# MOBCOM-MIdtermF2024

January 22, 2025

---
Mobile Communication Techniques
---

Petros Elia, elia@eurecom.fr
Midterm Exam
November 21st, 2024
Time: 9:00-10:00

---

Instructions

- Exercises fall in categories of 1-point and 2-point exercises.

- Total of $11 \times 1 + 2 \times 2 = 15$ points.

- NOTE!!! The exam will be evaluated, out of 13 points. Any points you get beyond 13 points, will be offered as extra bonus.

- Each answer should be clearly written, and the solution should be developed in detail.

- Mathematical derivations need to show all steps that lead to the answer.

- Complete as many exercises as you can. Don't spend too much time on an individual question.

- There is NO penalty for incorrect solutions.

- If in certain cases you are unable to provide rigorous mathematical proofs, go ahead and provide intuitive justification of your answers. Partial credit will be given.

- Calculators are not allowed.

- You are allowed your class notes and class book.

Hints - equations - conventions:

- Notation
  - SISO = single-input single-output, MISO = multiple-input single-output, SIMO = single-input multiple- output, MIMO = single-input multiple-output,
  - $R$ represents the rate of communication in bits per channel use (b.p.c.u),
  - $\rho$ represents the SNR (signal to noise ratio),
  - $w$ will denote additive noise which will be distributed as a circularly symmetric Gaussian random variable $\mathbb{CN}(0, N_0)$. If $N_0$ is not specified, then set $N_0 = 1$,

– $h_i$ will denote independent fading scalar coefficients which will be distributed as circularly symmetric Gaussian random variables $\mathbb{C}\mathcal{N}(0,1)$.
- GOOD LUCK!!

**1)** (1 point). In a multi-path fading scenario with delay spread $6\mu s$ and $L = 3$ channel taps, what is the operational bandwidth $W$?

---

In a multi-path fading scenario, the **operational bandwidth** $W$ is inversely proportional to the delay spread $T_{\max}$, which is the maximum delay between the arrival of multipath components. The operational bandwidth ensures that the channel is **frequency-flat**, meaning that the coherence bandwidth $B_c$ is greater than or equal to $W$.

The coherence bandwidth $B_c$ is given approximately by:

$B_c \approx \frac{1}{T_{\max}}$

Here, $T_{\max} = 6\,\mu s$. Substituting this value:

$B_c \approx \frac{1}{6 \times 10^{-6}} = 166.67 \,\text{kHz}$

Thus, the operational bandwidth $W$ should be:

$W \le B_c = 166.67 \,\text{kHz}$.

So, the **operational bandwidth** is approximately **166.67 kHz**.

**2)** (1 point). Imagine a given SNR equal to $\rho$, and imagine that we are operating over a (quasi-static) Rayleigh fading SISO channel. Can you describe a code that achieves probability of error approximately equal to $P_e \approx \rho^{-4}$, and rate equal to $R = 2$ bpcu.

---

To achieve $P_e \approx \rho^{-4}$ at $R = 2$ bpcu in a quasi-static Rayleigh fading SISO channel:

- **Code**: Use a 4-QAM constellation with a rate-2 space-time block code (e.g., Alamouti code or a 4-dimensional lattice code).
- **Diversity**: Achieve diversity order 4 by designing the code to span 4 independent dimensions.
- **Performance**: Ensures $P_e \sim \rho^{-4}$ with increasing SNR.

**3)** (1 point). How much time diversity will we get with the following SISO (time-diversity) channel model

$$[y_1 \ y_2 \ y_3] = [h_1 u_1 \quad h_2(u_1 + u_2) \quad h_3 u_2] + [w_1 \ w_2 \ w_3]$$

where the $u_1, u_2, u_3$ are independent PAM elements. Justify your answer.

---

To determine the **time diversity** in the given channel model:

**Channel Model** $[y_1 \ y_2 \ y_3] = [h_1 u_1 \quad h_2(u_1 + u_2) \quad h_3 u_2] + [w_1 \ w_2 \ w_3]$, where $u_1, u_2, u_3$ are independent PAM symbols, $h_1, h_2, h_3$ are the channel coefficients, and $w_1, w_2, w_3$ are noise terms.

**Analysis**

1. **Definition of Time Diversity**:
   - Time diversity is determined by the number of independently faded channel coefficients $(h_1, h_2, h_3)$ that affect the transmitted symbols.
2. **Observation of Dependencies**:
   - $y_1$ depends on $h_1 u_1$.
   - $y_2$ depends on $h_2(u_1 + u_2)$.
   - $y_3$ depends on $h_3 u_2$.
3. **Diversity Order**:
   - $u_1$ is present in both $y_1$ and $y_2$, thus contributing to diversity through $h_1$ and $h_2$.
   - $u_2$ is present in both $y_2$ and $y_3$, contributing to diversity through $h_2$ and $h_3$.

Since $u_1$ and $u_2$ are affected by two **independent channel coefficients** each, the effective **time diversity order** is:

Time Diversity Order = min(number of independent fades per symbol) = 2.

**Justification**    The system achieves a time diversity order of 2 because each transmitted symbol $u_1$ and $u_2$ is observed across two independently faded channels ($h_1, h_2$ for $u_1$; $h_2, h_3$ for $u_2$). The third symbol $u_3$ does not contribute additional diversity as it is only affected by $h_3$.

4) (1 point). In a SISO case, what is the degrees of freedom (DOF) if we have a time-diversity code (spanning three channel uses) of the form $\mathcal{X} = [u_1 + u_2 \quad u_1 + u\_3 \quad u_2 + u_3]$ where the $u_1, u_2, u_3, u_4$ are independent 16-PAM elements?

---

To determine the **Degrees of Freedom (DOF)** for the given time-diversity code:

**Code Representation**    The transmitted codeword over three channel uses is:

$$\mathcal{X} = \begin{bmatrix} u_1 + u_2 & u_1 + u_3 & u_2 + u_3 \end{bmatrix},$$

where $u_1, u_2, u_3$ are **independent symbols** from a 16-PAM constellation.

**Degrees of Freedom (DOF)**

- The **DOF** is the number of **independent information symbols** transmitted across the channel uses.
- In this case, $u_1, u_2, u_3$ are independent, and each symbol contributes one degree of freedom.

Since the code spans **three channel uses**, the DOF is:

$\text{DOF} = \frac{\text{Number of Independent Symbols}}{\text{Number of Channel Uses}} = \frac{3}{3} = 1.$

**Final Answer**

- **DOF**: 1.

5) (1 point). For the case of time diversity in the SISO (quasi-static) fading channel, what is the advantage and the disadvantage of the repetition code, compared to uncoded transmission.

---

- **Advantage**: Repetition code improves reliability by providing diversity gain, reducing the error probability in fading channels.

- **Disadvantage**: It reduces spectral efficiency by lowering the transmission rate due to redundant transmissions.

**6)** (1 point). In a SISO case, what is the DOF and the rate (in bpcu), of the following time-diversity code (three channel uses) that takes the form $\mathcal{X} = [u_1 + u_4 \quad u_2 \quad u_1 + u_2 + u_3]$ where the $u_1, u_2, u_3, u_4$ are independent 64-QAM elements?

---

To analyze the **Degrees of Freedom (DOF)** and **rate** for the given time-diversity code:

**Code Representation**   The transmitted codeword over three channel uses is:

$$X = [u_1 + u_4 \quad u_2 \quad u_1 + u_2 + u_3],$$

where $u_1, u_2, u_3, u_4$ are independent symbols from a 64-QAM constellation.

**1. Degrees of Freedom (DOF):**

- The **DOF** corresponds to the number of **independent information symbols** transmitted across the given channel uses.
- Here, $u_1, u_2, u_3, u_4$ are **independent symbols**, so there are **4 independent symbols** transmitted over **3 channel uses**.

$$\text{DOF} = \frac{\text{Number of Independent Symbols}}{\text{Number of Channel Uses}} = \frac{4}{3}.$$

**2. Rate (in bpcu):**

- Each symbol is from a 64-QAM constellation, which carries $\log_2(64) = 6$ bits per symbol.
- Since 4 symbols are transmitted over 3 channel uses, the rate $R$ is:

$$R = \frac{\text{Total Bits Transmitted}}{\text{Number of Channel Uses}} = \frac{4 \cdot 6}{3} = 8 \,\text{bpcu}.$$

**Final Answer:**

- **DOF**: $\frac{4}{3}$.

- **Rate**: $8 \,\text{bpcu}$.

**7)** (1 point). Imagine a SISO channel model with correlated fading, where the first fading coefficient (first transmission slot) is $h_1 = h_1' \times h_2'$, and the second fading coefficient (second transmission slot) is $h_2 = h_2'$, where $h_1', h_2' \sim i.i.d\ \mathbb{CN}(0,1)$. What is the maximum diversity we can achieve here?

[ ]:

**8)** (1 point). Describe the steps of converting a binary vector detection problem over a time diversity fading channel, into a scalar detection problem. Imagine that you are sending BPSK symbols using a repetition code, and consider $\mathbb{CN}(0, N_0)$ noise.

[ ]:

**9)** (1 point). Consider a deep-space communications scenario, where the received SNR is equal to 20dB. If you assume low rate communications, what do you expect the probability of error to be?

[ ]: 

**10)** (1 point). What is the approximate coherence time $T_c$ in a typical urban wireless network if you are driving approximately 20 kilometers per hour?

[ ]: 

**11)** (1 point). Consider communication over a SISO fading channel with a delay spread of $T_d = 3\mu s$ and a signal bandwidth of $W = 1$ MHz. - Write all the received signals, if we only send $x[0]$ and then we stop transmitting.

[ ]: 

**12)** *(2 points)*. What is the optimal diversity order over a 2×1 MISO channel $h = [h_1\ h_2], h_i \sim$ *i.i.d* $\mathbb{CN}(0,1)$? - In the same channel as above (again with no time diversity), consider a space time code whose matrices take the form

$$\begin{bmatrix} x_0 & x_1 \\ x_1 & x_0 \end{bmatrix}$$

where the $x_i$ are drawn independently from a QAM constellation. Will this code achieve optimal diversity order? (argue why or why not) - What is the diversity order achieved by the Alamouti code, over this $2 \times 1$ MISO channel? (again, you can just argue in words)

[ ]: 

**13)** *(EXTRA CREDIT: 2 points)*. Consider a setting where the transmit antenna array has length of 50 cm, the received antenna array has size 20cm, the transmission frequency is 1000 MHz, the signal bandwidth is 1 MHz, the channel coherence time is $T_c = 21$ ms, and the coding duration is $T_{coding} = 7$ms. - How much diversity can you get, in total?

---

To calculate the **total diversity**, we need to consider **spatial diversity**, **frequency diversity**, and **time diversity**. Let's analyze each component based on the given parameters:

**1. Spatial Diversity** Spatial diversity depends on the number of independent spatial paths between the transmit and receive antenna arrays, calculated using the formula: $D_{\text{spatial}} = \left\lfloor \frac{2L_t}{\lambda} \right\rfloor \cdot \left\lfloor \frac{2L_r}{\lambda} \right\rfloor$, where: - $L_t = 50\,\text{cm} = 0.5\,\text{m}$: Transmit antenna array length, - $L_r = 20\,\text{cm} = 0.2\,\text{m}$: Receive antenna array length, - $\lambda = \frac{c}{f}$: Wavelength of the transmitted signal.

At $f = 1000\,\text{MHz}$ ($f = 1\,\text{GHz}$): $\lambda = \frac{3\times10^8}{1\times10^9} = 0.3\,\text{m}$.

Substituting: $D_{\text{spatial}} = \left\lfloor \frac{2\cdot0.5}{0.3} \right\rfloor \cdot \left\lfloor \frac{2\cdot0.2}{0.3} \right\rfloor = \lfloor 3.33 \rfloor \cdot \lfloor 1.33 \rfloor = 3 \cdot 1 = 3$.

**2. Frequency Diversity** Frequency diversity depends on the relationship between the signal bandwidth ($W$) and the coherence bandwidth ($W_c$). The coherence bandwidth is inversely related to the channel delay spread ($T_d$): $W_c \approx \frac{1}{T_d}$.

Assuming the coherence bandwidth $W_c$ is smaller than the signal bandwidth $W = 1\,\text{MHz}$, the **frequency diversity** is approximately: $D_{\text{frequency}} = \frac{W}{W_c}$. Without explicit $T_d$, assume $W_c \approx 100\,\text{kHz}$: $D_{\text{frequency}} \approx \frac{1\,\text{MHz}}{100\,\text{kHz}} = 10$.

**3. Time Diversity**   Time diversity depends on the relationship between the **coding duration** ($T_{\text{coding}}$) and the **channel coherence time** ($T_c$): $D_{\text{time}} = \frac{T_c}{T_{\text{coding}}}$.

Given: $T_c = 21\,\text{ms}, \quad T_{\text{coding}} = 7\,\text{ms}, D_{\text{time}} = \frac{21}{7} = 3$.

**Total Diversity**   The total diversity is the product of the individual diversity components: $D_{\text{total}} = D_{\text{spatial}} \cdot D_{\text{frequency}} \cdot D_{\text{time}}$.

$$\boxed{D_{\text{total}} = 3 \cdot 10 \cdot 3 = 90}$$

Here's the breakdown of the **units** for each diversity component:

1. **Spatial Diversity ($D_{\text{spatial}}$):**
   - **Unitless**: It is a count of independent spatial paths, determined by the number of transmit and receive antenna elements relative to the wavelength.
2. **Frequency Diversity ($D_{\text{frequency}}$):**
   - **Unitless**: It represents the ratio of signal bandwidth ($W$) to coherence bandwidth ($W_c$), which both have units of Hz, making the ratio unitless.
3. **Time Diversity ($D_{\text{time}}$):**
   - **Unitless**: It is the ratio of channel coherence time ($T_c$) to coding duration ($T_{\text{coding}}$), both measured in seconds, so the ratio is unitless.

**Summary:**   All diversity components—spatial, frequency, and time—are **unitless** because they are ratios or counts, and the **total diversity** is also unitless.

[ ]: