Diversity :

Have seen that $P_{err} \to \frac{1}{e}$ with several encoding schemes.

$\Rightarrow$ Mitigate by
- Time diversity
- Space diversity: $T_x \uparrow$ same information from more than 1 location. Use spacing of $\frac{\lambda}{4}$ s.t channels are statistically indep.
- Frequency diversity.

(Recall context of $\Delta x_c \approx \frac{\lambda}{4} \approx$ few cm

$$\Delta T_c \approx \frac{\Delta x_c}{V} \approx \text{few ms.}$$

$$W_c \approx \frac{1}{T_d} \approx \frac{1}{\text{few } \mu s} \approx \text{few Hundred Hz}$$

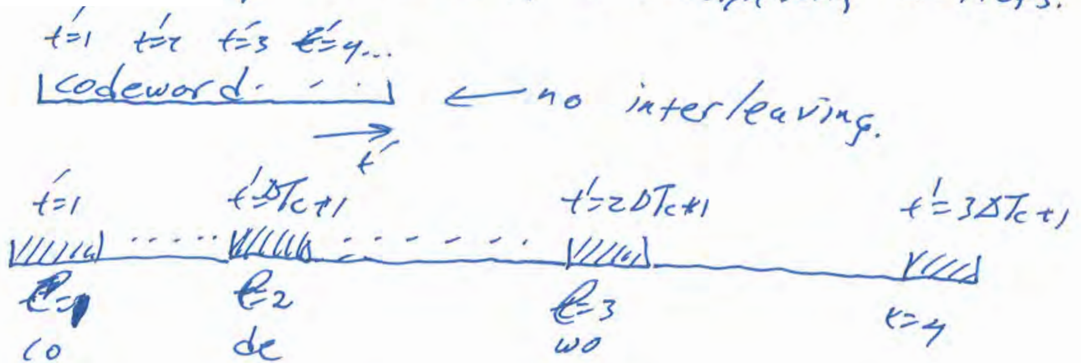
$\rightarrow 1$ MHz.

Time diversity:

First step: interleaving.

Instead of sending a codeword in consecutive time slot you place them in non-neighboring t-slots.

e.g.



⇒ After perfect interleaving, we have new channel model

$$y_\ell = h_\ell \cdot x_\ell + w_\ell \quad \ell = 1, 2, \dots, L$$ ← (Q: how many taps? 1)

- indep. fading coeffs h_ℓ : L-diversity branches we i.i.d. $\mathcal{CN}(0,1)$

Different time diversity methods.

Repetition coding & coherent decoding (Flat fading).

$$y_1 = h_1 x_1 + w_1 \quad y_2 = h_2 \cdot x_2 + w_2 \quad \cdots \quad y_L = h_L \cdot x_L + w_L .$$

$\xrightarrow{\text{repetition}}$

$$y_1 = h_1 x_1 + w_1 \quad y_2 = h_2 \cdot x_1 + w_2 \quad \cdots \quad y_L = h_L x_L + w_L$$

$$\Rightarrow \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_L \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_L \end{pmatrix} \cdot x_1 + \begin{pmatrix} w_1 \\ \vdots \\ \vdots \\ w_L \end{pmatrix} \quad \Rightarrow \underline{y} = \underline{h} \, x_1 + \underline{w}$$

Suff stat: $\tilde{y} = \frac{\underline{h}^H}{\|\underline{h}\|} \cdot \underline{y} = \frac{\underline{h}^H}{\|\underline{h}\|} \cdot \left(\underline{h} \cdot x + \underline{w}\right) = \frac{\|\underline{h}\|^2}{\|\underline{h}\|} \cdot x + \frac{\underline{h}^H}{\|\underline{h}\|} \underline{w}$

$\Rightarrow \tilde{y} = \|\underline{h}\| \cdot x + z$ where $z = \frac{\underline{h}^H}{\|\underline{h}\|} \underline{w} \sim \underbrace{\mathcal{CN}(0, N_0)}$.

Method called: matched filtering or maximal ratio combining

$\Rightarrow$ now scalar detection:

$\Rightarrow P_{err}|\underline{h} = Q\left(\frac{\text{half distance}}{\sigma_z}\right) =$

Say $w_i \sim \mathcal{CN}(0, \sigma_i^2)$

Let $z' = \underline{h}^H \cdot \underline{w} = \sum_{i=1}^{n} h_i^* \cdot w_i$

$\Rightarrow \sigma_{z'}^2 = \sum |h_i|^2 \cdot \sigma_i^2 = \sum |h_i|^2 \cdot N_0$

$= N_0 \cdot \|\underline{h}\|^2$

$\Rightarrow z = \frac{z'}{\|\underline{h}\|} \Rightarrow \boxed{\sigma_z^2 = N_0}$

$$\tilde{y} = \|\underline{h}\| \cdot x_1 + \underbrace{\frac{\underline{h}^H}{\|\underline{h}\|} \cdot \underline{w}} \quad \Rightarrow \quad \tilde{y} = \|\underline{h}\| \cdot x_1 + z \qquad z \sim \mathbb{C}N(0, N_0).$$

Let $\rho = \dfrac{E\{|x_1|^2\}}{E\{|z|^2\}} = \dfrac{P}{N_0}$. $\qquad P \triangleq E\{|x_1|^2\}$. (recall: measure SNR w.r.t complex dim).

$\Rightarrow$ Received SNR $= \dfrac{E\{|\|\underline{h}\| \cdot x_1|^2\}}{E\{|z|^2\}} = \|\underline{h}\|^2 \cdot \dfrac{P}{N_0} = \boxed{\|\underline{h}\|^2 \cdot \rho = SNR_{rx}}$

Assume BPSK:

$x_1 = \begin{cases} a \\ -a \end{cases} \qquad P = a^2 \qquad \Rightarrow \rho = \dfrac{a^2}{N_0}$

$\Rightarrow \tilde{\tilde{y}} = Re\{\tilde{y}\} =$

$\Rightarrow \tilde{\tilde{y}} = \|\underline{h}\| \cdot x_1 + Re\{z\} \qquad Re\{z\} \sim N\left(0, \frac{N_0}{2}\right)$ further reduced noise.

$\Rightarrow$ Now

$$P_{err|h} = Q\left(\frac{\overbrace{a \cdot \|\underline{h}\|}^{\text{half distance}}}{\underbrace{\sqrt{\frac{N_0}{2}}}_{\sigma_{Re}(z)}}\right) = Q\left(\sqrt{\frac{2a^2 \cdot \|\underline{h}\|^2}{N_0}}\right) = Q\left(\sqrt{2 \cdot \rho \cdot \|\underline{h}\|^2}\right).$$

$$P_{err} = E_{\underline{h}}\left\{ P_{err} | \underline{h} \right\}.$$

Recall statistics of $\|\underline{h}\|^2$

$$x = \|\underline{h}\|^2 = \sum_{\ell=1}^{L} |h_\ell|^2 \qquad f(x) = \frac{1}{(L-1)!} \cdot x^{L-1} \cdot e^{-x} \quad x \geq 0.$$

chi-squared. $2L$ - DOF.

$$\Rightarrow P_{err} = E_{x}\left\{ Q\left(\sqrt{2\rho \cdot x}\right)\right\} = \int_0^\infty \int_{\sqrt{2\rho \cdot x}}^\infty f_{\mathrm{Re}\,z}(\mathrm{Re}(z)) \, d_{\mathrm{Re}(z)} \, f_x(x) \, dx$$

$$= \int_0^\infty \int_{\sqrt{2\rho x}}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \cdot \frac{1}{(L-1)!} \cdot x^{L-1} e^{-x} dx$$

Using integration by parts.

$$P_{err} = E_h\{P_{err}|h\} = \left(\frac{1-\mu}{2}\right)^L \cdot \sum_{\ell=0}^{L-1} \binom{L-1-\ell}{\ell} \left(\frac{1+\mu}{2}\right)^\ell \qquad \mu \triangleq \sqrt{\frac{\rho}{\rho+1}}.$$

- Next step: Taylor of $\frac{1-\mu}{2} \xrightarrow{\rho\to\infty} \frac{1}{4\rho}$

$$\frac{1+\mu}{2} \xrightarrow{\rho\to\infty} \frac{1+\sqrt{\frac{\rho}{\rho+1}}}{2} \to 1$$

& $$\sum_{\ell=0}^{L-1} \binom{L-1-\ell}{\ell} = \binom{2L-1}{L} \leftarrow \text{combinatorial property:}$$

$$\Rightarrow P_{err} \approx \left(\frac{1-\mu}{2}\right)^L \cdot \sum_{\ell=0}^{L-1} \binom{L-1-\ell}{\ell}$$

$$\approx \frac{1}{(4\rho)^L} \cdot \binom{2L-1}{L}.$$

$$\Rightarrow \boxed{P_{err} \to \frac{1}{(4\rho)^L}\binom{2L-1}{L}} \leftarrow \boxed{\text{see } \underline{\text{diversity gain}} \quad P_{err} \approx \rho^{-1} \longrightarrow \rho^{-L}}$$

But this matches the prob. of deep fade (here)

$$\|\underline{h}\|^2 \ll \bar{e}^1$$

since

$P(\|\underline{h}\|^2 \ll \bar{e}^1)$; $\Rightarrow$ look for $P(x \ll \bar{e}^1)$ but $\bar{e}^1 \to 0$

To see this recall that $f_X(x) = \frac{1}{(L-1)!} \cdot x^{L-1} \cdot e^{-x}$: look for near-zero behavior of $f_X(x)$.

$$\Rightarrow f_X(x) \overset{x \to 0}{\approx} \frac{1}{(L-1)!} x^{L-1}.$$

$$\Rightarrow P(\|\underline{h}\|^2 < \bar{e}^1) = P(x < \bar{e}^1) \approx \int_0^{\bar{e}^1} \frac{1}{(L-1)!} x^{L-1} \cdot dx \qquad \frac{1}{(L-1)!} \cdot \left(\frac{x^L}{L}\right)\Bigg|_0^{\bar{e}^1} = \frac{1}{L(L-1)!} \cdot \left\{(\bar{e}^1)^L - 0\right\}$$

$$= \frac{1}{L!} \bar{e}^L$$

$\Rightarrow$ same slope L: $\Rightarrow$ main culprit is fading.
(div gain L).

(Q: is there a disadvantage & a solution ?!)

Time diversity at higher "rates"

- Repetition code utilize few DoF.

- Uncoded tx provides no diversity, since

$$y_1 = h_1 \cdot u_1 + w_1 \qquad y_2 = h_2 \cdot u_2 + w_2$$

u_1, u_2 indep

$\Rightarrow$ if $h_1 \ll \bar{e}^1 \Rightarrow x_1$ in trouble $\Rightarrow P_{err} \approx \bar{e}^1$ (no diversity)

- Consider rotation coding.

Consider case of encoding over 2 indep channels

$\Rightarrow$ as before $y_1 = h_1 x_1 + w_1$ $y_2 = h_2 x_2 + w_2$.

instead of sending $x_1 = u_1$ $x_2 = u_2$ as "uncoded" case,

let $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = R \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ where $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. is a rotation matrix.

Let $u_i \sim$ BPSK $u_i = \begin{cases} a \\ -a \end{cases}$

- $\Rightarrow \exists$ 4 codewords: $\mathcal{X} = \left\{ \underline{x}_A = R\begin{pmatrix} a \\ a \end{pmatrix}, \underline{x}_B = R\begin{pmatrix} a \\ -a \end{pmatrix}, \underline{x}_C = R\begin{pmatrix} -a \\ a \end{pmatrix}, \underline{x}_D = R\begin{pmatrix} -a \\ -a \end{pmatrix} \right\}$.

code.

e.g. $\underline{x}_A = \begin{bmatrix} x_{A1} \\ x_{A2} \end{bmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} a \\ a \end{pmatrix}$

- Assume $\underline{x}_A$ is tx.

- From union bound: $P_{err}|\underline{x}_A = P\left[(\underline{x}_A \to \underline{x}_B) \cup (\underline{x}_A \to \underline{x}_C) \cup (\underline{x}_A \to \underline{x}_D)\right]$.

$\leq P(\underline{x}_A \to \underline{x}_B) + P(\underline{x}_A \to \underline{x}_C) + P(\underline{x}_A \to \underline{x}_D)$

- Calc: $P(\underline{x}_A \to \underline{x}_B)$

calc $P(\underline{x}_A \to \underline{x}_B)$.

- see decision of decoder:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} h_1 \cdot x_{A1} \\ h_2 \ x_{A2} \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \quad \overset{\underline{x}_A \ \underline{x}_B}{\text{v.s}} \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} h_1 \cdot x_{B1} \\ h_2 \ x_{B2} \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$\Rightarrow$ two competing codewords after the action of the channel

$$\begin{pmatrix} h_1 \cdot x_{A1} \\ h_2 \ x_{A2} \end{pmatrix} \quad \text{v.s} \quad \begin{pmatrix} h_1 \cdot x_{B1} \\ h_2 \ x_{B2} \end{pmatrix}.$$

$\Rightarrow$ binary vector detection problem.

difference

$$\begin{pmatrix} h_1 \ x_{A1} \\ h_2 \ x_{A2} \end{pmatrix} - \begin{pmatrix} h_1 \ x_{B1} \\ h_2 \ x_{B2} \end{pmatrix} = \begin{pmatrix} h_1 (x_{A1} - x_{B1}) \\ h_2 (x_{A2} - x_{B2}) \end{pmatrix}$$

half distance $\left\| \begin{matrix} h_1 (x_{A1} - x_{B1}) \\ h_2 (x_{A2} - x_{B2}) \end{matrix} \right\| / 2$. ⊛

Let $d' = \underline{x}_A - \underline{x}_B$ commonly referred to as an unnormalized codeword difference ("also diff codeword").

Normalize so that we can give expressions w.r.t SNR. $\rho = a^2/N_0$ (*)

& let $\underline{d} := \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \frac{1}{a}\left(\underline{x}_A - \underline{x}_B\right) = \frac{1}{a}\underline{d}'$ ($P=1$ considering unit power tx vectors $\frac{1}{a}\underline{x}_A$ & $\frac{1}{a}\underline{x}_B$)

where $d_1 = \frac{1}{a}(x_{A_1} - x_{B_1})$, $d_2 = \frac{1}{a}(x_{A_2} - x_{B_2})$

$$\Rightarrow \underline{d} = \frac{1}{a}\left(\underline{x}_A - \underline{x}_B\right) = \frac{1}{a}\left[R\left(\begin{matrix} a \\ a \end{matrix} - \begin{pmatrix} a \\ -a \end{pmatrix}\right)\right] = \frac{1}{a}\left[R \cdot \begin{pmatrix} 0 \\ 2a \end{pmatrix}\right] = R\begin{pmatrix} 0 \\ 2 \end{pmatrix} \Rightarrow$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -2\sin\theta \\ 2\cos\theta \end{pmatrix}.$$

now write from #

$$\Rightarrow \text{half dist} = \frac{1}{2}\left\|\begin{pmatrix} h_1 . a . d_1 \\ h_2 . a . d_2 \end{pmatrix}\right\| = \frac{1}{2}\sqrt{a^2|h_1|^2 \cdot |d_1|^2 + a^2|h_2|^2|d_2|^2} = \frac{1}{2}a\sqrt{|h_1|^2|d_1|^2 + |h_2|^2|d_2|^2}$$

$$\Rightarrow P_{err}|h = Q\left(\frac{\text{half dist}}{\sigma}\right) = Q\left(\frac{\frac{1}{2}a\sqrt{|h_1|^2|d_1|^2 + |h_2|^2|d_2|^2}}{\sqrt{\frac{N_0}{2}}}\right) \leftarrow \text{since } \forall R.$$

$$= Q\left(\sqrt{\frac{a^2 \sum (|h_1|^2|d_1|^2 + |h_2|^2|d_2|^2)}{2N_0}}\right) = Q\left(\sqrt{\frac{\rho}{2}\sum_{i=1}^{2}|h_i|^2|d_i|^2}\right) = P\left(\underline{x}_A \to \underline{x}_B\right).$$

$$\Rightarrow \quad P(\underline{x}_A \to \underline{x}_B) = Q\left(\sqrt{\frac{\rho}{2} \sum_{i=1}^{2} |h_i|^2 |d_i|^2}\right).$$

recall that $Q(x) \le e^{-x^2/2} \quad x \ge 0.$

$$\Rightarrow P(\underline{x}_A \to \underline{x}_B \mid h_1, h_2) \le e^{-\rho\left(|h_1|^2 |d_1|^2 + |h_2|^2 |d_2|^2\right)/4}$$

$$\Rightarrow P(\underline{x}_A \to \underline{x}_B) \le \mathop{E}_{h_1, h_2}\left[e^{-\rho\left[\sum_{i=1}^{2} |h_i|^2 |d_i|^2\right]/4} \right].$$

- Use fact that M.G.F of unit-mean expon r.v is $\underline{E\{e^{st}\} = \frac{1}{1-s}, s<}$

$$\Rightarrow \boxed{P(\underline{x}_A \to \underline{x}_B) \le \frac{1}{1 + \rho \cdot \frac{|d_1|^2}{4}} \cdot \frac{1}{1 + \rho \cdot \frac{|d_2|^2}{4}}}$$

$\leftarrow$ remove it from $\forall$.

$$\Rightarrow \boxed{P(\underline{x}_A \to \underline{x}_B) \le \frac{16}{|d_1|^2 \cdot |d_2|^2} \cdot \rho^{-2}}$$

Note: if $|d_1|^2$ or $d_2 = 0$
$\Rightarrow$ div $= 1.$

Note: generally want $|d_1|^2 |d_2|^2$ to be large.

Let $|d_1|^2 \cdot |d_2|^2 =: \delta_{AB}$

$\Rightarrow P_{err}|_{x_A} \leq P(\underline{x}_A \to \underline{x}_B) + P(\underline{x}_A \to \underline{x}_C) + P(\underline{x}_A \to \underline{x}_D)$

$$\leq 16 \cdot e^{-2} \left(\frac{1}{\delta_{AB}} + \frac{1}{\delta_{AC}} + \frac{1}{\delta_{AD}} \right) \leq \frac{48 \cdot e^{-2}}{\min\{\delta_{AB}, \delta_{AC}, \delta_{AD}\}}$$

recall: $d_{x_A \to x_B} = \begin{pmatrix} -2\sin\theta \\ 2\cos\theta \end{pmatrix}$

$\Rightarrow \delta_{AB} = |-4\sin\theta\cos\theta|^2 + 4\sin^2\theta$: similar expressions for δ_{AC}, δ_{AD}.

$\Rightarrow$ Want to find θ to maximize $\min\{\delta_{AB}, \delta_{AC}, \delta_{AD}\}$.

turns out $\theta^* = \frac{1}{2}\arctan 2 \Rightarrow \min\{\delta_{AB}, \delta_{AC}, \delta_{AD}\} = \frac{16}{5}$

$\Rightarrow P_{err}|x_A \leq \frac{48}{\frac{16}{5}} e^{-2} = 15 \cdot e^{-2}$.

Similar for $P_{err}|x_B, x_C, x_D \Rightarrow \boxed{P_{err} \leq 15 \cdot e^{-2}} \leftarrow$ rotation code

Now we see that again, deep fade main cause of error.

$$P\left(e\left(|h_1|^2 \cdot |d_1|^2 + |h_2|^2 |d_2|^2\right)\right) \ll 1$$

deep fade brings two codewords too close to each other.

Calculate $P\left(|h_1|^2 \cdot |d_1|^2 + |h_2|^2 \cdot |d_2|^2 \leq e^{-1}\right).$

Two events are indep

$\Rightarrow \; p\left(|h_1|^2|d_1|^2 + |h_2|^2|d_2|^2 \le \rho^{-1}\right) \approx p\left(|h_1|^2|d_1|^2 \ll \rho^{-1} \;\&\; |h_2|^2|d_2|^2 \ll \rho^{-1}\right)$

$= p\left(|h_1|^2|d_1|^2 \ll \rho^{-1}\right) \cdot p\left(|h_2|^2|d_2|^2 \ll \rho^{-1}\right).$

near zero behavior of Gaussian/exponen

$P(|h|^2 \le \varepsilon) \approx \varepsilon$ for ε small

specifically $\lim_{\varepsilon \to 0} \frac{P(|h|^2 < \varepsilon)}{\varepsilon} \to$

$$\simeq p\left(|h_1|^2 \ll \frac{\rho^{-1}}{|d_1|^2}\right) \cdot p\left(|h_2|^2 \ll \frac{\rho^{-1}}{|d_2|^2}\right)$$

can be argued that $\frac{\rho^{-1}}{|d_1|^2} \ll 1$ (can be made).

$$\Rightarrow \quad P_{fade} \approx \frac{\rho^{-1}}{|d_1|^2}\,\frac{\rho^{-1}}{|d_2|^2} \approx \frac{\rho^{-2}}{|d_1|^2|d_2|^2}$$

$$P_{\text{rate}} \approx \frac{e^{-2}}{\underbrace{|d_1|^2 \cdot |d_2|^2}}$$

$|d_1|^2 \cdot |d_2|^2$ called the "coding gain" $\uparrow$ good.

- Let us connect, coding gain with DOF & dimensionalim.

In brief: we will see that as DOF $\uparrow$

$\Rightarrow$ coding gain $\uparrow$ & $P_{err} \downarrow$.

- Compare BPSK w/rotation coding $\left[P_{err} \leq 16 \cdot e^{-2} \right]$

to 4/PAM repetition code $\{-3b, -b, b, 3b\}$.

- Note rate is same 1 bpcu

Do as before, but now for 4-PAM: (repetition)

$$x = \Big[\underbrace{\begin{pmatrix} b \\ b \end{pmatrix}}_{x_A}, \underbrace{\begin{pmatrix} -b \\ -b \end{pmatrix}}_{x_B}, \underbrace{\begin{pmatrix} 3b \\ 3b \end{pmatrix}}_{x_C}, \underbrace{\begin{pmatrix} -3b \\ -3b \end{pmatrix}}_{x_D} \Big].$$

$$E\{\|4\text{-PAM}\|^2\} = \frac{1+1+9+9}{4} = \frac{20}{4} = 5b^2$$

→ Normalized code

$$x' = \frac{1}{\sqrt{5}b} x = \Big[\frac{1}{\sqrt{5}b}\begin{pmatrix} b \\ b \end{pmatrix}, \dots \frac{1}{\sqrt{5}b}\begin{pmatrix} -3b \\ -3b \end{pmatrix} \Big] = \frac{1}{\sqrt{5}} \Big[\underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{x_A}, \underbrace{\begin{pmatrix} -1 \\ -1 \end{pmatrix}}_{x_B}, \underbrace{\begin{pmatrix} 3 \\ 3 \end{pmatrix}}_{x_C}, \underbrace{\begin{pmatrix} -3 \\ -3 \end{pmatrix}}_{x_D} \Big].$$

$$d_{AB} = \frac{1}{\sqrt{5}}\Big[\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ -1 \end{pmatrix}\Big] = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 2 \end{bmatrix} \begin{matrix} \to d_{AB,1} \\ \to d_{AB,2} \end{matrix}, \quad d_{AC} = \frac{-1}{\sqrt{5}}\begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad d_{AD} = \frac{1}{\sqrt{5}}\begin{pmatrix} 4 \\ 4 \end{pmatrix}, \dots$$

$$\delta_{AB} = |d_{AB,1}|^2 \cdot |d_{AB,2}|^2 = \frac{1}{25} 2^2 \cdot 2^2 = \frac{16}{25}$$

Recall $P_e \le \underset{h_1 h_2}{E}\Big\{ Q\Big(\frac{\rho}{2}\big\{|h_1|^2|d_1|^2 + |h_2|^2|d_2|^2\big\}\Big)\Big\}$ & $P_{err|x_A} \le \dfrac{48\,\rho^{-2}}{\min\{\delta_{AB}, \delta_{AC}, \delta_{AD}\}}$

$$P_{err|x_A} \le \frac{48 \cdot \rho^{-2}}{\min\{\delta_{AB}, \delta_{AC}, \delta_{AD}\}} = \frac{48\cdot\rho^{-2}}{\underset{\uparrow \text{can see}}{\delta_{AB}}} = \underbrace{\frac{48\cdot\rho^{-2}}{16/25}}_{\uparrow\ \text{4-PAM rep}} \overset{\text{v.s}}{>} \underbrace{\frac{48\,\rho^{-2}}{16/5}}_{\uparrow\ \text{BPSK-rotation}}$$

4-PAM repetition: $(y_1 \;\; y_2) = (h_1 x_1 \;\; h_2 x_1) + \underline{w}^T \Rightarrow \underline{y} = \underline{h} \cdot x_1$

$\Rightarrow \dim(\mathrm{span}(\underline{y})) = 1$ (single basis vector $\underline{h}$).

$\Rightarrow$ 1 real dimension / 2 c.u

BPSK rotation: 2 real dimensions / 2 c.u

because

L.I with Prob $\to 1$

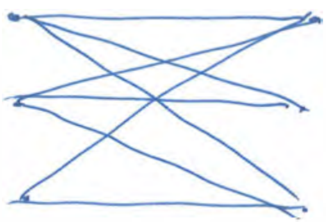
$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} h_1 & \\ & h_2 \end{pmatrix} R \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} h_1 \\ 0 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ h_2 \end{pmatrix} x_2 =$$

$$= \begin{pmatrix} h_1 x_1 \\ h_2 x_2 \end{pmatrix}$$

$\dim\left[\mathrm{span}(\underline{y})\right] = 2$ with prob 1

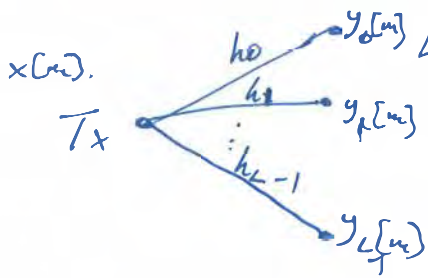
$\Rightarrow$ higher coding gain $\frac{16}{5}$ vs $\frac{16}{25}$

Antenna diversity



- offer rx & tx diversity: esp if time diversity is not an option.
- mimo communications & space-time codes (codes that "hedge" in space & time).

Receive diversity: (naturally in the presence of random fading).



L rx antenna. $\frac{\lambda}{4}$ route $\Rightarrow$ h_ℓ indep.

(one tap).

$$y_\ell[m] = h_\ell[m] \cdot x[m] + w_\ell[m] \qquad \ell = 0 \to L-1 \qquad w_\ell[m] \sim \mathcal{CN}(0, N_0).$$

- Quasi-static fading, no-time diversity.

$$y_\ell[m] = h_\ell \cdot x[m] + w_\ell[m] \qquad \ell = 0 \to L-1$$

$$\underline{y}[m] = \underline{h} \cdot x[m] + \underline{w}[m].$$

note that this is like (same l/o eq) ⇐ time diversity & rep. code

⇒ as before: $\tilde{y} = \dfrac{\underline{h}^H}{\|\underline{h}\|} \cdot \underline{y} = \|\underline{h}\| \cdot x + \dfrac{\underline{h}^H \cdot \underline{w}}{\|\underline{h}\|} = \|\underline{h}\| \cdot x + z \qquad z \sim \mathcal{CN}(0, N_0).$

Fix m

(recall from before
$\sum_{\ell=0} h_\ell^* w_\ell \sim \mathcal{CN}(0, \sum |h_\ell|^2 \cdot N_0)$
$\sim \mathcal{CN}(0, N_0 \|\underline{h}\|^2)$
⇒ $\dfrac{\underline{h}^H \underline{w}}{\|\underline{h}\|} \sim \mathcal{CN}(0, N_0)$)

Consider BPSK $x = \begin{cases} a \\ -a \end{cases}$ $\quad P = a^2$ ⇒ $\rho = \dfrac{P}{N_0} = \dfrac{a^2}{N_0}$ (complex)

since BPSK $\tilde{\tilde{y}} = \mathrm{Re}\{\tilde{y}\} = \|\underline{h}\| \cdot x + z' \qquad z' = \mathrm{Re}\{z\} \sim \mathcal{N}(0, \frac{N_0}{2}).$

$$P_{err|h} = Q\left(\frac{\|\underline{h}\| a}{\sqrt{\frac{N_0}{2}}}\right) = Q\left(\sqrt{\frac{2\|\underline{h}\|^2 \cdot a^2}{N_0}}\right) = Q\left(\sqrt{2\rho \|\underline{h}\|^2}\right).$$

(SIMO BPSK)

$$\Rightarrow P_{err|h} = Q\Big(\sqrt{2\cdot \underbrace{\tfrac{\|h\|^2}{L}}_{\text{diversity gain}} \cdot \underbrace{L\rho}_{\text{power gain}}}\Big) \rightarrow e^{-L}$$

- doubling L, provides 3dB power gain. (in front of e^{-L}).

- Power gain keeps on increasing as $L\uparrow$
  diversity gain diminishes (Law of diminishing returns).

since Note:

$$E_H\Big\{ Q\Big(\sqrt{2\cdot\tfrac{\|h\|^2}{L}\cdot L\rho}\Big)\Big\} \xrightarrow[\text{(since } \frac{\|h\|^2}{L} \xrightarrow{\text{a.s.}} 1)]{} E_H\Big[Q\big(\sqrt{2L\rho}\big)\Big] = Q\big(\sqrt{2L\rho}\big)$$

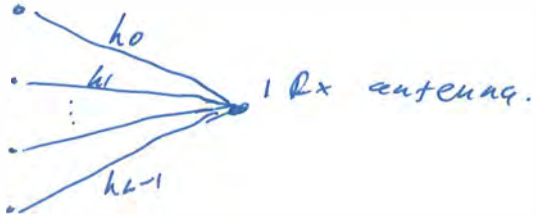
Just like AWGN, with better power.

Transmit Diversity

For now

L tx-antennas

$\frac{\lambda}{4}$ rule.



(stay in quasi-static
no time diversity).

- Tx same info from diff antennas.
- What is diversity that can be achieved?
  Easy probs.

	$t=0$	$t=1$	...	$t=L-1$
antenna 0	x	o	o ...	o
ant 1	o	x	o ...	o
⋮	⋮		x	o
⋮	⋮	o	...	
antenna $L-1$	.	.	. .	x

$$\underline{y}^T = [y_0 \; y_1 \cdots y_{L-1}] = [h_0 \; h_1 \; \cdots \; h_{L-1}] \begin{bmatrix} x & & 0 & \\ & x & & \\ & & x & \\ 0 & & & x & \\ & & & & \ddots \; x \end{bmatrix} + \underline{\omega}^T$$

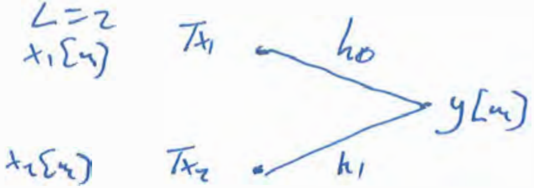
$\Rightarrow \underline{y} = \underline{h} \cdot x + \underline{\omega}$

$\Rightarrow$ Just like time diversity
with really low rate. $\left(\frac{1}{L}\right)$

Can do better.

- Explore the code in previous standards.

ALAMOUTI CODE : 2×1 MISO (flat-fading).



– Transmission over T time slots, & L antennas.

$\Rightarrow$ any such code is a collection of $L \times T$ matrices

$$\mathcal{X} = \left\{ \begin{bmatrix} x_1[1] & x_1[2] & \cdots & x_1[T] \\ x_2[1] & x_2[2] & \cdots & x_2[T] \\ \vdots & & & \\ & & & x_L[T] \end{bmatrix} \right\} \longleftarrow \text{General } L \times T \text{ ST-Codes}$$

In Alamouti code case, $T=2$ & $L=2$

$$\Rightarrow \mathcal{X} = \left\{ \begin{bmatrix} x_1[1] & x_1[2] \\ x_2[1] & x_2[2] \end{bmatrix} \right\}.$$

General 1×2 ST-code

$$\Rightarrow \begin{matrix} x_1[m] \xrightarrow{h_1} \\ x_2[m] \xrightarrow{h_2} \end{matrix} y[m] = \underline{h}^T \cdot \begin{bmatrix} x_1[m] \\ x_2[m] \end{bmatrix}$$

$$\Rightarrow \underline{y}^T_{1\times T} = \underline{h}^T \mathcal{X} + \underline{w}^T$$

with a specific structure

$$\mathcal{X}_{alam} = \left\{ \begin{vmatrix} u_1 & -u_2^* \\ u_2 & u_1^* \end{vmatrix} \quad u_1, u_2 \in \mathbb{C}, \text{ indep} \right\}.$$

$$[y[1] \quad y[2]] = [h_1 \quad h_2] \begin{bmatrix} u_1 & -u_2^* \\ u_2 & u_1^* \end{bmatrix} + [w[1] \quad w[2]].$$

Rewrite
$$\begin{bmatrix} y[1] \\ y^*[2] \end{bmatrix} = \begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} w[1] \\ w^*[2] \end{bmatrix}.$$

$$\Rightarrow \underline{y}' = \begin{bmatrix} y[1] \\ y^*[2] \end{bmatrix} = u_1 \begin{bmatrix} h_1 \\ h_2^* \end{bmatrix} + u_2 \begin{bmatrix} h_2 \\ -h_1^* \end{bmatrix} + \begin{bmatrix} w[1] \\ w^*[2] \end{bmatrix}.$$

but note that

$$\begin{bmatrix} h_1 \\ h_2^* \end{bmatrix} \;\&\; \begin{bmatrix} h_2 \\ -h_1^* \end{bmatrix} \text{ are } \perp$$

$\Rightarrow$ problem decomposes to two separate scalar problem

because

$$r_1 = \frac{\begin{pmatrix} h_1 \\ h_2^* \end{pmatrix}^H}{\left\|\begin{pmatrix} h_1 \\ h_2^* \end{pmatrix}\right\|} \cdot \begin{pmatrix} y[1] \\ y^*[2] \end{pmatrix} = \cdot \|(h_1\ h_2)\| \cdot u_1 + \overset{\sim \mathcal{CN}(0,}{w_1}$$

$$\& \; r_2 = \frac{\begin{pmatrix} h_2 \\ -h_1^* \end{pmatrix}^H}{\|\underline{h}\|^2} \cdot \underline{y}' = \|\underline{h}\| \cdot u_2 + w_2$$

$$\left(\text{just like if it was } \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ldots \right.$$

Let us compare Alamouti v.s Repetition coding.

Alamouti BPSK, repetition 4-PAM (both 1 bit/c.u).

Alamouti: equiv. scalar detection $r_i = \|\underline{h}\| . u_1 + \omega_1$

4-PAM: equiv " " $\tilde{y} = \|\underline{h}\| . x + \omega_1$

same rate

say BPSK $\{-a, a\}$ say 4-PAM $\{-3a, -a, a, 3a\}$

so that same minimum distance between codewords.

$\Rightarrow$ $\approx$ same Prob of error.

but main difference is in power used.

4-PAM rep $E\{|\underline{x}|^2\} = 5b^2 = 5a^2$ $\left[\begin{array}{l} x \to 5b^2 \\ x \to 5b^2 \end{array}\right]$

BPSK Alam: $E\{|\underline{x}|^2\} = 2a^2$ $\left[\begin{array}{l} x_1 \to a, -a \\ x_2 \to a, -a \end{array}\right]$

($\Rightarrow$ Alamouti 2.5 times less power for same rate & same)

Reason: Recall 4-PAM repetition

(signal part). $\underline{y} = \underline{h} \cdot x$ $\Rightarrow \dim\{\text{span}\{\underline{h}\,x\}\} = 1$ real dime

2 time-slots.

Alamouti–BPSK:

$$\underline{y} = \begin{pmatrix} h_1 \\ h_2^* \end{pmatrix} u_1 + \begin{pmatrix} h_2 \\ -h_1^* \end{pmatrix} u_2$$

$\perp$ $\Rightarrow$ span is 2 (2 basis vectors).

that is why you have power gain. $\Bigg\}$ $\Rightarrow$ 2 real dimensions in 2 time slots