
- $\mathbf{y}$ is on the periphery of a sphere, centered on $\mathbf{X}_A$, with radius $\sqrt{N\sigma^2}$

1 Sphere packing

- Try to pack max number of sphere of radius $\sqrt{N\sigma^2}$ in a bigger sphere of radius $\sqrt{N(P+\sigma^2)}$
- This max number is

$$\frac{Vol(Ball(rad=\sqrt{N(P+\sigma^2)}))}{Vol(Ball(rad=\sqrt{N\sigma^2}))} \approx \frac{(\sqrt{N(P+\sigma^2)})^N}{(\sqrt{N\sigma^2})^N} = (\frac{N(P+\sigma^2)}{N\sigma^2})^{\frac{N}{2}}$$

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- Thus max rate $R = \frac{1}{N}\log(\frac{N(P+\sigma^2)}{N\sigma^2})^{\frac{N}{2}} = \frac{1}{2}\log(1+\frac{P}{\sigma^2})$
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- The above is per real space. But we have complex space, thus

$$C = \log(1+\frac{P}{\sigma^2}) \text{ bits/complex dimension}$$

1 Parenthetical:

- Cap gives the challenge.
- Codes : another Story
- Decoders : another story.
- People relatively happy with various practical codes
  - turbo codes (interesting story)
  - LPPC codes (also interesting story)
- Commonly used soft decoders (complicated).

1 Capacity-based resource allocation

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- Total Power $\bar{P}$ (Watts)
- Total bandwidth W (Hz)
- Thus $\frac{\bar{P}}{W} \quad (J/s/Hz) = (Watts/Hz)$
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$$y[m] = x[m] + w[x]$$

  - Complex baseband representation (two independent uses of the real channel)
- $\frac{N_0}{2}$ noise power per real symbol $\Rightarrow N_0$ power of $w[x]$
- $\frac{\bar{P}}{2W}$ power constraint per real symbol.

1 Capacity-based resource allocation

- $C_{real} = \frac{1}{2}\log(1 + \frac{P_{real}}{\sigma^2_{real}}) = \frac{1}{2}\log(1 + \frac{\frac{\bar{P}}{w}}{\frac{N_0}{2}}) = \frac{1}{2}\log(1 + \frac{\bar{P}}{wN_0})$

$$\Rightarrow C_{AWGN} = 2C_{real} = \log(1 + \frac{\bar{P}}{wN_0})$$

- Consider $SNR = \frac{\bar{P}}{wN_0}$ (note: SNR wrt complex dimension)

$$\Rightarrow C_{complex} = \log(1 + SNR) \quad \text{(bits/complex symbol)}$$

- C_{AWGN}: called spectral efficiency.

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- C_{AWGN}: called spectral efficiency.
- Total capacity

$$C_{AWGN}(\bar{P}, W) = WC_{complex} = W\log(1+SNR) = W\log(1+\frac{\bar{P}}{WN_0}) \;\; bits/sec$$

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- Recall

$$C(\bar{P}, w) = W \log(1 + \frac{\bar{P}}{WN_0})$$

  - Choose (invest in) $\bar{P}$ and/or W

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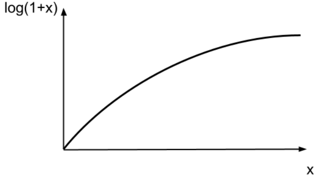
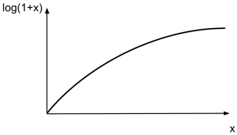
$$C(\bar{P}, w) = W \log(1 + \frac{\bar{P}}{WN_0})$$

  - Choose (invest in) $\bar{P}$ and/or W
- Note:
  - log is concave, i.e., $F^{''}(SNR) \leq 0$
  - $$\log(1+x) \approx x \log_2 e, \text{ as } x \to 0, \qquad \text{(linear)}$$
  - $$\log(1+x) \to \log(x), \text{ as } x \to \infty, \qquad \text{(logarithmic)}$$



1 Power limited regime

- low SNR: $\bar{P} \uparrow \Rightarrow \; C \uparrow$ linearly $\Rightarrow$ invest in $\bar{P}$
- high SNR: $\bar{P} \uparrow \Rightarrow \; C \uparrow$ logarithmically $\Rightarrow$ do NOT invest in $\bar{P}$



1 Bandwidth limited regime

- Fix $\bar{P}$
- C increasing function of W
- if W small $\Rightarrow$ SNR large $\Rightarrow W \uparrow \Rightarrow \log(\cdot) \downarrow$ a little

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- if W large $\Rightarrow$ SNR small

$$\Rightarrow \log(1+x) \downarrow \text{linearly} \Rightarrow \text{compensates for } W \uparrow \text{linearly}$$

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- if W small $\Rightarrow$ SNR large $\Rightarrow$ $W \uparrow \Rightarrow \log(\cdot) \downarrow$ a little

  $\Rightarrow W \log(\cdot) \uparrow$ almost linearly $\Rightarrow$ invest in W (bandwidth limited)

- if W large $\Rightarrow$ SNR small

  $\Rightarrow \log(1+x) \downarrow$ linearly $\Rightarrow$ compensates for $W \uparrow$ linearly

$$W \log(1 + \frac{\bar{P}}{N_0 W}) \approx W \log_2 e \frac{\bar{P}}{N_0 W} = \log_2 e \frac{\bar{P}}{N_0}$$

  > do not invest in W, but invest in $\bar{P}$ (linear increase) (power limited)

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- Minimize $\frac{E_b}{N_0}$: Power efficient region $\bar{P} \rightarrow 0$ (turns out)

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$$\approx \frac{\bar{P}}{N_0 W \log_2 e \frac{\bar{P}}{N_0 W}} = \frac{1}{\log_2 e}$$

$$\Rightarrow 10 \log_{10} \frac{1}{\log_2 e} = -1.59dB$$

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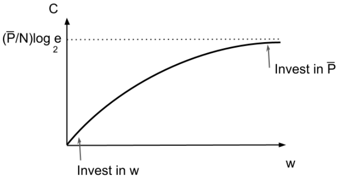

Capacity of LTI Gaussian channels

- The SIMO channel (L-receive antennas)

$$y_\ell[m] = h_\ell x[m] + w_\ell[m], \quad \ell = 1, ..., L, \quad w_\ell \sim \mathbb{C}N(0, N_0)$$

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  - Optimal since $\tilde{y}[m]$ is <u>sufficient statistic</u>

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- Thus

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- Thus capacity takes form

$$C_{\text{SIMO}} = \log(1 + \frac{P}{N_0}||\mathbf{h}||^2)$$

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- Thus MISO capacity is

$$C_{tx-beam} = C_{MISO} = \log(1 + \frac{P||\mathbf{h}||^2}{N_0}) \quad bits/s/Hz$$

1 Frequency Selective (and parallel) channels

- $$y[m] = \sum_{\ell=0}^{L-1} h_{[} m - \ell] + w[m], \quad E[|x[m]|^2] \leq P \quad (1)$$
  - L taps (convolution)

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- ith block

$$\tilde{y}_n[i] = \tilde{h}_n \tilde{d}_n[i] + \tilde{w}_n[i] \quad n = 0 \to N_c - 1 \quad (2)$$

$$\tilde{\mathbf{d}}[i] = [\tilde{d}_0[i] \to \tilde{d}_{N_c-1}[i]], \ \text{ DFT of input of } i\text{th block}, ||\tilde{\mathbf{d}}[i]||^2 \le N_c P \text{ (Pars}$$

$$\tilde{\mathbf{w}}[i] = [\tilde{w}_0[i] \to \tilde{w}_{N_c-1}[i]], \ \text{ DFT of noise: } \tilde{\mathbf{w}}[i] \sim \mathbb{C}N(0, N_0 I)$$

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- As $N_c \rightarrow \infty \;\Rightarrow\; N_c \gg L$

$$\Rightarrow \; C_{(1)} \approx C_{(2)}$$

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1 Frequency Selective (and parallel) channels

- Power allocation, then use capacity-achieving code over each parallel channel

$$C_{N_c} = \max_{P_0 \to P_{N_c-1}} \sum_{n=0}^{N_c-1} \log(1+\frac{P_n|\tilde{h}_n|^2}{N_0}), \quad s.t. \sum_{n=0}^{N_c-1} P_n = N_c P, \;\; P_n \geq 0, \forall n$$

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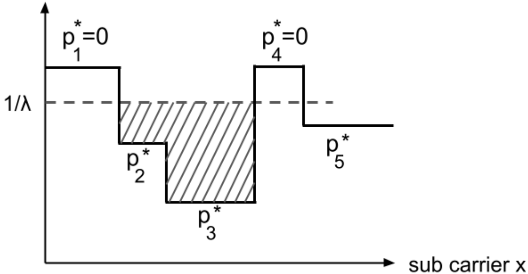
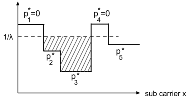
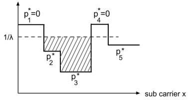
- Optimal $\{P_n\}$ via water-filling (y-axis is $\frac{N_0}{|\tilde{h}_n|^2}$)





- Write Lagrangian $\mathcal{L}(\lambda) = \sum_{n=0}^{N_c-1} \log(1 + \frac{P_n|\tilde{h}_n|^2}{N_0}) - \lambda \sum_{n=0}^{N_c-1} P_n$

1 Freq. Selective/Parallel channels: Lagrangian optimization



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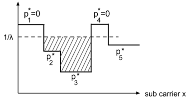
- Then evaluate $\frac{\delta \mathcal{L}}{\delta P_n} = \begin{cases} 0 & \text{if } P_n > 0 \\ \leq 0 & \text{if } P_n = 0 \end{cases}$
- Optimal power

$$P_n^* = \left(\frac{1}{\lambda} - \frac{N_0}{|\tilde{h}_n|^2}\right)^+, \quad \text{where } \lambda : \sum_{n=0}^{N_c-1} \overbrace{\left(\frac{1}{\lambda} - \frac{N_0}{|\tilde{h}_n|^2}\right)^+}^{P_n^*} = PN_c$$



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- Coding across parallel channels does not improve capacity (turns out)
  - but gives better performance if block length is small

1. Wireless Channels (AWGN and LTI)
2. Fading channels (epsilon, CSIT, freqSelec)
   - Capacity of fading channels
3. Multi-user Capacity

Capacities of Fading Channels

2 Capacity of fading channels

- Fading channel

$$y[m] = h[m]x[m] + w[m], \quad w[m] \sim \mathbb{C}N(0, N_0), \quad SNR = \frac{P}{N_0}$$

- h randomly drawn, with $E[|h[m]|^2] = 1$
- For now only CSIR (No CSIT)

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- h randomly drawn, with $E[|h[m]|^2] = 1$
- For now only CSIR (No CSIT)
- Slow fading "channel randomly drawn but here to stay"

$$\Rightarrow \; y[m] = hx[m] + w[m], \; \Rightarrow \quad SNR_{Rx} = |h|^2 SNR$$

$$C(h) = \log(1 + SNR|h|^2)$$

- No lower-limit ($C(h) \to 0$ maybe) (NO CSIT)
- Consider <u>outage</u> event

- ...Consider <u>outage</u> event

$$\text{Error} \iff \mathit{Capacity}(h) < \mathit{Rate}$$

2 Capacity of fading channels

- ...Consider outage event

$$\text{Error} \iff \mathit{Capacity}(h) < \mathit{Rate}$$

- Need another measure: $P(C(h) < R) = P(outage)$
  - applies when R is fixed

$$P_{out}(R) = P(\log(1 + SNR|h|^2) < R) \Rightarrow P(|h|^2 < \frac{2^R - 1}{SNR})$$

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$$P_{out}(R) = P(\log(1 + SNR|h|^2) < R) \Rightarrow P(|h|^2 < \frac{2^R - 1}{SNR})$$

  - Note

$$P(|h|^2 < \epsilon) \xrightarrow{\epsilon \to 0} \epsilon$$

$$\Rightarrow P_{out}(R) = P(|h|^2 < \frac{2^R - 1}{SNR}) \xrightarrow{SNR \to \infty} \frac{2^R - 1}{SNR}$$

$$\Rightarrow P_{out}(R) \to \frac{1}{SNR}$$

2 ϵ-outage capacity (quasi-static channel)

- Another measure of interest - ϵ-outage capacity

$$y = hx + w$$

- Recall probability of outage

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- Definition: ϵ-outage capacity

$$C_\epsilon = \max_R \ : \ P_{out}(R) \leq \epsilon$$

2 ϵ-outage capacity (quasi-static channel)

- Evaluate C_ϵ for any SNR

$$\text{Let} \quad x \triangleq \frac{2^R - 1}{\rho} \;\Rightarrow\; P(|h|^2 < x) = \epsilon \;\Rightarrow\; P(|h|^2 \geq x) = 1 - \epsilon$$

- and denote

$$F(x) \triangleq P(|h|^2 > x) \;\Rightarrow\; F(x) = 1 - \epsilon \Rightarrow x = F^{-1}(1-\epsilon)$$

$$\Rightarrow \; \frac{2^R - 1}{\rho} = F^{-1}(1-\epsilon) \Rightarrow 2^R = 1 + \rho F^{-1}(1-\epsilon)$$

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- Thus

$$C_\epsilon = \log(1 + SNR \cdot F^{-1}(1-\epsilon))$$

2 ϵ-outage capacity (high SNR)

- We have

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- At high SNR

$$C_\epsilon \approx \log(SNR \cdot F^{-1}(1-\epsilon)) = \log SNR + \log(F^{-1}(1-\epsilon))$$

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- Thus

$$C_\epsilon = \log SNR - \overbrace{\log(\frac{1}{F^{-1}(1-\epsilon)})}^{\text{constant offset}}$$

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- Low SNR

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$$C_\epsilon \approx \epsilon SNR \approx \epsilon C_{AWGN}$$

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- Big problem: Example

$$\epsilon \approx 1\% \rightarrow C_\epsilon \approx \frac{C_{AWGN}}{100}$$

- Thus diversity is important

2 Diversity for boosting ϵ-outage capacity

- Consider SIMO (R_x- diversity) to boost C_ϵ at low SNR

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$$\frac{R}{SNR} = \epsilon^{\frac{1}{L}} \Rightarrow R = \epsilon^{\frac{1}{L}} SNR \Rightarrow C_\epsilon = \epsilon^{\frac{1}{L}} C_{AWGN}$$

$$
\begin{aligned}
y_0[m] &= h_0 x_0[m] + w_0[m] \\
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- Fast-fading (ergodic) capacity

$$C_{FF} = E_h\{\log(1 + |h|^2 SNR)\}$$

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2 Compare C_{FF} with C_{AWGN}

- Employ Jensen's inequality $E\{f(u)\} \leq f(E\{u\})$

$E_h\{\log(1+\rho|h|^2)\}$

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- High SNR:

$$C_{FF} \approx E\{\log(\rho|h|^2)\} = \log\rho + E\{\log|h|^2\} \approx C_{AWGN} - \overbrace{0.83}^{2.5dB}$$

2 Capacity with CSIT

- TDD reciprocity, FDD feedback (expensive)

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- Slow fading: could do channel inversion, or rate adaptation i.e

$$\log(1 + \frac{P_t}{N_0}|h|^2) < R_t$$

  - $\Rightarrow$ change R_t or P_t (When bad channel)

- Fast Fading: Waterfilling

$$
\begin{aligned}
y_0 &= h_0 x_0 + w_0 \\
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$$\Rightarrow \max_{P_0 \to P_{L-1}} \frac{1}{L} \sum_{\ell=0}^{L-1} \log(1 + \frac{P_\ell}{N_0} |h_\ell|^2), \qquad \sum P_\ell = LP$$

- Seen

$$P_\ell^* = (\frac{1}{\lambda} - \frac{N_0}{|h_\ell|^2})^+, \quad \mathit{for} \quad \lambda \quad : \quad \frac{1}{L} \sum_{\ell=0}^{L-1} (\frac{1}{\lambda} - \frac{N_0}{|h_\ell|^2})^+ = P$$

2 Capacity with CSIT - waterfilling

- Causality an issue, BUT for large L

$$\frac{1}{L}\sum_{\ell=0}^{L-1}\left(\frac{1}{\lambda}-\frac{N_0}{|h_\ell|^2}\right)^+ \rightarrow E\left\{\left(\frac{1}{\lambda}-\frac{N_0}{|h|^2}\right)^+\right\}$$

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$$\Rightarrow C = E\{\log(1+P^*(h)\frac{|h|^2}{N_0})\}$$

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$$C_{AWGN} \approx C_{FF} \approx C_{CSIT}$$

- except (in theory) at low SNR where $C_{CSIT} >> C_{AWGN}$

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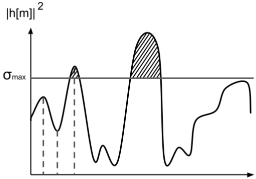
- In the end: For most SNR

$$C_{AWGN} \approx C_{FF} \approx C_{CSIT}$$

- except (in theory) at low SNR where $C_{CSIT} >> C_{AWGN}$ WARNING!!

- Send only if $|h[m]|^2 \approx G_{max}$, and store power while not sending



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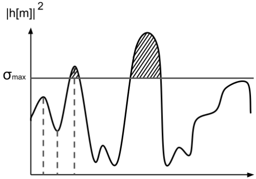

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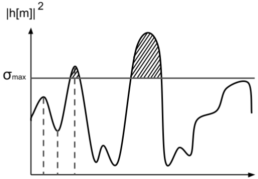

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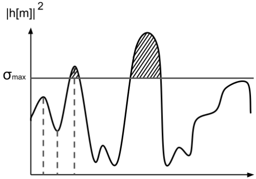

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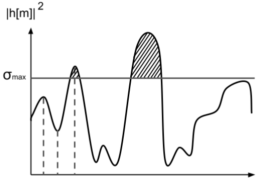

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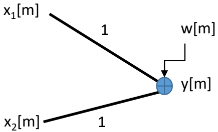
- Would not work in practice!!

1. Wireless Channels (AWGN and LTI)
2. Fading channels (epsilon, CSIT, freqSelec)
3. Multi-user Capacity

- Up-link AWGN (multiple-access channel)



3 Multi-user Capacity – Uplink

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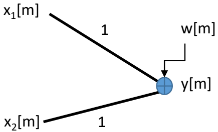

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  - $E\{|x_k[m]|^2 = P_k, k = 1, 2, \quad w[m] \sim \mathbb{C}\mathcal{N}(0, N_0)$ iid

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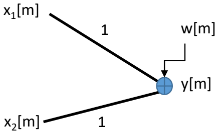

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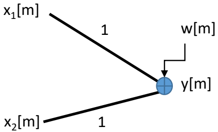

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$$R_1 + R_2 < \log(1 + \frac{P_1 + P_2}{N_0}), \quad \text{MISO bound with fixed } h_1 = h_2 = 1$$

> $E\{|\underline{x}|^2\} = P_1 + P_2$, where $\underline{x} = [x_1, x_2]^T$.

- Let us achieve these outer bounds.

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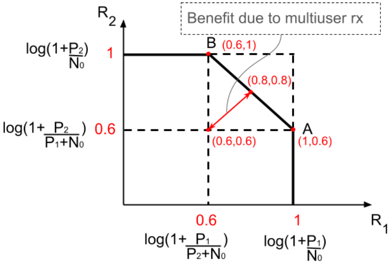

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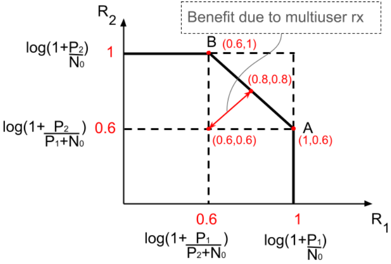

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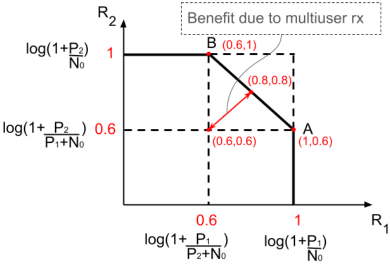

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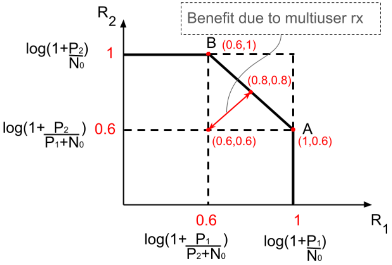

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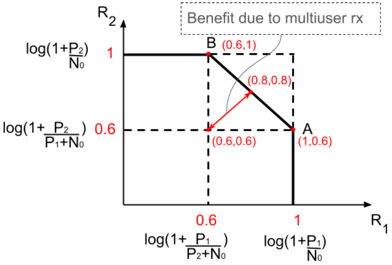

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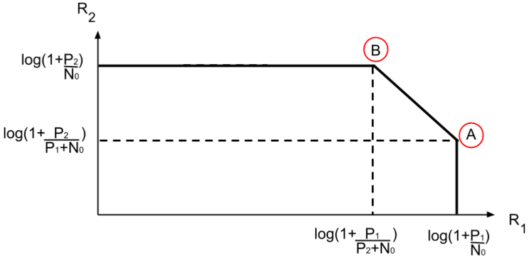
- Example $P_1 = P_2 = 1$

3 Non-Symmetric Multiple Access

- Considering the non-symmetric setting, say $P_1 >> P_2$.



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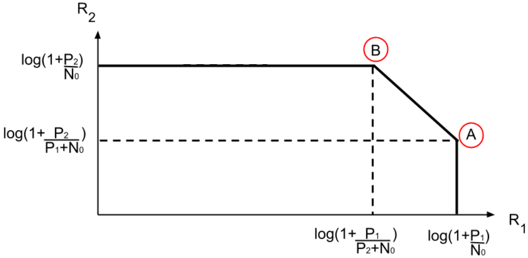

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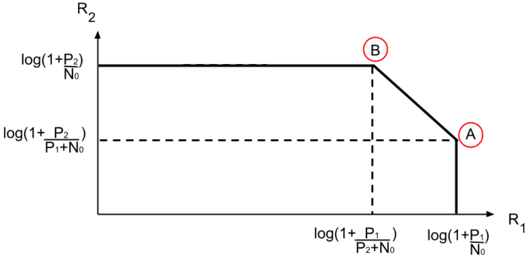

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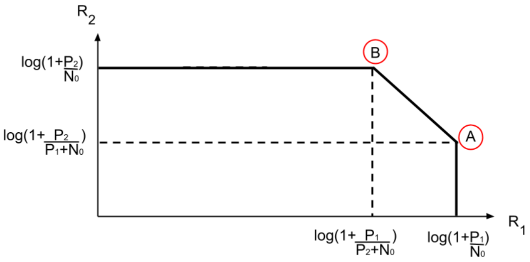

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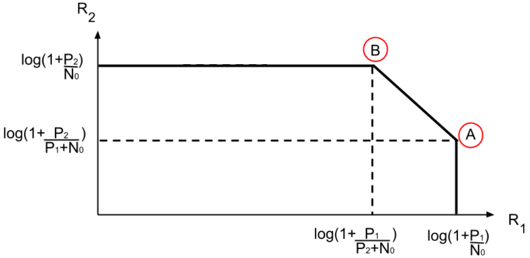

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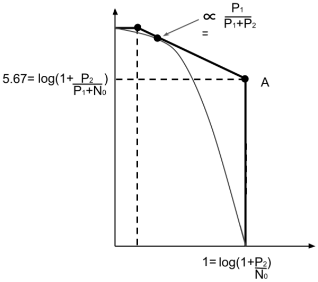
  - decode strongest first (small interference from weaker)
  - then fully decode weaker (point B)
  - Slightly bother U1, no interference for U2 (Reverse would kill U2)
- Strong users close to BS transmit fast, with no trouble to rest
  - No power control. Turns near-far issue into advantage.

3 Orthogonal Multiple Access

- Orthogonal Multiple access techniques Optimal? $(\alpha, 1-\alpha)$

$$R_1 = \alpha W \log(1+\frac{P_1}{\alpha N_0}), \quad R_2 = (1-\alpha) W \log(1+\frac{P_2}{(1-\alpha)N_0}) \text{ bits/s}$$

- $\exists$ only one optimal $\alpha: \ \alpha = \frac{P_1}{P_1+P_2}$ that achieves opt sum-capacity
  - Penalizes weak users ($P_1 \gg P_2 \Rightarrow \alpha \approx 1, \Rightarrow 1-\alpha \approx 0$)
  - Logarithmic increase vs. linear decrease
  - Giving high rate to weak user $\Rightarrow$ entirely sacrifice rate for strong user
  - Example $P_2 = 0dB, P_1 = 20dB$.



3 K-user Uplink Capacity

- for $P_i = P_j = P$

$$C_{sum} = \log(1 + \frac{kP}{N_0})$$

$$C_{sym} = \frac{1}{k}\log(1 + \frac{kP}{N_0}) \;\; \textit{max rate when all have the same rate}$$

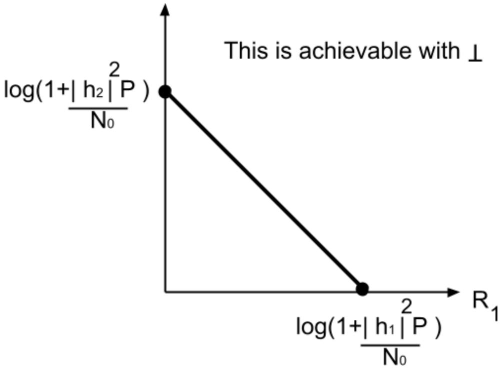
- achievable with ?? (say k=2 , $\alpha = \frac{1}{2}$, $\alpha\log(1 + \frac{P}{\alpha N_0})$, $(1-\alpha)\log(1 + \frac{P}{(1-\alpha)N_0})$) under equal received power
- OFDM ¿ CDMA , CDMA treats all others as noise

$$C_{sum} = k\log(1 + \frac{P}{(k-1)P + N_0}) \approx k\frac{P}{(k-1)P + N_0}\log_2 e \approx l\log_2 e \approx 1.44$$

$$\log(1 + \frac{kP}{N_0}) \uparrow \infty$$

3 Downlink AWGN with fixed fading

- $y_k[m] = h_k \quad x[m] + w_k[m] \quad k = 1??\ ??$ k fixed
- assume CSTIR $|X[m]|^2 \leq P \ \frac{Joules}{symbol}$
- Capacity region R_1, R_2
- $R_k \leq \log(1 + \frac{P|h_k|^2}{N_0})$ all power for user K and rate



3 Superposition-Symmetric

$$X[m] = X_1[m] + X_2[m] \quad \textit{iid Gaussian codes}$$

$$R_1 = \log(1 + \frac{P_1|h_1|^2}{P_2|h_1|^2 + N_0}) = \log(1 + \frac{(P_1 + P_2)|h_1|^2}{N_0}) - \log(1 + \quad)$$

$$R_2 = \log(1 + \frac{P_2|h_2|^2}{N_0})$$

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- User 2 can decode anything ??/

$$x = x_1 + x_2$$

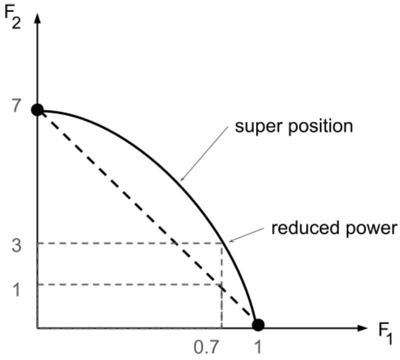
$$R_1 = \log(1 + \frac{P_1|h_1|^2}{|h_1|^2 P_2 + N_0}) \quad \textit{weakerseesinterference}$$

$$R_2 = \log(1 + \frac{P_2|h_2|^2}{N_0}) \; \textit{userseesnointerference}$$

- first, decode user 1 data and then Which is generally better than ?? (for all power splits $P_1 + P_2 = P$)

$$R_1 = \alpha \log(1 + \frac{P_1|h_1|^2}{\alpha N_0})$$

$$R_2 = (1-\alpha)\log(1 + \frac{P_2|h_2|^2}{(1-\alpha) N_0})$$



- In General

$$R_k = \log(1 + \frac{P_k|h_k|^2}{N_0 + \sum_{J=k+1}^{K} P_J|h_k|^2}) \quad P = \sum_{k=1}^{K} p_k$$

- sum capacity here (unlike uplink which all user R_x)

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