MOBCOM-MIdtermF2024

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Mobile Communication Techniques

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Midterm Exam

November 21st, 2024

Time: 9:00-10:00

Instructions

- Exercises fall in categories of 1-point and 2-point exercises.
- Total of 11 $\times 1$ + 2 $\times 2$ = 15 points.
- NOTE!!! The exam will be evaluated, out of 13 points. Any points you get beyond 13 points, will be offered as extra bonus.
- Each answer should be clearly written, and the solution should be developed in detail.
- Mathematical derivations need to show all steps that lead to the answer.
- Complete as many exercises as you can. Don't spend too much time on an individual question.
- There is NO penalty for incorrect solutions.
- If in certain cases you are unable to provide rigorous mathematical proofs, go ahead and provide intuitive justification of your answers. Partial credit will be given.
- Calculators are not allowed.
- You are allowed your class notes and class book.

Hints - equations - conventions:

- Notation
  - SISO = single-input single-output, MISO = multiple-input single-output, SIMO = single-input multiple- output, MIMO = single-input multiple-output,
  - R represents the rate of communication in bits per channel use (b.p.c.u),
  - ρ represents the SNR (signal to noise ratio),
  - w will denote additive noise which will be distributed as a circularly symmetric Gaussian random variable $\mathbb{CN}(0, N_0)$. If N_0 is not specified, then set $N_0 = 1$,

– h_i will denote independent fading scalar coefficients which will be distributed as circularly symmetric Gaussian random variables $\mathbb{C}\mathcal{N}(0,1)$.

- GOOD LUCK!!

1) (1 point). In a multi-path fading scenario with delay spread $6\mu s$ and $L = 3$ channel taps, what is the operational bandwidth W?

In a multi-path fading scenario, the **operational bandwidth** W is inversely proportional to the delay spread $T_{\max}$, which is the maximum delay between the arrival of multipath components. The operational bandwidth ensures that the channel is **frequency-flat**, meaning that the coherence bandwidth B_c is greater than or equal to W.

The coherence bandwidth B_c is given approximately by:

$B_c \approx \frac{1}{T_{\max}}$

Here, $T_{\max} = 6\,\mu s$. Substituting this value:

$B_c \approx \frac{1}{6\times10^{-6}} = 166.67\,\text{kHz}$

Thus, the operational bandwidth W should be:

$W \leq B_c = 166.67\,\text{kHz}$.

So, the **operational bandwidth** is approximately **166.67 kHz**.

2) (1 point). Imagine a given SNR equal to ρ, and imagine that we are operating over a (quasi-static) Rayleigh fading SISO channel. Can you describe a code that achieves probability of error approximately equal to $P_e \approx \rho^{-4}$, and rate equal to $R = 2$ bpcu.

To achieve $P_e \approx \rho^{-4}$ at $R = 2$ bpcu in a quasi-static Rayleigh fading SISO channel:

- **Code**: Use a 4-QAM constellation with a rate-2 space-time block code (e.g., Alamouti code or a 4-dimensional lattice code).
- **Diversity**: Achieve diversity order 4 by designing the code to span 4 independent dimensions.
- **Performance**: Ensures $P_e \sim \rho^{-4}$ with increasing SNR.

3) (1 point). How much time diversity will we get with the following SISO (time-diversity) channel model

$$[y_1\ y_2\ y_3] = [h_1 u_1 \quad h_2(u_1 + u_2) \quad h_3 u_2] + [w_1\ w_2\ w_3]$$

where the u_1, u_2, u_3 are independent PAM elements. Justify your answer.

To determine the **time diversity** in the given channel model:

Channel Model $[y_1\ \ y_2\ \ y_3] = [h_1 u_1 \quad h_2(u_1 + u_2) \quad h_3 u_2] + [w_1\ \ w_2\ \ w_3]$, where u_1, u_2, u_3 are independent PAM symbols, h_1, h_2, h_3 are the channel coefficients, and w_1, w_2, w_3 are noise terms.

Analysis

1. **Definition of Time Diversity**:
   - Time diversity is determined by the number of independently faded channel coefficients (h_1, h_2, h_3) that affect the transmitted symbols.
2. **Observation of Dependencies**:
   - y_1 depends on $h_1 u_1$.
   - y_2 depends on $h_2(u_1 + u_2)$.
   - y_3 depends on $h_3 u_2$.
3. **Diversity Order**:
   - u_1 is present in both y_1 and y_2, thus contributing to diversity through h_1 and h_2.
   - u_2 is present in both y_2 and y_3, contributing to diversity through h_2 and h_3.

Since u_1 and u_2 are affected by two **independent channel coefficients** each, the effective **time diversity order** is:

Time Diversity Order $=$ min(number of independent fades per symbol) $= 2$.

Justification The system achieves a time diversity order of 2 because each transmitted symbol u_1 and u_2 is observed across two independently faded channels (h_1, h_2 for u_1; h_2, h_3 for u_2). The third symbol u_3 does not contribute additional diversity as it is only affected by h_3.

4) (1 point). In a SISO case, what is the degrees of freedom (DOF) if we have a time-diversity code (spanning three channel uses) of the form $\mathcal{X} = [u_1 + u_2 \quad u_1 + u\_3 \quad u_2 + u_3]$ where the u_1, u_2, u_3, u_4 are independent 16-PAM elements?

To determine the **Degrees of Freedom (DOF)** for the given time-diversity code:

Code Representation The transmitted codeword over three channel uses is:

$\mathcal{X} = \left[u_1 + u_2 \quad u_1 + u_3 \quad u_2 + u_3\right]$,

where u_1, u_2, u_3 are **independent symbols** from a 16-PAM constellation.

Degrees of Freedom (DOF)

- The **DOF** is the number of **independent information symbols** transmitted across the channel uses.
- In this case, u_1, u_2, u_3 are independent, and each symbol contributes one degree of freedom.

Since the code spans **three channel uses**, the DOF is:

$\text{DOF} = \frac{\text{Number of Independent Symbols}}{\text{Number of Channel Uses}} = \frac{3}{3} = \boxed{1}$.

5) (1 point). For the case of time diversity in the SISO (quasi-static) fading channel, what is the advantage and the disadvantage of the repetition code, compared to uncoded transmission.

- **Advantage**: Repetition code improves reliability by providing diversity gain, reducing the error probability in fading channels.

- **Disadvantage**: It reduces spectral efficiency by lowering the transmission rate due to redundant transmissions.

6) (1 point). In a SISO case, what is the DOF and the rate (in bpcu), of the following time-diversity code (three channel uses) that takes the form $\mathcal{X} = [u_1 + u_4 \quad u_2 \quad u_1 + u_2 + u_3]$ where the u_1, u_2, u_3, u_4 are independent 64-QAM elements?

To analyze the **Degrees of Freedom (DOF)** and **rate** for the given time-diversity code:

Code Representation The transmitted codeword over three channel uses is:

$$X = \begin{bmatrix} u_1 + u_4 & u_2 & u_1 + u_2 + u_3 \end{bmatrix},$$

where u_1, u_2, u_3, u_4 are independent symbols from a 64-QAM constellation.

1. Degrees of Freedom (DOF):

- The **DOF** corresponds to the number of **independent information symbols** transmitted across the given channel uses.
- Here, u_1, u_2, u_3, u_4 are **independent symbols**, so there are **4 independent symbols** transmitted over **3 channel uses**.

$$\text{DOF} = \frac{\text{Number of Independent Symbols}}{\text{Number of Channel Uses}} = \boxed{\frac{4}{3}}.$$

2. Rate (in bpcu):

- Each symbol is from a 64-QAM constellation, which carries $\log_2(64) = 6$ bits per symbol.
- Since 4 symbols are transmitted over 3 channel uses, the rate R is:

$$R = \frac{\text{Total Bits Transmitted}}{\text{Number of Channel Uses}} = \frac{4}{3} \cdot 6 = \boxed{8 \text{ bpcu}}.$$

7) (1 point). Imagine a SISO channel model with correlated fading, where the first fading coefficient (first transmission slot) is $h_1 = h_1' \times h_2'$, and the second fading coefficient (second transmission slot) is $h_2 = h_2'$, where $h_1', h_2' \sim i.i.d\ \mathbb{C}\mathcal{N}(0, 1)$. What is the maximum diversity we can achieve here?

Maximum Diversity:

- Fading coefficients: $h_1 = h_1' \cdot h_2'$ and $h_2 = h_2'$.
- Independent components: h_1' and h_2' ($\mathbb{C}\mathcal{N}(0, 1)$, i.i.d.).
- **Diversity order** = Number of independent fading coefficients = $\boxed{2}$.

8) (1 point). Describe the steps of converting a binary vector detection problem over a time diversity fading channel, into a scalar detection problem. Imagine that you are sending BPSK symbols using a repetition code, and consider $\mathbb{C}\mathcal{N}(0, N_0)$ noise.

The correct process for converting the **binary vector detection problem** over a time diversity fading channel into a **scalar detection problem** is as follows:

Given:

- The **transmitted symbol** $x \in \{-1, +1\}$ (BPSK) is repeated N times using a repetition code.
- **Received signal** over N time slots: $y_i = h_i x + n_i, \quad i = 1, 2, \ldots, N$, where:
  - $h_i \sim \mathcal{CN}(0, 1)$ (time-varying fading coefficient),
  - $n_i \sim \mathcal{CN}(0, N_0)$ (AWGN noise).

Steps to Convert to Scalar Detection Problem:

1. **Combine All Observations**: To exploit the time diversity, combine the received signals from all N time slots into a single metric: $z = \sum_{i=1}^{N} h_i^* y_i$, where h_i^* is the conjugate of h_i. This is called **maximum ratio combining (MRC)**.

2. **Simplify the Combined Signal**: Substitute $y_i = h_i x + n_i$ into z: $z = \sum_{i=1}^{N} h_i^*(h_i x + n_i) = x \sum_{i=1}^{N} |h_i|^2 + \sum_{i=1}^{N} h_i^* n_i$.

3. **Interpret the Result**:
   - The first term, $x \sum_{i=1}^{N} |h_i|^2$, represents the signal component scaled by the channel gains.
   - The second term, $\sum_{i=1}^{N} h_i^* n_i$, is the noise term, which remains Gaussian with variance $N_0 \sum_{i=1}^{N} |h_i|^2$.

4. **Decision Rule**: The scalar detection problem is now: Decide $x = +1$ if $z > 0, \quad$ otherwise decide $x = -1$. This simplifies detection by collapsing the vector problem into a single scalar comparison.

Answer: By using **maximum ratio combining (MRC)**, the original binary vector detection problem is converted into a **scalar detection problem**: $z = x \sum_{i=1}^{N} |h_i|^2 + \sum_{i=1}^{N} h_i^* n_i$, with the decision rule: Decide $x = +1$ if $z > 0, \quad$ otherwise decide $x = -1$.

9) (1 point). Consider a deep-space communications scenario, where the received SNR is equal to 20dB. If you assume low rate communications, what do you expect the probability of error to be?

For a **deep-space communication scenario** with a received SNR of 20 dB, assuming **low-rate communication**, we can approximate the **probability of error** as follows:

1. Key Assumptions:

- **Low-Rate Communication**: Typically corresponds to Binary Phase Shift Keying (BPSK).
- **SNR in Linear Scale**: Convert 20 dB to linear scale: $\text{SNR}_{\text{linear}} = 10^{\frac{\text{SNR}_{\text{dB}}}{10}} = 10^{\frac{20}{10}} = 100$.

2. Probability of Error for BPSK: For BPSK in an AWGN channel, the probability of error (P_e) is given by: $P_e = Q\left(\sqrt{2 \cdot \text{SNR}_{\text{linear}}}\right)$, where $Q(x)$ is the tail probability of a standard normal distribution.

Substitute $\text{SNR}_{\text{linear}} = 100$: $P_e = Q\left(\sqrt{2 \cdot 100}\right) = Q(14.14)$.

3. Approximation of $Q(x)$ for Large x: For large x, $Q(x)$ can be approximated as: $Q(x) \approx \frac{1}{\sqrt{2\pi}x} e^{-\frac{x^2}{2}}$.

Substitute $x = 14.14$: $Q(14.14) \approx \frac{1}{\sqrt{2\pi} \cdot 14.14} e^{-\frac{14.14^2}{2}}$.

Compute: $\frac{14.14^2}{2} = 100, \quad e^{-100} \approx 3.72 \times 10^{-44}$.

Thus: $P_e \approx \frac{1}{\sqrt{2\pi} \cdot 14.14} \cdot 3.72 \times 10^{-44} \approx 1 \times 10^{-45}$.

The **probability of error** is extremely small: $\boxed{P_e \approx 10^{-45}}$.

In deep-space communication with high SNR and low rate, errors are nearly negligible.

10) (1 point). What is the approximate coherence time T_c in a typical urban wireless network if you are driving approximately 20 kilometers per hour?

To estimate the **coherence time** T_c in a typical urban wireless network, we use the following formula: $T_c \approx \frac{1}{f_d}$, where f_d is the **Doppler spread** given by $f_d = \frac{v}{\lambda} = \frac{v \cdot f_c}{c}$.

1. Given Parameters:

- Speed: $v = 20\,\text{km/h} = \frac{20 \times 1000}{3600} = 5.56\,\text{m/s}$,
- Carrier frequency: $f_c = 2\,\text{GHz} = 2 \times 10^9\,\text{Hz}$ (assumed typical urban value),
- Speed of light: $c = 3 \times 10^8\,\text{m/s}$.

2. Doppler Spread: $f_d = \frac{v \cdot f_c}{c} = \frac{5.56 \cdot 2 \times 10^9}{3 \times 10^8} = 37.1\,\text{Hz}$.

3. Coherence Time: $T_c \approx \frac{1}{f_d} = \frac{1}{37.1} \approx 0.027\,\text{seconds} = 27\,\text{ms}$.

The approximate coherence time is: $\boxed{27\,\text{ms}}$.

11) (1 point). Consider communication over a SISO fading channel with a delay spread of $T_d = 3\mu s$ and a signal bandwidth of $W = 1$ MHz. - Write all the received signals, if we only send $x[0]$ and then we stop transmitting.

To analyze this scenario, we need to consider the **SISO fading channel** with a **delay spread** $T_d = 3\,\mu s$ and a signal bandwidth $W = 1\,\text{MHz}$. The delay spread indicates the multipath environment, meaning the transmitted signal will arrive at the receiver through multiple delayed and scaled copies.

1. Transmitted Signal:

- Only $x[0]$ is transmitted, then the transmission stops. Thus: $x[n] = \begin{cases} x[0], & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$

2. Received Signal: The received signal is the convolution of the transmitted signal $x[n]$ with the channel impulse response $h(t)$: $y[n] = h[n] * x[n]$.

- The **channel impulse response** $h(t)$ is a sum of L multipath components: $h(t) = \sum_{l=0}^{L-1} h_l \delta(t - \tau_l)$, where:
  - h_l: Fading coefficient for the l-th path ($h_l \sim \mathcal{CN}(0, 1)$),
  - τ_l: Delay of the l-th path ($0 \le \tau_l \le T_d$).
- With $T_d = 3\,\mu s$, the maximum delay is $3\,\mu s$, corresponding to $L \approx W \cdot T_d = 1\,\text{MHz} \cdot 3\,\mu s = 3$ significant paths.

3. Writing the Received Signals: For $x[0]$ transmitted: - The received signal $y[n]$ consists of L delayed copies of $x[0]$, weighted by the fading coefficients h_l: $y[0] = h_0 x[0]$, $y[1] = h_1 x[0]$, $y[2] = h_2 x[0]$. - For $n > 2$, no further contributions occur, as $\tau_l \le T_d$.

Thus: $y[n] = \begin{cases} h_0 x[0], & n = 0, \\ h_1 x[0], & n = 1, \\ h_2 x[0], & n = 2, \\ 0, & n > 2. \end{cases}$

Final Answer: The received signals are: $y[0] = h_0 x[0], \quad y[1] = h_1 x[0], \quad y[2] = h_2 x[0], \quad y[n] = 0 \text{ for } n > 2.$

12) ***(2 points)***. What is the optimal diversity order over a 2×1 MISO channel $h = [h_1\ h_2], h_i \sim$ $i.i.d\ \mathbb{C}\mathcal{N}(0, 1)$? - In the same channel as above (again with no time diversity), consider a space time code whose matrices take the form

$$\begin{bmatrix} x_0 & x_1 \\ x_1 & x_0 \end{bmatrix}$$

where thex_i are drawn independently from a QAM constellation. Will this code achieve optimal diversity order? (argue why or why not) - What is the diversity order achieved by the Alamouti code, over this 2×1 MISO channel? (again, you can just argue in words)

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1. Optimal Diversity Order in a 2×1 MISO Channel In a 2×1 MISO channel, the **diversity order** is equal to the number of independent fading paths, which corresponds to the number of transmit antennas ($N_t = 2$) when there is 1 receive antenna.
Thus, the **optimal diversity order** is: $\boxed{2}$.

2. Diversity Order of the Given Space-Time Code The given code matrix is: $\mathbf{X} = \begin{bmatrix} x_0 & x_1 \\ x_1 & x_0 \end{bmatrix}$, where x_0 and x_1 are independent QAM symbols.

Key Analysis:

- **Rank Criterion**: For a space-time code to achieve full diversity, the difference between any two distinct code matrices $\mathbf{X}_1$ and $\mathbf{X}_2$ must result in a matrix of full rank.
- For this code: $\Delta \mathbf{X} = \mathbf{X}_1 - \mathbf{X}_2 = \begin{bmatrix} x_{01} - x_{02} & x_{11} - x_{12} \\ x_{11} - x_{12} & x_{01} - x_{02} \end{bmatrix}$.

– The rows of $\Delta\mathbf{X}$ are **linearly dependent** because the two rows are identical. This means $\Delta\mathbf{X}$ is **not full rank**.

Conclusion:

This code does **not achieve the optimal diversity order**, as it does not satisfy the rank criterion for full diversity.

3. Diversity Order of the Alamouti Code The Alamouti code for a 2×1 MISO channel is: $\mathbf{X}_{\text{Alamouti}} = \begin{bmatrix} x_0 & -x_1^* \\ x_1 & x_0^* \end{bmatrix}$.

Key Features:

- The Alamouti code satisfies the **rank criterion**, ensuring that $\Delta\mathbf{X} = \mathbf{X}_1 - \mathbf{X}_2$ is always full rank for distinct codewords $\mathbf{X}_1$ and $\mathbf{X}_2$.
- Each transmitted symbol experiences the full diversity of the channel, as it leverages both transmit antennas.

Conclusion:

The Alamouti code achieves the **optimal diversity order of 2** over the 2×1 MISO channel.

Final Answers:

1. Optimal diversity order in 2×1 MISO: $\boxed{2}$.
2. Given space-time code: **Does not achieve optimal diversity order** due to lack of full-rank property.
3. Alamouti code: **Achieves optimal diversity order of** $\boxed{2}$.

13) ***(EXTRA CREDIT: 2 points)***. Consider a setting where the transmit antenna array has length of 50 cm, the received antenna array has size 20cm, the transmission frequency is 1000 MHz, the signal bandwidth is 1 MHz, the channel coherence time is $T_c = 21$ ms, and the coding duration is $T_{coding} = 7$ms. - How much diversity can you get, in total?

To calculate the **total diversity**, we need to consider **spatial diversity**, **frequency diversity**, and **time diversity**. Let's analyze each component based on the given parameters:

1. Spatial Diversity Spatial diversity depends on the number of independent spatial paths between the transmit and receive antenna arrays, calculated using the formula: $D_{\text{spatial}} = \left\lfloor \frac{2L_t}{\lambda} \right\rfloor \cdot \left\lfloor \frac{2L_r}{\lambda} \right\rfloor$, where: - $L_t = 50\,\text{cm} = 0.5\,\text{m}$: Transmit antenna array length, - $L_r = 20\,\text{cm} = 0.2\,\text{m}$: Receive antenna array length, - $\lambda = \frac{c}{f}$: Wavelength of the transmitted signal.

At $f = 1000\,\text{MHz}$ ($f = 1\,\text{GHz}$): $\lambda = \frac{3\times10^8}{1\times10^9} = 0.3\,\text{m}$.

Substituting: $D_{\text{spatial}} = \left\lfloor \frac{2\cdot0.5}{0.3} \right\rfloor \cdot \left\lfloor \frac{2\cdot0.2}{0.3} \right\rfloor = \lfloor 3.33 \rfloor \cdot \lfloor 1.33 \rfloor = 3 \cdot 1 = 3$.

2. Frequency Diversity Frequency diversity depends on the relationship between the signal bandwidth (W) and the coherence bandwidth (W_c). The coherence bandwidth is inversely related to the channel delay spread (T_d): $W_c \approx \frac{1}{T_d}$.

Assuming the coherence bandwidth W_c is smaller than the signal bandwidth $W = 1\,\text{MHz}$, the **frequency diversity** is approximately: $D_{\text{frequency}} = \frac{W}{W_c}$. Without explicit T_d, assume $W_c \approx 100\,\text{kHz}$: $D_{\text{frequency}} \approx \frac{1\,\text{MHz}}{100\,\text{kHz}} = 10$.

3. Time Diversity Time diversity depends on the relationship between the **coding duration** ($T_{\textbf{coding}}$) and the **channel coherence time** (T_c): $D_{\text{time}} = \frac{T_c}{T_{\text{coding}}}$.

Given: $T_c = 21\,\text{ms}$, $T_{\text{coding}} = 7\,\text{ms}$, $D_{\text{time}} = \frac{21}{7} = 3$.

Total Diversity The total diversity is the product of the individual diversity components: $D_{\text{total}} = D_{\text{spatial}} \cdot D_{\text{frequency}} \cdot D_{\text{time}}$.

$\boxed{D_{\text{total}} = 3 \cdot 10 \cdot 3 = 90}$

Here's the breakdown of the **units** for each diversity component:

1. **Spatial Diversity ($D_{\textbf{spatial}}$)**:
    - **Unitless**: It is a count of independent spatial paths, determined by the number of transmit and receive antenna elements relative to the wavelength.
2. **Frequency Diversity ($D_{\textbf{frequency}}$)**:
    - **Unitless**: It represents the ratio of signal bandwidth (W) to coherence bandwidth (W_c), which both have units of Hz, making the ratio unitless.
3. **Time Diversity ($D_{\textbf{time}}$)**:
    - **Unitless**: It is the ratio of channel coherence time (T_c) to coding duration (T_{coding}), both measured in seconds, so the ratio is unitless.

Summary: All diversity components—spatial, frequency, and time—are **unitless** because they are ratios or counts, and the **total diversity** is also unitless.

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