Lecture $3 \rightarrow 5$

Detection in Gaussian noise:

Distinguish original / transmitted signal after it was corrupted by a) fading b) Noise. (additive, white Gaus)

Recall that we deal with discrete time complex model.

Scalar detection. (for now only noise).

$\boxed{y = x + w}$ y: r+, x: t+, $w \sim N(0, \frac{N_0}{2})$. $y, x, w \in \mathbb{R}$.

- $\boxed{x = x_A \text{ or } x_B}$ (1-bit) with equal probability

- Optimal detector (ML)

take y, & compare $\Rightarrow$ $P(x = x_A | y) \geq P(x = x_B | y)$ $\Rightarrow$ choose x_A (else x_B).

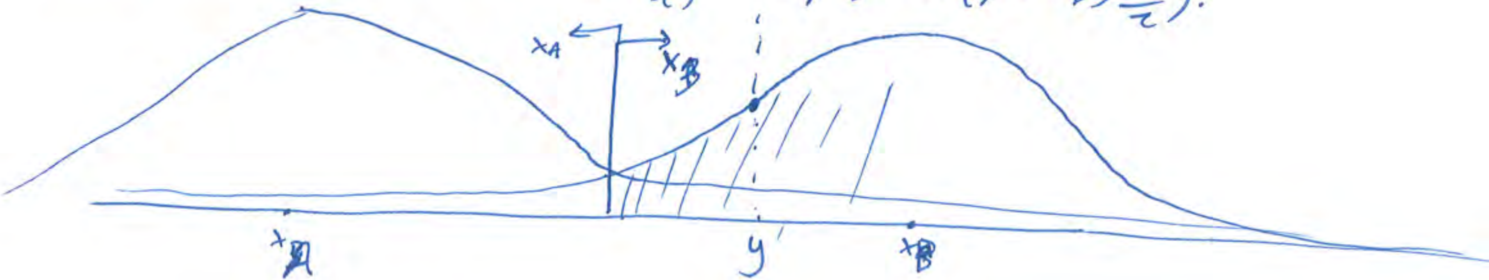
$$\Rightarrow \quad P(x = x_A | y) \cdot P(y) \geq P(x = x_B | y) P(y)$$

$$\Rightarrow \quad P(x_A, y) \geq P(x_B, y)$$

$$\Rightarrow \quad P(y | x_A) \cdot \cancel{P(x_A)} = P(y | x_B) \cancel{P(x_B)}$$

$$\Rightarrow \quad P(y | x_A) \overset{x_A}{\geq} P(y | x_B).$$

Note that $y/x_A \sim N(\mu = x_A, \frac{N_0}{2})$; $y/x_B \sim N(\mu = x_B, \frac{N_0}{2})$.



$$\Rightarrow \quad \frac{1}{\sqrt{\pi N_0}} e^{-(y-x_A)^2/N_0} \underset{x_B}{\overset{x_A}{\gtrless}} \frac{1}{\sqrt{\pi N_0}} \cdot e^{-(y-x_B)^2/N_0}$$

$$\Rightarrow \quad |y - x_A| \underset{x_B}{\gtrless} |y - x_B|$$

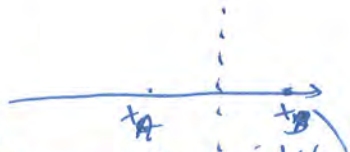
$\Rightarrow$ detector chooses nearest neighbor.

⇒ Optimal probability of error. (scalar detection).

error when $(y-x_A)^2 > (y-x_B)^2$ & $x = x_A$ $(y = x_A + w)$.

⇒ $\cancel{y^2} - 2x_A \cdot y + x_A^2 > \cancel{y^2} - 2x_B \cdot y + x_B^2$

⇒ $2y(x_B - x_A) > x_B^2 - x_A^2$

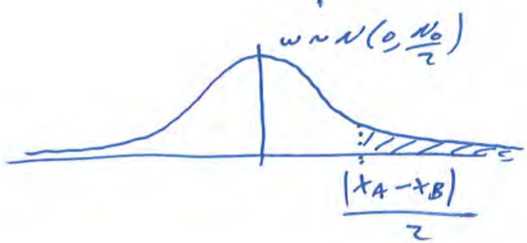


⇒ $\boxed{y > \frac{x_B + x_A}{2}}$ error event (if $x = x_A$). (passes the middle).

⇒ $y = x_A + w$ ⇒ $x_A + w > \frac{x_B + x_A}{2}$ ⇒ $w > \frac{x_B - x_A}{2}$

⇒ considering symmetry ⇒ error when $w > |x_A - x_B|/2$

⇒ $P_{err}|_{x = x_A} = P\left(y > \frac{x_B + x_A}{2} \,\Big|\, x_A\right) = P\left(w > \frac{|x_A - x_B|}{2}\right) = Q\left(\frac{|x_A - x_B|}{2\sqrt{N_0/2}}\right)$.



Vector Detection (Real).

$\underline{x} = \underline{x}_A$ or $\underline{x}_B$ 50/50.

$$\underline{y} = \underline{x} + \underline{w} \qquad \underline{w} \sim \mathcal{N}\left(0, \frac{N_0}{2} I\right).$$

$$\Rightarrow \quad P(\underline{x} = \underline{x}_A \mid \underline{y}) \overset{\underline{x}_A}{\gtrless} P(\underline{x} = \underline{x}_B \mid \underline{y}) \Rightarrow \underbrace{P(\underline{y} \mid \underline{x}_A)}_{\mathcal{N}\left(\underline{x}_A,\, I \frac{N_0}{2}\right)} \gtrless \underbrace{P(\underline{y} \mid \underline{x}_B)}_{\mathcal{N}\left(\underline{x}_B,\, I \cdot \frac{N_0}{2}\right)}.$$

$$\Rightarrow \quad \frac{1}{\sqrt{\pi N_0}^{\,n}} \cdot e^{-\|\underline{y} - \underline{x}_A\|^2 / N_0} \overset{\underline{x}_A}{\gtrless} \frac{1}{\sqrt{\pi N_0}^{\,n}} \cdot e^{-\|\underline{y} - \underline{x}_B\|^2 / N_0}$$

$$\Rightarrow \quad |\underline{y} - \underline{x}_A| \overset{\underline{x}_A}{<} |\underline{y} - \underline{x}_B|$$

when $\underline{x} = \underline{x}_A \Rightarrow$ error if $|\underline{y} - \underline{x}_A|^2 > |\underline{y} - \underline{x}_B|^2$

$\Rightarrow |\underline{x}_A + \underline{w} - \underline{x}_A|^2 > |\underline{x}_A + \underline{w} - \underline{x}_B|^2 \Rightarrow \|\underline{w}\|^2 > \|w + \underbrace{x_A - x_B}_{\gamma}\|^2$

$\Rightarrow \underline{w}^T \underline{w} > (\underline{w} + \underline{\gamma})^T (\underline{w} + \underline{\gamma})$

$\Rightarrow \cancel{w^T \underline{w}} > \cancel{w^T w} + \underline{w}^T \underline{\gamma} + (\underline{w}^T \underline{\gamma})^T + \gamma^T \gamma$

$\Rightarrow 2 Re\{w^T \underline{\gamma}\} < -\gamma \gamma^H$. but all real.

$P\left(w^H \cdot \gamma < -\frac{\|\gamma\|^2}{2}\right) = P\left((\underline{x}_A - \underline{x}_B)^T \cdot \underline{w} < -\frac{\|x_A - x_B\|^2}{2}\right)$.

$\Rightarrow$ $P_{err} = P\left((\underline{x}_B - \underline{x}_A)^T \cdot \underline{w} > \frac{\|x_A - x_B\|^2}{2}\right)$. This just means that error when the projection of noise in direction of signal exceeds the half-distance of the signal vectors.

Note that $(\underline{x}_B - \underline{x}_A)^T \cdot \underline{w} \sim \mathcal{N}\left(0, \|x_A - x_B\|^2 \cdot \frac{N_0}{2}\right)$.

(because recall: $x_i \sim \mathcal{N}(0, \sigma_i^2) \Rightarrow \sum_{i=1}^{n} c_i x_i \sim \mathcal{N}(0, \sum c_i^2 x_i^2) = \mathcal{N}(0, \sum_{i=1}^{n} c_i^2 \frac{N_0}{2})$

$\Rightarrow P(err) = Q\left(\frac{\|\underline{x}_A - \underline{x}_B\|^2}{2 \cdot \underbrace{\sqrt{\|x_A - x_B\|^2 N_0/2}}_{\sigma \text{ of noise.}}}\right) = Q\left(\frac{\|x_A - x_B\|}{2\sqrt{N_0/2}}\right) = P_{err}$

- Only function of Euclidean distance.

Let us provide an alternate view.

$\underline{y} = \underline{x} + \underline{w}$ $\quad \underline{w} \sim \mathcal{N}(0, \frac{N_0}{2} I)$ $\quad \underline{x} = \underline{x}_A$ or $\underline{x}_B$.

Let $x = \begin{cases} \frac{1}{2} & \text{if } \underline{x} = \underline{x}_A \\ -\frac{1}{2} & \text{if } \underline{x} = \underline{x}_B \end{cases}$ $\quad \Rightarrow \underline{x} = x\,(\underline{x}_A - \underline{x}_B) + \frac{1}{2}(\underline{x}_A + \underline{x}_B).$

NOTE (scalar).

$\underline{y} = \underline{x} + \underline{w}$

Let $\underline{y}' = \underline{y} - \frac{\underline{x}_A + \underline{x}_B}{2} = x\,(\underline{x}_A - \underline{x}_B) + \cancel{\left(\frac{\underline{x}_A + \underline{x}_B}{2}\right)} - \cancel{\left(\frac{\underline{x}_A + \underline{x}_B}{2}\right)} + \underline{w}$

(just subtract from both sides).

$\Rightarrow \underline{y}' = x\,(\underline{x}_A - \underline{x}_B) + \underline{w}.$ $\quad$ (*)

$\Rightarrow$ can see that "transmitted vector" only in direction

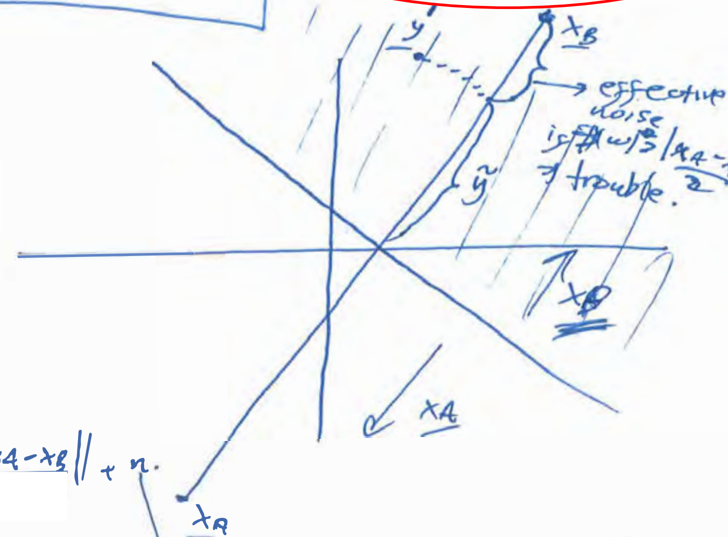
$$\underline{v} = \frac{\underline{x}_A - \underline{x}_B}{\|\underline{x}_A - \underline{x}_B\|}$$

- The projection of $\underline{y}'$ onto direction $\perp$ to $\underline{v}$ contains only noise, & that noise is $\perp$ (& $\exists$ independent) of the noise in direction of signal.

$\Rightarrow \quad \tilde{y} = \underline{v}^H \cdot \underline{y}' \quad \Rightarrow \quad \boxed{\tilde{y} = v^H \cdot \left(y - \tfrac{1}{2}(\underline{x}_A + \underline{x}_B)\right)}$

sufficient statistic

Equivalently:



note

$$\underline{v}^H \cdot \underline{y}' = \frac{(\underline{x}_A - \underline{x}_B)}{\|\underline{x}_A - \underline{x}_B\|} \cdot x \cdot (\underline{x}_A - \underline{x}_B) + \underline{v}^H \cdot \underline{w} = x\,\|\underline{x}_A - \underline{x}_B\| + n.$$

n: retains N_0 since $|\underline{v}| = 1$

$$\Rightarrow \quad O \cdot y' = \begin{bmatrix} x[\underline{x}_A - \underline{x}_B] \\ 0 \\ 0 \\ 0 \end{bmatrix} + O\underline{w} \rightarrow \begin{pmatrix} w \\ \vdots \\ d \end{pmatrix}$$

⇒ this approach is called "matched filter".

Project y in direction of signal space.

⇒ now scalar detection problem

$$\boxed{\tilde{y} = x \, \|\underline{x}_A - \underline{x}_B\| + w}$$

$x = \pm \frac{1}{2}$ send bit (1, 0)

⇒ effective half distance $\frac{\|\underline{x}_A - \underline{x}_B\|}{2}$

⇒ As before $\boxed{P_{er} = Q\left(\frac{\|\underline{x}_A - \underline{x}_B\|}{2\sqrt{\frac{N_0}{2}}}\right)}$.

- Argument Generalized $\{\underline{x}_1, \ldots, \underline{x}_m\}$: project $\underline{y}$ on $\langle \underline{x}_1, \ldots, \underline{x}_m \rangle$.

more signal options.

⇒ suff statistic.

- If $\underline{x}_1, \ldots, \underline{x}_m$ is colinear i.e $\underline{x}_i = x_i \, \underline{h}$

⇒ project $\underline{y}$ onto $\underline{h}$.

Vector Detection : complex

$$\underline{y} = \underline{x} + \underline{w} \qquad \underline{x} = \begin{cases} \underline{x}_A \\ \underline{x}_B \end{cases} \in \mathbb{C}^n. \qquad \underline{w} \sim \mathcal{CN}(0, I \cdot N_0)$$

$$\underline{x} = x\,(\underline{x}_A - \underline{x}_B) + \tfrac{1}{2}(\underline{x}_A + \underline{x}_B) \qquad x = \pm\tfrac{1}{2}.$$

signal direction $\quad \underline{v} = \dfrac{\underline{x}_A - \underline{x}_B}{\|\underline{x}_A - \underline{x}_B\|}$

$$\underline{y}' = \underline{y} - \tfrac{1}{2}(\underline{x}_A + \underline{x}_B)$$

suff stat $\quad \tilde{y} = \underline{v}^H \cdot \underline{y}' = \quad \boxed{\tilde{y} = x\,\|u_A - u_B\| + w'} \quad w' \sim \mathcal{CN}(0, N_0)$

Since $x \in \mathbb{R} \Rightarrow$ can get simpler suff stat

$$\mathrm{Re}\{\tilde{y}\} = x\,\|\underline{x}_A - \underline{x}_B\| + \mathrm{Re}\{w\} \qquad \Rightarrow \mathrm{Re}\{w\} \sim N(0, \tfrac{N_0}{2}).$$

$$\Rightarrow \quad P_{err} = Q\left(\frac{\|\underline{x}_A - \underline{x}_B\|}{2\sqrt{\frac{N_0}{2}}}\right) \qquad \text{eff. half distance } \frac{\|\underline{x}_A - \underline{x}_B\|}{2}$$

Generalization: if tx-vectors of form $\quad \underline{h} \cdot x_i,\ x_i \in \mathbb{C} \Rightarrow \underline{h}^H \cdot \underline{y}'$ suff stat

$\underline{h}\, x_i \quad x_i \in \mathbb{R} \Rightarrow \mathrm{Re}\{\underline{h}^H \underline{y}\}.$

Detection over fading channels

- Main diff over AWGN case: much higher $P_r(\text{err})$.

Recall $y[m] = \sum_{\ell=0}^{L-1} h_\ell[m]\, x[m-\ell] + w[m]$

Consider flat fading.

$$y[m] = h[m]\, x[m] + w[m]$$

$w[m] \sim \mathcal{CN}(0, N_0)$, $h[m] \sim \mathcal{CN}(0,1)$

random fading

Consider BPSK: $x[m] = \pm a$ (indep. in time)

Consider no knowledge of $h[m]$ at Rx.

$\Rightarrow P_{err} \geq \frac{1}{2}$

$\Rightarrow$ Need for coding.

Orthogonal signaling

$$\underline{x} = \begin{cases} \underline{x}_{A_1} = \begin{bmatrix} a \\ 0 \end{bmatrix} \\ \underline{x}_B = \begin{bmatrix} 0 \\ a \end{bmatrix}. \end{cases}$$

$$\underline{y} = \begin{bmatrix} y[0] \\ y[1] \end{bmatrix}$$

Q: say I send one of $\underline{x}_A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or $\underline{x}_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

say I receive. $\begin{bmatrix} y[0] \\ y[1] \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0.03 \end{bmatrix}$

tell me decision

ML - Detector

$$f(\underline{y} \mid \underline{x}_A) \overset{\underline{x}_A}{>} f(\underline{y} \mid \underline{x}_B) \Rightarrow$$

$$\Rightarrow \text{for } \Lambda(\underline{y}) = \log\left(\frac{f(\underline{y}/\underline{x}_A)}{f(\underline{y}/\underline{x}_B)}\right) \overset{\underline{x}_A}{>} 0.$$

$$\underline{y} \mid \underline{x}_A = [\underbrace{y[0] \mid \underline{x}_A}_{\mathcal{CN}(0,\, a^2 + N_0)},\ \underbrace{y[1] \mid \underline{x}_A}_{\mathcal{CN}(0,\, N_0)}].$$

$$\underline{y} \mid \underline{x}_B = [\underbrace{y[0] \mid \underline{x}_B}_{\mathcal{CN}(0,\, N_0)},\ \underbrace{y[1] \mid \underline{x}_B}_{\mathcal{CN}(0,\, a^2 + N_0)}]$$

mean zero

$$y[0] \mid \underline{x}_A = \underbrace{h \cdot x}_{\text{Power } a^2} + \underbrace{w}_{\text{Power } N_0}$$

$$y[1] \mid \underline{x}_A = 0 + w$$

$$E\{y[0] \mid x_A\} = 0 \text{ since } E\{h\,a + w\} = E\{h\}.E\{a\} + E\{w\} = 0$$

But $\underbrace{y[0]|\underline{x}_A}_{ha+w_1}$, $\quad \underbrace{y[1]|x_A}_{w_2}$ are indep (note $P(ha+w_1|w_2) = P(ha+w_1)$. ✓).

similarly $y[0]|\underline{x}_B$, $y[1]|\underline{x}_B$ indep

$$\Rightarrow \Lambda(\underline{y}) = \log\left(\frac{f(\underline{y}|\underline{x}_A)}{f(\underline{y}|\underline{x}_B)}\right) = \log\left(\frac{f(\{y[0], y[1]\}|\underline{x}_A)}{f(\{y[0], y[1]\}|\underline{x}_B)}\right)$$

$$= \log\left(\frac{f(y[0]|\underline{x}_A)\cdot f(y[1]|\underline{x}_A)}{f[y[0]|x_B\}\cdot f\{y[1]|x_B)}\right)$$

$$= \log\left[\frac{e^{-y^2[0]/a^2+N_0}\cdot e^{-y^2[1]/N_0}}{e^{-y^2[0]/N_0}\cdot e^{-y^2[1]/a^2+N_0}}\right] = \log\left[e^{(y^2[1]-y^2[0])/a^2+N_0} - (y^2[1]-y^2[0])/N_0\right]$$

$$= \left(y^2[1]-y^2[0]\right)\cdot\left(\frac{1}{a^2+N_0} - \frac{1}{N_0}\right) = \left(y^2[0]-y^2[1]\right)\cdot\left(\frac{1}{N_0} - \frac{1}{a^2+N_0}\right)$$

$$\Rightarrow \Lambda(\underline{y}) = \frac{\left(y^2[0]-y^2[1]\right)}{(a^2+N_0)\frac{N_0}{a^2}} \overset{x_A}{\gtrless} 0$$

$$\Rightarrow |y[0]| \overset{x_A}{\gtrless} |y[1]|$$

$$\Pr(\text{err} / \underline{x}_A) = P\left(|y[1]|^2 > |y[0]|^2 \,\middle|\, \underline{x}_A\right)$$

> **Bonus**
>
> Can someone offe[r] rigorous exposition??

To calculate this, note

$$|y[0]|^2 \sim \exp[a^2 + N_0], \quad |y[1]|^2 \sim \exp\{N_0\}$$

& then recall MGF of exp. distribution $\sim X$

$$E\{e^{sX}\} = \frac{1}{1+s} \qquad s < 1.$$

To get

$$P_{err|\underline{x}_A} = P\left(|y[1]|^2 > |y[0]|^2 \,\middle|\, \underline{x}_A\right) = \left(2 + \frac{a^2}{N_0}\right)^{-1} = P_{err|\underline{x}_A}$$

P_{err} for $\perp$ signaling & optimal decod[ing]

Received SNR

$$\rho := \frac{\text{average rx signal power per complex symbol}}{\text{average noise power per complex symbol}} = \frac{\frac{a^2}{2}}{N_0}$$

To deri[ve] SNR

$$\begin{matrix} a^2 \to \\ 0 \to \end{matrix} \begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \begin{matrix} \to \mathcal{CN}(0, N \ldots \\ \to \mathcal{CN}(0, N \ldots \end{matrix}$$

$$\Rightarrow P_{err} = \frac{1}{2 + \frac{a^2}{N_0}} = \frac{1}{2\left(1 + \frac{a^2}{2 N_0}\right)} = \frac{1}{2(1+\rho)} = P_{err}$$

$\rho = \frac{a^2}{N_0}$ → power/complex.
→ noise is there

Compare with detection with AWGN channel.

$y[m] = x[m] + w[m] \qquad x[m] = \pm a$

$\tilde{y} = Re\{y[m]\} \Rightarrow P_{err} = Q\left(\frac{a}{\sqrt{\frac{N_0}{2}}}\right) = Q\left(\sqrt{\frac{2a^2}{N_0}}\right) = Q(\sqrt{2\rho})$

but $Q(\sqrt{2\rho}) \approx e^{-\rho} << \rho^{-1}$ as ρ increases.

→ Huge difference between AWGN performance & Fading with non-coherent detection.

Ex: ASK $\left(P_e = 10^{-5} \Rightarrow P_e \approx \frac{1}{2(1+\rho)} \Rightarrow \rho_{min} = \frac{1}{2P_e} - 1 \approx 5 \cdot 10^4 \approx 50\,dB \right.$

$\left. \Rightarrow a^2 > 10^5 \quad a \approx 300,\ N_0 = 1 \right)$.

where as AWGN $e^{-\rho} \approx 10^{-5} \Rightarrow -\rho \cdot \log e = -5 \log_{10}$ (log P_e)

$\Rightarrow \rho \approx 10 \Rightarrow a^2 \approx 10 \Rightarrow (a \approx 3 \to 4.)$

Coherent detection:

Is detector to blame, or fading?

Let us try coherent detection.

BPSK & assume CSIR.

$y = hx + w$ $\quad$ h known to Rx. $\quad h \sim \mathcal{CN}(0,1)$

$w \sim \mathcal{CN}(0, N_0)$.

$x \in \{a, -a\}$ $\quad a \in \mathbb{R}$.

$$SNR = \rho = \frac{E\{|hx|^2\}}{E\{|w|^2\}} = \frac{a^2}{N_0}.$$

ASK THE

First do suff statistics: project onto dir of signal $\frac{h}{\|h\|} =: \underline{v}$

$\Rightarrow$ also take real. (ASK WHY)

$$\tilde{r} = \mathrm{Re}\Big\{ \underbrace{\frac{h^H}{\|h\|}}_{\underline{v}^H} \cdot y \Big\} = \mathrm{Re}\Big\{ \frac{h^H}{\|h\|} \cdot (hx + w) \Big\} = \mathrm{Re}\Big\{ \|h\| x + \underbrace{\frac{h^H}{\|h\|} w}_{\text{Just rotated } \mathcal{CN}(0, N_0)} \Big\}$$

$\Rightarrow \tilde{r} = \|h\| \cdot x + z$ $\qquad z = \mathrm{Re}\Big\{ \frac{h^H}{\|h\|} w \Big\}$ $\quad z \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

(half distance: $|h| \cdot a$)

$$\Rightarrow P_{err} = E_h\{P_{err}(err|h)\} = E_h\left\{Q\left(\frac{|h|.a}{\sqrt{N_0/2}}\right)\right\}, \quad \rho = \frac{a^2}{N_0}$$

$$= E_h\left\{Q\left(\sqrt{\frac{|h|^2 a^2}{N_0/2}}\right)\right\} = E_h\left\{Q\left(\sqrt{|h|^2 \cdot 2\rho}\right)\right\}.$$

$$= \int_{h \in \mathbb{C}} Q\left(\sqrt{|h|^2 \cdot 2\rho}\right) \underset{\text{by integration.}}{=} \frac{1}{2}\left[1 - \sqrt{\frac{\rho}{\rho+1}}\right] \underset{\text{taylor}}{=} \frac{1}{2}\left[1 - \left\{1 - \frac{1}{2\rho}\right\} + O(\rho^{-2})\right]$$

$$\boxed{\Rightarrow P_{err} \approx \frac{1}{4\rho}}$$

Note: only 3dB diff from $\perp$ (non-coh).

$\Rightarrow$ major cause of error. is "deep fade" : $|h|^2 \cdot \rho \ll 1$.

Deep fade: h: $|h|^2 \rho \ll 1$.

Strong association to event of error.

1) $|h|^2\rho \gg 1 \Rightarrow P_{err} \to Q(|h|^2\rho) \approx e^{-|h|^2\cdot\rho} \to 0$

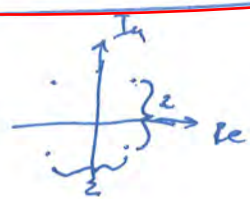
$\Rightarrow \overline{\text{fade}} \Rightarrow \overline{\text{error}}$

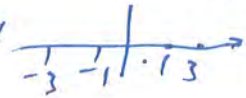
2) $|h|^2\rho \ll 1 \Rightarrow P(err) \to 1 \Rightarrow \text{fade} \xrightarrow{\text{prob}} \text{error}$.

$P(\text{deep fade}) \approx \rho^{-1} \approx P_{err}$.

- Intuitions 1) CSIR v.s. No CSIR not really the main cause of problem here.
  2) Problem caused by deep fade most often.

Exploiting degrees of freedom (DOF).

Question: what is better: send 

or send 

?

Let us try to use more dimensions than BPSK.

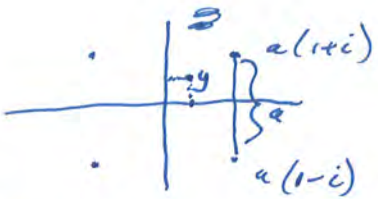
Use QPSK (Quadrature phase shift keying).

$$x[m] \in \{ a(1+j),\ a(1-j),\ a(-1,+j),\ a(-1,-j) \}.$$

- bits in I & Q dimensions are indep. detected. (due to noise indep.)

- Consider first AWGN case

$$y[m] = x[m] + w[m].$$



$$y = y_R + i \cdot y_I$$

⇒ have (due to noise indep) two indep BPSK detections.

$$y_R = x_R + w_R, \quad y_I = x_I + w_I \qquad x_R \in \{a, -a\} \quad x_I \in \{a, -a\}.$$

$$P_{err} = Q\left(\frac{\text{half distance}}{\sigma}\right) = Q\left(\frac{a}{\sqrt{\frac{N_0}{2}}}\right) = Q\left(\sqrt{\frac{2a^2}{N_0}}\right).$$

(matches BPSK case) (but double the rate).

- To be more fair in comparison:

BPSK: $e = \frac{a^2}{N_0} \Rightarrow P_{err} = Q(\sqrt{2e})$

QPSK $e = \frac{E\{|x|^2\}}{\sigma\{|w|^2\}} = \frac{2a^2}{N_0} \Rightarrow P_{err} = Q(\sqrt{e})$

Just note

$\Rightarrow e \to \frac{e}{2}$ (correction from BPSK $\to$ QPSK).

$\Rightarrow$ Recall $P_{err}(\text{BPSK, Rayleigh Fading}) \approx \frac{1}{2}\left(1 - \sqrt{\frac{e}{1+e}}\right) \approx \frac{1}{4e}$

$P_{err}(\text{QPSK, Rayleigh}) \overset{e \to \frac{e}{2}}{\approx} \frac{1}{2}\left(1 - \sqrt{\frac{2e}{1+2e}}\right) \approx \frac{1}{2e}$

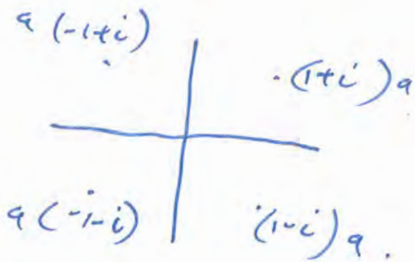
$\Rightarrow$

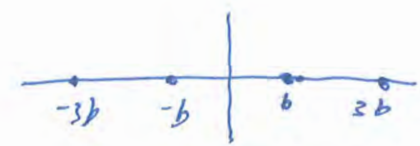
⇒ $P_{err}|_{QPSK} \approx P_{err}|_{BPSK}$ but double the rate.

- Due to more efficient packing.
  (* same P_{err}, double rate)

- Now take similar approach; same P_{err}, same Rate, less power, when using more dimensions

⇒ Compare QPSK (seen), with 4-PAM (fewer dimensions).





4-PAM: $P_{err}(4PAM, AWGN) \approx Q\left(\frac{\text{half dist}}{\sigma}\right) = Q\left(\frac{b}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{2b^2}{N_0}}\right)$

actually considers boundaries.

$$P_{err}(4PAM, AWGN) = \frac{2 \cdot Q\left(\sqrt{\frac{2b^2}{N_0}}\right) + \overbrace{2 \cdot \frac{1}{2} Q\left(\sqrt{\frac{2b^2}{N_0}}\right)}^{\text{edges}}}{4} = \frac{3}{4} Q\left(\sqrt{\frac{2b^2}{N_0}}\right)$$

Recall Q-PSK & AWGN $P_{er} = Q\left(\frac{a}{\sqrt{\frac{N_0}{2}}}\right) = Q\left(\sqrt{\frac{2a^2}{N_0}}\right)$

We compare, setting $P_{err}(\text{QPSK, AWGN}) \approx P_{er}(4\text{ PAM, AWGN})$

$$Q\left(\sqrt{\frac{2a^2}{N_0}}\right) \approx Q\left(\sqrt{\frac{2b^2}{N_0}}\right) \Rightarrow \boxed{a=b}$$

$\Rightarrow$ same prob of error, same rate (2 bits / c.u).

Power: QPSK: $E\{|x|^2\} = 2a^2$, 4-PAM: $\frac{b^2+b^2+9b^2+9b^2}{4} = 5b^2 = 5a^2$.

$$\Rightarrow \underbrace{E\{|x|^2\}}_{\text{QPSK}} = 2a^2 < 5a^2 = \underbrace{E\{|x|^2\}}_{4\text{-PAM}}$$

- Due to packing efficiency, we have 2.5 times less power (4 dB)

$$\text{dB} = 10 \log_{10} 2.5 \approx 10 \cdot (0.4) = 4 \text{ dB}$$

- Common problem: we are stuck $P_{err} \approx \frac{1}{\rho}$.

$\Rightarrow$ Need Diversity.