MOBCOM-MIdtermF2024

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Mobile Communication Techniques

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Midterm Exam

November 21st, 2024

Time: 9:00-10:00

Instructions

- Exercises fall in categories of 1-point and 2-point exercises.
- Total of $11 \times 1 + 2 \times 2 = 15$ points.
- NOTE!!! The exam will be evaluated, out of 13 points. Any points you get beyond 13 points, will be offered as extra bonus.
- Each answer should be clearly written, and the solution should be developed in detail.
- Mathematical derivations need to show all steps that lead to the answer.
- Complete as many exercises as you can. Don't spend too much time on an individual question.
- There is NO penalty for incorrect solutions.
- If in certain cases you are unable to provide rigorous mathematical proofs, go ahead and provide intuitive justification of your answers. Partial credit will be given.
- Calculators are not allowed.
- You are allowed your class notes and class book.

Hints - equations - conventions:

- Notation
  - SISO = single-input single-output, MISO = multiple-input single-output, SIMO = single-input multiple- output, MIMO = single-input multiple-output,
  - R represents the rate of communication in bits per channel use (b.p.c.u),
  - ρ represents the SNR (signal to noise ratio),
  - w will denote additive noise which will be distributed as a circularly symmetric Gaussian random variable $\mathbb{C}\mathcal{N}(0, N_0)$. If N_0 is not specified, then set $N_0 = 1$,
  - h_i will denote independent fading scalar coefficients which will be distributed as circularly symmetric Gaussian random variables $\mathbb{C}\mathcal{N}(0, 1)$.

- GOOD LUCK!!

1) (1 point). In a multi-path fading scenario with delay spread $6\mu s$ and $L = 3$ channel taps, what is the operational bandwidth W?

0.0.1 Answer:

Given $\tau_d = 6\,\mu s$ and $L = 3$: - Coherence bandwidth: $B_c \approx \frac{1}{\tau_d} = \frac{1}{6\times10^{-6}} \approx 166.67\,\text{kHz}$. - Operational bandwidth: $W \approx L \cdot B_c = 3 \cdot 166.67 = 500\,\text{kHz}$.

$\boxed{W = 500\,\text{kHz}}$

2) (1 point). Imagine a given SNR equal to ρ, and imagine that we are operating over a (quasi-static) Rayleigh fading SISO channel. Can you describe a code that achieves probability of error approximately equal to $P_e \approx \rho^4$, and rate equal to $R = 2$ bpcu.

Solution 1: Repetition Code (256-QAM)

- **Modulation**: 256-QAM (8 bits/symbol)
- **Code**: Repetition factor = 4 (diversity order = 4)
- **Rate**: $R = \frac{8}{4} = 2\,\text{bpcu}$
- **Error Probability**: $P_e \approx \rho^{-4}$
- **Complexity**: High due to 256-QAM

Solution 2: Rotated Code (4-QAM)

- **Modulation**: 4-QAM (2 bits/symbol)
- **Code**: Rotated constellation across 4 time slots (diversity order = 4)
- **Rate**: $R = 2\,\text{bpcu}$
- **Error Probability**: $P_e \approx \rho^{-4}$
- **Complexity**: Lower due to 4-QAM

Comparison

Feature	Solution 1: Repetition Code (256-QAM)	Solution 2: Rotated Code (4-QAM)
Modulation	256-QAM (8 bits/symbol)	4-QAM (2 bits/symbol)
Code Type	Repetition code	Rotated time-diversity code
Diversity Order	4	4
Rate	2 bpcu	2 bpcu

Feature	Solution 1: Repetition Code (256-QAM)	Solution 2: Rotated Code (4-QAM)
Error Probability	$P_e \approx \rho^{-4}$	$P_e \approx \rho^{-4}$
Complexity	Higher decoding complexity	Lower decoding complexity

Both solutions achieve **rate = 2 bpcu** and **diversity order 4**, but the rotated 4-QAM code offers **lower complexity**.

Note:

- Alamouti Code: Closely related to Solution 2 but typically designed for MIMO (2 transmit antennas).
- 4-Dimensional Lattice Code: Matches Solution 2 with rotation and symbol spreading across multiple time slots. This solution achieves diversity order 4, fitting the requirement perfectly.

Diversity Order Overview

- **Definition**:
  The number of independent signal paths used to combat fading. Error probability decreases as $P_e \approx \rho^{-d}$, where d is the **diversity order**.

Types of Diversity

1. **Time Diversity**: Transmit symbols across different time slots (e.g., repetition coding).
2. **Frequency Diversity**: Transmit across multiple frequencies (e.g., OFDM).
3. **Space Diversity**: Use multiple antennas (e.g., Alamouti code, MIMO).
4. **Code Diversity**: Spread symbol components across independent channels (e.g., rotated lattice codes).

Impact

- **Higher diversity order** reduces the likelihood of deep fades and improves error performance: $P_e \propto \rho^{-d}$

Examples

1. **Diversity Order 1**: SISO, $P_e \propto \rho^{-1}$.
2. **Order 2**: Alamouti code with 2 antennas, $P_e \propto \rho^{-2}$.
3. **Order 4**: Rotated 4-QAM or repetition with 4 paths, $P_e \propto \rho^{-4}$.

Higher diversity increases resilience against fading.

3) (1 point). How much time diversity will we get with the following SISO (time-diversity) channel model

$$[y_1\ y_2\ y_3] = [h_1 u_1 \quad h_2(u_1+u_2) \quad h_3 u_2] + [w_1\ w_2\ w_3]$$

where the u_1, u_2, u_3 are independent PAM elements. Justify your answer.

To determine the **time diversity** in the given channel model:

Channel Model $[y_1 \;\; y_2 \;\; y_3] = [h_1u_1 \quad h_2(u_1+u_2) \quad h_3u_2] + [w_1 \;\; w_2 \;\; w_3]$, where u_1, u_2, u_3 are independent PAM symbols, h_1, h_2, h_3 are the channel coefficients, and w_1, w_2, w_3 are noise terms.

Analysis

1. **Definition of Time Diversity**:
   - Time diversity is determined by the number of independently faded channel coefficients (h_1, h_2, h_3) that affect the transmitted symbols.
2. **Observation of Dependencies**:
   - y_1 depends on h_1u_1.
   - y_2 depends on $h_2(u_1+u_2)$.
   - y_3 depends on h_3u_2.
3. **Diversity Order**:
   - u_1 is present in both y_1 and y_2, thus contributing to diversity through h_1 and h_2.
   - u_2 is present in both y_2 and y_3, contributing to diversity through h_2 and h_3.

Since u_1 and u_2 are affected by two **independent channel coefficients** each, the effective **time diversity order** is:

Time Diversity Order = min(number of independent fades per symbol) = $\boxed{2}$.

Justification The system achieves a time diversity order of 2 because each transmitted symbol u_1 and u_2 is observed across two independently faded channels (h_1, h_2 for u_1; h_2, h_3 for u_2). The third symbol u_3 does not contribute additional diversity as it is only affected by h_3.

4) (1 point). In a SISO case, what is the degrees of freedom (DOF) if we have a time-diversity code (spanning three channel uses) of the form $\mathcal{X} = [u_1 + u_2 \quad u_1 + u\_3 \quad u_2 + u_3]$ where the u_1, u_2, u_3, u_4 are independent 16-PAM elements?

Step 1: Code Setup The time-diversity code is:

$\mathcal{X} = [u_1 + u_2, \quad u_1 + u_3, \quad u_2 + u_3]$

- u_1, u_2, u_3, u_4 are **independent complex numbers** from **16-PAM**, meaning each has a **real** and **imaginary** part.
- Since we are now counting **only the real part**, each complex symbol contributes **1 real degree of freedom**.

Step 2: Apply the Formula The formula is:

$\text{DOF} = \min\left(\frac{\text{\# of real symbols}}{T}, n_t\right)$

- **# of real symbols**: There are **3 complex symbols**, each contributing **1 real part**. So, the number of real symbols is **3**.
- T: Number of channel uses = 3
- n_t: Number of transmit antennas = 1 (SISO)

Calculate:

$\text{DOF} = \min\left(\frac{3}{3}, 1\right) = \min(1, 1) = 1$

Step 3: Adjust DOF for Real Parts Only Since we are counting only the **real parts**, the effective real DOF per channel use is:

Real DOF per channel use $= \frac{1}{2}$ (since each complex DOF is split between real and imaginary parts)

Final Answer: The **real degrees of freedom (DOF)** per channel use in this time-diversity SISO code is:

$$\boxed{\frac{1}{2} \text{ real DOF per channel use}}$$

5) (1 point). For the case of time diversity in the SISO (quasi-static) fading channel, what is the advantage and the disadvantage of the repetition code, compared to uncoded transmission.

- **Advantage**: Repetition code improves reliability by providing diversity gain, reducing the error probability in fading channels.

- **Disadvantage**: It reduces spectral efficiency by lowering the transmission rate due to redundant transmissions.

6) (1 point). In a SISO case, what is the DOF and the rate (in bpcu), of the following time-diversity code (three channel uses) that takes the form $\mathcal{X} = [u_1 + u_4 \quad u_2 \quad u_1 + u_2 + u_3]$ where the u_1, u_2, u_3, u_4 are independent 64-QAM elements?

To analyze the **Degrees of Freedom (DOF)** and **rate** for the given time-diversity code:

Code Representation The transmitted codeword over three channel uses is:

$X = \begin{bmatrix} u_1 + u_4 & u_2 & u_1 + u_2 + u_3 \end{bmatrix},$

where u_1, u_2, u_3, u_4 are independent symbols from a 64-QAM constellation.

1. Degrees of Freedom (DOF):

- The **DOF** corresponds to the number of **independent information symbols** transmitted across the given channel uses.
- Here, u_1, u_2, u_3, u_4 are **independent symbols**, so there are **4 independent symbols** transmitted over **3 channel uses**.

$\text{DOF} = \frac{\text{Number of Independent Symbols}}{\text{Number of Channel Uses}} = \boxed{\frac{4}{3}}.$

2. Rate (in bpcu):

- Each symbol is from a 64-QAM constellation, which carries $\log_2(64) = 6$ bits per symbol.
- Since 4 symbols are transmitted over 3 channel uses, the rate R is:

$R = \frac{\text{Total Bits Transmitted}}{\text{Number of Channel Uses}} = \frac{4}{3} \cdot 6 = \boxed{8\,\text{bpcu}}$.

When applying:

- n_t: Number of transmit antennas = 1 (SISO)

$\text{DOF} = \min\left(\frac{\text{Number of Independent Symbols}}{\text{Number of Channel Uses}}, n_t\right) = \min\left(\frac{4}{3}, 1\right)$ the answer is $\boxed{1}$

7) (1 point). Imagine a SISO channel model with correlated fading, where the first fading coefficient (first transmission slot) is $h_1 = h_1' \times h_2'$, and the second fading coefficient (second transmission slot) is $h_2 = h_2'$, where $h_1', h_2' \sim i.i.d\, \mathbb{CN}(0, 1)$. What is the maximum diversity we can achieve here?

Maximum Diversity:

- Fading coefficients: $h_1 = h_1' \cdot h_2'$ and $h_2 = h_2'$.
- Independent components: h_1' and h_2' ($\mathbb{CN}(0, 1)$, i.i.d.).
- **Diversity order** = Number of independent fading coefficients = $\boxed{2}$.

Note:

- If h_2' is bad everything is bad

8) (1 point). Describe the steps of converting a binary vector detection problem over a time diversity fading channel, into a scalar detection problem. Imagine that you are sending BPSK symbols using a repetition code, and consider $\mathbb{CN}(0, N_0)$ noise.

Steps to Convert to Real Scalar Detection

1. **Received signal model**:
   $y_i = h_i x + n_i, \quad n_i \sim \mathbb{CN}(0, N_0)$
2. **Combine the signals** using maximum ratio combining (MRC):
   $y_{\text{combined}} = \sum_{i=1}^{T} h_i^* y_i = \sum_{i=1}^{T} |h_i|^2 x + \sum_{i=1}^{T} h_i^* n_i$
3. **Take the real part**:
   $\tilde{y} = \text{Re}(y_{\text{combined}}) = \tilde{h}x + \tilde{n}, \quad \tilde{n} \sim \mathcal{N}(0, N_0\tilde{h})$
4. **Decision rule**:
   $$\boxed{\hat{x} = \begin{cases} +1, & \text{if } \tilde{y} > 0 \\ -1, & \text{if } \tilde{y} < 0 \end{cases}}$$

This reduces the vector detection problem to **real scalar detection**.

9) (1 point). Consider a deep-space communications scenario, where the received SNR is equal to 20dB. If you assume low rate communications, what do you expect the probability of error to be?

Step 1: Common Error Probability Expressions In certain cases, especially for large SNR in low-rate communication systems, error probability takes the form of:

$P_e \approx e^{-\gamma \cdot \text{SNR}}$

Here: - γ depends on the modulation scheme and coding structure. - This approximation is typical for systems with **diversity**, **strong coding**, or under certain approximations (e.g., union bounds for coded error probabilities).

Step 2: When Does $e^{-\mathbf{SNR}}$ Apply?

1. **Coded Systems**:
   For strong error-correcting codes, the probability of error often decreases exponentially with SNR:

$P_e \approx e^{-\text{coding gain}\cdot\text{SNR}}$

2. **Uncoded BPSK in AWGN**:
   The **bit error probability** for uncoded BPSK in AWGN is: $P_b = Q\left(\sqrt{2 \cdot \text{SNR}}\right)$

   For large SNR, using $Q(x) \approx \frac{1}{\sqrt{2\pi}x} e^{-\frac{x^2}{2}}$: $P_b \approx \frac{1}{\sqrt{2\pi}\cdot\sqrt{2\cdot\text{SNR}}} e^{-\text{SNR}}$

Step 3: Deep-Space Scenario In deep-space communications with **low-rate transmission**, **coding** and **interleaving** are critical for reliability. In such scenarios, error probability can behave as: $P_e \approx e^{-\text{SNR}}$

This results from: 1. **Effective coding gain**, which leads to rapid error decay. 2. **Low-rate transmissions** (few bits per channel use), allowing strong robustness against noise.

Step 4: Application Given **SNR = 20 dB** (or SNR = 100 in linear scale), if:

- **SNR in Linear Scale**: Convert 20 dB to linear scale: $\boxed{\text{SNR}_{\text{linear}} = 10^{\frac{\text{SNR}_{\text{dB}}}{10}} = 10^{\frac{20}{10}} = 100}$.
- The **probability of error** is extremely small: $\boxed{P_e \approx e^{-\text{SNR}} = e^{-100} \approx 3.72 \times 10^{-44}}$

This is consistent with the extremely low error probabilities observed in such scenarios.

Final Summary:

- In deep-space communication with low-rate coding, the error probability often follows an **exponential decay** form: $P_e \approx e^{-\text{SNR}}$
- For **SNR = 20 dB (100 linear)**, $P_e \approx e^{-100}$, giving an extremely small error probability, which aligns with robust, low-error communications in space missions.
- In deep-space communication with high SNR and low rate, errors are nearly negligible.

10) (1 point). What is the approximate coherence time T_c in a typical urban wireless network if you are driving approximately 20 kilometers per hour?

To estimate the **coherence time** T_c in a typical urban wireless network, we use the following formula: $T_c \approx \frac{1}{f_d}$, where f_d is the **Doppler spread** given by $f_d = \frac{v}{\lambda} = \frac{v \cdot f_c}{c}$.

1. Given Parameters:

- Speed: $v = 20\,\text{km/h} = \frac{20\times1000}{3600} = 5.56\,\text{m/s}$,
- Carrier frequency: $f_c = 2\,\text{GHz} = 2 \times 10^9\,\text{Hz}$ (assumed typical urban value),

- Speed of light: $c = 3 \times 10^8\,\text{m/s}$.

2. Doppler Spread: $f_d = \frac{v \cdot f_c}{c} = \frac{5.56 \cdot 2\times10^9}{3\times10^8} = 37.1\,\text{Hz}$.

3. Coherence Time: $T_c \approx \frac{1}{f_d} = \frac{1}{37.1} \approx 0.027\,\text{seconds} = 27\,\text{ms}$.

The approximate coherence time is: $\boxed{27\,\text{ms}}$.

11) (1 point). Consider communication over a SISO fading channel with a delay spread of $T_d = 3\mu s$ and a signal bandwidth of $W = 1$ MHz. - Write all the received signals, if we only send $x[0]$ and then we stop transmitting.

To analyze this scenario, we need to consider the **SISO fading channel** with a **delay spread** $T_d = 3\,\mu s$ and a signal bandwidth $W = 1\,\text{MHz}$. The delay spread indicates the multipath environment, meaning the transmitted signal will arrive at the receiver through multiple delayed and scaled copies.

1. Transmitted Signal:

- Only $x[0]$ is transmitted, then the transmission stops. Thus: $x[n] = \begin{cases} x[0], & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$

2. Received Signal: The received signal is the convolution of the transmitted signal $x[n]$ with the channel impulse response $h(t)$: $y[n] = h[n] * x[n]$.

- The **channel impulse response** $h(t)$ is a sum of L multipath components: $h(t) = \sum_{l=0}^{L-1} h_l \delta(t - \tau_l)$, where:
  - h_l: Fading coefficient for the l-th path ($h_l \sim \mathcal{CN}(0,1)$),
  - τ_l: Delay of the l-th path ($0 \leq \tau_l \leq T_d$).
- With $T_d = 3\,\mu s$, the maximum delay is $3\,\mu s$, corresponding to $L \approx W \cdot T_d = 1\,\text{MHz} \cdot 3\,\mu s = \boxed{3}$ significant paths.

3. Writing the Received Signals: For $x[0]$ transmitted: - The received signal $y[n]$ consists of L delayed copies of $x[0]$, weighted by the fading coefficients h_l: $y[0] = h_0 x[0]$, $y[1] = h_1 x[0]$, $y[2] = h_2 x[0]$. - For $n > 2$, no further contributions occur, as $\tau_l \leq T_d$.

Thus: $y[n] = \boxed{\begin{cases} h_0 x[0], & n = 0, \\ h_1 x[0], & n = 1, \\ h_2 x[0], & n = 2, \\ 0, & n > 2. \end{cases}}$

Final Answer: The received signals are: $y[0] = h_0 x[0], \quad y[1] = h_1 x[0], \quad y[2] = h_2 x[0], \quad y[n] = 0\,\text{for}\ n > 2$.

12) ***(2 points)***. What is the optimal diversity order over a 2×1 MISO channel $h = [h_1\ h_2], h_i \sim i.i.d\ \mathbb{CN}(0,1)$? - In the same channel as above (again with no time diversity), consider a space time code whose matrices take the form

$$\begin{bmatrix} x_0 & x_1 \\ x_1 & x_0 \end{bmatrix}$$

where thex_i are drawn independently from a QAM constellation. Will this code achieve optimal diversity order? (argue why or why not) - What is the diversity order achieved by the Alamouti code, over this 2×1 MISO channel? (again, you can just argue in words)

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1. Optimal Diversity Order in a 2×1 MISO Channel In a 2×1 MISO channel, the **diversity order** is equal to the number of independent fading paths, which corresponds to the number of transmit antennas ($N_t = 2$) when there is 1 receive antenna.
Thus, the **optimal diversity order** is: $\boxed{2}$.

2. Diversity Order of the Given Space-Time Code The given code matrix is: $\mathbf{X} = \begin{bmatrix} x_0 & x_1 \\ x_1 & x_0 \end{bmatrix}$, where x_0 and x_1 are independent QAM symbols.

Key Analysis:

- **Rank Criterion**: For a space-time code to achieve full diversity, the difference between any two distinct code matrices $\mathbf{X}_1$ and $\mathbf{X}_2$ must result in a matrix of full rank.
- For this code: $\Delta\mathbf{X} = \mathbf{X}_1 - \mathbf{X}_2 = \begin{bmatrix} x_{01} - x_{02} & x_{11} - x_{12} \\ x_{11} - x_{12} & x_{01} - x_{02} \end{bmatrix}$.
  - The rows of $\Delta\mathbf{X}$ are **linearly dependent** because the two rows are identical. This means $\Delta\mathbf{X}$ is **not full rank**.

Conclusion:

This code does **not achieve the optimal diversity order**, as it does not satisfy the rank criterion for full diversity.

3. Diversity Order of the Alamouti Code The Alamouti code for a 2×1 MISO channel is: $\mathbf{X}_{\text{Alamouti}} = \begin{bmatrix} x_0 & -x_1^* \\ x_1 & x_0^* \end{bmatrix}$.

Key Features:

- The Alamouti code satisfies the **rank criterion**, ensuring that $\Delta\mathbf{X} = \mathbf{X}_1 - \mathbf{X}_2$ is always full rank for distinct codewords $\mathbf{X}_1$ and $\mathbf{X}_2$.
- Each transmitted symbol experiences the full diversity of the channel, as it leverages both transmit antennas.

Conclusion:

The Alamouti code achieves the **optimal diversity order of 2** over the 2×1 MISO channel.

Final Answers:

1. Optimal diversity order in 2×1 MISO: $\boxed{2}$.
2. Given space-time code: **Does not achieve optimal diversity order** due to $\boxed{\text{lack of full-rank}}$ property.

3. Alamouti code: **Achieves optimal diversity order of** $\boxed{2}$.

13) ***(EXTRA CREDIT: 2 points)***. Consider a setting where the transmit antenna array has length of 50 cm, the received antenna array has size 20cm, the transmission frequency is 1000 MHz, the signal bandwidth is 1 MHz, the channel coherence time is $T_c = 21$ ms, and the coding duration is $T_{coding} = 7$ms. - How much diversity can you get, in total?

Explanation for selecting Space Diversity

1. **Only Space Diversity is usable**:
   - **Time diversity**: Not applicable since $T_{coding} = 7\,\text{ms}$ is much shorter than $T_c = 21\,\text{ms}$, so the channel does not change significantly.
   - **Frequency diversity**: Not effective as the bandwidth (1 MHz) is within the coherence bandwidth.
2. **Calculate Space Diversity**:
   - **Wavelength**: $\lambda = 0.3\,\text{m}$ (at 1000 MHz)
   - **Antenna Spacing**: $d = \frac{\lambda}{2} = 0.15\,\text{m}$
   - **Transmit Array**:
     $n_t = \frac{50\,\text{cm}}{15\,\text{cm}} \approx 4$
   - **Receive Array**:
     $n_r = \frac{20\,\text{cm}}{15\,\text{cm}} \approx 3$
3. **Total Space Diversity**: Space diversity $= n_t \times n_r = 4 \times 3 = 12$

Final Answer: $\boxed{\text{diversity} = 12}$