Space-Time Code design criterion.

Recall task: L tx-antennas, T-time slots.

Quasi-static block fading.

$T_c > T$ (no time diversity, hope for spatial div).

Looking for a ~~code~~ $L \times \overset{T}{N}$ code $\mathcal{X}$.

$$\begin{matrix} x_1[1] & x_1[2] & \cdots & x_1[T] \\ x_2[1] & x_2[2] & \cdots & x_2[T] \\ \vdots & & & \\ x_L[1] & x_L[2] & \cdots & x_L[T] \end{matrix}$$

$$\mathcal{X} = \Big\{ \underset{\substack{\uparrow \\ \text{message } 1}}{X_1} = \begin{bmatrix} \downarrow \\ \\ \end{bmatrix},\ \underset{\substack{\uparrow \\ \text{message } 2}}{X_2} = \begin{bmatrix} & \\ & \end{bmatrix}, \ldots, \underset{\substack{\uparrow \\ \text{message } |\mathcal{X}|}}{X_{|\mathcal{X}|}} = \begin{bmatrix} & \\ & \end{bmatrix} \Big\}.$$

Code defined by power constraint, rate, & of course the structure of the code.

Power constraint P $\quad E\{\|X\|_F^2\} \leq P.T$.

Rate R: $\quad |\mathcal{X}| = 2^{RT} \Leftarrow R$ bits/c.u.

Structure of code: recall e.g alamouti $\mathcal{X} = \left\{ \delta . \begin{bmatrix} s_0 & -s_1^* \\ s_1 & s_0^* \end{bmatrix}, s_i \in QAM \right\}$.

e.g repetition $\mathcal{X} = \left\{ \delta \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}, x \in \mathbb{C} \right\}$.

& finally recall channel model (MISO).

$$\boxed{\underset{1\times T}{\underline{y}^T} = \underset{1\times L}{\underline{h}^T} \cdot \underset{L\times T}{X} + \underset{1\times T}{\underline{w}^T}}$$

Example of bad & good code structure

Let $L=2$, $T=1$

let $R=1$ & $P=2$.

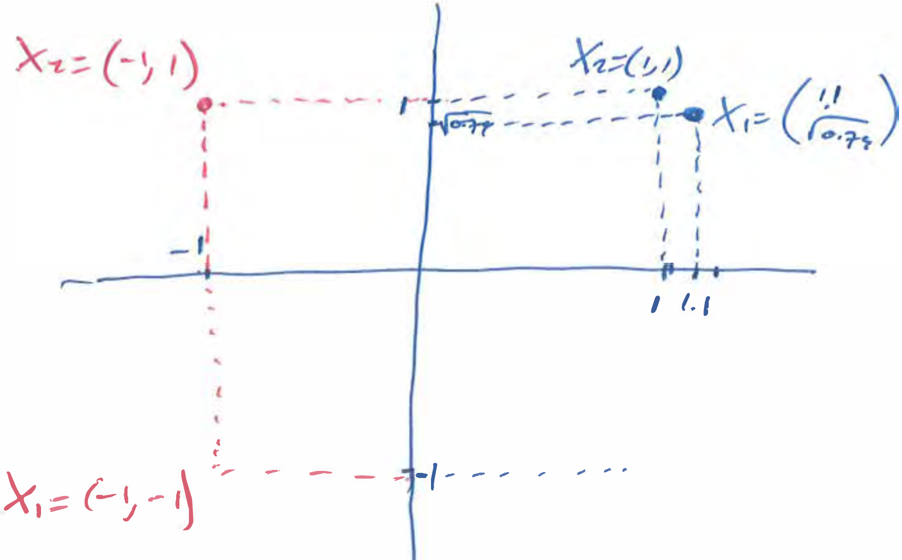
$$x = \begin{bmatrix} x_1[1] \\ x_2[1] \end{bmatrix}$$

Consider 2 codes:

$$\mathcal{X}_f = \left\{ x_1 = \begin{pmatrix} 1.1 \\ \sqrt{0.79} \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \qquad \mathcal{X}_s = \left\{ x_1 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, x_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}.$$

note $\|X\|_F^2 \leq 2$ $\qquad$ $\|X\|_F^2 = 2$.

$$\boxed{R = \frac{1}{T} \log_2 |\mathcal{X}| = \frac{1}{1} \log_2 2 = 1 \text{ b.p.c.u}}$$



Determinant criterion (MISO).

Consider $L\times T$ ST code $\mathcal{X} \subseteq \mathbb{C}^{L\times T}$.

Calculate Pairwise Err Prob.

⇒ Consider 2 codematrices $X_A, X_B \in \mathcal{X}$.

Consider a fixed channel $\underline{h} \in \mathbb{C}^{L\times 1}$.

⇒ Calculate $P(X_A \to X_B)$.

$$\underline{y}^T = \underline{h}^T . X + \underline{w}^T$$

where $\underline{y}^T = \underline{h}^T . X_A + \underline{w}^T$ or $\underline{y}^T = \underline{h}^T . \cancel{X_B} + \underline{w}^T$.

⇒ $\underline{y}^T = .\underline{x_A}^T + \underline{w}^T$ or $\underline{y}^T = \underline{x_B}^T + \underline{w}^T$

where $\underline{x_A}^T = \underline{h}^T . X_A$, $\underline{x_B}^T = \underline{h}^T X_B$

⇒ $\underline{y} = \underline{x_A} + \underline{w}$. or $\underline{y} = \underline{x_B} + \underline{w}$

This is exactly Gaussian vector detection problem, ~~with~~ $\underline{x_A}$ v.s $\underline{x_B}$.

- Recall steps:

$\underline{x} = x\,(\underline{x}_A - \underline{x}_B) + \frac{1}{2}(\underline{x}_A + \underline{x}_B)$ $\quad x = \frac{1}{2}$ or $-\frac{1}{2}$. & $\underline{y}' = \underline{y} - \frac{\underline{x}_A + \underline{x}_B}{2} = x(\underline{x}_A - \underline{x}_B) + \underline{w}$

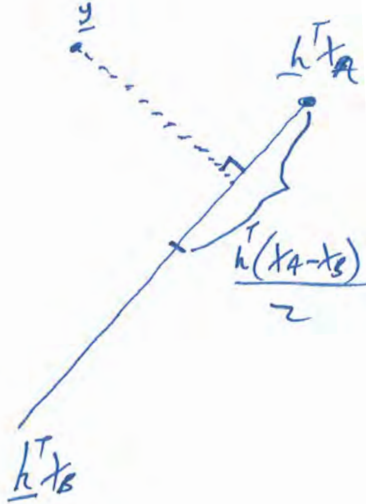
$\tilde{\tilde{y}} = \underline{v}^H \cdot \underline{y}' = x\,\|\underline{x}_A - \underline{x}_B\| + n'$ $\quad : \quad \underline{v} = \frac{(\underline{x}_A - \underline{x}_B)}{\|\underline{x}_A - \underline{x}_B\|} = \frac{(X_A - X_B)\cdot\underline{h}}{\|(X_A - X_B)\,\underline{h}\|}$

$\tilde{y} = \mathrm{Re}\{\tilde{\tilde{y}}\} = \underbrace{x\,\|\underline{x}_A - \underline{x}_B\|}_{\text{half distance}} + \underbrace{\mathrm{Re}\{n'\}}_{\mathcal{N}_0(0,\,\frac{N_0}{2})}.$

$\frac{1}{2}\|\underline{x}_A - \underline{x}_B\|$ $\qquad n''$

$$\Rightarrow \boxed{\tilde{y} = x\,\|\underline{h}^T(X_A - X_B)\| + n}$$

$$\Rightarrow P_e(\underline{h}^T X_A \to \underline{h}^T X_B) = Q\left(\frac{\frac{1}{2}\|h^T(X_A - X_B)\|}{\sqrt{\frac{N_0}{2}}}\right)$$



$$\Rightarrow P(x_A \rightarrow x_B \mid h) = Q\left(\frac{\| h^H \cdot (x_A - x_B) \|}{2\sqrt{\frac{N_0}{2}}}\right).$$

$$\underset{x \in \mathcal{X}}{E}\left\{\| X \|_F^2\right\} = P.T$$

& that $\rho = \frac{P}{N_0}$

P ← signal power/complex dim

N_0 ← noise power/complex dim

$\Rightarrow$ Power-normalize.

$$\mathcal{X} \rightarrow \mathcal{X}' = \frac{1}{\sqrt{P}}\mathcal{X} \quad \text{s.t } x_A', x_B' \in \mathcal{X}' \qquad \underset{x' \in \mathcal{X}'}{E} \| x' \|_F^2 = T.$$

with $\underline{x_A' \doteq \frac{1}{\sqrt{P}} x_A}$, $\underline{x_B' = \frac{1}{\sqrt{P}} x_B}$.

$$\Rightarrow P(x_A \rightarrow x_B \mid h) = Q\left(\frac{\sqrt{P}\, \| h^T \cdot (x_A' - x_B') \|}{\sqrt{N_0} \cdot \sqrt{2}}\right) = Q\left(\sqrt{\frac{\rho}{2} \| h^T (x_A' - x_B') \|^2}\right)$$

$$= Q\left(\sqrt{\frac{e}{2}\, h^T(x'_A - x'_B)(h^T(x'_A - x'_B))^T}\right) = Q\left(\sqrt{\frac{e}{2}\, h^T(x'_A - x'_B)(x'_A - x'_B)^T \underline{h}}\right)$$

Let $\Delta X' = x'_A - x'_B \Rightarrow P(x_A \to x_B | \underline{h}) = Q\left(\sqrt{\frac{e}{2}\, \underline{h}^T \Delta X' \Delta X'^H \underline{h}}\right).$

Unitary decomposition

$\Delta X . \Delta X'^H$ is hermitian $\Rightarrow$ it is diagonalizable

$$\Delta X' \Delta X'^H = U . \Lambda . U^H \quad \text{where } UU^H = I \quad \& \quad \Lambda = \begin{pmatrix} \lambda_1^2 & & \\ & \ddots & \\ & & \lambda_L^2 \end{pmatrix}, \lambda_\ell \text{ singular values of } \Delta X$$

$$\boxed{\lambda_\ell \text{ singular values of } \Delta X'}$$

$$\Rightarrow P(x_A \to x_B | \underline{h}) = Q\left(\sqrt{\frac{\rho}{2}\, \underline{h}^T \cdot U \cdot \Lambda \cdot U^H \cdot \underline{h}}\right).$$

Let $\underline{\tilde{h}} = U^H \cdot \underline{h} = \begin{pmatrix} \tilde{h}_1 \\ \tilde{h}_2 \\ \vdots \\ \tilde{h}_L \end{pmatrix}$ $\Rightarrow = Q\left(\sqrt{\frac{\rho}{2} \sum_{l=1}^{L} \lambda_l^2 \cdot |\tilde{h}_l|^2}\right) = P(x_A \to x_B / \underline{h})$

$\Rightarrow$

$$\Rightarrow P(x_A \to x_B) = \sum_{\underline{h}} P(x_A \to x_B | \underline{h}) = \sum_{h}\left[Q\left(\sqrt{\frac{\rho}{2} \sum_{l=1}^{L} \lambda_l^2 \cdot |\tilde{h}_l|^2}\right)\right].$$

note that $\tilde{h} \sim \underline{h} \sim \mathcal{CN}(0, I_L)$ because U^H is unitary.

but we have seen this exact problem before except that $\tilde{h}_l \to \lambda_l$ (normalized differences).

$$P(x_A \to x_B) \le \underset{h_1, h_2, .. h_L}{E}\left\{\exp\left[-\frac{\rho}{4} \sum_{l=1}^{L} |h_l|^2 \cdot \lambda_l^2\right]\right\}$$ recall $Q(x) \le e^{-x^2/2}$

As before, use MGF $E\{e^{sx}\} = \frac{1}{1-s}$ $s<1$

$$\boxed{\Rightarrow P(x_A \to x_B) \le \prod_{l=1}^{L} \frac{1}{1 + \rho \cdot \frac{\lambda_l^2}{4}} \le \frac{4^L}{\rho^L \cdot \prod_{l=1}^{L} \lambda_l^2} = \frac{4^L}{\rho^L \cdot \det(\Delta X' \Delta X^H)}}$$

Role of coding gain & degrees of freedom.

- Explore this using the determinant
- Consider MISO with $L = 2$ tx antennas.
- Look at two distinct example cases.

EXAMPLE

$\mathcal{X}_1$ is repetition code ($L = T = 2$), $\mathcal{X}_2$ is alamouti ($L = T = 2$).

(4-PAM) $\leftarrow$ ($R = 1$ bpcu). $\longrightarrow$ (BPSK).

- Repetition code

$$\mathcal{X} = \left\{ X_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, X_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, X_3 = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, X_4 = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix} \right\}.$$

$$E\{\|X\|^2\} = \frac{2+2+18+18}{4} = 10 = P.T \Rightarrow \underline{P = 5}$$

$$\Rightarrow \mathcal{X}' = \frac{1}{\sqrt{P}}\mathcal{X} = \left\{ X_1' = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, X_2' = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \ldots X_4' = \frac{1}{\sqrt{5}}\begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix} \right\}.$$

$$\Delta \mathcal{X}' = \left\{ \Delta X_{12}' \overset{= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}{=} \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \Delta X_{13} = \frac{1}{\ }\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \Delta X_{14} = \frac{1}{\sqrt{5}}\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \right.$$

$$\left. \Delta X_{23}' = \frac{1}{\sqrt{5}}\begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}, \Delta X_{24}' = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \Delta X_{34}' = \frac{1}{\sqrt{5}}\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} \right\}.$$

$$\Rightarrow \min_{\Delta X'} \det\{\Delta X' \Delta X'^H\} = \det\left[\frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}^H\right] = \det\left[\frac{1}{5}\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\right] = \frac{16}{25}$$

V.S Alamouti code

$$X=\left\{\begin{pmatrix}1&-1\\1&1\end{pmatrix},\begin{pmatrix}1&1\\-1&1\end{pmatrix},\begin{pmatrix}-1&-1\\1&-1\end{pmatrix},\begin{pmatrix}-1&1\\-1&-1\end{pmatrix}\right\},\ E\{\|X\|\}=2=P$$

$$X'=\frac{1}{\sqrt{P}}X=\frac{1}{\sqrt{2}}X \Rightarrow \Delta X'=\left\{\Delta X'_{12}=\frac{1}{\sqrt{2}}\begin{pmatrix}0&-2\\2&0\end{pmatrix},\ \Delta x_{13}=\frac{1}{\sqrt{2}}\begin{pmatrix}2&0\\0&2\end{pmatrix},\ \Delta x_{14}=\frac{1}{\sqrt{2}}\begin{pmatrix}2&-2\\2&2\end{pmatrix},\right.$$

$$\left.\Delta X'_{23}=\frac{1}{\sqrt{2}}\begin{pmatrix}2&2\\-2&2\end{pmatrix},\ \Delta X'_{24}=\frac{1}{\sqrt{2}}\begin{pmatrix}2&0\\0&2\end{pmatrix},\ \Delta X'_{34}=\frac{1}{\sqrt{2}}\begin{pmatrix}0&-2\\2&0\end{pmatrix}\right\}.$$

$$\det\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}2&0\\0&2\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}2&0\\0&2\end{pmatrix}^H\right\}=\det\left(\frac{1}{2}\begin{pmatrix}4&0\\0&4\end{pmatrix}\right)=\frac{1}{4}16=4$$

$$\det\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}2&-2\\2&2\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}2&2\\-2&2\end{pmatrix}\right\}=\frac{1}{4}\det\begin{pmatrix}8&0\\0&8\end{pmatrix}=\frac{1}{4}64=16$$

$$\boxed{\min_{\Delta X'}\det(\Delta X\Delta X')=\underline{\underline{4}}}$$

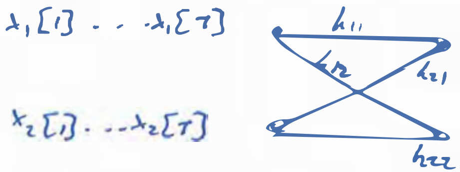
$$\Rightarrow \boxed{4>\frac{16}{25}}$$

- Alamouti allows 2 real dim/2 cu
- Repetit " 1 " "/2 cu

Packing ⇒ better coding gain ⇒ Per ↓.

Transition to mimo channel.

For now $L = n_r = 2$. 2×2 mimo



Let h_{ij} be spatially independent.

⇒ Diversity is 4:

Easy way to show.

Consider repetition code $X = \begin{pmatrix} x & \\ & x \end{pmatrix}$ $T = 2$.

$$\Rightarrow \underset{n_r \times T}{Y} = \underset{n_r \times n_t}{H} . \underset{n_t \times T}{X} + \underset{n_r \times T}{W}$$

$$\Rightarrow \boxed{\underset{2\times 2}{Y} = \underset{2\times 2}{H} . \underset{2\times 2}{X} + \underset{2\times 2}{W}}$$

$$\Rightarrow \begin{bmatrix} \underline{y}_1^T \\ \underline{y}_2^T \end{bmatrix} = \begin{bmatrix} - \underline{h}_1^T - \\ - \underline{h}_2^T - \end{bmatrix} \begin{bmatrix} x & \\ & x \end{bmatrix} + \begin{bmatrix} \underline{w}_1^T \\ \underline{w}_2^T \end{bmatrix}$$

$$= \begin{bmatrix} x . \underline{h}_1^T \\ x . \underline{h}_2^T \end{bmatrix} + \begin{bmatrix} \underline{w}_1^T \\ \underline{w}_2^T \end{bmatrix}$$

$$\Rightarrow \underline{y}' = \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \end{bmatrix}_{4\times1} = \begin{bmatrix} \underline{h}_1 \\ \underline{h}_2 \end{bmatrix}_{4\times1} \cdot x + \begin{bmatrix} \underline{w}_1 \\ \underline{w}_2 \end{bmatrix}_{4\times1} =$$

$$\Rightarrow \underset{4\times1}{\underline{y}'} = \underset{4\times1}{\underline{h}'} \cdot x + \underline{w}'$$

$$\Rightarrow \tilde{y} = \frac{\underline{h}'^H \cdot \underline{y}}{\|\underline{h}'\|} = \underbrace{\|\underline{h}'\|} \cdot x + w' \sim \mathcal{CN}(0, N_0)$$

$$P(x_1 \to x_2) = Q\left(\frac{\sqrt{P}}{\sqrt{\frac{N_0}{2}}} \frac{1}{2} \|\underline{h}'^H\| \left(\underset{\text{scalars}}{x_1' - x_2'}\right)\right) = Q\left(\sqrt{\frac{\rho}{2} \underbrace{\|\underline{h}\|^2}_{\downarrow} \cdot (x_1' - x_2\right.$$

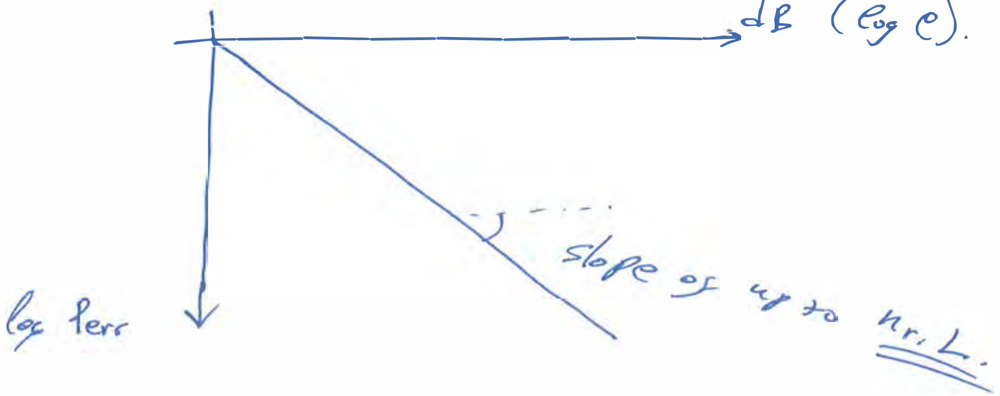
$$\|\underline{h}'\|^2 = \sum_{i=1}^{2} \sum_{j=1}^{2} |h_{ij}|^2 \sim \chi_2^{2\cdot 4}$$

$$\text{But note: } P\left(\sum_{i=1,j=1}^{4} |h_{ij}|^2 < \varepsilon\right) \xrightarrow[\approx]{\varepsilon \to 0} P\left(|h_{ij}|^2 < \varepsilon, \forall i,j\right) = \left[P\left(|h_{ij}|^2 < \varepsilon\right)\right]^4 \approx \varepsilon^4$$

$$\Rightarrow \varepsilon \approx \rho^{-1} \Rightarrow P(\text{deep fade}) \approx \rho^{-4} \quad (\text{diversity } 4).$$

Similarly in the $L\times nr$ mimo channel:

~~st~~. maximum diversity order $nr.L$ $\Rightarrow$ $P_{err} \rightarrow \rho^{-nr.L}$
(can ~~be~~ "scale" as low as $\rho^{-nr.L}$).



To see this, like before: ($nr = nt$)

$$\underset{nr\times T}{Y} = \underset{nr\times L}{H}\cdot\underset{L\times T}{X} + \underset{nr\times T}{W} = \begin{bmatrix} h_1^T \\ h_2^T \\ \vdots \\ h_{nr}^T \end{bmatrix}\begin{bmatrix} x & & \\ & x & \\ & & \ddots \\ & & & x \end{bmatrix} + \underline{W} = \begin{bmatrix} y_1^T \\ y_2^T \\ \vdots \\ y_{nr}^T \end{bmatrix}.$$

use repetition code $T=L$ $\quad X = \begin{pmatrix} x & & 0 \\ & x & \\ 0 & \ddots & x \end{pmatrix}$

$$\Rightarrow \underset{\substack{nr.T\times 1 \\ = nr.L\times 1}}{\begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \vdots \\ \underline{y}_{nr} \end{bmatrix}} = \underset{nr.L\times 1}{\begin{bmatrix} \underline{h}_1 \\ \underline{h}_2 \\ \vdots \\ \underline{h}_{nr} \end{bmatrix}}\cdot x + \underline{w} \Rightarrow \tilde{y} = \underline{h}'^H . \underline{y} = \underbrace{\|\underline{h}'\|}_{\chi_2^{2.nr.L}} . x + w$$

$\chi_2^{2.nr.L}$ - doF chi-~~sq~~squared

$$P(x_1 \rightarrow x_2) \approx P(\|\underline{h}\|^2 . \rho \ll 1) = P(\|\underline{h}\|^2 < \rho^{-1}) \approx \rho^{-L.nr}$$

DOF of mimo channel

note $\underline{y} = H . \underline{x} + \underline{w}$

since h_{ij} drawn from continuous distrib (e.g $h_{ij} \sim \mathcal{CN}(0,1)$)

$\Rightarrow P\left(\text{rank}(H) = \min(n_r, L)\right) = 1$

$\Rightarrow \dim\left(\text{span}(H \underline{x})\right) = \min($[illegible]$, n_r)$ with prob. 1.

BUT must make sure that $\underline{x}$ ~~is~~ (code) is chosen carefully.

Going back to 2x2 case : max DOF = $\min(n_r, L) = 2$ complex dim/c.u

- Alamouti 2 complex dim/ 2 c.u not enough.
- repetition 1 complex dim / 2 c.u even worse.

To explore D.O.F let us calculate
Prob of error in MIMO $(\underbrace{n_r \times L}_{L \times n_r.}$ MIMO, T chan uses$)$.

$$y = H\,X + w$$

$$\begin{bmatrix} y_1^T \\ y_2^T \\ \vdots \\ y_{n_r}^T \end{bmatrix} = \begin{bmatrix} \underline{h}_1^T \\ \underline{h}_2^T \\ \vdots \\ \underline{h}_{n_r}^T \end{bmatrix} X + w$$

$$\Rightarrow \underset{1\times n_r.T}{\begin{bmatrix} y_1^T & y_2^T & \cdots & y_{n_r}^T \end{bmatrix}} = \underset{1\times n_r.L}{\begin{bmatrix} \underline{h}_1^T & \underline{h}_2^T & \cdots & \underline{h}_{n_r}^T \end{bmatrix}} . \overset{\tilde{X}}{\overbrace{\underset{n_r.L \times n_r.T}{\begin{bmatrix} X & & & \\ & X & & \\ & & \ddots & \\ & & & X \end{bmatrix}}}} + \underline{\tilde{w}}^T$$

$$\Rightarrow \boxed{\underset{1\times n_r.T}{\underline{y}^T} = \underset{1\times L.n_r}{\underline{h}} \;.\; \tilde{X} + \underline{w}^T}$$

$$\Delta\tilde{X}' = \frac{1}{\sqrt{\rho}}\left(\tilde{X}_1 - \tilde{X}_2\right).$$

$$P\left(X_A \rightarrow X_B\right) = E_{\underline{h}}\left[Q\left(\sqrt{\frac{\rho}{2}\cdot \underline{h}\,\Delta\tilde{X}\,\Delta\tilde{X}^{H}\underline{h}}\right)\right\}$$

$$\le \prod_{i=1}^{nr.L} \frac{1}{1+\rho\cdot\frac{d_i^2}{4}} = \prod_{i=1}^{nr.\min(T,L)} \frac{1}{1+\rho\cdot\frac{d_i^2}{4}}$$

$$= \left(\prod_{i=1}^{\min(T,L)} \frac{1}{1+\rho\cdot\frac{d_i^2}{4}}\right)^{nr}$$ due to multiplicity of d's of $\tilde{X}$.

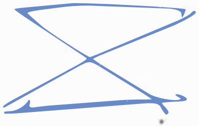
$$\boxed{P(X_A \to X_B) \le \left(\prod_{i=1}^{\min(T,L)} \frac{1}{\frac{4+\rho\cdot d_i^2}{4}}\right)^{nr} \le \left(\prod_{i=1}^{\min(T,L)} \frac{4}{\rho\cdot d_i^2}\right)^{nr} = \frac{4^{nr.\min(T,L)}}{\rho^{nr.\min(T,L)}\cdot\left(\prod_{i=1}^{\min(T,L)} d_i^2\right)^{nr}}}$$

Example:

Compare V-BLAST & alamouti.

V-BLAST $\quad \mathcal{X} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_L \end{pmatrix} \quad x_i \text{ uncoded} \right\}.$

Let $L = 2 = n_t$



$\Rightarrow \mathcal{X}_{VB} = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\} \qquad \mathcal{X}_{ala} = \left\{ X = \begin{pmatrix} x_1 & -x_2^* \\ x_2 & x_1^* \end{pmatrix} \quad x_1, x_2 \text{ indep} \right\}.$

Comparison: 2 bits per channel use

$\Rightarrow$ VB, BPSK & Alam, 4-PAM ..

VB:

$$\mathcal{X} = \left\{ \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\end{pmatrix} \right\}$$

$$\mathcal{X}' = \left\{ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \; - \; - \; ; \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\-1\end{pmatrix} \right\}.$$

$$\Delta x' = \frac{1}{\sqrt{2}}\left\{ \begin{pmatrix}0\\2\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}, \begin{pmatrix}2\\-2\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix} \right\}.$$

Recall:

$$P_{err} \le \frac{4^{n_r \cdot \min(T,L)}}{\rho^{n_r \cdot \min(T,L)} \cdot \left(\prod_{i=1}^{\min(T,L)} |\lambda_i|^2 \right)^{n_r}} = \frac{4^{2 \cdot 1}}{\rho^{2 \cdot 1} \cdot \left(\| \delta x \|^2 \right)^2} = \frac{16}{\rho^2 \left(\left\| \frac{1}{\sqrt{2}} \begin{pmatrix}2\\0\end{pmatrix} \right\|^2 \right)^2}$$

$$\boxed{\Rightarrow P_{er} \le \frac{16}{\rho^2 \cdot \frac{16}{4}} = 4 \cdot \rho^{-2}} \longleftarrow \text{V-BLAST.}$$

- For Alamouti, after setting 4-PAM & normalize for power constraint, above gives

$$P_{err} \le 1600 \cdot \rho^{-4}$$

→ V-BLAST no ^{tx} diversity (ρ^{-2} vs ρ^{-4}) but much better coding gain because better DOF.

Alamouti : better diversity (ρ^{-4}) but bad coding gain because reduced DOF.