REVIEW

January 25, 2025

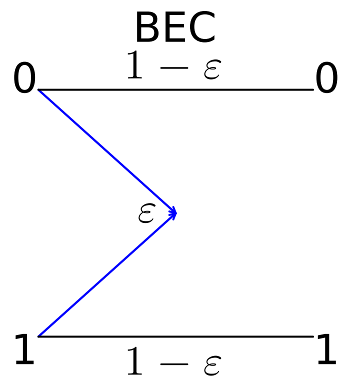
Quick Summary of P2P Channels

[1]:
```julia
using Plots, LaTeXStrings
```

[2]:
```julia
# Define coordinates for transmitted and received states
tx = [0, 1]
rx = [0, 1]

# Define probabilities as labels
p = [L"\epsilon", L"1 - \epsilon"];
```

[3]:
```julia
# Plot the BSC diagram
plot(grid=false
    , xaxis=false, yaxis=false
    , framestyle=:none, size = (200,200)
    , title = "BEC"
)

plot!([0, 0.5], [1, 0.5], arrow=:arrow, label="", color=:blue) # p line
plot!([0, 0.5], [0, 0.5], arrow=:arrow, label="", color=:blue) # p line
plot!([0, 1], [1, 1], label="", color=:black) # 1-p
plot!([0, 1], [0, 0], label="", color=:black)

# Annotate the graph
annotate!(-0.05, 1.05, tx[1]); annotate!(1.05, 1.05, rx[1])
annotate!(-0.05, -0.05, tx[2]); annotate!(1.05, -0.05, rx[2])
annotate!(0.4, 0.5, p[1]); annotate!(0.5, -0.1, p[2]); annotate!(0.5, 1.1, p[2])
```

[3]:



1 Binary Erasure Channel (BEC)

1. **Channel Model**:
   - Transmits binary symbols (0 or 1).
   - Each transmitted bit is either:
     - **Received correctly** with probability $1-\epsilon$, or
     - **Erased** with probability ϵ, represented as an erasure symbol (e).

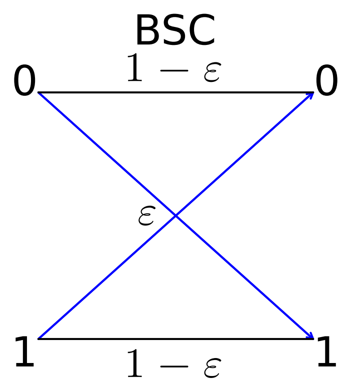
   Example:
   - $0 \to 0$ or e,
   - $1 \to 1$ or e.
2. **Capacity** (C): $\boxed{C = 1-\epsilon}$
   - 1: Maximum capacity with no erasures ($\epsilon = 0$).
   - ϵ: Fraction of bits erased by the channel, reducing capacity.
3. **Behavior**:
   - $\epsilon = 0$: Perfect channel, $C = 1$.
   - $\epsilon = 1$: Completely erasing channel, $C = 0$.
   - For $0 < \epsilon < 1$: Capacity decreases linearly as ϵ increases.

Compact Intuition: The **Binary Erasure Channel** (BEC) capacity is the fraction of bits successfully transmitted. Erasures (ϵ) reduce capacity by removing information from the channel.

[4]:
```
# Plot the BSC diagram
plot(grid=false
    , xaxis=false, yaxis=false
    , framestyle=:none, size = (200,200)
    , title = "BSC"
)

plot!([0, 1], [1, 0], arrow=:arrow, label="", color=:blue) # p line
plot!([0, 1], [0, 1], arrow=:arrow, label="", color=:blue) # p line
plot!([0, 1], [1, 1], label="", color=:black) # 1-p
```

```
plot!([0, 1], [0, 0], label="", color=:black)

# Annotate the graph
annotate!(-0.05, 1.05, tx[1]); annotate!(1.05, 1.05, rx[1])
annotate!(-0.05, -0.05, tx[2]); annotate!(1.05, -0.05, rx[2])
annotate!(0.4, 0.5, p[1]); annotate!(0.5, -0.1, p[2]); annotate!(0.5, 1.1, p[2])
```

[4]:



2 Binary Symmetric Channel (BSC)

1. **Channel Model**:
   - Transmits binary symbols (0 or 1).
   - Each bit has a probability ϵ of being flipped.
   - Error probability: $P(0 \to 1) = P(1 \to 0) = \epsilon$.
   - Correct transmission probability: $P(0 \to 0) = P(1 \to 1) = 1 - \epsilon$.
2. **Capacity** (C_{BSC}): $\boxed{C_{BSC} = 1 - H_2(\epsilon)}$
   - 1: Maximum capacity without errors.
   - $H_2(\epsilon)$: Binary entropy function, representing uncertainty due to errors.
3. **Binary Entropy Function** ($H_2(\epsilon)$): $\boxed{H_2(\epsilon) = -\epsilon \cdot \log_2(\epsilon) - (1 - \epsilon) \cdot \log_2(1 - \epsilon)}$
   - $H_2(0) = 0$: No errors, full capacity.
   - $H_2(0.5) = 1$: Maximum uncertainty, no capacity.
4. **Behavior**:
   - $\epsilon = 0$: Perfect channel, $C_{BSC} = 1$.
   - $\epsilon = 0.5$: Completely noisy, $C_{BSC} = 0$.
   - For $0 < \epsilon < 0.5$: Capacity decreases as ϵ increases.

Compact Intuition: The BSC capacity is the theoretical maximum rate of reliable data transmission, reduced by the uncertainty caused by errors. Lower ϵ means higher capacity, while higher ϵ reduces it.

2.0.1 AWGN Channel Summary

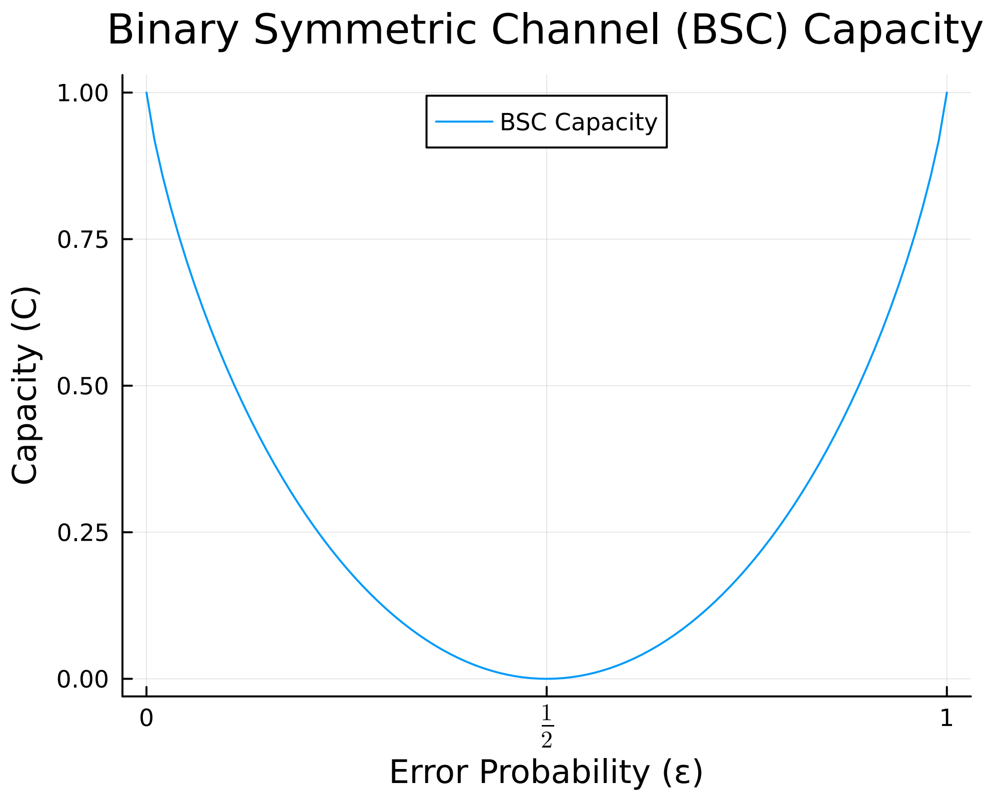
1. **Channel Model**: $\boxed{y = x + w, \quad w \sim N(0, N_0)}$
   - x: Transmitted signal.
   - y: Received signal.
   - w: Gaussian noise with zero mean and variance N_0 the noise power spectral density.
2. **Signal Power**: $\boxed{P = E[|x|^2]}$
3. **Signal-to-Noise Ratio (SNR)**: $\boxed{SNR = \frac{P}{N_0}}$
4. **Channel Capacity**: $\boxed{C = \log_2(1 + SNR) = \log_2\left(1 + \frac{P}{N_0}\right)}$
   - C: Maximum achievable data rate (in bps/Hz).
5. **Key Behavior**:
   - $P \uparrow$ (high signal power): $C \uparrow$ (more capacity).
   - $N_0 \uparrow$ (high noise): $C \downarrow$ (less capacity).

Insight: The AWGN channel capacity quantifies the theoretical limit of reliable communication over a noisy channel.

[5]:
```julia
using Plots

# Define the binary entropy function
function H(ε)
    if ε == 0 || ε == 1
        return 0.0
    end
    return -ε * log2(ε) - (1 - ε) * log2(1 - ε)
end

# Define the BSC capacity function
function C(ε)
    return 1 - H(ε)
end

# Generate values of ϵ from 0 to 1
ε_values = 0:0.01:1
capacities = C.(ε_values)

# Plot the capacity curve
plot(ε_values, capacities,
     label = "BSC Capacity"
    , xlabel = "Error Probability (ε)", ylabel = "Capacity (C)"
    , title = "Binary Symmetric Channel (BSC) Capacity"
    , legend = :top, size = (500,400)
```

```
    , xticks = (0:0.5:1, ["0", L"\frac{1}{2}", "1"])
)
```

[5]:



2.1 Communication System Components:

1. **Source**: Produces the information $\underline{x} = (x_1, \ldots, x_n)$.
2. **Encoder**: Transforms $\underline{x}$ into a codeword $\underline{c} = (c_1, \ldots, c_n)$, adding redundancy.
3. **Channel**: Transmits $\underline{c}$, introducing errors, resulting in $\underline{y} = (y_1, \ldots, y_n)$.
4. **Decoder**: Processes $\underline{y}$ to estimate $\hat{\underline{x}} = (\hat{x}_1, \ldots, \hat{x}_n)$, correcting errors.
5. **Sink**: Receives $\hat{\underline{x}}$, ideally matching $\underline{x}$.

2.1.1 Objective:

Ensure $\hat{\underline{x}} = \underline{x}$ despite channel errors.

2.1.2 Properties of Linear Block Code

1. **Linearity**:
   - The set of codewords $\mathcal{X}$ forms a **linear subspace** of $\mathbb{F}_2^n$.
   - Any linear combination of codewords is also a valid codeword: $\underline{v}_1 + \underline{v}_2 \in \mathcal{X},\ \forall \underline{v}_1, \underline{v}_2 \in \mathcal{X}$.
2. **Generator Matrix (G)**:
   - The $k \times n$ generator matrix G maps k-bit input vectors ($\underline{u} \in \mathbb{F}_2^k$) to n-bit codewords ($\underline{v} = \underline{u}^\top G$).
   - Defines the structure of the codebook $\mathcal{X}$.
3. **Code Rate (R)**:
   - The ratio of information bits to total bits: $R = \frac{k}{n}$
   - Indicates the efficiency of the code.
4. **Code Size ($|\mathcal{X}|$)**:
   - The number of unique codewords: $|\mathcal{X}| = 2^k$
5. **Minimum Hamming Distance ($d_{\min}$)**:
   - The smallest Hamming distance between any two distinct codewords.
   - Determines the error-detecting and error-correcting capability: $t = \left\lfloor \frac{d_{\min}-1}{2} \right\rfloor$
     - t: Maximum correctable errors.
     - $d_{\min} - 1$: Maximum detectable errors.
6. **Parity Check Matrix (H)**:
   - The H matrix defines the null space of G: $HG^\top = 0$
   - Used to verify codewords: $\underline{v}H^\top = 0 \implies \underline{v} \in \mathcal{X} \subseteq \mathbb{F}_2^n$
7. **Error Detection and Correction**:
   - Error detection: Capable of detecting up to $d_{\min} - 1$ errors.
   - Error correction: Can correct up to $\lfloor \frac{d_{\min}-1}{2} \rfloor$ errors.
8. **Redundancy**:
   - The number of redundant bits added for error correction is $n-k$, where n is the codeword length.

2.1.3 Compact Summary:

- **Linear subspace**: Codewords form a subspace of $\mathbb{F}_2^n$.
- **Size**: $|\mathcal{X}| = 2^k$.
- **Rate**: $R = \frac{k}{n}$.
- **Error capabilities**: Based on $d_{\min}$.
- Defined by:
  - **Generator matrix (G)**.
  - **Parity check matrix (H)**.

Error Correction Error Correction **Code** $\mathcal{X}$
Linear error corr. **Code** $\mathcal{X}$

Code Rate: $\boxed{R = \frac{k}{n}}$

Explanation:

- **Code Rate** (R) is the fraction of a codeword used for information:
  - k: Number of information bits.
  - n: Total number of bits in the codeword (information + redundancy).
- **Trade-off**:
  - Higher R ($R \to 1$): More efficient but less error protection.
  - Lower R ($R \to 0$): Less efficient but better error correction.

Linear Block Code Definition: $\boxed{\mathcal{X} = \{\underline{v} = \underline{u}^\top G,\ \underline{u} \in \mathbb{F}_2^k\}}$

Explanation:

- $\mathcal{X}$: The **set of all codewords** in the code (codebook).
- $\underline{u}$: A binary **message vector** of length k ($\underline{u} \in \mathbb{F}_2^k$).
- G: The **generator matrix** ($k \times n$) used to map $\underline{u}$ to a codeword.
- $\underline{v}$: A binary **codeword** of length n ($\underline{v} \in \mathbb{F}_2^n$).

Key Points:

1. Each $\underline{u}$ maps to a unique $\underline{v}$ via $\underline{v} = \underline{u}^\top G$.
2. The total number of codewords is 2^k (one for each $\underline{u}$).
3. $\mathcal{X}$ is a **linear subspace** of dimension k in $\mathbb{F}_2^n$.

Compact Summary: $\mathcal{X}$ is the **codebook** of a linear block code, where each codeword $\underline{v}$ is generated by multiplying a k-bit message vector $\underline{u}$ with the generator matrix G.

Code Size: $|\mathcal{X}| = 2^k$

- k-bit message vectors ($\underline{u} \in \mathbb{F}_2^k$) are mapped to n-bit codewords ($\underline{v} = \underline{u}^\top G$) via the generator matrix G.

- The codebook $\mathcal{X}$ forms a linear subspace with 2^k unique codewords, corresponding to the k-bit input combinations.

- **Minimum Hamming Distance**:
  $d_{\min} = \min\{d_H(x_i, x_j)\},\ \forall x_i, x_j \in \mathcal{X},\ x_i \neq x_j$ (Hamming distance)

- **Linear Code Minimum Distance**:
  $d_{\min} = \min_{\left(\substack{\underline{x}_i, \underline{x}_j \in \mathcal{X} \\ x_i \neq x_j}\right)} \left\{d_H(x_i, x_j)\right\} = \min_{x \in \mathcal{X}} \left\{W_H(\underline{x})\right\}$

- **Parity Check Relation**:
  $\underline{v} H^\top = 0 \implies \underline{v} \in \mathcal{X} \subseteq \mathbb{F}_2^n$

2.1.4 Parity Check Matrix

- $H \in \mathbb{F}_2^{(n-k) \times n}$
- $\dim(\mathrm{Im}(G)) = k$
- $H = \mathrm{null}(G^\top),\ \dim = n - k$

[6]:
```julia
using LinearAlgebra

# Define the parity check matrix H (size 3x7 for (7, 4) code)
H = [
    1 0 0 1 1 1 0;
    0 1 0 1 1 0 1;
    0 0 1 1 0 1 1
]

# Define the generator matrix G
G = [
    1 0 0 0 1 1 0;
    0 1 0 0 1 0 1;
    0 0 1 0 0 1 1;
    0 0 0 1 1 1 1
];
```

[7]:
```julia
# Message vector
u = [1 0 1 0]

# Generate codeword
v = mod.(u * G, 2)
println("Generated codeword: ", v)  # Should be a valid codeword
```

```
Generated codeword: [1 0 1 0 1 0 1]
```

[8]:
```julia
# Parity check validation
parity_check = mod.(v * H', 2)
println("Parity check result: ", parity_check)  # Should be [0, 0, 0]
```

```
Parity check result: [0 0 0]
```

[9]:
```julia
using LinearAlgebra

# Define the generator matrix G (4x7 for (7,4) code)
G = [
    1 0 0 0 1 1 0;
    0 1 0 0 1 0 1;
    0 0 1 0 0 1 1;
    0 0 0 1 1 1 1
]

# Generate all possible messages (4-bit binary combinations)
messages = [bitstring(i)[end-3:end] for i in 0:2^4-1]  # 4-bit binary strings
u = [parse.(Int, split(m, ""))' for m in messages]; @show u;  # Convert to row␣
↪vectors (1x4) u\underbar

# Generate all codewords using G
```

```
 = [mod.(u * G, 2) for u in u ]; @show 

# Define a function to compute Hamming distance
function hamming_distance(v , v)
    sum(v .!= v)  # Count differing elements
end

# Compute the minimum Hamming distance using a comprehension
d = minimum(
    hamming_distance([i], [j]) for (i, j)
        in Iterators.product(1:length(), 1:length()) if i < j
)

# Output the result
println("Minimum distance of the code: ", d)
```

```
u = Adjoint{Int64, Vector{Int64}}[[0 0 0 0], [0 0 0 1], [0 0 1 0], [0 0 1 1],
[0 1 0 0], [0 1 0 1], [0 1 1 0], [0 1 1 1], [1 0 0 0], [1 0 0 1], [1 0 1 0], [1
0 1 1], [1 1 0 0], [1 1 0 1], [1 1 1 0], [1 1 1 1]]
 = [[0 0 0 0 0 0 0], [0 0 0 1 1 1 1], [0 0 1 0 0 1 1], [0 0 1 1 1 0 0], [0 1 0
0 1 0 1], [0 1 0 1 0 1 0], [0 1 1 0 1 1 0], [0 1 1 1 0 0 1], [1 0 0 0 1 1 0], [1
0 0 1 0 0 1], [1 0 1 0 1 0 1], [1 0 1 1 0 1 0], [1 1 0 0 0 1 1], [1 1 0 1 1 0
0], [1 1 1 0 0 0 0], [1 1 1 1 1 1 1]]
Minimum distance of the code: 3
```

2.1.5 Parity Check Matrix (H):

1. **Dimensions**: $H \in \mathbb{F}_2^{(n-k)\times n}$
   - n: Codeword length.
   - $n-k$: Number of parity checks.
2. **Relation to** G:
   - G: Generates the code subspace with: $\dim(\text{Im}(G)) = k$
   - H: Defines the null space of $G^\top$: $H = \text{null}(G^\top)$
3. **Validation Condition**:
   - A valid codeword satisfies: $\underline{v}H^\top = 0$
4. **Null Space Dimension**: $\dim(\text{null}(G^\top)) = n-k$

2.1.6 Systematic Code:

A linear block code is called **systematic** if: 1. **Codeword Structure**:

$$\boxed{\underline{v} = [\text{information bits} \mid \text{parity bits}]}$$

- The first k bits are the unaltered information bits. - The last $n-k$ bits are the parity bits.

2. **Generator Matrix**:

$$\boxed{G = [I_k \vdots P_{k\times(n-k)}]_{k\times n}}$$

- I_k: Identity matrix for information bits.
- P: Parity matrix for redundancy.

3. **Parity Check Matrix**:

$$\boxed{H = [I_{n-k} \vdots P^{\top}_{(n-k)\times k}]}$$

4. **Benefits**:
   - Direct access to information bits.
   - Simplifies encoding and decoding.

2.1.7 The Generator Matrix (G):

1. **Definition**:
   - $G \in \mathbb{F}_2^{k\times n}$, maps k-bit messages $\underline{u}$ to n-bit codewords $\underline{v}$: $\underline{v} = \underline{u}^{\top} G$
2. **Structure** (for systematic codes):

$\boxed{G = [I_k \vdots P_{k\times(n-k)}]_{k\times n}}$ - I_k: $k \times k$ identity matrix (information bits). - P: $k \times (n-k)$ parity matrix (defines redundancy).

4. **Steps to Generate G**:
   - Determine code parameters: n, k, and $n-k$.
   - Design P to ensure $HG^{\top} = 0$, where H is the parity-check matrix.
   - Combine I_k and P to construct G.
5. **Example** ($(7,4)$ Hamming Code): The $(7,4)$ Hamming code has:

- $k = 4$ information bits.
- $n = 7$ total bits ($n - k = 3$ parity bits).

-> Steps:

- I_k is a 4×4 identity matrix.
- P is a 4×3 matrix: $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$
- Combine I_k and P to get G: $G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$

G encodes $k = 4$ information bits into $n = 7$ codeword bits.

2.1.8 The parity check matrix (H)

For systematic linear block codes like the Hamming code, the parity check matrix H is typically written in the form:

$$\boxed{H_{(n-k)\times n} = [I_{n-k} \vdots P^\top_{(n-k)\times k}]}$$

This is the **standard form** of the parity check matrix.

2.1.9 Formulation of H:

1. **Structure**:
   - I_{n-k}: Identity matrix of size $(n-k)\times(n-k)$, representing the parity bits.
   - $P^\top$: Transpose of the parity matrix P from $G = [I_k\ P]$.

   The matrix H ensures that: $HG^\top = 0$

2. **Dimensions**:
   - $H \in \mathbb{F}_2^{(n-k)\times n}$:
     - Rows: $n-k$ (number of parity check equations).
     - Columns: n (length of the codeword).

2.1.10 Example: (7, 4) Hamming Code

1. **Generator Matrix** G: $G = \begin{bmatrix} I_4 & P \end{bmatrix}$ where: $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

2. **Transpose of** P: $P^\top = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

3. **Identity Matrix** I_{n-k}: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

4. **Parity Check Matrix** H: Combine I_{n-k} and $P^\top$: $H = \begin{bmatrix} I_3 & P^\top \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$

2.1.11 Summary:

- For systematic codes, the parity check matrix is: $H = \begin{bmatrix} I_{n-k} & P^\top \end{bmatrix}$
- It is derived to satisfy: $HG^\top = 0$
- In the (7,4) Hamming code, H has dimensions 3×7, combining I_3 and $P^\top$.

- **Parity Check Matrix** is used to decode the RX vector when Decoding is **not optimal (generally)**, when you work with Tanner graph which representation of H.

2.1.12 Hamming Code:

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}^{\downarrow 7 \text{ variable nodes}}_{\rightarrow 3 \text{ check nodes}}$$

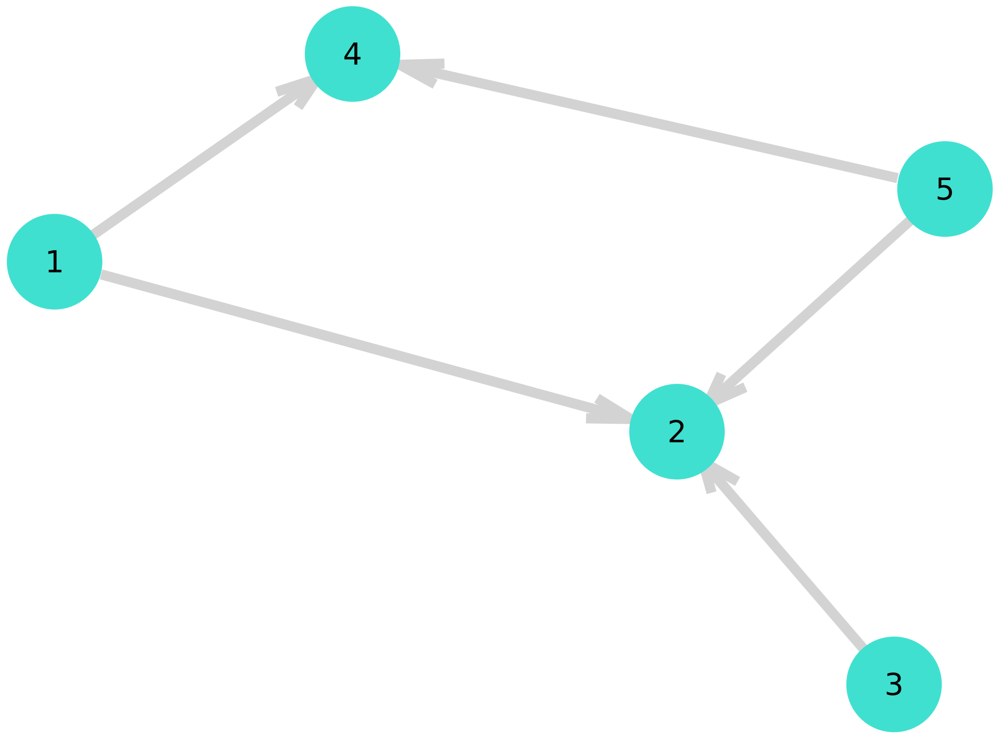
(n-k):

3 linear equations, that may valid, non-erroneous TX must satisfy

[56]:
```julia
using Graphs  # For graph representation

# Create a directed graph
tanner = DiGraph(5)  # A directed graph with 5 nodes

# Add edges to create an acyclic structure
add_edge!(tanner, 1, 2)
add_edge!(tanner, 1, 4)
add_edge!(tanner, 3, 2)
add_edge!(tanner, 5, 2)
add_edge!(tanner, 5, 4)

println("Is Bipartite: ", is_bipartite(tanner))
# Display basic properties of the graph
println("Number of nodes: ", nv(tanner))  # Number of vertices
println("Number of edges: ", ne(tanner))  # Number of edges

# Print edges of the graph
gplot(tanner, nodelabel=1:5)
```

```
Is Bipartite: true
Number of nodes: 5
Number of edges: 5
```

[56]:



When writing large codes:

Tanner Graphs (bipartite graph): | **Variable nodes** | **Check nodes** | | |-|-|-| | | | $C_1 + C_2 + C_3 + C_5 = 0$ | | | | $C_1 + C_3 + C_4 + C_6 = 0$ | | | $C_1 + C_2 + C_7 = 0$

Tanner Graph representation used in decoding codes in an **soft** manner. **Soft Decoding**: - [Passing **likelihood information** for each node votes]. - **Soft decoding** is generally **suboptimal**, unless it is such that the corresponding Tanner graph has **no cycles**.

This Cycle Consideration brought to the fore - **Low-Density Parity Check Codes (LDPC)**. Gallager 1960 - LDPC codes have a **sparse** H.

- **LDPC**
  - H is sparse.
  - Number of edges in the Tanner graph grows only linearly in n.
  - LDPC facilitates soft (iterative) decoding.
  - Iterative exchange of information.
  - Allows for simple local passing of information (at the node).
- **Regular LDPC**
  - n, J, k
    - n: Codeword length.
    - J: Degree of each variable node.
    - k: Degree of each check node.

Location of ones in H is chosen to have a certain randomness, subject to a structure that guarantees decoding performance.

Gallager $(n = 20, J = 3, k = 4)$ Construction

- $H = 15 \times 20$

Pick at Random

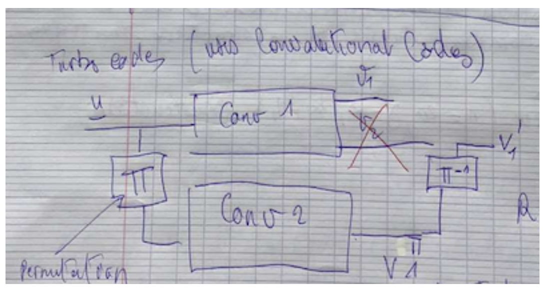
$$G = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1
\end{bmatrix}$$

Columns of first submatrix

$$\pi_1 = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0
\end{bmatrix}$$

$$\pi_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 \end{bmatrix}$$

Turbo Codes (uses Convolutional Codes)



- Three Main powerful Codes
  - Turbo Codes ≈ 1992
  - LDPC Codes Gallager ≈ 1960
  - Polar Codes
- **All 3 codes achieve Capacity (≈) with easier decoding**
  - LDPC easier to decode than Turbo
  - Polar has smaller error decay (a little bit slower)

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3 Polar Codes

3.0.1 Notation:

- $X, Y \sim P_{X,Y}$:
  - - Higher$H$: More uncertainty -$H(X|Y)$: Joint Entropy$I(X;Y) = \underbrace{H(X)}_{10} - \underbrace{H(X|Y)}_{-7}$ $\underbrace{\textbf{- Mutual Information}}_{= \text{3 Bits of Information}}$### Formulas: -
- $I(X;Y) = \sum\limits_{x,y} P(x,y) \log \frac{P(x,y)}{P(x)P(y)}$
- $X^n = (X_1, \ldots, X_n)$

- $Y^n = (Y_1, \ldots, Y_n) \qquad (X^n, Y^n) \approx P_{x^n, Y^n}$

3.0.2 Chain Rule:

$H(X^n|Y^n) = \sum_{i=1}^{n} H(X_i|Y^n, X^{i-1})$

Where $X^{i-1} = (X_1, \ldots, X_{i-1})$

$H(X_1|Y^n) + H(X_2|X_1, Y^n) + H(X_3|\underbrace{X_1, X_2}_{(\because)}, Y^n) + \ldots$

$(\because)$ know the 2 previous days, (this is the chain rule).

Result (math)

3.0.3 Principle of Polarization

- Let $X \sim P_X$, $\mathcal{X} = \{0, 1\}$
- Let $X_1, X_2, \ldots$ i.i.d. $\sim X$
- Let $N = 2^n$, $n > 1$
- Let $X^n = \{X_1, X_2, \ldots, X_n\}$
- Let: $F = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $F^{\otimes n}$= n-power Kronecker product of F

$$F^{\otimes 2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

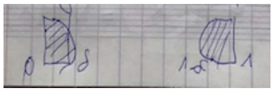
$$F^{\otimes 3} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

- Let $u^N = x^N \cdot F^{\otimes n}$

3.0.4 Theorem:

- For any fixed $\delta > 0$:
- $\boxed{\lim_{N \to \infty} \frac{1}{N} \left|\{i \in \{1, 2, \ldots, N\} \mid H(u_i|u^{i-1}) \in [\delta, 1-\delta]\}\right| = 0}$

- It says that: $H(u_i|u^{i-1}) \to \begin{cases} 0 & \boxed{\text{nothing in between them}} \\ 1 & \end{cases}$



no uncertainty

3.0.5 Polarization Kernel and Kronecker Power

1. **Base Matrix F:** $F = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

2. **Kronecker Power $F^{\otimes n}$:**
   - Recursive definition: $F^{\otimes n} = F^{\otimes(n-1)} \otimes F$
   - $\otimes$: Kronecker product.

3. **Examples:**
   - $F^{\otimes 1} = F$
   - $F^{\otimes 2}$: $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$
   - $F^{\otimes 3}$: $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

4. **Purpose:**
   - $F^{\otimes n}$ transforms $X^N \to U^N = X^N \cdot F^{\otimes n}$.
   - Enables **channel polarization**, splitting channels into highly reliable and unreliable sets for polar codes.

For any $\delta > 0$, in fact $\delta \approx 2^{-\sqrt{n}}$
(i.e. δ can be as low as?)

3.0.6 Example:

- $n = 2,\ F = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$(u_1, u_2) = (x_1, x_2) \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \left[x_1 \oplus x_2, x_2\right]$

- $n = 4,\ F^{\otimes 2} = \begin{bmatrix} 10 & 00 \\ 11 & 00 \\ 10 & 10 \\ 11 & 11 \end{bmatrix}$

$$(u_1, u_2, u_3, u_4) = (x_1, x_2, x_3, x_4) \cdot F^{\otimes 2} = [\underbrace{x_1 \oplus x_2 \oplus x_3 \oplus x_4}_{u_1}, \underbrace{x_2 \oplus x_4}_{u_2}, \underbrace{x_3 \oplus x_4}_{u_3}, \underbrace{x_4}_{u_4}]$$

3.0.7 Kronecker Product:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22}s & \\ \vdots & & \end{bmatrix}, B \qquad A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & \cdots \end{bmatrix}$$

3.0.8 key result in polar codes and channel polarization

$x_1, x_2, \ldots, x_n \qquad X \sim$ i.i.d.

$u^n = x^n \cdot F^n$

Corollary: $\frac{\left[\{i : H(u_i|u^{i-1}) > 1-\delta\}\right]}{N} = H(X)$

3.0.9 Proof:

$H(u^n) = H(x^n) = n\, H(x)(i.i.d)$

This describes a key result in **polar codes** and **channel polarization**, where the **entropy** and structure of the polar transformation are analyzed mathematically.

Explanation:

1. **Setup:**
   - $x_1, x_2, \ldots, x_n$: A sequence of independent and identically distributed (**i.i.d.**) random variables from the source X.
   - $u^n = x^n \cdot F^n$: The transformation of x^n using the **Kronecker power** of F (polarization matrix) to produce the vector u^n.
     - This defines the encoding operation in polar codes.
2. **Corollary:**
   - The fraction of indices i for which the **conditional entropy** $H(u_i|u^{i-1}) > 1 - \delta$ (i.e., where uncertainty is high) is proportional to the **source entropy** $H(X)$: $\frac{\left|\{i : H(u_i|u^{i-1}) > 1-\delta\}\right|}{N} = H(X)$.
   - This result demonstrates how the **polarization process** concentrates certain indices with high entropy (bad channels) and others with low entropy (good channels).
3. **Proof Outline:**
   - Since $x_1, x_2, \ldots, x_n$ are i.i.d., the total entropy of the source sequence x^n is: $H(x^n) = n \cdot H(X)$.
   - The polar transformation preserves entropy (it's a linear transformation), so: $H(u^n) = H(x^n) = n \cdot H(X)$.

- This shows that the entropy is distributed across the components of u^n, leading to the polarization effect.

Key Idea:

- **Channel Polarization**:
  - The transformation F^n polarizes the "channels" (indices i) into two categories:
    1. Channels with **high reliability** ($H(u_i|u^{i-1}) \approx 0$): These are the "good" channels used to carry information.
    2. Channels with **low reliability** ($H(u_i|u^{i-1}) \approx 1$): These are the "bad" channels, which are "frozen" (fixed to known values).
- The corollary quantifies this polarization, stating that the fraction of good channels corresponds to the source entropy $H(X)$.

Summary:

- **Transformation**: $u^n = x^n \cdot F^n$ is the polar transformation.
- **Result**: The fraction of "good" channels (low-entropy) matches the source entropy $H(X)$.
- **Proof**: Entropy is conserved, and the polarization process redistributes it across the indices of u^n.

This result is central to the design and analysis of **polar codes**, enabling efficient encoding and decoding.

$$H(u^N) = \sum_{i=1}^{N} \underbrace{H(u_i|u^{i-1})}_{\text{(chain rule)}} \quad = N \cdot H(X)(\because)$$

But recall (from theorem) that the # of terms where: $H(u_i|u^{i-1}) \in (\delta, 1-\delta)$ is near zero.

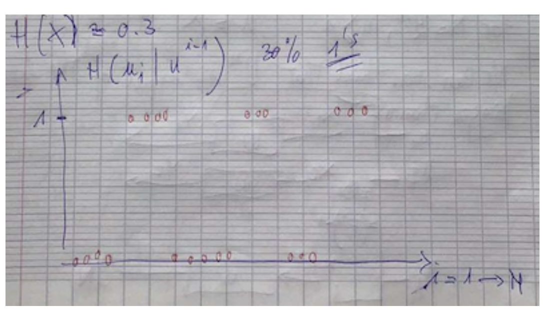
$$\approx 0 \cdot \left| \overbrace{H(u_i|u^{i-1})}^{\#\text{of times}} \approx 0 \right| + 1 \cdot \left| \overbrace{H(u_i|u^{i-1})}^{\#\text{of times}} \approx 1 \right|$$

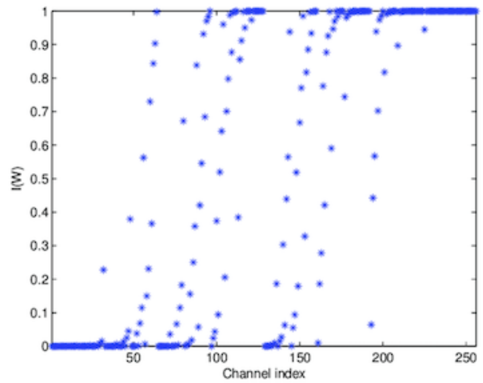
All the entropy accumulated in summation is due to K terms where: $H(u_i|u^{i-1}) \in (1-\delta, 1)$.

3.0.10 Accumulated Entropy:

$K \cdot 1 = N \cdot H(X)$ (from $\because$) $\implies \frac{K}{N} = H(X)$.

$H(X) \approx 0.3$ 30% of $H(u_i|u^{i-1}) \approx 1's$, 70% ≈ 0.





Channel polarization for a Polar code with length N=256 and rate R=0.5

Explanation of $K \cdot 1 = N \cdot H(X) \implies \frac{K}{N} = H(X)$ This equation arises in the context of **polar codes** and **channel polarization**, describing how the total entropy is distributed among the indices of the polar transformation.

Definitions and Context:

1. **Terms**:
   - K: The number of indices i where the **conditional entropy** $H(u_i|u^{i-1})$ contributes meaningfully to the total entropy (i.e., $H(u_i|u^{i-1}) \approx 1$).
   - N: The total block length (number of indices in u^N).
   - $H(X)$: The entropy of the source variable X, which determines the overall proportion of good (reliable) channels in the polarization process.
2. **Accumulated Entropy**:
   - The total entropy of the transformed vector u^N is: $H(u^N) = N \cdot H(X)$, since entropy is preserved during the linear transformation $u^N = x^N \cdot F^N$.

Key Idea:

- **Entropy Contribution**:
  - The entropy $H(u^N)$ is primarily accumulated in the K terms where $H(u_i|u^{i-1}) \approx 1$ (good channels).
  - For other indices, $H(u_i|u^{i-1}) \approx 0$, meaning they contribute negligible entropy.
- **Total Accumulated Entropy**:
  - If K terms contribute nearly 1 unit of entropy each, the total entropy contributed by these terms is: $K \cdot 1 = N \cdot H(X)$.
- **Proportion of Good Channels**:
  - Dividing through by N, we find that the proportion of indices i where $H(u_i|u^{i-1}) \approx 1$ is: $\frac{K}{N} = H(X)$.

Meaning:

- $\frac{K}{N} = H(X)$ indicates that the fraction of "good channels" (indices i where $H(u_i|u^{i-1})$ is significant) corresponds to the source entropy $H(X)$.
- For example:

- If $H(X) = 0.3$, then 30% of the channels are reliable, while 70% are unreliable (contributing $H(u_i|u^{i-1}) \approx 0$).

Importance in Polar Codes: This relationship quantifies **channel polarization**, where: - A fraction $H(X)$ of the channels becomes highly reliable (used for transmitting information). - The remaining fraction $1 - H(X)$ becomes unreliable (frozen).

This ensures that polar codes can efficiently encode and decode data based on the structure of F^N.

3.0.11 Channel Polarization

$X \rightarrow \boxed{W} \rightarrow Y \qquad X$ is Input to the memoryless channel W.

Recall: $(x, y) \sim P_{X,Y}(x, y)$

$

$$\begin{aligned} P(X,Y) &= P_X(x) \cdot P_{Y|X}(y|x) & (1) \\ &= P_X(x) \cdot W_{Y|X}(y|x) & (2) \end{aligned}$$

$

Using Bayes' Rule: $P_{X|Y}(x|y) = \frac{P_{Y|X}(y|x) \cdot P_X(x)}{P_Y(y)}$ (Channel Fixed).

Assume: - $X \sim X_1, X_2, \ldots, X_n$,
- X is discrete, $X \in \{0, 1\}$,
- Assume Binary Symmetric Channel (BSC).

For $N = 2^n$: $U^N = X^N \cdot F^n\ U^N$: Input of the following **bigger channel**:

3.0.12 Diagram:

- W: Memoryless channel,
- $U_1, U_2, \ldots, U_N$: Input,
- $Y_1, Y_2, \ldots, Y_N$: Output.

This process **creates bad and good channels**, enabling **channel polarization**.

[ ]: